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(a 5.10

$$\text{minimize} \quad \log\det(X^{-1})$$

$$\text{s.t.} \quad X = \sum_{i=1}^{p} x_i v_i v_i^T$$

$$x \succeq 0, \quad 1^T x = 1$$

$$L(x, Z, \lambda, w) = \log\det(X^{-1}) + \operatorname{tr}\left(Z\left(X - \sum_{i=1}^{p} x_i v_i v_i^T\right)\right) - \lambda^T x + w(1^T x - 1)$$

$$= \log\det(X^{-1}) + \operatorname{tr}(ZX) - \sum_{i=1}^{p} x_i \underbrace{\operatorname{tr}(Z v_i v_i^T)}_{\operatorname{tr}(AB) = \operatorname{tr}(BA)} - \lambda^T x + w(1^T x - 1)$$

$$= \log\det(X^{-1}) + \operatorname{tr}(ZX) + \sum_{i=1}^{p} x_i(-v_i^T Z v_i - \lambda_i + w) - w$$

if $w = v_i^T Z v_i + \lambda_i \rightsquigarrow \nabla_X L = 0 \Rightarrow -X^{-T} + Z^T = 0 \qquad Z = X^{-1}$

$$\Rightarrow g(Z, w) = \log\det(Z) + n - w$$

if $w \neq v_i^T Z v_i + \lambda_i \Rightarrow g(Z, w) = -\infty$

$$\Rightarrow \quad \text{maximize} \quad \log\det(Z) + n - w \qquad\qquad \text{maximize} \quad \log\det(wY) + n - w$$

slack variable $\leftarrow \lambda_i$

$$\text{s.t.} \quad w \geq v_i^T Z v_i \quad i = 1, \dots, p \Rightarrow \qquad \text{s.t.} \quad v_i^T Y v_i \leq 1 \quad i = 1, \dots, p$$

$$Z \succeq 0 \qquad\qquad wY \succeq 0$$

$$\Rightarrow \quad \text{maximize} \quad \log\det(Y) + \underbrace{n\log w + n - w}_{\leq n\log n} \rightsquigarrow \log\det(Y) + n\log n$$

$$\text{s.t.} \quad v_i^T Y v_i \leq 1$$

$$wY \succeq 0$$

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minimize $\operatorname{tr}(X^{-1})$

s.t. $X = \sum_{i=1}^{p} x_i v_i v_i^T$

$x \succcurlyeq 0 , \quad 1^T x = 1$

$$L(x,Z,\lambda,w) = \operatorname{tr}(X^{-1}) + \operatorname{tr}(ZX) - \sum_{i=1}^{p} x_i v_i^T Z v_i - \lambda^T x + w(1^T x - 1)$$

$$= \operatorname{tr}(X^{-1} + ZX) + \sum_{i=1}^{p} x_i (-v_i^T Z v_i - \lambda_i + w) - w$$

$$g(Z,\lambda,w) = \inf_x L = \begin{cases} 2\operatorname{tr}(Z^{\frac{1}{2}}) - w \;* & w = v_i^T Z v_i + \lambda , \; Z \succcurlyeq 0 \\ -\infty & \text{o.w.} \end{cases}$$

$$\nabla_X L = 0 \rightsquigarrow Z^T + X^{-1} X^{-T} = (X^{-T})^2 + Z^T = 0 \qquad X = Z^{-\frac{1}{2}} \quad *$$

$$L \rightsquigarrow 2\operatorname{tr}(Z^{\frac{1}{2}}) - w$$

$\Rightarrow$ maximize $2\operatorname{tr}(Z^{\frac{1}{2}}) - w$

s.t. $v_i^T Z v_i \succcurlyeq w \quad i=1,\dots,p$

$Z \succcurlyeq 0$

$\Rightarrow$ maximize $2\operatorname{tr}(Y^{\frac{1}{2}} w^{\frac{1}{2}}) - w$

s.t. $v_i^T Y v_i \succcurlyeq 1 \quad i=1,\dots,p$

$Y \succcurlyeq 0$

$\nabla_w \text{obj} = 0 \rightsquigarrow w = (\operatorname{tr}(Y^{\frac{1}{2}}))^2$

$\Rightarrow$ maximize $(\operatorname{tr}(Y^{\frac{1}{2}}))^2$

s.t. $v_i^T Y v_i \succcurlyeq 1 \quad i=1,\dots,p$

$Y \succcurlyeq 0$

5.40

minimize $\frac{1}{t}$

s.t. $\sum_{i=1}^{p} x_i v_i v_i^T \succeq tI$ (λ_{min})

$x \succeq 0$, $1^T x = 1$

$L(x, t, Z, \lambda, w) = \frac{1}{t} + tr(tZ) - \sum x_i v_i^T Z v_i - \lambda^T x + w(1^T x - 1)$

$= \frac{1}{t} + tr(tZ) + \sum_{i=1}^{p} x_i (-v_i^T Z v_i - \lambda_i + w) - w$

$$g(Z, \lambda, w) = \inf L = \begin{cases} 2\sqrt{tr(Z)} - w & * \quad w = v_i^T Z v_i + \lambda_i \\ -\infty & \text{o.w.} \end{cases}$$

$\inf \frac{1}{t} + tr(tZ) \rightsquigarrow \frac{-1}{t^2} + tr(Z) = 0 \quad t = \frac{1}{\sqrt{tr Z}}$ *

$\Rightarrow = tr(Z)^{\frac{1}{2}} + tr(Z)^{\frac{1}{2}} = 2\sqrt{tr(Z)}$

$\Rightarrow$ maximize $2\sqrt{tr(Z)} - w$ $\qquad$ maximize $2\sqrt{tr(wy)} - w$

s.t. $v_i^T Z v_i \leq w \quad i = 1, \dots, p \rightsquigarrow$ $\qquad$ s.t. $v_i^T y v_i \leq 1$

$Z \succeq 0$ $\qquad$ $Z \succeq 0$

$\nabla_w 2\sqrt{w}\sqrt{tr(y)} - w = 0 \rightsquigarrow w^{-\frac{1}{2}}\sqrt{tr(y)} - 1 = 0 \quad w = tr(y)$

$\Rightarrow$ maximize $tr(y)$

s.t. $v_i^T y v_i \leq 1$

$Z \succeq 0$

5.12 (2)

$$\text{minimize} \quad -\sum_{i=1}^{m} \log y_i$$

$$\text{s.t.} \quad y = b - Ax$$

$y \succ 0 \rightsquigarrow$ log شرط $\rightarrow$ صرف نظر

$$L(x, y, \nu) = -\sum_{i=1}^{m} \log y_i + \nu^T (y - b + Ax)$$

$$g(\nu) = \begin{cases} \sum_{i=1}^{m} \log \nu_i + m - \nu^T b^{*} & \nu^T A = 0,\ \nu \succ 0 \\ -\infty & \text{o.w.} \end{cases}$$

$\nu^T A \neq 0 \quad x \to$ unbounded $\qquad \nu \succ 0 \Rightarrow -\dfrac{1}{y_i} + \nu_i = 0 \quad i = 1, \dots, m$ ☆

$$\Rightarrow \nu_i = \frac{1}{y_i} \qquad \nu^T y = m$$

$$\Rightarrow \text{maximize} \quad \sum_{i=1}^{m} \log \nu_i + m - \nu^T b$$

$$\text{s.t.} \quad \nu^T A = 0$$

5.14 (3)

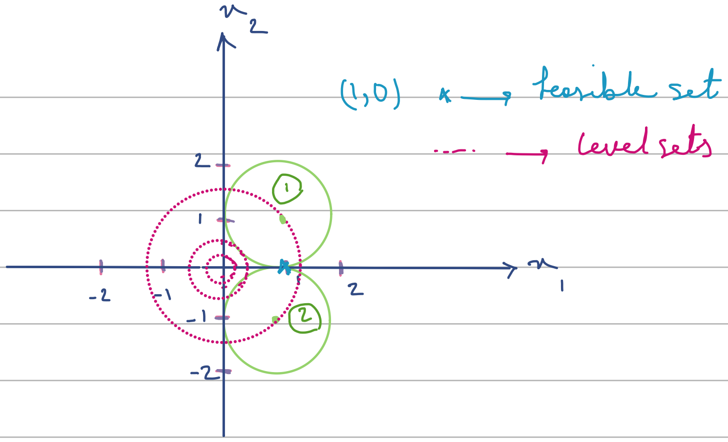
$f_0 \rightsquigarrow$ convex & differentiable $\Rightarrow \phi(x) = f_0(x) + \alpha \|Ax - b\|_2^2 \rightsquigarrow$ convex & differentiable $\Rightarrow \tilde{x} \to$ solution : $\nabla_x \phi(\tilde{x}) = \nabla_x f_0(\tilde{x}) + 2A^T \alpha (Ax - b) = 0$

$L(x, \nu) = f_0(x) + \nu^T (Ax - b) \longrightarrow g(\nu) = \inf \left(f_0(x) + \nu^T (Ax - b) \right) \leq f_0(x) \quad \forall \nu, x$

$\Rightarrow g(\alpha (A\tilde{x} - b)) = \inf \left(f_0(x) + \alpha (A\tilde{x} - b)^T (Ax - b) \right) = \phi(\tilde{x}) \leq f_0(x)$

for all x in feasible set

(a $\overset{4}{5.26}$



(b

$(x_1-1)^2+(x_2-1)^2 \leq 1$ $\qquad (x_1-1)^2+(x_2+1)^2 \leq 1$ $\qquad \lambda_1, \lambda_2 \geq 0$

$\lambda_1((x_1-1)^2+(x_2-1)^2-1)=0$ $\qquad \lambda_2((x_1-1)^2+(x_2+1)^2-1)=0$

$2x_1+2\lambda_1(x_1-1)+2\lambda_2(x_1-1)=0$ $\qquad 2x_2+2\lambda_1(x_2-1)+2\lambda_2(x_2+1)=0$

$\Rightarrow x_1=1 \quad x_2=0 \quad \underbrace{2=0}_{\times} \quad \lambda_1=\lambda_2 \quad \lambda_1,\lambda_2 \geq 0 \Rightarrow$ جواب ندارد

(c

$L(x_1,x_2,\lambda_1,\lambda_2)=x_1^2+x_2^2+\lambda_1((x_1-1)^2+(x_2-1)^2-1)+\lambda_2((x_1-1)^2+(x_2+1)^2-1)$

$g(\lambda_1,\lambda_2)=\inf_x L(x_1,x_2,\lambda_1,\lambda_2)$ $\qquad \nabla L=0 \Rightarrow (1+\lambda_1+\lambda_2)x_1=\lambda_1+\lambda_2$

$, \ (1+\lambda_1+\lambda_2)x_2=\lambda_1-\lambda_2 \Rightarrow x_1=\dfrac{\lambda_1+\lambda_2}{1+\lambda_1+\lambda_2} \qquad x_2=\dfrac{\lambda_1-\lambda_2}{1+\lambda_1+\lambda_2}$

$$g(\lambda_1,\lambda_2)=\begin{cases} \lambda_1+\lambda_2-\dfrac{(\lambda_1+\lambda_2)^2+(\lambda_1-\lambda_2)^2}{1+\lambda_1+\lambda_2} & 1+\lambda_1+\lambda_2>0 \\ -\infty & \text{o.w.} \end{cases}$$

$\Rightarrow$ maximize $\lambda_1+\lambda_2-\dfrac{(\lambda_1+\lambda_2)^2+(\lambda_1-\lambda_2)^2}{1+\lambda_1+\lambda_2}$

s.t. $1+\lambda_1+\lambda_2>0$

$\Rightarrow \nabla_\lambda g(\lambda_1,\lambda_2)=0 \xrightarrow{\text{Symmetric}} \lambda_1=\lambda_2 \rightsquigarrow g(\lambda_1,\lambda_2)=2\lambda_1-\dfrac{4\lambda_1^2}{1+2\lambda_1}=\dfrac{2\lambda_1}{1+2\lambda_1}$

$\lambda_1 \to \infty : g(\lambda_1,\lambda_2)=1$ $\qquad p^*=d^* \rightsquigarrow$ not attained $\rightsquigarrow$ KKT not holds

KKT $\longrightarrow$ $\quad Xs - y = 0 \qquad X \succ 0 \qquad I - X^{-1} + \frac{1}{2}(vs^T + sv^T)^{*} = 0$ $\quad$ 5.30

$$v^T(Xs - y) = \underbrace{v^TXs}_{\text{scalar}} - v^Ty = tr(v^TXs) - v^Ty \rightsquigarrow tr(v^TXs) = tr(sv^TX) = tr(Xsv^T) \; *$$

$$= tr(X^Tvs^T) = tr(Xvs^T) = \frac{1}{2} tr(X(\underbrace{vs^T + sv^T}_{\in S^n})) \quad \Rightarrow \nabla v^T(Xs - y) =$$

$$\frac{1}{2}\nabla tr((vs^T + sv^T)X) = \frac{1}{2}(vs^T + sv^T) \quad \checkmark$$

$$X^{-1} = I + \frac{1}{2}(vs^T + sv^T) \qquad {}^{*}s = (I + \frac{1}{2}(sv^T + vs^T))y = y + \frac{1}{2}sv^Ty + \frac{1}{2}v$$

$$y^Ts = 1 = y^Ty + \frac{1}{2}v^Ty + \frac{1}{2}y^Tv = y^Ty + y^Tv \qquad y^Tv = 1 - \|y\|_2^2$$

$$* \; s = y + \frac{1}{2}s(1 - \|y\|_2^2) + \frac{1}{2}v \qquad v = -2y + (\|y\|_2^2 + 1)s$$

$$X^{-1} = I + \frac{1}{2}(-2ys^T + (\|y\|_2^2 + 1)ss^T - 2sy^T + (\|y\|_2^2 + 1)ss^T)$$

$$= I + (\|y\|_2^2 + 1)ss^T - ys^T - sy^T$$

$$X^{-1}X = I = X + (\|y\|_2^2 + 1)ss^TX - ys^TX - sy^TX \rightsquigarrow X = I - \frac{ss^T}{\|s\|_2^2} + yy^T$$

$$= \left\| I + \frac{sy^T}{\|s\|_2} - \frac{ss^T}{\|s\|_2^2} \right\|_2^2 \succeq 0$$

$$L(v, Z) = 1^Tv - tr(Z(W + diag(v))) = tr(WZ) + \sum_{i=1}^{n} v_i(1 - Z_{ii})$$ (a) 5.39

$$g(Z) = \begin{cases} -tr(WZ) & Z_{ii} = 1 \\ -\infty & \text{o.w.} \end{cases}$$

$\Rightarrow$ dual dual $\rightarrow$ maximize $-tr(WZ)$ $\rightsquigarrow$ minimize $tr(WZ)$

s.t. $\quad Z_{ii} = 1 \quad i = 1, \ldots, n$

$Z \succeq 0$ $\qquad \Rightarrow$ اثبات lower bound بودن

using hint: if $X \succeq 0$ and $\text{rank}(X) = 1 \Rightarrow X = xx^T$

$$X_{ii} = 1 \Rightarrow x_i^2 = 1 \qquad x_i = \pm 1$$

$$\text{tr}(WX) = \text{tr}(Wxx^T) = \text{tr}(x^T W x) = x^T W x \quad \checkmark$$

$$\text{if} \quad x_i \in \{-1, 1\} \Rightarrow x^T W x = \text{tr}(x^T W x) = \text{tr}(W x x^T)$$

$$X_{ii} = (xx^T)_{ii} = x_i^2 = 1$$

$$X = xx^T \Rightarrow \text{rank} X = 1, \quad X \succeq 0 \quad \checkmark$$

b) همانطور که در بخش قبل نشان دادیم این مسئله، dual مسئله dual ما است پس یک lower bound برای مسئله اصلی است. همچنین می‌توان گفت اگر معادله مثبت a را relax کنیم به یک جواب کوچکتر مساوی آن می‌رسیم

چون هر متغیری که در a، feasible باشد در b هم هست پس b یک lower bound است.

c) در بخش a ثابت کردم:

$$L(v, Z) = 1^T v - \text{tr}(Z(W + \text{diag}(v))) = \text{tr}(WZ) + \sum_{i=1}^{n} v_i (1 - Z_{ii})$$

$$g(Z) = \begin{cases} -\text{tr}(WZ) & Z_{ii} = 1 \\ -\infty & \text{o.w.} \end{cases}$$

$$\Rightarrow \text{dual dual} \rightarrow \text{maximize } -\text{tr}(WZ) \quad \rightsquigarrow \quad \text{minimize } \text{tr}(WZ)$$

$$\text{s.t.} \quad Z_{ii} = 1 \quad i = 1, \dots, n$$

$$Z \succeq 0$$

KKT $\longrightarrow$ $-y \preceq 0$ $\quad \beta^T y = 0$ $\quad \lambda \succeq 0$ $\quad -\lambda y = 0$

$$\alpha - \frac{|\gamma|}{\sqrt{y}} - \lambda + \nu\beta = 0$$

$$\alpha_i - \frac{|\gamma|_i}{\sqrt{y_i}} - \lambda_i + \nu\beta_i = 0 \longrightarrow \sqrt{y_i}\,\alpha_i - |\gamma|_i - \sqrt{y_i}\,\lambda_i + \sqrt{y_i}\,\nu\beta_i = 0$$

$\sqrt{y_i}\,\lambda_i = 0$ $\qquad$ if $y_i = 0$: $\alpha_i\sqrt{y_i} - |\gamma|_i + \sqrt{y_i}\,\nu\beta_i = 0$ $\qquad |\gamma|_i = 0$

if $y_i > 0$: $\alpha_i\sqrt{y_i} - |\gamma|_i + \sqrt{y_i}\,\nu\beta_i = 0$ $\qquad |\gamma|_i = \alpha_i\sqrt{y_i} + \sqrt{y_i}\,\nu\beta_i$

$$\gamma_i^2 = \alpha_i^2 y_i + y_i\nu^2\beta_i^2 + 2\alpha_i y_i \nu\beta_i \qquad y_i = \frac{\gamma_i^2}{\alpha_i^2 + \nu^2\beta_i^2 + 2\alpha_i\nu\beta_i} = \frac{\gamma_i^2}{(\alpha_i + \nu\beta_i)^2}$$

$$\beta^T y = \sum_{i\in C} \beta_i y_i = \gamma_i^2 \beta_i \times (\alpha_i + \nu\beta_i)^{-2} = 0 \xrightarrow{\times \prod_{j\in C}(\alpha_i + \nu\beta_i)} = \sum_{i\in C} \gamma_i^2 \beta_i \prod_{\substack{j\in C \\ i\neq j}} (\alpha_j + \nu\beta_j)^2 = 0$$

$$\Rightarrow \lambda = -\nu \Rightarrow \sum_{i\in C} \gamma_i^2 \beta_i \prod_{\substack{j\in C \\ i\neq j}} (\alpha_j - \lambda\beta_j)^2 = P(\lambda) = 0 \Rightarrow$$ [illegible]

$C = \{1, 2, \dots, n\}$