

Disturbance observer-based feedback linearization control of an unmanned quadrotor helicopter

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Abstract

Feedback linearization is widely used for the purpose of quadrotor control. Unfortunately, feedback linearization is highly sensitive to any quadrotor model uncertainties. This paper provides feedback linearization-based control with robustness by integrating it with a disturbance observer. The proposed approach maintains the simplicity of the control structure without ignoring the high nonlinearities existing in the model by considering these nonlinearities as disturbances to be attenuated by the disturbance observer. Thus, the requirement to include complex high-order Lie derivatives in the controller is eliminated even in the presence of the high nonlinearities. Simulation results show that the proposed controller successfully force the quadrotor to follow the desired position and heading trajectories in the presence of different types of disturbances including ignored nonlinear dynamics, wind disturbances and partial actuator failure.

Keywords

Disturbance observer, Dryden wind, feedback linearization, quadrotor

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Introduction

During the previous years, there has been a growing interest in quadrotor helicopters due to their wide variety of applications. Unfortunately, quadrotors are found to be unstable systems. Thus, the potential to use these vehicles has raised the interest of the control systems community to design controllers that guarantee their stability and improve their performance during flight.

In the literature, both linear and nonlinear control designs have been presented. Although linear control techniques have been implemented in several research projects and proved capable of regulating the quadrotor system, they are only valid while operating near the hovering state. Otherwise, nonlinear control methods are used such as feedback linearization-based controllers.

In 2004, Mokhtari and Benallegue,¹ feedback linearization-based control is designed to control the quadrotor rotational and altitude dynamics. Whereas, open loop control is used to control the horizontal position dynamics. Thus, the position dynamics stabilization is not guaranteed. In other words, open loop control cannot drive the position error to zero if either disturbing forces or initial position errors occur.

Moreover, in the studies conducted by Das et al.² and Zhou et al.,³ the quadrotor model is divided into two loops. The inner attitude loop is controlled using feedback linearization based control in which the horizontal position dynamics is considered to be unstable zero dynamics. Thus, a closed loop control is developed for the outer position loop dynamics to overcome the problems of the open loop control. Although the position loop dynamics is nonlinear, the outer loop is controlled using linear controllers.

In the work by Voos,⁴ a cascaded controller is designed where the quad-rotor model is divided into an inner attitude loop, controlled by feedback linearization, and an outer velocity loop, controller using a proportional controller and a nonlinear transformation. Cascaded control approach has to respect the time-scale

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separation between the two loops. In other words, the outer loop dynamics has to be much slower than the inner loop dynamics because the inner loop dynamics is ignored while designing the outer loop controller.

Lee et al.,⁵ proposed a feedback linearization based controller with dynamic extension. Although this controller overcomes the problem of the cascaded control techniques, some highly nonlinear dynamics are ignored when designing the feedback linearization-based controller. Moreover, high order derivatives of the output vector are required. To get these derivatives, they are either measured using sensors or calculated by differentiating the measured output vector.

As mentioned above, feedback linearization is widely used for the application of quadrotor control. Unfortunately, feedback linearization techniques suffer from their high sensitivity to disturbances that may be applied to the system.⁶ These disturbances may be exogenous such as those due to the different types of wind. Whereas others may be endogenous such as the ones due to parameter uncertainties and unmodeled dynamics. Several approaches are developed to provide feedback linearization with robustness against disturbances. Mokhtari and Benallegue¹ utilized an estimator to estimate the wind parameter affecting the quadrotor. Zhou et al.³ utilized the integral terms of the PID controllers to add robustness to the feedback linearization based controller. In 2005, Mokhtari et al.,⁷ presented a robust feedback linearization augmented with a linear GH_∞ controller which provides robustness against model uncertainties. The developed approach is evaluated in the presence of constant external disturbance, parameter uncertainties and measurement noise. Benallegue et al.⁸ designed a feedback linearization based controller integrated with a sliding mode observer to estimate and compensate for external wind disturbances applied to the quadrotor. The developed controller is evaluated in the presence of sinusoidal disturbances.

One robust method used to attenuate the effect of disturbances applied to quadrotors is the disturbance observer-based control. A sliding mode disturbance observer augmented with a sliding mode controller is designed by Besnard et al.⁹ for quadrotor trajectory tracking in the presence of constant disturbances. Hancer et al.,¹⁰ utilized a frequency domain disturbance observer where the quadrotor is divided into six single input single output subsystems; one for each degree of freedom. In each subsystem, a PD controller is augmented with a disturbance observer to stabilize the quadrotor's position and attitude in the presence of Dryden wind. Wang and Chen¹¹ designed a time domain disturbance observer with a sliding mode controller for the attitude control in the case of constant and sinusoidal disturbances.

The main aim of this paper is to provide feedback linearization with robustness against different types of

endogenous and exogenous disturbances by augmenting it with a disturbance observer. Feedback linearization aims at stabilizing the quadrotor dynamics in the absence of disturbances. Whereas, the disturbance observer aims at estimating the existing disturbances influencing the quadrotor. Finally, a linear state feedback controller is designed to stabilize the quadrotor feedback linearized dynamics and compensate for the disturbances using the disturbance observer estimates.

Utilizing the disturbance observer maintains the simplicity of the control structure without ignoring any of the high nonlinearities occurring in the quadrotor model. This is done by considering these nonlinearities as part of the disturbances to be estimated by the disturbance observer. Hence, the requirement to calculate complex higher order Lie derivatives is avoided even though no model simplifications are assumed. Moreover, no high order output derivatives are required to be measured and used in the linear controller.

Finally, the developed control approach is evaluated in simulations by applying it to a quadrotor model in the presence of high nonlinearities acting as ignored dynamics, wind disturbances and partial actuator failure disturbances. Wind disturbances are generated using real-life model combining constant wind model, discrete wind gust and dryden wind turbulence model. Actuator failure disturbances are generated by assuming that the thrust force generated by one of the propellers at a certain angular speed is lower than those generated by the other propellers at the same speed.

The paper is organized as follows: the quadrotor's nonlinear dynamic model including disturbances due to unmodeled dynamics, wind and actuator failure is presented in the second section. Then, the disturbance observer-based feedback linearization controller is designed in the third section. In the fourth section, the simulation results are shown and discussed followed by the conclusion in the fifth section.

Dynamic model

Quadrotor nonlinear model

The equations of motion of quadrotors have been derived and analyzed by several researchers in the literature.^{12,13} So we only discuss the quadrotor model briefly in this paper.

A quadrotor is a highly-nonlinear underactuated system with six DOF and four control inputs. In order to derive its model, two reference frames are defined; an inertial reference frame $I: \{O_I, X_I, Y_I, Z_I\}$ fixed to the earth and a body-fixed frame $B: \{O_B, X_B, Y_B, Z_B\}$ attached to the quadrotor center of mass. These two reference frames are shown in Figure 1. The rotation matrix transforming the body-fixed axes to the inertial

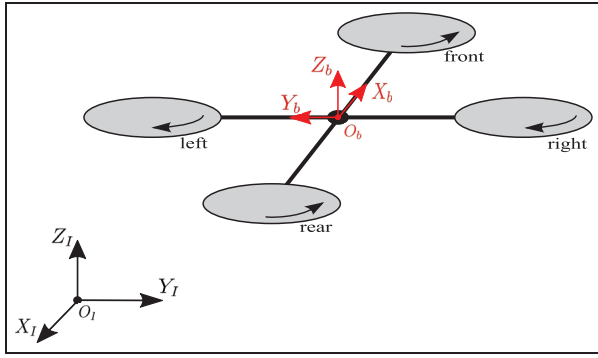


Figure 1. Reference Frames.

axes is obtained using the XYZ rotation sequence and is found to be

$$\mathbf{R} = \begin{bmatrix} c_\theta c_\psi & c_\theta s_\psi & -s_\theta \\ s_\phi s_\theta c_\psi - c_\phi s_\psi & s_\phi s_\theta s_\psi + c_\phi c_\psi & s_\phi c_\theta \\ c_\phi s_\theta c_\psi + s_\phi s_\psi & c_\phi s_\theta s_\psi - s_\phi c_\psi & c_\phi c_\theta \end{bmatrix} \quad (1)$$

where ϕ , θ and ψ are the roll, pitch and yaw angles, respectively, defining the quadrotor attitude. The terms $c_{(\cdot)}$ and $s_{(\cdot)}$ represent the cosine and sine functions respectively. These three Euler angles are considered to be 3DOF of the quadrotor. The other 3DOF are defined to be the position of the quadrotor center of mass along the X_I , Y_I and Z_I axes and denoted, respectively, as x , y and z . On the other hand, the quadrotor four control inputs are the collective thrust force U_1 , the rolling moment U_2 , the pitching moment U_3 and the yawing moment U_4 . By denoting the body angular velocities as p , q and r , the relation between these body angular rates and the Euler rates $\dot{\phi}$, $\dot{\theta}$ and $\dot{\psi}$ according to the XYZ sequence is expressed as

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} c_\psi/c_\theta & -s_\psi/c_\theta & 0 \\ s_\psi & c_\psi & 0 \\ -c_\psi t_\theta & s_\psi t_\theta & 1 \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} \quad (2)$$

where $t_{(\cdot)}$ denotes the tangent function.

By applying the Newton–Euler formalism to the quadrotor system, the translation dynamics is found to be

$$m\mathbf{a} = \mathbf{f} \quad (3)$$

where m is the quadrotor mass, $\mathbf{a} = [\ddot{x} \ \ddot{y} \ \ddot{z}]^T$ is the acceleration of the quadrotor center of mass in the inertial frame and $\mathbf{f} = [f_x \ f_y \ f_z]^T$ is the total force vector applied to the quadrotor in the inertial frame. There are three main sources contributing to the total force vector; the thrust force, the gravitational force and the disturbing force. Thus, the total force vector is represented as

$$\mathbf{f} = U_1 \mathbf{R} \bar{\mathbf{e}}_3 - mg \bar{\mathbf{e}}_3 + \mathbf{d}_f \quad (4)$$

where g is the gravitational acceleration, $\bar{\mathbf{e}}_3 = [0 \ 0 \ 1]^T$ and \mathbf{d}_f is the disturbing force vector in the inertial frame. Notice that the thrust force U_1 is multiplied by

the unit vector $\bar{\mathbf{e}}_3$ because this force points in the positive direction of Z_B axis. Then, this vector is multiplied by the rotation matrix \mathbf{R} to transform the thrust force to the inertial frame. Similarly, the gravitational force vector is multiplied by the unit vector $\bar{\mathbf{e}}_3$ because this force points in the positive direction of Z_I axis. Thus, the translation dynamics can be written in a detailed form as

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} -s_\theta \frac{U_1}{m} + \frac{d_x}{m} + \frac{d_{ux}}{m} \\ s_\phi c_\theta \frac{U_1}{m} + \frac{d_y}{m} + \frac{d_{uy}}{m} \\ c_\phi c_\theta \frac{U_1}{m} - g + \frac{d_z}{m} + \frac{d_{uz}}{m} \end{bmatrix} \quad (5)$$

where the disturbing force \mathbf{d}_f is expressed as $\mathbf{d}_f = [d_x + d_{ux}, d_y + d_{uy}, d_z + d_{uz}]^T$. The terms d_x , d_y and d_z denote the disturbing forces applied to the quadrotor due to wind. The terms d_{ux} , d_{uy} and d_{uz} denote the disturbing forces applied due to actuator failure.

According to the Newton–Euler formalism, the quadrotor rotational dynamics can be represented as

$$\mathbf{I} \dot{\boldsymbol{\Omega}} = -\boldsymbol{\Omega} \times \mathbf{I} \boldsymbol{\Omega} - I_p (\boldsymbol{\Omega} \times \bar{\mathbf{e}}_3) \omega_T + \boldsymbol{\tau} \quad (6)$$

where $\mathbf{I} = \text{diag}\{I_x, I_y, I_z\}$ is the inertia matrix, $\boldsymbol{\Omega} = [p \ q \ r]^T$ is the angular velocity vector, I_p is the rotor inertia around its axis of rotation, $\boldsymbol{\tau}$ is the total torque vector applied to the quadrotor, and $\omega_T = -\omega_f + \omega_l - \omega_b + \omega_r$ is the algebraic sum of the propellers' angular velocities such that the subscripts f, l, b, r denote the front, left, back and right propellers respectively. Thus, the rotational dynamics can be written in a detailed form as

$$\begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} \frac{1}{I_x} [(I_y - I_z)qr - I_p q \omega_T + U_2 + d_{u2}] \\ \frac{1}{I_y} [(I_z - I_x)pr + I_p p \omega_T + U_3 + d_{u3}] \\ \frac{1}{I_z} [(I_x - I_y)pq + U_4 + d_{u4}] \end{bmatrix} \quad (7)$$

where the torque $\boldsymbol{\tau}$ is expressed as $\boldsymbol{\tau} = [U_2 + d_{u2}, U_3 + d_{u3}, U_4 + d_{u4}]^T$. The terms d_{u2} , d_{u3} and d_{u4} denote the disturbing moments applied on the quadrotor due to actuator failure.

It is worth mentioning that the rotation matrix used to derive this quadrotor model is based on the sequence XYZ as in the work of Das et al.,² and not the conventional ZYX sequence, to simplify the equations of motion of the quadrotor. If the sequence ZYX is used, then the first term in each equation in (5) is replaced as follows

$$\begin{aligned} -s_\theta \frac{U_1}{m} &\Rightarrow s_\phi s_\psi + c_\phi s_\theta c_\psi \frac{U_1}{m} \\ s_\phi c_\theta \frac{U_1}{m} &\Rightarrow -s_\theta c_\psi + c_\phi s_\theta s_\psi \frac{U_1}{m} \\ c_\phi c_\theta \frac{U_1}{m} &\Rightarrow c_\phi c_\theta \frac{U_1}{m} \end{aligned}$$

As will be shown in the controller section, using the XYZ sequence will simplify the controller structure.

Finally, the relation between the propellers angular speeds and the generated aerodynamic forces and moments due to the propellers is expressed as

$$\begin{bmatrix} \omega_f^2 \\ \omega_l^2 \\ \omega_h^2 \\ \omega_r^2 \end{bmatrix} = \begin{bmatrix} \frac{1}{4b} & 0 & -\frac{1}{2bl_a} & -\frac{1}{4d} \\ \frac{1}{4b} & -\frac{1}{2bl_a} & 0 & \frac{1}{4d} \\ \frac{1}{4b} & 0 & \frac{1}{2bl_a} & -\frac{1}{4d} \\ \frac{1}{4b} & \frac{1}{2bl_a} & 0 & \frac{1}{4d} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} \quad (8)$$

where b is the propeller thrust coefficient, d is the propeller drag coefficient, l_a is the distance between the quadrotor center line and a propeller's axis of rotation.

Wind disturbance model

In this subsection, the wind model, considered in this work, is presented. Wind contributes to the drag forces acting on the quadrotor's body. The components of the drag forces influencing the quadrotor during flight are represented for each axis $j \in \{x, y, z\}$ as follows

$$d_j = -\frac{1}{2}\rho C_{d_j} A_j (v_j - v_{wj})^2 \text{sgn}(v_j - v_{wj}) \quad (9)$$

where ρ is the air density, C_{d_x} , C_{d_y} and C_{d_z} are the drag coefficients along the inertial axes, A_x , A_y and A_z are the quadrotor's projected areas on the planes perpendicular to the inertial axes X_I , Y_I and Z_I respectively and $\text{sgn}(\cdot)$ denotes the usual sign function. The terms v_x , v_y and v_z are the quadrotor velocities along the inertial axes with respect to the ground (i.e., $v_x = \dot{x}$, $v_y = \dot{y}$, $v_z = \dot{z}$). The terms v_{wx} , v_{wy} and v_{wz} are the wind velocities along the inertial axes with respect to ground. The areas A_x , A_y and A_z are calculated as

$$\mathbf{A}_i = \bar{\mathbf{R}} \mathbf{A}_b \quad (10)$$

where $\mathbf{A}_i = [A_x \ A_y \ A_z]^T$ is a vector representing the projected areas on the planes perpendicular to the inertial axes X_I , Y_I and Z_I respectively, $\mathbf{A}_b = [A_u \ A_v \ A_w]^T$ is a vector representing the projected areas on the planes perpendicular to the body fixed axes X_B , Y_B and Z_B respectively. The matrix $\bar{\mathbf{R}}$ is the matrix containing the absolute values of the rotation matrix and is expressed as

$$\bar{\mathbf{R}} = \begin{bmatrix} |c_\theta c_\psi| & |c_\theta s_\psi| & |-s_\theta| \\ |s_\theta s_\psi c_\psi - c_\theta s_\psi| & |s_\theta s_\psi s_\psi + c_\theta c_\psi| & |s_\theta c_\theta| \\ |c_\theta s_\psi c_\psi + s_\theta s_\psi| & |c_\theta s_\psi s_\psi - s_\theta c_\psi| & |c_\theta c_\theta| \end{bmatrix} \quad (11)$$

The idea behind using the absolute values is not only to guarantee that the components of A_i are all positive. This operator is used to ensure that the contribution of each component in A_b to each component in A_i is positive, in other words, to ensure that projecting any of the non-negative areas given in the body-fixed frames

does not result in a negative area in the inertial frame. The wind velocities v_{wx} , v_{wy} and v_{wz} comprise two terms

$$v_{wj} = \bar{v}_{wj} + \hat{v}_{wj} \quad (12)$$

where the over-bar denotes the mean wind velocity generated based on the discrete wind gust model,¹⁴ while the over-hat denotes the deviations from the mean velocity generated based on the Dryden wind turbulence model.^{15,16}

According to the discrete wind gust model, the wind velocity changes from one constant value to another based on a (1-minus-cosine) function as follows

$$\bar{v}_{wj} = \begin{cases} v_{j1}, & 0 < t < t_1 \\ \frac{(v_{j2} + v_{j1})}{2} - \frac{(v_{j2} - v_{j1})}{2} c\left(\frac{\pi(t - t_1)}{t_2 - t_1}\right), & t_1 < t < t_2, \\ v_{j2}, & t_2 < t \end{cases} \quad (13)$$

where v_{j1} is the wind velocity before the wind gust, v_{j2} is the wind velocity after the wind gust, t_1 is the time at which the wind gust starts and t_2 is the time at which the wind gust stops.

According to the Dryden wind turbulence model, the wind velocity is generated by passing band-limited white noise through the following forming filters

$$\begin{aligned} G_u(s) &= \frac{\hat{v}_{wx}(s)}{\eta_u(s)} = \sigma_u \sqrt{\frac{2L_u}{\pi V}} \frac{1}{1 + \frac{L_u}{V}s} \\ G_v(s) &= \frac{\hat{v}_{wy}(s)}{\eta_v(s)} = \sigma_v \sqrt{\frac{L_v}{\pi V}} \frac{1 + \frac{\sqrt{3}L_v}{V}s}{(1 + \frac{L_v}{V}s)^2} \\ G_w(s) &= \frac{\hat{v}_{wz}(s)}{\eta_w(s)} = \sigma_w \sqrt{\frac{L_w}{\pi V}} \frac{1 + \frac{\sqrt{3}L_w}{V}s}{(1 + \frac{L_w}{V}s)^2} \end{aligned} \quad (14)$$

respectively where η_u , η_v , η_w denote zero-mean white Gaussian noise with unity standard deviation, σ_u , σ_v and σ_w are the turbulence intensities, L_u , L_v and L_w are the turbulence scale lengths, $V = \sqrt{\sum_j (v_j - v_{wj})^2}$ is the mean wind speed and s is the Laplace variable. At low altitudes the turbulence intensities are calculated as¹⁶

$$\sigma_w = 0.1 W_{20}, \quad \sigma_u = \sigma_v = \frac{\sigma_w}{(0.177 + 0.000823z)^{0.4}} \quad (15)$$

where W_{20} represents the wind speed at 6m altitude. Whereas, the turbulence scale lengths depend on the altitude of the quadrotor helicopter as follows

$$L_w = z, \quad L_u = L_v = \frac{z}{(0.177 + 0.000823z)^{1.2}} \quad (16)$$

Actuator failure uncertainties

An actuator failure is modeled by assuming that the thrust force, generated by the propeller whose actuator has failed, does not reach the required value. Without loss of generality, the case in which the front rotor has

partially failed is considered. In this case, the actuator failure is represented mathematically as

$$\begin{bmatrix} \bar{U}_f \\ \bar{U}_l \\ \bar{U}_b \\ \bar{U}_r \end{bmatrix} = \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} U_f \\ U_l \\ U_b \\ U_r \end{bmatrix} \quad (17)$$

where $0 \leq \lambda \leq 1$ represents the effectiveness of the front rotor ($\lambda = 1$ indicates that the front rotor is undamaged), $\bar{U}_i (i \in \{f, l, b, r\})$ represents the actual propeller thrust force and U_i represents the required propeller thrust force. The actual control inputs $(\bar{U}_1, \bar{U}_2, \bar{U}_3, \bar{U}_4)^T$ are calculated from the actual thrust forces of the propellers as

$$\begin{bmatrix} \bar{U}_1 \\ \bar{U}_2 \\ \bar{U}_3 \\ \bar{U}_4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -l_a & 0 & l_a \\ -l_a & 0 & l_a & 0 \\ -d/b & d/b & -d/b & d/b \end{bmatrix} \begin{bmatrix} \bar{U}_f \\ \bar{U}_l \\ \bar{U}_b \\ \bar{U}_r \end{bmatrix} \quad (18)$$

The required thrust forces of the propellers are calculated from the required control inputs as

$$\begin{bmatrix} U_f \\ U_l \\ U_b \\ U_r \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & 0 & -\frac{1}{2l_a} & -\frac{b}{4d} \\ \frac{1}{4} & -\frac{1}{2l_a} & 0 & \frac{b}{4d} \\ \frac{1}{4} & 0 & \frac{1}{2l_a} & -\frac{b}{4d} \\ \frac{1}{4} & \frac{1}{2l_a} & 0 & \frac{b}{4d} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} \quad (19)$$

Notice that the relation in (19) is the inverse of the relation in (18). From (17), (18), (19) respectively, the relation between the actual control inputs and the required control inputs are found to be

$$\begin{bmatrix} \bar{U}_1 \\ \bar{U}_2 \\ \bar{U}_3 \\ \bar{U}_4 \end{bmatrix} = \begin{bmatrix} \frac{3+\lambda}{4} & 0 & \frac{1-\lambda}{2l_a} & \frac{b-b\lambda}{4d} \\ 0 & 1 & 0 & 0 \\ \frac{l_a-l_a\lambda}{4} & 0 & \frac{1+\lambda}{2l_a} & \frac{bl_a(-1+\lambda)}{4d} \\ \frac{d-d\lambda}{4b} & 0 & \frac{2}{2bl_a} & \frac{3+\lambda}{4} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} \quad (20)$$

The effect of the actuator failure introduces a cross-coupling in the system dynamics as implied by (20). This cross-coupling occurs between the collective thrust force U_1 , the yawing moment U_4 and the pitching moment U_3 . It is worth mentioning that the forthcoming developed controller is independent of which rotor fails.

In the case of actuator failure, the actual control inputs replace the required control inputs in (5), (7). The disturbing forces and moments, in this case, are considered to be the difference between the actual control inputs and the required control inputs as follows

$$\begin{aligned} d_{u1} &= \bar{U}_1 - U_1 \\ d_{u2} &= \bar{U}_2 - U_2 \\ d_{u3} &= \bar{U}_3 - U_3 \\ d_{u4} &= \bar{U}_4 - U_4 \end{aligned} \quad (21)$$

The first equation in (21) represents the disturbing force along the direction of the thrust generation and can be resolved with respect to the inertial axes as follows

$$\begin{aligned} d_{ux} &= -s_\theta d_{u1} \\ d_{uy} &= s_\phi c_\theta d_{u1} \\ d_{uz} &= c_\phi c_\theta d_{u1} \end{aligned} \quad (22)$$

Whereas the second three equations in (21) represent the disturbing moments around the inertial axes.

State space model

In this subsection, a state space model is presented. The states of this model are chosen to be

$$\mathbf{x} = [x \ y \ z \ \dot{x} \ \dot{y} \ \dot{z} \ \phi \ \theta \ \psi \ p \ q \ r \ U_1 \ \dot{U}_1]^T \quad (23)$$

While deriving the control law, the thrust force second derivative \ddot{U}_1 appears. Thus, it is considered as a control input instead of the thrust force U_1 which is considered, together with its first derivative \dot{U}_1 , as states. Although the thrust force second derivative \ddot{U}_1 is the control input to be computed using the control law, the thrust force U_1 is the force to be applied to the quadrotor in the real world. In other words, after calculating the second derivative, this signal can be integrated twice to obtain the thrust force needed. Then, this thrust force is used to compute the required speeds of the propellers as in (8). Finally, the required speeds are sent to the control units of the motors.

Using (2), (5), (7), the quadrotor nonlinear state space model is found to be

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ -s_\theta \frac{U_1}{m} + \frac{d_x + d_{ux}}{m} \\ s_\phi c_\theta \frac{U_1}{m} + \frac{d_y + d_{uy}}{m} \\ c_\phi c_\theta \frac{U_1}{m} - g + \frac{d_z + d_{uz}}{m} \\ c_{\psi_o} p - s_{\psi_o} q + \dot{\phi} \\ s_{\psi_o} p + c_{\psi_o} q + \dot{\theta} \\ r + \dot{\psi} \\ \frac{1}{I_x} U_2 + \frac{1}{I_x} (d_p + d_{u2}) \\ \frac{1}{I_y} U_3 + \frac{1}{I_y} (d_q + d_{u3}) \\ \frac{1}{I_z} U_4 + \frac{1}{I_z} (d_r + d_{u4}) \\ \dot{U}_1 \end{bmatrix} \quad (24)$$

where ψ_o is the heading around which the first two equations in (2) are linearized. The terms d_ϕ , d_θ and d_ψ are the disturbances affecting the angular velocities of the quadrotor due to the ignored nonlinear dynamics. While

$$\begin{aligned} d_p &= (I_y - I_z)qr - I_{p_z}q\omega_T \\ d_q &= (I_z - I_x)pr + I_{p_z}p\omega_T \\ d_r &= (I_x - I_y)pq \end{aligned} \quad (25)$$

are the disturbing moments affecting the quadrotor due to the ignored nonlinear dynamics. Notice that the disturbances $d_x, d_y, d_z, d_{ux}, d_{uy}, d_{uz}, d_{u2}, d_{u3}, d_{u4}$ are the disturbance due to wind and actuator failure introduced in the previous two subsections. For the rest of the paper, the following notation is used.

$$\begin{aligned} p_n &= c_{\psi_o}p - s_{\psi_o}q \\ q_n &= s_{\psi_o}p + c_{\psi_o}q \end{aligned} \quad (26)$$

Control System Design

In this section, the controller design is presented in four steps. First, a feedback linearization-based control law is derived to convert the quadrotor nonlinear model into a linear model. Then, a linear state feedback control law is designed to stabilize the resulting linear model. Then, a disturbance observer is designed to estimate the exogenous and endogenous disturbances affecting the quadrotor model. Finally, these disturbance estimates are used within the control law to attenuate the effect of the actual disturbances.

Feedback linearization

The goal of this subsection is to represent the quadrotor nonlinear model assuming zero disturbances using the four differential equations

$$\bar{y} = \alpha(\mathbf{x}) + \beta(\mathbf{x})\mathbf{u} \quad (27)$$

where

$$\mathbf{u} = [\ddot{U}_1 \ U_2 \ U_3 \ U_4]^T \quad (28)$$

is the control input vector

$$\bar{y} = [x^{(4)} \ y^{(4)} \ z^{(4)} \ \ddot{\psi}]^T, \quad (29)$$

is a vector including the output derivatives $\alpha(\mathbf{x})$ is a 4×1 vector represented as

$$\alpha(\mathbf{x}) = \begin{bmatrix} \alpha_x(\mathbf{x}) \\ \alpha_y(\mathbf{x}) \\ \alpha_z(\mathbf{x}) \\ \alpha_\psi(\mathbf{x}) \end{bmatrix} \quad (30)$$

and $\beta(\mathbf{x})$ is an invertible 4×4 matrix expressed as

$$\beta(\mathbf{x}) = \begin{bmatrix} \beta_{11}(\mathbf{x}) & \beta_{12}(\mathbf{x}) & \beta_{13}(\mathbf{x}) & 0 \\ \beta_{21}(\mathbf{x}) & \beta_{22}(\mathbf{x}) & \beta_{23}(\mathbf{x}) & 0 \\ \beta_{31}(\mathbf{x}) & \beta_{32}(\mathbf{x}) & \beta_{33}(\mathbf{x}) & 0 \\ 0 & 0 & 0 & \beta_{44}(\mathbf{x}) \end{bmatrix} \quad (31)$$

Thus, a control law is designed in the form

$$\mathbf{u} = \beta^{-1}(\mathbf{x})(-\alpha(\mathbf{x}) + \mathbf{v}) \quad (32)$$

to feedback linearize the quadrotor nonlinear model where

$$\mathbf{v} = [v_x \ v_y \ v_z \ v_\psi]^T \quad (33)$$

is the new control input vector to be designed using linear control theory. To reach (27), a new state vector \mathbf{w} is defined such that

$$\mathbf{w} = \mathbf{h}(\mathbf{x}) \quad (34)$$

where $\mathbf{h}(\mathbf{x})$ is a nonlinear transformation describing the relation between the old state vector \mathbf{x} and the new state vector \mathbf{w} . The transformation $\mathbf{h}(\mathbf{x})$ is chosen to be

$$\mathbf{h}(\mathbf{x}) = \begin{bmatrix} L_f^0 x \\ L_f^1 x \\ L_f^2 x \\ L_f^3 x \\ L_f^0 y \\ L_f^1 y \\ L_f^2 y \\ L_f^3 y \\ L_f^0 z \\ L_f^1 z \\ L_f^2 z \\ L_f^3 z \\ L_f^0 \psi \\ L_f^1 \psi \end{bmatrix} = \begin{bmatrix} x \\ \dot{x} \\ -s_\theta \frac{U_1}{m} \\ -q_n c_\theta \frac{U_1}{m} - s_\theta \frac{\dot{U}_1}{m} \\ y \\ \dot{y} \\ s_\phi c_\theta \frac{U_1}{m} \\ p_n c_\phi c_\theta \frac{U_1}{m} - q_n s_\phi s_\theta \frac{U_1}{m} + s_\phi c_\theta \frac{\dot{U}_1}{m} \\ z \\ \dot{z} \\ c_\phi c_\theta \frac{U_1}{m} - g \\ -p_n s_\phi c_\theta \frac{U_1}{m} - q_n c_\phi s_\theta \frac{U_1}{m} + c_\phi c_\theta \frac{\dot{U}_1}{m} \\ \psi \\ r \end{bmatrix} \quad (35)$$

where $L_f^n h$ denotes the n^{th} Lie derivative of $h \in \{x, y, z, \psi\}$ with respect to $\mathbf{f}(\mathbf{x})$ shown in (24) assuming zero disturbances. Notice that the resulting Lie derivatives are not complex due to the fact that the disturbances, whose expressions are relatively complex, are ignored while calculating these Lie derivatives. As shown in (35), the new state vector includes the quadrotor's position, velocity, acceleration and jerk vectors in the inertial frame assuming zero disturbances. In addition, the new state vector includes the heading ψ as well as the angular velocity r . By differentiating the new state vector \mathbf{w} , the quadrotor nonlinear model is written as (for the detailed derivation of the nonlinear model (36)–(42), see Appendix 2)

$$\dot{\mathbf{w}} = \begin{bmatrix} w_2 \\ w_3 + \bar{d}_{x2} \\ w_4 + \bar{d}_{x3} \\ \alpha_x(\mathbf{x}) + \beta_{11}(\mathbf{x})\ddot{U}_1 + \beta_{12}(\mathbf{x})U_2 + \beta_{13}(\mathbf{x})U_3 + \bar{d}_{x4} \\ w_6 \\ w_7 + \bar{d}_{y2} \\ w_8 + \bar{d}_{y3} \\ \alpha_y(\mathbf{x}) + \beta_{21}(\mathbf{x})\ddot{U}_1 + \beta_{22}(\mathbf{x})U_2 + \beta_{23}(\mathbf{x})U_3 + \bar{d}_{y4} \\ w_{10} \\ w_{11} + \bar{d}_{z2} \\ w_{12} + \bar{d}_{z3} \\ \alpha_z(\mathbf{x}) + \beta_{31}(\mathbf{x})\ddot{U}_1 + \beta_{32}(\mathbf{x})U_2 + \beta_{33}(\mathbf{x})U_3 + \bar{d}_{z4} \\ w_{14} + \bar{d}_{\psi} \\ \alpha_{\psi}(\mathbf{x}) + \beta_{44}(\mathbf{x})U_4 + \bar{d}_r \end{bmatrix} \quad (36)$$

where

$$\begin{aligned} \alpha_x(\mathbf{x}) &= q_n^2 s_{\theta} \frac{U_1}{m} - 2q_n c_{\theta} \frac{\dot{U}_1}{m}, \\ \alpha_y(\mathbf{x}) &= -(p_n^2 + q_n^2) s_{\phi} c_{\theta} \frac{U_1}{m} - 2p_n q_n c_{\phi} s_{\theta} \frac{U_1}{m} \\ &\quad + 2(p_n c_{\phi} c_{\theta} - q_n s_{\phi} s_{\theta}) \frac{\dot{U}_1}{m}, \\ \alpha_z(\mathbf{x}) &= -(p_n^2 + q_n^2) c_{\phi} c_{\theta} \frac{U_1}{m} + 2p_n q_n s_{\phi} s_{\theta} \frac{U_1}{m} \\ &\quad - 2(p_n s_{\phi} c_{\theta} + q_n c_{\phi} s_{\theta}) \frac{\dot{U}_1}{m}, \\ \alpha_{\psi}(\mathbf{x}) &= 0 \end{aligned} \quad (37)$$

and

$$\begin{aligned} \beta_{11}(\mathbf{x}) &= \frac{-s_{\theta}}{m} \\ \beta_{12}(\mathbf{x}) &= \frac{-s_{\psi_o} c_{\theta} U_1}{m I_x} \\ \beta_{13}(\mathbf{x}) &= \frac{-c_{\psi_o} c_{\theta} U_1}{m I_y} \\ \beta_{21}(\mathbf{x}) &= \frac{s_{\phi} c_{\theta}}{m} \\ \beta_{22}(\mathbf{x}) &= \frac{(c_{\psi_o} c_{\phi} c_{\theta} - s_{\psi_o} s_{\phi} s_{\theta}) U_1}{m I_x} \\ \beta_{23}(\mathbf{x}) &= \frac{-(s_{\psi_o} c_{\phi} c_{\theta} + c_{\psi_o} s_{\phi} s_{\theta}) U_1}{m I_y} \\ \beta_{31}(\mathbf{x}) &= \frac{c_{\phi} c_{\theta}}{m} \\ \beta_{32}(\mathbf{x}) &= \frac{-(c_{\psi_o} s_{\phi} c_{\theta} + s_{\psi_o} c_{\phi} s_{\theta}) U_1}{m I_x} \\ \beta_{33}(\mathbf{x}) &= \frac{(s_{\psi_o} s_{\phi} c_{\theta} - c_{\psi_o} c_{\phi} s_{\theta}) U_1}{m I_y} \\ \beta_{44}(x) &= \frac{1}{I_z} \end{aligned} \quad (38)$$

Moreover, the expressions of the disturbances \bar{d}_{x2} , \bar{d}_{x3} and \bar{d}_{x4} influencing the quadrotor's acceleration, jerk and snap (position fourth derivative) along the inertial axis X_I respectively are found to be

$$\begin{aligned} \bar{d}_{x2} &= \frac{d_x + d_{ux}}{m} \\ \bar{d}_{x3} &= -d_{\theta} c_{\theta} \frac{U_1}{m} \\ \bar{d}_{x4} &= -\left(\frac{s_{\psi_o}}{I_x} (d_p + d_{u_2}) + \frac{c_{\psi_o}}{I_y} (d_q + d_{u_3}) \right) c_{\theta} \frac{U_1}{m} \\ &\quad + q_n d_{\theta} s_{\theta} \frac{U_1}{m} - d_{\theta} c_{\theta} \frac{\dot{U}_1}{m}, \end{aligned} \quad (39)$$

the expressions of the disturbances \bar{d}_{y2} , \bar{d}_{y3} and \bar{d}_{y4} influencing the quadrotor's acceleration, jerk and snap (position fourth derivative) along the inertial axis Y_I respectively are found to be

$$\begin{aligned} \bar{d}_{y2} &= \frac{d_y + d_{uy}}{m} \\ \bar{d}_{y3} &= d_{\phi} c_{\phi} c_{\theta} \frac{U_1}{m} - d_{\theta} s_{\phi} s_{\theta} \frac{U_1}{m} \\ \bar{d}_{y4} &= \left(\frac{c_{\psi_o}}{I_x} (d_p + d_{u_2}) - \frac{s_{\psi_o}}{I_y} (d_q + d_{u_3}) \right) c_{\phi} c_{\theta} \frac{U_1}{m} \\ &\quad - \left(\frac{s_{\psi_o}}{I_x} (d_p + d_{u_2}) + \frac{c_{\psi_o}}{I_y} (d_q + d_{u_3}) \right) s_{\phi} s_{\theta} \frac{U_1}{m} \\ &\quad - (p_n d_{\theta} + q_n d_{\phi}) c_{\phi} s_{\theta} \frac{U_1}{m} - (p_n d_{\phi} + q_n d_{\theta}) s_{\phi} c_{\theta} \frac{U_1}{m} \\ &\quad + (d_{\phi} c_{\phi} c_{\theta} - d_{\theta} s_{\phi} s_{\theta}) \frac{\dot{U}_1}{m} \end{aligned} \quad (40)$$

the expressions of the disturbances \bar{d}_{z2} , \bar{d}_{z3} and \bar{d}_{z4} influencing the quadrotor's acceleration, jerk and snap (position fourth derivative) along the inertial axis Z_I respectively are found to be

$$\begin{aligned} \bar{d}_{z2} &= \frac{d_z + d_{uz}}{m} \\ \bar{d}_{z3} &= -d_{\phi} s_{\phi} c_{\theta} \frac{U_1}{m} - d_{\theta} c_{\phi} s_{\theta} \frac{U_1}{m} \\ \bar{d}_{z4} &= -\left(\frac{c_{\psi_o}}{I_x} (d_p + d_{u_2}) - \frac{s_{\psi_o}}{I_y} (d_q + d_{u_3}) \right) s_{\phi} c_{\theta} \frac{U_1}{m} \\ &\quad - \left(\frac{s_{\psi_o}}{I_x} (d_p + d_{u_2}) + \frac{c_{\psi_o}}{I_y} (d_q + d_{u_3}) \right) c_{\phi} s_{\theta} \frac{U_1}{m} \\ &\quad + (p_n d_{\theta} + q_n d_{\phi}) s_{\phi} s_{\theta} \frac{U_1}{m} - (p_n d_{\phi} + q_n d_{\theta}) c_{\phi} c_{\theta} \frac{U_1}{m} \\ &\quad - (d_{\phi} s_{\phi} c_{\theta} + d_{\theta} c_{\phi} s_{\theta}) \frac{\dot{U}_1}{m} \end{aligned} \quad (41)$$

the expressions of the disturbances \bar{d}_{ψ} and \bar{d}_r influencing the quadrotor's heading dynamics are found to be

$$\bar{d}_{\psi} = d_{\psi}, \quad \bar{d}_r = \frac{d_r + d_{u_4}}{m} \quad (42)$$

Ignoring the disturbances (39), (40), (41), (42) affecting the quadrotor, the nonlinear model (36), (37), (38) is written as

$$\dot{\mathbf{w}} = \begin{bmatrix} w_2 \\ w_3 \\ w_4 \\ \alpha_x(\mathbf{x}) + \beta_{11}(\mathbf{x})\ddot{U}_1 + \beta_{12}(\mathbf{x})U_2 + \beta_{13}(\mathbf{x})U_3 \\ w_6 \\ w_7 \\ w_8 \\ \alpha_y(\mathbf{x}) + \beta_{21}(\mathbf{x})\ddot{U}_1 + \beta_{22}(\mathbf{x})U_2 + \beta_{23}(\mathbf{x})U_3 \\ w_{10} \\ w_{11} \\ w_{12} \\ \alpha_z(\mathbf{x}) + \beta_{31}(\mathbf{x})\ddot{U}_1 + \beta_{32}(\mathbf{x})U_2 + \beta_{33}(\mathbf{x})U_3 \\ w_{14} \\ \alpha_\psi(\mathbf{x}) + \beta_{44}(\mathbf{x})U_4 \end{bmatrix} \quad (43)$$

By referring to the physical meanings of the new state vector \mathbf{w} stated after (35), it is found that \dot{w}_4 , \dot{w}_8 , \dot{w}_{12} and \dot{w}_{14} represent the fourth derivative of the position along the inertial X_I , Y_I and Z_I axes, and the heading's second derivative respectively. Thus, the nonlinear model can be described as mentioned in (27) where (37) and (38) represent the nonlinear functions $\alpha(\mathbf{x})$ and $\beta(\mathbf{x})$ respectively. Hence, the control law (32) can be used to feedback linearize the quadrotor model. The determinant of the nonlinear matrix $\beta(\mathbf{x})$ is found to be

$$|\beta(\mathbf{x})| = \frac{c_\theta U_1^2}{m^3 I_x I_y I_z} \quad (44)$$

Thus, the matrix $\beta(\mathbf{x})$ is invertible only if $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ and $U_1 \neq 0$ which is naturally satisfied during the normal operation of the quadrotor. In this case, the resulting differential equations are represented as

$$\bar{\mathbf{y}} = \mathbf{v} \quad (45)$$

Taking into consideration the disturbance terms, the nonlinear model (36)–(42) is converted to a linear model represented as

$$\begin{aligned} \dot{\mathbf{w}} &= \mathbf{A}\mathbf{w} + \mathbf{B}\mathbf{v} + \mathbf{B}_d\mathbf{d} \\ \mathbf{y} &= \mathbf{C}\mathbf{x} \end{aligned} \quad (46)$$

where

$$\mathbf{d} = [\bar{d}_{x2} \ \bar{d}_{x3} \ \bar{d}_{x4} \ \bar{d}_{y2} \ \bar{d}_{y3} \ \bar{d}_{y4} \ \bar{d}_{z2} \ \bar{d}_{z3} \ \bar{d}_{z4} \ \bar{d}_\psi \ \bar{d}_r]^T \quad (47)$$

and

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_p & \mathbf{0}_{4 \times 4} & \mathbf{0}_{4 \times 4} & \mathbf{0}_{4 \times 2} \\ \mathbf{0}_{4 \times 4} & \mathbf{A}_p & \mathbf{0}_{4 \times 4} & \mathbf{0}_{4 \times 2} \\ \mathbf{0}_{4 \times 4} & \mathbf{0}_{4 \times 4} & \mathbf{A}_p & \mathbf{0}_{4 \times 2} \\ \mathbf{0}_{2 \times 4} & \mathbf{0}_{2 \times 4} & \mathbf{0}_{2 \times 4} & \mathbf{A}_a \end{bmatrix} \quad (48)$$

$$\mathbf{B} = [\hat{\mathbf{e}}_4 \ \hat{\mathbf{e}}_8 \ \hat{\mathbf{e}}_{12} \ \hat{\mathbf{e}}_{14}] \quad (49)$$

$$\mathbf{B}_d = [\hat{\mathbf{e}}_2 \ \hat{\mathbf{e}}_3 \ \hat{\mathbf{e}}_4 \ \hat{\mathbf{e}}_6 \ \hat{\mathbf{e}}_7 \ \hat{\mathbf{e}}_8 \ \hat{\mathbf{e}}_{10} \ \hat{\mathbf{e}}_{11} \ \hat{\mathbf{e}}_{12} \ \hat{\mathbf{e}}_{13} \ \hat{\mathbf{e}}_{14}], \quad (50)$$

$$\mathbf{C} = [\hat{\mathbf{e}}_1 \ \hat{\mathbf{e}}_5 \ \hat{\mathbf{e}}_9 \ \hat{\mathbf{e}}_{13}]^T \quad (51)$$

such that

$$\mathbf{A}_p = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{A}_a = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad (52)$$

$\mathbf{0}_{i \times j}$ is a zero matrix of size $i \times j$ and $\hat{\mathbf{e}}_k$ is a column unit vector of size 14 whose k th element is 1.

Linear state feedback control

The eigenvalues of the feedback linearized model (46) are all zeros. Thus, the linearized system is not asymptotically stable. To asymptotically stabilize this system, a linear state feedback control law needs to be designed. The feedback linearized system consists of four decoupled subsystems; the x -axis position subsystem including the states w_1, w_2, w_3, w_4 , the y -axis position subsystem including the states w_5, w_6, w_7, w_8 , the altitude subsystem including the states $w_9, w_{10}, w_{11}, w_{12}$, and the heading subsystem including the states w_{13}, w_{14} . Therefore, the stabilizing control law is designed to be

$$\mathbf{v} = -\mathbf{K}\mathbf{w} + \mathbf{B}_r\mathbf{r} \quad (53)$$

where $\mathbf{r} = [x_d \ y_d \ z_d \ \psi_d]^T$ is the desired output trajectories, the matrices \mathbf{K} and \mathbf{B}_r are the controller gain matrices designed to be

$$\mathbf{K} = \begin{bmatrix} \mathbf{K}_x & \mathbf{0}_{1 \times 4} & \mathbf{0}_{1 \times 4} & \mathbf{0}_{1 \times 2} \\ \mathbf{0}_{1 \times 4} & \mathbf{K}_y & \mathbf{0}_{1 \times 4} & \mathbf{0}_{1 \times 2} \\ \mathbf{0}_{1 \times 4} & \mathbf{0}_{1 \times 4} & \mathbf{K}_z & \mathbf{0}_{1 \times 2} \\ \mathbf{0}_{1 \times 4} & \mathbf{0}_{1 \times 4} & \mathbf{0}_{1 \times 4} & \mathbf{K}_\psi \end{bmatrix} \quad (54)$$

$$\mathbf{B}_r = \begin{bmatrix} k_{x0} & 0 & 0 & 0 \\ 0 & k_{y0} & 0 & 0 \\ 0 & 0 & k_{z0} & 0 \\ 0 & 0 & 0 & k_{\psi 0} \end{bmatrix} \quad (55)$$

such that

$$\begin{aligned} \mathbf{K}_x &= [k_{x0} \ k_{x1} \ k_{x2} \ k_{x3}] \\ \mathbf{K}_y &= [k_{y0} \ k_{y1} \ k_{y2} \ k_{y3}] \\ \mathbf{K}_z &= [k_{z0} \ k_{z1} \ k_{z2} \ k_{z3}] \\ \mathbf{K}_\psi &= [k_{\psi 0} \ k_{\psi 1}] \end{aligned} \quad (56)$$

and k_{ij} are the controller gains chosen to place the eigenvalues of the feedback linearized model at the desired locations. In this case, the closed loop dynamics is found to be

$$\dot{\mathbf{w}} = (\mathbf{A} - \mathbf{BK})\mathbf{w} + \mathbf{BB}_r\mathbf{r} + \mathbf{B}_d\mathbf{d} \quad (57)$$

Disturbance observer

The control law (32), (53) stabilizes the quadrotor dynamics. However, this control law does not attenuate the effect of the disturbances (39), (40), (41), (42) influencing the quadrotor. In this section, a disturbance

observer¹⁷ is designed to estimate the values of the disturbances affecting the quadrotor. From (46), the disturbance vector \mathbf{d} is calculated as

$$\mathbf{d} = \mathbf{B}_d^+ (\dot{\mathbf{w}} - \mathbf{A}\mathbf{w} - \mathbf{B}\mathbf{v}) \quad (58)$$

where \mathbf{B}_d^+ denotes the pseudo-inverse of the matrix \mathbf{B}_d . However, an algebraic loop may exist if the calculated disturbance vector (58) is used within the feedback control law immediately since the control input vector \mathbf{v} is used in the calculation of the disturbance vector. Therefore, a disturbance estimate vector $\hat{\mathbf{d}}$ is defined whose dynamics is represented as

$$\dot{\hat{\mathbf{d}}} = -\mathbf{L}\mathbf{B}_d(\hat{\mathbf{d}} - \mathbf{d}) \quad (59)$$

where \mathbf{L} is the disturbance observer gain matrix to be designed later. By substituting (58) in (59), the disturbance estimate dynamics is found to be

$$\dot{\hat{\mathbf{d}}} = -\mathbf{L}\mathbf{B}_d\hat{\mathbf{d}} + \mathbf{L}(\dot{\mathbf{w}} - \mathbf{A}\mathbf{w} - \mathbf{B}\mathbf{v}) \quad (60)$$

To overcome the requirement of the state vector derivative to estimate the values of the disturbance vector, a new auxiliary vector \mathbf{z} is defined to be

$$\mathbf{z} = \hat{\mathbf{d}} - \mathbf{L}\mathbf{w} \quad (61)$$

By differentiating (61) and substituting with (60), the auxiliary variable \mathbf{z} dynamics is found to be

$$\begin{aligned} \dot{\mathbf{z}} &= \dot{\hat{\mathbf{d}}} - \mathbf{L}\dot{\mathbf{w}} \\ &= -\mathbf{L}\mathbf{B}_d\hat{\mathbf{d}} + \mathbf{L}(\dot{\mathbf{w}} - \mathbf{A}\mathbf{w} - \mathbf{B}\mathbf{v}) - \mathbf{L}\dot{\mathbf{w}} \\ &= -\mathbf{L}\mathbf{B}_d(\mathbf{z} + \mathbf{L}\mathbf{w}) - \mathbf{L}(\mathbf{A}\mathbf{w} + \mathbf{B}\mathbf{v}) \\ &= -\mathbf{L}\mathbf{B}_d\mathbf{z} - \mathbf{L}(\mathbf{A}\mathbf{w} + \mathbf{B}\mathbf{v} + \mathbf{B}_d\mathbf{L}\mathbf{w}) \end{aligned} \quad (62)$$

Then, the disturbance estimate is calculated from (61). The error between the actual disturbance and the disturbance estimate is defined to be

$$\mathbf{e}_d = \mathbf{d} - \hat{\mathbf{d}} \quad (63)$$

By differentiating (63) and substituting with (59), the error dynamics is found to be

$$\begin{aligned} \dot{\mathbf{e}}_d &= \dot{\mathbf{d}} - \dot{\hat{\mathbf{d}}} \\ &= -\mathbf{L}\mathbf{B}_d\mathbf{e}_d + \dot{\mathbf{d}} \end{aligned} \quad (64)$$

Therefore, the matrix \mathbf{L} is designed such that the matrix $\mathbf{L}\mathbf{B}_d$ is positive definite and thus the error dynamics is asymptotically stable. To maintain the disturbance error dynamics decoupled, the matrix \mathbf{L} is designed to be

$$\mathbf{L} = \mathbf{L}_d\mathbf{B}_d^+ \quad (65)$$

where $\mathbf{L}_d = \text{diag}(L_{d1}, \dots, L_{d11})$ is a positive definite diagonal matrix. In this case, the error dynamics is modified to be

$$\dot{\mathbf{e}}_d = -\mathbf{L}_d\mathbf{e}_d + \dot{\mathbf{d}} \quad (66)$$

For constant disturbances (i.e., $\dot{\mathbf{d}} = 0$), the error \mathbf{e}_d asymptotically converges to zero. If the derivative of the disturbance is bounded (i.e., $|\dot{d}_i| \leq \eta_i$ where d_i is the i th term of the disturbance vector \mathbf{d} and $\eta_i > 0$), then the estimation error is bounded. Since the disturbance error e_{di} is the output of a first order filter driven by \dot{d}_i as implied by equation (66), the disturbance error is bounded by

$$|e_{di}| \leq \frac{\eta_i}{L_{di}} \quad (67)$$

Disturbance compensation

Since the disturbance vector \mathbf{d} is not affecting the quadrotor plant through the same channel as the control input vector \mathbf{v} , the disturbances are called mismatched. In this case, the disturbance estimate vector $\hat{\mathbf{d}}$ is multiplied by a disturbance compensation gain \mathbf{K}_d before being used within the control law.¹⁷ Therefore, the control law (53) is modified to be

$$\mathbf{v} = -\mathbf{K}\mathbf{w} + \mathbf{B}_r\mathbf{r} - \mathbf{K}_d\hat{\mathbf{d}} \quad (68)$$

The compensation gain is calculated as follows. By substituting the control law (68) in the dynamics (46), the closed loop dynamics is found to be

$$\dot{\mathbf{w}} = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{w} + \mathbf{B}\mathbf{B}_r\mathbf{r} - \mathbf{B}\mathbf{K}_d\hat{\mathbf{d}} + \mathbf{B}_d\mathbf{d} \quad (69)$$

Therefore, the state vector is rewritten as

$$\mathbf{w} = (\mathbf{A} - \mathbf{B}\mathbf{K})^{-1}(\dot{\mathbf{w}} - \mathbf{B}\mathbf{B}_r\mathbf{r} + \mathbf{B}\mathbf{K}_d\hat{\mathbf{d}} - \mathbf{B}_d\mathbf{d}) \quad (70)$$

By substituting (70) in the output equation $\mathbf{y} = \mathbf{C}\mathbf{w}$ where \mathbf{C} is defined in (51), the output vector is found to be

$$\begin{aligned} \mathbf{y} &= \mathbf{C}(\mathbf{A} - \mathbf{B}\mathbf{K})^{-1}\dot{\mathbf{w}} - \mathbf{C}(\mathbf{A} - \mathbf{B}\mathbf{K})^{-1}\mathbf{B}\mathbf{B}_r\mathbf{r} \\ &\quad + \mathbf{C}(\mathbf{A} - \mathbf{B}\mathbf{K})^{-1}\mathbf{B}\mathbf{K}_d\hat{\mathbf{d}} - \mathbf{C}(\mathbf{A} - \mathbf{B}\mathbf{K})^{-1}\mathbf{B}_d\mathbf{d} \end{aligned} \quad (71)$$

Therefore, by choosing the compensation gain matrix \mathbf{K}_d to be

$$\mathbf{K}_d = (\mathbf{C}(\mathbf{A} - \mathbf{B}\mathbf{K})^{-1}\mathbf{B})^{-1}(\mathbf{C}(\mathbf{A} - \mathbf{B}\mathbf{K})^{-1}\mathbf{B}_d) \quad (72)$$

the output vector equation in (71) becomes

$$\begin{aligned} \mathbf{y} &= \mathbf{C}(\mathbf{A} - \mathbf{B}\mathbf{K})^{-1}\dot{\mathbf{w}} - \mathbf{C}(\mathbf{A} - \mathbf{B}\mathbf{K})^{-1}\mathbf{B}\mathbf{B}_r\mathbf{r} \\ &\quad - \mathbf{C}(\mathbf{A} - \mathbf{B}\mathbf{K})^{-1}\mathbf{B}_d\mathbf{e}_d \end{aligned} \quad (73)$$

Thus, as the disturbance observer estimation error approaches zero, its effect on the output decreases.

Stability analysis

The augmented plant is formed by augmenting the quadrotor closed loop dynamics (69) and the disturbance observer error dynamics (66). The augmented plant is found to be

$$\begin{bmatrix} \dot{\mathbf{w}} \\ \dot{\mathbf{e}}_d \end{bmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{BK} & \mathbf{BK}_d \\ \mathbf{0}_{11 \times 14} & -\mathbf{L}_d \end{bmatrix} \begin{bmatrix} \mathbf{w} \\ \mathbf{e}_d \end{bmatrix} + \begin{bmatrix} \mathbf{BB}_r & \mathbf{B}_d - \mathbf{BK}_d & \mathbf{0}_{14 \times 11} \\ \mathbf{0}_{11 \times 4} & \mathbf{0}_{11 \times 11} & \mathbf{I}_3 \end{bmatrix} \begin{bmatrix} \mathbf{r} \\ \mathbf{d} \\ \dot{\mathbf{d}} \end{bmatrix} \quad (74)$$

which implies that the disturbance observer error dynamics is decoupled from the quadrotor closed loop dynamics. Therefore, their eigenvalues can be chosen independently. The eigenvalues of the quadrotor closed loop dynamics can be placed in the desired locations by tuning the gain matrix \mathbf{K} . By choosing the non-zero elements of the matrix \mathbf{K} such that

$$\begin{aligned} s^4 + k_{x3}s^3 + k_{x2}s^2 + k_{x1}s + k_{x0} &= 0 \\ s^4 + k_{y3}s^3 + k_{y2}s^2 + k_{y1}s + k_{y0} &= 0 \\ s^4 + k_{z3}s^3 + k_{z2}s^2 + k_{z1}s + k_{z0} &= 0 \\ s^2 + k_{\psi1}s + k_{\psi0} &= 0 \end{aligned} \quad (75)$$

are Hurwitz, the matrix $\mathbf{A} - \mathbf{BK}$ is Hurwitz. Similarly, the eigenvalues of the disturbance observer dynamics can be placed in the desired locations by tuning the diagonal matrix \mathbf{L}_d . By choosing all the non-zero elements of the matrix \mathbf{L}_d to be positive, then, $-\mathbf{L}_d$ is Hurwitz. The eigenvalues of the augmented plant are those of the matrix $\mathbf{A} - \mathbf{BK}$ and those of the matrix $-\mathbf{L}_d$. Thus, the augmented plant is asymptotically stable if the matrices $\mathbf{A} - \mathbf{BK}$ and $-\mathbf{L}_d$ are Hurwitz.

Simulation results

In this section, the disturbance observer-based feedback linearization controller is implemented and applied to the quadrotor model in simulation. The simulation results are presented below.

The quadrotor's parameters used in the simulations are selected as follows: $m = 2.25 \text{ Kg}$, $g = 9.81 \text{ m/s}^2$, $I_x = 41.3 \text{ g.m}^2$, $I_y = 42.2 \text{ g.m}^2$, $I_z = 75.9 \text{ g.m}^2$, $I_p = 0.041 \text{ g.m}^2$, $b = 2.55 \times 10^{-5} \text{ N.s}^2$, $l_a = 0.2 \text{ m}$ and $d = 0.64 \times 10^{-6} \text{ N.m.s}^2$. The quadrotor is required to follow the desired trajectories given by $x_d = \sin(\pi t/10)$, $y_d = \cos(\pi t/10)$, $z_d = 5 + \sin(\pi t/10)$ and $\psi_d = 50 \text{ deg}$. The initial conditions of the quadrotor's outputs are chosen to be $x(0) = 0 \text{ m}$, $y(0) = 1 \text{ m}$, $z(0) = 5 \text{ m}$ and $\psi(0) = 40 \text{ deg}$. The initial conditions of the other states are selected to be all zeros. The gains of the linear state feedback controller are chosen to be $k_{x0} = k_{y0} = k_{z0} = 625$, $k_{x1} = k_{y1} = k_{z1} = 500$, $k_{x2} = k_{y2} = k_{z2} = 150$, $k_{x3} = k_{y3} = k_{z3} = 20$, $k_{\psi0} = 1$ and $k_{\psi1} = 2$. Whereas, the disturbance observer gains are selected to be $L_{di} = 10$ ($i \in \{1, \dots, 11\}$).

Test scenario 1: Actuator failure

This test scenario is divided into two periods; each of which lasts for 50 seconds. First, the ignored nonlinear dynamics d_ϕ , d_θ , d_ψ , d_p , d_q and d_r as well as that due to the drag forces in (9), assuming zero wind velocity, are applied. The drag force equation parameters are

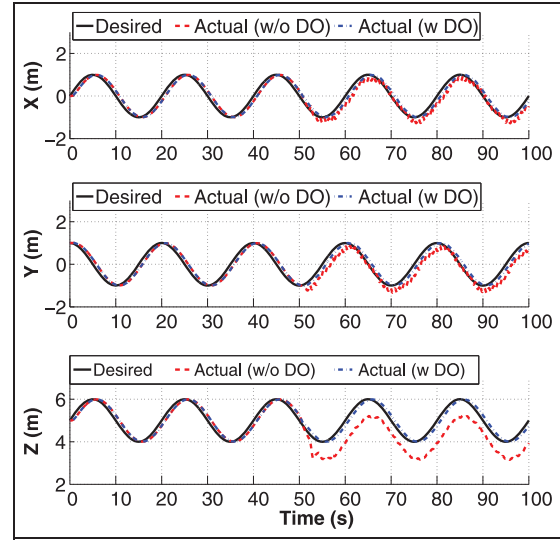


Figure 2. Test scenario 1: position trajectories.

selected to be $\rho = 1.2 \text{ kg/m}^3$, $C_{dx} = C_{dy} = 0.6$, $C_{dz} = 1$,

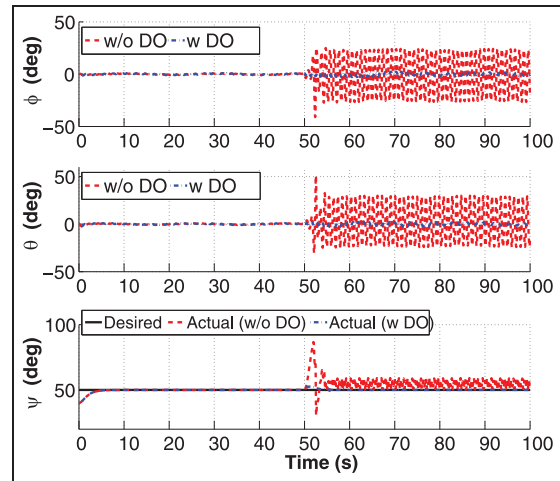


Figure 3. Test scenario 1: attitude trajectories.

$A_u = A_v = 0.1 \text{ m}^2$ and $A_w = 0.2 \text{ m}^2$. Second, besides the disturbances of the first period, a partial actuator failure occurs in which the front rotor is simulated to lose 40% of its effectiveness (i.e. $\lambda = 0.6$). The simulation results of this test scenario, included in Figures (2)–(4), present a comparison between the quadrotor's performance with and without the disturbance observer. Figure 2 presents the quadrotor position trajectories. Figure 3 shows the attitude trajectories. Figure 4 presents the control inputs.

During the first period, the quadrotor's position shown in Figure 2 follows the desired trajectories with and without the disturbance observer due to the very small effect of the ignored nonlinear dynamics on the quadrotor position at low velocities.

During the second period, the quadrotor follows the desired position trajectories only if the disturbance observer is applied as in Figure 2. In the absence of the disturbance observer, a steady state error occurs in the

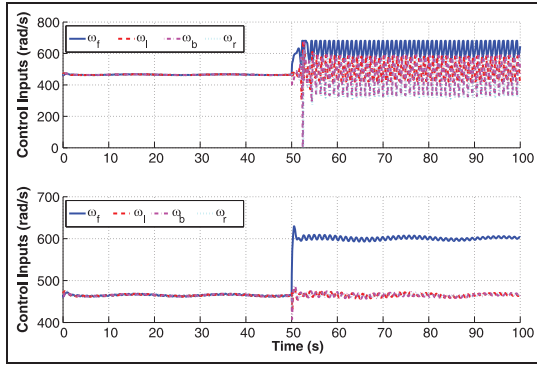


Figure 4. Test scenario 1: control inputs (upper: without disturbance observer, lower: with disturbance observer).

position trajectories. In the presence of the disturbance observer, some transients occur in the heading trajectory in Figure 3 at the beginning of the second period until the disturbance estimates converge to the values of the actual disturbances. The roll and pitch angles are slightly affected. However, in the absence of the disturbance observer, the oscillations of the roll and pitch angles increase tremendously. Moreover, a steady state error accompanied with fluctuations occurs in the heading trajectory. In the case of the disturbance observer, the angular speed of the first rotor increases without reaching saturation to compensate for the loss of its effectiveness and generate the thrust required so that the quadrotor can follow the desired output trajectories. Whereas, in the absence of the disturbance observer, the first rotor reaches saturation due to the actuator failure. In other words, the angular speed of the first rotor, whose effectiveness is reduced, reaches its maximum value.

Test scenario 2: Wind disturbances

In this test scenario, wind disturbances are applied to the quadrotor according to the wind models mentioned in the modeling section in addition to the disturbances occurring due to the ignored nonlinear dynamics. Two discrete wind gusts occur in this test scenario. The parameters of the first discrete wind gust model are selected to be $t_1 = 20s$, $t_2 = 40s$, $v_{z1} = -5m/s$, $v_{z2} = -8m/s$, $v_{x1} = v_{y1} = -10m/s$ and $v_{x2} = v_{y2} = -12m/s$. Whereas, the parameters of the second discrete wind gust are chosen to be $t_1 = 60s$, $t_2 = 80s$, $v_{x1} = v_{y1} = -12m/s$, $v_{x2} = v_{y2} = -10m/s$, $v_{z1} = -8m/s$ and $v_{z2} = -5m/s$. The parameter W_{20} of the Dryden wind turbulence model is chosen to be $W_{20} = 15m/s$. The simulation results of the second scenario are included in Figures (5)–(7) that presents the quadrotor position trajectories, attitude trajectories and control inputs respectively.

As shown in Figure 5, the quadrotor follows the desired position trajectories in the presence of the disturbance observer only. On the other hand, the desired position trajectories are not followed in the absence of the disturbance observer. The position error increases

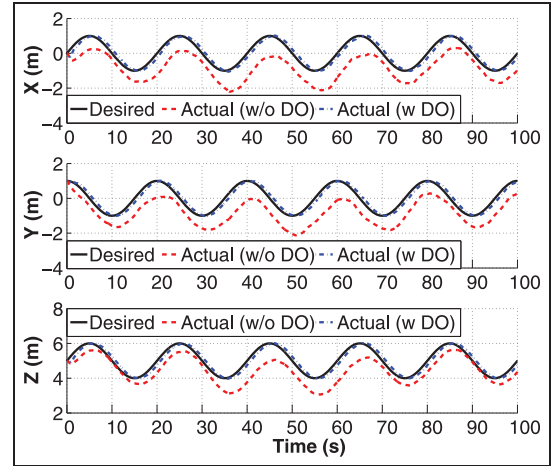


Figure 5. Test scenario 2: position trajectories.

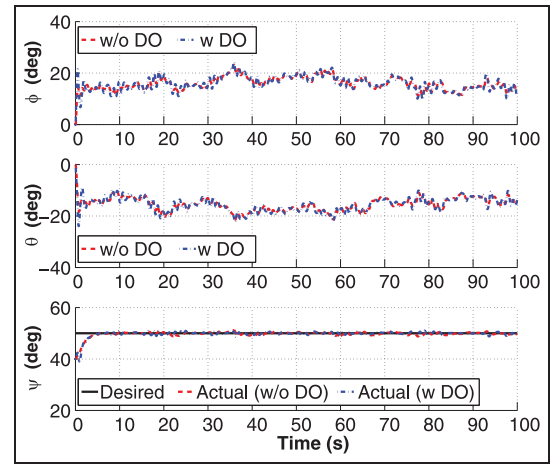


Figure 6. Test scenario 2: attitude trajectories.

after approximately 30 seconds due to the occurrence of the first discrete wind gust where the mean wind velocity increases. Then, the position error decreases after approximately 70 seconds due to the occurrence of the second discrete wind gust where the mean wind velocity decreases. In the absence of the disturbance observer, the quadrotor output trajectory is not following pure sinusoidal trajectories due to the dryden wind effect. As shown in Figure 6, the quadrotor attitude in both cases are similar since the disturbances in this test scenario are mainly force disturbances affecting the translation dynamics. Finally, as shown in Figure 7, the control inputs in both cases are similar even though the presence of the disturbance observer results in better position trajectory tracking as in Figure 5.

Conclusion

In this paper, the robustness of feedback linearization against disturbances is introduced by augmenting it with a disturbance observer so that the quadrotor can follow the desired output trajectories in the presence of disturbances. Two test scenarios are implemented and

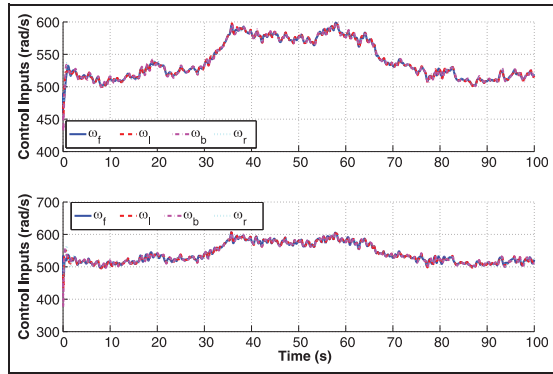


Figure 7. Test scenario 2: control inputs (upper: without disturbance observer, lower: with disturbance observer).

applied to the quadrotor in simulation. The first one includes disturbances due to ignored nonlinear dynamics as well as actuator failure. Whereas, the second one includes wind disturbances in addition to ignored dynamics effects. The effectiveness of the proposed control approach is validated by the simulation results which show that the quadrotor is able to follow the desired trajectories in the presence of the above-mentioned disturbances.

Declaration of Conflicting Interests

The author(s) declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

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Appendix I

Nomenclature

a

quadrotor acceleration vector

$\mathbf{A}_a, \mathbf{A}_p$

controller matrices

$\mathbf{A}_b = (\mathbf{A}_u \ \mathbf{A}_v \ \mathbf{A}_w)^T$

quadrotor projected areas on the planes perpendicular to the body-fixed frames

$\mathbf{A}_i = (\mathbf{A}_x \ \mathbf{A}_y \ \mathbf{A}_z^T)$

quadrotor projected areas on the planes perpendicular to the inertial frames

B, B_d

controller matrices

B_r

controller matrix

b

thrust coefficient

C

controller matrix

$C_{di}, i \in \{x, y, z\}$

drag coefficient

$\hat{\mathbf{d}}$	disturbance estimate	$\hat{v}_{wj}, j \in \{x, y, z\}$	Dryden wind velocity
d	drag coefficient	\mathbf{w}	transformed state vector
d_{u2}, d_{u3}, d_{u4}	actuator failure torque	x	position along the inertial X_I axis
d_{ux}, d_{uy}, d_{uz}	actuator failure force	$\bar{\mathbf{y}}$	output derivative vector
d_x, d_y, d_z	disturbance	y	position along the inertial Y_I axis
$\bar{d}_\psi, \bar{d}_r, \bar{d}_{ij}, i \in \{x, y, z\}, j \in \{2, 3, 4\}$	wind force disturbance	\mathbf{z}	disturbance observer
\mathbf{e}_d	actuator failure torque	z	internal state
$\hat{\mathbf{e}}_i$	disturbance		position along the inertial Z_I axis
$\bar{\mathbf{e}}_3$	disturbance observer estimation error	α	quadrotor nonlinear model function
$\mathbf{f} = (f_x f_y f_z)^T$	14×1 unit vector whose i^{th} element is 1	β	quadrotor nonlinear model matrix
g	3×1 unit vector whose 3^{rd} element is 1	$\eta_i, i \in \{1, \dots, 11\}$	disturbance derivative
$I = \text{diag}(I_x, I_y, I_z)$	total force vector affecting the quadrotor	$\eta_i, i \in \{u, v, w\}$	upper bound
I_p	gravitational acceleration	θ	white noise
\mathbf{K}	inertia matrix	λ	Pitch angle
\mathbf{L}	propeller's moment of inertia	ρ	front rotor effectiveness
l_a	controller matrix	$\sigma_i, i \in \{u, v, w\}$	air density
$L_{di}, i \in \{1, \dots, 12\}$	disturbance observer gain matrix	τ	turbulence intensity
$L_i, i \in \{u, v, w\}$	arm length	ϕ	total torque vector affecting the quadrotor
m	disturbance observer gains	ψ	roll angle
p	turbulence scale length	ψ_o	yaw angle
q	quadrotor mass	Ω	yaw angle equilibrium point
r	quadrotor angular velocity around the body-fixed X_B axis	$\omega_i, i \in \{f, l, b, r\}$	quadrotor angular velocity vector
\mathbf{R}	position along the body-fixed Y_B axis	ω_T	propellers' angular speeds
\mathbf{r}	rotation matrix		algebraic sum of the propellers' angular speeds
r	reference trajectory		
s	position along the body-fixed Z_B axis		
t_1	Laplace variable		
t_2	discrete wind starting time		
\mathbf{u}	discrete wind final time		
$\bar{\mathbf{U}} = (\bar{U}_1 \bar{U}_2 \bar{U}_3 \bar{U}_4)^T$	nonlinear control input vector		
$\mathbf{U} = (\bar{U}_i, i \in \{f, l, r, b\})$	quadrotor actual control inputs		
$\mathbf{U} = (U_1 U_2 U_3 U_4)^T$	propeller's actual thrust force		
$\mathbf{U} = (U_i, i \in \{f, l, r, b\})$	quadrotor calculated control inputs		
\mathbf{v}	propeller's calculated thrust force		
V	linear control input vector		
$\bar{v}_{j1}, j \in \{x, y, z\}$	mean wind speed		
$\bar{v}_{j2}, j \in \{x, y, z\}$	discrete wind starting velocity		
$v_{wj}, j \in \{x, y, z\}$	discrete wind final velocity		
$\bar{v}_{wj}, j \in \{x, y, z\}$	wind velocity		
	discrete wind velocity		

Appendix 2

This appendix shows a detailed derivation of the quadrotor dynamics using the state vector \mathbf{w} . The derivation uses the equations mentioned in (24), (26), (35)–(42). Some parts of the derivation are underlined using different line types to facilitate following the derivation.

Position subsystem along the inertial X_I axis

$$\dot{w}_1 = \dot{x} = w_2$$

$$\begin{aligned} \dot{w}_2 = \ddot{x} &= -s_\theta \frac{U_1}{m} + \frac{d_x + d_{ux}}{m} = \underline{w_3} + \underline{\bar{d}_{x2}} \\ \dot{w}_3 &= \frac{d}{dt} \left(-s_\theta \frac{U_1}{m} \right) \\ &= -s_\theta \frac{\dot{U}_1}{m} - \dot{\theta} c_\theta \frac{U_1}{m} \\ &= -s_\theta \frac{\dot{U}_1}{m} - q_n c_\theta \frac{U_1}{m} - d_\theta c_\theta \frac{U_1}{m} \\ &= \underline{w_4} + \underline{\bar{d}_{x3}} \end{aligned}$$

$$\begin{aligned}
\dot{w}_4 &= \frac{d}{dt} \left(-q_n c_\theta \frac{U_1}{m} - s_\theta \frac{\dot{U}_1}{m} \right) \\
&= -\dot{q}_n c_\theta \frac{U_1}{m} + q_n \dot{\theta} s_\theta \frac{U_1}{m} - q_n c_\theta \frac{\dot{U}_1}{m} - \dot{\theta} c_\theta \frac{\dot{U}_1}{m} - s_\theta \frac{\ddot{U}_1}{m} \\
&= -(s_{\psi_o} \dot{p} + c_{\psi_o} \dot{q}) c_\theta \frac{U_1}{m} + q_n (q_n + d_\theta) s_\theta \frac{U_1}{m} \\
&\quad - q_n c_\theta \frac{\dot{U}}{m} - (q_n + d_\theta) c_\theta \frac{\dot{U}}{m} - s_\theta \frac{\ddot{U}_1}{m} \\
&= - \left(\frac{s_{\psi_o} U_2}{I_x} + \frac{s_{\psi_o} (d_p + d_{u_2})}{I_x} \right) c_\theta \frac{U_1}{m} \\
&\quad - \left(\frac{c_{\psi_o} U_3}{I_y} + \frac{c_{\psi_o} (d_q + d_{u_3})}{I_y} \right) c_\theta \frac{U_1}{m} \\
&\quad + \frac{q_n^2 s_\theta U_1}{m} + \frac{d_\theta q_n s_\theta U_1}{m} - \frac{q_n c_\theta \dot{U}_1}{m} \\
&\quad - \frac{q_n c_\theta \dot{U}_1}{m} - \frac{d_\theta c_\theta \dot{U}}{m} - \frac{s_\theta \ddot{U}_1}{m} \\
&= \frac{q_n^2 s_\theta U_1}{m} - 2q_n c_\theta \frac{\dot{U}_1}{m} - \frac{s_\theta \ddot{U}_1}{m} \\
&\quad - \frac{1}{m I_x} s_{\psi_o} c_\theta U_1 U_2 - \frac{1}{m I_y} c_{\psi_o} c_\theta U_1 U_3 \\
&\quad - \frac{1}{m I_x} s_{\psi_o} c_\theta U_1 (d_p + d_{u_2}) - \frac{1}{m I_y} c_{\psi_o} c_\theta U_1 (d_q + d_{u_3}) \\
&\quad + \frac{q_n s_\theta d_\theta U_1}{m} - \frac{c_\theta d_\theta \dot{U}_1}{m} \\
&= \alpha_x(\mathbf{x}) + \beta_{11}(\mathbf{x}) \dot{U}_1 + \beta_{12}(\mathbf{x}) U_2 + \beta_{13}(\mathbf{x}) U_3 + \bar{d}_{x4},
\end{aligned}$$

Position subsystem along the inertial Y_I axis

$$\dot{w}_5 = \dot{y} = w_6$$

$$\dot{w}_6 = \ddot{y} = s_\phi c_\theta \frac{U_1}{m} + \frac{d_y + d_{u_y}}{m} = \bar{w}_7 + \bar{d}_{y2}$$

$$\begin{aligned}
\dot{w}_7 &= \frac{d}{dt} \left(s_\phi c_\theta \frac{U_1}{m} \right) \\
&= \dot{\phi} c_\phi c_\theta \frac{U_1}{m} - \dot{\theta} s_\phi s_\theta \frac{U_1}{m} + s_\phi c_\theta \frac{\dot{U}_1}{m} \\
&= (p_n + d_\phi) c_\phi c_\theta \frac{U_1}{m} - (q_n + d_\theta) s_\phi s_\theta \frac{U_1}{m} + s_\phi c_\theta \frac{\dot{U}_1}{m} \\
&= \frac{p_n c_\phi c_\theta U_1}{m} - \frac{q_n s_\phi s_\theta U_1}{m} + \frac{s_\phi c_\theta \dot{U}_1}{m} \\
&\quad + \frac{d_\phi c_\phi c_\theta U_1}{m} - \frac{d_\theta s_\phi s_\theta U_1}{m} \\
&= \bar{w}_8 + \bar{d}_{y3}
\end{aligned}$$

$$\begin{aligned}
\dot{w}_8 &= \frac{d}{dt} \left(p_n c_\phi c_\theta \frac{U_1}{m} - q_n s_\phi s_\theta \frac{U_1}{m} + s_\phi c_\theta \frac{\dot{U}_1}{m} \right) \\
&= \dot{p}_n c_\phi c_\theta \frac{U_1}{m} - p_n \dot{\phi} s_\phi c_\theta \frac{U_1}{m} - p_n \dot{\theta} c_\phi s_\theta \frac{U_1}{m} + p_n c_\phi c_\theta \frac{\dot{U}_1}{m} \\
&\quad - \dot{q}_n s_\phi s_\theta \frac{U_1}{m} - q_n \dot{\phi} c_\phi s_\theta \frac{U_1}{m} - q_n \dot{\theta} s_\phi c_\theta \frac{U_1}{m} - q_n s_\phi s_\theta \frac{\dot{U}_1}{m} \\
&\quad + \dot{\phi} c_\phi c_\theta \frac{\dot{U}_1}{m} - \dot{\theta} s_\phi s_\theta \frac{\dot{U}_1}{m} + s_\phi c_\theta \frac{\ddot{U}_1}{m} \\
&= (c_{\psi_o} \dot{p} - s_{\psi_o} \dot{q}) c_\phi c_\theta \frac{U_1}{m} - p_n (p_n + d_\phi) s_\phi c_\theta \frac{U_1}{m} \\
&\quad - p_n (q_n + d_\theta) c_\phi s_\theta \frac{U_1}{m} + p_n c_\phi c_\theta \frac{\dot{U}_1}{m} \\
&\quad - (s_{\psi_o} \dot{p} + c_{\psi_o} \dot{q}) s_\phi s_\theta \frac{U_1}{m} - q_n (p_n + d_\phi) c_\phi s_\theta \frac{U_1}{m} \\
&\quad - q_n (q_n + d_\theta) s_\phi c_\theta \frac{U_1}{m} - q_n s_\phi s_\theta \frac{\dot{U}_1}{m} \\
&\quad + (p_n + d_\phi) c_\phi c_\theta \frac{\dot{U}_1}{m} - (q_n + d_\theta) s_\phi s_\theta \frac{\dot{U}_1}{m} + s_\phi c_\theta \frac{\ddot{U}_1}{m} \\
&= \left(c_{\psi_o} \frac{U_2 + d_p + d_{u_2}}{I_x} - s_{\psi_o} \frac{U_3 + d_q + d_{u_3}}{I_y} \right) c_\phi c_\theta \frac{U_1}{m} \\
&\quad - \frac{p_n^2 s_\phi c_\theta U_1}{m} - \frac{p_n d_\phi s_\phi c_\theta U_1}{m} \\
&\quad - \frac{p_n q_n c_\phi s_\theta U_1}{m} - \frac{p_n d_\theta c_\phi s_\theta U_1}{m} + \frac{p_n c_\phi c_\theta \dot{U}_1}{m} \\
&\quad - \left(s_{\psi_o} \frac{U_2 + d_p + d_{u_2}}{I_x} + c_{\psi_o} \frac{U_3 + d_q + d_{u_3}}{I_y} \right) s_\phi s_\theta \frac{U_1}{m} \\
&\quad - \frac{q_n p_n c_\phi s_\theta U_1}{m} - \frac{q_n d_\phi c_\phi s_\theta U_1}{m} \\
&\quad - \frac{q_n^2 s_\phi c_\theta U_1}{m} - \frac{q_n d_\theta s_\phi c_\theta U_1}{m} - \frac{q_n s_\phi s_\theta \dot{U}_1}{m} \\
&\quad + (p_n + d_\phi) c_\phi c_\theta \frac{\dot{U}_1}{m} - (q_n + d_\theta) s_\phi s_\theta \frac{\dot{U}_1}{m} + \frac{s_\phi c_\theta \ddot{U}_1}{m} \\
&= - (p_n^2 + q_n^2) s_\phi c_\theta \frac{U_1}{m} - 2p_n q_n c_\phi s_\theta \frac{U_1}{m} \\
&\quad + 2(p_n c_\phi c_\theta - q_n s_\phi s_\theta) \frac{\dot{U}_1}{m} + \frac{s_\phi c_\theta \ddot{U}_1}{m} \\
&\quad + \frac{(c_{\psi_o} c_\phi c_\theta - s_{\psi_o} s_\phi s_\theta) U_1 U_2}{m I_x} \\
&\quad + \frac{-(s_{\psi_o} c_\phi c_\theta + c_{\psi_o} s_\phi s_\theta) U_1 U_3}{m I_y}
\end{aligned}$$

$$\begin{aligned}
& + \left(\frac{c_{\psi_o}}{I_x} (d_p + d_{u_2}) - \frac{s_{\psi_o}}{I_y} (d_q + d_{u_3}) \right) c_{\phi} c_{\theta} \frac{U_1}{m} \\
& - \left(\frac{s_{\psi_o}}{I_x} (d_p + d_{u_2}) + \frac{c_{\psi_o}}{I_y} (d_q + d_{u_3}) \right) s_{\phi} s_{\theta} \frac{U_1}{m} \\
& - (p_n d_{\phi} + q_n d_{\phi}) c_{\phi} s_{\theta} \frac{U_1}{m} - (p_n d_{\phi} + q_n d_{\theta}) s_{\phi} c_{\theta} \frac{U_1}{m} \\
& + (d_{\phi} c_{\phi} c_{\theta} - d_{\theta} s_{\phi} s_{\theta}) \frac{\dot{U}_1}{m} \\
& = \alpha_y(\mathbf{x}) + \beta_{21}(\mathbf{x}) \ddot{U}_1 + \beta_{22}(\mathbf{x}) U_2 + \beta_{23}(\mathbf{x}) U_3 + \ddot{d}_{y4},
\end{aligned}$$

Position subsystem along the inertial Z_I axis

$$\begin{aligned}
\dot{w}_9 &= \dot{z} = w_{10} \\
\dot{w}_{10} &= \ddot{z} = c_{\phi} c_{\theta} \frac{U_1}{m} + \frac{d_z + d_{u_z}}{m} = \underline{w}_{11} + \ddot{d}_{z2} \\
\dot{w}_{11} &= \frac{d}{dt} \left(c_{\phi} c_{\theta} \frac{U_1}{m} \right) \\
&= -\dot{\phi} s_{\phi} c_{\theta} \frac{U_1}{m} - \dot{\theta} c_{\phi} s_{\theta} \frac{U_1}{m} + c_{\phi} c_{\theta} \frac{\dot{U}_1}{m} \\
&= -(p_n + d_{\phi}) s_{\phi} c_{\theta} \frac{U_1}{m} - (q_n + d_{\theta}) c_{\phi} s_{\theta} \frac{U_1}{m} + c_{\phi} c_{\theta} \frac{\dot{U}_1}{m} \\
&= -p_n s_{\phi} c_{\theta} \frac{U_1}{m} - q_n c_{\phi} s_{\theta} \frac{U_1}{m} + c_{\phi} c_{\theta} \frac{\dot{U}_1}{m} \\
&\quad - d_{\phi} s_{\phi} c_{\theta} \frac{U_1}{m} - d_{\theta} c_{\phi} s_{\theta} \frac{U_1}{m} \\
&= \underline{w}_{12} + \ddot{d}_{z3}, \\
\dot{w}_{12} &= \frac{d}{dt} \left(-p_n s_{\phi} c_{\theta} \frac{U_1}{m} - q_n c_{\phi} s_{\theta} \frac{U_1}{m} + c_{\phi} c_{\theta} \frac{\dot{U}_1}{m} \right) \\
&= -\dot{p}_n s_{\phi} c_{\theta} \frac{U_1}{m} - p_n \dot{\phi} c_{\phi} c_{\theta} \frac{U_1}{m} \\
&\quad + p_n \dot{\theta} s_{\phi} s_{\theta} \frac{U_1}{m} - p_n s_{\phi} c_{\theta} \frac{\dot{U}_1}{m} \\
&\quad - \dot{q}_n c_{\phi} s_{\theta} \frac{U_1}{m} + q_n \dot{\phi} s_{\phi} s_{\theta} \frac{U_1}{m} \\
&\quad - q_n \dot{\theta} c_{\phi} c_{\theta} \frac{U_1}{m} - q_n c_{\phi} s_{\theta} \frac{\dot{U}_1}{m} \\
&\quad - \dot{\phi} s_{\phi} c_{\theta} \frac{\dot{U}_1}{m} - \dot{\theta} c_{\phi} s_{\theta} \frac{\dot{U}_1}{m} + c_{\phi} c_{\theta} \frac{\ddot{U}_1}{m} \\
&= -(c_{\psi_o} \dot{p} - s_{\psi_o} \dot{q}) s_{\phi} c_{\theta} \frac{U_1}{m} - p_n (p_n + d_{\phi}) c_{\phi} c_{\theta} \frac{U_1}{m} \\
&\quad + p_n (q_n + d_{\theta}) s_{\phi} s_{\theta} \frac{U_1}{m} - p_n s_{\phi} c_{\theta} \frac{\dot{U}_1}{m} \\
&\quad - (s_{\psi_o} \dot{p} + c_{\psi_o} \dot{q}) c_{\phi} s_{\theta} \frac{U_1}{m} + q_n (p_n + d_{\phi}) s_{\phi} s_{\theta} \frac{U_1}{m} \\
&\quad - q_n (q_n + d_{\theta}) c_{\phi} c_{\theta} \frac{U_1}{m} - q_n c_{\phi} s_{\theta} \frac{\dot{U}_1}{m}
\end{aligned}$$

$$\begin{aligned}
& - (p_n + d_{\phi}) s_{\phi} c_{\theta} \frac{\dot{U}_1}{m} - (q_n + d_{\theta}) c_{\phi} s_{\theta} \frac{\dot{U}_1}{m} + c_{\phi} c_{\theta} \frac{\ddot{U}_1}{m} \\
& = - \left(c_{\psi_o} \frac{U_2 + d_p + d_{u_2}}{I_x} - s_{\psi_o} \frac{U_3 + d_q + d_{u_3}}{I_y} \right) s_{\phi} c_{\theta} \frac{U_1}{m} \\
& \quad - p_n^2 c_{\phi} c_{\theta} \frac{U_1}{m} - p_n d_{\phi} c_{\phi} c_{\theta} \frac{U_1}{m} \\
& \quad + p_n q_n s_{\phi} s_{\theta} \frac{U_1}{m} + p_n d_{\theta} s_{\phi} s_{\theta} \frac{U_1}{m} - p_n s_{\phi} c_{\theta} \frac{\dot{U}_1}{m} \\
& \quad - \left(s_{\psi_o} \frac{U_2 + d_p + d_{u_2}}{I_x} + c_{\psi_o} \frac{U_3 + d_q + d_{u_3}}{I_y} \right) c_{\phi} s_{\theta} \frac{U_1}{m} \\
& \quad + q_n p_n s_{\phi} s_{\theta} \frac{U_1}{m} + q_n d_{\phi} s_{\phi} s_{\theta} \frac{U_1}{m} \\
& \quad - q_n^2 c_{\phi} c_{\theta} \frac{U_1}{m} - q_n d_{\theta} c_{\phi} c_{\theta} \frac{U_1}{m} - q_n c_{\phi} s_{\theta} \frac{\dot{U}_1}{m} \\
& \quad - (p_n + d_{\phi}) s_{\phi} c_{\theta} \frac{\dot{U}_1}{m} - (q_n + d_{\theta}) c_{\phi} s_{\theta} \frac{\dot{U}_1}{m} + c_{\phi} c_{\theta} \frac{\ddot{U}_1}{m} \\
& = -(p_n^2 + q_n^2) c_{\phi} c_{\theta} \frac{U_1}{m} + 2p_n q_n s_{\phi} s_{\theta} \frac{U_1}{m} \\
& \quad - 2(p_n s_{\phi} c_{\theta} + q_n c_{\phi} s_{\theta}) \frac{\dot{U}_1}{m} + \frac{c_{\phi} c_{\theta}}{m} \ddot{U}_1 \\
& \quad - \frac{(c_{\psi_o} s_{\phi} c_{\theta} + s_{\psi_o} c_{\phi} s_{\theta}) U_1}{m I_x} U_2 \\
& \quad + \frac{(s_{\psi_o} s_{\phi} c_{\theta} - c_{\psi_o} c_{\phi} s_{\theta}) U_1}{m I_y} U_3 \\
& \quad - \left(\frac{c_{\psi_o}}{I_x} (d_p + d_{u_2}) - \frac{s_{\psi_o}}{I_y} (d_q + d_{u_3}) \right) s_{\phi} c_{\theta} \frac{U_1}{m} \\
& \quad - \left(\frac{s_{\psi_o}}{I_x} (d_p + d_{u_2}) + \frac{c_{\psi_o}}{I_y} (d_q + d_{u_3}) \right) c_{\phi} s_{\theta} \frac{U_1}{m} \\
& \quad + (p_n d_{\theta} + q_n d_{\phi}) s_{\phi} s_{\theta} \frac{U_1}{m} - (p_n d_{\phi} + q_n d_{\theta}) c_{\phi} c_{\theta} \frac{U_1}{m} \\
& \quad - (d_{\phi} s_{\phi} c_{\theta} + d_{\theta} c_{\phi} s_{\theta}) \frac{\dot{U}_1}{m}
\end{aligned} \tag{76}$$

$$= \alpha_z(\mathbf{x}) + \beta_{31}(\mathbf{x}) \ddot{U}_1 + \beta_{32}(\mathbf{x}) U_2 + \beta_{33}(\mathbf{x}) U_3 + \ddot{d}_{z4}$$

Heading subsystem

$$\begin{aligned}
\dot{w}_{13} &= \dot{\psi} = r + d_{\psi} = w_{14} + \ddot{d}_{\psi} \\
\dot{w}_{14} &= \dot{r} = \frac{1}{I_z} U_4 + \frac{1}{I_z} (d_r + d_{u_4}) = \beta_{44}(\mathbf{x}) U_4 + \ddot{d}_r
\end{aligned}$$