400104264 تمرین سوم آیلین رستمی

3.1 a

طبق تعریف داریم:

$$P = \mathrm{conv}(v_1, v_2, \dots, v_k) = \{\theta_1 v_1 + \dots + \theta_k v_k \mid \theta \succeq 0,\ 1^T\theta = 1\}$$

$$\forall x \in P \Rightarrow x = \theta_1 v_1 + \dots + \theta_k v_k \qquad f \to \text{convex}$$

$$\Rightarrow f(x) = f(\theta_1 v_1 + \dots + \theta_k v_k) \leq \theta_1 f(v_1) + \dots + \theta_k f(v_k)$$ ⟿ نابرابری جنسن

$$\sup_{x\in P} f(x) \leq \sup_{\theta}(\theta_1 f(v_1) + \theta_2 f(v_2) + \dots + \theta_k f(v_k)) \leq \max_{1\leq i\leq k} f(v_i) = f(v_j)$$

از آنجایی که $f(v_j) \leq \sup f(x)$ و از طرفی در بالا اثبات کردیم $f(v_j) \geq \sup f(x)$ در نتیجه:

$$f(v_j) = \max_{1\leq i\leq k} f(v_i) = \sup_{x\in P} f(x) \quad \checkmark$$

حالت خطی آن با برهان خلف:

$$\exists\, z \in S :\ f(z) > f(v_i) \quad i = 1, \dots, k \qquad 1^T\lambda = 1,\ \lambda \succeq 0$$

نابرابری جنسن

$$\Rightarrow f(z) = f(\lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_n v_n) \leq \lambda_1 f(v_1) + \dots + \lambda_k f(v_k) < f(z)\ \times$$

3.17 1-2

$$\frac{\partial f(x)}{\partial x_i} = p x_i^{p-1} \times \frac{1}{p}\left(\sum_{i=1}^n x_i^p\right)^{\frac{1}{p}-1} = \left(\frac{(\sum_{i=1}^n x_i^p)^{\frac{1}{p}}}{x_i}\right)^{1-p} = \left(\frac{f(x)}{x_i}\right)^{1-p}$$

$$\frac{\partial^2 f(x)}{\partial x_i^2} = (1-p)\left(\frac{f(x)}{x_i}\right)^{-p}\left(\frac{(\frac{f(x)}{x_i})^{1-p} x_i - f(x)}{x_i^2}\right) = \frac{1-p}{f(x)}\left(\left(\frac{f(x)^2}{x_i^2}\right)^{1-p} - \left(\frac{f(x)}{x_i}\right)^{2-p}\right)$$

$$\frac{\partial^2 f(x)}{\partial x_i \partial x_j} = (1-p)\left(\frac{f(x)}{x_i}\right)^{-p} \times \frac{1}{x_i}\left(\frac{f(x)}{x_j}\right)^{1-p} = \frac{1-p}{f(x)}\left(\frac{f(x)^2}{x_i x_j}\right)^{1-p}$$

$$h_{ij} = \frac{\partial^2 f(x)}{\partial x_i \partial x_j}$$

باید ثابت کنیم $H \preceq 0$ یعنی $\forall v \in R^n : v^T H v$. داریم:

$$\forall v \in R^n :\ v^T \nabla^2 f(x) v = \sum_{i=1}^n v_i^2 h_{ii} + 2\sum_{i<j} v_i v_j h_{ij}$$

$$= \frac{1-p}{f(x)}\left(\sum_{i=1}^n \sum_{j=1}^n v_i v_j \left(\frac{f(x)^2}{x_i x_j}\right)^{1-p} - \sum_{i=1}^n v_i^2 \left(\frac{f(x)}{x_i}\right)^{2-p}\right) \quad *$$

$$a_i = v_i\left(\frac{f(x)}{x_i}\right)^{1-\frac{p}{2}} \qquad b_i = \left(\frac{f(x)}{x_i}\right)^{-\frac{p}{2}} \qquad \|b\|_2 = \frac{\sum x^p}{\sum x^p} = 1$$

$$* \Rightarrow \frac{1-p}{f(x)}\left((a^T b)^2 - \|a\|_2^2 \|b_2^2\|\right) \xrightarrow[\text{کوشی شوارز}]{1-p>0,\ f(x)>0} v^T H v \leq 0$$

(a 2-2 3.18

$$g(t)=f(Z+tV)=\operatorname{tr}\left((Z+tV)^{-1}\right)=\operatorname{tr}\left(\left(Z^{\frac12}(I+tZ^{-\frac12}VZ^{-\frac12})Z^{\frac12}\right)^{-1}\right)\qquad Z,\ Z+tV\in S^n_{++}$$

$$=\operatorname{tr}\left(Z^{-\frac12}(I+tZ^{-\frac12}VZ^{-\frac12})^{-1}Z^{-\frac12}\right)=\operatorname{tr}\left(Z^{-1}(I+tZ^{-\frac12}VZ^{-\frac12})^{-1}\right)\rightarrow Z^{-\frac12}VZ^{-\frac12}\in S^n$$

$$\Rightarrow Z^{-\frac12}VZ^{-\frac12}=Q^T\Lambda Q\quad\Rightarrow g(t)=\operatorname{tr}\left(QZ^{-1}Q^T(I+t\Lambda)^{-1}\right)=\sum_{i=1}^{n}\frac{(QZ^{-1}Q^T)_{ii}}{1+t\lambda_i}$$

$$\underset{t>0}{\frac{1}{t}}:\text{Convex}\ \xrightarrow[\lambda_i t+1>0]{\text{affine}}\ \frac{1}{\lambda_i t+1}:\text{Convex}\ \xrightarrow[q_i^TZ^{-1}q_i>0]{Z^{-1}\in S^n_{++}}\ \sum_{i=1}^{n}\frac{(QZ^{-1}Q^T)_{ii}}{1+t\lambda_i}:\text{Convex}\Rightarrow\operatorname{tr}(X^{-1}):\text{Convex}$$

(b

$$g(t)=f(Z+tV)=\det(Z+tV)^{\frac1n}=\det\left(Z^{\frac12}(I+tZ^{-\frac12}VZ^{-\frac12})Z^{\frac12}\right)^{\frac1n}\qquad Z,\ Z+tV\in S^n_{++}$$

$$=\det\left(Z(I+tZ^{-\frac12}VZ^{-\frac12})\right)^{\frac1n}=\det(Z)^{\frac1n}\times\det(I+tZ^{-\frac12}VZ^{-\frac12})^{\frac1n}$$

$$\det(I+tZ^{-\frac12}VZ^{-\frac12})=\prod\lambda_{i\,(I+tZ^{-\frac12}VZ^{-\frac12})}=\prod(1+t\lambda_i)\rightsquigarrow\lambda_i=\text{eigenvalue }Z^{-\frac12}VZ^{-\frac12}$$

$$\Rightarrow g(t)=\underbrace{\det(Z)^{\frac1n}}_{Z\in S^n_{++}\Rightarrow>0}\times\left(\prod(1+t\lambda_i)\right)^{\frac1n}$$

می‌دانیم میانگین هندسی یک تابع concave است : $\left(\prod_{i=1}^{n}t\right)^{\frac1n}:\text{Concave}\ \xrightarrow{\text{affine}}\left(\prod(1+t\lambda_i)\right)^{\frac1n}:\text{Concave}$

$$\Rightarrow g(t):\text{Concave}\ \Rightarrow\ \det(X)^{\frac1n}:\text{Concave}$$

(a 3 3.5 a

$$l(t)=tf\left(\frac{x}{t}\right)$$

می‌خواهیم نشان دهیم اگر $0<t_1\leqslant t_2$ آنگاه $l(t_1)\geqslant l(t_2)$. تعریف می‌کنیم

$$\frac{x}{t_2},\ \frac{x}{t_1}\in D_f,\ t_2>t_1>0\qquad f\left(\frac{x}{t_2}\right)\leqslant\frac{t_1}{t_2}f\left(\frac{x}{t_1}\right)+\left(1-\frac{t_1}{t_2}\right)\underbrace{f(0)}_{f(0)\leqslant0}\leqslant\frac{t_1}{t_2}f\left(\frac{x}{t_1}\right)$$

$$\Rightarrow t_2f\left(\frac{x}{t_2}\right)\leqslant t_1f\left(\frac{x}{t_1}\right)\Rightarrow l(t_2)\leqslant l(t_1)\ \checkmark$$

(b

$$h(x)=l(g(x))\qquad h(\theta x_1+(1-\theta)x_2)=l(g(\theta x_1+(1-\theta)x_2))\xrightarrow{g\text{ is Concave}}\qquad 0\leqslant\theta\leqslant1$$

$$\underbrace{g(\theta x_1+(1-\theta)x_2)}_{t_2}\geqslant\underbrace{\theta g(x_1)+(1-\theta)g(x_2)}_{t_1}>0$$

در بخش قبل نشان دادیم اگر $t_2\geqslant t_1>0$ باشد آنگاه $l(t_2)\leqslant l(t_1)$. درنتیجه:

$$h(\theta x_1+(1-\theta)x_2)\leqslant l(\theta g(x_1)+(1-\theta)g(x_2))\leqslant\theta l(g(x_1))+(1-\theta)l(g(x_2))=\theta h(x_1)+(1-\theta)h(x_2)$$

ℓ is Convex

c) $g(t) = (\prod_{k=1}^{n} x_k)^{\frac{1}{n}}$ $\quad h(x) = \ell(g(x)) = g(x)\, f\left(\frac{x}{g(x)}\right)$

$(\prod x_k)^{\frac{1}{n}} f\left(\frac{x}{(\prod x_k)^{\frac{1}{n}}}\right) = \frac{x^T x}{(\prod x_k)^{\frac{1}{n}}}$ $\quad f\left(\frac{x}{(\prod x_k)^{\frac{1}{n}}}\right) = \frac{x^T x}{(\prod x_k)^{\frac{2}{n}}}$

$\leadsto f(x) = x^T x$

3.26 a) با استفاده از راهنمایی

$\sum_{i=1}^{k} \lambda_i(x) = \sup \{ tr(V^T X V) \mid V \in R^{n\times k}, V^T V = I \}$

$tr(V^T X V)$ یک تابع خطی برحسب X است. pointwise sup یک تابع خطی، convex است.

b) طبق راهنمایی

$(\prod_{i=n-k+1}^{n} \lambda_i(X))^{\frac{1}{k}} = \frac{1}{k} \inf \{ tr(V^T X V) \mid V \in R^{n\times k}, \det V^T V = 1 \}$

$tr(V^T X V)$ یک تابع خطی برحسب X است. pointwise inf یک تابع خطی، concave است. *

اثبات *:

$\underset{\text{affine}}{\inf f(X)} = -\sup \overbrace{-f(X)}^{\text{affine}} = -\underset{\text{convex}}{g(x)} \longrightarrow \text{concave}$

c) طبق راهنمایی داریم:

$\prod_{i=n-k+1}^{n} \lambda_i(X) = \inf \{ \prod_{i=1}^{k} (V^T X V)_{ii} \mid V \in R^{n\times k}, V^T V = I \}$

$\sum_{i=n-k+1}^{n} \log \lambda_i(x) = \log \prod_{i=n-k+1}^{n} \lambda_i(x) = \log (\inf \prod_{i=1}^{k} (V^T X V)_{ii}) \xrightarrow[\text{است}]{\text{log صعودی}} \inf \{ \log \prod_{i=1}^{k} (V^T X V)_{ii} \}$

$= \inf \sum_{i=1}^{k} \log (V^T X V)_{ii}$ $\quad \log X : \text{concave} \xrightarrow{\text{خطی}} \log v_i^T X v_i : \text{concave} \xrightarrow{\Sigma} \sum \log v_i^T X v_i$

$: \text{concave} \xrightarrow{\text{pointwise inf}} \inf \sum_{i=1}^{k} \log v_i^T X v_i$ * $: \text{concave}$

اثبات *:

$g(x) \to \text{concave}$ $\quad \underset{\text{concave}}{\inf g(x)} = -\sup \overbrace{\underbrace{-g(x)}_{\text{convex}}}^{\text{convex}} \longrightarrow \text{concave}$

3.49 a)

$f(x) = \frac{e^x}{1+e^x}$ $\quad \log f(x) = \log \frac{e^x}{1+e^x} = x^{**} - \log(1+e^x)^{*}$

$* \log(1+e^x) = \log(e^0 + e^x) \longrightarrow \text{log-sum-exp} \Rightarrow \text{convex} \leadsto -\log(1+e^x) \Rightarrow \text{concave}$

$** \; x \longrightarrow \text{linear} \Rightarrow \text{convex and concave}$ $\Rightarrow x - \log(1+e^x) \leadsto \text{concave}$ $\quad f(x) \to \text{log concave}$

(b

$$f(x) = \frac{1}{\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}} \qquad x \in R^n_{++}$$

$$g(x) = \log f(x) = -\log\left(\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}\right) \qquad \frac{\partial g(x)}{\partial x_i} = \frac{\frac{1}{x_i^2}}{\frac{1}{x_1}+\cdots+\frac{1}{x_n}}$$

$$\frac{\partial g(x)}{\partial x_i^2} = \frac{\frac{-2}{x_i^3}\left(\frac{1}{x_1}+\cdots+\frac{1}{x_n}\right)+\frac{1}{x_i^4}}{\left(\frac{1}{x_1}+\cdots+\frac{1}{x_n}\right)^2} \qquad \frac{\partial g(x)}{\partial x_i \partial x_j} = \frac{\frac{1}{x_i^2 x_j^2}}{\left(\frac{1}{x_1}+\cdots+\frac{1}{x_n}\right)^2}$$

$$h_{ij} = \frac{\partial g(x)}{\partial x_i \partial x_j}$$

حال باید ثابت کنیم $H \preceq 0$ یعنی $\forall v: v^T H v \preceq 0$ درنتیجه داریم:

$$\forall v \in R^n, \quad v^T H v = \sum_{i=1}^n \sum_{j=1}^n v_i v_j h_{ij} = \sum_{i=1}^n v_i^2 h_{ii} + \sum_{i \neq j} v_i v_j h_{ij}$$

$$= -2\sum_{i=1}^n \frac{v_i^2}{x_i^3} \times \frac{1}{\frac{1}{x_1}+\cdots+\frac{1}{x_n}} + \sum_{i=1}^n \sum_{j=1}^n \frac{\frac{v_i v_j}{x_i^2 x_j^2}}{\left(\frac{1}{x_1}+\cdots+\frac{1}{x_n}\right)^2}$$

$$= \frac{1}{\left(\frac{1}{x_1}+\cdots+\frac{1}{x_n}\right)^2}\left(\left(\sum_{i=1}^n \frac{v_i}{x_i^2}\right)^2 - 2\sum_{i=1}^n \frac{1}{x_i}\sum_{i=1}^n \frac{v_i^2}{x_i^3}\right) \qquad a_i = \frac{1}{\sqrt{x_i}} \qquad b_i = \frac{v_i}{x_i\sqrt{x_i}}$$

کوشی شوارتز

$$= \frac{1}{\left(\frac{1}{x_1}+\cdots+\frac{1}{x_n}\right)^2}\left((a^Tb)^2 - 2\|a\|^2\|b\|^2\right) \preceq 0 \implies g(x): \text{concave}, \quad f(x): \text{log concave}$$

(c

$$f(x) = \frac{\prod_{i=1}^n x_i}{\sum_{i=1}^n x_i} \qquad x \in R^n_{++} \qquad g(x) = \log f(x) = \sum_{i=1}^n \log x_i - \log \sum_{i=1}^n x_i$$

non decreasing

$$\frac{\partial g(x)}{\partial x_i} = \frac{1}{x_i} - \frac{1}{\sum_{i=1}^n x_i} > 0 \qquad \frac{\partial g(x)}{\partial x_i^2} = \frac{-1}{x_i^2} + \frac{1}{\left(\sum_{i=1}^n x_i\right)^2} \qquad \frac{\partial g(x)}{\partial x_i \partial x_j} = \frac{1}{\left(\sum_{i=1}^n x_i\right)^2}$$

$$h_{ij} = \frac{\partial g(x)}{\partial x_i \partial x_j}$$

باید ثابت کنیم $H \preceq 0$ است یعنی $\forall v: v^T H v \preceq 0$ است.

$$v^T H v = \sum_{i=1}^n v_i^2 h_{ii} + \sum_{i \neq j} v_i v_j h_{ij} = -\sum_{i=1}^n \frac{v_i^2}{x_i^2} + \frac{1}{\left(\sum_{i=1}^n x_i\right)^2}\sum_{i=1}^n \sum_{j=1}^n v_i v_j$$

$$= \frac{\left(\sum_{i=1}^n v_i\right)^2}{\left(\sum_{i=1}^n x_i\right)^2} - \sum_{i=1}^n \frac{v_i^2}{x_i^2} \quad *$$

اگر $x>0$ باشد داریم:

$$\left(\sum_{i=1}^n x_i\right)^2 = \sum_{i=1}^n x_i^2 + \sum_{i=1}^n \sum_{j=1}^n x_i x_j \geq \sum_{i=1}^n x_i^2$$

$$* \quad \frac{(\sum_{i=1}^n v_i)^2}{(\sum_{i=1}^n x_i)^2} - \sum_{i=1}^n \frac{v_i^2}{x_i^2} = \frac{1}{(\sum_{i=1}^n x_i)^2}\left((\sum_{i=1}^n v_i)^2 - (\sum_{i=1}^n x_i)^2 \sum_{i=1}^n (\frac{v_i}{x_i})^2\right)$$

$$\leq \frac{1}{(\sum_{i=1}^n x_i)^2}\left((\sum_{i=1}^n v_i)^2 - \sum_{i=1}^n x_i^2 \sum_{i=1}^n (\frac{v_i}{x_i})^2\right) \qquad a_i = x_i \quad b_i = \frac{v_i}{x_i}$$

$$\xrightarrow[(a^Tb)^2 \leq \|a\|^2\|b\|^2]{} \forall v: \ v^T H v \leq 0 \implies \frac{\prod x_i}{\sum x_i} \rightsquigarrow \text{log Concave}$$

(d

$$f(X) = \frac{\det(X)}{tr(X)} \qquad h(X) = \log f(X) = \log\det X - \log tr(X)$$

$$g(t) = h(Z+tV) \overset{Z,\,Z+tV \in S_{++}^n}{=} \log\det(Z+tV) - \log tr(Z+tV)$$

$$= \log\det(Z(I+tZ^{-\frac{1}{2}}VZ^{-\frac{1}{2}})) - \log tr(Z(\underbrace{I+tZ^{-\frac{1}{2}}VZ^{-\frac{1}{2}}}_{Q\Lambda Q^T})) = \log\det Z + \sum_{i=1}^n \log(1+t\lambda_i)$$

$$-\log\sum_{i=1}^n q_i^T Z q_i (1+t\lambda_i) = \underbrace{\log\det Z - \sum_{i=1}^n \log q_i^T Z q_i}_{c} + \sum_{i=1}^n \log \overset{t>0}{\underbrace{q_i^T Z q_i (1+t\lambda_i)}_{\text{affine} \leftarrow a_i}}$$

$$-\log\sum_{i=1}^n q_i^T Z q_i (1+t\lambda_i) = C + \underbrace{\sum_{i=1}^n \log a_i - \log\sum_{i=1}^n a_i}_{\text{log Concave} \leftarrow c \text{ ثابت}} \rightarrow \text{non decreasing}$$

در نتیجه با ثابت شدن آن تابع یک تابع log concave است.

(a $\boxed{\overset{6\ a}{3.26}}$

$$Y, Z \in S_{++}^n \rightsquigarrow Y \preceq Z \iff 0 \preceq Z - Y \iff \forall v \in R^n: v^T(Z-Y)v \geq 0$$

$$\iff \forall v \in R^n: \underbrace{v^T Z^{\frac{1}{2}}}_{u^T}(I - Z^{-\frac{1}{2}}YZ^{-\frac{1}{2}})\underbrace{Z^{\frac{1}{2}}v}_{u} \geq 0 \xrightarrow[Z^{\frac{1}{2}} \to \text{full rank}]{Z \to \text{full rank}} \forall u \in R^n: u^T(I - Z^{-\frac{1}{2}}YZ^{-\frac{1}{2}})u \geq 0$$

$$\iff I - Z^{-\frac{1}{2}}YZ^{-\frac{1}{2}} \succeq 0 \iff \lambda_{max}(Z^{-\frac{1}{2}}YZ^{-\frac{1}{2}}) \leq 1 \implies |Z^{-\frac{1}{2}}YZ^{-\frac{1}{2}}| = \prod_{i=1}^n \lambda_i \leq 1 \iff$$

$$\det(Z^{-\frac{1}{2}}YZ^{-\frac{1}{2}}) = \det(Z^{-1}Y) = \frac{\det(Y)}{\underbrace{\det(Z)}_{\det(Z^{-1}) = \frac{1}{\det Z}}} \leq 1 \xrightarrow{Z \succ 0 \Rightarrow \det Z > 0} \det(Y) \leq \det(Z) \checkmark$$

(b

$$\begin{bmatrix} Y & 0 \\ 0 & 0 \end{bmatrix} \preceq \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix} \implies \begin{bmatrix} X_{11}-Y & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix} \succeq 0$$

از آنجایی که X_{22} متقارن است و وارون دارد پس PD است. طبق Schur complement: $\implies X_{11} - Y - X_{12}X_{22}^{-1}X_{12}^T \succeq 0$

به تقارن X

$$\Rightarrow 0 \prec Y \preceq X_{11} - X_{12}X_{22}^{-1}X_{12}^T \xrightarrow{\text{طبق } a^{-1}} |Y| \leq |X_{11} - X_{12}X_{22}^{-1}X_{12}^T|$$

$$\min \log \underbrace{\det Y^{-1}}_{1/\det Y} = \min -\log \det Y = \max \underbrace{\log \det Y}_{\text{concave}} \xrightarrow{\log \to \text{صعودی}} \leq \log\det(X_{11}-X_{12}X_{22}^{-1}X_{12}^T)$$

$$\xrightarrow{\text{reachable}} \max \log\det Y = \log\det(X_{11}-X_{12}X_{22}^{-1}X_{12}^T) \rightsquigarrow \arg\max \log\det Y = X_{11}-X_{12}X_{22}^{-1}X_{12}^T$$

c) تابع $g(X,Y)$ را به صورت رو به رو تعریف می‌کنیم: $g(X,Y) = -\log\det Y \mid Y \preceq X_{11}-X_{12}X_{22}^{-1}X_{12}^T,\ X,Y \in S_{++}^n$

طبق قسمت b می‌دانیم داریم که:

$$f(X) = \inf_{Y \in D_g} g(X,Y)$$

می‌دانیم $\log\det Y$ یک تابع concave است پس منفی آن یک تابع محدب است. همچنین داریم که

inf ترابع محدب، تابع convex می‌شود پس $f(X)$ محدب است درصورتی که دامنه g محدب باشد

که D_g برابر $X \succeq 0$ و $X_{11}-X_{12}X_{22}^{-1}X_{12}^T \succeq Y \succ 0$ است که محدب هستند.

d) طبق راهنمایی داریم:

$$X^{-1} = \begin{bmatrix} X_{11}-X_{12}X_{22}^{-1}X_{12}^T & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

پس اگر $P = \begin{bmatrix} I \\ 0 \end{bmatrix}$ باشد داریم: $P^TX^{-1}P = X_{11}-X_{12}X_{22}^{-1}X_{12}^T \Rightarrow \log\det P^TX^{-1}P \rightsquigarrow \text{convex}$

3.17 a)

$$D_g = \{x \mid \underbrace{x_1 + x_2 + \dots + x_m}_{X = [x_1 \dots x_m]} = x\} = \{x \mid X\mathbf{1} = x\}$$

$$X = \{[x_1 \dots x_m] \mid \forall i:\ x_i \in \underset{\text{Convex}}{D_{f_i}}\} \Rightarrow \text{Convex} \xrightarrow[\text{affine}]{\text{inverse image}} D_g \to \text{Convex} \ *$$

$$f_i(x_i) \to \text{Convex} \qquad f_1(x_1) + \dots + f_m(x_m) = \sum \text{convex function} \rightsquigarrow \text{Convex} \ **$$

$$*,** \ h(X,x) = f_1(x_1) + \dots + f_m(x_m) \mid x_1 + \dots + x_m = x \Rightarrow \text{Convex} \rightsquigarrow \text{convex}$$ دامنه و خود تابع

$$g(x) = \inf_{\substack{X \\ \rightsquigarrow X\mathbf{1}=x}} h(X,x) \xrightarrow{\inf \text{ Convex}} \text{Convex} \ \checkmark$$

یک راه دیگر هم عوض کردن تعریف تابع به شکل $g(x) = \inf_{x_1,\dots,x_m} \left(f_1(x_1) + \dots + f_m(x_m) + I_{D_h}\right)$ است که محدب است.

$$\hookrightarrow \text{indicator} = \begin{cases} 0 & X,x \in D_h \rightsquigarrow x = x_1 + \dots + x_m \\ \infty & \text{o.w.} \end{cases}$$

$$g^*(y) = \sup_x \{ y^Tx - f(x) \} = \sup_x \{ y^Tx - \inf_X h(X,x) \} \qquad -\inf x = \sup x$$ (b

$$= \sup_x \{ y^Tx + \sup_X -h(X,x) \} = \sup_{\substack{X \\ X1=x}} \{ y^Tx_1 + \dots + y^Tx_m - f_1(x_1) - \dots - f_m(x_m) \}$$

$$= \sup_{\substack{X \\ X1=x}} \{ y^Tx_1 - f_1(x_1) + \dots + y^Tx_m - f_m(x_m) \} = \sup_{x_1} \{ y^Tx_1 - f_1(x_1) \} + \dots + \sup_{x_m} \{ y^Tx_m - f_m(x_m) \}$$

$$= f_1^*(y) + f_2^*(y) + \dots + f_m^*(y)$$

2-7 a
3.30

در سوال قبلی نشان دادیم که infimal convolution یک تابع Convex است. همچنین اثبات کردیم که

$h(x) = \inf_{x_1,x_2} \bar{l}(x_1,x_2,x)$ برای به دست آوردن $\bar{l}(x_1,x_2,x) = \{ f_1(x_1) + f_2(x_2) \mid x_1 + x_2 = x \}$ محدب است. پس

مشتق آن را برابر صفر قرار می دهیم: $\quad h(x) = \inf_y l(y,x) \quad l(y,x) = \bar{l}(y, x-y, x)$

$$l(y,x) = \|y\|_1 + \frac{1}{2}\|x-y\|_2^2 = \sum_{i=1}^n |y_i| + \frac{1}{2}\sum_{i=1}^n (x_i - y_i)^2$$

$$h(x) = \inf_y \underbrace{\sum_{i=1}^n |y_i| + \frac{1}{2}\sum_{i=1}^n (x_i-y_i)^2}_{\sum_{i=1}^n |y_i| + \frac{1}{2}(x_i-y_i)^2} = \sum_{i=1}^n \inf_{y_i} \underbrace{|y_i| + \frac{1}{2}(y_i - x_i)^2}_{a_i}$$

برای اینکه a_i مینیمم شود دو حالت وجود دارد: 1) $y_i = 0$ باشد که در این نقطه $|y_i|$ مشتق ناپذیر

نباشد و مقادیر تابع باید در $y_i = 0$ و $y_i \neq 0$ مقایسه شوند و کمترین برداشته شود. 2) $y_i \neq 0$

$$\Rightarrow \frac{\partial a_i}{\partial y_i} = \frac{y_i}{|y_i|} + (y - x_i) = 0 \qquad x_i = y_i + \frac{y_i}{|y_i|} \rightarrow |x_i| = \left|y_i + \frac{y_i}{|y_i|}\right| = |y_i| + 1 > 1$$

هم علامت

$$h(x) = \sum_{i=1}^n \min \left\{ \frac{x_i^2}{2}, |x_i| - 1 + \frac{1}{2}\underbrace{|x_i - y_i|^2}_{1} \right\} = \sum_{i=1}^n \min \left\{ \frac{x_i^2}{2}, |x_i| - \frac{1}{2} \right\}$$

$$\Rightarrow \text{if } |x_i| \leq 1 \rightarrow \min \left\{ \frac{x_i^2}{2}, |x_i| - \frac{1}{2} \right\} = \frac{x_i^2}{2}$$ به دلیل محدودیت دامنه x_i

$$\text{if } |x_i| > 1 \rightarrow \min \left\{ \frac{x_i^2}{2}, |x_i| - \frac{1}{2} \right\} = |x_i| - \frac{1}{2}$$

$$\Rightarrow h(x) = \sum_{i=1}^n \phi(x_i)$$

نابرابری ۸

تابع $f(x)=\frac{1}{\sqrt{x}}$ را برای $x>0$ در نظر می‌گیریم. با مشتق دوم ثابت می‌شود که محدب است:

$$f(x)=x^{-\frac{1}{2}} \qquad f'(x)=-\frac{1}{2}x^{-\frac{3}{2}} \qquad f''(x)=\frac{3}{4}x^{-\frac{5}{2}} \xrightarrow{x>0} f''(x)>0 \to \text{convex}$$

حال بدون کم شدن از مسأله، فرض می‌کنیم $a+b+c=1$ است زیرا تمام جمله‌های صورت و مخرج هادرجه یک هستند و با ضرب λ کردن $a+b+c$ صورت و مخرج در یک عدد ضرب می‌شوند و ساده می‌شوند. حال تعریف می‌کنیم:

$$x_1=a^2+8bc \qquad x_2=b^2+8ac \qquad x_3=c^2+8ab \qquad \overset{a,b,c>0}{a+b+c=1},\ f\to\text{convex}$$

$$f(ax_1+bx_2+cx_3)=\frac{1}{\sqrt{a^3+b^3+c^3+24abc}}\leq\frac{a}{\sqrt{a^2+8bc}}+\frac{b}{\sqrt{b^2+8ac}}+\frac{c}{\sqrt{c^2+8ab}}$$

$$(a+b+c)^3=a^3+b^3+c^3+3(a^2b+a^2c+b^2a+b^2c+c^2a+c^2b)+6abc$$

چون نامساوی حسابی هندسی داریم:

$$\frac{a^2b+a^2c+b^2a+b^2c+c^2a+c^2b}{6}\geq\sqrt[6]{a^6b^6c^6}=abc$$

$$\Rightarrow 1=(a+b+c)^3\geq a^3+b^3+c^3+3(6abc)+6abc=a^3+b^3+c^3+24abc$$

$$\Rightarrow\sqrt{a^3+b^3+c^3+24abc}\leq 1 \Rightarrow 1\leq\frac{1}{\sqrt{a^3+b^3+c^3+24abc}}\leq\frac{a}{\sqrt{a^2+8bc}}+\frac{b}{\sqrt{b^2+8ac}}+\frac{c}{\sqrt{c^2+8ab}}\ \checkmark$$