Steenrod Operations and Steenrod Algebra

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Contents

1 Introduction

In this report we will study ...

2 Background

2.1 Theorem environments. Labels and references

Theorem 2.1. *Let M be an abelian group. Then*

$$\operatorname{Ker}(\operatorname{Id}\colon M \to M) = 0.$$

Definition 2.2. Let R be a ring and let $\operatorname{Mod} R$ be the category of left R-modules. Define ...

Theorem 2.3. *Assume that ... Then ...*

Proof. See [4], [1, Theorem 5], [2, §3.4], [3, Lemma 2.1] – **Citations**. □

Lemma 2.4. *We have*

$$1 + 1 = 2. \tag{1}$$

Proof. Eq. (1) follows from Theorem 2.3. See also Eq. (2) from Section 2.4. – **References**. □

Remark 2.5. Note that ... ◊

2.2 Lists

1.
2.
3.

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2.3 Fonts

Mathcal:

$$\mathcal{A}, \mathcal{B}, \mathcal{C}.$$

Mathbb:

$$\mathbb{A}, \mathbb{B}, \mathbb{C}, \mathbb{N}, \mathbb{Q}, \mathbb{R}.$$

Greek letters:

$$\alpha, \beta, \gamma$$

2.4 Math formulas

Sets

$$\mathbb{N} = \{\, n \in \mathbb{Z} \mid n \geq 0 \,\}.$$

Norms

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|, \qquad x \in \mathbb{R}^n.$$

Large brackets

$$2 \cdot \left(\sum_{i=1}^{n} i \right) = n(n+1), \qquad \sum_{i=1}^{n} i = \binom{n+1}{2}.$$

Equation with a number

$$2 \cdot 2 = 4. \tag{2}$$

Multiline equation

$$\begin{aligned}\sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1} \right) \\ = \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{n} - \frac{1}{n+1} = 1 - \frac{1}{n+1} = \frac{n}{n+1}\end{aligned}$$

2.5 Cases

$$(-1)^n = \begin{cases} 1 & n \text{ is even,} \\ -1 & n \text{ is odd.} \end{cases}$$

2.6 Commutative diagrams

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d] \arrow[rr, bend left, "\alpha"] & B \arrow[r, "\psi"] \arrow[d, "y"'] & B' \arrow[dl] \\ C \arrow[r, "x"'] & D \arrow[loop right, "f"] & \end{tikzcd}$$

2.7 Yondeda Lemma

Cohomology operations are operations between cohomology groups that commute with homomorphisms induced by continuous maps. Thus, they provide us with another means of distinguishing spaces.

Definition 2.6 (*Cohomology Operation*)**.** A *cohomology operation* is a map $\Theta = \Theta_X \colon H^m(X;G) \to H^n(X;H)$ between cohomology groups, for any space X, and fixed m, n, G, H, satisfying the naturality property that for any map $f \colon X \to Y$ between spaces, the following diagram commutes:

$$\begin{tikzcd} H^m(Y;G) \arrow[r, "\theta_Y"] \arrow[d, "f^*"] & H^n(Y;H) \arrow[d, "f^*"] \\ H^m(X;G) \arrow[r, "\Theta_X"'] & H^n(X;H) \end{tikzcd}$$

Theorem 2.7. *[?, Proposition 4L.1] Fix m, $n \in \mathbb{Z}$, G, H groups. For a space Z, there is a bijection between the set of cohomology operations $\Theta \colon H^m(Z;G) \to H^n(Z;H)$ and $H^n(K(G,m);H)$ given by $\Theta \mapsto \Theta(\iota)$, where ι is a fundamental class in $H^m(K(G,m);G)$.*

We will state and prove a more general result in category theory, the *Yoneda Lemma*, and apply this result to prove Theorem 2.7. We first introduce the more general concept of a natural transformation between contravariant functors.

Definition 2.8 (*Natural Transformation*)**.** If F and G are contravariant functors between categories C and D, a *natural transformation* $\eta \colon F \to G$ is a transformation such that for all maps $g \colon X \to Y$ in C, the following diagram commutes:

$$\begin{tikzcd} X \arrow[d, "g"'] & FY \arrow[r, "\eta_Y"] \arrow[d, "Fg"] & GY \arrow[d, "Gg"] \\ Y & FX \arrow[r, "\eta_X"] & GX \end{tikzcd}$$

Proposition 2.9. *Let C be the category of CW complexes and morphisms homotopy classes of continuous maps. Then cohomology operations are natural transformations from C to C.*

Proof. Immediate from definitions. □

We give a contravariant argument of the Yoneda lemma, so as to apply it to cohomology operations directly. The covariant argument is analogous.

Theorem 2.10 (*Yoneda Lemma*)**.** *Let C be a category, and X an object of C. Let $h^X\colon C^{\mathrm{op}} \to$ **Set** be the contravariant functor $h^X = \mathrm{Hom}(-, X)$. Then for any contravariant set-valued functor $F\colon C^{\mathrm{op}} \to$ **Set**, we have a bijection between the natural transformations from h^X to F and $FX \in$ **Set**, that is,*

$$FX \simeq \mathrm{Nat}(h^X, F)$$

Proof. Consider a natural transformation $\eta\colon h^X \to F$. Then for any object Y in C and a map $g\colon Y \to X$, the following square commutes, where $(h^X g)(\beta) = \beta \circ g$ for $\beta \in h^X X$, and η_X, η_Y are the components of η at X, Y respectively.

$$\begin{tikzcd} Y \arrow[d, "g"'] & h^X X \arrow[r, "\eta_X"] \arrow[d, "h^X g"] & FX \arrow[d, "Fg"] \\ X & h^X Y \arrow[r, "\eta_Y"] & FY \end{tikzcd}$$

Let $1_X \in h^X X$ be the identity map. Then $\eta_X(1_X) \in FX$ and

$$\eta_Y(h^X g)(1_X) = \eta_Y(g) = Fg(\eta_X(1_X)).$$

Thus for every object Y, η_Y is determined by $\eta_X(1_X)$. We define $\tau\colon \mathrm{Nat}(h^X, F) \to FX$ by $\tau(\eta) = \eta_X(1_X)$.

Conversely, any $g\colon Y \to X$, gives rise to $Fg\colon FX \to FY$. Let $x \in FX$. We wish to define a natural transformation $\lambda(x)\colon h^X \to F$. We define components map $(\lambda(x))_Y\colon h^X Y \to FY$ given by

$$(\lambda(x))_Y(g) = Fg(x).$$

Then $\lambda(x)$ is a natural transformation, that is given any $f\colon Z \to Y$, we claim we have the following commuting:

$$\begin{tikzcd} Z \arrow[d, "f"'] & h^X Y \arrow[r, "(\lambda(x))_Y"] \arrow[d, "h^X f"] & FY \arrow[d, "Ff"] \\ Y & h^X Z \arrow[r, "(\lambda(x))_Z"] & FZ \end{tikzcd}$$

Indeed, $g\colon Y \to X$, $g \in h^X Y$ has $Ff(Fg(x)) = F(g \circ f)(x) = F(h^X f)(g)(x)$. Therefore we can define $\lambda\colon FX \to \mathrm{Nat}(h^X, F)$, $x \mapsto \lambda(x)$.

Finally, we need to show that τ and λ are inverses. For $x \in FX$, we have $\tau(\lambda(x)) = (\lambda(x))_X(1_X) = F(1_X)(x) = 1_{FX}(x)$, so that $\tau \circ \lambda = 1_{FX}$. For $\eta \in \mathrm{Nat}(h^X, F)$, we have $\lambda(\tau(\eta)) = \lambda(\eta_X(1_X))$. Then for any object Y and any $g\colon Y \to X$, $\lambda(\eta_X(1_X))_Y(g) = Fg(\eta_X(1_X)) = \eta_Y(g)$ by above. Thus $\lambda(\eta_X(1_X)) = \eta$, so that $\lambda \circ \tau = 1_{\mathrm{Nat}(h^X, F)}$.

□

Corollary 2.11. *Let C be a category, X, Y objects in C. Then,*

$$\mathrm{Hom}(X,Y) \simeq \mathrm{Nat}\left(\mathrm{Hom}(-,X),\mathrm{Hom}(-,Y)\right)$$

Proof. Let $F = h^Y$, and apply Yoneda lemma. □

We are now in a position to prove Theorem 2.7.

Proof of Theorem 2.7. By CW-approximation, it suffices to prove the statememt for the case of Z a CW-complex. Then we can identify $H^m(Z;G)$ with $[Z, K(G,m)]$ and likewise $H^n(Z;H)$ with $[Z, K(H;n)]$.

By Corollary 2.11,

$$\mathrm{Hom}(K(G,m),K(H,n)) \simeq \mathrm{Nat}\left(\mathrm{Hom}(Z,K(G,m)),\mathrm{Hom}(Z,K(H,n)\right) \simeq \mathrm{Nat}(H^m(Z;G),H^n(Z;H)),$$

but the natural transformations between the cohomology groups are cohomology operations by Proposition 2.9, and $\mathrm{Hom}(K(G,m),K(H,n))$ is $H^n(K(G,m);H)$.

Let $K = K(G,m))$. The map τ from the proof of the Yoneda lemma sends a cohomology operation Θ to $\Theta_K(1_K)$, where 1_K is the identity map on K. Then $\Theta_K(1_K) = \Theta(\iota)$ for $\iota \in H^m(K,G) = H^m(K(G,m);G)$ with ι a fundamental class since $1_K = 1^*\iota = \iota$. □

References

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[4] Edward Witten, *Topological quantum field theory*, Comm. Math. Phys. **117** (1988), no. 3, 353–386.