A GEOMETRIC PROOF OF THE QUASI-LINEARITY OF THE WATER-WAVES SYSTEM

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ABSTRACT. In the first part of this paper we prove that the flow associated to the Burgers equation with a non local term of the form $H\langle \mathrm{D}\rangle^{\alpha} u$ fails to be uniformly continuous from bounded sets of $H^s(\mathbb{D})$ to $C^0([0,T], H^s(\mathbb{D}))$ for $T>0$ $s>\frac{1}{2}+\max(\alpha,1)$, $0\leq\alpha<2$, $\mathbb{D}=\mathbb{R}$ or $\mathbb{T}$ and H is the Hilbert transform. Furthermore we show that the flow can not be C^1 from bounded sets of $H^s(\mathbb{D})$ to $C^0([0,T], H^{s-1+(\alpha-1)^{+}+\epsilon}(\mathbb{D}))$ for $\epsilon>0$. We generalize this result to a large class of non linear transport-dispersive equations in any dimension, that in particular contains the Whitham equation and the paralinearisation of the water waves system with and without surface tension. The current result is optimal in the sense that for $\alpha=2$ and $\mathbb{D}=\mathbb{T}$ the flow associated to the Benjamin-Ono equation is Lipschitz. In the second part of this paper we apply this method to deduce the quasi-linearity of the water waves system, which is the main result of this paper.

CONTENTS

1. INTRODUCTION

A commonly found definition is that a partial differential equation is said to be quasi-linear if it is linear with respect to all the highest order derivatives of the unknown function, for example equations of the form:

$$\partial_t u+\sum A_j(u)\partial_j u=F(u).$$

Which we compare to the definition of semi-linearity as a partial differential equation whose highest order terms are linear, for example equations of the form:

$$\partial_t u+\sum A_j\partial_j u=F(u).$$

This distinction is supposed to classify the equations in accordance to how one solves their respective Cauchy problems. For example, semi-linear equations are expected to be solved locally by a Picard iteration scheme and thus the associated flow is expected to depend regularly on the data. On the other hand quasi-linear equations are expected to be solved by a compactness method and no more information than continuity can be recovered on the flow. The problem of those broad definitions with the count of derivatives is that they fail to classify the equations according to this simple criteria on their Cauchy problem. Indeed by those definitions the (KPI)

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and (KPII) equations are semi-linear by the count of the derivatives, and are given by:

$$(u_t + u_x + u_{xxx})_x + u_{yy} = 0, \tag{KPI}$$

$$(u_t + u_x + u_{xxx})_x - u_{yy} = 0. \tag{KPII}$$

Indeed Bourgain showed in [19] that (KPII) can be solved by an iteration scheme and that the flow is regular. But Moulinet, Saut and Tzvetkov showed in [10] that the flow associated to (KPI) cannot be C^2 and that it cannot be solved by a Picard iteration scheme. Thus it seems that the adequate definitions to quasi-linearity and semi-linearity is the one given in [10] and that we will use here are given by:

- A partial differential equation is said to be semi-linear if its flow is regular (at least C^1).
- A partial differential equation is said to be quasi-linear if its flow is not C^1.

It is well known that the flow associated to the Burgers equation:

$$\partial_t u + u\partial_x u = 0 \quad \text{on} \quad \mathbb{R},$$

fails to be uniformly continuous, given the equation its quasi-linear nature, as for example shown in [9]. An important class of equations that arises in the study of asymptotic models of the water waves equations is Burgers type equation with a dispersive term, for example the Benjamin-Ono equation:

$$\partial_t u + u\partial_x u + H\partial_x^2 u = 0 \quad \text{on} \quad \mathbb{R}, \tag{BO}$$

and Korteweg-de Vries equation:

$$\partial_t u + u\partial_x u + \partial_x^3 u = 0 \quad \text{on} \quad \mathbb{R}. \tag{KdV}$$

It was also shown in [11], that the flow associated to the Benjamin-Ono equation on $H^s(\mathbb{R}), s > \frac{3}{2}$ fails to be uniformly continuous. The proof relies heavily on the dimension, the structure of the equation and on some interactions between small and high frequencies thus it does not generalize to the case of $\mathbb{T}$. More generally in [9], it is shown that the flow fails to be C^2 (thus the equations are unsolvable by a Picard fixed point scheme) for equations of the form:

$$\partial_t u + u\partial_x u + \omega(D)\partial_x u = 0, \text{ with } |\omega(\xi)| \leq |\xi|^\gamma, \ \gamma < 2.$$

Here the proof relies heavily on the Duhamel formula, on the explicit solvability of the linear part using the Fourier transform and again on some interactions between small and high frequencies thus it does not generalize to the case of $\mathbb{T}$.

In [9], for the KdV equation, using Strichartz type dispersive estimates the Cauchy problem is solved by a Picard fixed point scheme and thus the flow is regular, showing a change in nature for the problem. This shows that an interesting phenomena happening where the dispersive term can dominate the nonlinearity. On $\mathbb{R}$, the previous examples show that this change of regime happens for a dispersive term of order 3. Thus the result obtained in [9] is optimal in $d = 1$.

In this paper we improve these results in several directions:

- we prove the result for a generic dispersive perturbation of order $\alpha < 2$,
- we prove the strongest result possible by proving that the flow is not uniformly continuous,
- for $\epsilon > 0$ we prove that the flow cannot be
$$C^1(H^s(\mathbb{D}), C^0([0,T], H^{s-1+(\alpha-1)^+ + \epsilon}(\mathbb{D}))).$$
- we prove the result in any dimension.

For the sake of clarity we begin by stating a result in dimension 1.

Theorem 1.1. *Consider three real numbers $\alpha \in [0,2[$, $s \in]\max(1,\alpha)+\frac{1}{2},+\infty[$, $r>0$ and $u_0 \in H^s(\mathbb{D})$. Then there exists $T>0$ such that for all v_0 in the ball $B(u_0,r) \subset H^s(\mathbb{D})$ there exists a unique $v \in C([0,T],H^s(\mathbb{D}))$ solving the Cauchy problem:*

$$\begin{cases} \partial_t v + v\partial_x v + H\langle \mathrm{D}\rangle^\alpha v = 0 \\ v(0,\cdot) = v_0(\cdot), \end{cases} \tag{1.1}$$

where H is the Hilbert transform defined by it's symbol[1]*:*

$$H(\xi) = -i\,\mathrm{sgn}(\xi).$$

Moreover the flow map:

$$\begin{aligned} B(u_0,r) &\to C([0,T],H^s(\mathbb{D})) \\ v_0 &\mapsto v \end{aligned}$$

is not uniformly continuous.
Considering a weaker control norm we get, for all $\epsilon>0$ the flow map:

$$\begin{aligned} B(u_0,r) &\to C([0,T],H^{s-1+(\alpha-1)^++\epsilon}(\mathbb{D})) \\ v_0 &\mapsto v \end{aligned}$$

is not C^1.

We shall prove a stronger result (see Theorem 3.1) showing that for a dispersive perturbation of order $\alpha<2$, the non-linear transport term dominates the flow's evolution locally and this happens independently of the dimension. This limited regularity of the flow implies that the Cauchy problem can not be solved by a Picard fixed point scheme and thus those equations are quasi-linear.

- Let us also notice that the result obtained in this paper on the regularity is optimal since in [15], L. Molinet proves that the flow has Lipschitz regularity for the Benjamin-Ono equation on the torus. This same result tells us that for $\epsilon>0$ the flow cannot be

$$C^1(H^s(\mathbb{T}),C^0([0,T],H^{s-1+(\alpha-1)^++\epsilon}(\mathbb{T}))),$$

  which shows that our result is optimal for $\alpha=2$.
- In our recent work [22], we generalize the result on the Benjamin-Ono equation and prove that the flow associated to the Burgers equation with a non local term of the form $D^{\alpha-1}\partial_x u$, $\alpha \in]1,2]$ is Lipschitz from bounded sets of $H^s(\mathbb{T};\mathbb{R})$ to $C^0([0,T],H^{s-1+(\alpha-1)^+}(\mathbb{T};\mathbb{R}))$. Thus proving that the result is optimal for $\alpha\in]1,2]$.
- The case on $\mathbb{R}$ is more subtle because by the results in [9] the change to the semi-linear type equations happens for $\alpha=3$, and the flow associated to the Benjamin-Ono equation on $\mathbb{R}$ fails to be uniformly continuous as shown in [11]. Again in [22] we show that the flow associated to the following equation

$$\partial_t v + \mathrm{Re}(v)\partial_x v + i\langle \mathrm{D}\rangle^2 v = 0,$$

  is Lipschitz from bounded sets of $H^s(\mathbb{R};\mathbb{R})$ to $C^0([0,T],H^s(\mathbb{R};\mathbb{R}))$. Showing that the lack of regularity obtained in [9] for $\alpha\geq 2$ is essentially due to the lack of invertibility of the low frequency component of the dispersive term and that the result obtained here is optimal on $\mathbb{R}$ as well.

[1] The use of the Hilbert transform insures that we always work with real valued functions when the initial data is real valued.

Finally Theorem 3.1 contains applications to different classes of equations:
-Firstly the Whitham equation on $\mathbb{R}$:

$$\begin{cases} \partial_t u + u\partial_x u - Lu_x = 0, \\ Lf(x) = \int e^{ix\cdot\xi} p(x,\xi)\hat{f}(\xi)d\xi, \end{cases}$$

is quasi-linear for $p \in S^\alpha, \alpha < 1$ and such that $\mathrm{Im}(p) \in S^0$ (See (A.3) for the definition of the symbol classes).

-The second and main application is the water waves system with and without surface tension. We follow here the presentation in [4] and [7].

1.1. **Assumptions on the domain.** We consider a domain with free boundary, of the form:

$$\left\{(t,x,y) \in [0,T] \times \mathbb{R}^d \times \mathbb{R} : (x,y) \in \Omega_t\right\},$$

where Ω_t is the domain located between a free surface

$$\Sigma_t = \left\{(x,y) \in \mathbb{R}^d \times \mathbb{R} : y = \eta(t,x)\right\}$$

and a given (general) bottom denoted by $\Gamma = \partial\Omega_t \setminus \Sigma_t$. More precisely we assume that initially $(t = 0)$ we have the hypothesis H_t given by:

- The domain Ω_t is the intersection of the half space, denoted by $\Omega_{1,t}$, located below the free surface Σ_t,

  $$\Omega_{1,t} = \left\{(x,y) \in \mathbb{R}^d \times \mathbb{R} : y < \eta(t,x)\right\}$$

  and an open set $\Omega_2 \subset \mathbb{R}^{d+1}$ such that Ω_2 contains a fixed strip around Σ_t, which means that there exists $h > 0$ such that,

  $$\left\{(x,y) \in \mathbb{R}^d \times \mathbb{R} : \eta(t,x) - h \leq y \leq \eta(t,x)\right\} \subset \Omega_2.$$

  We shall assume that the domain Ω_2 (and hence the domain $\Omega_t = \Omega_{1,t} \cap \Omega_2$) is connected.

1.2. **The equations.** We consider an incompressible inviscid liquid, having unit density. The equations of motion are given by the Euler system of equations on the velocity field v:

$$\partial_t v + v \cdot \nabla v + \nabla P = -ge_y \ , \quad \mathrm{div}\, v = 0 \ \text{ in } \ \Omega_t, \tag{1.2}$$

where $-ge_y$ is the acceleration of gravity $(g > 0)$ and where the pressure term P can be recovered from the velocity by solving an elliptic equation. The problem is then coupled with the boundary conditions:

$$\begin{cases} v \cdot n = 0 & \text{on } \ \Gamma, \\ \partial_t \eta = \sqrt{1 + |\nabla\eta|^2} v \cdot \nu & \text{on } \ \Sigma_t, \\ P = -\kappa H(\eta) & \text{on } \ \Sigma_t, \end{cases} \tag{1.3}$$

where n and ν are the exterior normals to the bottom Γ and the free surface Σ_t, κ is the surface tension and $H(\eta)$ is the mean curvature of the free surface:

$$H(\eta) = \mathrm{div}\left(\frac{\nabla\eta}{\sqrt{1 + |\nabla\eta|^2}}\right).$$

We take $\kappa = 1$ for the case with surface tension and $\kappa = 0$ in the case of gravity water waves (without surface tension). The first condition in (1.3) expresses in fact that the particles in contact with the rigid bottom remain in contact with it. As no hypothesis is made on the regularity of Γ, this condition is shown to make sense in a weak variational meaning due to the hypothesis H_t, for more details on this we refer to Section 2 in [4] and Section 3 in [7].

The fluid motion is supposed to be irrotational and Ω_t is supposed to be simply connected thus the velocity v field derives from some potential ϕ i.e $v = \nabla\phi$ and:

$$\Delta\phi = 0 \text{ in } \Omega, \quad \partial_n\phi = 0 \text{ on } \Gamma.$$

The boundary condition on ϕ becomes:

$$\begin{cases} \partial_n\phi = 0 & \text{on } \Gamma, \\ \partial_t\eta = \partial_y\phi - \nabla\eta\cdot\nabla\phi & \text{on } \Sigma_t, \\ \partial_t\phi = -g\eta + \kappa H(\eta) - \frac{1}{2}\left|\nabla_{x,y}\phi\right|^2 & \text{on } \Sigma_t. \end{cases} \tag{1.4}$$

Following Zakharov [17] and Craig-Sulem [18] we reduce the analysis to a system on the free surface Σ_t. If ψ is defined by

$$\psi(t,x) = \phi(t,x,\eta(t,x)),$$

then ϕ is the unique variational solution of

$$\Delta\phi = 0 \text{ in } \Omega_t, \quad \phi_{|y=\eta} = \psi, \quad \partial_n\phi = 0 \text{ on } \Gamma.$$

Define the Dirichlet-Neumann operator by

$$\begin{aligned} (G(\eta)\psi)(t,x) &= \sqrt{1+|\nabla\eta|^2}\partial_n\phi_{|y=\eta} \\ &= (\partial_y\phi)(t,x,\eta(t,x)) - \nabla\eta(t,x)\cdot(\nabla\phi)(t,x,\eta(t,x)). \end{aligned}$$

For the case with rough bottom we refer to [3], [4] and [7] for the well posedness of the variational problem and Dirichlet-Neumann operator. Now (η,ψ) (see for example [18]) solves:

$$\begin{aligned} &\partial_t\eta = G(\eta)\psi, \\ &\partial_t\psi = -g\eta + \kappa H(\eta) + \frac{1}{2}\left|\nabla\psi\right|^2 + \frac{1}{2}\frac{\nabla\cdot\eta\nabla\psi + G(\eta)\psi}{1+|\nabla\eta|^2}. \end{aligned} \tag{1.5}$$

We see that in the case of $\kappa = 0$, the pressure "disappears" from the current formulation and an additional work is needed to redefine it.

1.3. **Gravity water waves: Pressure and Taylor Coefficients.** Here we give a quick review of the ideas in [6]. Recall that by definition for gravity water waves we work with $\kappa = 0$ and we define the Taylor coefficient

$$a(t,x) = -(\partial_y P)(t,x,\eta(t,x)).$$

The stability of the waves is dictated by the Taylor sign condition, which is the assumption that there exists a positive constant c such that

$$a(t,x) \geq c > 0. \tag{1.6}$$

In [7] this condition is needed in the proof of the well posedness of the Cauchy problem and it is shown to be locally propagated by the flow.

Now we will show how to define P from the Zakharov formulation. Let R be the variational solution of

$$\Delta R = 0 \text{ in } \Omega_t, \quad R_{|y=\eta} = \eta g + \frac{1}{2}\left|\nabla_{x,y}\phi\right|^2_{|y=\eta}.$$

We define the pressure P in the domain Ω by

$$P(x,y) = R(x,y) - gy - \frac{1}{2}\left|\nabla_{x,y}\phi(x,y)\right|^2.$$

In [6] Alazard, Burq, and Zuily show that to a solution $(\eta, \psi) \in C([0,T], H^{s+\frac{1}{2}(\mathbb{R}^d)} \times H^{s+\frac{1}{2}(\mathbb{R}^d)})$, $s > \frac{d}{2}+\frac{1}{2}$ of the Zakharov/Craig-Sulem system (1.5) corresponds a unique solution v of the Euler system.

1.4. **Quasi-linearity of the water Wave system.** In [4] and [7], Alazard, Burq, and Zuily preform a paralinearisation and symmetrisation of the the water waves system that take the form:

$$\partial_t u + T_V.\nabla u + iT_\gamma u = f,$$

where γ is of order $\frac{3}{2}$ in the case with surface tension and $\frac{1}{2}$ in the case without. The terms V and γ verify the conditions required by Theorem 3.1 and thus the paralinearisation of the water-waves system are quasi-linear in the considered thresholds of regularity. From this we will deduce the following two theorems.

First in the case of water waves with surface tension, i.e $\kappa = 1$, where the well posedness of the Cauchy problem is proved in [4] we complete it by the following.

Theorem 1.2. *Fix the dimension $d \geq 1$ and consider two real numbers $r > 0$, $s \in]2+\frac{d}{2}, +\infty[$ and $(\eta_0, \psi_0) \in H^{s+\frac{1}{2}}(\mathbb{R}^d) \times H^s(\mathbb{R}^d)$ such that*

$$\forall (\eta_0', \psi_0') \in B((\eta_0, \psi_0), r) \subset H^{s+\frac{1}{2}}(\mathbb{R}^d) \times H^s(\mathbb{R}^d)$$

the assumption $H_{t=0}$ is satisfied. Then there exists $T > 0$ such that the Cauchy problem (1.5) with initial data $(\eta_0', \psi_0') \in B((\eta_0, \psi_0), r)$ has a unique solution

$$(\eta', \psi') \in C^0([0,T]; H^{s+\frac{1}{2}}(\mathbb{R}^d) \times H^s(\mathbb{R}^d))$$

and such that the assumption H_t is satisfied for $t \in [0,T]$. Moreover the flow map:

$$\begin{aligned} B((\eta_0, \psi_0), r) &\to C([0,T], H^{s+\frac{1}{2}}(\mathbb{R}^d) \times H^s(\mathbb{R}^d)) \\ (\eta_0', \psi_0') &\mapsto (\eta', \psi') \end{aligned}$$

is not uniformly continuous.
Considering a weaker control norm we get: For all $\epsilon > 0$, the flow map:

$$\begin{aligned} B((\eta_0, \psi_0), r) &\to C([0,T], H^{s+\epsilon}(\mathbb{R}^d) \times H^{s-\frac{1}{2}+\epsilon}(\mathbb{R}^d)) \\ (\eta_0', \psi_0') &\mapsto (\eta', \psi') \end{aligned}$$

is not C^1.

Now we turn to gravity water waves, i.e $\kappa = 0$ where the well posedness of the Cauchy problem is proved in [7]. It is well known that the vertical and horizontal traces of the velocity on the free boundary play an important role in the well posedness of the Cauchy problem and are given by:

$$\begin{aligned} B &= (\partial_y \phi)_{|y=\eta} = \frac{\nabla \eta \cdot \nabla \psi + G(\eta)\psi}{1 + |\nabla \eta|^2}, \\ V &= (\nabla_x \phi)_{|y=\eta} = \nabla \psi - B \nabla \eta. \end{aligned} \tag{1.7}$$

Theorem 1.3. *Fix the dimension $d \geq 1$ and consider two real numbers $r > 0$, $s \in]1+\frac{d}{2}, +\infty[$ and $(\eta_0, \psi_0) \in H^{s+\frac{1}{2}}(\mathbb{R}^d) \times H^{s+\frac{1}{2}}(\mathbb{R}^d)$ and consider*

$$(\eta_0', \psi_0') \in B((\eta_0, \psi_0), r) \subset H^{s+\frac{1}{2}}(\mathbb{R}^d) \times H^{s+\frac{1}{2}}(\mathbb{R}^d)$$

such that we have:

(1) $V_0' \in H^s(\mathbb{R}^d)$, $B_0' \in H^s(\mathbb{R}^d)$,
(2) $H_{t=0}$ is satisfied,
(3) there exits a positive constant c such that, $\forall x \in \mathbb{R}^d, a_0'(x) \geq c > 0$.

Then there exists $T > 0$ such that the Cauchy problem (1.5) with initial data (η'_0, ψ'_0) has a unique solution

$$(\eta', \psi') \in C^0([0,T]; H^{s+\frac{1}{2}}(\mathbb{R}^d) \times H^{s+\frac{1}{2}}(\mathbb{R}^d))$$

such that for $t \in [0,T]$ the assumption H_t is satisfied, $\forall x \in \mathbb{R}^d, a'(t,x) \geq \frac{c}{2}$ and

$$(V', B') \in C^0([0,T]; H^s(\mathbb{R}^d) \times H^s(\mathbb{R}^d)).$$

Moreover the flow map:

$$\begin{aligned} B((\eta_0, \psi_0), r) \cap H^{s+\frac{1}{2}}(\mathbb{R}^d) \times H^{s+\frac{1}{2}}(\mathbb{R}^d) &\to C([0,T], H^{s+\frac{1}{2}}(\mathbb{R}^d) \times H^{s+\frac{1}{2}}(\mathbb{R}^d)) \\ (\eta'_0, \psi'_0) &\mapsto (\eta', \psi') \end{aligned}$$

is not uniformly continuous.
Considering a weaker control norm we get: For all $\epsilon > 0$, the flow map:

$$\begin{aligned} B((\eta_0, \psi_0), r) \cap H^{s+\frac{1}{2}}(\mathbb{R}^d) \times H^{s+\frac{1}{2}}(\mathbb{R}^d) &\to C([0,T], H^{s+\epsilon}(\mathbb{R}^d) \times H^{s+\epsilon}(\mathbb{R}^d)) \\ (\eta'_0, \psi'_0) &\mapsto (\eta', \psi') \end{aligned}$$

is not C^1.

Remark 1.1. *It is worth noticing that a previous result was obtained on the regularity of the flow for the two dimensional gravity-capillary water waves (i.e with surface tension) in [2] where they have proved that the flow is not C^3 with respect to initial data $(\eta_0, \psi_0) \in H^{s+\frac{1}{2}}(\mathbb{R}^d) \times H^s(\mathbb{R}^d)$ for $s < 3$.*

This result is in contrast with our result which holds for $s > 3$ and this can indeed be seen in the fact that in [2] the lack of regularity of the flow is shown to be primarily due to the influence of surface tension. Though in our work the lack of regularity of the flow is shown to be due to the hydrodynamic term (the non-linear transport term).

1.5. **Strategy of the proof.** We explain the key ideas at the level of the equation (1.1),

$$\partial_t v + v\partial_x v + H \langle \mathrm{D} \rangle^\alpha v = 0.$$

The point of start is to adapt the classic proof of the quasi-linearity of the Burgers equation, presented to me in a personal note of C. Zuily [1], that we will recall here.

1.5.1. *Quasi-linearity of the Burgers equation.* The result of quasi-linearity of the Burgers equation is that the flow map taken point-wise in time fails to be uniformly continuous. Such a result is obtained by constructing two families of solutions u and v from some initial data $u^0$and v^0 depending on parameters λ and ϵ such that

$$\lim_{\substack{\lambda \to +\infty \\ \epsilon \to 0}} \left\| u^0 - v^0 \right\|_{H^s} = 0 \text{ and } \left\| (u-v)(t,\cdot) \right\|_{H^s} \geq c > 0, \text{ with } t > 0.$$

To show how to construct such families we start by recalling the usual geometric construction of the graph of a function $u(t,\cdot)$ solution to the Burgers equation with initial data u^0. Put

$$\chi(t,x) = x + tu^0(x)$$

the characteristic flow associated to the problem, which is a diffeomorphism in the x variable. Then,

$$u(t,\cdot) = u^0 \circ \chi(t,x)^{-1}.$$

The action of χ^{-1} on the graph of u^0 is given by the following Figure (1) that also shows the shock formation phenomena.
Then u^0 and v^0 are chosen as a high frequency compactly supported ansatz depending on (λ, ϵ):

$$u^0(x) = \lambda^{\frac{1}{2}-s}\omega(\lambda x), \quad v^0(x) = u^0(x) + \epsilon\omega(x), \quad \text{with } \omega \in C_0^\infty,$$

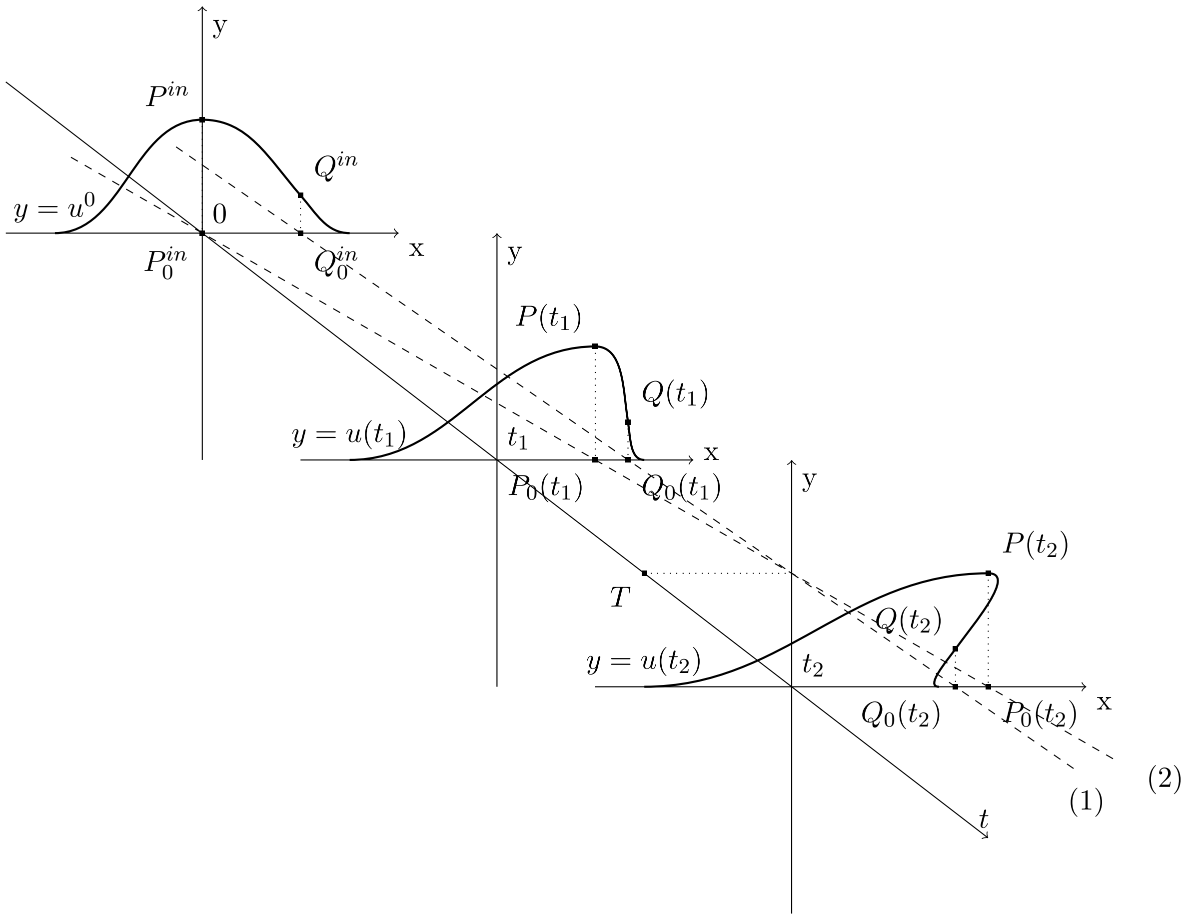

FIGURE 1. The lines (1) and (2) are the characteristic curves from Q_0^{in} and P_0^{in}. T is the time of formation of the shock wave.

where ϵ represents a change in the initial speed of transport, and (ϵ, λ) verify:

$$\epsilon \to 0,\ \lambda \to +\infty,\ \epsilon\lambda \to +\infty.$$

Now if we put χ and $\tilde{\chi}$ to be the characteristic flow maps associated to the solutions

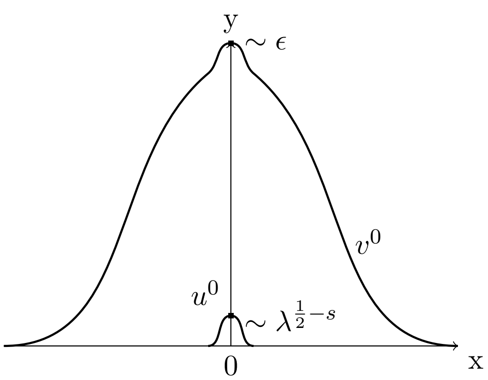

FIGURE 2. Graph of the ansatz.

u^0 and v^0 then:

$$\begin{aligned}(u-v)(t,x) &= u^0(\chi(t,x)^{-1}) - v^0(\tilde{\chi}(t,x)^{-1}) \\ &= u^0(\chi(t,x)^{-1}) - u^0(\tilde{\chi}(t,x)^{-1}) + O_{H^s}(\epsilon).\end{aligned}$$

Then using the compactly supported property of u^0 and the change of speed we prove that $u^0(\chi(t,x)^{-1})$ and $u^0(\tilde{\chi}(t,x)^{-1})$ have disjoint supports which is illustrated by Figure (3). We then prove that $\left\|u^0(\chi(t,x)^{-1})\right\|_{H^s} \geq c > 0$ which finishes the

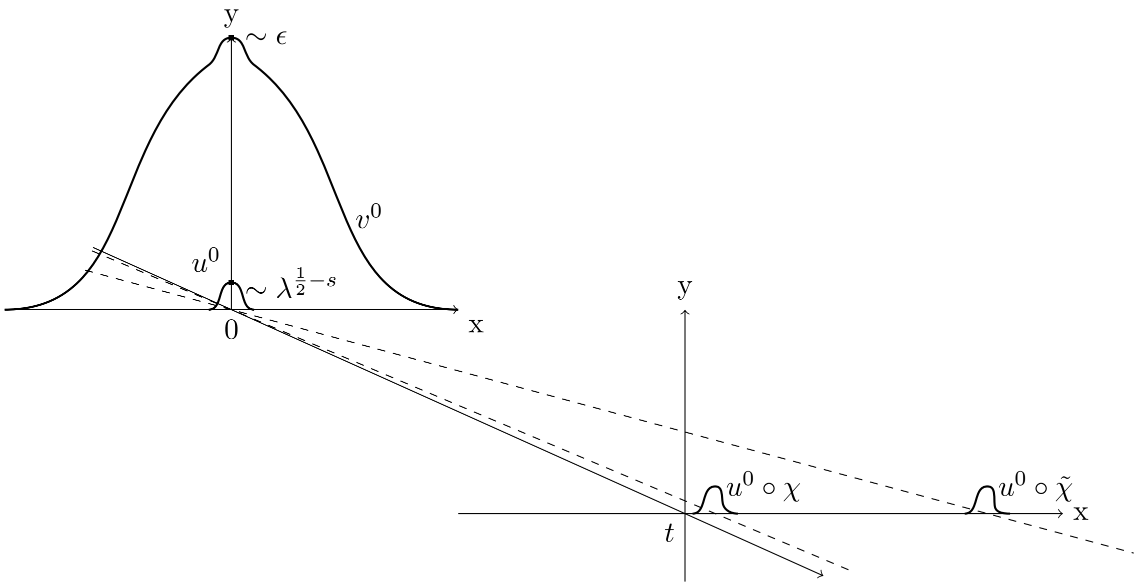

FIGURE 3. Transport of the ansatz.

proof of the non uniform continuity of the flow. For the control in a weaker norm, that is the flow cannot be $C^1(H^s(\mathbb{D}), C^0([0,T], H^{s-1+\epsilon}(\mathbb{D})))$, we get it from the estimate $\left\|u^0(\chi(t,x)^{-1})\right\|_{H^{s-\mu}} \geq c\lambda^{-\mu}$.

1.5.2. *Quasi-linearity of problem* (2.1). Now if we adapt the proof to our current problem (1.1) we get:

$$\begin{aligned}(u-v)(t,x) &= f(t,\chi(t,x)^{-1}) - g(t,\tilde{\chi}(t,x)^{-1}) \\ &= f(t,\chi(t,x)^{-1}) - f(t,\tilde{\chi}(t,x)^{-1}) + O_{H^s}(C(\tau\lambda^{\alpha-1})\epsilon),\end{aligned}$$

where f and g are solutions to

$$\partial_t f + (H\langle \mathrm{D}\rangle^\alpha)^* f = 0 \tag{1.8}$$

$$\partial_t g + \widetilde{(H\langle \mathrm{D}\rangle^\alpha)}^* g = 0 \tag{1.9}$$

and $(\cdot)^*$ and $\widetilde{(\cdot)}^*$ are the change of variables by the characteristic flows defined for a symbol a by

$$\mathrm{Op}(a^*)(u\circ\chi) = (\mathrm{Op}(a)u)\circ\chi \;\text{ i.e }\; \mathrm{Op}(a^*)(u) = (\mathrm{Op}(a)[u\circ\chi^{-1}])\circ\chi,$$

and analogously for $\widetilde{(\cdot)}^*$, which we prove that they are well posed in Appendix B.

The first immediate problem we face is the extra term $t\lambda^{\alpha-1}$ which diverges, to remedy this we give up control of the flow punctually in time and use a conveniently chosen sequence of small time (τ) to control $\tau\lambda^{\alpha-1}$:

$$\tau \to 0, \;\; \lambda\epsilon\tau \to +\infty.$$

The second, deeper problem we face is that we lose control over the support of the solution. Indeed (1.8) and (1.9) are obtained by pull back of the linear equation

$$\partial w + H\langle \mathrm{D}\rangle^\alpha w = 0 \tag{1.10}$$

which is a non local-dispersive equation that is expected to disperse the support of the solution and the L^∞ norm. This phenomena is thus expected to oppose the phenomena illustrated by the previous Figures (1) and (2) and indeed does so for the Benjamin-Ono and KdV equations.

To remedy this, the idea is not to use u^0 and v^0 as initial data but by profiting of the time reversibility [2] of the equations use the backward in time solutions u^1 and v^1 defined by:

$$\begin{cases} \omega \ \text{ solution of } (1.10), \\ \omega(\tau,\cdot) = u^0, \\ \omega(0,\cdot) = u^1, \end{cases} \qquad \begin{cases} \omega' \ \text{ solution of } (1.10), \\ \omega'(\tau,\cdot) = v^0, \\ \omega'(0,\cdot) = v^1. \end{cases}$$

This gives us:

$$(u-v)(\tau,x) = u^0(\chi(0,\tau,x)) - u^0(\tilde{\chi}(0,\tau,x)) + O_{H^s}(C(\tau\lambda^{(\alpha-1)^+})\epsilon + \tau\lambda^{(\alpha-1)^+}).$$

We then prove that this gives the desired result, in the threshold $\alpha \in [0,2[$, by proving analogously to the Burgers equation: $\left\|u^0(\chi^{-1}(t,x))\right\|_{H^s} \geq c > 0$ and then using the compactly supported property of u^0 and the change of speed we prove that $u^0(\chi(0,\tau,x))$ and $u^0(\tilde{\chi}(0,\tau,x))$ have disjoint supports.

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2. Proof of Theorem 1.1

2.0.1. *Prerequisites on the Cauchy Problems.* For a real number $\alpha \in [0,2[$, we consider the Cauchy problem:

$$\begin{cases} \partial_t u + u\partial_x u + H\langle \mathrm{D}\rangle^\alpha u = 0 \\ u(0,\cdot) = u_0(\cdot) \in H^s(\mathbb{D}), \ s > \frac{3}{2}, \end{cases} \tag{2.1}$$

It is well known that the problem is well posed in Sobolev spaces, this can be summarized in the following Theorem:

Theorem 2.1. *Consider two real numbers, $s \in]\frac{3}{2}, +\infty[$ and $r > 0$. Fix $u_0 \in H^s(\mathbb{D})$. Then there exist $T > 0$, such that $\forall v_0 \in B(u_0, r) \subset H^s(\mathbb{D})$, the problem (2.1) with initial data v_0 has a unique solution $v \in C^0([0,T], H^s(\mathbb{D}))$, the map $v_0 \mapsto v$ is continuous from $B(u_0,r)$ to $C^0([0,T], H^s(\mathbb{D}))$ and maps real functions into real functions. Moreover we have the estimates:*

$$\forall 0 \leq \mu \leq s, \|v(t)\|_{H^\mu(\mathbb{D})} \leq C_\mu \|v_0\|_{H^\mu(\mathbb{D})}. \tag{2.2}$$

Taking Two different solution u, v, such that $u_0 \in H^{s+1}(\mathbb{D})$:

$$\forall 0 \leq \mu \leq s, \|(u-v)(t)\|_{H^\mu(\mathbb{D})} \leq \|u_0 - v_0\|_{H^\mu(\mathbb{D})} e^{C_\mu \int_0^t \|u(s)\|_{H^{\mu+1}(\mathbb{D})}} ds. \tag{2.3}$$

We will also need to remark that fixing the initial data at 0 is an arbitrary choice, that is all of the previous conclusions hold for the Cauchy problem defined for $t_0 \leq T$:

$$\begin{cases} \partial_t v + v\partial_x v + H\langle \mathrm{D}\rangle^\alpha v = 0 \\ v(t_0,\cdot) = v_0(\cdot) \in H^s(\mathbb{D}), \ s > \frac{3}{2}. \end{cases} \tag{2.4}$$

[2]This idea fundamentally depends on the local reversibility in time of the linearised equations and thus fails for the fractional Burgers equation.

Remark 2.1. *Note that the previous Theorem holds for the Cauchy problem associated to the Burgers equation:*

$$\begin{cases} \partial_t u + u\partial_x u = 0 \\ u(0,\cdot) = u_0(\cdot) \in H^s(\mathbb{D}), \ \ s > \frac{3}{2}, \end{cases} \tag{2.5}$$

Though we have some extra estimates in Hölder type spaces:

$$\forall 0 \leq k < s - \frac{1}{2}, \|u(t)\|_{W^{k,\infty}(\mathbb{D})} \leq C_k \|u_0\|_{W^{k,\infty}(\mathbb{D})}, \tag{2.6}$$

Taking Two different solution u, v, such that $u_0 \in H^{s+1}(\mathbb{D})$:

$$\forall 0 \leq k < s - \frac{1}{2}, \|(u-v)(t)\|_{W^{k,\infty}(\mathbb{D})} \leq \|u_0 - v_0\|_{W^{k,\infty}(\mathbb{D})} e^{C_k \int_0^t \|u(s)\|_{W^{k+1,\infty}(\mathbb{D})} ds}.$$

Remark 2.2. *The evolution PDE (2.1), does not have a scaling because of the inhomogeneous term $H \langle \mathrm{D} \rangle^\alpha$. But for the purpose of our study of the local Cauchy problem, the small frequency part doesn't play an important role. So in order to have a better idea on the main terms that locally drive the evolution we can heuristically replace it with $H|D|^\alpha$ in order to compute the scaling. By doing so we get that the change of scale,*

$$\text{for} \ \ \lambda > 0, u_0 \mapsto \lambda^{\alpha-1} u_0(\lambda x)$$

gives the solution

$$\lambda^{\alpha-1} u_0(\lambda x) \mapsto \lambda^{\alpha-1} u(\lambda^\alpha t, \lambda x).$$

Thus giving the scaling in Sobolev spaces: $s_c = 1 + \frac{1}{2} - \alpha$, thus we prove quasi-linearity in the subcritical regime of the problem.

Notation 2.1. *In order not to be confused with the pull back symbol, henceforth the conjugate of a symbol a will be written as $a^\top$.*

As the linearised equation is a hyperbolic pseudo-differential equation we recall the result on the Cauchy problem associated to this type of equations:

Theorem 2.2. *Consider $(a_t)_{t\in\mathbb{R}}$ a family of symbols in $S^\beta(\mathbb{D}^d)$,$\beta \in \mathbb{R}$, such that $t \mapsto a_t$ is continuous and bounded from $\mathbb{R}$ to $S^\beta(\mathbb{D}^d)$ and such that $\mathrm{Re}(a_t) = \frac{a_t + a_t^\top}{2}$ is bounded in $S^0(\mathbb{D}^d)$, and take $T > 0$. Then for all $s \in \mathbb{R}$, $u_0 \in H^s(\mathbb{D}^d)$ and $f \in C^0([0,T]; H^s(\mathbb{D}^d))$ the Cauchy problem:*

$$\begin{cases} \partial_t u + \mathrm{Op}(a)u = f \\ \forall x \in \mathbb{D}^d, u(0,x) = u_0(x) \end{cases} \tag{2.7}$$

has a unique solution $u \in C^0([0,T]; H^s(\mathbb{D}^d)) \cap C^1([0,T]; H^{s-\beta}(\mathbb{D}^d))$ which verifies the estimates:

$$\|u(t)\|_{H^s(\mathbb{D}^d)} \leq e^{Ct} \|u_0\|_{H^s(\mathbb{D}^d)} + 2\int_0^t e^{C(t-t')} \|f(t')\|_{H^s(\mathbb{D}^d)} dt',$$

where C depends on a finite symbol semi-norm of $\mathrm{Re}(a_t)$. We will also need to remark that fixing the initial data at 0 is an arbitrary choice. More precisely, $\forall 0 \leq t_0 \leq T$ and all data $u_0 \in H^s(\mathbb{D}^d)$ the Cauchy problem:

$$\begin{cases} \partial_t u + \mathrm{Op}(a)u = f \\ \forall x \in \mathbb{D}^d, u(t_0,x) = u_0(x) \end{cases} \tag{2.8}$$

has a unique solution $u \in C^0([0,T]; H^s(\mathbb{D}^d)) \cap C^1([0,T]; H^{s-\beta}(\mathbb{D}^d))$ which verifies the estimate:

$$\|u(t)\|_{H^s(\mathbb{D}^d)} \leq e^{C|t-t_0|} \|u_0\|_{H^s(\mathbb{D}^d)} + 2\left|\int_{t_0}^t e^{C(t-t')} \|f(t')\|_{H^s(\mathbb{D}^d)} dt'\right|.$$

2.1. **Proof of Theorem 1.1.** Here we will give the proof when $u_0 = 0$, the general case just adds harmless terms and is treated in the generalization in the next section.

2.1.1. *Definition of the Ansatz.*

- For $\mathbb{D} = \mathbb{R}$, take $\omega \in C_0^\infty(\mathbb{R}), \omega(x) = 1 \text{ if } |x| \leq \frac{1}{2}, \omega(x) = 0 \text{ if } |x| \geq 1$.
- For $\mathbb{D} = \mathbb{T}$, we see functions on $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ as 2π periodic function on $\mathbb{R}$ and we take $\omega \in C_0^\infty(\mathbb{T})$ as the periodic extension of the function defined above.

Let (λ, ϵ) be two positive real sequences such that

$$\lambda \to +\infty, \quad \epsilon \to 0, \quad \lambda\epsilon \to +\infty. \tag{2.9}$$

Put

- for $\mathbb{D} = \mathbb{R}$,
$$u^0(x) = \lambda^{\frac{1}{2}-s}\omega(\lambda x), \quad v^0(x) = u^0(x) + \epsilon\omega(x),$$
- for $\mathbb{D} = \mathbb{T}$, u^0 and v^0 as the periodic extensions of the functions defined above.

Take $t_0 > 0$ smaller than a harmless constant which will be fixed later, and $(\tau), 0 < \tau \leq t_0$ and $\tau \to 0$.
Now let l, l' be the solutions to the Cauchy problem on $[0, t_0]$:

$$\begin{cases} \partial l + H \langle \mathrm{D} \rangle^\alpha l = 0, \\ l(\tau, \cdot) = u^0, \end{cases} \quad \begin{cases} \partial l' + H \langle \mathrm{D} \rangle^\alpha l' = 0, \\ l'(\tau, \cdot) = v^0. \end{cases}$$

Put $u^1(x) = l(0, x)$ and define analogously $v^1(x) = l'(0, x)$.

Define u and v as the solution given by Theorem 2.1 with initial data u^1 and v^1 on the intervals $[0, T]$ and $[0, T']$. Taking $0 < \delta < s - \frac{3}{2}$, u^0 and v^0 are uniformly bounded in $H^{\frac{3}{2}+\delta}(\mathbb{D})$ when $\lambda \to +\infty$ and thus by Theorem 2.2, u^1 and v^1 are also uniformly bounded in $H^{\frac{3}{2}+\delta}(\mathbb{D})$ and thus by the Sobolev injection Theorems they are bounded in $\dot{W}^{1,\infty}(\mathbb{D})$. Thus we can take a uniform $0 < T$ on which all the solutions are well defined and we take $0 < t_0 \leq T$ [3].

2.1.2. *Change of variables by transport.* Put

$$\begin{cases} \frac{d}{dt}\chi(t, s, x) = u(t, \chi(t, s, x)) , \\ \chi(s, s, x) = x, \end{cases}$$

and define analogously $\tilde{\chi}$ from v, κ from w and κ' from w'.

We recall that from the Cauchy-Lipschitz Theorem we have $\chi, \tilde{\chi} \in C^1([0, T]^2, W^{s-\frac{1}{2}-\delta,\infty}(\mathbb{D}))$ [4] and they are both diffeomorphisms in the x variable.

By the estimate (2.2) u and v are uniformly bounded in $\dot{W}^{1,\infty}(\mathbb{D})$ because their Sobolev norms are dominated by those of u^1 and v^1 thus by those of u^0 and v^0 by Theorem 3.3. By classic manipulations of ODEs we get the estimates:

$$\begin{cases} \exists C > 0, \forall t', t \leq t_0, \forall x, C^{-1} \leq |\partial_x \chi(t, t', x)| \leq C \\ \forall 2 \leq k \leq \lfloor s - \frac{1}{2} - \delta \rfloor, \left\| \partial_x^k \chi(t, t', x) \right\|_{L^\infty} \leq C \left\| u \right\|_{W^{k,\infty}} . \end{cases} \tag{2.10}$$

[3] Heuristically, if the existence time of the solution ω is $[0, T]$ then the existence time of u is $\sim T\lambda^{s-\frac{3}{2}}$ which tends to infinity with λ, thus we are "dilating" the time scale of the problem with initial data ω and "zooming" for short time and in the $\dot{H}^s(\mathbb{D})$ norm. In this part of the evolution, we prove that the Burgers transport term is more important and gives this quasi-linear character to the PDE.

[4] We actually have better regularities, as u^0 and v^0 are $H^{+\infty}(\mathbb{D})$ functions, then u^1, v^1 are $H^{+\infty}(\mathbb{D})$ and u and v are $H^{+\infty}(\mathbb{D})$ with respect to the x variable thus $\chi, \tilde{\chi} \in C^1([0, T]^2, C^\infty)$.

Analogous estimates hold for $\tilde{\chi}$ using v.
The classic transport computation reads:

$$\begin{cases} \partial_t(u(t,\chi(t,0,x))) = \partial_t(u)(t,\chi(t,0,x)) + \partial_t(\chi(t,0,x))\partial_x(u)(t,\chi(t,0,x)) \\ \qquad\qquad\qquad\qquad\quad = -(H\langle \mathrm{D}\rangle^\alpha u)(t,\chi(t,0,x)) \\ \qquad\qquad\qquad\qquad\quad = -(H\langle \mathrm{D}\rangle^\alpha)^* u(t,\chi(t,0,x)), \\ u(0,\chi(0,0,x)) = u(0,x) = u^1(x). \end{cases}$$

where $(\cdot)^*$ is the change of variables by $\chi(t,0,x)$ as presented in Theorem A.3. Thus if we put f the solution to the following Cauchy problem, which is well posed by Appendix B:

$$\begin{cases} \partial_t w + (H\langle \mathrm{D}\rangle^\alpha)^* w = 0 \\ \forall x\in\mathbb{D}, w(0,x) = u^1(x) \end{cases} \tag{2.11}$$

we get:

$$u(t,x) = f(t,\chi(0,t,x)) \ \Leftrightarrow\ u(t,\chi(t,0,x)) = f(t,x). \tag{2.12}$$

Analogously, if we put g the solution to the well posed Cauchy problem,

$$\begin{cases} \partial_t w + \widetilde{(H\langle \mathrm{D}\rangle^\alpha)}^* w = 0 \\ \forall x\in\mathbb{D}, w(0,x) = v^1(x) \end{cases} \tag{2.13}$$

where $\widetilde{(\cdot)}^*$ is the change of variables by $\tilde{\chi}(t,0,x)$, we get

$$v(t,x) = g(t,\tilde{\chi}(0,t,x)) \ \Leftrightarrow\ v(t,\tilde{\chi}(t,0,x)) = g(t,x). \tag{2.14}$$

Returning to the ODEs defining χ and $\tilde{\chi}$ we get:

$$\begin{cases} \chi(t,t',x) = x + \int_{t'}^t f(s,x)ds, \\ \tilde{\chi}(t,t',x) = x + \int_{t'}^t g(s,x)ds, \end{cases} \tag{2.15}$$

Proposition 2.1. *Their exists $C>0$ independent of (τ,ϵ,λ) such that:*
$\forall h\in H^s(\mathbb{D}), \forall (t,t')\le t_0,$

$$C^{-1}\|h\|_{H^s} \le \|h\circ\chi(t,t',x)\|_{H^s} \le C\|h\|_{H^s},$$

$$C^{-1}\|h\|_{H^s} \le \|h\circ\tilde{\chi}(t,t',x)\|_{H^s} \le C\|h\|_{H^s}.$$

Proof. We will start by proving the upper bound for the estimate on the composition with χ. As u is bounded in (τ,ϵ,λ) on $C([0,T],H^s(\mathbb{D}))$ then there exist unique solution $H\in C([0,T],H^s(\mathbb{D}))$ to

$$\begin{cases} \partial_t H + u\partial_x H = 0 \\ H(t,x) = h(x) \end{cases}$$

and H is bounded in (τ,ϵ,λ) on $C([0,T],H^s(\mathbb{D}))$. The desired bounds come from the fact that we have the explicit formula for H:

$$H(t',x) = h\circ\chi(t,t',x).$$

Now to get the lower bound it suffices to write by the upper bound computations:

$$\begin{aligned} \|h\|_{H^s} &= \|h\circ\chi(t,t',x)\circ\chi(t',t,x)\|_{H^s} \\ &\le C\|h\circ\chi(t,t',x)\|_{H^s}. \end{aligned}$$

We get analogously the estimates on the composition with $\tilde{\chi}$. ∎

2.1.3. *Key Lemma and proof of the Theorem.*

Lemma 2.1. *Take $\epsilon' > 0$ sufficiently small, as $0 \leq \alpha < 2$ and $0 < \epsilon' << 1$ we can find* [5] *a sequence $(\tau, \epsilon, \lambda)$ such that:*

$$\begin{cases} \tau \to 0, \\ \epsilon \to 0, \\ \lambda \to +\infty, \end{cases} \qquad \begin{cases} \tau\lambda^{(\alpha-1)^+} \to 0, \\ \epsilon^{-1}\lambda^{-1+(\alpha-1)^+ +\epsilon'} \to +\infty, \\ \lambda\epsilon\tau \to +\infty. \end{cases} \tag{2.16}$$

Then there exists $c > 0$ such that:

(1) *For $\nu \geq 0$ and $\forall(\tau, \epsilon, \lambda), \left\|u^0 \circ \chi(0, \tau, x) - u^0 \circ \tilde{\chi}(0, \tau, x)\right\|_{H^s} > c\lambda^{-\nu}$.*
(2) *For $\nu \geq 0$*

$$u(\tau, x) - v(\tau, x) = u^0 \circ \chi(0, \tau, x) - u^0 \circ \tilde{\chi}(0, \tau, x) + O_{H^{s-\nu}}(C(\tau\lambda^{(\alpha-1)^+})\epsilon + \tau\lambda^{(\alpha-1)^+}\lambda^{-\nu}).$$

We will now show that this Lemma implies the Theorem. We have $\tau \leq t_0$ is such that $\tau\epsilon\lambda \to +\infty$ and $\tau\lambda^{(\alpha-1)^+} \to 0$ which gives:

$$\forall(\tau, \epsilon, \lambda), \|u(\tau, x) - v(\tau, x)\|_{H^s} > \frac{c}{2} > 0 \ \text{ thus } \ \sup_{\tau,\epsilon,\lambda} \|u(\tau, x) - v(\tau, x)\|_{H^s} > \frac{c}{2} > 0.$$

Also by Theorem 2.2:

$$\exists C > 0, \left\|u^1(x) - v^1(x)\right\|_{H^s} \leq C\epsilon \ \text{ thus } \ \left\|u^1(x) - v^1(x)\right\|_{H^s} \to 0,$$

which gives the non uniform continuity in the desired norms.
Now for the control in a weaker norm we write:

$$\frac{\|u(\tau, x) - v(\tau, x)\|_{H^{s-1+(\alpha-1)^+ +\epsilon'}}}{\|u^1(x) - v^1(x)\|_{H^s}} \geq c\epsilon^{-1}\lambda^{-1+(\alpha-1)^+ +\epsilon'} \to +\infty,$$

which gives the desired result.

2.1.4. *Proof of point 1 of Lemma 2.1.* We first prove that $\exists c > 0$ such that $\left\|u^0 \circ \chi(0, \tau, x)\right\|_{H^{s-\nu}} > c\lambda^{-\nu}$, indeed by Proposition 2.1 and change of variable:

$$\left\|u^0 \circ \chi(0, \tau, x)\right\|_{H^{s-\nu}} \geq C^{-1}\left\|u^0\right\|_{H^{s-\nu}} \geq C^{-1}\lambda^{-\nu}\|\omega\|_{H^{s-\nu}}. \tag{2.17}$$

Now we will show that $u^0 \circ \chi(0, \tau, x)$ and $u^0 \circ \tilde{\chi}(0, \tau, x)$ have disjoint supports which will suffice to conclude given (2.17). Put $y = \chi(0, \tau, x)$, thus $x = \chi(\tau, 0, y)$. On the support of $u^0 \circ \chi(0, \tau, x)$ we have:

- If $\mathbb{D} = \mathbb{R}$, $\lambda|y| \leq 1$.
- If $\mathbb{D} = \mathbb{T}$, $\forall k \in \mathbb{N}, 2\pi k - 1 \leq \lambda|y| \leq 2\pi k + 1$.

We then compute by the Taylor formula:

$$\begin{aligned} \tilde{\chi}(0, \tau, x) &= \tilde{\chi}(0, \tau, \tilde{\chi}(\tau, 0, y)) \\ &\quad + (\chi(\tau, 0, y) - \tilde{\chi}(\tau, 0, y)) \int_0^1 \partial_y\tilde{\chi}(0, \tau, r\chi(\tau, 0, y) + (1-r)\tilde{\chi}(\tau, 0, y))dr \\ &= y + (\chi(\tau, 0, y) - \tilde{\chi}(\tau, 0, y)) \int_0^1 \partial_y\tilde{\chi}(0, \tau, r\chi(\tau, 0, y) + (1-r)\tilde{\chi}(\tau, 0, y))dr. \end{aligned} \tag{2.18}$$

[5] Indeed, take $0 < \delta << 1$ and put:

$$\begin{cases} \tilde{\alpha} = (\alpha - 1)^+, \quad \epsilon = n^{-(1-\tilde{\alpha}-\epsilon')\frac{1-\delta}{2-\epsilon'} - \delta}, \\ \tau = n^{-\tilde{\alpha}\frac{1-\delta}{2-\epsilon'} - \delta}, \quad \lambda = 2\pi n^{\frac{1-\delta}{2-\epsilon'}} + 2(\tau\epsilon)^{-1}. \end{cases}$$

First,

$$\begin{aligned}\partial_y\tilde{\chi}(0,\tau,r\chi(\tau,0,y)+(1-r)\tilde{\chi}(\tau,0,y))\\ =1+\int_0^\tau \partial_y[g(t,r\chi(\tau,0,y)+(1-r)\tilde{\chi}(\tau,0,y))]dt.\end{aligned}$$

Thus by estimates of Theorem B.1, taking $0<\delta<s-\frac{3}{2}$:

$$\begin{aligned}\partial_y\tilde{\chi}(0,\tau,r\chi(\tau,0,y)+(1-r)\tilde{\chi}(\tau,0,y)) &= 1+O_{L^\infty}(\tau[1+\left\|v^1\right\|_{H^{\frac{3}{2}+\delta}}+\left\|u^1\right\|_{H^{\frac{3}{2}+\delta}}])\\ &=1+O_{L^\infty}(\tau),\end{aligned}$$

Which gives

$$\int_0^1 \partial_y\tilde{\chi}(0,\tau,r\chi(\tau,0,y)+(1-r)\tilde{\chi}(\tau,0,y))dr=1+O_{L^\infty}(\tau). \tag{2.19}$$

Now we estimate $\chi(\tau,0,y)-\tilde{\chi}(\tau,0,y)$, by (2.15) :

$$\chi(\tau,0,y)-\tilde{\chi}(\tau,0,y)=\int_0^\tau f(t,y)-g(t,y)dt. \tag{2.20}$$

Taking $0<\delta<s-\frac{3}{2}$, by estimates of Theorem B.1:

$$\begin{aligned}f(t,y)=f(0,y)+\int_0^t \partial_t f(r,y)dr=u^1(y)+tO_{L^\infty}(\left\|u^1\right\|_{H^{\frac{1}{2}+\alpha+\delta}})\\ =u^1(y)+O_{L^\infty}(t).\end{aligned}$$

Analogously we get:

$$g(t,y)=v^1(y)+O_{L^\infty}(t).$$

Consider μ the solution to the Cauchy problem:

$$\begin{cases}\partial_t w+H\left\langle \mathrm{D}\right\rangle^\alpha w=0\\ \forall y\in\mathbb{D}, w(\tau,y)=\omega(y).\end{cases} \tag{2.21}$$

By definition:

$$\begin{aligned}u^1(y)-v^1(y)=-\epsilon\mu(0,y)=-\epsilon\omega(y)+\epsilon\int_\tau^0 \partial_t\mu(t,y)dt\\ =-\epsilon\omega(y)+O_{L^\infty}(\epsilon\tau).\end{aligned}$$

Thus,

$$\chi(\tau,0,y)-\tilde{\chi}(\tau,0,y)=-\epsilon\tau\omega(y)+O_{L^\infty}(\tau^2),$$

and finally we get in (2.18),

$$\tilde{\chi}(\tau,0,x)-y=-\epsilon\tau\omega(y)+O_{L^\infty}(\tau^2).$$

We get for $x\in$ supp $u^0\circ\chi(0,\tau,\cdot)$:

- For $\mathbb{D}=\mathbb{R}$:

$$\lambda\left|\tilde{\chi}(0,\tau,x)\right|\geq\tau\epsilon\lambda-1+o_{L^\infty}(\tau\epsilon\lambda)\geq 2,$$

  by hypothesis $\tau\epsilon\lambda\to+\infty$, which gives the desired result.
- For $\mathbb{D}=\mathbb{T}$ given an adequate choice of τ,ϵ and λ:

$$2n\pi+1\leq\lambda\left|\tilde{\chi}(0,\tau,x)\right|\leq 2(n+1)\pi-1,$$

  Which again gives the desired result.

2.1.5. *Proof of point 2 of Lemma 2.1.* We start by writing:

$$\begin{aligned} u(t,x) - v(t,x) &= f(t,\chi(0,t,x)) - g(t,\tilde\chi(0,t,x)) \\ &= \underbrace{f(t,\chi(0,t,x)) - f(t,\tilde\chi(0,t,x))}_{(1)} + (f-g)(t,\tilde\chi(0,t,x)). \end{aligned}$$

Term (1) resembles the main term in the usual transport estimates we used in point 1 of the Lemma [6] but with a main difference of f being some dispersed data and not compactly supported. The main trick here was to construct from u^0, v^0 the defocused data in the past u^1, v^1 and use this as the initial data for f and g.

$$\begin{aligned} u(\tau,x) - v(\tau,x) &= u^0 \circ \chi(0,\tau,x) - u^0 \circ \tilde\chi(0,\tau,x) \\ &+ (f-u^0)(\tau,\chi(0,\tau,x)) - (f-u^0)(\tau,\tilde\chi(0,t,x)) + (f-g)(\tau,\tilde\chi(0,\tau,x)). \end{aligned}$$

The idea is then to see that by definition of l: $l(\tau,x) = u^0(x)$ and we get:

$$\begin{aligned} u(\tau,x) - v(\tau,x) &= u^0 \circ \chi(0,\tau,x) - u^0 \circ \tilde\chi(0,\tau,x) \\ &+ \underbrace{(f-l)(\tau,\chi(0,\tau,x)) - (f-l)(\tau,\tilde\chi(0,t,x))}_{(1)} + \underbrace{(f-g)(\tau,\tilde\chi(0,\tau,x))}_{2}. \end{aligned}$$

We start by estimating (1), by Proposition 2.1:

$$\|(f-l)(\tau,\chi(0,\tau,x))\|_{H^s} \leq C\,\|(f-l)(\tau,\cdot)\|_{H^s}.$$

Now $f-l$ solve:

$$\begin{cases} \partial_t(f-l) + H\,\langle \mathrm{D}\rangle^{\alpha}(f-l) = H(\langle \mathrm{D}\rangle^{\alpha} - \langle \mathrm{D}\rangle^{\alpha *})f \\ \forall x \in \mathbb{D}, (f-l)(0,x) = 0. \end{cases} \tag{2.22}$$

Thus we have the estimates:

$$\begin{aligned} \|f - l(\tau,\cdot)\|_{H^\nu} &\leq C\,\|(\langle \mathrm{D}\rangle^{\alpha} - \langle \mathrm{D}\rangle^{\alpha *})f\|_{L^1([0,\tau],H^\nu)} \\ &\leq C\tau\,\|(\langle \mathrm{D}\rangle^{\alpha} - \langle \mathrm{D}\rangle^{\alpha *})f\|_{L^\infty([0,\tau],H^\nu)} \end{aligned}$$

By Theorem A.3,

$$\begin{aligned} \|f - l(\tau,\cdot)\|_{H^\nu} &\leq C\tau \left\|(Id - D\chi(0,t,\chi(t,0,x)))\alpha D\,\langle \mathrm{D}\rangle^{\alpha-2} f\right\|_{L^\infty([0,\tau],H^\nu)} \\ &\leq C\tau\,\|Id - D\chi(0,t,\chi(t,0,x))\|_{L^\infty([0,\tau],W^{\nu,\infty})} \left\|D\,\langle \mathrm{D}\rangle^{\alpha-2} f\right\|_{L^\infty([0,\tau],L^2)} \\ &+ C\tau\,\|Id - D\chi(0,t,\chi(t,0,x))\|_{L^\infty([0,\tau],L^\infty)} \left\|D\,\langle \mathrm{D}\rangle^{\alpha-2} f\right\|_{L^\infty([0,\tau],H^\nu)} \end{aligned}$$

Using Theorem B.3 and taking $0 < \delta < s - \frac{3}{2}$:

$$\|f - l(\tau,\cdot)\|_{H^\nu} \leq C\tau\lambda^{(\alpha-1)^+ - s}\lambda^{\nu+\frac{1}{2}+\delta-s} + \tau\lambda^{(\alpha-1)^+ + \nu - s}, \tag{2.23}$$

Thus,

$$\|(f-l)(\tau,\chi(0,\tau,x))\|_{H^\nu} \leq C\tau\lambda^{(\alpha-1)^+ + \nu - s}.$$

Analogously we get,

$$\|(f-l)(\tau,\tilde\chi(0,\tau,x))\|_{H^\nu} \leq C\tau\lambda^{(\alpha-1)^+ + \nu - s},$$

which gives

$$\|(1)\|_{H^\nu} \leq C\tau\lambda^{(\alpha-1)^+ + \nu - s}. \tag{2.24}$$

Now we estimate (2) in the same manner, by Proposition 2.1:

$$\|(f-g)(\tau,\tilde\chi(0,\tau,x))\|_{H^\nu} \leq \|(f-g)(\tau,\cdot)\|_{H^\nu}$$

[6] Like the ones used in proving the quasi-linearity of the Burgers equation.

$f-g$ solve:

$$\begin{cases} \partial_t(f-g) + H\langle \mathrm{D}\rangle^{\alpha *}(f-g) + (H\langle \mathrm{D}\rangle^{\alpha *} - \widetilde{H\langle \mathrm{D}\rangle^{\alpha}}^{*})g = 0 \\ \forall x \in \mathbb{D}, (f-g)(0,x) = (u^1 - v^1)(x). \end{cases} \tag{2.25}$$

By Theorem A.3,

$$\begin{aligned} G(f-g) &= (H\langle \mathrm{D}\rangle^{\alpha *} - \widetilde{H\langle \mathrm{D}\rangle^{\alpha}}^{*})g \\ &\sim \alpha(D\chi(0,t,\chi(t,0,x)) - D\tilde{\chi}(0,t,\tilde{\chi}(t,0,x)))F(f,g)g \\ &\sim \alpha(D\int_0^t f(s,\cdot) - g(s,\cdot)ds)F(f,g)g, \end{aligned}$$

With,

$$F(f,g) = \int_0^1 \partial_\xi(H\langle \mathrm{D}\rangle^{\alpha})(sD\chi(0,t,\chi(t,0,x)) - (1-s)D\tilde{\chi}(0,t,\tilde{\chi}(t,0,x)))ds. \tag{2.26}$$

Thus we have the estimates by Theorem B.2 and B.3:

$$\begin{aligned} \|f - g(\tau,\cdot)\|_{H^\nu} &\le C(\tau\lambda^{(\alpha-1)^+}) \left\|u^1 - v^1\right\|_{H^\nu} \\ &\le C(\tau\lambda^{(\alpha-1)^+})\epsilon \end{aligned}$$

which gives

$$\|(2)\|_{H^\nu} \le C(\tau\lambda^{(\alpha-1)^+})\epsilon, \tag{2.27}$$

finishing the proof of Lemma 2.1 and Theorem 1.1.

3. Theorem 3.1: A technical generalization

The techniques used in the previous proof will be generalized directly to give the following Theorem:

Theorem 3.1. *Consider five numbers $\alpha \in [0,2[$, $s \in]\max(1,\alpha) + \frac{d}{2}, +\infty[$, $T > 0$ and $(\beta,k) \in \mathbb{R}^+$ verifying:*

$$\begin{cases} k \ge 1, \quad \beta \le \alpha, \\ \beta < 2 - \frac{1}{k}. \end{cases}$$

Consider an elliptic C^1 symbol $a : [0,T] \times H^s(\mathbb{D}^d) \to \Gamma_1^\alpha(\mathbb{D}^d)$ skew symmetric[7]

$$i.e \text{ such that } \quad \mathrm{Re}(a_t) = \frac{a_t + a_t^\top}{2} \quad \text{is bounded in } \Gamma_1^0(\mathbb{D}^d),$$

and that $D_u a$ verifies:

$$\forall \mu \in \mathbb{R}, \forall g \in H^\mu(\mathbb{D}^d), \left\|T_{D_u a} g + \overline{T_{D_u a} g}\right\|_{H^\mu} \le M_0^\beta(D_u a + \overline{D_u a}) \|g\|_{H^{\mu+\beta}}. \tag{3.1}$$

Consider a regular function[8] *$V(t,x,u) : [0,T] \times \mathbb{D}^d \times \mathbb{C} \to \mathbb{R}^d$ and a function F C^1 in u and L^∞ in t, $F : [0,T] \times H^s(\mathbb{D}^d) \to H^s(\mathbb{D}^d)$.*
Suppose that the following hypothesis ***H1*** *and* ***H2*** *are verified:*

$$\begin{cases} \forall (t,u), M_1^\alpha(a) \le C(1 + \|u\|_{W^{1,\infty}}), \; M_0^\beta(D_u a + \overline{D_u a}) \le C\|u\|_{L^\infty}^{k-1}, \\ \forall (t,u), k \in \{0,1\}, \left\|D^k F(t,u)\right\|_{H^s} \le C(1 + \|u\|_{H^s}), \\ \forall (t,x,u), \|V(t,x,u)\| \le C\|u\|, \forall k \ge 1, \left\|D^k V(t,x,u)\right\| \le C. \end{cases} \tag{\textbf{H1}}$$

$$\forall (t,x,u), 0 < C^{-1} \le \|D_u V(t,x,u)\| \le C. \tag{\textbf{H2}}$$

[7] Recall the notation $a^\top$ for the adjoin of an operator a.

[8] V could be taken in a more relaxed setting, specifically the Zygmund space C_*^s but we will avoid this technicality in this current work.

Fix $u_0 \in H^s(\mathbb{D}^d)$ and take $r > 0$, then there exists $T > 0$ such that for all v_0 in the ball $B(u_0, r) \subset H^s(\mathbb{D}^d)$ the Cauchy problem:

$$\begin{cases} \partial_t v + V(t,x,v) \cdot \nabla v + T_{a(t,v)} v = F(t,v) \\ v(0,\cdot) = v_0(\cdot), \end{cases}$$

has a unique solution $v \in C([0,T], H^s(\mathbb{D}^d))$. Moreover the flow map:

$$\begin{aligned} B(u_0, r) \to & C([0,T], H^s(\mathbb{D}^d)) \\ v_0 \mapsto & v \end{aligned}$$

is not uniformly continuous. Considering a weaker control norm we get, for all $\epsilon > 0$ the flow map:

$$\begin{aligned} B(u_0, r) \to & C([0,T], H^{s-1+(\alpha-1)^+ + \epsilon}(\mathbb{D}^d)) \\ v_0 \mapsto & v \end{aligned}$$

is not C^1.

Remark 3.1. *The parameters (β, k) are here to quantify the degree of the non linearity in the dispersive term. The previous result is optimal in (α, β, k), indeed writing*

$$\partial_t u = 0 = \partial_t u + u \partial_x u - u \partial_x u,$$

on which the result clearly fails we see that the hypothesis on (β, k) are optimal. We also remark that for the paralinearised water waves system with surface tension we have

$$\alpha = \frac{3}{2}, \quad \beta = 1 \quad \text{and} \quad k = 2,$$

$\beta = 1$ is the main cancellation property needed for the application of the Theorem we prove in Lemma 4.3.

3.1. **Prerequisites on the Cauchy problem.** We consider the Cauchy problem associated to Theorem 3.1:

$$\begin{cases} \partial_t u + V(t,x,u) \cdot \nabla u + T_a u = F(t,u) \\ u(0,\cdot) = u_0(\cdot) \in H^s(\mathbb{D}^d), \ s > 1 + \frac{d}{2}, \end{cases} \tag{3.2}$$

Theorem 3.2. *Consider $0 \le \alpha < 2$, $\beta \le \alpha$, $T > 0$, an elliptic C^1 symbol $a : [0,T] \times H^s(\mathbb{D}^d) \to \Gamma^\alpha_1(\mathbb{D}^d)$ skew symmetric*

$$i.e \text{ such that } \ \mathrm{Re}(a_t) = \frac{a_t + a_t^\top}{2} \ \text{ is bounded in } \ \Gamma^0_1(\mathbb{D}^d),$$

and that $D_u a$ verifies:

$$\forall \mu \in \mathbb{R}, \forall g \in H^\mu(\mathbb{D}^d), \left\| T_{D_u a} g + \overline{T_{D_u a} g} \right\|_{H^\mu} \le \|g\|_{H^{\mu+\beta}}. \tag{3.3}$$

Consider a regular function $V(t,x,u) : [0,T] \times \mathbb{D}^d \times \mathbb{C} \to \mathbb{R}^d$ and a function F C^1 in u and L^∞ in t, $F : [0,T] \times H^s(\mathbb{D}^d) \to H^s(\mathbb{D}^d)$ verifying:

$$\begin{cases} \forall (t,u), M^\alpha_1(a) \le C(1 + \|u\|_{H^s}) \ , \ M^\beta_0(D_u a + \overline{D_u a}) \le C \|u\|^{k-1}_{L^\infty} , \\ \forall (t,u), k \in \{0,1\} , \left\| D^k F(t,u) \right\|_{H^s} \le C(1 + \|u\|_{H^s}), \\ \forall (t,x,u), \|V(t,x,u)\| \le C, \|u\| \, \forall k \ge 1, \left\| D^k V(t,x,u) \right\| \le C. \end{cases} \tag{\textbf{H1}}$$

Consider $s > 1 + \dfrac{d}{2}$, $r > 0$ and $u_0 \in H^s(\mathbb{D}^d)$ such that :

$$\forall v_0 \in B(u_0, r), T \|\nabla_x v_0\|_{L^\infty} < 1.$$

Then the problem (3.2) *with initial data* v_0 *has a unique solution* $v \in C^0([0,T], H^s(\mathbb{D}^d))$ *and the map* $v_0 \mapsto v$ *is continuous from* $B(u_0, r)$ *to* $C^0([0,T], H^s(\mathbb{D}^d))$*. Moreover we have the estimates:*

$$\forall 0 \leq \mu \leq s, \|v(t)\|_{H^\mu(\mathbb{D}^d)} \leq C_\mu(\|v_0\|_{H^\mu(\mathbb{D}^d)} + t). \tag{3.4}$$

Taking Two different solution v, v'*, such that* $v_0 \in H^{s+\beta}(\mathbb{D}^d)$*:*

$$\forall 0 \leq \mu \leq s, \left\|(v - v')(t)\right\|_{H^\mu(\mathbb{D}^d)} \leq \left\|v_0 - v_0'\right\|_{H^\mu(\mathbb{D}^d)} e^{C_\mu \int_0^t \|v(s)\|_{H^{\mu+\beta}(\mathbb{D}^d)}} ds. \tag{3.5}$$

We will also work with hyperbolic paradifferential equations and we summarize the properties needed in the following Theorem:

Theorem 3.3. *Consider*$(a_t)_{t\in\mathbb{R}}$ *a family of symbols in* $\Gamma_1^\beta(\mathbb{D}^d)$ *with* $\beta \in \mathbb{R}$*, such that* $t \mapsto a_t$ *is continuous and bounded from* $\mathbb{R}$ *to* $\Gamma_1^\beta(\mathbb{D}^d)$ *and such that* $\mathrm{Re}(a_t) = \frac{a_t + a_t^\top}{2}$ *is bounded in* $\Gamma_1^0(\mathbb{D}^d)$*, and take* $T > 0$*. Then for all initial data* $u_0 \in H^s(\mathbb{D}^d)$*, and* $f \in C^0([0,T]; H^s(\mathbb{D}^d))$ *the Cauchy problem:*

$$\begin{cases} \partial_t u + T_a u = f \\ \forall x \in \mathbb{D}^d, u(0,x) = u_0(x) \end{cases} \tag{3.6}$$

has a unique solution $u \in C^0([0,T]; H^s(\mathbb{D}^d)) \cap C^1([0,T]; H^{s-\beta}(\mathbb{D}^d))$ *which verifies the estimates:*

$$\|u(t)\|_{H^s(\mathbb{D}^d)} \leq e^{Ct} \|u_0\|_{H^s(\mathbb{D}^d)} + 2\int_0^t e^{C(t-t')} \left\|f(t')\right\|_{H^s(\mathbb{D}^d)} dt',$$

where C *depends on a finite symbol semi-norm* $M_1^0(\mathrm{Re}(a_t))$*. We will also need to remark that fixing the initial data at 0 is an arbitrary choice. More precisely,* $\forall 0 \leq t_0 \leq T$ *and all data* $u_0 \in H^s(\mathbb{D}^d)$ *the Cauchy problem:*

$$\begin{cases} \partial_t u + T_a u = f \\ \forall x \in \mathbb{D}^d, u(t_0,x) = u_0(x) \end{cases} \tag{3.7}$$

has a unique solution $u \in C^0([0,T]; H^s(\mathbb{D}^d)) \cap C^1([0,T]; H^{s-\beta}(\mathbb{D}^d))$ *which verifies the estimate:*

$$\|u(t)\|_{H^s(\mathbb{D}^d)} \leq e^{C|t-t_0|} \|u_0\|_{H^s(\mathbb{D}^d)} + 2\left|\int_{t_0}^t e^{C(t-t')} \left\|f(t')\right\|_{H^s(\mathbb{D}^d)} dt'\right|.$$

3.2. **Proof of Theorem 3.1.**

3.2.1. *Reduction of the problem to* $u_0 = 0$. First as Theorem 3.1 is of local nature, we see that it's sufficient to prove the Theorem for $r > 0$ small. Now Fixing $u_0 \in H^s(\mathbb{D}^d)$ and writing for $v_0 \in \mathrm{B}(u_0, r)$,

$$v = u + \epsilon, \quad \text{and} \quad \epsilon_0 = v_0 - u_0$$

we see that ϵ solves a an equation of the form (3.2) with a modified F and a but that still verify the hypothesis **H1** and **H2** and the problem is then reduced to a small ball around 0 and $F(t, 0) = 0$.

The rest of the proof will follow mainly as the proof of Theorem 1.1.

3.2.2. *Definition of the Ansatz.*

- For $\mathbb{D}^d = \mathbb{R}^d$, take $\omega \in C_0^\infty(\mathbb{R}^d), \omega(x) = 1$ if $|x| \leq \frac{1}{2}, \omega(x) = 0$ if $|x| \geq 1$.
- For $\mathbb{D}^d = \mathbb{T}^d$, we see functions on $\mathbb{T}^d = \mathbb{R}^d/2\pi\mathbb{Z}$ as 2π periodic function in each direction on $\mathbb{R}^d$ and we take $\omega \in C_0^\infty(\mathbb{T}^d)$ as the periodic extension of the functions defined above.

Let (λ, ϵ) be two positive real sequences such that

$$\lambda \to +\infty, \quad \epsilon \to 0, \quad \lambda\epsilon \to +\infty. \tag{3.8}$$

Put

- on $\mathbb{R}^d$,
$$u^0(x) = \lambda^{\frac{d}{2}-s}\omega(\lambda x), \quad v^0(x) = u^0(x) + \epsilon\omega(x),$$
- on $\mathbb{T}^d$, u^0 and v^0 as the periodic extensions of the functions defined above.

Take $t_0 > 0$ smaller than a harmless constant which will be fixed later, and (τ), $0 < \tau \leq t_0$.
Now let l be the solutions to the Cauchy problem on $[0, t_0]$:

$$\begin{cases} \partial_t l + T_{a(t,l)} l = F(t, l) \\ \forall x \in \mathbb{D}^d, l(\tau, x) = u^0(x). \end{cases} \tag{3.9}$$

Put $u^1(x) = l(0, x)$ and define l' to be the solutions to the Cauchy problem on $[0, t_0]$:

$$\begin{cases} \partial_t l' + T_{a(t,l)} l' = F(t, l') \\ \forall x \in \mathbb{D}^d, l(\tau, x) = v^0(x). \end{cases} \tag{3.10}$$

and put $v^1 = l(0, x)$.

Remark 3.2. *It's important to notice that we use the same term $T_{a(t,l)}$ in 3.9 and 3.10 and thus (l, l') have Lipschitz dependence on the data (u^0, v^0).*

Define u and v as the solution given by Theorem 3.2 with initial data u^1 and v^1 on the intervals $[0, T]$, $[0, T']$. Taking $0 < \delta < s-1-\frac{d}{2}$, u^0 and v^0 are uniformly bounded in $H^{1+\frac{d}{2}+\delta}(\mathbb{D}^d)$ and thus by Theorem 3.3, u^1 and v^1 are also uniformly bounded in $H^{1+\frac{d}{2}+\delta}(\mathbb{D}^d)$ and thus by the Sobolev injection Theorems they are bounded in $\dot{W}^{1,\infty}(\mathbb{D}^d)$. Thus we can take a uniform $0 < T$ on which all the solutions are well defined and we take $0 < t_0 \leq T$ [9].

3.2.3. *Change of variables by transport.* Put

$$\begin{cases} \frac{d}{dt}\chi(t, s, x) = V(t, \chi(t, s, x), u(t, \chi(t, s, x))) \ , \\ \chi(s, s, x) = x, \end{cases}$$

and define analogously $\tilde{\chi}$ from v. We recall that from the Cauchy-Lipschitz Theorem we have $\chi, \tilde{\chi} \in C^1([0, T]^2, W^{s-\frac{d}{2}-\delta,\infty}(\mathbb{D}^d))$ [10] and they are both diffeomorphisms in the x variable.

By the estimate (2.2) u and v are uniformly bounded in $\dot{W}^{1,\infty}(\mathbb{D}^d)$ say by $M > 0$ and there Sobolev norms are dominated by those of u^1 and v^1 thus by those of u^0 and v^0 by Theorem 3.3. By classic manipulations of ODEs we get the estimates:

$$\begin{cases} \exists C > 0, \forall t', t \leq t_0, \forall x, C^{-1} \leq |D\chi(t, t', x)| \leq C \\ \forall 2 \leq k \leq \lfloor s - \frac{d}{2} - \delta \rfloor, \left\| D^k\chi(t, t', x) \right\|_{L^\infty} \leq C \left\| D^k u \right\|_{W^{k,\infty}} \end{cases} \tag{3.11}$$

[9] Again heuristically, if the existence time of the solution ω is $[0, T]$ then the existence time of u is $\sim T\lambda^{s-1-\frac{d}{2}}$ which tends to infinity with n, thus we are "dilating" the time scale of the problem with initial data ω and "zooming" for short time and in the $H^s(\mathbb{D}^d)$ norm. In this part of the evolution, we prove that the Burgers type transport term is more important and gives this quasi-linear character to the PDE.

[10] We actually have better regularities, as u^0 and v^0 are $H^{+\infty}(\mathbb{D}^d)$ functions, then u^1, v^1 are $H^{+\infty}$ and u and v are $H^{+\infty}(\mathbb{D}^d)$ with respect to the x variable thus $\chi, \tilde{\chi} \in C^1([0, T]^2, C^\infty(\mathbb{D}^d))$.

Analogous estimates hold for $\tilde{\chi}$ using v.
The classic transport computation reads:

$$\begin{cases} \partial_t(u(t,\chi(t,0,x))) = \partial_t(u)(t,\chi(t,0,x)) + \partial_t(\chi(t,0,x))\cdot\nabla(u)(t,\chi(t,0,x)) \\ \qquad\qquad\qquad\quad = -(T_{a(t,u)}u)(t,\chi(t,0,x)) + F(t,u)\circ\chi(t,0,x) \\ \qquad\qquad\qquad\quad = -T_{a(t,u)^*}u(t,\chi(t,0,x)) + F(t,u)\circ\chi(t,0,x), \\ u(0,\chi(0,0,x)) = u(0,x) = u^1(x). \end{cases}$$

where $(\cdot)^*$ is the change of variables by $\chi(t,0,x)$ as presented in Theorem A.7.
Thus if we put f the solution to the Cauchy problem, which is well posed by Appendix B:

$$\begin{cases} \partial_t w + T_{a(t,u)^*}w = F(t,u)\circ\chi(t,0,x) \\ \forall x\in\mathbb{D}^d, w(0,x) = u^1(x) \end{cases} \tag{3.12}$$

we get:

$$u(t,x) = f(t,\chi(0,t,x)) \;\Leftrightarrow\; u(t,\chi(t,0,x)) = f(t,x). \tag{3.13}$$

Analogously, if we put g the solution to the well posed Cauchy problem,

$$\begin{cases} \partial_t w + T_{\widetilde{a(t,v)}^*}w = F(t,v)\circ\tilde{\chi}(t,0,x) \\ \forall x\in\mathbb{D}^d, w(0,x) = v^1(x) \end{cases} \tag{3.14}$$

where $\widetilde{(\cdot)}^*$ is the change of variables by $\tilde{\chi}(t,0,x)$, we get

$$v(t,x) = g(t,\tilde{\chi}(0,t,x)) \;\Leftrightarrow\; v(t,\tilde{\chi}(t,0,x)) = g(t,x). \tag{3.15}$$

Returning to the ODEs defining χ and $\tilde{\chi}$ we get:

$$\begin{cases} \chi(t,t',x) = x + \int_{t'}^t V(s,\chi(s,t',x),f(s,x)))ds, \\ \tilde{\chi}(t,t',x) = x + \int_{t'}^t V(s,\tilde{\chi}(s,t',x),g(s,x))ds. \end{cases} \tag{3.16}$$

Proposition 3.1. *Their exists $C>0$ independent of (τ,ϵ,λ) such that:*
$\forall h\in H^s(\mathbb{D}^d), \forall(t,t')\leq t_0,$

$$C^{-1}\|h\|_{H^s} \leq \left\|h\circ\chi(t,t',x)\right\|_{H^s} \leq C\|h\|_{H^s},$$

$$C^{-1}\|h\|_{H^s} \leq \left\|h\circ\tilde{\chi}(t,t',x)\right\|_{H^s} \leq C\|h\|_{H^s}.$$

Proof. We will start by proving the upper bound for the estimate on the composition with χ. As u is bounded in (τ,ϵ,λ) on $C([0,T],H^s(\mathbb{D}^d))$ then there exist unique solution $H\in C([0,T],H^s(\mathbb{D}))$ to

$$\begin{cases} \partial_s H(s,x) + V(s,x,u)\cdot\nabla H(s,x) = 0 \\ H(t,x) = h(x) \end{cases}$$

and H is bounded in (τ,ϵ,λ) on $C([0,T],H^s(\mathbb{D}^d))$. The desired bounds come from the fact that we have the explicit formula for H:

$$H(t',x) = h\circ\chi(t,t',x).$$

Now to get the lower bound it suffices to write by the upper bound computations:

$$\begin{aligned} \|h\|_{H^s} &= \left\|h\circ\chi(t,t',x)\circ\chi(t',t,x)\right\|_{H^s} \\ &\leq C\left\|h\circ\chi(t,t',x)\right\|_{H^s}. \end{aligned}$$

We get analogously the estimates on the composition with $\tilde{\chi}$. $\square$

3.2.4. *Key Lemma and proof of the Theorem.*

Lemma 3.1. *As* $0 \le \alpha < 2$ *and by hypothesis* ***H2*** *we can find a sequence* $(\tau, \epsilon, \lambda)$*, with an adequate choice of* ω *such that for all* $0 < \epsilon'$ *sufficiently small:*

$$\begin{cases} \tau \to 0, \\ \epsilon \to 0, \\ \lambda \to +\infty, \end{cases} \qquad \begin{cases} \tau\lambda^{\alpha-1} \to 0, \\ \tau\epsilon^k\lambda^\beta \to 0, \\ \epsilon^{-1}\lambda^{-1+(\alpha-1)^+ +\epsilon'} \to +\infty, \end{cases} \tag{3.17}$$

$$\lambda\epsilon \left\| \int_0^\tau \int_0^1 D_u V(t, \chi(t,0,y), rf(t,y) + (1-r)g(t,y)) dr dt [\omega(y)] \right\| \to +\infty.$$

Then there exists $c > 0$ *such that:*

(1) $\forall(\tau,\epsilon,\lambda), \left\| u^0 \circ \chi(0,\tau,x) - u^0 \circ \tilde{\chi}(0,\tau,x) \right\|_{H^s} > c.$

(2) *For* δ *such that* $0 < \delta < s - 1 - \frac{d}{2}$*:*

$$\begin{aligned} u(\tau,x) - v(\tau,x) &= u^0 \circ \chi(0,\tau,x) - u^0 \circ \tilde{\chi}(0,\tau,x) \\ &\quad + O_{H^{s-\nu}}(C(\tau\lambda^{(\alpha-1)^+})\epsilon + \tau\lambda^{(\alpha-1)^+}\lambda^{-\nu} + \tau\epsilon^k\lambda^\beta\lambda^{-\nu}). \end{aligned}$$

We will now show that this Lemma implies the Theorem. We have $\tau \le t_0$ is such that $\tau\epsilon\lambda \to +\infty$ and $\tau\lambda^{\alpha-1} \to 0$ which gives:

$$\forall(\tau,\epsilon,\lambda), \|u(\tau,x) - v(\tau,x)\|_{H^s} > \frac{c}{2} > 0 \quad \text{thus} \quad \sup_{\tau,\epsilon,\lambda} \|u(\tau,x) - v(\tau,x)\|_{H^s} > \frac{c}{2} > 0$$

Also by Theorem 3.3 and Remark 3.2:

$$\exists C > 0, \left\|u^1(x) - v^1(x)\right\|_{H^s} \le C\epsilon \quad \text{thus} \quad \left\|u^1(x) - v^1(x)\right\|_{H^s} \to 0,$$

which gives the non uniform continuity in the desired norms. Now for the control in a weaker norm we write:

$$\frac{\|u(\tau,x) - v(\tau,x)\|_{H^{s-1+(\alpha-1)^+ +\epsilon'}}}{\|u^1(x) - v^1(x)\|_{H^s}} \ge c\epsilon^{-1}\lambda^{-1+(\alpha-1)^+ +\epsilon'} \to +\infty,$$

which gives the desired result.

3.2.5. *Proof of point 1 of Lemma 3.1.* We first prove that $\exists c > 0$ such that $\left\|u^0 \circ \chi(0,\tau,x)\right\|_{H^s} > c$, indeed by Proposition 2.1 and change of variable:

$$\left\|u^0 \circ \chi(0,\tau,x)\right\|_{H^s} \ge C^{-1}\left\|u^0\right\|_{H^s} \ge C^{-1}\|\omega\|_{H^s}. \tag{3.18}$$

Now we will show that $u^0 \circ \chi(0,\tau,x)$ and $u^0 \circ \tilde{\chi}(0,\tau,x)$ have disjoint supports which will suffice to conclude given (3.18). Put $y = \chi(0,\tau,x)$,
thus $x = \chi(\tau,0,y)$. On the support of $u^0 \circ \chi(0,\tau,x)$ we have:

- If $\mathbb{D}^d = \mathbb{R}^d$, $\lambda|y| \le 1$.
- If $\mathbb{D}^d = \mathbb{T}^d$, $\forall k \in \mathbb{N}, 2\pi k - 1 \le \lambda|y| \le 2\pi k + 1$.

We then compute by the Taylor formula:

$$\begin{aligned} \tilde{\chi}(0,\tau,x) &= \tilde{\chi}(0,\tau,\tilde{\chi}(\tau,0,y)) \\ &\quad + \int_0^1 D_y\tilde{\chi}(0,\tau,r\chi(\tau,0,y) + (1-r)\tilde{\chi}(\tau,0,y)) dr[\chi(\tau,0,y) - \tilde{\chi}(\tau,0,y)] \end{aligned} \tag{3.19}$$

$$= y + \int_0^1 D_y\tilde{\chi}(0,\tau,r\chi(\tau,0,y) + (1-r)\tilde{\chi}(\tau,0,y)) dr[\chi(\tau,0,y) - \tilde{\chi}(\tau,0,y)].$$

First,

$$D_y\tilde{\chi}(0,\tau,r\chi(\tau,0,y)+(1-r)\tilde{\chi}(\tau,0,y))$$
$$= Id+\int_0^\tau D_y[V(t,\tilde{\chi}(0,\tau,r\chi(\tau,0,y)+(1-r)\tilde{\chi}(\tau,0,y)),g(t,r\chi(\tau,0,y)+(1-r)\tilde{\chi}(\tau,0,y)))]dt.$$

Thus by estimates of Theorem B.1 and hypothesis **H1**, taking $0<\delta<s-\frac{d}{2}-1$:

$$\begin{aligned} D_y\tilde{\chi}(0,\tau,r\chi(\tau,0,y)+(1-r)\tilde{\chi}(\tau,0,y)) &= Id+O_{L^\infty}(\tau(1+\|v^1\|_{H^{1+\frac{d}{2}+\delta}}+\|u^1\|_{H^{1+\frac{d}{2}+\delta}})) \\ &= Id+O_{L^\infty}(\tau), \end{aligned}$$

Which gives

$$\int_0^1 D_y\tilde{\chi}(0,\tau,r\chi(\tau,0,y)+(1-r)\tilde{\chi}(\tau,0,y))dr = Id+O_{L^\infty}(\tau). \tag{3.20}$$

Now we estimate $\chi(\tau,0,y)-\tilde{\chi}(\tau,0,y)$, by (3.16) :

$$\chi(\tau,0,y)-\tilde{\chi}(\tau,0,y) = \int_0^\tau V(t,\chi(t,0,y),f(t,y))-V(t,\tilde{\chi}(t,0,y),g(t,y))dt \tag{3.21}$$
$$= \int_0^\tau\int_0^1 D_uV(t,\chi(t,0,y),rf(t,y)+(1-r)g(t,y))[f(t,y)-g(t,y)]dtdr$$
$$+\int_0^\tau\int_0^1 D_xV(t,r\chi(t,0,y)+(1-r)\tilde{\chi}(t,0,y),g(t,y))[\chi(t,0,y)-\tilde{\chi}(t,0,y)]dtdr.$$

Taking $0<\delta<s-1-\frac{d}{2}$,by estimates of Theorem B.1:

$$\begin{aligned} f(t,y) &= f(0,y)+\int_0^t \partial_t f(r,y)dr \\ &= u^1(y)+O_{L^\infty}(t(\|u^1\|_{H^{\frac{d}{2}+\alpha+\delta}}+1)) = u^1(y)+O_{L^\infty}(t). \end{aligned}$$

Analogously we get:

$$g(t,y) = v^1(y)+O_{L^\infty}(t).$$

Now $(u^1-v^1)(y)=(l-l')(0,y)$ is the evaluation of the solution of the following Cauchy problem at $t=0$:

$$\begin{cases} \partial_t(l-l')+T_{a(t,u^0)}(l-l') = F(t,l)-F(t,l')+(T_{a(t,l')}-T_{a(t,l)})l' \\ \forall y\in\mathbb{D}^d, (l-l')(\tau,y) = -\epsilon\omega(y). \end{cases} \tag{3.22}$$

thus by estimates of Theorem B.1 and hypothesis **H1**:

$$\begin{aligned} u^1(y)-v^1(y) = (l-l')(0,y) &= -\epsilon\omega(y)+\int_\tau^0 \partial_t(l-l')(t,y)dt \\ &= -\epsilon\omega(y)+O_{L^\infty}(\tau(1+\|v^1\|_{H^{\alpha+\frac{d}{2}+\delta}}+\|u^1\|_{H^{\alpha+\frac{d}{2}+\delta}})) \\ &= -\epsilon\omega(y)+O_{L^\infty}(\tau). \end{aligned}$$

Thus,

$$\chi(\tau,0,y)-\tilde{\chi}(\tau,0,y)$$
$$= -\epsilon[\int_0^\tau\int_0^1 D_uV(t,\chi(t,0,y),rf(t,y)+(1-r)g(t,y))dtdr][\omega(y)]+O_{L^\infty}(\tau^2)$$
$$+\int_0^\tau\int_0^1 D_xV(t,r\chi(t,0,y)+(1-r)\tilde{\chi}(t,0,y),g(t,y))dr[\underbrace{\chi(t,0,y)-\tilde{\chi}(t,0,y)}_{*}]dt.$$

Iterating the computation in (*):

$$\chi(\tau,0,y)-\tilde{\chi}(\tau,0,y)$$
$$=-\epsilon[\int_0^\tau\int_0^1 D_uV(t,\chi(t,0,y),rf(t,y)+(1-r)g(t,y))dtdr][\omega(y)]+O_{L^\infty}(\tau^2).$$

and finally we get in (3.19),

$$\tilde{\chi}(\tau,0,x)-y$$
$$=-\epsilon[\int_0^\tau\int_0^1 D_uV(t,\chi(t,0,y),rf(t,y)+(1-r)g(t,y))dtdr][\omega(y)]+O_{L^\infty}(\tau^2).$$

We get for $x\in$ supp $u^0\circ\chi(0,\tau,\cdot)$:

- For $\mathbb{D}^d=\mathbb{R}^d$:

$$\begin{aligned}&\lambda\,|\tilde{\chi}(0,\tau,x)|\\&\geq\epsilon\lambda\left\|\int_0^\tau\int_0^1 D_uV(t,\chi(t,0,y),rf(t,y)+(1-r)g(t,y))drdt[\omega(y)]\right\|-1+o_{L^\infty}(\tau\lambda\epsilon)\\&\geq 2,\end{aligned}$$

  as by hypothesis

$$\epsilon\lambda\left\|\int_0^\tau\int_0^1 D_uV(t,\chi(t,0,y),rf(t,y)+(1-r)g(t,y))drdt[\omega(y)]\right\|\geq C\tau\lambda\epsilon\to+\infty,$$

  which gives the desired result.

- For $\mathbb{D}^d=\mathbb{T}^d$ given an adequate choice of τ,ϵ and λ we get:

$$2n\pi+1\leq\lambda\,|\tilde{\chi}(0,\tau,x)|\leq 2(n+1)\pi-1,$$

  Which again gives the desired result.

3.2.6. *Proof of point 2 of Lemma 3.1.* We start by writing:

$$\begin{aligned}u(t,x)-v(t,x)&=f(t,\chi(0,t,x))-g(t,\tilde{\chi}(0,t,x))\\&=\underbrace{f(t,\chi(0,t,x))-f(t,\tilde{\chi}(0,t,x))}_{(1)}+(f-g)(t,\tilde{\chi}(0,t,x)).\end{aligned}$$

Term (1) resembles the main term in the usual transport estimates we used in point 1 of the Lemma [11] but with a main difference of f being some dispersed data and not compactly supported. Again, the main trick here was to construct from u^0,v^0 the defocused data in the past u^1,v^1 and use this as the initial data for f and g.

$$\begin{aligned}u(\tau,x)-v(\tau,x)&=u^0\circ\chi(0,\tau,x)-u^0\circ\tilde{\chi}(0,\tau,x)\\&+(f-u^0)(\tau,\chi(0,\tau,x))-(f-u^0)(\tau,\tilde{\chi}(0,t,x))\\&+(f-g)(\tau,\tilde{\chi}(0,\tau,x)).\\&=u^0\circ\chi(0,\tau,x)-u^0\circ\tilde{\chi}(0,\tau,x)\\&+\underbrace{(f-l)(\tau,\chi(0,\tau,x))-(f-l)(\tau,\tilde{\chi}(0,t,x))}_{(1)}+\underbrace{(f-g)(\tau,\tilde{\chi}(0,\tau,x))}_{2}.\end{aligned}$$

Where l is defined by (3.9). We start by estimating (1), by Proposition 3.1:

$$\|(f-l)(\tau,\chi(0,\tau,x))\|_{H^s}\leq C\,\|(f-l)(\tau,\cdot)\|_{H^s}\,.$$

Now $f-l$ solve:

$$\begin{cases}\partial_t(f-l)+T_{a(t,l)}(f-l)=(T_{a(t,l)}-T_{a(t,u)^*})f-F(t,l)+F(t,f)\\\forall x\in\mathbb{D}^d,(f-l)(0,x)=0.\end{cases}\tag{3.23}$$

[11] Like the ones used in proving the quasi-linearity of the Burgers equation.

Here will need to be more careful in treating the nonlinearity in the dispersive term, which was not there in the previous section. More precisely we write:

$$F(t,l)-F(t,f)=\int_0^1 DF(t,rl+(1-r)f)dr[l-f],$$

$$(T_{a(t,l)}-T_{a(t,u)^*})f=(T_{a(t,l)}-T_{a(t,l)^*})f-G_1(f-l),$$

where G_1 is defined by:

$$G_1(f-l)=(T_{a(t,u)^*}-T_{a(t,l)^*})f=-T_{(f-l)(\int_0^1\partial_u a(t,su+(1-s)l)ds)^*}f.$$

The key estimate here comes from hypothesis of skew symmetry and noticing that by symbolic calculus rules and commutator estimates:

$$\mathrm{Re}(G_1(f-l))=\frac{G_1(f-l)+G_1^\top(f-l)}{2}=R_1,$$

$$\text{with } \|R_1\|_{H^\nu\to H^\nu}\leq C\|(f,l)\|_{L^\infty}M_0^\beta(\partial_u a+\overline{\partial_u a})\|f\|_{H^{\nu+\beta}}$$

Thus we have the estimates by Theorem B.1:

$$\|f-l(\tau,\cdot)\|_{H^\nu} \tag{3.24}$$

$$\leq C(\tau\lambda^{\alpha-1})[\left\|(T_{a(t,l)}-T_{a(t,l)^*})f\right\|_{L^1([0,\tau],H^\nu)} \tag{3.25}$$

$$\begin{aligned}&\leq C(\tau\lambda^{\alpha-1})\tau\left\|(T_{a(t,l)}-T_{a(t,l)^*})f\right\|_{L^\infty([0,\tau],H^\nu)}\\&\leq C(\tau\lambda^{\alpha-1})\tau\|(Id-\chi(0,t,\chi(t,0,x)))\|_{L^\infty([0,\tau],L^\infty)}\|f\|_{L^\infty([0,\tau],H^{\nu+\alpha})}\\&+C(\tau\lambda^{\alpha-1})\tau\|(Id-D\chi(0,t,\chi(t,0,x)))\|_{L^\infty([0,\tau],L^\infty)}\|f\|_{L^\infty([0,\tau],H^{\nu+\alpha-1})}.\end{aligned}$$

Taking $0<\delta<s-\frac{d}{2}$ we get:

$$\|f-l(\tau,\cdot)\|_{H^\nu}\leq C(\tau\lambda^{\alpha-1})\tau[\lambda^{\frac{d}{2}+\delta-s}\lambda^{\nu-s+\alpha}+\lambda^{\nu-s+(\alpha-1)^+}]$$

Thus we get in (3.24) by keeping the dominating terms,

$$\|f-l(\tau,\cdot)\|_{H^\nu}\leq C(\tau\lambda^{\alpha-1})\tau\lambda^{(\alpha-1)^+}\lambda^{\nu-s},$$

which gives

$$\|(f-l)(\tau,\chi(0,\tau,x))\|_{H^\nu}\leq C(\tau\lambda^{\alpha-1})\tau\lambda^{(\alpha-1)^+}\lambda^{\nu-s}.$$

Analogously we get,

$$\|(f-l)(\tau,\tilde\chi(0,\tau,x))\|_{H^\nu}\leq C(\tau\lambda^{\alpha-1})\tau\lambda^{(\alpha-1)^+}\lambda^{\nu-s},$$

which gives

$$\|(1)\|_{H^\nu}\leq C(\tau\lambda^{\alpha-1})\tau\lambda^{(\alpha-1)^+}\lambda^{\nu-s}. \tag{3.26}$$

Now we estimate (2) in the same manner, by Proposition 3.1:

$$\|(f-g)(\tau,\tilde\chi(0,\tau,x))\|_{H^\nu}\leq C\|(f-g)(\tau,\cdot)\|_{H^\nu}$$

$f-g$ solve:

$$\begin{cases}\partial_t(f-g)+T_{a(t,u)^*}(f-g)-(T_{a(t,v)^*}-T_{\widetilde{a(t,v)}^*})g\\-F(t,f)+F(t,g)=(T_{a(t,u)^*}-T_{a(t,v)^*})g\\\forall x\in\mathbb{D}^d,(f-g)(0,x)=(u^1-v^1)(x).\end{cases} \tag{3.27}$$

Again Here will need to be more careful in treating the nonlinearity in the dispersive term, which was not there in the previous section. More precisely we write:

$$F(t,f)-F(t,g)=\int_0^1 DF(t,rf+(1-r)g)dr[f-g],$$

$$G_2(f-g)=(T_{a(t,u)^*}-T_{a(t,v)^*})g=T_{(f-g)(\int_0^1\partial_u a(t,su+(1-s)v)ds)^*}g.$$

$$
\begin{aligned}
(T_{a(t,v)^*} - T_{\widetilde{a(t,v)}^*})g &= T_{(\chi-\tilde{\chi})(\int_0^1 \partial_x v(t,s\chi+(1-s)\tilde{\chi})ds)(\int_0^1 \partial_u a(t,sv\circ\chi+(1-s)v\circ\tilde{\chi})ds)} g \\
&\quad + T_{(D\chi-D\tilde{\chi})(\int_0^1 \partial_\xi a(t,\cdot,[sD\chi+(1-s)D\tilde{\chi}]^\top \xi)ds)} g \\
&= -G_3(f-g) + T_{(D\chi-D\tilde{\chi})(\int_0^1 \partial_\xi a(t,\cdot,[sD\chi+(1-s)D\tilde{\chi}]^\top \xi)ds)} g.
\end{aligned}
$$

The key estimate here comes from:

$$
\begin{aligned}
\mathrm{Re}((G_2(f-g), f-g)_{H^\nu}) &\leq C \|f-g\|_{H^\nu} M_0^\beta(D_u a + \overline{D_u a}) \|f-g\|_{L^\infty} \|g\|_{H^{\nu+\beta}} \\
&\leq C \|f-g\|_{H^\nu} \epsilon^k \|g\|_{H^{\nu+\beta}}, \quad (3.28)
\end{aligned}
$$

and analogously the estimate for G_3:

$$
\mathrm{Re}((G_3(f-g), f-g)_{H^\nu}) \leq C \|f-g\|_{H^\nu} \epsilon^k \|g\|_{H^{\nu+\beta}},
$$

Thus we have the estimates by Theorem B.1:

$$
\|f - g(\tau,\cdot)\|_{H^\nu} \leq C(\tau\lambda^{(\alpha-1)^+})(\epsilon + \tau\epsilon^k\lambda^{\nu-s+\beta}). \quad (3.29)
$$

which gives

$$
\|(2)\|_{H^\nu} \leq C(\tau\lambda^{(\alpha-1)^+})(\epsilon + \tau\epsilon^k\lambda^{\nu-s+\beta}), \quad (3.30)
$$

finishing the proof of Lemma 3.1 and Theorem 3.1. In the proof of quasi-linearity of the water waves systems we will need the following slight generalization to systems given by the following corollary.

Corollary 3.1. *Consider a positive integer $n \geq 1$ and five numbers $\alpha \in [0,2[$, $s \in]\max(1,\alpha) + \frac{d}{2}, +\infty[$, $T > 0$ and $(\beta, k) \in \mathbb{R}^+$ verifying:*

$$
\begin{cases}
k \geq 1, & \beta \leq \alpha, \\
\beta < 2 - \frac{1}{k}.
\end{cases}
$$

Consider a C^1 elliptic symbol $a : [0,T] \times H^s(\mathbb{D}^d;\mathbb{R}^n) \to \Gamma_1^\alpha(\mathbb{D}^d; M_n(\mathbb{R}))$ skew symmetric[12]

$$
\text{i.e such that} \quad \mathrm{Re}(a_t) = \frac{a_t + a_t^\top}{2} \quad \text{is bounded in} \quad \Gamma_1^0(\mathbb{D}^d; M_n(\mathbb{R})).
$$

Consider a regular function $V(t,x,u) : [0,T] \times \mathbb{D}^d \times \mathbb{C}^n \to \mathbb{R}^d$ and a function F C^1 in u and L^∞ in t, $F : [0,T] \times H^s(\mathbb{D}^d;\mathbb{R}^n) \to H^s(\mathbb{D}^d;\mathbb{R}^n)$ verifying ***H1***, ***H2*** *and* ***H3*** *given by:*

H1:

$$
\begin{cases}
\forall (t,u), M_1^\alpha(a) \leq C(1 + \|u\|_{W^{1,\infty}}), \\
\forall (t,u), k \in \{0,1\}, \left\|D^k F(t,u)\right\|_{H^s} \leq C(1 + \|u\|_{H^s}), \\
\forall (t,x,u), \|V(t,x,u)\| \leq C\|u\|, \forall k \geq 1, \left\|D^k V(t,x,u)\right\| \leq C.
\end{cases} \quad (\mathbf{H1})
$$

H2:

$$
\forall (t,x,u), 0 < C^{-1} \leq \|D_u V(t,x,u)\| \leq C. \quad (\mathbf{H2})
$$

H3: *Fix $u_0 \in H^s(\mathbb{D}^d;\mathbb{R}^n)$ and take $r > 0$, then there exists $T > 0$ such that for all v_0 in the ball $B(u_0, r) \subset H^s(\mathbb{D}^d;\mathbb{R}^n)$ the Cauchy problem:*

$$
\begin{cases}
\forall i \in [1,..,n], \ \partial_t v_i + V(t,x,v)\cdot\nabla v_i + (T_{a(t,v)} v)_i = 0, \\
v(0,\cdot) = v_0(\cdot),
\end{cases}
$$

has a unique solution $v \in C([0,T], H^s(\mathbb{D}^d;\mathbb{R}^n))$. Moreover the flow map verifies the estimate:

$$
\begin{aligned}
\|u - v(t,\cdot)\|_{H^s} &\leq C\|u_0 - v_0\|_{H^s} \\
&\quad + Ct\|u_0 - v_0\|_{L^\infty} (\|u(t,\cdot)\|_{L^\infty} + \|v(t,\cdot)\|_{L^\infty})^{k-1} \|u(t,\cdot)\|_{H^{s+\beta}}. \quad (\mathbf{H3})
\end{aligned}
$$

[12] Recall the notation $a^\top$ for the adjoint of an operator a.

Fix $u_0 \in H^s(\mathbb{D}^d;\mathbb{R}^n)$ and take $r>0$, then there exists $T>0$ such that for all v_0 in the ball $B(u_0,r)\subset H^s(\mathbb{D}^d;\mathbb{R}^n)$ the Cauchy problem:

$$\begin{cases}\forall i\in[1,..,n],\ \partial_t v_i + V(t,x,v)\cdot\nabla v_i + (T_{a(t,v)}v)_i = F_i(t,v),\\ v(0,\cdot)=v_0(\cdot),\end{cases}$$

has a unique solution $v\in C([0,T],H^s(\mathbb{D}^d;\mathbb{R}^n))$. Moreover the flow map:

$$\begin{aligned} B(u_0,r) &\to C([0,T],H^s(\mathbb{D}^d;\mathbb{R}^n))\\ v_0 &\mapsto v\end{aligned}$$

is not uniformly continuous. Considering a weaker control norm we get, for all $\epsilon>0$ the flow map:

$$\begin{aligned} B(u_0,r) &\to C([0,T],H^{s-1+(\alpha-1)^+ +\epsilon}(\mathbb{D}^d;\mathbb{R}^n))\\ v_0 &\mapsto v\end{aligned}$$

is not C^1.

4. Quasi-linearity of the Water-Waves system with surface tension

In this section we always have $\kappa=1$.

4.1. Prerequisites from the Cauchy problem.

We start by recalling the apriori estimates given by Proposition 5.2 of [4]. We keep the notations of Theorem 1.2.

Proposition 4.1. *(From [4]) Let $d\geq 1$ be the dimension and consider a real number $s>2+\frac{d}{2}$. Then there exists a non decreasing function C such that, for all $T\in]0,1]$ and all solution (η,ψ) of (1.5) such that*

$$(\eta,\psi)\in C^0([0,T];H^{s+\frac12}(\mathbb{R}^d)\times H^s(\mathbb{R}^d))\ \ \text{and } H_t \text{ is verified for } t\in[0,T],$$

we have

$$\|(\eta,\psi)\|_{L^\infty(0,T;H^{s+\frac12}\times H^s)} \leq C((\eta_0,\psi_0)_{H^{s+\frac12}\times H^s}) + TC(\|(\eta,\psi)\|_{L^\infty(0,T;H^{s+\frac12}\times H^s)}).$$

The proof will rely on the para-linearised and symmetrized version of (1.5) given by Proposition 4.8 and corollary 4.9 of [4]. Before we recall this, for clarity as in [4] we introduce a special class of operators $\Sigma^m\subset\Gamma^m_0$ given by:

Definition 4.1. *(From [4]) Given $m\in\mathbb{R}$, Σ^m denotes the class of symbols a of the form*

$$a=a^{(m)}+a^{(m-1)}$$

with

$$a^{(m)} = F(\nabla\eta(t,x),\xi)$$
$$a^{(m-1)} = \sum_{|k|=2} G_\alpha(\nabla\eta(t,x),\xi)\partial_x^k\eta(t,x),$$

such that

1. *T_a maps real valued functions to real-valued functions;*
2. *F is of class C^∞ real valued function of $(\zeta,\xi)\in\mathbb{R}^d\times(\mathbb{R}^d\setminus 0)$, homogeneous of order m in ξ; and such that there exists a continuous function $K=K(\zeta)>0$ such that*
$$F(\zeta,\xi)\geq K(\zeta)\,|\xi|^m,$$
for all $(\zeta,\xi)\in\mathbb{R}^d\times(\mathbb{R}^d\setminus 0)$;
3. *G_α is a C^∞ complex valued function of $(\zeta,\xi)\in\mathbb{R}^d\times(\mathbb{R}^d\setminus 0)$, homogeneous of order $m-1$ in ξ.*

Σ^m enjoys all the usual symbolic calculus properties modulo acceptable reminders that we define by the following:

Definition-Notation 4.1. *(From [4]) Let $m \in \mathbb{R}$ and consider two families of operators of order m,*

$$\{A(t) : t \in [0,T]\}, \quad \{B(t) : t \in [0,T]\}.$$

We shall say that $A \sim B$ if $A - B$ is of order $m - \frac{3}{2}$ and satisfies the following estimate: for all $\mu \in \mathbb{R}$, there exists a continuous function C such that for all $t \in [0,T]$,

$$\|A(t) - B(t)\|_{H^\mu \to H^{\mu - m + \frac{3}{2}}} \leq C(\|\eta(t)\|_{H^{s+\frac{1}{2}}}).$$

In the next Proposition we recall the different symbols that appear in the para-linearisation and symmetrisation of the equations.

Proposition 4.2. *(From [4])We work under the hypothesis of Proposition 4.1. Put*

$$\lambda = \lambda^{(1)} + \lambda^{(0)}, \quad l = l^{(2)} + l^{(1)} \quad \text{with},$$

$$\begin{cases} \lambda^{(1)} = \sqrt{(1+|\nabla\eta|^2)\,|\xi|^2 - (\nabla\eta\cdot\xi)^2}, \\ \lambda^{(0)} = \frac{1+|\nabla\eta|^2}{2\lambda^{(1)}}\left\{ div\left(\alpha^{(1)}\nabla\eta\right) + i\partial_\xi\lambda^{(1)}\cdot\nabla\alpha^{(1)}\right\}, \\ \alpha^{(1)} = \frac{1}{\sqrt{1+|\nabla\eta|^2}}\left(\lambda^{(1)} + i\nabla\eta\cdot\xi\right). \end{cases} \tag{4.1}$$

$$\begin{cases} l^{(2)} = (1+|\nabla\eta|^2)^{-\frac{1}{2}}\left(|\xi|^2 - \frac{(\nabla\eta\cdot\xi)^2}{1+|\nabla\eta|^2}\right), \\ l^{(1)} = -\frac{i}{2}(\partial_x\cdot\partial_\xi)l^{(2)}. \end{cases} \tag{4.2}$$

Now let $q \in \Sigma^0, p \in \Sigma^{\frac{1}{2}}, \gamma \in \Sigma^{\frac{3}{2}}$ be defined by

$$\begin{aligned} q &= (1+|\nabla\eta|^2)^{-\frac{1}{2}}, \\ p &= (1+|\nabla\eta|^2)^{-\frac{5}{4}}\sqrt{\lambda^{(1)}} + p^{(-\frac{1}{2})}, \\ \gamma &= \sqrt{l^{(2)}\lambda^{(1)}} + \sqrt{\frac{l^{(2)}}{\lambda^{(1)}}}\frac{Re\lambda^{(0)}}{2} - \frac{i}{2}(\partial_\xi\cdot\partial_x)\sqrt{l^{(2)}\lambda^{(1)}}, \\ p^{(-\frac{1}{2})} &= \frac{1}{\gamma^{(\frac{3}{2})}}\left\{ q l^{(1)} - \gamma^{(\frac{1}{2})}p^{(\frac{1}{2})} + i\partial_\xi\gamma^{(\frac{3}{2})}\cdot\partial_x p^{(\frac{1}{2})}\right\}. \end{aligned}$$

Then

$$T_qT_\lambda \sim T_\gamma T_q, \quad T_qT_l \sim T_\gamma T_p, \quad T_\gamma \sim (T_\gamma)^\top.$$

Now we can write the para-linearisation and symmetrisation of the equations (1.5) after a change of variable:

Corollary 4.1. *(From [4])Under the hypothesis of Proposition 4.1, introduce the unknowns*

$$U = \psi - T_B\eta^{13}, \quad \Phi_1 = T_p\eta \quad \text{and} \quad \Phi_2 = T_qU.$$

Then $\Phi_1, \Phi_2 \in C^0([0,T]; H^s(\mathbb{R}^d))$ and

$$\begin{cases} \partial_t\Phi_1 + T_V\cdot\nabla\Phi_1 - T_\gamma\Phi_2 = f_1, \\ \partial_t\Phi_2 + T_V\cdot\nabla\Phi_2 + T_\gamma\Phi_1 = f_2, \end{cases} \tag{4.3}$$

with $f_1, f_2 \in L^\infty(0,T; H^s(\mathbb{R}^d))$, and f_1, f_2 have C^1 dependence on (U,θ) verifying:

$$\|(f_1,f_2)\|_{L^\infty(0,T;H^s(\mathbb{R}^d))} \leq C(\|(\eta,\psi)\|_{L^\infty(0,T;H^{s+\frac{1}{2}}\times H^s(\mathbb{R}^d))}).$$

[13] Recall B and V are defined by (1.7). U is commonly called the "good" unknown of Alinhac.

4.2. **Proof of Theorem 1.2.** Corollary 4.1 shows that the para-linearisation and symmetrisation of the equations (1.5) are of the form of the equations treated in Theorem 3.1. The goal of the proof is thus to mainly show that the previous change of unknowns preserves the quasi-linear structure of the equations. This we will be proved but with a slightly different change of unknowns that will satisfy the same type of equations.

4.2.1. *Reducing the problem around 0.*
Fix $T>0$, $r>0$ and $(\eta,\psi)\in C^0([0,T];H^{s+\frac{1}{2}}(\mathbb{R}^d)\times H^{s+\frac{1}{2}}(\mathbb{R}^d))$ a solution to (1.5) such that H_t is verified on $[0,T]$. As in the proof of Theorem 3.1, given the local nature of the result we see that first we can work on balls with radius r small, and by translation the equation is only modified by a term of order 0 and skew-symmetric term of order 1 and we stay in the same form of equations treated in Theorem 3.1. Henceforth we will be working on $\mathrm{B}(0,r)\subset C^0([0,T];H^{s+\frac{1}{2}}(\mathbb{R}^d)\times H^s(\mathbb{R}^d))$ and without loss of generality we suppose that H_t is always verified on $[0,T]$ on that set.

4.2.2. *New change of unknowns.*

Lemma 4.1. *Under the hypothesis of Proposition 4.1, fix $\epsilon>0$ and introduce the unknowns*

$$U=\psi-T_B\eta,\quad \tilde{\Phi}_1=[T_p+\epsilon(I-T_I)]\eta \quad \textit{and} \quad \tilde{\Phi}_2=[T_q+\epsilon(I-T_I)]U.$$

Then $\tilde{\Phi}_1,\tilde{\Phi}_2\in C^0([0,T];H^s(\mathbb{R}^d))$ and

$$\begin{cases}\partial_t\tilde{\Phi}_1+V\cdot\nabla\tilde{\Phi}_1-T_\gamma\tilde{\Phi}_2=\tilde{f}_1,\\ \partial_t\tilde{\Phi}_2+V\cdot\nabla\tilde{\Phi}_2+T_\gamma\tilde{\Phi}_1=\tilde{f}_2,\end{cases}\tag{4.4}$$

with $\tilde{f}_1,\tilde{f}_2\in L^\infty(0,T;H^s(\mathbb{R}^d))$, and $\tilde{f}_1,\tilde{f}_2$ have C^1 dependence on (U,θ) verifying:

$$\left\|(\tilde{f}_1,\tilde{f}_2)\right\|_{L^\infty(0,T;H^s(\mathbb{R}^d))}\leq C(\|(\eta,\psi)\|_{L^\infty(0,T;H^{s+\frac{1}{2}}\times H^s(\mathbb{R}^d))}).$$

Proof. The Lemma simply follows from the fact that $I-T_I$ is a regularizing operator and that symbolic calculus rules applied to $V-T_V$ specifically A.5. $\square$

4.2.3. *The new change of unknowns locally preserves the structure of the equations:*
To apply Theorem 3.1 the only non trivial hypothesis to verify is **H3** with $\beta=1,k=2$, which we do in Lemma 4.3, indeed we note that $DV(0,0)(h,k)=\nabla h$.

The proof of Theorem 1.2 in the threshold $s>2+\frac{d}{2}$ will then follow form Theorem 3.1 combined with Lemma 4.1 and the following Lemma.

Lemma 4.2. *Let $d\geq 1$ and $s>2+\frac{d}{2}$. There exists $r,\epsilon>0$ such that:*

$$\begin{aligned}\tilde{\Phi}:B(0,r)&\to C^0([0,T];H^s(\mathbb{R}^d))\\ (\eta,\psi)&\mapsto(\tilde{\Phi}_1,\tilde{\Phi}_2)\end{aligned}$$

is a C^∞ diffeomorphism upon it's image and $\tilde{\Phi}(0)$=0.

Proof.

$$\tilde{\Phi}(\eta,\psi)=\underbrace{\begin{pmatrix}T_p+\epsilon(I-T_I) & 0\\ 0 & T_q+\epsilon(I-T_I)\end{pmatrix}}_{(1)}\underbrace{\begin{pmatrix}I & 0\\ -T_B & I\end{pmatrix}}_{(2)}\begin{pmatrix}\eta\\ \psi\end{pmatrix}$$

(2) being clearly a diffeomorphism we will concentrate on (1).
First we see that for r small enough $T_q+\epsilon(I-T_I)$ is a perturbation of the identity,

indeed by symbolic calculus rules:

$$\begin{aligned}\|T_q+\epsilon(I-T_I)-I\|_{\mathcal{L}(H^s)} &= \|T_q-T_I+(\epsilon-1)(I-T_I)\|_{\mathcal{H}^f} \\ &\le 1-\epsilon+M_0^0(q-1) \\ &\le 1-\epsilon+C(\|\eta\|_{W^{1,\infty}})\,\|\eta\|_{W^{1,\infty}} \\ &\le 1-\epsilon+C(\|\eta\|_{H^s})\,\|\eta\|_{H^s}\end{aligned}$$

which gives the desired result.
Now we turn to $T_p+\epsilon(I-T_I)$. First notice that for $\epsilon>0$:

$$T_{|\xi|^{\frac{1}{2}}}+\epsilon(I-T_I): C^0([0,T];H^{s+\frac{1}{2}}(\mathbb{R}^d))\to C^0([0,T];H^s(\mathbb{R}^d))$$

is a C^∞ diffeomorphism. And now we see that $T_p+\epsilon(I-T_I)$ is a perturbation of $T_{|\xi|^{\frac{1}{2}}}+\epsilon(I-T_I)$ indeed by symbolic calculus rules:

$$\begin{aligned}\left\|T_p-T_{|\xi|^{\frac{1}{2}}}\right\|_{\mathcal{L}(H^{s+\frac{1}{2}},H^s)} &\le C(\|\eta\|_{W^{1,\infty}})\,\|\eta\|_{W^{1,\infty}} \\ &\le C(\|\eta\|_{H^s})\,\|\eta\|_{H^s}\end{aligned}$$

□

Now to conclude the proof of Theorem 1.2, we see that by Lemma 4.2, the equations (4.4) verify the hypothesis of Corollary 3.1 in the threshold $s>2+\frac{d}{2}$ thus we have two sequences

$$\begin{cases}\exists(\tilde\Phi_1^0,\tilde\Phi_2^0)\in C^0([0,T];H^s(\mathbb{R}^d)) & \text{solution of (4.4)},\\ \exists(\tilde\Phi_1^1,\tilde\Phi_2^1)\in C^0([0,T];H^s(\mathbb{R}^d)) & \text{solution of (4.4)},\end{cases}$$

such that $\exists c>0$

$$\begin{cases}\left\|(\tilde\Phi_1^0,\tilde\Phi_2^0)(0,\cdot)-(\tilde\Phi_1^1,\tilde\Phi_2^1)(0,\cdot)\right\|_{H^s}\to 0,\\ \left\|(\tilde\Phi_1^0,\tilde\Phi_2^0)-(\tilde\Phi_1^1,\tilde\Phi_2^1)\right\|_{L^\infty([0,T],H^s)}>c.\end{cases}$$

Now putting $(\eta^0,\psi^0)=\tilde\Phi^{-1}(\tilde\Phi_1^0,\tilde\Phi_2^0)$ and $(\eta^1,\psi^1)=\tilde\Phi^{-1}(\tilde\Phi_1^1,\tilde\Phi_2^1)$ we get from Lemmas 4.1 and 4.2:

$$\begin{cases}(\eta^0,\psi^0)\in C^0([0,T];H^{s+\frac{1}{2}}(\mathbb{R}^d)\times H^s(\mathbb{R}^d)) & \text{is a solution of (1.5)},\\ (\eta^1,\psi^1)\in C^0([0,T];H^{s+\frac{1}{2}}(\mathbb{R}^d)\times H^s(\mathbb{R}^d)) & \text{is a solution of (1.5)},\end{cases}$$

such that

$$\begin{cases}\left\|(\eta^0,\psi^0)(0,\cdot)-(\eta^1,\psi^1)(0,\cdot)\right\|_{H^{s+\frac{1}{2}}\times H^s}\to 0,\\ \left\|(\eta^0,\psi^0)-(\eta^1,\psi^1)\right\|_{L^\infty([0,T],H^{s+\frac{1}{2}}\times H^s)}>c.\end{cases}$$

thus giving us the desired result. As the change of unknowns is a diffeomorphism (thus is Lipschitz) we get analogously the result on the control in weaker norms.

4.2.4. *Verification of hypothesis* ***H3***. Writing

$$\gamma=\gamma^{(\frac{3}{2})}+\gamma^{(1)}+\gamma^{(\frac{1}{2})},$$

we see that to verify hypothesis **H3** with $\beta=1$ and $\kappa=2$, the only cancellation we have to prove is on $\gamma^{(\frac{3}{2})}$ which we do in the following Lemma:

Lemma 4.3. *Fix $(\tilde\Phi_1^0,\tilde\Phi_2^0)\in H^s(\mathbb{D}^d;\mathbb{R}^2)$ and take $r>0$, then there exists $T>0$ such that for all $(\tilde\Psi_1^0,\tilde\Psi_2^0)$ in the ball $B((\tilde\Phi_1^0,\tilde\Phi_2^0),r)\subset H^s(\mathbb{D}^d;\mathbb{R}^2)$ the Cauchy problem:*

$$\begin{cases}\partial_t\tilde\Psi_1-T_{\gamma^{(\frac{3}{2})}}\tilde\Psi_2=0,\\ \partial_t\tilde\Psi_2+T_{\gamma^{(\frac{3}{2})}}\tilde\Psi_1=0,\end{cases}\tag{4.5}$$

has a unique solution $(\tilde{\Psi}_1^0, \tilde{\Psi}_2^0) \in C([0,T], H^s(\mathbb{D}^d;\mathbb{R}^2))$*, where* γ *are defined via the diffeomorphism* $\tilde{\Phi}$*.*

Moreover the flow map verifies the estimate:

$$\begin{aligned}
&\left\|(\tilde{\Phi}_1,\tilde{\Phi}_2)-(\tilde{\Psi}_1,\tilde{\Psi}_2)(t,\cdot)\right\|_{H^s}\\
&\leq C\left(t\left\|(\tilde{\Phi}_1^0,\tilde{\Phi}_2^0)\right\|_{H^{s+\frac{1}{2}}}, t\left\|(\tilde{\Psi}_1^0,\tilde{\Psi}_2^0)\right\|_{H^{s+\frac{1}{2}}}\right)\left\|(\tilde{\Phi}_1^0,\tilde{\Phi}_2^0)-(\tilde{\Psi}_1^0,\tilde{\Psi}_2^0)\right\|_{H^s}\\
&+Ct\left\|(\tilde{\Phi}_1^0,\tilde{\Phi}_2^0)-(\tilde{\Psi}_1^0,\tilde{\Psi}_2^0)\right\|_{W^{1,\infty}}(\left\|(\tilde{\Phi}_1^0,\tilde{\Phi}_2^0)\right\|_{W^{1,\infty}}+\left\|(\tilde{\Psi}_1^0,\tilde{\Psi}_2^0)\right\|_{W^{1,\infty}})\left\|(\tilde{\Phi}_1^0,\tilde{\Phi}_2^0)\right\|_{H^{s+1}}.
\end{aligned}$$

Proof. the key idea is to see that there exists a diffeomorphism κ that depends quadratically on $(\tilde{\Psi}_1,\tilde{\Psi}_2)(t,\cdot)$ and it's first derivative conjugates $\gamma^{(\frac{3}{2})}$ to $|\xi|^{\frac{3}{2}}$. This a new generalization of a similar idea in dimension 1 applied in [5] that we do by exploiting the following algebraic structure of $\gamma^{(\frac{3}{2})}$:

$$\gamma^{(\frac{3}{2})}=[(A\xi)^\top A\xi]^{\frac{3}{2}}$$

where,

$$A(t,x,\eta,\psi)=\frac{1}{|\nabla\eta|^2}\times$$
$$\begin{pmatrix}
\frac{(\partial_1\eta)^2}{\sqrt{1+|\nabla\eta|^2}}+\sum\limits_{i\neq1}(\partial_i\eta)^2 & \frac{\partial_1\eta\partial_2\eta}{\sqrt{1+|\nabla\eta|^2}}-\partial_1\eta\partial_2\eta & \cdots & \frac{\partial_1\eta\partial_n\eta}{\sqrt{1+|\nabla\eta|^2}}-\partial_1\eta\partial_n\eta\\
\frac{\partial_1\eta\partial_2\eta}{\sqrt{1+|\nabla\eta|^2}}-\partial_1\eta\partial_2\eta & \frac{(\partial_2\eta)^2}{\sqrt{1+|\nabla\eta|^2}}+\sum\limits_{i\neq2}(\partial_i\eta)^2 & \cdots & \frac{\partial_n\eta\partial_2\eta}{\sqrt{1+|\nabla\eta|^2}}-\partial_n\eta\partial_2\eta\\
\vdots & \vdots & \ddots & \vdots\\
\frac{\partial_1\eta\partial_n\eta}{\sqrt{1+|\nabla\eta|^2}}-\partial_1\eta\partial_n\eta & \frac{\partial_n\eta\partial_2\eta}{\sqrt{1+|\nabla\eta|^2}}-\partial_n\eta\partial_2\eta & \cdots & \frac{(\partial_n\eta)^2}{\sqrt{1+|\nabla\eta|^2}}+\sum\limits_{i\neq n}(\partial_i\eta)^2
\end{pmatrix}\tag{4.6}$$

and (η,ψ) are defined via the diffeomorphism $\tilde{\Phi}$. to better understand A we make the following computation:

$$A(t,x,\eta,\psi)=\begin{cases}Id_d \text{ if } \nabla\eta=0,\\ \mathrm{Diag}_d(\frac{1}{\sqrt{1+|\nabla\eta|^2}},1,\cdot\cdot,1), \text{ in an orthonormal basis starting with } \frac{\nabla\eta}{|\nabla\eta|},\end{cases}$$

Put

$$\kappa(t,x,\eta,\psi)=\int_0^1 A(t,rx,\eta,\psi)xdr.$$

As $s>2+\frac{d}{2}$ and using spherical coordinates we see that A is defined on $\mathbb{R}^n$, $A\in W^{1,\infty}$ and $\kappa\in W^{2,\infty}$, moreover $D\kappa=A$ is positive definite thus κ is one to one and thus is a diffeomorphism.

Now we put

$$\begin{cases}U=(u_1,u_2)=\kappa^*_{(\tilde{\Phi}_1,\tilde{\Phi}_2)}(\tilde{\Phi}_1,\tilde{\Phi}_2)=\kappa^*_{\tilde{\Phi}}\tilde{\Phi},\\ V=(v_1,v_2)=\kappa^*_{(\tilde{\Psi}_1,\tilde{\Psi}_2)}(\tilde{\Psi}_1,\tilde{\Psi}_2)=\kappa^*_{\tilde{\Psi}}\tilde{\Psi},\end{cases}$$

where we use Alinhac's paracomposition operator for which we recall the definition in Appendix A. Computing the equations verified by (u_1,u_2) and (v_1,v_2) we get by definition of κ:

$$\begin{cases}\partial_t u_1-T_{|\xi|^{\frac{3}{2}}}u_2+T_{R_{\tilde{\Phi}}}u_1=0,\\ \partial_t u_2+T_{|\xi|^{\frac{3}{2}}}u_1+T_{R_{\tilde{\Phi}}}u_2=0,\end{cases}\tag{4.7}$$

$$\begin{cases}\partial_t v_1 p-T_{|\xi|^{\frac{3}{2}}}v_2+T_{R_{\tilde{\Psi}}}v_1=0,\\ \partial_t v_2+T_{|\xi|^{\frac{3}{2}}}v_1+T_{R_{\tilde{\Psi}}}v_2=0,\end{cases}\tag{4.8}$$

where $T_{R_{\tilde{\Phi}}} \in \Gamma_1^1(\mathbb{D}^d)$ verifies

$$M_0^1(D_{\tilde{\Phi}}R) \le C \left\|(\tilde{\Phi}_1, \tilde{\Phi}_2)\right\|_{W^{1,\infty}}. \tag{4.9}$$

Put

$$B(t,\xi) = \frac{i}{2}\begin{pmatrix} -e^{it\psi(\xi)|\xi|^{\frac{3}{2}}} - ie^{-it\psi(\xi)|\xi|^{\frac{3}{2}}} & e^{it\psi(\xi)|\xi|^{\frac{3}{2}}} - ie^{-it\psi(\xi)|\xi|^{\frac{3}{2}}} \\ e^{it\psi(\xi)|\xi|^{\frac{3}{2}}} - ie^{-it\psi(\xi)|\xi|^{\frac{3}{2}}} & -e^{it\psi(\xi)|\xi|^{\frac{3}{2}}} - ie^{-it\psi(\xi)|\xi|^{\frac{3}{2}}} \end{pmatrix}$$

where ψ is the the low frequency cutoff in the paradifferential operators definition. Solving the equations (4.7) and (4.7) we get:

$$U = B(t,D)U_0 + \int_0^t B(t-s)T_{R_{\tilde{\Phi}}}U(s,\cdot)ds, \tag{4.10}$$

$$V = B(t,D)V_0 + \int_0^t B(t-s)T_{R_{\tilde{\Psi}}}V(s,\cdot)ds. \tag{4.11}$$

Getting back to $\tilde{\Phi}, \tilde{\Psi}$ we get

$$(Id)^*\tilde{\Phi} = \kappa_{\tilde{\Phi}}^{-1*}B(t,D)\kappa_{\tilde{\Phi}(0)}{}^*\tilde{\Phi}(0) + \kappa_{\tilde{\Phi}}^{-1*}\int_0^t B(t-s)T_{R_{\tilde{\Phi}}}\kappa_{\tilde{\Phi}}{}^*\tilde{\Phi}(s,\cdot)ds,$$

$$(Id)^*\tilde{\Psi} = \kappa_{\tilde{\Psi}}^{-1*}B(t,D)\kappa_{\tilde{\Psi}(0)}{}^*\tilde{\Psi}(0) + \kappa_{\tilde{\Psi}}^{-1*}\int_0^t B(t-s)T_{R_{\tilde{\Psi}}}\kappa_{\tilde{\Psi}}{}^*\tilde{\Psi}(s,\cdot)ds,$$

thus by the calculus rules on the paracomposition operator,

$$(Id)^*\tilde{\Phi} = (\kappa_{\tilde{\Phi}}^{-1}\circ\kappa_{\tilde{\Phi}(0)})^*(\kappa_{\tilde{\Phi}(0)}^{-1}{}^*B)(t,D)\tilde{\Phi}(0) + \kappa_{\tilde{\Phi}}^{-1*}\int_0^t B(t-s)T_{R_{\tilde{\Phi}}}\kappa_{\tilde{\Phi}}{}^*\tilde{\Phi}(s,\cdot)ds, \tag{4.12}$$

$$(Id)^*\tilde{\Psi} = (\kappa_{\tilde{\Psi}}^{-1}\circ\kappa_{\tilde{\Psi}(0)})^*(\kappa_{\tilde{\Psi}(0)}^{-1}{}^*B)(t,D)\tilde{\Psi}(0) + \kappa_{\tilde{\Psi}}^{-1*}\int_0^t B(t-s)T_{R_{\tilde{\Psi}}}\kappa_{\tilde{\Psi}}{}^*\tilde{\Psi}(s,\cdot)ds. \tag{4.13}$$

Now by Theorem A.7 we have that

$$\kappa_{\tilde{\Phi}(0)}^{-1}{}^*B - \kappa_{\tilde{\Psi}(0)}^{-1}{}^*B \in \Gamma_1^{\frac{1}{2}}(\mathbb{D}^d)$$

and,

$$M_0^{\frac{1}{2}}(\kappa_{\tilde{\Phi}(0)}^{-1}{}^*B - \kappa_{\tilde{\Psi}(0)}^{-1}{}^*B) \le C\left\|(\tilde{\Phi}_1^0, \tilde{\Phi}_2^0) - (\tilde{\Psi}_1^0, \tilde{\Psi}_2^0)\right\|_{W^{1,\infty}}. \tag{4.14}$$

Computing the difference between (4.12) and (4.12), by the mean value theorem and the estimates (4.9) and (4.14) we get the desired estimate by noticing that:

$$\left\|(\tilde{\Phi}_1, \tilde{\Phi}_2) - (\tilde{\Psi}_1, \tilde{\Psi}_2)(t,\cdot)\right\|_{H^s} \le C\left\|(Id)^*[(\tilde{\Phi}_1, \tilde{\Phi}_2) - (\tilde{\Psi}_1, \tilde{\Psi}_2)(t,\cdot)]\right\|_{H^s}.$$

□

5. Quasi-Linearity of the Gravity Water Waves

In this section we always have $\kappa = 0$. The proof will follow as in the previous section but with some extra care, taking into account the lower regularity framework.

5.1. **Prerequisites from the Cauchy problem.** We start by recalling the apriori estimates given by Proposition 4.1 of [7], we keep the notations of Theorem 1.3.

Proposition 5.1. *(From [7]) Let $d \geq 1$ be the dimension and consider a real number $s > 1 + \frac{d}{2}$. Then there exists a non decreasing function C such that, for all $T \in]0,1]$ and all solution (η, ψ) of (1.5) such that:*

$$\begin{cases} (\eta, \psi) \in C^0([0,T]; H^{s+\frac{1}{2}}(\mathbb{R}^d) \times H^{s+\frac{1}{2}}(\mathbb{R}^d)), \\ H_t \textit{ is verified for } t \in [0,T], \\ \exists c_0 > 0, \forall t \in [0,T], a(t,x) \geq c_0, \end{cases}$$

we have [14]

$$\begin{aligned} &\|(\eta, \psi, V, B)\|_{L^\infty([0,T]; H^{s+\frac{1}{2}} \times H^{s+\frac{1}{2}} \times H^s \times H^s)} \\ &\leq C(\|(\eta_0, \psi_0, V_0, B_0)\|_{H^{s+\frac{1}{2}} \times H^{s+\frac{1}{2}} \times H^s \times H^s}) \\ &+ TC(\|(\eta, \psi, V, B)\|_{L^\infty(0,T; H^{s+\frac{1}{2}} \times H^{s+\frac{1}{2}} \times H^s \times H^s)}). \end{aligned}$$

The proof will rely on the para-linearised and symmetrized version of (1.5) given by Proposition 4.8 and 4.10 of [7]. Given the low regularity threshold, η and thus Ω_t are in $W^{\frac{3}{2},\infty}(\mathbb{R}^d)$ for the gravity water waves by contrast to $W^{\frac{5}{2},\infty}(\mathbb{R}^d)$ frame work for the case with surface tension, the para-linearisation of (1.5) is done with the variables V and B. This will only add a technical level to our proof of quasi-linearity.

Proposition 5.2. *(From [7])Under the hypothesis of Proposition (5.1), suppose moreover that $\|(V_0, B_0)\|_{H^s \times H^s} < +\infty$ thus by Proposition (5.1) this regularity is propagated on $[0,T]$. Now introduce the unknowns*

$$\begin{cases} \zeta = \nabla\eta, \\ U = V + T_\zeta B, \end{cases} \quad \textit{where,} \quad \begin{cases} B = (\partial_y \phi)_{|y=\eta} = \frac{\nabla\eta \cdot \nabla\psi + G(\eta)\psi}{1+|\nabla\eta|^2}, \\ V = (\nabla_x \phi)_{|y=\eta} = \nabla\psi - B\nabla\eta. \end{cases}$$

Now define the symbols:

$$\begin{cases} \lambda = \sqrt{(1+|\nabla\eta|^2)\,|\xi|^2 - (\nabla\eta \cdot \xi)^2}, \\ \gamma = \sqrt{a\lambda}, \\ q = \sqrt{\frac{a}{\lambda}}. \end{cases}$$

Set $\theta = T_q \zeta$. Then $\theta, U \in C^0([0,T]; H^s(\mathbb{R}^d))$ and

$$\begin{cases} \partial_t U + T_V \cdot \nabla U + T_\gamma \theta = f_1, \\ \partial_t \theta + T_V \cdot \nabla\theta - T_\gamma U = f_2, \end{cases} \tag{5.1}$$

with $f_1, f_2 \in L^\infty(0,T; H^s(\mathbb{R}^d))$, and f_1, f_2 have C^1 dependence on (U, θ) verifying:

$$\|(f_1, f_2)\|_{L^\infty(0,T;H^s)} \leq C(\|(\eta, \psi, V, B)\|_{L^\infty(0,T; H^{s+\frac{1}{2}} \times H^{s+\frac{1}{2}} \times H^s \times H^s)})$$

5.2. **Proof of Theorem 1.3.** As in the proof of Theorem 1.2, Proposition (5.2) shows that the para-linearisation and symmetrisation of the Equations (1.5) are of the form of the equations treated in Theorem 3.1. Thus again, the goal of the proof is thus to mainly show that the previous change of unknowns preserves the quasi-linear structure of the equations. This we will be proved but with a slightly different change of unknowns that will satisfy the same type of equations but where we take into account the low frequencies. For concision we will omit the $(\mathbb{R}^d)$ when writing the functional spaces.

[14] Recall B and V are defined by (1.7).

5.2.1. *Reducing the problem around 0.*
Fix $T>0$, $r>0$ and $(\eta,\psi)\in C^0([0,T];H^{s+\frac{1}{2}}\times H^{s+\frac{1}{2}})$ a solution to (1.5) verifying the hypothesis of Theorem 1.3. As in the proof of Theorem 3.1 and 1.2, given the local nature of the result we see that first we can work on balls with radius r small, and by translation the equation is only modified by a term of order 0 and skew-symmetric term of order 1 and we stay in the same form of equations treated in Theorem 3.1. Put

$$I_{s,T}=\left\{(\eta,\psi)\in C^0([0,T];H^{s+\frac{1}{2}}\times H^{s+\frac{1}{2}}),(V,B)\in C^0([0,T];H^s\times H^s),\exists c>0,a\geq c\right\},$$
$$I_{s,0}=\left\{(\eta_0,\psi_0)\in H^{s+\frac{1}{2}}\times H^{s+\frac{1}{2}},(V_0,B_0)\in H^s\times H^s,\exists c>0,a\geq c\right\},$$

henceforth we will be working on $\mathrm{B}(0,r)\subset I_{s,T}$ and without loss of generality we suppose that H_t is always verified on $[0,T]$, on that set.

5.2.2. *New change of unknowns.*

Lemma 5.1. *Consider $\epsilon>0$ and $\omega\in C_0^\infty(\mathbb{R}^d)$ such that $\omega=1$ on $B(0,1)$ and $\omega=0$ on $\mathbb{R}^d\setminus B(0,2)$. Under the hypothesis of Proposition (5.2), introduce the unknowns*

$$\begin{cases}\tilde{\zeta}=(1-\omega(D))\nabla\eta,\\ \tilde{U}=(1-\omega(D))(V+T_\zeta B),\\ aux_1=\omega(D)\psi,\\ aux_2=\omega(D)\eta,\end{cases}\quad \textit{where,}\quad \begin{cases}B=(\partial_y\phi)_{|y=\eta}=\frac{\nabla\eta\cdot\nabla\psi+G(\eta)\psi}{1+|\nabla\eta|^2},\\ V=(\nabla_x\phi)_{|y=\eta}=\nabla\psi-B\nabla\eta.\end{cases}$$

and set $\tilde{\theta}=T_q\tilde{\zeta}+\epsilon(I-T_I)$, where q is defined in Proposition (5.2).
Then $\tilde{\theta},U,aux_1,aux_2\in C^0([0,T];H^s)$ and

$$\begin{cases}\partial_t\tilde{U}+V\cdot\nabla\tilde{U}+T_\gamma\tilde{\theta}=f_1',\\ \partial_t\tilde{\theta}+V\cdot\nabla\tilde{\theta}-T_\gamma\tilde{U}=f_2',\end{cases}\tag{5.2}$$

with $f_1',f_2'\in L^\infty(0,T;H^s)$, and f_1',f_2' have C^1 dependence on (U,θ) verifying:

$$\left\|(f_1',f_2')\right\|_{L^\infty(0,T;H^s)}\leq C(\|(\eta,\psi,V,B)\|_{L^\infty(0,T;H^{s+\frac{1}{2}}\times H^{s+\frac{1}{2}}\times H^s\times H^s)})$$

Proof. Again the lemma simply follows from the fact that $I-T_I$ and $\omega(D)$ are regularizing operator and that symbolic calculus rules applied to $V-T_V$ specifically A.5. □

5.2.3. *Decomposing the change of variable:* Set

$$\begin{aligned}\Phi:I_{s,T}\times H^s&\to C^0([0,T];H^s) & \Phi:I_{s,0}\times H^s&\to H^s\\ (\eta,\psi)&\mapsto(\tilde{U},\tilde{\theta},aux_1,aux_2) & (\eta,\psi)&\mapsto(\tilde{U},\tilde{\theta},aux_1,aux_2)\end{aligned}$$

The goal is to prove that Φ is locally invertible and then the proof will follow from Theorem 3.1.
We write $\Phi=\Phi_1\circ\Phi_2$ with

$$\begin{aligned}\Phi_1:I_{s,T}&\to C^0([0,T];H^s\times H^{s-\frac{1}{2}}\times H^s\times H^s)\\ (\eta,\psi)&\mapsto(\tilde{U},\tilde{\zeta},aux_1,aux_2)\end{aligned}$$

and,

$$\begin{aligned}\Phi_2:C^0([0,T];H^s\times H^{s-\frac{1}{2}}\times H^s\times H^s)&\to C^0([0,T];H^s)\\ (\tilde{U},\tilde{\zeta},aux_1,aux_2)&\mapsto(\tilde{U},\tilde{\theta},aux_1,aux_2)\end{aligned}$$

We define Φ_1 and Φ_2 analogously when Φ is defined on $I_{s,0}$.

Lemma 5.2. *There exists $r, r_1, \epsilon > 0$ such that:*

$$\Phi_1 : B(0,r) \cap I_{s,T} \to C^0([0,T]; H^s \times H^{s-\frac{1}{2}} \times H^s \times H^s)$$

is a C^∞ diffeomorphism upon it's image.

$$\Phi_2 : B(0,r_1) \cap C^0([0,T]; H^s \times H^{s-\frac{1}{2}} \times H^s \times H^s) \to C^0([0,T]; H^s)$$

is a C^∞ diffeomorphism upon it's image.
Analogous result hold when Φ is defined on $I_{s,0}$.

The proof of Theorem 1.3 follows as in the previous section from Corollary 3.1 and the previous Lemma combined with the fact that $\Phi_1(0) = 0$ thus we have

$$\mathrm{B}(0,r_1)\cap C^0([0,T]; H^s\times H^{s-\frac{1}{2}}\times H^s\times H^s) \subset \Phi_1\Big(\mathrm{B}(0,r)\cap C^0([0,T]; H^{s+\frac{1}{2}}\times H^{s+\frac{1}{2}})\Big).$$

Also $\Phi_2(0) = 0$ thus there exists r_2:

$$\mathrm{B}(0,r_2)\cap C^0([0,T]; H^s) \subset \Phi_2\Big(\mathrm{B}(0,r_1)\cap C^0([0,T]; H^s\times H^{s-\frac{1}{2}}\times H^s\times H^s\Big).$$

We now turn to the proof of the lemma.

Proof. As all of the estimates used are punctual in time thus the proof is the same for $I_{s,T}$ and $I_{s,0}$ and we only write the one for $I_{s,T}$. We start by Φ_1, first the part $\eta \mapsto (\tilde{\zeta}, aux_2)$ is invertible with inverse

$$\mathscr{F}[\Phi_1^{-1}(\tilde{\zeta}, aux_2)](\xi) = \frac{1}{d}\sum_j (1-\omega(\xi))\frac{\mathscr{F}[\partial_i\tilde{\zeta}](\xi)}{i\xi_j} + \omega(\xi)\mathscr{F}[aux_2].$$

By the same argument $\psi \mapsto ((1-\omega(D))\nabla\psi, \omega(D)\psi)$ is invertible and we see that $(\tilde{U}, aux_1)$ is a perturbation of that map indeed:

$$\begin{aligned}\Big\|((1-\omega(D))\nabla\psi, \omega(D)\psi) - (\tilde{U}, aux_1)\Big\|_{\mathcal{L}(H^{s+\frac{1}{2}}, H^s)} &\leq C(\|B\|_{W^{\frac{1}{2},\infty}})\,\|\eta\|_{H^{s+\frac{1}{2}}} \\ &\leq C(\|(\eta,\psi)\|_{H^{s+\frac{1}{2}}})\,\|\eta\|_{H^{s+\frac{1}{2}}}\end{aligned}$$

thus for r small enough we get the desired result.
Now we turn to Φ_2. This operator is the identity on $\tilde{U}, aux_1, aux_2$ thus we only have to work on $\tilde{\theta}$. Put a_0 as the Taylor coefficient associated to the solution of the problem (0,0). Now notice that for $\epsilon > 0$:

$$T_{\sqrt{a_0}|\xi|^{-\frac{1}{2}}} + \epsilon(I - T_I) : C^0([0,T]; H^{s-\frac{1}{2}}) \to C^0([0,T]; H^s)$$

is a C^∞ diffeomorphism. And now we see that $T_q + \epsilon(I - T_I)$ is a perturbation of $T_{\sqrt{a_0}|\xi|^{-\frac{1}{2}}}$ indeed by symbolic calculus rules:

$$\Big\|T_q - T_{\sqrt{a_0}|\xi|^{-\frac{1}{2}}}\Big\|_{\mathcal{L}(H^{s-\frac{1}{2}}, H^s)} \leq C(\|\eta\|_{H^s})\,\|\eta\|_{H^s},$$

which gives the result by taking r small. $\square$

Appendix A. Pseudodifferential and Paradifferential operators

In this paragraph we review classic notations and results about pseudodifferential and paradifferential calculus that we need in this paper. We follow the presentations in [12] , [14], and [13] which give an accessible presentation.

A.1. **Notations and functional analysis.** We present the definitions of the functional spaces that will be used.
We will use the usual definitions and standard notations for the regular functions C^k, C_0^k for those with compact support, the distribution space $\mathbb{D}'$,$\mathscr{E}'$ for those with compact support, $\mathbb{D}'^k$,$\mathscr{E}'^k$ for distributions of order k, Lebesgue spaces (L^p), Sobolev spaces ($H^s, W^{p,q}$) and the Schwartz class $\mathscr{S}$ and it's dual $\mathscr{S}'$. All of those spaces are equipped with their standard topology.

Definition A.1 (Littlewood-Paley decomposition)**.** *Pick $P_0 \in C_0^\infty(\mathbb{R}^d)$ so that $P_0(\xi) = 1$ for $|\xi| < 1$ and 0 for $|\xi| > 2$. We define a dyadic decomposition of unity by:*

$$for\ k \geq 1, P_{\leq k}(\xi) = \Phi_0(2^{-k}\xi), \quad P_k(\xi) = P_{\leq k}(\xi) - P_{\leq k-1}(\xi).$$

Thus,

$$P_{\leq k}(\xi) = \sum_{0 \leq j \leq k} P_j(\xi) \quad and \quad 1 = \sum_{j=0}^{\infty} P_j(\xi).$$

Introduce the operator acting on $\mathscr{S}'(\mathbb{R}^d)$:

$$P_{\leq k} u = \mathscr{F}^{-1}(P_{\leq k}(\xi) u) \quad and \quad u_k = \mathscr{F}^{-1}(P_k(\xi) u).$$

Thus,

$$u = \sum_k u_k.$$

Finally put $\{k \geq 1, C_k = \mathrm{supp}\ \ P_k\}$ the set of rings associated to this decomposition.

Remark A.1. *An interesting property of the Littlewood-Paley decomposition is that even if the decomposed function is merely a distribution the terms of the decomposition are regular, indeed they all have compact spectrum and thus are entire functions. On classical functions spaces this regularization effect can be "measured" by the following inequalities due to Bernstein.*

Proposition A.1 (Bernstein's inequalities)**.** *Suppose that $a \in L^p(\mathbb{R}^d)$ has its spectrum contained in the ball $\{|\xi| \leq \lambda\}$. Then $a \in C^\infty$ and for all $\alpha \in \mathbb{N}^d$ and $q \geq p$, there is $C_{\alpha,p,q}$ (independent if λ) such that*

$$\|\partial_x^\alpha a\|_{L^q} \leq C_{\alpha,p,q} \lambda^{|\alpha| + \frac{d}{p} - \frac{d}{q}} \|a\|_{L^p}.$$

In particular,

$$\|\partial_x^\alpha a\|_{L^q} \leq C_\alpha \lambda^{|\alpha|} \|a\|_{L^p}, \quad and\ for\ p = 2,\ p = \infty$$

$$\|a\|_{L^\infty} \leq C \lambda^{\frac{d}{2}} \|a\|_{L^2}.$$

Proposition A.2. *For all $\mu > 0$, there is a constant C such that for all $\lambda > 0$ and for all $\alpha \in W^{\mu,\infty}$ with spectrum contained in $\{|\xi| \geq \lambda\}$. one has the following estimate:*

$$\|a\|_{L^\infty} \leq C \lambda^{-\mu} \|a\|_{W^{\mu,\infty}}.$$

Definition A.2 (Sobolev spaces on $\mathbb{R}^d$)**.** *It is also a classical result that for $s \in \mathbb{R}$:*

$$H^s(\mathbb{R}^d) = \left\{ u \in \mathscr{S}'(\mathbb{R}^d), |u|_s = \left(\sum_q 2^{2qs} \|u_q\|_{L^2}{}^2 \right)^{\frac{1}{2}} < \infty \right\}$$

with the right hand side equipped with its canonical topology giving it a Hilbert space structure and $|\ |_s$ is equivalent to the usual norm on $\|\ \|_{H^s}$.

Proposition A.3. *Let B be a ball with center 0. There exists a constant C such that for all $s > 0$ and for all $(u_q)_{\in \mathbb{N}} \in \mathscr{S}'(\mathbb{R}^d)$ verifying:*

$$\forall q, \text{supp } \hat{u_q} \subset 2^q B \textit{ and } (2^{qs} \|u_q\|_{L^2})_{q\in\mathbb{N}} \textit{ is in } L^2(\mathbb{N})$$

$$\textit{then, } u = \sum_q u_q \in H^s(\mathbb{R}^d) \textit{ and } |u|_s \leq \frac{C}{1-2^{-s}} \left(\sum_q 2^{2qs} \|u_q\|_{L^2}{}^2 \right)^{\frac{1}{2}}.$$

Remark A.2. *The previous definition and properties of the Littlewood-Paley decomposition and Sobolev spaces carries out naturally to $\mathbb{T}^d$.*

A.2. **Pseudodifferential operators.** We introduce here the basic definitions and symbolic calculus results. We first introduce the classes of regular symbols.

Definition A.3. *Given $m \in \mathbb{R}, 0 \leq \rho \leq 1$ and $0 \leq \sigma \leq 1$ we denote the symbol class $S^m_{\rho,\sigma}(\mathbb{R}^d \times \mathbb{R}^d)$ the set of all $a \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ such that for all multi-orders α, β we have the estimate:*

$$\left|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)\right| \leq C_{\alpha,\beta}(1+|\xi|)^{m-\rho\beta+\sigma\alpha}.$$

$S^m_{\rho,\sigma}(\mathbb{R}^d \times \mathbb{R}^d)$ is a Fréchet space with the topology defined by the family of semi-norms:

$$M^m_{\beta,\alpha}(a) = \sup_{\mathbb{R}^d\times\mathbb{R}^d} \left|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)(1+|\xi|)^{\rho\beta-m-\sigma\alpha}\right|.$$

Set

$$S^m(\mathbb{R}^d \times \mathbb{R}^d) = S^m_{1,0}(\mathbb{R}^d \times \mathbb{R}^d),$$

$$S^{-\infty}(\mathbb{R}^d \times \mathbb{R}^d) = \bigcap_{m\in\mathbb{R}} S^m(\mathbb{R}^d \times \mathbb{R}^d) \quad \textit{and} \quad S^{+\infty}(\mathbb{R}^d \times \mathbb{R}^d) = \bigcup_{m\in\mathbb{R}} S^m(\mathbb{R}^d \times \mathbb{R}^d)$$

equipped with their canonically induced topology.

Given a symbol $a \in S^m(\mathbb{R}^d \times \mathbb{R}^d)$, we define the pseudodifferential operator:

$$\text{Op}(a)u(x) = a(x,D)u(x) = (2\pi)^{-n} \int_{\mathbb{R}^d} e^{ix.\xi} a(x,\xi)\hat{u}(\xi)d\xi.$$

For $u \in \mathscr{S}(\mathbb{R}^d)$ we have

$$\begin{aligned}
\text{Op}(a)u(x) &= (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ix.\xi} a(x,\xi)\hat{u}(\xi)d\xi \\
&= (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ix.\xi} a(x,\xi) \int_{\mathbb{R}^d} e^{-iy.\xi} u(y)dyd\xi \\
&= \int_{\mathbb{R}^d} \left((2\pi)^{-n} \int_{\mathbb{R}^d} e^{i(x-y).\xi} a(x,\xi)d\xi \right) u(y)dy
\end{aligned}$$

Thus giving us the following Proposition.

Proposition A.4. *For $a \in S^m(\mathbb{R}^d \times \mathbb{R}^d)$, $\text{Op}(a)$ has a kernel K defined by*

$$K(x,y) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i(x-y).\xi} a(x,\xi)d\xi = (2\pi)^{-n} \mathscr{F}_\xi a(x, y-x). \tag{A.1}$$

Which can be inverted to give:

$$\begin{aligned}
a(x,\xi) &= \mathscr{F}_{y\to\xi} K(x, x-y) = \int_{\mathbb{R}^d} e^{-iy.\xi} K(x,x-y)dy \\
&= (-1)^d e^{-ix.\xi} \int_{\mathbb{R}^d} e^{iy.\xi} K(x,y)dy
\end{aligned} \tag{A.2}$$

Definition A.4. *Let $m \in \mathbb{R}$, an operator T is said to be of order m if, and only if, for all $\mu \in \mathbb{R}$, it is bounded from $H^\mu(\mathbb{R}^d)$ to $H^{\mu-m}(\mathbb{R}^d)$.*

Theorem A.1. *If $a \in S^m(\mathbb{R}^d \times \mathbb{R}^d)$, then $a(x, D)$ is an operator of order m. Moreover we have the norm estimate:*

$$\|a(x,D)\|_{H^\mu \to H^{\mu-m}} \leq C M^m_{\mu,m+d/2+1}(a)$$

We will now present the main results in symbolic calculus associated to pseudodifferential operators.

Theorem A.2. *Let $m, m' \in \mathbb{R}^d$, $a \in S^m(\mathbb{R}^d \times \mathbb{R}^d)$ and $b \in S^{m'}(\mathbb{R}^d \times \mathbb{R}^d)$.*

- *Composition: Then $\mathrm{Op}(a) \circ \mathrm{Op}(b)$ is a pseudodifferential operator of order $m + m'$ with symbol $a\#b$ defined by:*

$$a\#b(x,\xi) = (2\pi)^{-d} \int_{\mathbb{R}^d \times \mathbb{R}^d} e^{i(x-y).(\xi-\eta)} a(x,\eta) b(y,\xi) dy d\eta$$

  Moreover,

$$\mathrm{Op}(a) \circ \mathrm{Op}(b)(x,\xi) - \mathrm{Op}\Big(\sum_{|\alpha|<k} \frac{1}{i^{|\alpha|}\alpha!} (\partial_\xi^\alpha a(x,\xi))(\partial_x^\alpha b(x,\xi))\Big) \text{ is of order } m + m' - k$$

  for all $k \in \mathbb{N}$.
- *Adjoint: The adjoint operator of $\mathrm{Op}(a)$, $\mathrm{Op}(a)^\top$ is a pseudodifferential operator of order m with symbol $a^\top$ defined by:*

$$a^\top(x,\xi) = (2\pi)^{-d} \int_{\mathbb{R}^d \times \mathbb{R}^d} e^{-iy.\xi} a(x-y, \xi-\eta) dy d\eta$$

  Moreover,

$$\mathrm{Op}(a^\top)(x,\xi) - \mathrm{Op}\Big(\sum_{|\alpha|<k} \frac{1}{i^{|\alpha|}\alpha!} (\partial_\xi^\alpha \partial_x^\alpha \bar{a}(x,\xi))\Big) \text{ is of order } m - k$$

  for all $k \in \mathbb{N}$.

Definition A.5. *Let $(a_j) \in S^{m_j}(\mathbb{R}^d \times \mathbb{R}^d)$ be a series of symbols with $(m_j) \in \mathbb{R}^d$ decreasing to $-\infty$. We say that $a \in S^{m_0}(\mathbb{R}^d \times \mathbb{R}^d)$ is the asymptotic sum of (a_j) if*

$$\forall k \in \mathbb{N}, a - \sum_{j=0}^{k} a_j \in S^{m_{k+1}}(\mathbb{R}^d).$$

We denote $a \sim \sum a_j$

Remark A.3. *We can now write simply:*

$$a\#b \sim \sum_{|\alpha|} \frac{1}{i^{|\alpha|}\alpha!} (\partial_\xi^\alpha a(x,\xi))(\partial_x^\alpha b(x,\xi))$$

and

$$a^\top \sim \sum_{|\alpha|} \frac{1}{i^{|\alpha|}\alpha!} (\partial_\xi^\alpha \partial_x^\alpha \bar{a}(x,\xi)).$$

Pseudodifferential operators have an interesting property of spectral localization called the Pseudolocal property.

Proposition A.5. *Let $a \in S^m(\mathbb{R}^d \times \mathbb{R}^d)$ and let K be its kernel. Then K is C^∞ for $x \neq y$. In particular, for all $u \in \mathscr{S}'$:*

$$\text{sing supp}\, a(x,D)u \subset \text{sing supp}\, u,$$

where we recall the definition of singular support: $f \in \mathscr{S}'(\mathbb{R}^d)$ is said to be C^∞ in a neighborhood of x, if there exists a neighborhood ω of x such that for all $\psi \in C_0^\infty(\omega)$ we have $\psi f \in C^\infty(\mathbb{R}^d)$.
The singular support of a distribution f, $\text{sing supp}\, f$, is defined as the complementary of such points and is clearly closed.

Proof. Let $x \neq y$, $\psi, \theta \in C_0^\infty(\mathbb{R}^d)$,$\psi = 1$ near x, $\theta = 1$ near y and supp $\psi \cap$supp $\theta = \emptyset$. Then $\tilde{K}(x, y) = \psi(x)K(x, y)\theta(y)$ is the kernel of the operator $\psi a\theta$. By Theorem A.2, $\psi a\theta \sim 0$ thus is of order $-\infty$ which finishes the proof. □

Let Ω be an open subset of $\mathbb{R}^d$. We will now define the notion of local symbols and operators in an open set.

Definition A.6. *We define $S^m(\Omega \times \mathbb{R}^d)$ to be the set of $a \in C^\infty(\Omega \times \mathbb{R}^d)$ such that for all multi-orders α, β we have the estimate:*

$$\left|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)\right| \leq C_{\alpha,\beta}(1 + |\xi|)^{m-\rho\beta+\sigma\alpha}.$$

$S^m(\Omega^d \times \mathbb{R}^d)$ is a Fréchet space with the topology defined by the family of semi-norms:

$$M^m_{\beta,\alpha}(a) = \sup_{\Omega\times\mathbb{R}^d} \left|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)(1 + |\xi|)^{\rho\beta-m-\sigma\alpha}\right|.$$

We define the local spaces:

$$S^m_{loc}(\Omega \times \mathbb{R}^d) = \left\{a \in C^\infty(\Omega \times \mathbb{R}^d), \forall\psi \in C_0^\infty(\Omega), \psi a \in S^m(\Omega \times \mathbb{R}^d)\right\},$$

equipped with it's canonical topology giving it a Fréchet space structure.

If $a \in S^m(\Omega \times \mathbb{R}^d)$ or $S^m_{loc}(\Omega \times \mathbb{R}^d)$, the usual formula

$$Au(x) = a(x, D)u(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ix.\xi} a(x, \xi)\hat{u}(\xi)d\xi$$

defines an operator respectively from $\mathscr{S}'(\mathbb{R}^d)$, $\mathscr{E}'(\Omega)$ to $\mathbb{D}'(\Omega)$, which can be restricted to an operator $\mathscr{E}'(\Omega) \to \mathbb{D}'(\Omega)$ and $C_0^\infty(\Omega) \to C^\infty(\Omega)$.
The link between such operators and the operators obtained by cut-off from global operators is given by the following Proposition:

Proposition A.6. *Let $A : v \to C^\infty(\Omega)$ be a continuous linear operator such that for all $\psi, \theta \in C_0^\infty(\Omega)$, $\psi A\theta \in \mathrm{Op}(S^m)$. Then there exists $a' \in S^m(\Omega \times \mathbb{R}^d)$ with A=a'(x,D)+R, where R is an operator with kernel in $C^\infty(\Omega \times \Omega)$.*

Proof. Let $(\psi_j) \in C_0^\infty(\Omega)$ be a partition of unity locally finite over Ω. Put $\psi_j A\psi_k = A_{jk} \in \mathrm{Op}(S^m)$ then

$$Au = \sum_{j,k} \psi_j A\psi_k = \sum_{\substack{j,k \\ \text{supp } \psi_j\cap\psi_k\neq\emptyset}} A_{jk} + \sum_{\substack{j,k \\ \text{supp } \psi_j\cap\psi_k=\emptyset}} A_{jk}.$$

Then

$$a' = \sum_{\substack{j,k \\ \text{supp } \psi_j\cap\psi_k\neq\emptyset}} A_{jk} \in S^m(\Omega \times \mathbb{R}^d)$$

because for $\forall\psi \in C_0^\infty(\Omega), \psi a'$ is a finite sum by definition of a partition of unity locally finite.
The remainder has a kernel:

$$\sum_{\substack{j,k, \\ \text{supp } \psi_j\cap\psi_k=\emptyset}} \psi_j(x)K(x, y)\psi_k(y) \in C^\infty(\Omega \times \Omega)$$

by the pseudo-local property, Proposition A.5. □

We see from the previous definition that there is a subtlety with the support of the functions if one wants for example to define $A^\top$. The following class of local operators clarifies that problem:

Definition A.7. *A continuous linear operator $A : C_0^\infty(\Omega) \to C^\infty(\Omega)$ is said to be properly supported if, for any compact subset $K \subset \Omega$, there exists a compact subset $K' \subset \Omega$ with:*

$$\text{supp }\ u \subset K \Longrightarrow \text{supp }\ Au \subset K' \quad \textit{and} \quad u = 0 \quad \textit{on} \quad K' \Longrightarrow Au = 0 \quad \textit{on} \quad K$$

We see that such an operator maps C_0^∞ to C_0^∞ and for example $A^\top$ can be extended in a standard way to an operator from $\mathbb{D}'(\Omega)$ to itself.

Proposition A.7. *Let $A = a(x, D)$ where $a \in S^m_{loc}(\Omega \times \mathbb{R}^d)$. There exists an operator R with kernel in $C^\infty(\Omega \times \Omega)$ such that $A+R$ is properly supported.*

Proof. This is the same proof as Proposition A.6 because

$$\sum_{\substack{j,k,\\ \text{supp}\, \psi_j \cap \psi_k = \emptyset}} A_{jk}$$

is properly supported. $\square$

Remark A.4. *The previous Proposition tells us that for local regularity considerations we can essentially work with properly supported operators for local symbols (modulo a C^∞ kernel) and by Proposition A.6 we can do the same for operators obtained by cut-off.*

Now we present the classic results of change of variables in pseudodifferential operators.

Theorem A.3. *Let $\chi : \Omega \to \Omega'$ be a C^∞ diffeomorphism and $A = a(x, D) \in S^m_{loc}(\Omega' \times \mathbb{R}^d)$ a properly supported pseudodifferential operator with kernel K.*
Then the operator A^* defined by K^* i.e:

$$\forall u \in C_0^\infty(\Omega), A^* u = \int_\Omega K(\chi(x), \chi(y)) u(y) |det D\chi(y)| dy$$

is a properly supported pseudodifferential operator with symbol

$$a^*(x, \xi) = (-1)^d e^{-ix.\xi} \int_{\Omega \times \mathbb{R}^d} a(\chi(x), \eta) e^{i(\chi(x) - \chi(y).\eta + iy.\xi} |det D\chi(y)| dy d\eta \in S^m_{loc}(\Omega \times \mathbb{R}^d),$$

and verifies

$$(\mathrm{Op}(a)u) \circ \chi = \mathrm{Op}(a^*)(u \circ \chi).$$

An expansion of a^* is given by:

$$a^*(x, \xi) \sim \sum_\alpha \frac{1}{\alpha} \partial^\alpha a(\chi(x), D\chi^{-1}(\chi(x))^\top \xi) P_\alpha(\chi(x), \xi), \tag{A.3}$$

where,

$$P_\alpha(x', \xi) = D^\alpha_{y'}(e^{i(\chi^{-1}(y') - \chi^{-1}(x') - D\chi^{-1}(x')(y' - x')).\xi})_{|y' = x'}$$

and P_α is polynomial in ξ of degree $\leq \frac{|\alpha|}{2}$, with $P_0 = 1, P_1 = 0$.

Remark A.5. *A standard implication of Theorem A.3, is the generalization of pseudodifferential calculus to manifolds and thus to $\mathbb{D}^d$.*

A.3. **Paradifferential operators.** We start by the definition of symbols with limited spatial regularity. Let $\mathscr{W} \subset \mathscr{S}'$ be a Banach space.

Definition A.8. *Given $m \in \mathbb{R}$, $\Gamma^m_{\mathscr{W}}(\mathbb{R}^d)$ denotes the space of locally bounded functions $a(x,\xi)$ on $\mathbb{R}^d \times (\mathbb{R}^d \setminus 0)$, which are C^∞ with respect to ξ for $\xi \neq 0$ and such that, for all $\alpha \in \mathbb{N}^d$ and for all $\xi \neq 0$, the function $x \mapsto \partial_\xi^\alpha a(x,\xi)$ belongs to $\mathscr{W}$ and there exists a constant C_α such that,*

$$\forall\, |\xi| > \frac{1}{2}, \left\|\partial_\xi^\alpha a(.,\xi)\right\|_{\mathscr{W}} \leq C_\alpha (1+|\xi|)^{m-|\alpha|}$$

Given a symbol a, define the paradifferential operator T_a by

$$\widehat{T_a u}(\xi) = (2\pi)^{-d} \int_{\mathbb{R}^d} \theta(\xi-\eta,\eta)\hat{a}(\xi-\eta,\eta)\psi(\eta)\hat{u}(\eta)d\eta,$$

where $\hat{a}(\eta,\xi) = \int e^{-ix.\eta} a(x,\xi)dx$ is the Fourier transform of a with respect to the first variable; θ and ψ are two fixed C^∞ functions such that:

$$\psi(\eta) = 0 \quad for \quad |\eta| \leq 1, \qquad \psi(\eta) = 1 \quad for \quad |\eta| \geq 2,$$

and $\theta(\xi,\eta)$ is homogeneous of degree 0 and satisfies for $0 < \epsilon_1 < \epsilon_2$ small enough,

$$\theta(\xi,\eta) = 1 \quad if \quad |\xi| \leq \epsilon_1 |\eta|, \quad \theta(\xi,\eta) = 0 \quad if \quad |\xi| \geq \epsilon_2 |\eta|.$$

For quantitative estimates we introduce as in [13]:

Definition A.9. *For $m \in \mathbb{R}$, $\rho \geq 0$ and $a \in \Gamma^m_{\mathscr{W}}(\mathbb{R}^d)$, we set*

$$M^m_{\mathscr{W}}(a) = \sup_{|\alpha| \leq \frac{d}{2}+1+c} \sup_{|\xi| \geq \frac{1}{2}} \left\| (1+|\xi|)^{m-|\alpha|} \partial_\xi^\alpha a(.,\xi) \right\|_{\mathscr{W}}, \quad with \quad c > 0.$$

We will essentially work with $\mathscr{W} = W^{\rho,\infty}$ and write $M^m_{W^{\rho,\infty}}(a) = M^m_\rho(a)$ with $c = \rho$.

The main features of symbolic calculus for paradifferential operators are given by the following Theorems.

Theorem A.4. *Let $m \in \mathbb{R}$. if $a \in \Gamma^m_0(\mathbb{R}^d)$, then T_a is of order m. Moreover, for all $\mu \in \mathbb{R}$ there exists a constant K such that*

$$\|T_a\|_{H^\mu \to H^{\mu-m}} \leq K M^m_0(a).$$

Theorem A.5. *Let $m, m' \in \mathbb{R}^d$, and $\rho > 0$, $a \in \Gamma^m_\rho(\mathbb{R}^d)$and $b \in \Gamma^{m'}_\rho(\mathbb{R}^d)$.*

- *Composition: Then $T_a T_b$ is a paradifferential operator of order $m+m'$ and $T_a T_b - T_{a\#b}$ is of order $m+m'-\rho$ where $a\#b$ is defined by:*

$$a\#b = \sum_{|\alpha|<\rho} \frac{1}{i^{|\alpha|}\alpha!} \partial_\xi^\alpha a \partial_x^\alpha b$$

  Moreover, for all $\mu \in \mathbb{R}$ there exists a constant K such that

$$\|T_a T_b - T_{a\#b}\|_{H^\mu \to H^{\mu-m-m'+\rho}} \leq K M^m_\rho(a) M^{m'}_\rho(b).$$

- *Adjoint: The adjoint operator of T_a, $T_a^\top$ is a paradifferential operator of order m with symbol $a^\top$ defined by:*

$$a^\top = \sum_{|\alpha|<\rho} \frac{1}{i^{|\alpha|}\alpha!} \partial_\xi^\alpha \partial_x^\alpha \bar{a}$$

  Moreover, for all $\mu \in \mathbb{R}$ there exists a constant K such that

$$\left\| T_a^\top - T_{a^\top} \right\|_{H^\mu \to H^{\mu-m+\rho}} \leq K M^m_\rho(a).$$

If $a = a(x)$ is a function of x only, the paradifferential operator T_a is called a paraproduct. It follows from Theorem A.5 and the Sobolev embeddings that:

- If $a \in H^\alpha(\mathbb{R}^d)$ and $b \in H^\beta(\mathbb{R}^d)$ with $\alpha, \beta > \frac{d}{2}$, then
$$T_a T_b - T_{ab} \text{ is of order } \quad -\left(min\,\{\alpha, \beta\} - \frac{d}{2}\right).$$
- If $a \in H^\alpha(\mathbb{R}^d)$ with $\alpha > \frac{d}{2}$, then
$$T_a^\top - T_{a^\top} \text{ is of order } \quad -\left(\alpha - \frac{d}{2}\right).$$
- If $a \in W^{r,\infty}(\mathbb{R}^d)$, $r \in \mathbb{N}$ then:
$$\|au - T_a u\|_{H^r} \leq C \|a\|_{W^{r,\infty}} \|u\|_{L^2}.$$

An important feature of paraproducts is that they are well defined for function $a = a(x)$ which are not L^∞ but merely in some Sobolev spaces H^r with $r < \frac{d}{2}$.

Proposition A.8. *Let $m > 0$. If $a \in H^{\frac{d}{2}-m}(\mathbb{R}^d)$ and $u \in H^\mu(\mathbb{R}^d)$ then $T_a u \in H^{\mu-m}(\mathbb{R}^d)$. Moreover,*
$$\|T_a u\|_{H^{\mu-m}} \leq K \|a\|_{H^{\frac{d}{2}-m}} \|u\|_{H^\mu}$$

A main feature of paraproducts is the existence of paralinearisation Theorems which allow us to replace nonlinear expressions by paradifferential expressions, at the price of error terms which are smoother than the main terms.

Theorem A.6. *Let $\alpha, \beta \in \mathbb{R}$ be such that $\alpha, \beta > \frac{d}{2}$, then*

- *Bony's Linearization Theorem*[15] *For all C^∞ function F, if $a \in H^\alpha(\mathbb{R}^d)$ then*
$$F(a) - F(0) - T_{F'(a)} a \in H^{2\alpha - \frac{d}{2}}(\mathbb{R}^d).$$
- *If $a \in H^\alpha(\mathbb{R}^d)$ and $b \in H^\beta(\mathbb{R}^d)$, then $ab - T_a b - T_b a \in H^{\alpha+\beta-\frac{d}{2}}(\mathbb{R}^d)$. Moreover there exists a positive constant K independent of a and b such that:*
$$\|ab - T_a b - T_b a\|_{H^{\alpha+\beta-\frac{d}{2}}} \leq K \|a\|_{H^\alpha} \|b\|_{H^\beta}.$$

And finally we give the transformation of paradifferential operators under a change of variables with limited regularity, we refer to [21] for a more detailed presentation.

Theorem A.7. *Let $\chi : \Omega \to \Omega'$ be a $W^{1+\rho,\infty}_{loc}$ diffeomorphism with $D\chi \in W^{\rho,\infty}$ and $\rho \geq 0$. Consider $a \in \Gamma^m_r(\mathbb{R}^d)$ a properly supported paradifferential operator. Then there exists a property supported $a^* \in \Gamma^m_{min(r,\rho)}(\mathbb{R}^d)$ defined by:*
$$T_{a^*}(u \circ \chi) = (T_a u) \circ \chi.$$
Moreover a^* has the local expansion:
$$a^*(x,\xi) \sim \sum_{\substack{\alpha \\ |\alpha| \leq \lfloor min(r,\rho) \rfloor}} \frac{1}{\alpha} \partial^\alpha a(\chi(x), D\chi^{-1}(\chi(x))^\top \xi) P_\alpha(\chi(x), \xi), \tag{A.4}$$
where,
$$P_\alpha(x', \xi) = D^\alpha_{y'}(e^{i(\chi^{-1}(y') - \chi^{-1}(x') - D\chi^{-1}(x')(y'-x')).\xi})_{|y'=x'}$$
and P_α is polynomial in ξ of degree $\leq \frac{|\alpha|}{2}$, with $P_0 = 1, P_1 = 0$.

Remark A.6. *A standard implication of theorem A.7, is the generalization of paradifferential calculus to manifolds and thus to $\mathbb{D}^d$.*

[15] In our recent work [21] we give a generalization to this Theorem.

A.4. **Paracomposition.** We recall the main properties of the paracomposition operator first introduced by S. Alinhac in [20] to treat low regularity change of variables. Here we present the results we reviewed and generalized in some cases in [21].

Theorem A.8. *Let $\chi : \mathbb{R}^d \to \mathbb{R}^d$ be a $W^{1+r,\infty}$ diffeomorphism with $r > 0$ and take $s \in \mathbb{R}$ then the following maps are continuous:*

$$H^s(\mathbb{R}^d) \to H^s(\mathbb{R}^d)$$
$$u \mapsto \chi^* u = \sum_{k\geq 0} \sum_{\substack{l\geq 0 \\ l\leq k+N}} P_l(D)u_k \circ \chi.$$

Taking $\tilde{\chi} : \mathbb{R}^d \to \mathbb{R}^d$ a $W^{1+\tilde{r},\infty}$ map with $\tilde{r} > 0$, then the previous operation has the natural fonctorial property:

$$\forall u \in H^s(\mathbb{R}^d), \chi^* \tilde{\chi}^\star u = (\chi \circ \tilde{\chi})^* u + Ru,$$
$$\text{with,} \quad R : H^s(\mathbb{R}^d) \to H^{s+min(r,\tilde{r})}(\mathbb{R}^d) \quad \text{continous.}$$

We now give the key paralinearization theorem taking into account the paracomposition operator.

Theorem A.9. *Let u be a $W^{1,\infty}(\mathbb{R}^d)$ map and χ be a $H^{1+r}(\mathbb{R}^d)$ diffeomorphism with $r > \frac{d}{2}$. Then:*

$$u \circ \chi(x) = \chi^* u(x) + T_{u'\circ\chi}\chi(x) + R_0(x) + R_1(x) + R_2(x)$$

where the paracomposition given in the previous Theorem verifies the estimates:

$$\forall s \in \mathbb{R}, \|\chi^* u(x)\|_{H^s} \leq C(\|D\chi\|_\infty) \|u(x)\|_{H^s},$$
$$u' \circ \chi \in \Gamma^0_{W^{0,\infty}(\mathbb{R}^d)}(\mathbb{R}^d) \quad \text{for } u \text{ Lipchitz},$$

and the remainders verify the estimates:

$$\|R_0\|_{H^{1+r+min(1+r-\frac{d}{2},s-\frac{d}{2})}} \leq C \|\chi\|_{H^{1+r}} \|u\|_{H^{1+s}}$$
$$\|R_1\|_{H^{1+r+s-\frac{d}{2}}} \leq C(\|D\chi\|_\infty) \|\chi\|_{H^{1+r}} \|u\|_{H^{1+s}}.$$
$$\|R_2\|_{H^{1+r+s-\frac{d}{2}}} \leq C(\|D\chi\|_\infty, \|D\chi^{-1}\|_\infty) \|\chi\|_{H^{1+r}} \|u\|_{H^{1+s}}.$$

Finally the commutation between a paradifferential operator $a \in \Gamma^m_\beta(\mathbb{R}^d)$ and a paracomposition operator χ^* is given by the following

$$\chi^* T_a u = T_{a^*}\chi^* u + T_{q^*}\chi^* u \quad \text{with} \quad q \in \Gamma^{m-\beta}_0(\mathbb{R}^d),$$

where a^* is given by A.7.

Appendix B. Energy estimates and well-posedness of some pulled back linear hyperbolic equations

Theorem B.1. *Let $T > 0$, $\chi \in W^{1,\infty}([0,T], W^{1,\infty}_{loc}(\mathbb{D}^d))$ with $D_x\chi \in W^{1,\infty}([0,T], L^\infty)$ and consider $(a_t)_{t\in\mathbb{R}}$ a family of symbols in $\Gamma^\beta_1(\mathbb{D}^d)$ with $\beta \in \mathbb{R}$, such that $t \mapsto a_t$ is continuous and bounded from $\mathbb{R}$ to $\Gamma^\beta_1(\mathbb{D}^d)$ and such that $Re(a_t) = \frac{a_t + a_t^\top}{2}$ is bounded in $\Gamma^0_1(\mathbb{D}^d)$. Suppose moreover that $\chi(t,\cdot)$ is a diffeomorphism between open sets of $\mathbb{D}^d$ and that we have the bounds:*

$$\exists C > 0, \forall t \leq T, \forall x, C^{-1} \leq |D_x\chi(t,x)| \leq C. \tag{B.1}$$

Put $(\cdot)^*$ is the change of variables by χ as presented in Theorem A.7.
Then for all initial data $u_0 \in H^s(\mathbb{D}^d)$ and $f \in C^0([0,T]; H^s(\mathbb{D}^d))$ the Cauchy problem:

$$\begin{cases} \partial_t u + T_{a^*}u = f \\ \forall x \in \mathbb{D}^d, u(0,x) = u_0(x) \end{cases} \tag{B.2}$$

has a unique solution $u \in C^0([0,T]; H^s(\mathbb{D}^d)) \cap C^1([0,T]; H^{s-\beta}(\mathbb{D}^d))$ *which verifies the estimates:*

$$\begin{aligned}\|u(t)\|_{H^s} \leq & e^{C(\|D_x\chi\|_{L^\infty L^\infty}, \|D_x\chi^{-1}\|_{L^\infty L^\infty}, M_0^0(Re(a)))t} \|u_0\|_{H^s} \\ & + 2\int_0^t e^{C(\|D_x\chi\|_{L^\infty L^\infty}, \|D_x\chi^{-1}\|_{L^\infty L^\infty}, M_0^0(Re(a)))(t-t')} \|f(t')\|_{H^s} dt'.\end{aligned} \tag{B.3}$$

Again fixing the initial data at 0 is an arbitrary choice. More precisely, $\forall 0 \leq t_0 \leq T$ *and all data*
$u_0 \in H^s(\mathbb{D}^d)$ *the Cauchy problem:*

$$\begin{cases} \partial_t u + T_{a^*} u = f \\ \forall x \in \mathbb{D}^d, u(t_0, x) = u_0(x) \end{cases} \tag{B.4}$$

has a unique solution $u \in C^0([0,T]; H^s(\mathbb{D}^d)) \cap C^1([0,T]; H^{s-\beta}(\mathbb{D}^d))$ *which verifies the estimate:*

$$\begin{aligned}\|u(t)\|_{H^s} \leq & e^{C(\|D_x\chi\|_{L^\infty L^\infty}, \|D_x\chi^{-1}\|_{L^\infty L^\infty}, M_0^0(Re(a)))|t-t_0|} \|u_0\|_{H^s} \\ & + 2\left|\int_{t_0}^t e^{C(\|D_x\chi\|_{L^\infty L^\infty}, \|D_x\chi^{-1}\|_{L^\infty L^\infty}, M_0^0(Re(a)))(t-t')} \|f(t')\|_{H^s} dt'\right|.\end{aligned}$$

Proof. The existence of a solution follows from standard compacity arguments after regularization given the priory estimates (B.3). Also, the equation being linear those estimates give the unicity immediately. Thus we will only show the desired priory estimates.
Put $\Gamma_s = \langle \mathrm{D} \rangle^s$, we will compute $\frac{d}{dt}({\Gamma_s}^* u, {\Gamma_s}^* u)_{L^2(\mathbb{D}^d, |D_x\chi(t,x)|dx)}$ in two different ways.
Method 1. First notice that by Theorem A.7

$${\Gamma_s}^*(x,\xi) \sim ([D\chi^{-1}(t,\chi(t,x))]^t \xi)^s + R$$

Where R is of order $s - \frac{1}{2}$.
Thus using the lower and upper bound on $|D\chi(t,x)|$ combined with upper bound on $\frac{d}{dt}|D\chi(t,x)|$ we have:

$$C(\|D_x\chi^{-1}\|_{L^\infty L^\infty})\frac{d}{dt}[(\Gamma_s u, \Gamma_s u)_{L^2}] - C(\|D_x\chi\|_{L^\infty L^\infty})\|\Gamma_s u\|_{L^2}^2 \leq \frac{d}{dt}({\Gamma_s}^* u, {\Gamma_s}^* u)_{L^2(|D_x\chi(t,x)|dx)}.$$

Method 2.

$$\begin{aligned}&\frac{d}{dt}({\Gamma_s}^* u, {\Gamma_s}^* u)_{L^2(|D_x\chi(t,x)|dx)} \\ &= 2Re((\partial_t {\Gamma_s}^* u, {\Gamma_s}^* u))_{L^2(|D\chi(t,x)|dx)}) + ({\Gamma_s}^* u, {\Gamma_s}^* u)_{L^2(\frac{d}{dt}|D\chi(t,x)|dx)} \\ &= -2Re(({\Gamma_s}^* T_{a^*} u, {\Gamma_s}^* u)_{L^2(|D\chi(t,x)|dx)}) + 2Re(({\Gamma_s}^* f, {\Gamma_s}^* u))_{L^2(|D\chi(t,x)|dx)} \\ &+ 2Re(([\partial_t {\Gamma_s}^*]u, {\Gamma_s}^* u)_{L^2(|D\chi(t,x)|dx)}) + ({\Gamma_s}^* u, {\Gamma_s}^* u)_{L^2(\frac{d}{dt}|D\chi(t,x)|dx)}\end{aligned}$$

By the hyperbolic character of the symbol a,

$$\begin{aligned}&\frac{d}{dt}({\Gamma_s}^* u, {\Gamma_s}^* u)_{L^2(|D_x\chi(t,x)|dx)} \\ &= -2(\Gamma_s T_{Re(a)}[u \circ \chi^{-1}], \Gamma_s[u \circ \chi^{-1}])_{L^2} + 2Re(({\Gamma_s}^* f, {\Gamma_s}^* u)_{L^2(|D\chi(t,x)|dx)}) \\ &+ 2Re(([\partial_t {\Gamma_s}^*]u, {\Gamma_s}^* u)_{L^2(|D\chi(t,x)|dx)}) + ({\Gamma_s}^* u, {\Gamma_s}^* u)_{L^2(\frac{d}{dt}|D\chi(t,x)|dx)}.\end{aligned}$$

By the upper bound on $|D\chi(t,x)|$ and $|D\chi^{-1}(t,x)|$:

$$\begin{aligned}&(\Gamma_s T_{Re(a)}[u \circ \chi^{-1}], \Gamma_s[u \circ \chi^{-1}])_{L^2(\mathbb{D}^d)} \\ &= ({\Gamma_s}^* T_{Re(a)^*} u, {\Gamma_s}^* u)_{L^2(|D\chi(t,x)|dx)} \\ &\leq C M_0^0(Re(a))(\|D_x\chi\|_{L^\infty L^\infty}, \|D_x\chi^{-1}\|_{L^\infty L^\infty})\|\Gamma_s u\|_{L^2}^2.\end{aligned}$$

Now by the upper bound on $\frac{d}{dt}\left|D\chi(t,x)\right|$ and $\frac{d}{dt}\left|D\chi^{-1}(t,x)\right|$ we have:

$$({\Gamma_s}^* u, {\Gamma_s}^* u)_{L^2(\frac{d}{dt}|D\chi(t,x)|dx)} \leq C(\|D_x\chi\|_{L^\infty L^\infty}) \|{\Gamma_s}^* u\|_{L^2}^2$$

Now using the upper bound on $|D\chi(t,x)|$:

$$({\Gamma_s}^* u, {\Gamma_s}^* u)_{L^2(\frac{d}{dt}|D\chi(t,x)|dx)} \leq C(\|D_x\chi\|_{L^\infty L^\infty}, \left\|D_x\chi^{-1}\right\|_{L^\infty L^\infty}) \|\Gamma_s u\|_{L^2}^2 .$$

Analogously we get:

$$({\Gamma_s}^* f, {\Gamma_s}^* u)_{L^2(|D\chi(t,x)|dx)} \leq C(\|D_x\chi\|_{L^\infty L^\infty(\mathbb{D}^d)}, \left\|D_x\chi^{-1}\right\|_{L^\infty L^\infty}) \|\Gamma_s u\|_{L^2} \|\Gamma_s f\|_{L^2} ,$$

$$([\partial_t {\Gamma_s}^*]u, {\Gamma_s}^* u)_{L^2(|D\chi(t,x)|dx)} \leq C(\|D_x\chi\|_{L^\infty L^\infty}, \left\|D_x\chi^{-1}\right\|_{L^\infty L^\infty}) \|\Gamma_s u\|_{L^2}^2 .$$

Combining the 2 computations we get:

$$\begin{aligned}\frac{d}{dt}[(\Gamma_s u, \Gamma_s u)_{L^2}] \leq\ & C(\|D_x\chi\|_{L^\infty L^\infty}, \left\|D_x\chi^{-1}\right\|_{L^\infty L^\infty}, M_0^0(Re(a))) \|\Gamma_s u\|_{L^2}^2 \\ & + C(\|D_x\chi\|_{L^\infty L^\infty}, \left\|D_x\chi^{-1}\right\|_{L^\infty L^\infty}) \|\Gamma_s u\|_{L^2} \|\Gamma_s f\|_{L^2} .\end{aligned}$$

The result then follows from the Gronwall Lemma. □

We see that the proof depends essentially on symbolic calculus rules and those still clearly hold in the case of pseudodifferential operators as presented in Appendix A.

Theorem B.2. *Let $T > 0$, $\chi \in W^{1,\infty}([0,T], C^\infty(\mathbb{D}^d))$ such that $D_x\chi \in C_b^\infty(\mathbb{D}^d)$ and consider $(a_t)_{t\in\mathbb{R}}$ a family of symbols in $S^\beta(\mathbb{D}^d)$ with $\beta \in \mathbb{R}$, such that $t \mapsto a_t$ is continuous and bounded from $\mathbb{R}$ to $S^\beta(\mathbb{D}^d)$ and such that $Re(a_t) = \frac{a_t + a_t^\top}{2}$ is bounded in $S^0(\mathbb{D}^d)$. Suppose moreover that $\chi(t,\cdot)$ is a diffeomorphism between open sets of $\mathbb{D}^d$ and that we have the bounds:*

$$\exists C > 0, \forall t \leq T, \forall x, C^{-1} \leq |D_x\chi(t,x)| \leq C. \tag{B.5}$$

Put $(\cdot)^*$ is the change of variables by χ as presented in Theorem A.3.
Then for all initial data $u_0 \in H^s(\mathbb{D}^d)$ and $f \in C^0([0,T]; H^s(\mathbb{D}^d))$ the Cauchy problem:

$$\begin{cases} \partial_t u + \mathrm{Op}(a^*)u = f \\ \forall x \in \mathbb{D}^d, u(0,x) = u_0(x) \end{cases} \tag{B.6}$$

has a unique solution $u \in C^0([0,T]; H^s(\mathbb{D}^d)) \cap C^1([0,T]; H^{s-\beta}(\mathbb{D}^d))$ which verifies the estimates:

$$\begin{aligned}\|u(t)\|_{H^s} \leq\ & e^{C(\|D_x\chi\|_{L^\infty L^\infty}, \left\|D_x\chi^{-1}\right\|_{L^\infty L^\infty})t} \|u_0\|_{H^s} \\ & + 2\int_0^t e^{C(\|D_x\chi\|_{L^\infty L^\infty}, \left\|D_x\chi^{-1}\right\|_{L^\infty L^\infty})(t-t')} \left\|f(t')\right\|_{H^s} dt',\end{aligned} \tag{B.7}$$

where C depends also on a finite symbol semi-norm of $Re(a_t)$. Again fixing the initial data at 0 is an arbitrary choice. More precisely, $\forall 0 \leq t_0 \leq T$ and all data $u_0 \in H^s(\mathbb{D}^d)$ the Cauchy problem:

$$\begin{cases} \partial_t u + \mathrm{Op}(a^*)u = f \\ \forall x \in \mathbb{D}^d, u(t_0,x) = u_0(x) \end{cases} \tag{B.8}$$

has a unique solution $u \in C^0([0,T]; H^s(\mathbb{D}^d)) \cap C^1([0,T]; H^{s-\beta}(\mathbb{D}^d))$ which verifies the estimate:

$$\begin{aligned}\|u(t)\|_{H^s} \leq\ & e^{C(\|D_x\chi\|_{L^\infty L^\infty}, \left\|D_x\chi^{-1}\right\|_{L^\infty L^\infty})|t-t_0|} \|u_0\|_{H^s} \\ & + 2\left|\int_{t_0}^t e^{C(\|D_x\chi\|_{L^\infty L^\infty}, \left\|D_x\chi^{-1}\right\|_{L^\infty L^\infty})(t-t')} \left\|f(t')\right\|_{H^s} dt'\right| .\end{aligned}$$

We finally show a general regularizing effect due to integration in time. We present the result and prove it in the paradifferential case but the result holds clearly in the pseudodifferential case.

Theorem B.3. *Consider$(a_t)_{t\in\mathbb{R}}$ a family of symbols in $S^\beta(\mathbb{D}^d)$ with $\beta \in \mathbb{R}$, such that $t \mapsto a_t$ is continuous and bounded from $\mathbb{R}$ to $S^\beta(\mathbb{D}^d)$ and such that $\mathrm{Re}(a_t) = \frac{a_t + a_t^\top}{2}$ is bounded in $S^0(\mathbb{D}^d)$, and take $T > 0$. Then for all initial data $u_0 \in H^s(\mathbb{D}^d)$, and $f \in C^0([0,T]; H^s(\mathbb{D}^d))$ the Cauchy problem:*

$$\begin{cases} \partial_t u + op(a)u = f \\ \forall x \in \mathbb{D}^d, u(0,x) = u_0(x) \end{cases} \tag{B.9}$$

has a unique solution $u \in C^0([0,T]; H^s(\mathbb{D}^d)) \cap C^1([0,T]; H^{s-\beta}(\mathbb{D}^d))$ which verifies the estimates:

$$\|u(t)\|_{H^s(\mathbb{D}^d)} \leq e^{Ct} \|u_0\|_{H^s(\mathbb{D}^d)} + 2\int_0^t e^{C(t-t')} \|f(t')\|_{H^s(\mathbb{D}^d)} dt',$$

where C depends on a finite symbol semi-norm $M_1^0(\mathrm{Re}(a_t))$.
Suppose moreover that a is elliptic that is:

$$\forall (x,\xi) \in \mathbb{R}^{2d}, |a(x,\xi)| \geq C\langle \xi \rangle^\beta.$$

Then $\forall t \in [0,T]$:

$$\left\| \int_0^t u(s,\cdot)ds \right\|_{H^s} \leq C(\|u_0\|_{H^{s-1}} + \|u_0\|_{H^{s-\beta}} + \|f\|_{L^\infty([0,T],H^{s-1})} + \|f\|_{L^\infty([0,T],H^{s-\beta})}).$$

Proof. We start by writing:

$$\partial_t u + \mathrm{Op}(a)u = f$$

we then apply $\mathrm{Op}(a^{-1})$:

$$\mathrm{Op}(a^{-1})\partial_t u + u = \mathrm{Op}(a^{-1})f + Ru$$

with $\mathrm{R} \in S^{-1}(\mathbb{D}^d)$,

$$\partial_t \mathrm{Op}(a^{-1})u + u = \mathrm{Op}(a^{-1})f + Ru + \mathrm{Op}(\partial_t a^{-1})u = \mathrm{Op}(a^{-1})f + Ru + \mathrm{Op}(\frac{\partial_t a}{a^2})u.$$

the proof then follows by integration in time and the usual elliptic estimates. □

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