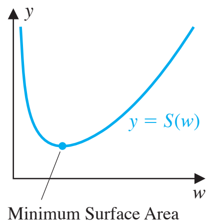


This can be substituted into the equation for surface area to produce

$$S(w) = w^2 + 4w\left(\frac{V}{w^2}\right) = w^2 + \frac{4V}{w}.$$

So the surface area S of the box is a *function* of the width w; each positive value of w gives a specific amount of required material $S(w)$. We need to choose w so that $S(w)$ is a minimum, which is easy to do using calculus. When we have this value of w, the equation $V = w^2h$ gives us the appropriate value of h. The figure shows a typical graph of $y = S(w)$, which illustrates that a number w does exist with $S(w)$ a minimum.

We can approximate the solution using the graph, and the exact solution can easily be found using calculus. The difficult part of the problem is not the calculus solution, it is the construction of the function $S(w)$. This is a precalculus problem of the type we will consider throughout the book. In this chapter we see what functions are and what they look like, and begin to examine in detail the functions that are commonly used in calculus and beyond.

1.1 INTRODUCTION

The functions we will study in precalculus—because they are needed in calculus—are transformations of the set of real numbers. This section contains a short development of the *real numbers*. We begin with the most basic set of numbers, the *natural numbers*.

The set of **natural numbers** consists of the counting numbers, 1, 2, 3, . . . , and is denoted by the symbol $\mathbb{N}$. Any pair of natural numbers can be added or multiplied and the result is another natural number. This is sometimes expressed by saying that set $\mathbb{N}$ is *closed* under the operations of addition and multiplication.

The operation of subtraction is needed for many purposes, but the set of natural numbers is *not closed* under subtraction. For example, if we subtract the natural number 3 from the natural number 7, we get the natural number 4, but subtracting 7 from 3 is impossible if we must stay within the set of natural numbers. To permit subtraction, we expand the set of natural numbers to the set of *integers*, a set that remains closed under addition and multiplication, and is closed under subtraction as well.

The set of **integers** consists of the natural numbers, the negative of each natural number, and the number 0. The set of integers is denoted using the symbol $\mathbb{Z}$ (the German word *zählen* means "number"). Addition, multiplication, and subtraction of two integers results in an integer, so the integers are closed under all these operations.

The set of *rational numbers* is introduced to permit division by nonzero numbers. A **rational number** has the form p/q, where p and q are integers and $q \neq 0$. The set of rational numbers is denoted $\mathbb{Q}$ (for quotient) and is closed under addition, multiplication, subtraction, and, except when the denominator is 0, under division.

The rational numbers satisfy all the common arithmetic properties but fail to have a property called *completeness*; that is, there are some essential numbers missing from the set. At least as early as 400 B.C.E. Greek mathematicians of the Pythagorean school recognized that $\sqrt{2}$, the length of a diagonal of a square with sides of length 1, is not a rational number. (See Figure 1.)

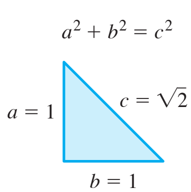


FIGURE 1

The nonrational numbers, or **irrational numbers**, were originally said to b *incommensurable* because they could not be directly compared to the familiar ration numbers. There are many irrational numbers, including $\sqrt{5}$, π, and $-\sqrt[3]{3}+1$. Th discovery of irrational numbers resulted in a profound change in ancient mathematic thinking. Before that time it was assumed that all quantities could be expressed a proportions of integers, and integers in some cultures had a mystical property.

A precise definition of completeness needs the concept of limits, an important topic in calculus.

The set $\mathbb{R}$ of **real numbers** consists of the rational numbers together with th irrational numbers. This set is described most easily by considering the set of a numbers that are expressed as infinite decimals. The rational numbers are those wi expansions that terminate or that eventually repeat in sequence, such as

$$\tfrac{1}{2}=0.5, \qquad \tfrac{1}{3}=0.\overline{3}, \qquad \tfrac{16}{11}=1.\overline{45}, \quad \text{or} \quad -\tfrac{123}{130}=-0.9\overline{461538},$$

where the bar indicates that the digit or block of digits is repeated indefinitely. Num bers with decimal expansions that have no repeating blocks are irrational.

1.2 THE REAL LINE

The material in the next few sections will likely be familiar from your previou mathematics courses. But in these sections we establish notation and a number o definitions that will be used throughout the text.

The fact that each real number can be written uniquely as a decimal gives u a way to associate each real number with a distinct point on a **coordinate line**. W first choose a point on a horizontal line as the origin and associate the real number with the origin. We designate some point to the right of 0 as the real number 1. Th positive integers are then marked with equal spacing consecutively to the right of The negative integers are marked, with this same spacing, to the left of 0. Nonintegr real numbers are placed on the line according to their decimal expansions.

Figure 1 shows an x-coordinate line and the points associated with a few of th real numbers.

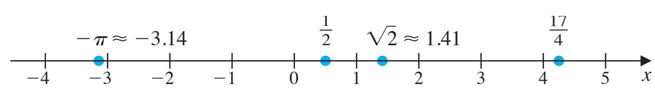
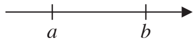


FIGURE 1

The coordinate-line representation of the real numbers is so convenient that w frequently do not explicitly distinguish between the points on the line and the re numbers that these points represent. Both are called the set of real numbers and a denoted $\mathbb{R}$.

Inequalities



$a < b$ or $b > a$

FIGURE 2

The relative position of two points on a coordinate line is used to define an inequali relationship on the set of real numbers. We say that a is less than b, written $a <$ when the real number a lies to the left of the real number b on the coordinate lin This is also expressed by stating that b is greater than a, written $b > a$, as shown Figure 2.

The notation $a \le b$, or $b \ge a$, is used to express that a is either less than or equal to b. The following properties of inequalities can be verified by referring to the coordinate-line representation of the real numbers a, b, and c.

Inequality Properties

- Precisely one of $a < b$, $b < a$, or $a = b$ holds.
- If $a > b$, then $a + c > b + c$.
- If $a > b$ and $c > 0$, then $ac > bc$.
- If $a > b$ and $c < 0$, then $ac < bc$.

Note that the fourth Inequality Property states that the inequality sign must be reversed when both sides of the inequality are multiplied by a negative number. For example,

$$4 > 3, \quad \text{but} \quad -8 = (-2)4 < (-2)3 = -6.$$

These rules are used to solve problems involving inequality relations.

EXAMPLE 1 Find all real numbers satisfying $2x - 1 < 4x + 3$.

Solution First add -3 to both sides of

$$2x - 1 < 4x + 3 \quad \text{to produce} \quad 2x - 4 < 4x.$$

Now add $-2x$ to each side to isolate x on one side, giving

$$-4 < 2x.$$

$x > -2$

FIGURE 3

Finally, multiply both sides by $\frac{1}{2}$, which gives the solution, $-2 < x$. This can also be written as $x > -2$, as shown in Figure 3. ■

The steps in the solution to Example 1 are not the only way to proceed. For example, if we first add 1 to both sides of $2x - 1 < 4x + 3$ and then subtract $4x$, the inequality becomes

$$-2x < 4.$$

Now divide both sides by -2. The division is by a negative number, so we reverse the inequality to obtain $x > -2$.

EXAMPLE 2 Find all real numbers x satisfying $-1 < 2x + 3 \le 5$.

Solution This inequality relation is a compact way of expressing that x must satisfy both of the inequalities

$$-1 < 2x + 3 \quad \text{and} \quad 2x + 3 \le 5.$$

Proceeding as in Example 1, we add -3 to both sides of each inequality to produce

$$-4 < 2x \quad \text{and} \quad 2x \le 2.$$

$-2 < x \le 1$

FIGURE 4

Multiplying these inequalities by $\frac{1}{2}$ gives

$$-2 < x \quad \text{and} \quad x \le 1.$$

This last set of inequalities can be expressed compactly as $-2 < x \le 1$, which shown in Figure 4.

Intervals

Interval notation is preferred in calculus to describe sets of real numbers lying between two given numbers.

Interval notation is a convenient way to represent certain important sets of real numbers. For real numbers a and b with $a < b$, the **open interval** (a, b) is defined

the set of

$$(a, b) = \{x \mid a < x < b\},$$

such that

and read "the set of real numbers x such that x is greater than a and less than b."

When the *endpoints*, a and b, of the interval are included in the set, it is called **closed interval** and denoted

$$[a, b] = \{x \mid a \le x \le b\}.$$

An interval that contains one endpoint but not the other is said to be a *half-ope* interval (although it could just as well be called half-closed). So the intervals

$$(a, b] = \{x \mid a < x \le b\}$$

and

$$[a, b) = \{x \mid a \le x < b\}$$

are both half-open.

The *interior* of an interval consists of all the numbers in the interval that are n endpoints. The intervals (a, b), $[a, b]$, $(a, b]$, and $[a, b)$ all have the same interio which is the open interval (a, b).

In addition to the intervals with finite endpoints, the **infinity symbol**, ∞, used to indicate that an interval extends indefinitely. Such an interval is said to b **unbounded**. The intervals

$$[a, \infty) = \{x \mid x \ge a\}$$

and

$$(a, \infty) = \{x \mid x > a\}$$

are *unbounded above* since they contain no largest real number. The intervals

$$(-\infty, a] = \{x \mid x \le a\}$$

and

$$(-\infty, a) = \{x \mid x < a\}$$

are *unbounded below*. The interval $(-\infty, \infty)$, which represents the set $\mathbb{R}$ of all re numbers, is unbounded both above and below.

In general, a square bracket indicates that the real number next to it is in the interval. A real number next to a parenthesis indicates the number is *not* in the interval. The symbols $-\infty$ and ∞ are *never* next to a square bracket, since they are only symbols and do not represent real numbers. Notice also that

- In any use of interval notation, the symbol on the left must be less than the symbol on the right.

Table 1 summarizes the interval notation.

TABLE 1

Interval Notation	Set Notation	Graphic Representation
(a, b)	$\{x \mid a < x < b\}$	
$[a, b]$	$\{x \mid a \le x \le b\}$	
$[a, b)$	$\{x \mid a \le x < b\}$	
$(a, b]$	$\{x \mid a < x \le b\}$	
(a, ∞)	$\{x \mid a < x\}$	
$[a, \infty)$	$\{x \mid a \le x\}$	
$(-\infty, b)$	$\{x \mid x < b\}$	
$(-\infty, b]$	$\{x \mid x \le b\}$	
$(-\infty, \infty)$	$\mathbb{R} = \{x \mid -\infty < x < \infty\}$	

EXAMPLE 3

a. Express the interval $[-3, 2)$ using inequalities.

b. Express the inequality $-\infty < x \le 2$ using interval notation.

Solution

a. The interval $[-3, 2)$ represents all real numbers between -3 and 2, including -3, indicated by the square bracket, and excluding 2, indicated by the parenthesis. So $-3 \le x$ and $x < 2$. Both conditions hold so we can write $-3 \le x < 2$. To graph the inequality $-3 \le x < 2$ we shade all the points on the real line between -3 and 2, use a solid dot at -3 to indicate that it is included, and use an open circle at 2 to indicate that it is not included, as shown in Figure 5.

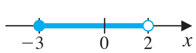


FIGURE 5

b. The inequality $-\infty < x \le 2$ represents all real numbers less than or equal to 2. This is equivalent to the interval notation $(-\infty, 2]$, whose graph is shown in Figure 6. ■

FIGURE 6

Interval notation can be used to give an alternative expression for the answers
Examples 1 and 2.

- In Example 1, the inequality was satisfied when $-2 < x$, that is, for x in $(-2, \infty$
- In Example 2, the inequality was satisfied when $-2 < x \leq 1$, that is, for x
  $(-2, 1]$.

The next example involves a quadratic inequality that can be solved algebraical in a manner similar to the method used in Examples 1 and 2. It is easier, however, use a graphical technique involving the coordinate line. This is the method of choi for solving the numerous inequalities that arise in a calculus problem.

EXAMPLE 4 Find all values of x that satisfy the inequality $(x - 1)(x - 3) > 0$.

Solution Figure 7 is a **sign chart** for the inequality $(x - 1)(x - 3) > 0$. It is used to determi where the inequality is positive and where it is negative.

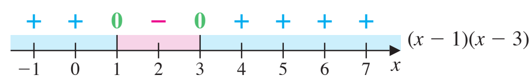


FIGURE 7

Solving inequalities is often needed in calculus, when determining where functions are defined or where graphs are increasing and decreasing.

To construct the sign chart, first determine where each of the factors is 0, namel at $x = 1$ and $x = 3$. The inequality will be always positive or always negative on t intervals separated by the values where the factors are 0.

Select any number inside one of the intervals. For example, choose $x = 0$, $(-\infty, 1)$, and evaluate the left side, giving

$$(0 - 1)(0 - 3) = (-1)(-3) = 3 > 0.$$

The value of the inequality is positive when $x = 0$, so it is positive on the enti interval $(-\infty, 1)$. The value of the inequality is negative when $x = 2$ because

$$(2 - 1)(2 - 3) = (1)(-1) = -1 < 0,$$

so the inequality is negative on the entire interval $(1, 3)$. Similarly, the value is positi when $x = 4$ because

$$(4 - 1)(4 - 3) = (3)(1) > 0.$$

So the inequality is positive on $(3, \infty)$.

EXAMPLE 5 Find all values of x that satisfy the inequality $x^2 - 4x + 5 > 2$.

Solution This problem is solved by first changing the inequality into one that has 0 on the rig side, and then factoring the term on the left side. It will then be in a form similar the inequality in Example 4.

Subtracting 2 from both sides of the inequality, we see that

$$x^2 - 4x + 5 > 2 \quad \text{implies that} \quad x^2 - 4x + 3 > 0.$$

We now factor the quadratic $x^2 - 4x + 3$ by determining constants a and b so th $x^2 - 4x + 3 = (x + a)(x + b)$. Because

$$x^2 - 4x + 3 = (x + a)(x + b) = x^2 + (a + b)x + a \cdot b,$$

we must have $a + b = -4$ and $a \cdot b = 3$.

This implies that $x^2-4x+3 = (x-1)(x-3)$. So to solve the original inequality, we must have $(x-1)(x-3) > 0$. Since this factored form is the same as the inequality in Example 4, the solution is as shown in Figure 7. ■

The sign chart in Figure 7 also tells us that

$$x^2 - 4x + 3 = (x - 1)(x - 3) < 0 \text{ when } 1 < x < 3.$$

Unions and Intersections

The answer to Examples 4 and 5 can also be expressed using interval notation, but it requires the introduction of the union symbol. The **union** of two sets A and B, written $A \cup B$, is the set of all elements that are in either A or in B (or in both). So in interval notation the answer in Example 4 is

$$(-\infty, 1) \cup (3, \infty).$$

The **intersection** of A and B, written $A \cap B$, is the set of elements that are in both A and B.

Factoring the quadratic in Example 5 is relatively easy because there are only two integer possibilities for a product of the constant term $3 = 1 \cdot 3 = (-1) \cdot (-3)$. Once we determine this, it is clear which one to use because the coefficient of x in the quadratic $x^2 - 4x + 3$ must be -4.

When there are more factoring possibilities for the constant term, experimentation is needed. For example, suppose that we want to factor the quadratic $x^2 + 7x + 12$. Since

$$12 = 1 \cdot 12 = 2 \cdot 6 = 3 \cdot 4 = (-1) \cdot (-12) = (-2) \cdot (-6) = (-3) \cdot (-4)$$

we have many more possibilities. However, the coefficient of x is positive, so we can disregard all the negative factors. In addition, $7 = 3 + 4$, so the only possibility is $(x+3)$ and $(x+4)$. Notice that we do, in fact, have $(x+3)(x+4) = x^2+7x+12$.

There are other techniques that can be used to simplify the factoring process, but we will postpone these until Chapter 3.

EXAMPLE 6 Find all values of x for which $(x - 1)(x - 3)(5 - x) \le 0$.

Solution The three factors separate the real line into four separate regions,

$$x < 1, \quad 1 < x < 3, \quad 3 < x < 5, \quad \text{and} \quad x > 5.$$

To check the region where $x < 1$, substitute $x = 0$ in the inequality to obtain

$$(0 - 1)(0 - 3)(5 - 0) = (-1)(-3)(5) = 15 > 0.$$

Since the inequality is positive at $x = 0$, the inequality is positive on the entire region $x < 1$. Checking the values when x is 2, 4, and 6 shows that the inequality alternates on the other regions as shown in the sign chart in Figure 8.

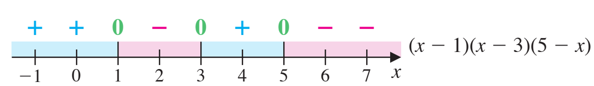


FIGURE 8

The solution to this inequality is written in interval notation as

$$[1, 3] \cup [5, \infty).$$

The sign chart technique can also be applied to factored quotients.

EXAMPLE 7 Find all values of x for which $\dfrac{x^2 - 4}{x(4 - x)} \geq 0.$

Solution The numerator of the quotient can be factored to give

$$\frac{x^2 - 4}{x(4 - x)} = \frac{(x + 2)(x - 2)}{x(4 - x)}.$$

This quotient is zero when the numerator $x^2 - 4 = 0$, which occurs at $x = 2$ ar $x = -2$. The quotient is undefined when the denominator is 0, which occurs at $x =$ and $x = 4$.

The four factors in the numerator and the denominator of the quotient separa the real line into five separate regions, $x < -2, -2 < x < 0, 0 < x < 2, 2 < x <$ and $x > 4$.

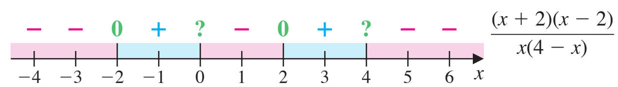


FIGURE 9

The sign chart in Figure 9 shows the possibilities. We have used the symbol as a shortcut to indicate that the quotient is undefined at $x = 0$ and $x = 4$. The cha gives the solution as $[-2, 0) \cup [2, 4)$.

Absolute Values

The **absolute value** of a real number x, denoted $|x|$, describes the distance on tl coordinate line from the number x to the number 0. The definition follows.

Absolute Value

$$|x| = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x < 0. \end{cases}$$

A number and its negative have the same absolute value. For example, $|2| = 2$ ar $|-2| = -(-2) = 2$. This means when solving an equation of the form

$$|\square| = a$$

we should determine the values that satisfy either

$$\square = a \quad \text{or} \quad \square = -a.$$

EXAMPLE 8 Determine all values of x for which $\left|\dfrac{x - 2}{2x + 1}\right| = 3.$

Solution We need to find all values of x that satisfy either

$$\frac{x-2}{2x+1} = 3 \quad \text{or} \quad \frac{x-2}{2x+1} = -3.$$

The first equation implies that when $x \neq -\frac{1}{2}$, we have

$$x - 2 = 6x + 3.$$

So $-5 = 5x$ and $x = -1$ is one solution.

The second equation implies that when $x \neq -\frac{1}{2}$, we have

$$x - 2 = -6x - 3.$$

So $7x = -1$ and $x = -\frac{1}{7}$ is the only other solution. ■

The absolute value of x is $|x| = \sqrt{x^2}$, so $d(x_1, x_2) = \sqrt{(x_1 - x_2)^2}$. A similar formula for the distance between points in the plane is given in the next section.

The absolute value of a number gives a measure of its distance from 0, as shown in Figure 10(a). Using this as a guide, we can define the **distance** $d(x_1, x_2)$ between the real number x_1 and the real number x_2, as shown in Figure 10(b).

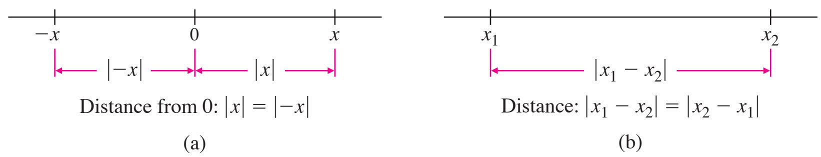


FIGURE 10

The Distance from x_1 to x_2

The distance between real numbers x_1 and x_2 is $d(x_1, x_2) = |x_1 - x_2|$.

For example,

$$d(5, 1) = d(1, 5) = 4 \quad \text{and} \quad d(-3.2, 4.5) = d(4.5, -3.2) = 7.7.$$

The distance formula is also used to determine a formula for the *midpoint* of two numbers, as illustrated in Figure 11.

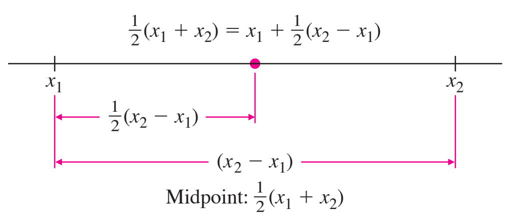


FIGURE 11

The midpoint of two numbers is just their average.

The Midpoint Formula

The midpoint of the line segment joining x_1 and x_2 is $\frac{1}{2}(x_1 + x_2)$.

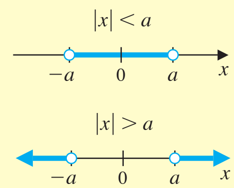
For example, the midpoint of the interval $[-1.2, 5.6]$ is

$$\frac{1}{2}(5.6 + (-1.2)) = \frac{1}{2}(4.4) = 2.2.$$

The third condition will be used frequently in calculus. It implies that $|x| < a$ precisely when the distance from x to the origin is less than a.

Absolute Value Properties

- $|ab| = |a|\,|b|$ and, if $b \neq 0$, $\left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|}$.
- $||a| - |b|| \leq |a + b| \leq |a| + |b|$.
- $|x| < a$ if and only if $-a < x < a$.
- $|x| > a \geq 0$ if and only if $x < -a$ or $x > a$.



EXAMPLE 9 Find all values of x that satisfy $|2x - 1| < 3$.

Solution Suppose we replace x with $2x - 1$ and a with 3 in the third Absolute Value Propert Then we have

$$|2x - 1| < 3 \quad \text{if and only if} \quad -3 < 2x - 1 < 3.$$

This implies that we need to solve both the inequalities

$$-3 < 2x - 1 \quad \text{and} \quad 2x - 1 < 3.$$

Adding 1 to both sides of each of the inequalities gives

$$-2 < 2x \quad \text{and} \quad 2x < 4.$$

Multiplying the inequalities by $\frac{1}{2}$ gives the solution

$$-1 < x \quad \text{and} \quad x < 2,$$

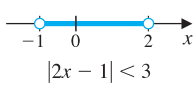


FIGURE 12

as shown in Figure 12.

Note that the numbers that are outside the interval shown in Figure 12 satisfy tl opposite inequality $|2x - 1| \geq 3$.

The problem in Example 9 can also be solved using the coordinate line and tl distance property of the absolute value of the difference of two real numbers. Sinc

$$|2x - 1| = \left|2\left(x - \tfrac{1}{2}\right)\right| = 2\left|x - \tfrac{1}{2}\right|,$$

the inequality $|2x - 1| < 3$ is equivalent to the inequality

$$\left|x - \tfrac{1}{2}\right| < \tfrac{3}{2}.$$

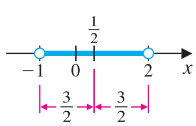


FIGURE 13

So, as shown in Figure 13, x satisfies the condition $|2x - 1| < 3$ precisely when

$$d\left(x, \tfrac{1}{2}\right) < \tfrac{3}{2}.$$

EXAMPLE 10 Find all values of x that satisfy $|-3x + 2| > 8$.

Solution The Absolute Value Properties imply that

$$|-3x + 2| > 8 \quad \text{if and only if} \quad -3x + 2 > 8 \text{ or } -3x + 2 < -8.$$

FIGURE 14

Subtracting 2 from both sides of each of the latter inequalities gives

$$-3x > 6 \quad \text{or} \quad -3x < -10.$$

Multiplying these inequalities by $-\frac{1}{3}$ (remembering to reverse the inequality) gives the solution, shown in Figure 14,

$$x < -2 \quad \text{or} \quad x > \tfrac{10}{3}.$$ ■

Applications

EXAMPLE 11 When a charity conducts a fund-raising campaign, it is common to describe the progress with respect to the midpoint value (half the goal). Find an expression for the amount raised relative to the midpoint value.

Solution Let R be the amount the charity wants to raise. Then the midpoint value is $M = R/2$. We need an expression describing A, the amount raised, relative to M.

Before considering the general question, let's look at a specific example. If the amount the charity wants to raise is $R = \$200{,}000$, then the midpoint amount is $M = \$100{,}000$. When the current amount raised is \$78,000, the charity reports that it is \$22,000 from half the goal. When the current amount raised is \$150,000, they report that they are \$50,000 beyond half the goal.

Principles of Problem Solving:

- Analyze some examples.
- Consider different cases.

In the general situation, let C be the amount currently raised. There are two cases that can occur.

Case 1: If $C \le M$, they report that they are below half the goal by

$$A = M - C \text{ dollars.}$$

Case 2: If $C \ge M$, they report that they are above half their goal by

$$A = C - M \text{ dollars.}$$

Because $C - M = -(M - C)$, in either case this can be expressed as

$$A = |M - C| = \left|\frac{R}{2} - C\right|.$$ ■

EXERCISE SET 1.2

In Exercises 1–8, express the interval using inequalities, and sketch the numbers in the interval.

1. $[-1, 5]$ **2.** $[-3, -1)$

3. $\left(-\sqrt{3}, \sqrt{2}\right]$ **4.** $\left[-\sqrt{5}, -\sqrt{2}\right)$

5. $(-\infty, 3)$ **6.** $(-\infty, 0]$

7. $[\sqrt{2}, \infty)$ **8.** $(-2, \infty)$

In Exercises 9–16, express the inequalities using interval notation, and sketch the numbers in the interval.

9. $-2 \le x \le 3$ **10.** $2 < x \le 6$

11. $2 \le x < 5$ **12.** $-2 \le x < 4$

13. $x < 3$ **14.** $x \le -2$

15. $x \ge 3$ **16.** $x > -2$

*In Exercises 17–20, find **(a)** the distance between the points and **(b)** the midpoint of the line segment connecting them.*

17. 3 and 7 **18.** 4 and 7

19. −3 and 5 **20.** −4 and −1

In Exercises 21–26, factor the quadratic equation.

21. $x^2 + 3x + 2$ **22.** $x^2 + 7x + 6$

23. $x^2 + 5x + 6$ **24.** $x^2 - 9x + 8$

25. $x^2 + 4x - 12$

26. $x^2 - 4x - 12$

*In Exercises 27–30, find **(a)** the intersection, and **(b)** the union of the intervals.*

27. $[-1, 3]$ and $(0, 4)$

28. $[-3, -1]$ and $[-2, 2)$

29. $(-\infty, 0)$ and $(-2, 3]$

30. $[1, \infty)$ and $[-3, 3)$

In Exercises 31–56, use interval notation to list the values of x that satisfy the inequality.

31. $x + 3 < 5$

32. $x - 4 < 9$

33. $2x - 2 \geq 8$

34. $3x + 2 \geq 8$

35. $-3x + 4 < 5$

36. $-2x - 4 \geq 10$

37. $2x + 9 \leq 5 + x$

38. $-3x - 2 < 3 - x$

39. $-1 < 3x - 3 < 6$

40. $-3 < 2x + 1 \leq 2$

41. $(x + 1)(x - 2) \geq 0$

42. $(x - 1)(x + 3) < 0$

43. $x^2 - 4x + 3 \leq 0$

44. $x^2 - 2x - 3 > 0$

45. $(x - 1)(x - 2)(x + 1) \leq 0$

46. $(x - 1)(x + 2)(x - 3) \geq 0$

47. $x^3 - 3x^2 + 2x \geq 0$

48. $x^3 - 3x^2 - 4x < 0$

49. $x^3 - 2x^2 < 0$

50. $x^3 - 2x^2 + x > 0$

51. $\dfrac{x + 3}{x - 1} \geq 0$

52. $\dfrac{x - 2}{x + 1} \leq 0$

53. $\dfrac{x(x + 2)}{x - 2} \leq 0$

54. $\dfrac{x + 2}{x(x - 2)} > 0$

55. $\dfrac{(1 - x)(x + 2)}{x(x + 1)} > 0$

56. $\dfrac{(1 - x)(x + 3)}{(x + 1)(2 - x)} \leq 0$

In Exercises 57–60, solve the inequality. (Hint: First rewrite the inequality by setting one side to 0.)

57. $\dfrac{1}{x} \leq 5$

58. $-2 \leq \dfrac{1}{x}$

59. $\dfrac{2}{x - 1} \geq \dfrac{3}{x + 2}$

60. $\dfrac{2}{x - 1} - \dfrac{x}{x + 1} \leq -1$

In Exercises 61–64, find all values of x that solve the equation.

61. $|5x - 3| = 2$

62. $|2x + 3| = 1$

63. $\left|\dfrac{x - 1}{2x + 3}\right| = 2$

64. $\left|\dfrac{2x + 1}{x - 3}\right| = 4$

In Exercises 65–72, solve the inequality and write the solution using interval notation.

65. $|x - 4| \leq 1$

66. $|4x - 1| < 0.01$

67. $|3 - x| \geq 2$

68. $|2x - 1| > 5$

69. $\dfrac{1}{|x + 5|} > 2$

70. $\left|\dfrac{3}{2x + 1}\right| < 1$

71. $|x^2 - 4| > 0$

72. $|x^2 - 4| \leq 1$

73. Show that if $0 < a < b$, then $a^2 < b^2$. (*Hint*: Show first that $a^2 < ab$ and $ab < b^2$.)

74. Solve the equation $|x - 1| = |2x + 2|$.

75. Degrees Celsius (C) and degrees Fahrenheit (F) are related by the formula $C = \frac{5}{9}(F - 32)$.

   a. What is the temperature range in the Celsius scal corresponding to a temperature range in Fahrenheit of $20 \leq F \leq 50$?

   b. What is the temperature range in the Fahrenheit scale corresponding to a temperature range in Celsius of $20 \leq C \leq 50$?

76. Calculus can be used to show that a ball thrown straight upward (neglecting air resistance) from the top of a building 128 feet high, with an initial velocity of 48 feet/second, has a height, in feet, of

$$h(t) = -16t^2 + 48t + 128$$

t seconds later. For what time interval will the ball b at least 64 feet above the ground?

77. You are contemplating investing \$50,000 between two different investments. One of the investments, a stable one, returns 5% annually. A more risky one returns 9% annually. You need a return of at least 7.5%. What is the maximum amount of the money that you can invest at the 5% rate?

1.3 THE COORDINATE PLANE

Calculus is the study of change between variables, and graphs help us visualize this change. To sketch graphs we first need a coordinate system on which to place the graphs.

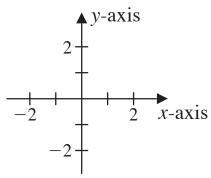


FIGURE 1

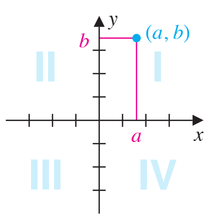


FIGURE 2

In Section 1.2 we saw how the set of real numbers relates to points on a coordinate line and how this relationship permits us to solve problems more easily. In this section we consider ordered pairs of real numbers and their relation to the coordinate plane.

Each point in the plane is associated with an ordered pair of real numbers. First, an arbitrary point in the plane is associated with $(0, 0)$ and designated the *origin*. Then horizontal and vertical lines are drawn, intersecting at the origin. The horizontal line is called the first-coordinate axis, or, commonly, the *x-axis*. The vertical line is called the second-coordinate axis, or the *y-axis*. (See Figure 1.)

A scale is placed on both axes. Since the x-axis is the same as the coordinate line introduced in the previous section, it is again labeled with positive numbers to the right of the origin and negative numbers to the left. Labeling on the y-axis is similar. The numbers above the origin are labeled as positive and the ones below the origin as negative.

The ordered pair (a, b) is associated with the point of intersection of the vertical line drawn through the point a on the x-axis and the horizontal line drawn through b on the y-axis. (See Figure 2.) We will not generally make a distinction between an ordered pair and the point it represents in a coordinate plane.

The x- and y-axes divide the plane into four regions, or *quadrants*, labeled as shown in Figure 2. The set of all ordered pairs of real numbers is denoted $\mathbb{R} \times \mathbb{R}$, or $\mathbb{R}^2$, and the plane determined by the x- and y-axes is called the *xy-plane*.

The coordinate plane is also called the Cartesian plane, and a rectangular coordinate system is called a Cartesian coordinate system. These names honor the versatile mathematician, philosopher, and physicist René Descartes (1596–1650), whose name in Latin was Renatus Cartesius. He introduced analytic geometry to the mathematical world in *La géométrie*, an appendix to his treatise on universal science.

EXAMPLE 1 Sketch the points in the coordinate plane associated with the ordered pairs $(1, 2)$, $(-1, 3)$, $(-2, -\pi)$, and $(\sqrt{2}, -\sqrt{3})$.

Solution To plot a point in the plane we locate the intersection of the vertical line through the first coordinate on the x-axis and the horizontal line through the second coordinate on the y-axis. These points are shown in Figure 3. ■

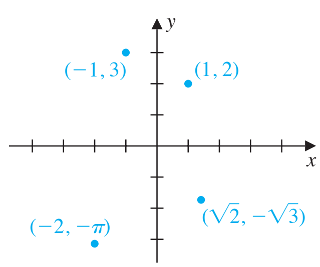


FIGURE 3

EXAMPLE 2 Sketch the points in the xy-plane that satisfy both $x > 2$ and $y \leq 1$.

Solution The points satisfying the single inequality $x > 2$ are shown in Figure 4(a). The dashe line at $x = 2$ indicates that these points are not in the set. The points satisfying $y \leq$ are shown in Figure 4(b). The points satisfying both conditions are in the shade region shown in Figure 4(c).

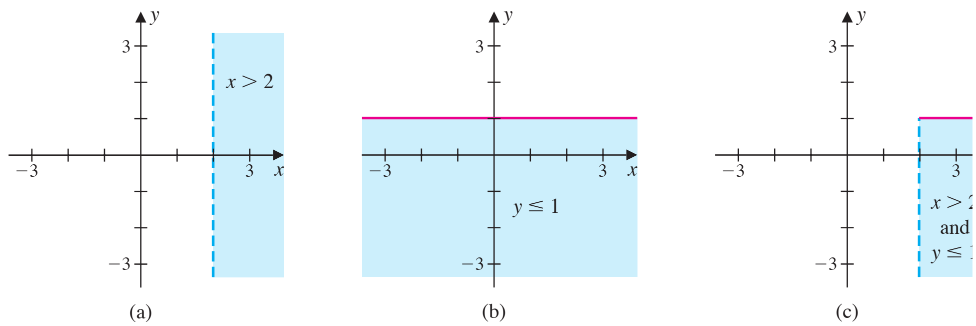


FIGURE 4

EXAMPLE 3 Sketch the points in the xy-plane that satisfy both the inequalities

$$1 \leq |x| \leq 2 \quad \text{and} \quad -1 < y < 3.$$

Solution The inequality $1 \leq |x| \leq 2$ implies that the distance from the x-coordinate of tl point to the y-axis is between 1 and 2 units. These are the points that lie on or betwee the vertical lines $x = -2$ and $x = -1$, together with those that lie on or between tl vertical lines $x = 1$ and $x = 2$. This region is shown shaded in Figure 5(a).

The points whose y-coordinates satisfy $-1 < y < 3$ lie strictly between tl dashed horizontal lines $y = -1$ and $y = 3$, shown in Figure 5(b). The poin satisfying both conditions are in the shaded regions shown in Figure 5(c).

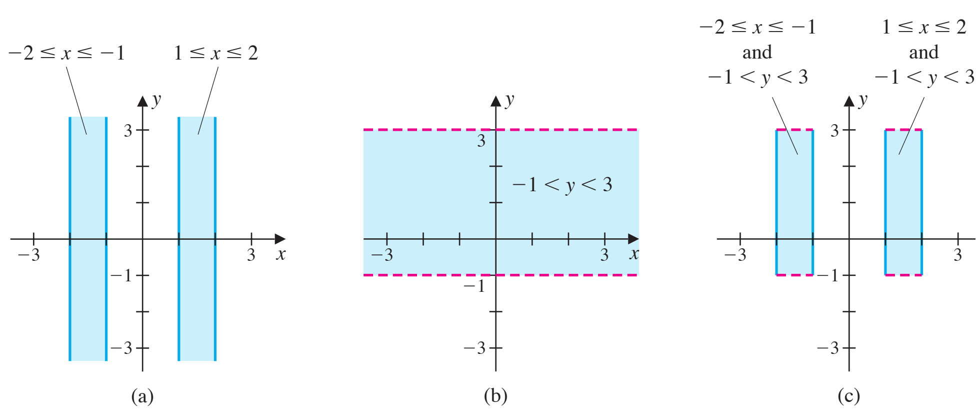
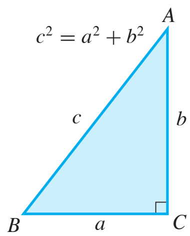


FIGURE 5

Distance between Points in the Plane



Notice the similarity between the distance between points in the plane and the distance between points on a line. In calculus you will see a similar formula for the distance between points in space.

The distance between two ordered pairs of real numbers (x_1, y_1) and (x_2, y_2) is found by introducing the third point (x_2, y_1) and applying the Pythagorean Theorem, as shown in Figure 6.

On the x-axis we have $d(x_1, x_2) = |x_1 - x_2|$, so the distance between (x_1, y_1) and (x_2, y_1) is $d((x_1, y_1), (x_2, y_1)) = |x_1 - x_2|$.

Similarly, $d((x_2, y_1), (x_2, y_2)) = |y_1 - y_2|$. Applying the Pythagorean Theorem gives

$$\begin{aligned} d((x_1, y_1), (x_2, y_2)) &= \sqrt{[d((x_1, y_1), (x_2, y_1))]^2 + [d((x_2, y_1), (x_2, y_2))]^2} \\ &= \sqrt{|x_1 - x_2|^2 + |y_1 - y_2|^2}. \end{aligned}$$

Since $|x_1 - x_2|^2 = (x_1 - x_2)^2$ and $|y_1 - y_2|^2 = (y_1 - y_2)^2$, we have the following result.

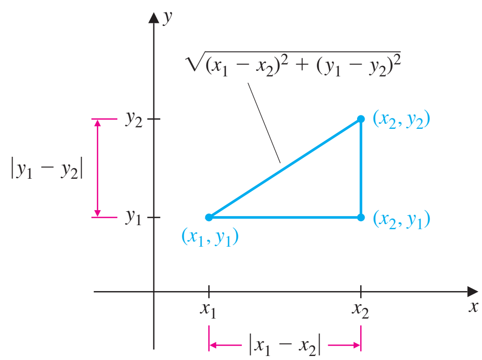


FIGURE 6

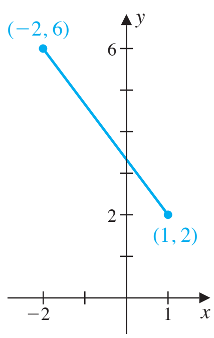


FIGURE 7

Distance between Points in the Plane

The distance between (x_1, y_1) and (x_2, y_2) is

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

For example, the distance between the points $(1, 2)$ and $(-2, 6)$, as shown in Figure 7, is

$$d((1, 2), (-2, 6)) = \sqrt{(1 - (-2))^2 + (2 - 6)^2} = \sqrt{9 + 16} = 5.$$

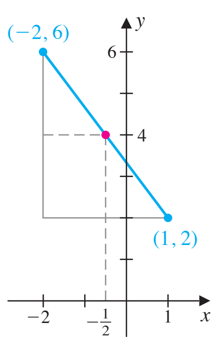


FIGURE 8

In Section 1.2 we used the distance formula between two numbers on a coordinate line to determine the midpoint of the numbers, which is the average of the endpoints. We use this same result to determine the midpoint of a line segment in the plane. An example is shown in Figure 8. It shows that the midpoint of the line segment with endpoints $(-2, 6)$ and $(1, 2)$ is

$$\left(\tfrac{1}{2}(-2 + 1), \tfrac{1}{2}(6 + 2)\right) = \left(-\tfrac{1}{2}, 4\right).$$

The Midpoint Formula

The midpoint of the line segment with endpoints (x_1, y_1) and (x_2, y_2) is

$$\left(\tfrac{1}{2}(x_1 + x_2), \tfrac{1}{2}(y_1 + y_2)\right).$$

Circles in the Plane

The distance formula for points in the plane can also be used to obtain the equation
a circle. A **circle** is the set of all points whose distance from a given point, the *cente*
is a fixed distance, the *radius*. Figure 9 shows that a point (x, y) will be on the circ
with center (h, k) having radius r precisely when

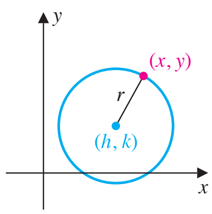


FIGURE 9

$$r = d((x, y), (h, k)) = \sqrt{(x-h)^2 + (y-k)^2}.$$

Squaring both sides produces the formula in its most familiar, or *standard*, form.

Circles in the Plane

The point (x, y) lies on the circle with center (h, k) and radius r precisely when

$$(x-h)^2 + (y-k)^2 = r^2.$$

This is the **standard form** of the equation of a circle.

The circle with center (h, k) is simply the unit circle scaled by the factor r and then moved h units horizontally and k units vertically.

We call a circle whose radius is 1 a **unit circle**. The applications involving the ur
circle whose center is at $(0, 0)$ are so extensive that this circle is frequently known
the unit circle. The points (x, y) on this unit circle are those that satisfy the equatic
$x^2 + y^2 = 1$.

EXAMPLE 4 Determine an equation for the circle of radius 3 centered at $(-1, 2)$.

Solution The equation has the form

$$(x - (-1))^2 + (y-2)^2 = 9 \quad \text{or} \quad (x+1)^2 + (y-2)^2 = 9.$$

Expanding the factors on the left side of the equation gives

$$x^2 + 2x + 1 + y^2 - 4y + 4 = 9.$$

This simplifies to $x^2 + y^2 + 2x - 4y = 4$.

EXAMPLE 5 Sketch the set of points that satisfies the inequality

$$1 < (x-2)^2 + (y-1)^2 \le 4.$$

Solution The equation

$$(x-2)^2 + (y-1)^2 = 1$$

describes the circle with center $(2, 1)$ and radius 1. The equation

$$(x-2)^2 + (y-1)^2 = 4$$

describes the circle with center $(2, 1)$ and radius 2. A point satisfies

$$(x-2)^2 + (y-1)^2 > 1$$

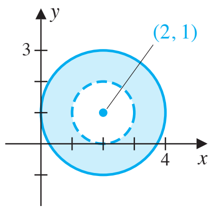


FIGURE 10

if its distance from the point $(2, 1)$ is greater than 1. Similarly, a point satisfies

$$(x-2)^2 + (y-1)^2 \le 4$$

if it lies inside or on the circle. So the points satisfying

$$1 < (x-2)^2 + (y-1)^2 \le 4$$

lie inside the circle of radius 4, including the points on the circle, and outside the circle of radius 1, excluding the points on the inner circle, as shown in Figure 10. ■

EXAMPLE 6 Find the equation of the circle with center $(-2, -1)$ that passes through the point $(-3, 1)$.

Solution The circle has center $(-2, -1)$, so its equation has the form

$$r^2 = (x-(-2))^2 + (y-(-1))^2 = (x+2)^2 + (y+1)^2.$$

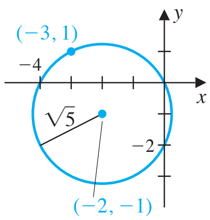


FIGURE 11

It remains to find the radius r. Since $(-3, 1)$ is on the circle, the equation is satisfied when $x = -3$ and $y = 1$. So

$$r = \sqrt{(-3+2)^2 + (1+1)^2} = \sqrt{1+4} = \sqrt{5}.$$

The circle shown in Figure 11 has equation

$$(x+2)^2 + (y+1)^2 = 5.$$ ■

Completing the Square

Completing the square follows from the fact that

$$(x + a/2)^2 = x^2 + ax + (a/2)^2.$$

All circles in the plane have equations of the form

$$x^2 + y^2 + ax + by = c.$$

We use a technique called **completing the square** to determine the center and radius of the circle. First group the x-terms and the y-terms separately and rewrite the equation as

$$x^2 + ax \quad + y^2 + by \quad = c.$$

A space is left after the terms ax and by because we will be adding constants that will make the x- and y-terms perfect squares.

The constants we need are $(a/2)^2$ and $(b/2)^2$, respectively. These terms must also be subtracted to ensure the equation has not changed. So the equation becomes

$$x^2 + ax + \left(\frac{a}{2}\right)^2 - \left(\frac{a}{2}\right)^2 + y^2 + by + \left(\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 = c.$$

This simplifies to

$$\left(x + \frac{a}{2}\right)^2 - \left(\frac{a}{2}\right)^2 + \left(y + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 = c$$

or to

$$\left(x+\frac{a}{2}\right)^2+\left(y+\frac{b}{2}\right)^2=c+\frac{a^2}{4}+\frac{b^2}{4}=\frac{4c+a^2+b^2}{4}.$$

Hence, the circle has center $(-a/2, -b/2)$ and radius $\frac{1}{2}\sqrt{4c+a^2+b^2}$, provided, o course, that $4c+a^2+b^2>0$.

There is no need to memorize the form of the resulting equation; just follow th procedure illustrated in Example 7.

EXAMPLE 7 Sketch the graph of the circle with equation

$$x^2+y^2+4x-6y=3.$$

Solution The equation can be regrouped as

$$x^2+4x \quad +y^2-6y \quad =3.$$

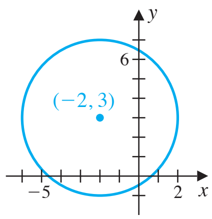


FIGURE 12

Principles of Problem Solving:

- Draw a picture to show the unknowns.
- Use equations to describe the problem.

To complete the square on the x-terms and y-terms, we add and subtract, respectivel

$$\left(\tfrac{4}{2}\right)^2=2^2=4 \quad \text{and} \quad \left(-\tfrac{6}{2}\right)^2=(-3)^2=9.$$

This produces the equation

$$(x^2+4x+4)-4+(y^2-6y+9)-9=3$$

or

$$(x+2)^2+(y-3)^2=3+4+9=16.$$

The equation of the circle in standard form is

$$(x-(-2))^2+(y-3)^2=4^2,$$

which describes the circle with center $(-2, 3)$ and radius 4 that is shown in Figure 1

Applications

EXAMPLE 8 A gas pipeline is to be constructed between points A and B on opposite sides of river, as shown in Figure 13. A surveyor first determines a third point C on the san side of the river as point A so that the lines between A and C and between B and are perpendicular. The distance from A to C is 300 feet and the distance from B to is 750 feet. How much pipe is required?

Solution The problem asks for the distance between the points A and B. Let z denote th distance, and $x=300$ and $y=750$ be the measured distances.

Because $\triangle ACB$ is a right triangle we can use the Pythagorean Theorem to fir z. This gives

$$\begin{aligned} z^2 &= x^2+y^2 \\ &= (300)^2+(750)^2 \\ &= 652500, \quad \text{so} \\ z &= \sqrt{652500} \approx 808 \text{ feet,} \end{aligned}$$

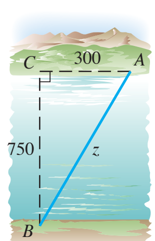


FIGURE 13

and the project requires approximately 808 feet of pipe.

EXERCISE SET 1.3

In Exercises 1–4, sketch the listed points in the same coordinate plane.

1. $(1, 0)$, $(0, 1)$, $(-1, 0)$, $(0, -1)$

2. $(0, 3)$, $(-1, 3)$, $(3, -2)$, $(-3, -1)$

3. $(2, 3)$, $(-2, -3)$, $(2, -3)$, $(-2, 3)$

4. $(5, -10)$, $(10, 20)$, $(-20, 10)$, $(-20, -10)$

*In Exercises 5–8, find **(a)** the distance between the points and **(b)** the midpoints of the line segments joining the points.*

5. $(2, 4)$, $(-1, 3)$

6. $(-3, 8)$, $(5, 4)$

7. $(\pi, 0)$, $(-1, 2)$

8. $(\sqrt{3}, \sqrt{2})$, $(\sqrt{2}, \sqrt{3})$

In Exercises 9–22, indicate on an xy-plane those points (x, y) for which the statement holds.

9. $x = 5$

10. $y = -2$

11. $x > 1$

12. $x < -2$

13. $x \geq 1$ and $y \geq 2$

14. $x < -3$ and $y < -4$

15. $-3 < y \leq 1$

16. $-1 \leq x \leq 2$

17. $2 \leq |x|$

18. $|y + 1| > 2$

19. $-1 \leq x \leq 2$ and $2 < y < 3$

20. $-3 < x \leq 1$ and $-1 \leq y \leq 2$

21. $|x - 1| < 3$ and $|y + 1| < 2$

22. $|x - 2| \leq 4$ and $|y + 3| < 7$

In Exercises 23–28, find the standard form of the equation of the circle, and sketch the graph.

23. center $(2, 0)$; radius 3

24. center $(0, 2)$; radius 3

25. center $(-2, 3)$; radius 2

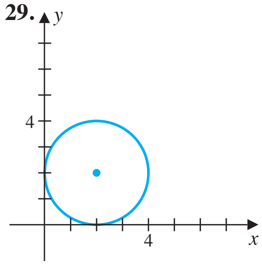
26. center $(-1, 4)$; radius 4

27. center $(-1, -2)$; radius 2

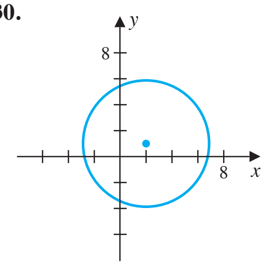
28. center $(-2, -1)$; radius 3

In Exercises 29 and 30, find the equation of the circle shown in the figure.

29.



30.



*In Exercises 31–36, **(a)** find the center and radius of each circle, and **(b)** sketch its graph.*

31. $x^2 + y^2 = 9$

32. $x^2 + y^2 = 2$

33. $x^2 + (y - 1)^2 = 1$

34. $(x + 1)^2 + y^2 = 9$

35. $(x - 2)^2 + (y + 1)^2 = 9$

36. $(x - 1)^2 + (y + 2)^2 = 16$

In Exercises 37–42, complete the square on the x and y terms to find the center and radius of the circle.

37. $x^2 - 2x + y^2 = 3$

38. $x^2 + y^2 + 4y = -3$

39. $x^2 + 2x + y^2 - 4y = -4$

40. $x^2 - 2x + y^2 + 4y = 4$

41. $x^2 - 4x + y^2 - 2y - 4 = 0$

42. $x^2 + 4x + y^2 + 6y + 9 = 0$

In Exercises 43–48, sketch the region in the xy-plane.

43. $\{(x, y) \mid x^2 + y^2 \leq 1\}$

44. $\{(x, y) \mid (x - 1)^2 + y^2 > 2\}$

45. $\{(x, y) \mid 1 < x^2 + y^2 < 4\}$

46. $\{(x, y) \mid 4 \leq (x - 1)^2 + (y - 1)^2 \leq 9\}$

47. $\{(x, y) \mid x^2 + y^2 \leq 4 \text{ and } y \geq x\}$

48. $\{(x, y) \mid x^2 + y^2 \geq 4 \text{ and } y \leq x\}$

49. Which of the points $(-7, 2)$ and $(6, 3)$ is closer to the origin?

50. Which of the points $(1, 6)$ and $(7, 4)$ is closer to $(3, 2)$?

51. **a.** Find the distances between the points $(-1, 4)$, $(-3, -4)$, and $(2, -1)$, and show that they are vertices of a right triangle.

b. At which vertex is the right angle?

52. Show that the points (2, 1), (−1, 2), and (2, 6) are vertices of an isosceles triangle.

53. Find a fourth point that will form the vertices of a rectangle when added to the points in Exercise 51. Is the point unique?

54. Find a fourth point that will form the vertices of a parallelogram when added to the points in Exercise 52. Is the point unique?

55. Find an equation of the circle with center (0, 0) that passes through (2, 3).

56. Find an equation of the circle with center (1, 3) that passes through (−2, 4).

57. Find an equation of the circle with center (3, 7) that is tangent to the y-axis.

58. Find a point on the y-axis that is equidistant from the points (2, 1) and (4, −3).

59. Find an equation of the circle whose center lies in the second quadrant, that has radius 3, and that is tangent to both the x-axis and the y-axis.

60. Find an equation for the circle whose center lies in the first quadrant, that has radius 2, and that is tangent to the lines $x = 1$ and $y = 2$.

61. Find the area of the region that lies outside the circle $x^2 + y^2 = 1$ and inside the circle $x^2 + y^2 = 9$. (*Note*: Area formulas are on the back inside cover.)

62. Find the area of the region that lies outside the circle $(x-1)^2 + y^2 = 1$ and inside the circle $(x-2)^2 + y^2 = 4$.

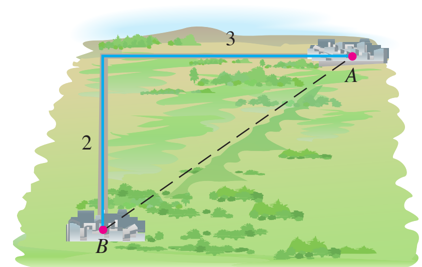
63. Indicate on an xy-plane those points for which $|x| + |y| \le 4$.

64. Indicate on an xy-plane those points for which $|x-1| + |y+2| \le 2$.

65. Find the area of the region containing the points satisfying $|x| + |y| \ge 1$ and $x^2 + y^2 \le 1$.

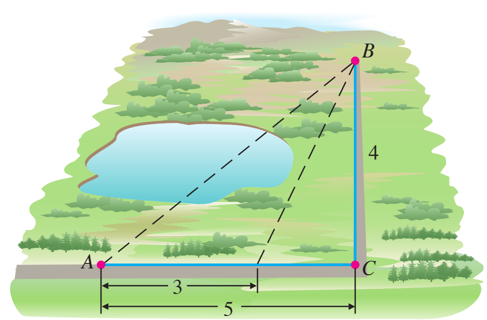
66. Find the area of the region containing the points satisfying $|x-1| + |y+2| \ge 2$ and $(x-1)^2 + (y+2)^2 \le 4$.

67. Towns A and B are connected by two roads that intersect at a right angle, as shown in the figure. The highway department needs to repair the roads and has decided instead to construct a new road that connects the two towns directly. The cost of eliminating the two existing sections of road is estimated at $50,00(and the cost of constructing the new road is $200,0(per mile. Estimate the cost of the project.



68. Two ships leave port 2 hours apart. Ship A leaves fi and travels due east at 15 miles per hour and ship B leaves 2 hours later due south at 8 miles per hour. When ship A is 5 hours out of port, estimate the distance between the two ships.

69. A gas pipeline has to be constructed between points and B, as shown in the figure. Roads connect points A and C and points C and B and the line can be buried alongside the road at $200,000 per mile. The pipe can also be buried directly between A and B at cost of $150,000 per mile, but the construction mus avoid the swamp shown in the figure. The decision has been made to bury the pipe along the road for 3 miles and then run it directly to point B. Estimate tl savings using this alternate route. Estimate the cost per mile to bury the line off-road that would make tl route along the entire existing road more economica



1.4 EQUATIONS AND GRAPHS

We use an **equation** to indicate that two mathematical expressions are equivalent. Sometimes the equation is an **identity**, which means that the equation is true for all values of the variable for which the equation is defined. For example, the equations

$$|x^2 - 4| = |x - 2||x + 2|, \quad |x| = \sqrt{x^2}, \quad \text{and} \quad \frac{1}{x^2 - 4} = \frac{1}{(x - 2)(x + 2)}$$

are identities. The first two hold for all real numbers x. The third holds only when $x \neq \pm 2$, because the denominator must be nonzero for the expression to be defined.

More often an equation is **conditional**, which means that it is true for some values of a variable, but not all. For conditional equations we need to determine and express the set of those values of the variable for which the equation is true. Equations such as

$$3x + 2 = 5, \quad x^2 - 3x + 2 = 0, \quad \text{and} \quad x^2 = -1$$

are conditional. The first has the single solution $x = 1$. The second is solved by factoring

$$0 = x^2 - 3x + 2 = (x - 1)(x - 2)$$

to give the solutions $x = 1$ and $x = 2$. The third conditional equation, $x^2 = -1$, has no real number solutions, because the right side is negative and the left side cannot be negative.

EXAMPLE 1 Determine all values of x that satisfy the following conditional equations.

a. $x^2 - 5x - 6 = 0$

b. $\dfrac{x - 3}{x^2 + x + 1} = 0$

c. $\dfrac{1}{x + 1} + 1 = \dfrac{1}{x^2 - x - 2}$

d. $\sqrt{x - 6} + \sqrt{x - 1} = 5$

Solution

a. The quadratic in the equation $x^2 - 5x - 6 = 0$ is first factored to give

$$0 = x^2 - 5x - 6 = (x + 1)(x - 6), \quad \text{which implies} \quad x + 1 = 0 \quad \text{or} \quad x - 6 = 0.$$

This gives the solutions $x = -1$ and $x = 6$.

b. The quotient can be 0 only when the numerator $x - 3$ is 0. In this case

$$x - 3 = 0 \quad \text{and the solution is} \quad x = 3.$$

c. First factor the quadratic as $x^2 - x - 2 = (x - 2)(x + 1)$. The factored form contains the terms of all the denominators in the equation, so we multiply both sides of the equation by $(x - 2)(x + 1)$ to produce

$$(x - 2)(x + 1)\left(\frac{1}{x + 1} + 1\right) = 1.$$

This simplifies to

$$x - 2 + x^2 - x - 2 = 1, \quad \text{so} \quad x^2 = 5 \quad \text{and} \quad x = \pm\sqrt{5}.$$

d. Rewriting the equation with one radical on each side gives

$$\sqrt{x-6} = 5 - \sqrt{x-1}.$$

Squaring both sides produces

$$x - 6 = 25 - 10\sqrt{x-1} + x - 1, \quad \text{so} \quad 10\sqrt{x-1} = 30,$$

and $\sqrt{x-1} = 3$. Squaring both sides once more gives

$$x - 1 = 9, \quad \text{so} \quad x = 10.$$

Notice that $x = 10$ is indeed a solution since $\sqrt{10-6} = \sqrt{4} = 2$, and th
$5 - \sqrt{10-1} = 5 - 3 = 2$.

We needed to verify that the value of x obtained in part (d) of Example 1 w
truly a solution, because the squaring operations could have produced solutions
the final equation that were not solutions to the original equation. These are call
extraneous solutions. For example, suppose that the equation in part (d) was instea

$$\sqrt{6-x} - \sqrt{1-x} = -5,$$

and that the operations similar to those in the example were performed. Square bo
sides of

$$\sqrt{6-x} = \sqrt{1-x} - 5 \quad \text{to obtain} \quad 6 - x = 1 - x - 10\sqrt{1-x} + 25,$$

which simplifies to $\sqrt{1-x} = 2$. Squaring again produces

$$1 - x = 4, \quad \text{so} \quad x = -3.$$

However, when we substitute this value back into the original equation we find th
it is not actually a solution because

$$\sqrt{6-(-3)} - \sqrt{1-(-3)} = \sqrt{9} - \sqrt{4} = 3 - 2 \neq 5.$$

As a consequence, there are no real numbers that satisfy the original equatio
$\sqrt{6-x} - \sqrt{1-x} = -5$.

Conditional equations might involve more than one variable, but the objective
the same. In the case of two variables, the objective is to determine which collecti
of ordered pairs satisfies the equation and gives a representation of the solution. F
example, the conditional equation in the two variables x and y given as $y = 2x +$
has as its solutions those pairs of real numbers of the form $(x, 2x+1)$, where x c
be any real number. A few of the solutions to this equation, then, are $(0, 1)$, $(2, 5$
and $(-3, -5)$.

Graphs of Equations

Graphing can add to your understanding of a problem. Draw a picture as the first step to solving a problem.

The **graph of an equation** in the variables x and y consists of the set of points $(x,$
in the xy-plane whose coordinates satisfy the equation.

EXAMPLE 2 Sketch the graph of the equation $2x + 3y = 6$.

Solution One way to obtain a rough sketch of the graph of an equation is to first find t
coordinates of several points, as we have done in Table 1. Then plot the points in t

TABLE 1

x	y
0	2
1	$\frac{4}{3}$
2	$\frac{2}{3}$
3	0
-1	$\frac{8}{3}$

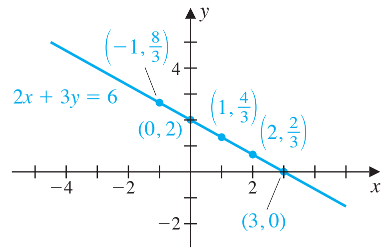


FIGURE 1

coordinate plane and connect those points, as best you can, with a curve. Figure 1 indicates that this process produces a straight line for the graph of the equation $2x + 3y = 6$. In Section 1.7 we will see that all equations of the form $Ax + By = C$ have graphs that are straight lines. ■

The points (0, 2) and (3, 0) shown on the graph of the line in Example 2 are the **axis intercepts** of the graph. They are found by setting one of the variables to zero and solving for the other. The ***x*-intercept** of the graph of $2x + 3y = 6$ occurs when $y = 0$. Hence

$$x\text{-intercept:} \qquad 2x = 6 \quad \text{so} \quad x = 3.$$

The ***y*-intercept** of the graph of $2x + 3y = 6$ occurs when $x = 0$. Hence

$$y\text{-intercept:} \qquad 3y = 6 \quad \text{so} \quad y = 2.$$

In general, all points of the form $(0, y)$ that satisfy an equation are called the y-intercepts of its graph, and the points of the form $(x, 0)$ that satisfy an equation are called the x-intercepts. When these points can be easily determined, they should be plotted on the graph.

EXAMPLE 3 Sketch the graph of the equation

$$y = \frac{x^2 + x - 2}{x - 1}.$$

Solution First notice that the numerator of the fraction can be factored to give

$$y = \frac{x^2 + x - 2}{x - 1} = \frac{(x - 1)(x + 2)}{x - 1} = x + 2, \quad \text{provided that } x \neq 1.$$

This means that the graph of

$$y = \frac{x^2 + x - 2}{x - 1}$$

is the same as the graph of $y = x + 2$, except it is not defined when $x = 1$.

As in Example 2, we plot several points on the graph of the equation and connect the points with a curve. The x-intercept occurs when $x = -2$, and the y-intercept occurs when $y = 2$. The result is the straight line shown in Figure 2.

The open circle at (1, 3) indicates that this point is missing from the graph. ■

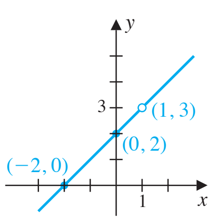


FIGURE 2

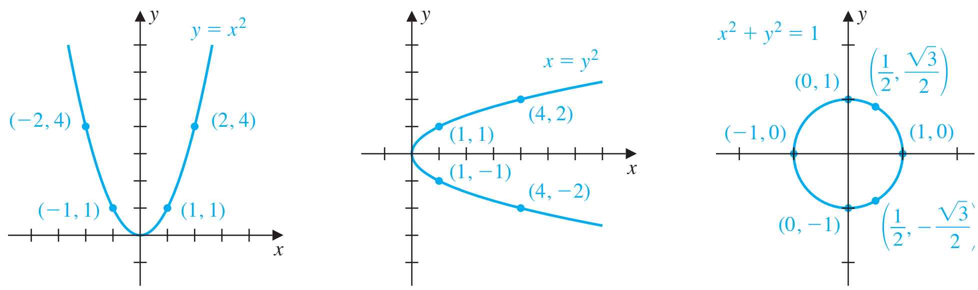
EXAMPLE 4 Consider the graphs of the following equations.

a. $y = x^2$ **b.** $x = y^2$ **c.** $x^2 + y^2 = 1$

Solution **a.** The graph of $y = x^2$ is the parabola shown in Figure 3(a). This graph was obtaine by plotting representative points that satisfy the equation and then connecting th points with a smooth curve.

b. This equation is the same as the equation in part (a) except that the roles of th two variables are interchanged. So the graph of $x = y^2$ is the parabola shown Figure 3(b).

c. We know from Section 1.3 that the graph of the equation $x^2 + y^2 = 1$ is the ur circle shown in Figure 3(c).



(a) (b) (c)

FIGURE 3

EXAMPLE 5 Use the graph of $x = y^2$ shown in Figure 3(b) to sketch the graph of $y = \sqrt{x}$.

Solution Solving the equation $x = y^2$ for y gives two solutions

$$y = \sqrt{x} \quad \text{and} \quad y = -\sqrt{x}.$$

The graph of $y = \sqrt{x}$ is the part of the graph of $x = y^2$ shown in Figure 3(b) th lies above the x-axis. Recall that the square root is defined only for positive valu of x, and that when $x > 0$ the value $\sqrt{x}$ is the positive square root of x. The graph shown in Figure 4.

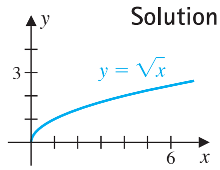


FIGURE 4

Symmetry of a Graph

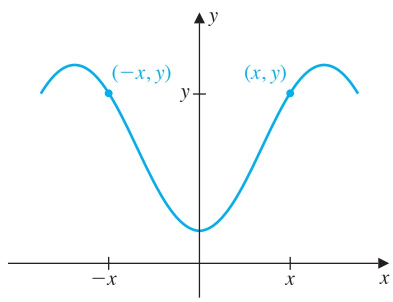
The graphs of the equations in Example 4 illustrate a feature known as *symmetry a line*. A graph is **symmetric to a line** when the portion of the graph on one side the line is the mirror image of the portion on the other side. Symmetry of a graph a line is easy to determine when the line is one of the coordinate axes.

Axis symmetry is also called *symmetry with respect to the axis*.

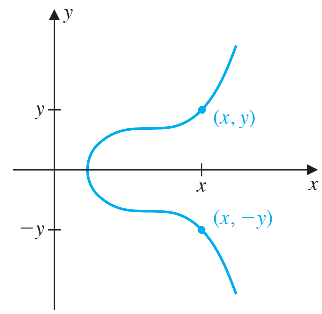
Coordinate Axis Symmetry of a Graph (see Figure 5)

- The graph of an equation has ***y*-axis symmetry** if $(-x, y)$ is on the graph whenever (x, y) is on the graph.
- The graph of an equation has ***x*-axis symmetry** if $(x, -y)$ is on the graph whenever (x, y) is on the graph.

The graph of $y = x^2$ in Figure 3(a) has y-axis symmetry because $(-x)^2 = x^2$. The graph in Figure 3(b) of $x = y^2$ has x-axis symmetry because $(-y)^2 = y^2$. The graph of the circle with equation $x^2 + y^2 = 1$ in Figure 3(c) has both x-axis and y-axis symmetry.



y-axis symmetry



x-axis symmetry

FIGURE 5

EXAMPLE 6 Use symmetry to sketch the graph of $y = |x|$.

Solution When $x > 0$, the graph is the same as $y = x$. In addition, the absolute value satisfies $|-x| = |x|$. This means that $(-x, y)$ is on the graph whenever (x, y) is on the graph, and the graph has the y-axis symmetry shown in Figure 6. ■

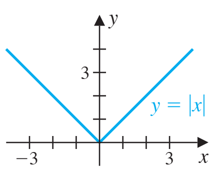


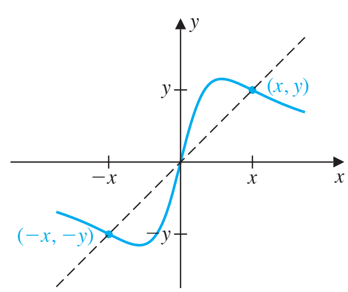
FIGURE 6

Symmetry is also defined *with respect to a point* in the plane. In this case, the graph has a mirror reflection property with respect to the point. This feature might be difficult to detect for arbitrary points in the plane, but it is easy when the point is the origin. A graph having this symmetry is shown in Figure 7. The unit circle shown in Figure 3(c) also has this property.

Origin symmetry is also called *symmetry with respect to the origin*.

Origin Symmetry of a Graph

The graph of an equation has **origin symmetry** if $(-x, -y)$ is on the graph whenever (x, y) is on the graph.



Origin symmetry

FIGURE 7

EXAMPLE 7 Determine any symmetry properties of the *cubing* function $y = x^3$.

Solution Suppose (x, y) satisfies the equation $y = x^3$. Since $(-x)^3 = -x^3 = -y$, the poi $(-x, -y)$ also satisfies the equation, and the graph has origin symmetry. For exampl the point $(2, 8)$ is on the graph of $y = x^3$, as is $(-2, -8)$ since $(-2)^3 = -8$.

Figure 8 shows a graph of $y = x^3$ along with several points on the graph. Lat in this chapter we will introduce more aids to graphing. You will learn other valuab techniques when studying calculus.

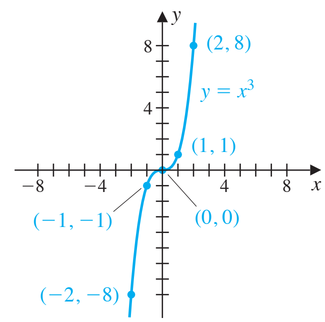


FIGURE 8

Applications

EXAMPLE 8 A homeowner wants to create a garden surrounded by a brick border contained in plot with an area of 384 square feet. The garden is to be rectangular with a leng twice its width. The brick border is 4 feet wide on all sides. **(a)** Find the dimensio of the garden, and **(b)** estimate the number of 10-inch by 4-inch bricks required f the border.

Solution

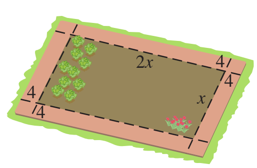


FIGURE 9

a. The situation is shown in Figure 9, where x denotes the width of the garden. Th length of the garden is $2x$ and the border is 4 feet, so the area, in square feet, the garden with its brick border is

$$(x + 8)(2x + 8) = 2x^2 + 24x + 64.$$

This is specified to be 384, so

$$2x^2 + 24x + 64 = 384, \quad \text{which simplifies to} \quad x^2 + 12x - 160 = 0.$$

Factoring the quadratic equation $x^2 + 12x - 160$ gives

$$0 = x^2 + 12x - 160 = (x - 8)(x + 20), \quad \text{so} \quad x = 8 \quad \text{or} \quad x = -20.$$

Since x must be positive, the garden has width 8 feet and length $2 \cdot 8 = 16$ feet

b. The area of the garden is $8 \cdot 16 = 128$ square feet, and the garden together with i border has area 384 square feet, so the area of the walkway is $384 - 128 = 25$ square feet. Each 10-inch by 4-inch brick has a face area of

$$10 \text{ inches}\left(\frac{1 \text{ foot}}{12 \text{ inches}}\right) \cdot 4 \text{ inches}\left(\frac{1 \text{ foot}}{12 \text{ inches}}\right) = \frac{5}{18} \text{ square feet.}$$

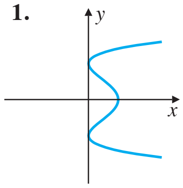
So the required number of bricks is

$$\frac{256}{5/18} = \frac{256 \cdot 18}{5} \approx 922. \qquad \blacksquare$$

EXERCISE SET 1.4

In Exercises 1–8, specify any axis or origin symmetry of the graphs.

1.



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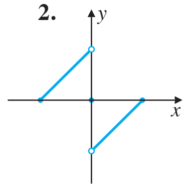

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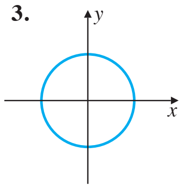

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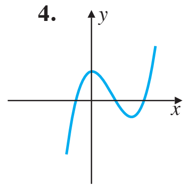

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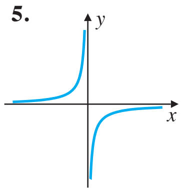

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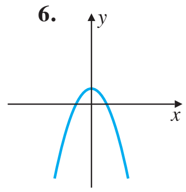

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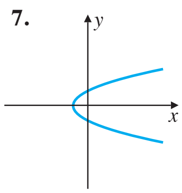

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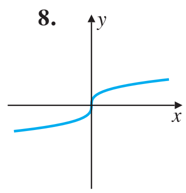


In Exercises 9–38, determine any axis intercepts and describe any axis or origin symmetry.

9. $y = x + 3$

10. $y = 2x - 3$

11. $x + y = 1$

12. $-2x - y = 2$

13. $y = x^2 - 3$

14. $y = x^2 + 2$

15. $y = 1 - x^2$

16. $2y = x^2$

17. $y = -2x^2$

18. $y = 3 - 3x^2$

19. $x = y^2 - 1$

20. $x = y^2 - 4$

21. $y = x^3 + 1$

22. $3y = x^3$

23. $y = -x^3$

24. $y = -x^3 + 1$

25. $y = \dfrac{(x+3)(x-3)}{x-3}$

26. $y = \dfrac{(x+3)(x+1)}{x+1}$

27. $y = \dfrac{x^2 - x - 6}{x+2}$

28. $y = \dfrac{x^2 + 2x - 3}{x+3}$

29. $y = \sqrt{x} + 2$

30. $y = \sqrt{x - 1}$

31. $x^2 + y^2 = 4$

32. $(x-1)^2 + y^2 = 1$

33. $y = \sqrt{9 - x^2}$

34. $y = -\sqrt{9 - x^2}$

35. $y = |x|$

36. $y = |x - 1|$

37. $y = |x| - 1$

38. $y = 2 - |x|$

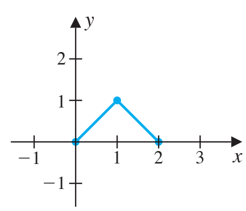
In Exercises 39–42, complete the graph in the figure so the curve has the specified symmetry.

39. x-axis symmetry

40. y-axis symmetry

41. origin symmetry

42. x-axis and y-axis symmetry



43. Find the distance between the points of intersection of the graphs $y = x^2 + 1$ and $y = 2$.

44. Find the distance between the points of intersection of the graphs $y = x^2 - 3$ and $y = x + 3$.

45. Determine three consecutive positive integers whose sum is 156.

46. Determine two consecutive positive integers whose squares sum to 925.

47. Find the dimensions of the rectangle whose area is 12 and whose perimeter is 14.

48. Find the area of a square whose side length is one fourth the value of its area.

49. A automobile radiator has 10 quarts of water with a 10% concentration of antifreeze. How much of the liquid in the radiator should be drained and replaced with pure antifreeze to ensure that the radiator contains water that has a 30% concentration of antifreeze?

50. Metal alloy A contains 20% copper, and metal alloy B contains 45% copper. How many pounds of each alloy should be combined to produce 100 pounds of new alloy that contains 35% copper?

51. Symmetry was discussed in the text with respect to the x-axis, the y-axis, and the origin. Show that if a graph has x-axis and y-axis symmetry, it must also have origin symmetry.

52. You are contemplating investing $10,000 between two bond funds. One fund is less risky, and expecte to return 5% annually. The riskier fund is expected t return 8% annually. If you would like an overall return of 6%, how much should you place in each fund?

1.5 USING TECHNOLOGY TO GRAPH EQUATIONS

Calculators with extensive graphing capabilities cost no more than a standard textboo and are easily worth their price. In addition, powerful computer algebra systems suc as Maple, Mathematica, Mathcad, and Derive are available on most campuses, ar all these contain sophisticated graphing techniques. Technology is becoming mo advanced and cheaper, and portions of these computer algebra systems are now inco porated into calculators and even into phones. In this book we take a generic approac indicating where technology can be useful without giving instructions specific to ar particular device. Technology is not necessary for an understanding of the materi in this book, but it will be helpful if you use it intelligently.

Graphing calculators and computer algebra systems sketch graphs of equatior quickly by plotting as many points on the graph as the resolution of the screen w permit. When using a graphing device to plot an equation, you must be careful ensure that all the interesting aspects of the graph have been displayed. Using tecl nology without an understanding of the underlying concepts can result in acceptir misleading information. This is particularly true in the case of plotting curves.

In calculus you will need a familiarity with the graphs of a library of functions (se the inside front cover). Technology can help in visualizing curves but an understandir of the concepts underlying how graphs are constructed is essential.

The plots in this section and throughout this book were generated using a cor puter algebra system. A rectangular portion of the plane is called a **viewing rectang** for a plot and is defined by specifying the range of values for x and the range of valu for y. We denote a viewing rectangle specified by the inequalities

$$a \le x \le b \quad \text{and} \quad c \le y \le d \qquad \text{as } [a, b] \times [c, d].$$

Choosing an appropriate viewing rectangle is essential for obtaining a represer tative plot of an equation, and understanding the concepts of graphing is essential f recognizing when you have a good representation. The following examples show ho to use technology to plot curves and how to choose an appropriate viewing rectang to maximize the information.