ℓ-parity results over global function fields

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Global Function Fields

- $K = \mathbb{F}_q(C)$ is the function field of a smooth, projective, geometrically irreducible curve $C/\mathbb{F}_q$, q a p-power.
- A closed point of C is a $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$-orbit of a point $P \in C(\overline{\mathbb{F}}_q)$, and

$$\text{Places of } K \leftrightarrow \text{closed points of } C.$$

- $\mathcal{O}_x =$ local ring of functions that are regular at a closed point x, with maximal ideal $\mathfrak{m}_x =$ functions vanishing at x.
- Residue field: $\kappa_x = \mathcal{O}_x/\mathfrak{m}_x$, with $\deg(x) := [\kappa_x : \mathbb{F}_q]$, $\quad |\kappa_x| = q^{\deg(x)}$.
- Completion: K_x.

Example

Consider $C = \mathbb{P}^1_{\mathbb{F}_q}$ with function field $K = \mathbb{F}_q(t)$.

- The closed points of C correspond to monic irreducible polynomials in $\mathbb{F}_q[t]$, together with the point at infinity.
- For a closed point x corresponding to an irreducible polynomial $f(t)$:

$$\mathcal{O}_x = \left\{ \frac{g}{h} \in \mathbb{F}_q(t) : f \nmid h \right\}, \quad \mathfrak{m}_x = (f), \quad \kappa_x = \mathbb{F}_q[t]/(f) \simeq \mathbb{F}_{q^{\deg(f)}}, \quad K_x = \kappa_x((f))$$

- The point at infinity corresponds to the valuation $v_\infty(f/g) = \deg(g) - \deg(f)$.

Abelian Varieties

- An abelian variety A/K is a complete connected algebraic group.
- In other words, it's an elliptic curve, the Jacobian of a smooth projective curve, or something else.
- Mordell-Weil theorem: its group of K-rational points $A(K)$ is a finitely generated abelian group:

$$A(K) \simeq A(K)_{\mathrm{tors}} \oplus \mathbb{Z}^r, \quad r = \mathrm{rk}(A/K).$$

- For $\ell \neq p$, the ℓ-adic Tate module:

$$T_\ell(A) = \varprojlim_n A[\ell^n](K^{\mathsf{sep}}), \quad V_\ell(A) = T_\ell(A) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell,$$

carries a continuous $G_K = \mathrm{Gal}(K^{\mathsf{sep}}/K)$-action.

L-functions and BSD

Given an abelian variety A/K, its Hasse–Weil L-function is given by an Euler product ($\ell \neq p$):

$$L(A/K,s) = \prod_x P_x\left(q^{(1-s)\cdot\deg(x)}\right)^{-1}, \qquad P_x(T) = \det(1-\varphi_x T \mid V_\ell(A)^{I_x}).$$

where

- x ranges over all closed points of C,
- $I_x \leq G_K$ is the inertia subgroup at x,
- φ_x is the geometric Frobenius at x, acting on $\overline{\kappa}_x$ by $y \mapsto y^{1/\#\kappa_x}$.

Converges absolutely for $\mathrm{Re}(s) > \frac{3}{2}$ (Weil bounds).

Birch–Swinnerton-Dyer Conjecture

$$\mathrm{rk}\, A/K = \mathrm{ord}_{s=1} L(A/K,s).$$

BSD over function fields fact file

- Unbounded rank: There exist explicit families of elliptic curves over function fields with arbitrarily large rank.
- Tate: $\mathrm{rk}\, A/K \leq \mathrm{ord}_{s=1} L(A/K, s)$.
- BSD for Jacobians $\implies$ BSD for all abelian varieties.
- Kato–Trihan (2003): BSD is equivalent to showing the finiteness of any ℓ-primary part of the Tate-Shafarevich group $\text{Ш}(A/K)$.
- Kato–Trihan (2003): Weak BSD implies refined BSD.
- BSD for Jacobians: look at surface over $\mathbb{F}_q$ corresponding to curve. Néron-Severi groups, Ш is a Brauer group, Artin–Tate conjecture for surfaces . . .

Example

Example

Consider $E\colon y^2 = x^3 + t^2x + t^2$ over $K = \mathbb{F}_3(t) = \mathbb{F}_3(\mathbb{P}^1)$. Then E has bad reduction at t and at ∞, and good reduction elsewhere. The L-funtion of E can be computed using the formula[a]

$$L(E/K, T) = \frac{Z(C,T)Z(C,qT)}{Z(\mathcal{E},T)} \prod_{\text{bad } v} \frac{(1-T)^{a_v+1}(1+T)^{b_v}}{(1-q_v T^{\deg(v)})^{f_v-1}(1+q_v T^{\deg(v)})^{g_v}},$$

where $\mathcal{E}/\mathbb{F}_3$ is the elliptic surface associated to E. One computes

$$L(E/K, T) = 1 - 9T^2, \qquad L(E/K, 3^{-s}) = 2\log 3 \cdot (s-1) - 2(\log 3)^2 \cdot (s-1)^2 + \cdots$$

Thus $\mathrm{ord}_{s=1} L(E/K, s) = 1$. The point $(t, 0)$ has infinite order.

[a] Proposition 6.1, *Elliptic curves over function fields*, Douglas Ulmer.

L-functions are rational functions

Theorem (Grothendieck)

$$L(A/K,s) = \frac{L_1(s)}{L_0(s)\cdot L_2(s)}, \quad L_i(s) = \det(1 - q^{1-s}\varphi_\ell, \mathcal{H}^i_{\mathbb{Q}_\ell}) \in \mathbb{Q}[q^{-s}],$$

where $\mathcal{H}^i_{\mathbb{Q}_\ell}$ are étale cohomology groups with an action of φ_ℓ induced by geometric Frobenius.

Idea: Representations $\leftrightarrow$ sheaves correspondence and Grothendieck-Lefschetz trace formula.

- It follows that $L(A/K,s)$ is a rational function in q^{-s} and admits a meromorphic continuation to $\mathbb{C}$.
- Duality results $\rightsquigarrow$ $L(A/K,s)$ satisfies a functional equation centered at $s=1$.
- Deligne: Eigenvalues of φ_ℓ acting on $\mathcal{H}^i_{\mathbb{Q}_\ell}$ have absolute value $q^{(i-1)/2}$.
- Thus

$$\begin{aligned}\mathrm{ord}_{s=1} L(A/K,s) &= \text{multiplicity of 1 as a root of characteristic polynomial of } \varphi_\ell \\ &= \text{dimension of part of } \mathcal{H}^i_{\mathbb{Q}_\ell} \text{ on which } \varphi_\ell \text{ acts unipotently.}\end{aligned}$$

The Parity Conjecture

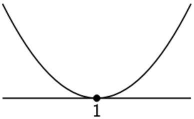
BSD

$$\operatorname{rk} A/K = \operatorname{ord}_{s=1} L(A/K, s).$$

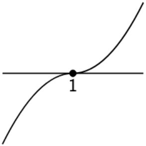
Functional equation

$$L(A/K, 2-s) = w(A/K) \cdot q^{\alpha(s)} \cdot L(A/K, s).$$

$\omega(E/K) = +1$



$\omega(E/K) = -1$



Parity conjecture

$$(-1)^{\operatorname{rk} A/K} = w(A/K), \quad w(A/K) = \pm 1.$$

The global root number $w(A/K)$ is the product of local root numbers. Because the functional equation holds in the global function field setting,

$$w(A/K) = (-1)^{\operatorname{ord}_{s=1} L(A/K,s)}.$$

Thus the parity conjecture can be restated as

$$\operatorname{ord}_{s=1} L(A/K, s) \equiv \operatorname{rk} A/K \mod 2.$$

ℓ-parity conjecture

Define the ℓ^∞-Selmer group $\mathrm{Sel}_{\ell^\infty}(A/K) = \varinjlim_n \mathrm{Sel}_{\ell^n}(A/K)$ where

$$\mathrm{Sel}_{\ell^n}(A/K) = \ker(H^1(G_K, A[\ell^n]) \to \prod_v H^1(G_{K_v}, A(K_v^{\mathrm{sep}})).$$

This fits into the following exact sequence of cofinitely generated $\mathbb{Z}_\ell$-modules.

$$0 \to A(K) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell \to \mathrm{Sel}_{\ell^\infty}(A/K) \to Ш(A/K)[\ell^\infty] \to 0.$$

Set $\mathcal{X}_\ell(A/K) = \mathrm{Hom}(\mathrm{Sel}_{\ell^\infty}(A/K), \mathbb{Q}_\ell/\mathbb{Z}_\ell) \otimes \mathbb{Q}_\ell$. The dimension of $\mathcal{X}_\ell(A/K)$ is the Mordell–Weil rank of A/K plus the number of copies of $\mathbb{Q}_\ell/\mathbb{Z}_\ell$ in $Ш(A/K)$.

ℓ-parity conjecture

$$(-1)^{\dim_{\mathbb{Q}_\ell} \mathcal{X}_\ell(A/K)} = w(A/K).$$

If $|Ш(A/K)[\ell^\infty]| < \infty$ then ℓ-parity conjecture $\implies$ parity conjecture.

Semi-simplicity. What, like it's hard?

Let $r_\ell = \dim_{\mathbb{Q}_\ell} \mathcal{X}_\ell(A/K)$. By work of Kato and Trihan, it turns out that

$$r_\ell = \dim_{\mathbb{Q}_\ell} \ker(1 - \varphi_\ell \mid \mathcal{H}^1_{\mathbb{Q}_\ell}).$$

Set

$$\mathcal{I}_{2,\ell} = \ker(1 - \varphi_\ell \mid \mathcal{H}^1_{\mathbb{Q}_\ell}), \qquad \mathcal{I}_{3,\ell} = \text{part of } \mathcal{H}^1_{\mathbb{Q}_\ell} \text{ where } \varphi_\ell \text{ acts unipotently.}$$

Recall that $\mathrm{ord}_{s=1} L(A/K, s) = \dim_{\mathbb{Q}_\ell} \mathcal{I}_{3,\ell}$. One has $\mathcal{I}_{2,\ell} \subset \mathcal{I}_{3,\ell}$ with equality iff. φ_ℓ acts semi-simply on $\mathcal{H}^1_{\mathbb{Q}_\ell}$.

Theorem (Trihan–Yasuda (2014))

The ℓ-parity conjecture for abelian varieties is true over global function fields for any prime ℓ. In other words,

$$\dim_{\mathbb{Q}_\ell} \mathcal{I}_{2,\ell} \equiv \dim_{\mathbb{Q}_\ell} \mathcal{I}_{3,\ell} \mod 2.$$

Overview of proof

Trihan and Yasuda construct a perfect pairing

$$(\cdot,\cdot)_\ell \colon \mathcal{I}_{3,\ell} \times \mathcal{I}_{3,\ell} \to \mathbb{Q}_\ell.$$

that is symmetric and compatible with Frobenius action φ_ℓ. This comes from the Weil-pairing on $V_\ell(A)$.

Thus one can view φ_ℓ as a unipotent element of the orthogonal group $O((\cdot,\cdot)_\ell)$. Linear algebra says that such a unipotent element satisfies

$$\det(-\varphi_\ell) = (-1)^{\dim \ker(1-\varphi_\ell)}.$$

Trihan-Yasuda's more general parity result

The exact same argument to deduce the ℓ-parity conjecture for abelian varieties yields the following result.

Theorem

Let F_ℓ be a smooth $\overline{\mathbb{Q}}_\ell$-sheaf on an open subset $U \subset C$. Assume that F_ℓ is pure of weight -1 and equipped with a skew-symmetric non-degenerate pairing $F_\ell \times F_\ell \to \overline{\mathbb{Q}}_\ell(1)$. Then

$$r_{an}(F_\ell) \equiv r(F_\ell) \mod 2.$$

Here $r_{an}(F_\ell)$ is the order of vanishing at $s = 1$ of the L-function $L(U, F_\ell, s)$, and $r(F_\ell) = \ker(1 - \varphi_\ell \mid \mathcal{H}^1_{\mathbb{Q}_\ell}(F_\ell))$.

In other words, if you have

- A continuous ℓ-adic representation $\tau\colon G_K \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ that is unramified outside a finite number of places.
- For all places v where τ is unramified, the eigenvalues of $\tau(\varphi_v)$ have absolute value $\kappa_v^{-1/2}$.
- τ has a G_K-equivariant skew-symmetric pairing $\tau \times \tau \to \overline{\mathbb{Q}}_\ell(1)$.

Then you get a parity-like result for τ.

Application: ℓ-parity for twists

Consider a Galois extension F/K with Galois group G and an abelian variety A/K. Then the L-function $L(A/F, s)$ factors into a product of twisted L-functions

$$L(A/F, s) = \prod_{\rho \in \mathrm{Irr}_{\mathbb{C}}(G)} L(A, \rho, s)^{\dim \rho}.$$

The group $A(F) \otimes_{\mathbb{Z}} \mathbb{C}$ is a G-representation, and the analogue of BSD for twists conjectures that

$$\langle \rho, A(F) \otimes_{\mathbb{Z}} \mathbb{C} \rangle = \mathrm{ord}_{s=1} L(A, \rho, s),$$

for a representation ρ of G.

Similarly, $\mathcal{X}_\ell(A/F)$ inherits a G-action, and the ℓ-parity conjecture for twists states that

$$(-1)^{\langle \rho, \mathcal{X}_\ell(A/F) \rangle} = w(A, \rho).$$

Corollary

The ℓ-parity conjecture for twists of abelian varieties by orthogonal Artin representations holds for all primes $\ell \neq p$.

Proof sketch

Proof:

1. Let ρ be an orthogonal Artin representation factoring through a finite Galois extension F/K, with $G = \mathrm{Gal}(F/K)$. View this as a continuous ℓ-adic representation.
2. Set $W = \mathrm{Res}_K^F A$, the Weil restriction of A/F. This is an abelian variety of dimension $[L\colon K] \cdot \dim A$. The group G acts on W by K-automorphisms, and we can consider $V_\ell(W)$ as a $G \times G_K$-representation.
3. Consider the G_K-representation $\sigma = \mathrm{Hom}_G(\rho, V_\ell(W))$, where G_K acts trivially on ρ.
4. Then σ is unramified outside a finite set of places, and when v is unramified, $\sigma(\varphi_v)$ has eigenvalues of absolute value $\kappa_v^{-1/2}$.
5. Since ρ is orthogonal it is equipped with a symmetric G_K-equivariant pairing $\rho \times \rho \to \overline{\mathbb{Q}}_\ell$ and so σ has a G_K-equivariant skew-symmetric pairing $\sigma \times \sigma \to \overline{\mathbb{Q}}_\ell(1)$.

Therefore

$$r_{\mathrm{an}}(F_{\sigma,\ell}) \equiv r(F_{\sigma,\ell}) \mod 2,$$

where $F_{\sigma,\ell}$ is the sheaf corresponding to σ.

Proof sketch

Proof:

6. Claim: $r_{\mathrm{an}}(F_{\sigma,\ell}) = \mathrm{ord}_{s=1} L(A, \rho, s)$.
7. Claim: $r(F_{\sigma,\ell}) = \langle \rho, \mathcal{X}_\ell(W/K) \rangle$.
8. $\mathcal{X}_\ell(W/K) \simeq \mathcal{X}_\ell(A/F)$ as G-representations.

So

$$\mathrm{ord}_{s=1} L(A, \rho, s) \equiv \langle \rho, \mathcal{X}_\ell(A/F) \rangle \mod 2.$$

Why did I want this?

Local global methods for local fields of positive characteristic.

Thank you for your attention!