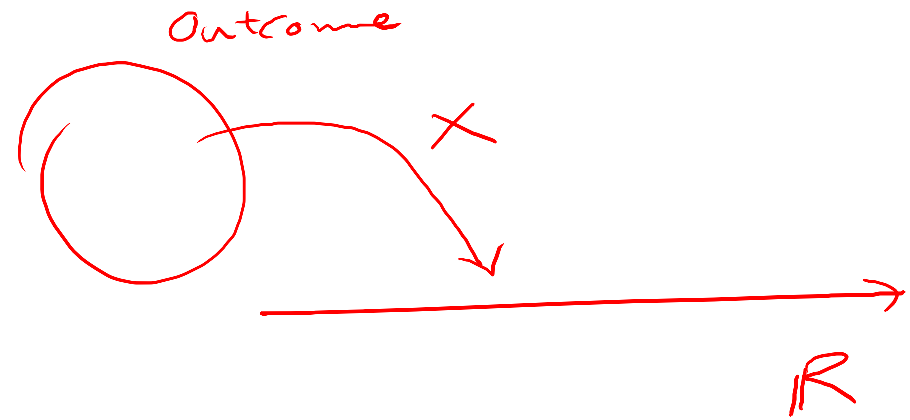
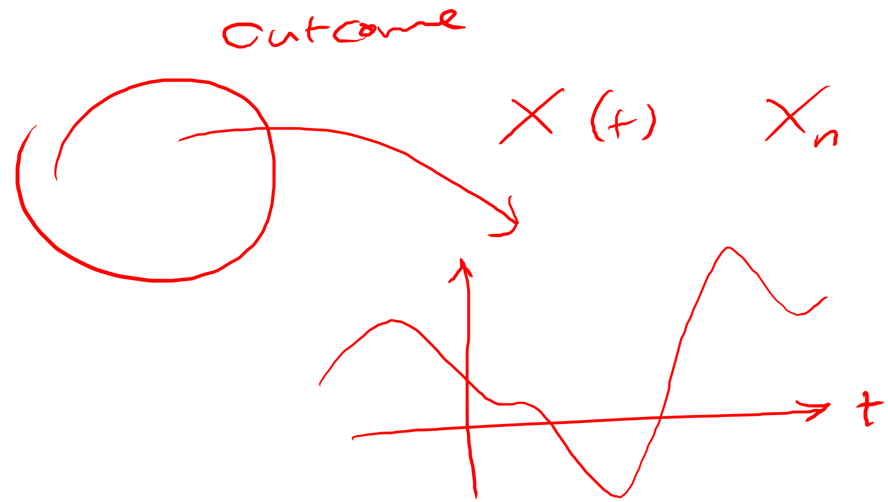
Stochastic Process

Markov Process





$$\{S_1, S_2 \dots, S_n, \dots\}$$

$$\mathbb{P}(S_n \mid S_{n-1}, \dots S_1) = \mathbb{P}(S_n \mid S_{n-1})$$

$$\mathbb{P}(S_n) =: \pi_n$$

π_0

trans. prob. matrix

$$P = \begin{array}{c} \\ i \end{array} \begin{array}{c} j \\ \begin{bmatrix} & \\ & p_{ij} \end{bmatrix} \end{array}$$

$$p_{ij} = \mathbb{P}(S_n = j \mid S_{n-1} = i)$$

$$\pi_n = \pi_0 P^n$$

Does $\lim_{n \to \infty} \pi_n$ exists?

Stationary prob. π^* : $\underline{\pi^* = \pi^* P}$

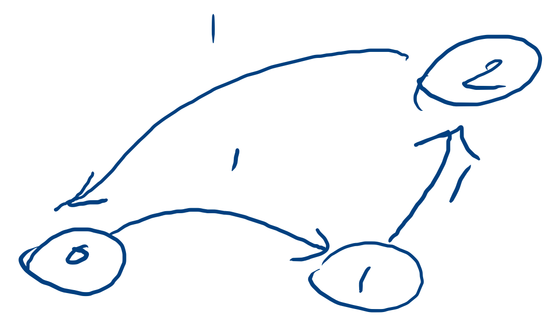
Under certain conditions

$\lim_{n\to\infty} \pi_n$ exists & it's equal

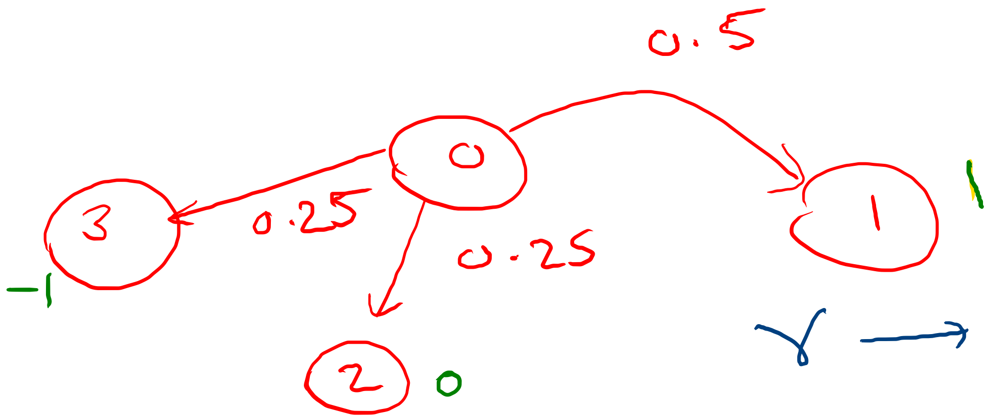
π^*, which is unique.

Conditions:

- Finite State
- Irreducible
- Aperiodic



Markov Reward Process



$\gamma \le 1$

$\gamma \longrightarrow$ discount factor

S_n

R_n

$$\underbrace{V(s)}_{\text{Value function}} := \mathbb{E}\left(\sum_{n=0}^{\infty} \gamma^n R_n \,\middle|\, S_0 = s \right)$$

$$= \mathbb{E}\left(R_0 + \gamma \sum_{n=0}^{\infty} \gamma^n R_{n+1} \,\middle|\, S_0 = s \right)$$

$$= \mathbb{E}(R_0 \mid S_0 = s) + \mathbb{E}\left(\gamma \sum_{n=0}^{\infty} \gamma^n R_{n+1} \,\middle|\, S_0 = s\right)$$

$V(s_1)$

Recall :

$$\mathbb{E}(\underbrace{\mathbb{E}(X \mid Y)}_{g(Y)}) = \mathbb{E}(X)$$

$$\mathbb{E}\left(\mathbb{E}\left(\gamma \sum_{n=0}^{\infty} \gamma^n R_{n+1} \,\middle|\, S_1 = s_1\right) \,\middle|\, S_0 = s\right)$$

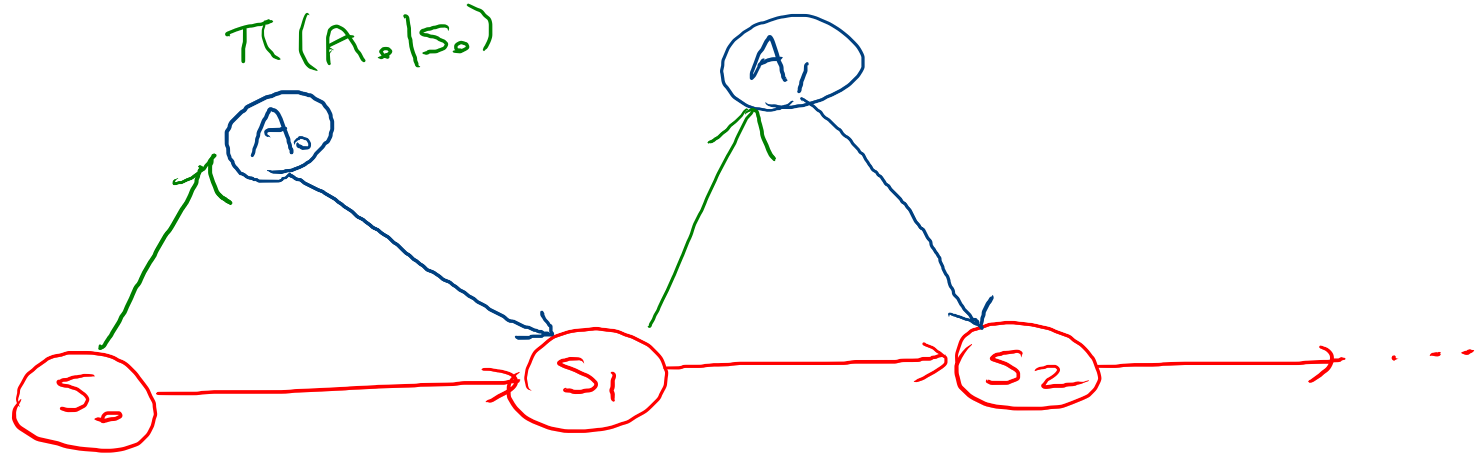
trans. prob.

$$= \mathbb{E}(R_0 + \gamma V(s_1) \mid S = s_0)$$

$V(s_0)$

Bellman's Eq.

Markov Decision Process



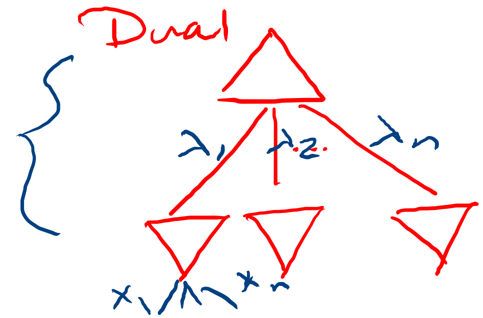
$$\mathbb{P}(S_0, A_0, S_1, A_1 \cdots) = \overbrace{\mathbb{P}(S_0)}\,\pi(A_0|S_0)\,\overbrace{\mathbb{P}(S_1|S_0, A_0)}$$

$$\pi(A_1|S_1)\cdot\mathbb{P}(S_2|S_1, A_1)\cdots$$

$$\max_{\pi}\ \mathbb{E}\left\{\sum \gamma^n R(S_n, A_n, S_{n+1})\right\}$$

$$\min_{x} \ f(x)$$

$$\text{s.t. } \forall_i \ g_i(x) \le 0$$



Primal

$$\min_{x} \ \max_{\lambda_i \ge 0} \ f(x) + \sum_{i=1}^{n} \lambda_i g_i(x)$$

$g_i(x) < 0$

$\lambda_i^* = 0$

$0 = g_i(x)$

λ_i^* could be pos.

$\rightarrow$ Duality Gap

Dual

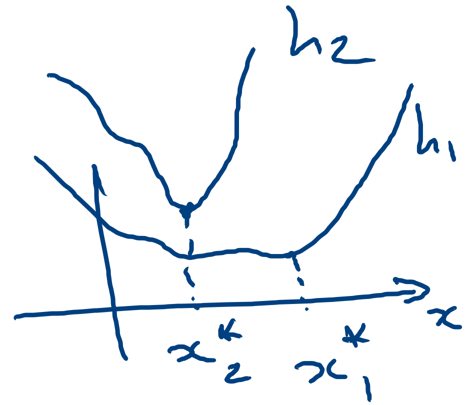
$$\rightarrow \max_{\lambda_i \ge 0} \ \min_{x} \ f(x) + \sum_{i=1}^{n} \lambda_i g_i(x)$$

$0 = \lambda_i^* g_i(x^*)$

λ_i^*'s are the optimal Lagrange Multipliers of the dual problem.

$$\forall x \quad \max_{\lambda_i \geq 0} \min_{x} \overbrace{f(x) + \sum_{i=1}^{n} \lambda_i g_i(x)}^{h_1(x)} \leq$$

$$\longrightarrow \quad \min_{x} \max_{\lambda_i \geq 0} \overbrace{f(x) + \sum_{i=1}^{n} \lambda_i g_i(x)}^{h_2(x)}$$

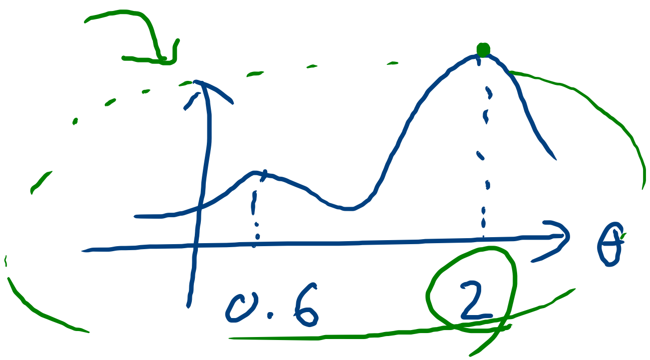


$$\forall x \quad h_1(x) \leq h_2(x) \implies h_1(x_1^*) \leq h_2(x_2^*)$$

How to estimate unknown prams?

$X_1 \dots X_n \overset{iid}{\sim} f_{\theta}$

point estimation



$P(\theta \mid X_1 \dots X_n)$ full posterior

$P(\theta)$

MLE: $\max_{\theta} \underbrace{P(X_1 \dots X_n \mid \theta)}_{\text{Likelihood}}$

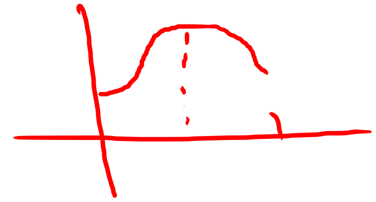
MAP
Bayesian

$\theta \sim P_{\theta}$

$\max_{\theta} \underbrace{P(\theta \mid X_1 \dots X_n)} = \frac{P(X_1 \dots X_n \mid \theta) \cdot P(\theta)}{P(X_1 \dots X_n)}$

$$X_1 \dots X_n \overset{iid}{\sim} \text{Bernoulli}(\theta)$$

$$\theta \sim \underbrace{C\,\theta^{\alpha}(1-\theta)^{1-\alpha}} \qquad 0 \le \theta \le 1$$



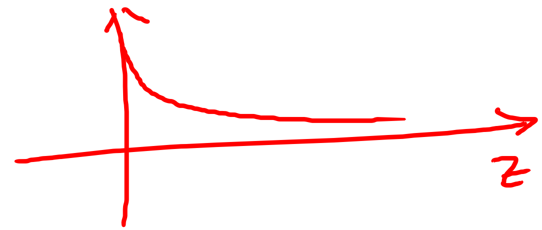
$$P(\theta \mid X_1 \dots X_n) \propto \underbrace{P(X_1 \dots X_n \mid \theta)}_{\prod_{i=1}^{n} \underbrace{P(X_i \mid \theta)}_{\theta^{X_i}(1-\theta)^{1-X_i}}} \cdot \underbrace{P(\theta)}_{C\theta^{\alpha}(1-\theta)^{1-\alpha}}$$

$$= C\,\theta^{\sum X_i + \alpha}(1-\theta)^{n - \sum X_i + 1 - \alpha}$$

$$x_1 \overset{iid}{\sim} \mathcal{N}(z;1)$$

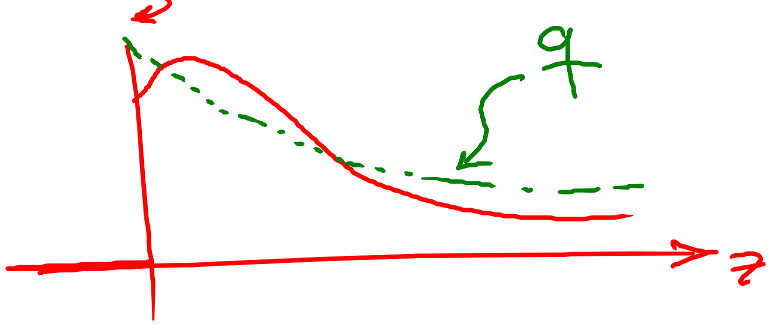
$$Z \sim e^{-z}\,\mathbb{I}(z \geq 0)$$

$$\lambda e^{-\lambda z}\,\mathbb{I}(z \geq 0)$$

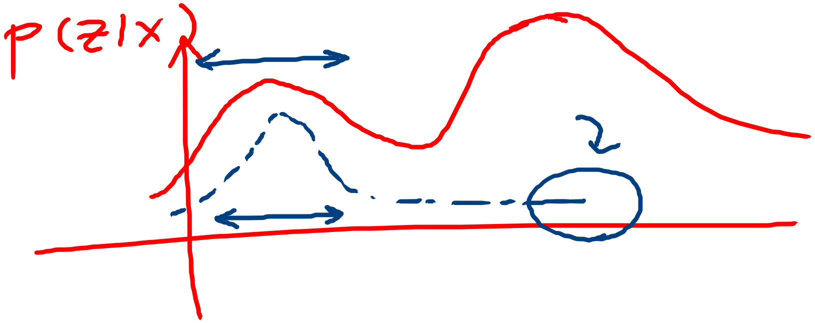


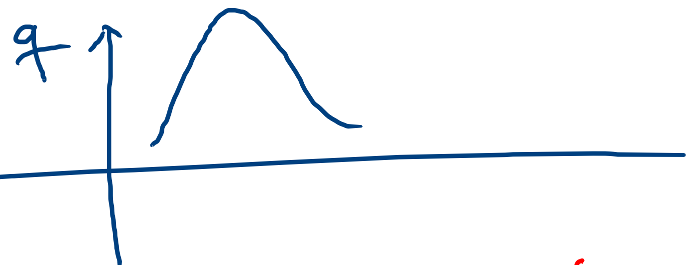
$$P(z \mid x_1) = P(x_1 \mid z)\cdot P(z)$$

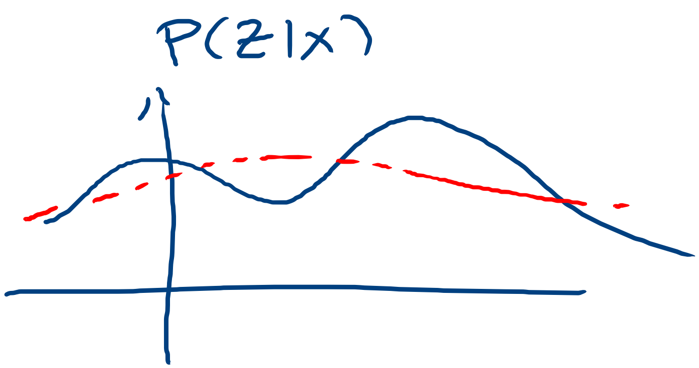
$$= \frac{\frac{1}{\sqrt{2\pi}}\, e^{-\frac{(x_1-z)^2}{2}} \cdot e^{-z}\,\mathbb{I}(z \geq 0)}{\int \frac{1}{\sqrt{2\pi}}\, e^{-\frac{(x_1-z)^2}{2}}\, e^{-z}\,\mathbb{I}(z \geq 0)\, dz}$$



$$\min_{q} D(q \,\|\, p(z \mid x_1))$$







$$D\left(q \,\|\, p(z|x)\right) = \mathbb{E}_{\underbrace{z \sim q}}\left[\log \frac{q}{p(z|x)}\right]$$

$$\log\left\{C\,\theta^{\sum x_i+\alpha}(1-\theta)^{n-\sum x_i+1-\alpha}\right\}$$

$$\log C + \left(\sum x_i+\alpha\right)\log\theta + \left(n-\sum x_i+1-\alpha\right)\log(1-\theta)$$

$$\frac{\sum x_i+\alpha}{\theta} + \frac{-\left(n-\sum x_i+1-\alpha\right)}{1-\theta} = 0$$

$$(1-\theta)\left(\cancel{\sum x_i}\cancel{+\alpha}\right) = \theta\left(n-\cancel{\sum x_i}+1-\cancel{\alpha}\right)$$

$$\sum x_i+\alpha = (n+1)\theta \Rightarrow$$

$$\theta = \frac{1}{n+1}\left\{\underbrace{\sum x_i+\alpha}\right\}$$

$-D(q \| p(z|x))$

LHS

evidence

$$D(q \| p(z/x))$$

$$= \mathbb{E}_{z\sim q}\left[\log \frac{q(z)}{p(z|x)}\right]$$

$$p(z|x) = p(z,x) / p(x)$$

$$= \mathbb{E}_{z\sim q}\left[\log \frac{q(z)}{p(x,z)} + \log p(x)\right]$$

a lower bound on $\log P(x)$

ELBO

$$= \underbrace{-\mathbb{E}_{z\sim q}\left[\log \frac{P(x,z)}{q(z)}\right]}_{\text{ELBO}} + \underbrace{\log P(x)}_{\text{evidence}}$$

$$-D(q \| p(z|x))$$

$$\text{ELBO} = \mathbb{E}_{\underset{\text{proposal}}{z \sim q}}\left[\log \frac{P(x,z)}{q(z)}\right]$$

$$q_\theta(z) = \theta e^{-\theta z}\, \mathbb{I}(z \geq 0)$$

$$\text{ELBO} = \mathbb{E}_{z\sim q}\Bigg[\log \overbrace{\frac{1}{\sqrt{2\pi}} e^{-\frac{(x-z)^2}{2}} e^{-z}\, \cancel{\mathbb{I}(z \geq 0)}}^{-\frac{(x-z)^2}{2} - z}$$

$$- \underbrace{\log \theta e^{-\theta z}}_{-\log\theta + \theta z}\, \cancel{\mathbb{I}(z \geq 0)}\Bigg]$$

$$\mathbb{E}_{z\sim q}(z) = 1/\theta$$

$$\mathrm{Var}(z) = 1/\theta^2$$

$$\mathbb{E}_{z\sim q}\left\{-\frac{x^2 - z^2}{2} + zx - z - \log\theta + \theta z\right]$$

$$\frac{\partial}{\partial\theta}\left\{-\frac{1}{\theta^2} + \frac{x}{\theta} - \frac{1}{\theta} - \log\theta + 1\right\} = 0$$

$$\frac{+2}{\theta^3} - \frac{x}{\theta^2} + \frac{1}{\theta^2} - \frac{1}{\theta} = 0$$

$$2 - \theta x + \theta - \theta^2 = 0 \rightarrow \theta^2 + (x-1)\theta - 2 = 0$$

$$\theta = \frac{(1-x) + \sqrt{(x-1)^2 + 8}}{2}$$