

Steenrod Operations and Steenrod Algebra

Edwina Aylward

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Introduction and Motivation

In this report we will study ...

1 Background

1.1 Theorem environments. Labels and references

Theorem 1.1. *Let M be an abelian group. Then*

$$\text{Ker}(\text{Id}: M \rightarrow M) = 0.$$

Definition 1.2. Let R be a ring and let $\text{Mod } R$ be the category of left R -modules. Define ...

Theorem 1.3. *Assume that ... Then ...*

Proof. See [4], [1, Theorem 5], [2, §3.4], [3, Lemma 2.1] – **Citations**. □

Lemma 1.4. *We have*

$$1 + 1 = 2. \tag{1}$$

Proof. Eq. (1) follows from Theorem 1.3. See also Eq. (2) from Section 1.4. – **References**. □

Remark 1.5. Note that ... ◊

1.2 Lists

- 1.
- 2.
- 3.
-
-

1.3 Fonts

Mathcal:

$$\mathcal{A}, \mathcal{B}, \mathcal{C}.$$

Mathbb:

$$\mathbb{A}, \mathbb{B}, \mathbb{C}, \mathbb{N}, \mathbb{Q}, \mathbb{R}.$$

Greek letters:

$$\alpha, \beta, \gamma$$

1.4 Math formulas

Sets

$$\mathbb{N} = \{ n \in \mathbb{Z} \mid n \geq 0 \}.$$

Norms

$$\|x\|_1 = \sum_{i=1}^n |x_i|, \quad x \in \mathbb{R}^n.$$

Large brackets

$$2 \cdot \left(\sum_{i=1}^n i \right) = n(n+1), \quad \sum_{i=1}^n i = \binom{n+1}{2}.$$

Equation with a number

$$2 \cdot 2 = 4. \tag{2}$$

Multiline equation

$$\begin{aligned}\sum_{k=1}^n \frac{1}{k(k+1)} &= \sum_{k=1}^n \left(\frac{1}{k} - \frac{1}{k+1} \right) \\ &= \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{n} - \frac{1}{n+1} = 1 - \frac{1}{n+1} = \frac{n}{n+1}\end{aligned}$$

1.5 Cases

$$(-1)^n = \begin{cases} 1 & n \text{ is even,} \\ -1 & n \text{ is odd.} \end{cases}$$

1.6 Commutative diagrams

$$\begin{array}{ccccc} & & \alpha & & \\ & A & \xrightarrow{\phi} & B & \xrightarrow{\psi} B' \\ & \downarrow & & \downarrow y & \\ C & \xrightarrow{x} & D & \xleftarrow{f} & \end{array}$$

1.7 Brown Representability Theorem

Consider the category of pointed CW-complexes, with morphisms that are homotopy equivalence classes of continuous maps between objects. Let X be an object in our category. Brown's representability theorem states that any reduced cohomology theory $h^*(X)$ is of the form $h^n(X) = [X, E_n]$, where $[-, -]$ denotes the set of basepoint preserving maps up to homotopy equivalence, and $\{E_n\}_{n \in \mathbb{Z}}$ is a sequence of spaces that is a Ω -spectrum.

Definition 1.6. A sequence $\{E_n\}_{n \in \mathbb{Z}}$ of spaces is a Ω -spectrum if $\Sigma E_n \xrightarrow{\sim} E_{n+1}$ is a weak homotopy equivalence, or considering the adjoint, $E_n \xrightarrow{\sim} \Omega E_{n+1}$ is a weak homotopy equivalence.

Definition 1.7. A reduced cohomology theory for a space X is

Theorem 1.8. Let X be a pointed CW-complex, and let $\{E_n\}$ be a Ω -spectrum. Then $h^*(X)$ with $h^n(X) = [X, E_n]$ for all $n \in \mathbb{Z}$ defines a reduced cohomology theory.

Proof. yo □

Theorem 1.9 (Brown's Representability theorem). Let X be a pointed CW-complex, and let $h^*(X)$ be a reduced cohomology theory on X . Then there exists a Ω -spectrum of [Eilenberg MacLane Spaces](#) $\{E_n\}_{n \in \mathbb{Z}}$ such that $h^n(X) = [X, E_n]$ for all $n \in \mathbb{Z}$.

1.8 Yoneda Lemma

Cohomology operations are operations between cohomology groups that commute with homomorphisms induced by continuous maps. Thus, they provide us with another means of distinguishing spaces.

Definition 1.10 (Cohomology Operation). A cohomology operation is a map $\Theta = \Theta_X: H^m(X; G) \rightarrow H^n(X; H)$ between cohomology groups, for any space X , and fixed m, n, G, H , satisfying the

naturality property that for any map $f: X \rightarrow Y$ between spaces, the following diagram commutes:

$$\begin{array}{ccc} H^m(Y; G) & \xrightarrow{\theta_Y} & H^n(Y; H) \\ \downarrow f^* & & \downarrow f^* \\ H^m(X; G) & \xrightarrow{\Theta_X} & H^n(X; H) \end{array}$$

Theorem 1.11. [5, Proposition 4L.1] Fix $m, n \in \mathbb{Z}$, G, H groups. For a space Z , there is a bijection between the set of cohomology operations $\Theta: H^m(Z; G) \rightarrow H^n(Z; H)$ and $H^n(K(G, m); H)$ given by $\Theta \mapsto \Theta(\iota)$, where ι is a fundamental class in $H^m(K(G, m); G)$.

We will state and prove a more general result in category theory, the *Yoneda Lemma*, and apply this result to prove Theorem 1.11. We first introduce the more general concept of a natural transformation between contravariant functors.

Definition 1.12 (*Natural Transformation*). If F and G are contravariant functors between categories C and D , a *natural transformation* $\eta: F \rightarrow G$ is a transformation such that for all maps $g: X \rightarrow Y$ in C , the following diagram commutes:

$$\begin{array}{ccc} X & & FY \xrightarrow{\eta_Y} GY \\ g \downarrow & & \downarrow Fg \\ Y & & FX \xrightarrow{\eta_X} GX \end{array}$$

Proposition 1.13. Let C be the category of CW complexes and morphisms homotopy classes of continuous maps. Then cohomology operations are natural transformations from C to C .

Proof. Immediate from definitions. □

We give a contravariant argument of the Yoneda lemma, so as to apply it to cohomology operations directly. The covariant argument is analogous.

Theorem 1.14 (*Yoneda Lemma*). Let C be a category, and X an object of C . Let $h^X: C^{\text{op}} \rightarrow \mathbf{Set}$ be the contravariant functor $h^X = \text{Hom}(-, X)$. Then for any contravariant set-valued functor $F: C^{\text{op}} \rightarrow \mathbf{Set}$, we have a bijection between the natural transformations from h^X to F and $FX \in \mathbf{Set}$, that is,

$$FX \simeq \text{Nat}(h^X, F)$$

Proof. Consider a natural transformation $\eta: h^X \rightarrow F$. Then for any object Y in C and a map $g: Y \rightarrow X$, the following square commutes, where $(h^X g)(\beta) = \beta \circ g$ for $\beta \in h^X Y$, and η_X, η_Y are the components of η at X, Y respectively.

$$\begin{array}{ccc} Y & & h^X X \xrightarrow{\eta_X} FX \\ g \downarrow & & \downarrow h^X g \\ X & & h^X Y \xrightarrow{\eta_Y} FY \end{array}$$

Let $1_X \in h^X X$ be the identity map. Then $\eta_X(1_X) \in FX$ and

$$\eta_Y(h^X g)(1_X) = \eta_Y(g) = Fg(\eta_X(1_X)).$$

Thus for every object Y , η_Y is determined by $\eta_X(1_X)$. We define $\tau: \text{Nat}(h^X, F) \rightarrow FX$ by $\tau(\eta) = \eta_X(1_X)$.

Conversely, any $g: Y \rightarrow X$, gives rise to $Fg: FX \rightarrow FY$. Let $x \in FX$. We wish to define a natural transformation $\lambda(x): h^X \rightarrow F$. We define components map $(\lambda(x))_Y: h^X Y \rightarrow FY$ given by

$$(\lambda(x))_Y(g) = Fg(x).$$

Then $\lambda(x)$ is a natural transformation, that is given any $f: Z \rightarrow Y$, we claim we have the following commuting:

$$\begin{array}{ccc} Z & & h^X Y \xrightarrow{(\lambda(x))_Y} FY \\ f \downarrow & & \downarrow h^X f \\ Y & & h^X Z \xrightarrow{(\lambda(x))_Z} FZ \end{array}$$

Indeed, $g: Y \rightarrow X$, $g \in h^X Y$ has $Ff(Fg(x)) = F(g \circ f)(x) = F(h^X f)(g)(x)$. Therefore we can define $\lambda: FX \rightarrow \text{Nat}(h^X, F)$, $x \mapsto \lambda(x)$.

Finally, we need to show that τ and λ are inverses. For $x \in FX$, we have $\tau(\lambda(x)) = (\lambda(x))_X(1_X) = F(1_X)(x) = 1_{FX}(x)$, so that $\tau \circ \lambda = 1_{FX}$. For $\eta \in \text{Nat}(h^X, F)$, we have $\lambda(\tau(\eta)) = \lambda(\eta_X(1_X))$. Then for any object Y and any $g: Y \rightarrow X$, $\lambda(\eta_X(1_X))_Y(g) = Fg(\eta_X(1_X)) = \eta_Y(g)$ by above. Thus $\lambda(\eta_X(1_X)) = \eta$, so that $\lambda \circ \tau = 1_{\text{Nat}(h^X, F)}$. \square

Corollary 1.15. *Let C be a category, X, Y objects in C . Then,*

$$\text{Hom}(X, Y) \simeq \text{Nat}(\text{Hom}(-, X), \text{Hom}(-, Y))$$

Proof. Let $F = h^Y$, and apply Yoneda lemma. \square

We are now in a position to prove Theorem 1.11.

Proof of Theorem 1.11. By CW-approximation, it suffices to prove the statement for the case of Z a CW-complex. Then we can identify $H^m(Z; G)$ with $[Z, K(G, m)]$ and likewise $H^n(Z; H)$ with $[Z, K(H; n)]$.

By Corollary 1.15,

$$\text{Hom}(K(G, m), K(H, n)) \simeq \text{Nat}(\text{Hom}(Z, K(G, m)), \text{Hom}(Z, K(H, n))) \simeq \text{Nat}(H^m(Z; G), H^n(Z; H)),$$

but the natural transformations between the cohomology groups are cohomology operations by Proposition 1.13, and $\text{Hom}(K(G, m), K(H, n))$ is $H^n(K(G, m); H)$.

Let $K = K(G, m)$. The map τ from the proof of the Yoneda lemma sends a cohomology operation Θ to $\Theta_K(1_K)$, where 1_K is the identity map on K . Then $\Theta_K(1_K) = \Theta(\iota)$ for $\iota \in H^m(K, G) = H^m(K(G, m); G)$ with ι a fundamental class since $1_K = 1^*\iota = \iota$. \square

2 Steenrod Operations

2.1 Introduction, Applications, Examples

2.2 Adem relations

Theorem 2.1. *The Adém relations hold for Steenrod squares*

Proof. We have the diagonal action $\mathbb{Z}_p \times \mathbb{Z}_p \curvearrowright S^\infty \times S^\infty$, with quotient isomorphic to $L^\infty \times L^\infty$. Also $\mathbb{Z}_p \times \mathbb{Z}_p \curvearrowright X^{\wedge p^2}$ by writing points of $X^{\wedge p^2}$ as a $p \times p$ matrix and letting the first \mathbb{Z}_p act on the rows, and the second \mathbb{Z}_p act on the columns.

Therefore, $\mathbb{Z}_p \times \mathbb{Z}_p \curvearrowright S^\infty \times S^\infty \times X^{\wedge p^2}$ with quotient space $\Gamma_z X$ \square

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