REGULARITY RESULTS ON THE FLOW MAPS OF PERIODIC DISPERSIVE BURGERS TYPE EQUATIONS AND THE GRAVITY-CAPILLARY EQUATIONS

AYMAN RIMAH SAID

ABSTRACT. In the first part of this paper we prove that the flow associated to a dispersive Burgers equation with a non local term of the form $|D|^{\alpha-1}\partial_x u$, $\alpha \in [1,+\infty[$ is Lipschitz from bounded sets of $H_0^s(\mathbb{T};\mathbb{R})$ to $C^0([0,T],H_0^{s-(2-\alpha)^+}(\mathbb{T};\mathbb{R}))$ for $T>0$ and $s>1+\frac{2-\alpha}{\alpha-1}+\frac{1}{2}$, where H_0^s are the Sobolev spaces of functions with 0 mean value, proving that the result obtained in [21] is optimal on the torus. The proof relies on a paradifferential generalization of a complex Cole-Hopf gauge transformation introduced by T.Tao in [24] for the Benjamin-Ono equation.

For this we prove a generalization of the Baker-Campbell-Hausdorff formula for flows of hyperbolic paradifferential equations and prove the stability of the class of paradifferential operators modulo more regular remainders, under conjugation by such flows. For this we prove a new characterization of paradifferential operators in the spirit of Beals [9].

In the second part of this paper we use a paralinearization version of the previous method to prove that a re-normalization of the flow of the one dimensional periodic gravity capillary equation is Lipschitz from bounded sets of H^s to $C^0([0,T],H^{s-\frac{1}{2}})$ for $T>0$ and $s>3+\frac{1}{2}$. This proves that the result obtained in [21] is optimal for the water waves system.

***Keywords*— Flow map, Regularity, Quasi-linear, nonlinear Burgers type dispersive equations, Water Waves system, Gravity-Capillary equations, Cole-Hopf Gauge transform.**

CONTENTS

1. INTRODUCTION

In our study of the quasi-linearity of the water waves system in [21] we studied the flow map regularity for some model nonlinear dispersive equations of the form:

$$\partial_t u + u\partial_x u + |D|^{\alpha-1}\partial_x u = 0 \text{ on } \mathbb{D}, \tag{1.1}$$

where $\mathbb{D}=\mathbb{R}$ or $\mathbb{T}$, $\alpha \in [0,2[$ and $|D|$ is the Fourier multiplier with symbol $|\xi|$. We proved that they are quasi-linear. We based our work on the following distinction between semi-linearity and quasi-linearity given in [18]:

- A partial differential equation is said to be semi-linear if its flow map is regular (at least C^1).
- A partial differential equation is said to be quasi-linear if its flow map is not Lipschitz.

PhD student at l'IMO, Paris Sud University and Centre Borelli, ENS Paris Saclay. email: aymanrimah@gmail.com.

More precisely we proved that:

- the flow map associated to (1.1) fails to be uniformly continuous from bounded sets of $H^s(\mathbb{D})$ to $C^0([0,T],H^s(\mathbb{D}))$ for $T>0$ and $s>2+\frac{1}{2}$.

The drawback of this test of quasi-linearity is that it does not show the effect of the dispersive term. The natural question was then to ask if one can see the effect of the dispersive term by analyzing more precisely the regularity of the flow map.

For this we can start by noticing that independently of α the flow map is Lipschitz from bounded sets of $H^s(\mathbb{D})$ to $C^0([0,T],H^{s-1}(\mathbb{D}))$ and ask: can the space $H^{s-1}(\mathbb{D})$ be replaced by $H^{s-\mu}(\mathbb{D})$ with $\mu<1$ depending on α? Again in [21] we proved that the best μ one can hope for is $\mu=1-(\alpha-1)^+$, more precisely we showed that:

- the flow map cannot be Lipschitz from bounded sets of $H^s(\mathbb{D})$ to $C^0([0,T],H^{s-1+(\alpha-1)^++\epsilon}(\mathbb{D}))$ for $\epsilon>0$.

Looking to the literature to assess the optimality of the result, first in [23] the equation (1.1) is actually shown to be quasi-linear for $\alpha\in[0,3[$ and becomes semi-linear for $\alpha=3$, *i.e* the Korteweg-de Vries equation, when $\mathbb{D}=\mathbb{R}$ suggesting that our results are sub-optimal. Then when $\mathbb{D}=\mathbb{T}$, in [19], for the case $\alpha=2$ and the Benjamin-Ono equation, the flow map is shown to be Lipschitz (and even has analytic regularity) on bounded sets of H^s_0 the (Sobolev spaces of functions with mean value 0). Which suggests that our results could be optimal but with a subtlety in the low frequencies.

The aim of the current paper is to prove that the results obtained in [21] are optimal on the torus and for the full periodic Water Waves system with surface tension, i.e the Gravity Capillary equation, while clarifying in all of those cases the effect brought on by the low frequencies.

Remark 1.1. *In Appendix B, we look to the problem on $\mathbb{R}$ and use the same Gauge transform to show that the lack of regularity obtained in [23] for $\alpha\geq 2$ is essentially due to the lack of control of the L^1 norm in Sobolev spaces.*

1.1. **On the torus.** We show that the flow map associated to (1.1) is Lipschitz from bounded sets of $H^s_0(\mathbb{T})$ to $C^0([0,T],H_0^{s-1+(\alpha-1)^+}(\mathbb{T}))$. We begin by recalling a classical result.

Theorem 1.1. *Consider three real numbers $\alpha\in[0,+\infty[$, $s\in]1+\frac{1}{2},+\infty[$, $r>0$ and $u_0\in H^s(\mathbb{T};\mathbb{R})$. Then there exists $T>0$ such that for all $v_0\in B(u_0,r)\subset H^s(\mathbb{T};\mathbb{R})$, there exists a unique $v\in C([0,T],H^s(\mathbb{T};\mathbb{R}))$ solving the Cauchy problem:*

$$\begin{cases}\partial_t v+v\partial_x v+|D|^{\alpha-1}\partial_x v=0\\ v(0,\cdot)=v_0(\cdot)\end{cases}. \tag{1.2}$$

Moreover, for all $\delta>0$ and all of $\mu\in[0,s]$, there is $C:\mathbb{R}_+\to\mathbb{R}_+$ non decreasing such that:

$$\|v(t)\|_{H^\mu(\mathbb{T})}\leq C\big(\|v_0\|_{H^{\frac{3}{2}+\delta}(\mathbb{T})}\big)\|v_0\|_{H^\mu(\mathbb{T})}. \tag{1.3}$$

Taking two different solutions u,v, and assuming moreover that $u_0\in H^{s+1}(\mathbb{T})$ then:

$$\|(u-v)(t)\|_{H^s(\mathbb{T})}\leq C_s(\|(u,v)\|_{H^s(\mathbb{T})};t\|u\|_{H^{s+1}(\mathbb{T})})\|u_0-v_0\|_{H^s(\mathbb{T})}. \tag{1.4}$$

In [21] we studied the regularity of the flow map and proved the following.

Theorem 1.2 ([21])**.** *Consider three real numbers $\alpha\in[0,2[$, $s\in]2+\frac{1}{2},+\infty[$, $r>0$ and $u_0\in H^s(\mathbb{T};\mathbb{R})$.*

- *Then the flow map associated to the Cauchy problem (1.2):*

$$\begin{aligned}B(u_0,r)&\to C([0,T],H^s(\mathbb{T};\mathbb{R}))\\ v_0&\mapsto v\end{aligned}$$

is continuous but not uniformly continuous.

- *Moreover for all $\epsilon > 0$ the flow map:*

$$\begin{aligned} B(u_0, r) \to & C([0,T], H^{s-1+(\alpha-1)^+ + \epsilon}(\mathbb{T};\mathbb{R})) \\ v_0 \mapsto & v \end{aligned}$$

is not C^1.

Here we prove that those results are essentially optimal on the Torus more precisely we prove the following.

Theorem 1.3 ([21])**.** *Consider three real numbers $\alpha \in [1, +\infty[$, $s \in]1 + \frac{2-\alpha}{\alpha-1} + \frac{1}{2}, +\infty[$, $r > 0$ and $u_0 \in H_0^s(\mathbb{T};\mathbb{R})$. Then the flow map associated to the Cauchy problem (1.2):*

$$\begin{aligned} B(u_0, r) \cap H_0^s(\mathbb{T};\mathbb{R}) \to & C([0,T], H_0^{s-(2-\alpha)^+}(\mathbb{T};\mathbb{R})) \\ v_0 \mapsto & v \end{aligned}$$

is Lipschitz.

Several remarks are in order.

Remark 1.2. *(1) As a corollary of Theorem 1.3 we prove in Section 2.2 the following.*

Corollary 1.1. *Consider three real numbers $\alpha \in [0, +\infty[$, $s \in]1 + \frac{2-\alpha}{\alpha-1} + \frac{1}{2}, +\infty[$, $r > 0$ and $u_0 \in H^s(\mathbb{T};\mathbb{R})$.*

- *Then the flow map associated to the Cauchy problem (1.2):*

$$\begin{aligned} B(u_0, r) \to & C([0,T], H^s(\mathbb{T};\mathbb{R})) \\ v_0 \mapsto & v \end{aligned}$$

is continuous but not uniformly continuous.

- *For all $\epsilon > 0$ the flow map:*

$$\begin{aligned} B(u_0, r) \to & C([0,T], H^{s-1+\epsilon}(\mathbb{T};\mathbb{R})) \\ v_0 \mapsto & v \end{aligned}$$

is not C^1.

(2) The case $\alpha = \frac{3}{2}$ is closely related to the system obtained after reduction and para-linearization of the periodic Water Waves system in dimension 1 obtained in [4] Proposition 3.3 by T. Alazard, N. Burq and C. Zuily, which we will treat in the second part of this paper.

(3) The case $\alpha = 2$ and the Benjamin-Ono equation on the circle was obtained by Molinet in [19]. Though Molinet's result extends to the Cauchy problem on $L^2(\mathbb{T})$ and only studied the flow map regularity for data with 0 mean value.

1.2. **The periodic Gravity Capillary equation.** We follow here the presentation in [4], [3] and [2].

1.2.1. *Assumptions on the domain.* We consider a domain with free boundary, of the form:

$$\{(t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R} : (x, y) \in \Omega_t\},$$

where Ω_t is the domain located between a free surface:

$$\Sigma_t = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y = \eta(t, x)\}$$

and a given (general) bottom denoted by $\Gamma = \partial\Omega_t \setminus \Sigma_t$. More precisely we assume that initially ($t = 0$) we have the hypothesis (H_t) given by:

- The domain Ω_t is the intersection of the half space, denoted by $\Omega_{1,t}$, located below the free surface Σ_t,

$$\Omega_{1,t} = \{(x,y) \in \mathbb{R} \times \mathbb{R} : y < \eta(t,x)\} \tag{H_t}$$

and an open set $\Omega_2 \subset \mathbb{R}^{1+1}$ such that Ω_2 contains a fixed strip around Σ_t, which means that there exists $h > 0$ such that,

$$\{(x,y) \in \mathbb{R} \times \mathbb{R} : \eta(t,x) - h \leq y \leq \eta(t,x)\} \subset \Omega_2. \tag{H_t}$$

We shall assume that the domain Ω_2 (and hence the domain $\Omega_t = \Omega_{1,t} \cap \Omega_2$) is connected.

1.2.2. *The equations.* We consider an incompressible inviscid liquid, having unit density. The equations of motion are given by the Euler system on the velocity field v:

$$\begin{cases} \partial_t v + v \cdot \nabla v + \nabla P = -g e_y \\ \operatorname{div} v = 0 \end{cases} \quad \text{in } \Omega_t, \tag{1.5}$$

where $-ge_y$ is the acceleration of gravity ($g > 0$) and where the pressure term P can be recovered from the velocity by solving an elliptic equation. The problem is then coupled with the boundary conditions:

$$\begin{cases} v \cdot n = 0 & \text{on } \Gamma, \\ \partial_t \eta = \sqrt{1 + |\nabla \eta|^2} v \cdot \nu & \text{on } \Sigma_t, \\ P = -\kappa H(\eta) & \text{on } \Sigma_t, \end{cases} \tag{1.6}$$

where n and ν are the exterior normals to the bottom Γ and the free surface Σ_t, κ is the surface tension and $H(\eta)$ is the mean curvature of the free surface:

$$H(\eta) = \operatorname{div}\left(\frac{\nabla \eta}{\sqrt{1 + |\nabla \eta|^2}}\right).$$

We are interested in the case with surface tension and take $\kappa = 1$. The first condition in (1.6) expresses in fact that the particles in contact with the rigid bottom remain in contact with it. As no hypothesis is made on the regularity of Γ, this condition is shown to make sense in a weak variational meaning due to the hypothesis (H_t), for more details on this we refer to Section 2 in [4].

The fluid motion is supposed to be irrotational and Ω_t is supposed to be simply connected thus the velocity v field derives from some potential ϕ i.e $v = \nabla \phi$ and:

$$\begin{cases} \Delta \phi = 0 \text{ in } \Omega, \\ \partial_n \phi = 0 \text{ on } \Gamma. \end{cases}$$

The boundary condition on ϕ becomes:

$$\begin{cases} \partial_n \phi = 0 & \text{on } \Gamma, \\ \partial_t \eta = \partial_y \phi - \nabla \eta \cdot \nabla \phi & \text{on } \Sigma_t, \\ \partial_t \phi = -g\eta + H(\eta) - \frac{1}{2} |\nabla_{x,y} \phi|^2 & \text{on } \Sigma_t. \end{cases} \tag{1.7}$$

Following Zakharov [27] and Craig-Sulem [12] we reduce the analysis to a system on the free surface Σ_t. If ψ is defined by

$$\psi(t,x) = \phi(t,x,\eta(t,x)),$$

then ϕ is the unique variational solution of

$$\Delta \phi = 0 \text{ in } \Omega_t, \ \phi_{|y=\eta} = \psi, \ \partial_n \phi = 0 \text{ on } \Gamma.$$

Define the Dirichlet-Neumann operator by

$$\begin{aligned}(G(\eta)\psi)(t,x) &= \sqrt{1+|\nabla\eta|^2}\partial_n\phi_{|y=\eta} \\ &= (\partial_y\phi)(t,x,\eta(t,x)) - \nabla\eta(t,x)\cdot(\nabla\phi)(t,x,\eta(t,x)).\end{aligned}$$

For the case with rough bottom we refer to [6], [4] and [3] for the well-posedness of the variational problem and the Dirichlet-Neumann operator. Now (η,ψ) (see for example [12]) solves:

$$\begin{aligned}&\partial_t\eta = G(\eta)\psi, \\ &\partial_t\psi = -g\eta + H(\eta) + \frac{1}{2}|\nabla\psi|^2 + \frac{1}{2}\frac{\nabla\eta\cdot\nabla\psi + G(\eta)\psi}{1+|\nabla\eta|^2}.\end{aligned} \tag{1.8}$$

The system is completed with initial data

$$\eta(0,\cdot) = \eta_{in},\ \ \psi(0,\cdot) = \psi_{in}.$$

We consider the case when η, ψ are 2π-periodic in the space variable x.

1.2.3. *Flow map regularity.* In [4] and [3], Alazard, Burq, and Zuily perform a paralinearization and symmetrization of the the water waves system that takes the form:

$$\partial_t u + T_V.\nabla u + iT_\gamma u = f,$$

where γ is elliptic of order $\frac{3}{2}$ which closely resemble the model problem we presented on $\mathbb{T}$ but with an extra non linearity in γ. The paralinearization and symmetrization of the system was used to prove the well-posedness of the Cauchy problem in the optimal threshold $s > 2+\frac{1}{2}$ in which the velocity field v is Lipschitz. We will complete this and our result in [21] by giving the precise regularity of the flow map. First we recall some previously known results on the Cauchy problem from [3, 4].

Theorem 1.4 (From [3, 4])**.** *Consider two real numbers $r>0$, $s\in]2+\frac{1}{2},+\infty[$ and $(\eta_0,\psi_0)\in H^{s+\frac{1}{2}}(\mathbb{T})\times H^s(\mathbb{T})$ such that,*

$$\forall(\eta_0',\psi_0')\in B((\eta_0,\psi_0),r)\subset H^{s+\frac{1}{2}}(\mathbb{T})\times H^s(\mathbb{T})$$

the assumption $(H_t)_{t=0}$ is satisfied. Then there exists $T>0$ such that the Cauchy problem (1.8) with initial data $(\eta_0',\psi_0')\in B((\eta_0,\psi_0),r)$ has a unique solution

$$(\eta',\psi')\in C^0([0,T];H^{s+\frac{1}{2}}(\mathbb{T})\times H^s(\mathbb{T}))$$

and such that the assumption (H_t) is satisfied for $t\in[0,T]$. Moreover the flow map $(\eta_0',\psi_0')\mapsto(\eta',\psi')$ is continuous.

In [21] we completed this by the following.

Theorem 1.5. *Consider two real numbers $r>0$, $s\in]2+\frac{1}{2},+\infty[$ and $(\eta_0,\psi_0)\in H^{s+\frac{1}{2}}(\mathbb{T})\times H^s(\mathbb{T})$ such that,*

$$\forall(\eta_0',\psi_0')\in B((\eta_0,\psi_0),r)\subset H^{s+\frac{1}{2}}(\mathbb{T})\times H^s(\mathbb{T})$$

the assumption $(H_t)_{t=0}$ is satisfied.

Then for all $R>0$ the flow map associated to the Cauchy problem (1.8):

$$\begin{aligned}B(0,R) &\to C([0,T],H^{s+\frac{1}{2}}(\mathbb{T})\times H^s(\mathbb{T})) \\ (\eta_0',\psi_0') &\mapsto (\eta',\psi')\end{aligned}$$

is not uniformly continuous.

And at least a loss of $\frac{1}{2}$ derivative is necessary to have Lipschitz control over the flow map, i.e for all $\epsilon' > 0$ the flow map

$$\begin{aligned} B(0, R) &\to C([0, T], H^{s+\epsilon'}(\mathbb{T}) \times H^{s-\frac{1}{2}+\epsilon'}(\mathbb{T})) \\ (\eta_0', \psi_0') &\mapsto (\eta', \psi') \end{aligned}$$

is not Lipschitz.

Here it is shown that those results are sufficient after suitable re-normalization of the flow map.

Theorem 1.6. *Consider two real numbers $r > 0$, $s \in]3 + \frac{1}{2}, +\infty[$ and $(\eta_0, \psi_0) \in H^{s+\frac{1}{2}}(\mathbb{T}) \times H^s(\mathbb{T})$ such that,*

$$\forall (\eta_0', \psi_0') \in B((\eta_0, \psi_0), r) \subset H^{s+\frac{1}{2}}(\mathbb{T}) \times H^s(\mathbb{T})$$

the assumption $(H_t)_{t=0}$ is satisfied. Define (η, ψ) and (η', ψ') as the solutions to the Cauchy problem (1.8) on $[0, T]$, $T > 0$. Define the following change of variables:

$$\begin{aligned} \chi(t, x) &= \int_0^x \frac{1}{\sqrt{1 + (\partial_x \eta(t, y))^2}} dy - \int_0^t \int_\Sigma \left[\frac{1}{1 + (\partial_x \eta(t, y))^2} + \partial_x \phi \right] d\Sigma \qquad (1.9) \\ &= \int_0^x \sqrt{1 + (\partial_x \eta(t, y))^2} dy - \int_0^t \int_0^{2\pi} \left[\frac{1}{1 + (\partial_x \eta(t, y))^2} + \partial_x \phi \right] \sqrt{1 + (\partial_x \eta)^2} dy, \end{aligned}$$

and χ' is defined analogously from (η', ψ').

Then for r sufficiently small and $t \in [0, T]$ we have:

$$\begin{aligned} &\left\| (\eta, \psi)^* - (\eta', \psi')^{*'}(t, \cdot) \right\|_{H^s \times H^{s-\frac{1}{2}}} \\ &\leq C(\left\| (\eta_0, \psi_0, \eta_0', \psi_0') \right\|_{H^{s+\frac{1}{2}} \times H^s}) \left\| (\eta_0, \psi_0)^* - (\eta_0', \psi_0')^{*'} \right\|_{H^s \times H^{s-\frac{1}{2}}}, \qquad (1.10) \end{aligned}$$

where * and $^{*'}$ are the paracomposition by χ and χ', which we recall it's definition in A.3.

Remark 1.3. *The time integral in the re-normalization 1.9 is to insure that the mean value of the transport term vanishes. This re-normalization is needed here to compensate the non-linearity in the dispersive term of order $\frac{3}{2}$ here.*

1.3. **Strategy of the proof.** For Theorem 1.3, we first work on H_0^s and the main idea is to conjugate (1.1) to a semi-linear dispersive equation of the form:

$$\partial_t w + |D|^{\alpha-1} \partial_x w = Ru,$$

where R is continuous from H^s to itself. For the viscous Burgers equation such a result is obtained by the Cole-Hopf transformation that reduces the problem to a one dimensional heat equation. In [24], T.Tao used a complex version of the Cole-Hopf transformation to reduce the problem on the Benjamin-Ono equation to a one dimensional Schrödinger type equation, this idea was extensively used to lower the regularity needed for the well-posedness of the Cauchy problem as in Molinet's work in [19]. A generalized pseudodifferential form of this transformation was used in [1] to reduce the one dimensional water waves system to a a one dimensional semi-linear Schrödinger type system.

Formally if we follow the same lines of those previous papers, the transformation we will have to use is a pseudodifferential transformation of the form:

$$\begin{cases} w = \mathrm{Op}(a) u, \\ a = e^{\frac{1}{i\alpha} \xi |\xi|^{1-\alpha} U}, \end{cases} \qquad (1.11)$$

where U is a real valued periodic primitive of u that exists because u has mean value 0. The main problem is that such an operator belongs to a Hörmander symbol class of the form $S^0_{\alpha-1,2-\alpha}$, which for $\alpha = \frac{3}{2}$ becomes $S^0_{\frac{1}{2},\frac{1}{2}}$ which is a "bad" symbol class with no general symbolic calculus rules. Thus we have to treat this transformation with care.

The idea here is inspired by the particular form of the formal computation, we express the desired operator as the time one of a flow map associated to a hyperbolic equation, i.e $a = A_1$ where $(A_\tau)_{\tau\in\mathbb{R}}$ is defined as the group generated by the paradifferential operator iT_p where p is a real valued symbol of order smaller than 1. This is inspired by previous results of Alazard, Baldi and P.Gérard [7].

Take a different operator T_b. The main new idea is to apply a Baker-Campbell-Hausdorff formula. Formally this allows one to express $A_\tau T_b A_{-\tau}$ as a series of successive Lie derivatives $[iT_p, \cdots, [iT_p, T_b]$. The same kind of computations go for $[A_\tau, T_b]$. The convergence of such a series is a non trivial problem, equivalent to solving a linear ODE in the Fréchet space of paradifferential symbol classes $\Gamma^m_{+\infty}$ defined in Appendix A.2. Such an ODE is not generally well posed and to solve such a problem one usually has to look at a Nash-Moser type scheme. Though in our case we have an explicit ODE that can be solved locally with loss of derivative thus inspired by Hörmander's [15] and Beals in [9], we prove the existence of a symbol b^τ such that $A_\tau T_b A_{-\tau} = T_{b^\tau}$, moreover b^τ is shown to have the asymptotic expansion given by the Baker-Campbell-Hausdorff formula. The use of paradifferential operators is key here, as in Hörmander's [15], because the continuity of paradifferential operators given by Theorem A.2 insures that we do not need to control an infinite number of semi-norms as would have been the case for pseudodifferential operators.

Finally the transformation defined in this way helps us reduce the transport term of order 1 to a term of order $2-\alpha$ which is enough for our problem.

Passing from H^s_0 to H^s we use the following gauge transform:

$$\tilde{u}(t,x) = u(t, x - t\fint u_0) - \fint u_0, \text{ where } \frac{1}{|\mathbb{T}|}\int_{\mathbb{T}} u_0,$$

which we prove is continuous on H^s but not uniformly continuous and C^1 only from H^s to H^{s-1}.

For the Gravity-Capillary equation the problem is more delicate. Indeed the model problems we study are for the paralinearized and symmetrized system, though the change of variable from the original system to the paralinearized and symmetrized one is known to be Lipschitz on H^s for $s > 2 + \frac{1}{2}$. Thus the problem is reduced to the study of the flow map regularity of an equation of the form

$$\partial_t u + T_V.\nabla u + iT_\gamma u = f.$$

In the same spirit as [1, 2] we preform a para-change of variable, i.e we para-compose with χ defined by (1.9), to get:

$$\partial_t[u^*] + T_W.\nabla u^* + iT_{|\xi|^{\frac{3}{2}}} u^* = f, \text{ with } \int_{\mathbb{T}} W = 0.$$

We then proceed exactly as for Equation (1.1) (with the 0 mean value hypothesis insured by the choice of χ).

Remark 1.4.
- *Transformation (1.11), in which we use a primitive of the solution is called a gauge transform in the literature.*
- *As for the Cole-Hopf transformation, this gauge transform (1.11) is essentially one dimensional.*
- *It is interesting to know that the same type of transformation can be iterated and get at the step of order k a remainder of order $k+1-k\alpha$ which is acceptable for k sufficiently large as $\alpha > 1$ but the price to pay is $s > 1+\frac{1}{\alpha-1}$.*

In [22] we use this iteration to prove that for possible for $2 < \alpha < 3$, the paradifferential version of (1.1) can be transformed to a semi-linear equation with a regularizing remainder, i.e:

$$\partial_t u + T_u \partial_x u + \partial_x |D|^{\alpha-1} u = 0 \Rightarrow \partial_t A u + \partial_x |D|^{\alpha-1} A u = R_\infty(u),$$

where the operator norm of $R_\infty(u)$ is controlled by $\|u\|_{L^\infty([0,T],C_*^{2-\alpha,\infty})}$.

1.4. **Acknowledgement.** I would like to express my sincere gratitude to my thesis advisor Thomas Alazard.

2. Study of the model problems

2.1. **Proof of Theorem 1.3, the estimates on H_0^s.** We keep the notations of Theorem 1.3, fixing $u_0 \in H_0^s(\mathbb{T};\mathbb{R})$ and $r > 0$ and taking:

$$v_0, w_0 \in \mathrm{B}(u_0, r) \subset H_0^s(\mathbb{T};\mathbb{R}).$$

As the mean value is conserved by the flow of (1.2) we consider the solutions $u, v, w \in C^0([0,T]; H_0^s(\mathbb{T};\mathbb{R}))$ to (1.2) with initial data u_0, v_0, w_0 and on a uniform small interval $[0,T]$.

The main goal of the proof is to show the following estimate:

$$\|v(t,\cdot) - w(t,\cdot)\|_{H^{s-(2-\alpha)^+}} \leq C(\|(v_0,w_0)\|_{H^s}) \|v_0 - w_0\|_{H^{s-(2-\alpha)^+}}, \tag{2.1}$$

with the following tame control,

$$C(\|(v_0,w_0)\|_{H^s}) \leq C'(\|(v_0,w_0)\|_{H^{s-(2-\alpha)^+}})[\|(v_0,w_0)\|_{H^s} + 1], \tag{2.2}$$

where C and C' are non decreasing positive functions.

The final simplification we make in this paragraph is that given the well-posedness of the Cauchy problem in H^s, and the density of $H^{+\infty}$ in H^s, it suffice to prove (2.1) for $v_0, w_0 \in H^{+\infty}$, which henceforth we will suppose.

We start by applying the paralinearization Theorem A.3 to the term $u\partial_x v$ to get:

$$\begin{cases} \partial_t v + T_{vi\xi} v + T_{i|\xi|^{\alpha-1}\xi} v = R_1(v) v, \\ v(0,\cdot) = v_0(\cdot), \end{cases} \tag{2.3}$$

where R_1 verifies as $s > 1 + \frac{1}{2}$:

$$\|R_1(v)\|_{H^{s-(2-\alpha)^+} \to H^{s-(2-\alpha)^+}} \leq C(\|v\|_{W^{1,\infty}}) \leq C(\|v\|_{H^s}),$$
$$\|[R_1(v) - R_1(w)]v\|_{H^{s-(2-\alpha)^+} \to H^{s-(2-\alpha)^+}} \leq C \|v - w\|_{H^{s-(2-\alpha)^+}} \|v\|_{H^s},$$

where C verifies 2.2. Now we reduce $H^{s-(2-\alpha)^+}$ estimates to L^2 ones by defining $f_1 = \langle \mathrm{D} \rangle^{s-(2-\alpha)^+} v$. Commuting $\langle \mathrm{D} \rangle^{s-(2-\alpha)^+}$ with (2.3), using the symbolic calculus rules of Theorem A.3, we get that:

$$\begin{cases} \partial_t f_1 + T_{vi\xi} f_1 + T_{i|\xi|^{\alpha-1}\xi} f_1 = R_1(v) f_1 \\ f_1(0,\cdot) = \langle \mathrm{D} \rangle^{s-(2-\alpha)^+} v_0(\cdot), \end{cases} \tag{2.4}$$

where R_1 was modified to include terms verifying the same estimate i.e:

$$\|R_1(v)\|_{L^2 \to L^2} \leq C(\|v\|_{H^s}),$$
$$\|[R_1(v) - R_1(w)] f_1\|_{L^2} \leq C \|v - w\|_{H^{s-(2-\alpha)^+}} \|f_1\|_{H^{(2-\alpha)^+}}.$$

We define analogously $g_1 = \langle \mathrm{D} \rangle^{s-(2-\alpha)^+} w$ and notice that by definition:

$$\|f_1 - g_1\|_{L^2} = \|v - w\|_{H^{s-(2-\alpha)^+}},$$

thus the problem is reduced to getting L^2 estimates on $f_1 - g_1$.

Here we explain the scheme of the proof, using some estimates which will be proved in Section 4.

2.1.1. *Gauge transform and Energy estimate.* The goal of this section is to find an operator A_v such that

$$\partial_t[A_v f_1] + A_v T_{i|\xi|^{\alpha-1}\xi} f_1 + A_v T_{vi\xi} f_1 + [A_v, T_{i|\xi|^{\alpha-1}\xi}] f_1 = (\partial_t A_v) f_1 + A_v R_1(f_1) f_1,$$

and $A_v T_{vi\xi} + [A_v, T_{i|\xi|^{\alpha-1}\xi}]$ is a hyperbolic operator of order $(2-\alpha)^+ < 1$.
If we define $V = \partial_x^{-1} v$ which is the periodic zero mean value primitive of v, then

$$\hat{V}(0) = 0 \text{ and } \hat{V}(\xi) = \frac{\hat{v}(\xi)}{i\xi}, \text{ for } \xi \in \mathbb{Z}^*,$$

and we define analogously W from w. Then a formal computation shows that one can choose $A_v = T_{e^{\frac{1}{i\alpha}\xi|\xi|^{1-\alpha}V}} \in S^0_{\alpha-1,2-\alpha}(\mathbb{T}\times\mathbb{Z})$ which is a symbol class with no general symbolic calculus rules. Here we will define A_v differently [1].

A_v is defined as the time one of the flow map generated by:

$$p_v = \frac{1}{\alpha}\xi\,|\xi|^{1-\alpha} V \in \Gamma_2^{2-\alpha}(\mathbb{T}),$$

which is well defined by Proposition 4.1. We define analogously A_w and p_w from w. Now introduce:

$$f_2 = A_v f_1, \quad g_2 = A_w g_1. \tag{2.5}$$

The study of the symbolic calculus associated to this very specific form of symbols is given by Proposition 4.1 and the change of variable (2.5) is Lipschitz from L^2 to L^2 but under $H^{(2-\alpha)^+}$ control on (f_2, g_2). Indeed we write:

$$\begin{aligned}\|f_2 - g_2\|_{L^2} &= \|A_v f_1 - A_w g_1\|_{L^2}\\ &\leq \|A_v[f_1 - g_1]\|_{L^2} + \|(A_v - A_w) g_2\|_{L^2}.\end{aligned}$$

Applying estimate (1) of Proposition 4.1 and estimate (4.4):

$$\begin{aligned}\|f_2 - g_2\|_{L^2} &\leq C(\|v\|_{H^s}) \|f_1 - g_1\|_{L^2} + \|V - W\|_{L^\infty} \|g_2\|_{H^{(2-\alpha)^+}}\\ &\leq C(\|v\|_{H^s}) \|f_1 - g_1\|_{L^2} + \|v - w\|_{H^{s-(2-\alpha)^+}} \|g_2\|_{H^{(2-\alpha)^+}}\\ &\leq C(\|(v,w)\|_{H^s}) \|f_1 - g_1\|_{L^2},\end{aligned}$$

where C verifies the estimate 2.2. As A_v^{-1} and A_w^{-1} are the time -1 generated by the flow map p_v, p_w respectively which is well defined by Proposition 4.1. We get by symmetry:

$$\|f_1 - g_1\|_{L^2} \leq C(\|(v,w)\|_{H^s}) \|f_2 - g_2\|_{L^2},$$

thus,

$$C^{-1}(\|(v,w)\|_{H^s}) \|f_1 - g_1\|_{L^2} \leq \|f_2 - g_2\|_{L^2} \leq C(\|(v,w)\|_{H^s}) \|f_1 - g_1\|_{L^2},$$

and the problem is reduced to getting L^2 estimates on $f_2 - g_2$.

To get the equations on f_2 and g_2 we commute A_v and A_w with (2.4), we make the computations for f_2, those for g_2 are obtained by symmetry:

$$A_v \partial_t f_1 + A_v T_{vi\xi} f_1 + A_v T_{i|\xi|^{\alpha-1}\xi} f_1 = A_v R_1(v) f_1, \text{ thus,}$$

$$\partial_t\big(A_v f_1\big) + T_{i|\xi|^{\alpha-1}\xi} A_v f_1 + (A_v T_{vi\xi} - [T_{i|\xi|^{\alpha-1}\xi}, A_v]) f_1 + [A_v, \partial_t] f_1 = A_v R_1(v) f_1.$$

By definition of p_v and Proposition 4.4 we have:

$$\partial_\xi(\xi\,|\xi|^{\alpha-1})\partial_x p_v = v\xi \text{ and } [A_v, \partial_t] = -A_v \int_0^1 A^p_{-r} T_{i\partial_t p_v} A^p_r dr.$$

[1] Similar ideas were used in Appendix C of [1] to get estimates on a change of variable operator which are still in the usual symbol classes $S^m_{1,0}$, the difficulty here being that we are no longer in those symbol classes.

Thus by Corollary 4.2 we have:

$$\partial_t f_2 + T_{i|\xi|^{\alpha-1}\xi} f_2 = R_2(v) f_2 + A_v R_1(v) A_v^{-1} f_2, \tag{2.6}$$

where R_2 and $A_v R_1(v) A_v^{-1}$ verify:

$$\begin{aligned}
\|\mathrm{Re}(R_2(v))\|_{L^2\to L^2} &\leq C(\|v\|_{H^s}), \\
\|[R_2(v)-R_2(w)]g_2\|_{L^2} &\leq C\,\|v-w\|_{H^{s-(2-\alpha)^+}} \|g_2\|_{H^{(2-\alpha)^+}}, \\
\left\|[A_v R_1(v) A_v^{-1} - A_w R_1(w) A_w^{-1}]g_2\right\|_{L^2} &\leq C\,\|v-w\|_{H^{s-(2-\alpha)^+}} \|g_2\|_{H^{(2-\alpha)^+}}.
\end{aligned}$$

We get analogously on g_2,

$$\partial_t g_2 + T_{i|\xi|^{\alpha-1}\xi} g_2 = R_2(w) g_2 + A_w R_1(w) A_w^{-1} g_2. \tag{2.7}$$

Taking the difference between (2.6) and (2.7) yields:

$$\partial_t(f_2-g_2)+T_{i|\xi|^{\alpha-1}\xi}(f_2-g_2) = [R_2(w)-R_2(v)-A_v R_1(v) A_v^{-1}+A_w R_1(w) A_w^{-1}]g_2.$$

Thus the usual energy estimate combined with the Gronwall lemma on f_2-g_2 gives for $0\leq t\leq T$:

$$\begin{aligned}
\|f_2-g_2\|_{L^2} &\leq C(\|(u_0,v_0)\|_{H^s}, \|(f_2,g_2)(0,\cdot)\|_{H^{(2-\alpha)^+}}) \|(f_2-g_2)(0,\cdot)\|_{L^2} \\
&\leq C(\|(u_0,v_0)\|_{H^s}) \|(f_2-g_2)(0,\cdot)\|_{L^2},
\end{aligned}$$

with C verifying (2.2), which concludes the proof.

2.2. **Proof of Corollary 1.1, the estimates on H^s.** The starting point is noticing that the mean value is preserved by (1.2) and by doing the change of unknowns:

$$\begin{cases} \tilde{u}(t,x) = u(t, x - t⨍ u_0) - ⨍ u_0 \\ \tilde{v}(t,x) = v(t, x - t⨍ v_0) - ⨍ v_0 \end{cases}, \tag{2.8}$$

where $⨍ u_0 = \frac{1}{2\pi}\int_{\mathbb{T}} u_0$ is the mean value. We can reduce the Cauchy problem for general data to ones with 0 mean value by verifying that $\tilde{u}, \tilde{v} \in H_0^s$ still solve (1.2). Thus the main goal is to prove that the change of variable (2.8) is not regular. More precisely we will show that there exists a positive constant C and two sequences (u_ϵ^λ) and (v_ϵ^λ) solutions of 1.2 in $C^0([0,1], H^s(\mathbb{T}))$ such that for every $t\leq T$, where T is a uniform small time,

$$\sup_{\lambda,\epsilon} \left\|u_\epsilon^\lambda\right\|_{H^s(\mathbb{T})(t,\cdot)} + \left\|v_\epsilon^\lambda(t,\cdot)\right\|_{H^s(\mathbb{T})} \leq C,$$

$(u_{\epsilon,\tau}^\lambda)$ and $(v_{\epsilon,\tau}^\lambda)$ satisfy initially:

$$\lim_{\substack{\lambda\to+\infty \\ \epsilon\to 0}} \left\|u_\epsilon^\lambda(0,\cdot) - v_\epsilon^\lambda(0,\cdot)\right\|_{H^s(\mathbb{T})} = 0,$$

but,

$$\liminf_{\substack{\lambda\to+\infty \\ \epsilon\to 0}} \left\|u_\epsilon^\lambda(t,\cdot) - v_\epsilon^\lambda(t,\cdot)\right\|_{H^s(\mathbb{T})} \geq c > 0.$$

Which proves the non uniform continuity. Considering a weaker control norm we want to get, for all $\delta>0$ and for $t>0$:

$$\liminf_{\substack{\lambda\to+\infty \\ \epsilon\to 0}} \frac{\left\|u_\epsilon^\lambda(t,\cdot) - v_\epsilon^\lambda(t,\cdot)\right\|_{H^{s-1+\delta}(\mathbb{T})}}{\|u_\epsilon^\lambda(0,\cdot) - v_\epsilon^\lambda(0,\cdot)\|_{H^s(\mathbb{T})}} = +\infty.$$

2.2.1. *Definition of the Ansatz.* Take $\omega \in C_0^\infty(\mathbb{T})$ such that for $x \in [0, 2\pi]$:

$$\omega(x) = 1 \text{ if } |x| \leq \frac{1}{2},\ \omega(x) = 0 \text{ if } |x| \geq 1.$$

Let (λ, ϵ) be two positive real sequences such that:

$$\lambda \to +\infty,\ \epsilon \to 0,\ \lambda\epsilon \to +\infty. \tag{2.9}$$

Put for $x \in [0, 2\pi]$,

$$u^0(x) = \lambda^{\frac{1}{2}-s}\omega(\lambda x),\ v^0(x) = u^0(x) + \epsilon\omega(x),$$

and extend u^0 and v^0 periodically. The main trick here will be to use the time reversibility of equation (1.2) by defining $\tilde{u}, \tilde{v}$ as the solution of (1.2) with data fixed at time $t > 0$ given by

$$\begin{cases} \tilde{u}(t,x) = u_0 - \fint u_0 \\ \tilde{v}(t,x) = v_0 - \fint v_0 \end{cases}, \tag{2.10}$$

where $t \leq t_0$ is chosen small enough for the equations to be well-posed. Finally, define u and v by (2.8).

2.2.2. *Main estimates.* First the estimates at time 0, for $0 \leq \nu \leq s$:

$$\|u(0,x) - v(0,x)\|_{H^\nu} = \left\|\tilde{u}(0,x) - \tilde{v}(0,x) + \fint u_0 - \fint v_0\right\|_{H^\nu}$$

By the estimate (2.1), the tame control (2.2) and the Cauchy-Schwartz inequality,

$$\begin{aligned} \|u(0,x) - v(0,x)\|_{H^\nu} &\leq C(\|(u_0, v_0)\|_{H^{\nu+(2-\alpha)^+}})\,\|u_0 - v_0\|_{H^\nu} \\ &\leq C[1 + \lambda^{\nu-s+(2-\alpha)^+}]\epsilon. \end{aligned} \tag{2.11}$$

Now the estimates at a fixed time $t > 0$, by construction:

$$\begin{aligned} \|u(t,x) - v(t,x)\|_{H^\nu} &= \left\|u_0(x + t\fint u_0) - v_0(x + t\fint v_0)\right\|_{H^\nu} \\ &= \left\|u_0(x + t\fint u_0) - u_0(x + t\fint v_0)\right\|_{H^\nu} + O_{H^\nu}(\epsilon) \end{aligned}$$

Now by hypothesis $\lambda\epsilon \to +\infty$ and $t\fint \omega > 0$, thus $u_0(\cdot + t\fint u_0)$ and $u_0(\cdot + t\fint v_0)$ have disjoint supports, thus

$$\begin{aligned} \|u(t,x) - v(t,x)\|_{H^\nu} &= \left\|u_0(x + t\fint u_0)\right\|_{H^\nu} + \left\|u_0(x + t\fint v_0)\right\|_{H^\nu} + O_{H^\nu}(\epsilon) \\ &= C\lambda^{\nu-s} + O_{H^\nu}(\epsilon). \end{aligned} \tag{2.12}$$

Now to conclude the proof we differentiate the cases:

- in the case of non uniform continuity we take ϵ such that $\epsilon\lambda^{(2-\alpha)^+} \to 0$ and apply the previous estimates with $\nu = s$.
- In the case of non Lipschitz control we take ϵ such that $\lambda^{-1+\delta}\epsilon^{-1} \to +\infty$ and apply the previous estimates with $\nu = s - 1 + \delta$.

3. Flow map regularity for the periodic Gravity Capillary equation

3.1. **Prerequisites from the Cauchy problem.** We start by recalling the apriori estimates given by Proposition 5.2 of [4] combined with the results of [3]. We keep the notations of Theorem 1.6.

Proposition 3.1. *(From [4] and [3]) Consider a real number $s > 2 + \frac{1}{2}$. Then there exists a non decreasing function C such that, for all $T \in]0, 1]$ and all solution (η, ψ) of (1.8) such that:*

$$(\eta, \psi) \in C^0([0, T]; H^{s+\frac{1}{2}}(\mathbb{T}) \times H^s(\mathbb{T})) \text{ and } (H_t) \text{ is verified for } t \in [0, T],$$

we have:

$$\|(\eta, \psi)\|_{L^\infty(0,T;H^{s+\frac{1}{2}} \times H^s)} \le C((\eta_0, \psi_0)_{H^{s+\frac{1}{2}} \times H^s}) + TC(\|(\eta, \psi)\|_{L^\infty(0,T;H^{s+\frac{1}{2}} \times H^s)}).$$

The proof will rely on the para-linearised and symmetrized version of (1.8) given by Proposition 4.8 and corollary 4.9 of [4] which are valid on $\mathbb{T}$ as shown in [3]. Before we recall this, for clarity as in [4], we introduce a special class of operators $\Sigma^m \subset \Gamma_0^m$ given by:

Definition 3.1. *(From [4]) Given $m \in \mathbb{R}$, Σ^m denotes the class of symbols a of the form*

$$a = a^{(m)} + a^{(m-1)},$$

with,

$$a^{(m)} = F(\partial_x \eta(t, x), \xi),$$

$$a^{(m-1)} = \sum_{|k|=2} G_\alpha(\partial_x \eta(t, x), \xi) \partial_x^k \eta(t, x),$$

such that

(1) T_a maps real valued functions to real-valued functions;
(2) F is of class C^∞ real valued function of $(\zeta, \xi) \in \mathbb{R} \times (\mathbb{Z} \setminus 0)$, homogeneous of order m in ξ; and such that there exists a continuous function $K = K(\zeta) > 0$ such that

$$F(\zeta, \xi) \ge K(\zeta) |\xi|^m,$$

for all $(\zeta, \xi) \in \mathbb{R} \times (\mathbb{Z} \setminus 0)$;
(3) G_α is a C^∞ complex valued function of $(\zeta, \xi) \in \mathbb{R} \times (\mathbb{Z} \setminus 0)$, homogeneous of order $m - 1$ in ξ.

Σ^m enjoys all the usual symbolic calculus properties modulo acceptable remainders that we define by the following:

Definition-Notation 3.1. *(From [4]) Let $m \in \mathbb{R}$ and consider two families of operators of order m,*

$$\{A(t) : t \in [0, T]\}, \quad \{B(t) : t \in [0, T]\}.$$

We shall say that $A \sim B$ if $A - B$ is of order $m - \frac{3}{2}$ and satisfies the following estimate: for all $\mu \in \mathbb{R}$, there exists a continuous function C such that for all $t \in [0, T]$,

$$\|A(t) - B(t)\|_{H^\mu \to H^{\mu-m+\frac{3}{2}}} \le C(\|\eta(t)\|_{H^{s+\frac{1}{2}}}).$$

In the next Proposition we recall the different symbols that appear in the para-linearization and symmetrization of the equations.

Proposition 3.2. *(From [4])We work under the hypothesis of Proposition 3.1. Put*

$$\lambda = \lambda^{(1)} + \lambda^{(0)}, \ l = l^{(2)} + l^{(1)} \text{ with,}$$

$$\begin{cases} \lambda^{(1)} = |\xi|\,, \\ \lambda^{(0)} = \frac{1+|\partial_x\eta|^2}{2|\xi|}\left\{\partial_x\Big(\alpha^{(1)}\partial_x\eta\Big) + i\frac{\xi}{|\xi|}\partial_x\alpha^{(1)}\right\}, \\ \alpha^{(1)} = \frac{1}{\sqrt{1+|\partial_x\eta|^2}}\Big(\,|\xi| + i\partial_x\eta\xi\Big). \end{cases} \tag{3.1}$$

$$\begin{cases} l^{(2)} = (1+|\partial_x\eta|^2)^{-\frac{3}{2}}\xi^2, \\ l^{(1)} = -\frac{i}{2}(\partial_x\cdot\partial_\xi)l^{(2)}. \end{cases} \tag{3.2}$$

Now let $q \in \Sigma^0, p \in \Sigma^{\frac{1}{2}}, \gamma \in \Sigma^{\frac{3}{2}}$ be defined by

$$q = (1+|\partial_x\eta|^2)^{-\frac{1}{2}},$$
$$p = (1+|\partial_x\eta|^2)^{-\frac{5}{4}}\,|\xi|^{\frac{1}{2}} + p^{(-\frac{1}{2})},$$
$$\gamma = \sqrt{l^{(2)}\lambda^{(1)}} + \sqrt{\frac{l^{(2)}}{\lambda^{(1)}}}\frac{\operatorname{Re}\lambda^{(0)}}{2} - \frac{i}{2}(\partial_\xi\cdot\partial_x)\sqrt{l^{(2)}\lambda^{(1)}},$$
$$p^{(-\frac{1}{2})} = \frac{1}{\gamma^{(\frac{3}{2})}}\left\{ql^{(1)} - \gamma^{(\frac{1}{2})}p^{(\frac{1}{2})} + i\partial_\xi\gamma^{(\frac{3}{2})}\cdot\partial_x p^{(\frac{1}{2})}\right\}.$$

Then

$$T_qT_\lambda \sim T_\gamma T_q,\ \ T_qT_l \sim T_\gamma T_p,\ \ T_\gamma \sim (T_\gamma)^*,$$

where $(T_\gamma)^*$ is the adjoint of T_γ.

Now we can write the paralinearization and symmetrization of the equations (1.8) after a change of variable:

Corollary 3.1. *(From [4])Under the hypothesis of Proposition 3.1, introduce the unknowns*[2]

$$U = \psi - T_B\eta,\ \ \Phi_1 = T_p\eta \text{ and } \Phi_2 = T_qU,$$

where we recall,

$$\begin{cases} B = (\partial_y\phi)_{|y=\eta} = \frac{\partial_x\eta\cdot\partial_x\psi + G(\eta)\psi}{1+(\partial_x\eta)^2}, \\ V = (\partial_x\phi)_{|y=\eta} = \partial_x\psi - B\partial_x\eta. \end{cases}$$

Then $\Phi_1, \Phi_2 \in C^0([0,T];H^s(\mathbb{T}))$ and

$$\begin{cases} \partial_t\Phi_1 + T_V\times\partial_x\Phi_1 - T_\gamma\Phi_2 = f_1, \\ \partial_t\Phi_2 + T_V\times\partial_x\Phi_2 + T_\gamma\Phi_1 = f_2, \end{cases} \tag{3.3}$$

with $f_1, f_2 \in L^\infty(0,T;H^s(\mathbb{T}))$, and f_1, f_2 have C^1 dependence on (Φ_1,Φ_2) verifying:

$$\|(f_1,f_2)\|_{L^\infty(0,T;H^s(\mathbb{T}))} \leq C(\|(\eta,\psi)\|_{L^\infty(0,T;H^{s+\frac{1}{2}}\times H^s(\mathbb{T}))}).$$

3.2. **Proof of Theorem 1.6.** Corollary 3.1 shows that the paralinearization and symmetrization of the equations (1.8) are of the form of the equations treated in Theorem 1.3, so the proof will follow the same lines but with more care in treating the non linearity in the dispersive term.

We keep the notations of Theorem 1.6, fixing $(\eta_0,\psi_0) \in H^{s+\frac{1}{2}}(\mathbb{T}) \times H^s(\mathbb{T})$ and $r > 0$. We begin by taking $(\tilde{\eta}_0,\tilde{\psi}_0) \in \mathrm{B}((\eta_0,\psi_0),r) \subset H^{s+\frac{1}{2}}(\mathbb{T})\times H^s(\mathbb{T})$ and consider the solutions $(\eta,\psi),(\tilde{\eta},\tilde{\psi}) \in C^0([0,T];H^{s+\frac{1}{2}}(\mathbb{T})\times H^s(\mathbb{T}))$ to (1.2) with initial data

[2] U is commonly called the "good" unknown of Alinhac. Introduced by Alazard-Metivier in [6], following earlier works by Lannes in [16].

$(\eta_0,\psi_0),(\tilde\eta_0,\tilde\psi_0)$, on a uniform small interval $[0,T]$ where the hypothesis (H_t) is also supposed to be verified. Define the following change of variables:

$$\begin{aligned}\chi(t,x)&=\int_0^x \frac{1}{\sqrt{1+(\partial_x\eta(t,y))^2}}dy-\int_0^t\int_\Sigma\left[\frac{1}{1+(\partial_x\eta(t,y))^2}+\partial_x\phi\right]d\Sigma\\&=\int_0^x \sqrt{1+(\partial_x\eta(t,y))^2}dy-\int_0^t\int_0^{2\pi}\left[\frac{1}{1+(\partial_x\eta(t,y))^2}+\partial_x\phi\right]\sqrt{1+(\partial_x\eta)^2}dy,\end{aligned} \tag{3.4}$$

and $\tilde\chi$ is defined analogously from $(\tilde\eta,\tilde\psi)$.

The main goal of the proof is to show the following estimate:

$$\begin{aligned}&\left\|(\eta,\psi)^*(t,\cdot)-(\tilde\eta,\tilde\psi)^{\tilde *}(t,\cdot)\right\|_{H^s\times H^{s-\frac12}}\\&\le C\Big(\left\|(\eta_0,\psi_0,\tilde\eta_0,\tilde\psi_0)\right\|_{H^{s+\frac12}\times H^s}\Big)\left\|(\eta_0,\psi_0)^*-(\tilde\eta_0,\tilde\psi_0)^{\tilde *}\right\|_{H^s\times H^{s-\frac12}},\end{aligned} \tag{3.5}$$

where * and $^{\tilde *}$ are the paracomposition operators defined by χ and $\tilde\chi$ respectively. We recall that the definition of the paracomposition operator is given in Section A.3.

Put $\Phi=(\Phi_1,\Phi_2)$ the unknowns obtained from (η,ψ) after paralinearization and symmetrization of the equations as in Corollary 3.1. Define analogously $\tilde\Phi=(\tilde\phi_1,\tilde\phi_2)$ from $(\tilde\eta,\tilde\psi)$. Let us notice that, in order to prove 3.5, it suffice to get estimates on $\Phi-\tilde\Phi$. Indeed by the ellipticity of the symbols p and q combined with the immediate L^2 estimates(as $s>2+\frac12$) we have:

$$\begin{aligned}&\left\|(\eta,\psi)^*(t,\cdot)-(\tilde\eta,\tilde\psi)^{\tilde *}(t,\cdot)\right\|_{H^s\times H^{s-\frac12}}\\&\le C\Big(\left\|(\eta_0,\psi_0,\tilde\eta_0,\tilde\psi_0)\right\|_{H^{s+\frac12}\times H^s}\Big)\left\|\Phi^*(t,\cdot)-\tilde\Phi^{\tilde *}(t,\cdot)\right\|_{H^{s-\frac12}\times H^{s-\frac12}},\end{aligned} \tag{3.6}$$

$$\begin{aligned}&\left\|\Phi^*(t,\cdot)-\tilde\Phi^{\tilde *}(t,\cdot)\right\|_{H^{s-\frac12}\times H^{s-\frac12}}\\&\le C\Big(\left\|(\eta_0,\psi_0,\tilde\eta_0,\tilde\psi_0)\right\|_{H^{s+\frac12}\times H^s}\Big)\left\|(\eta,\psi)^*(t,\cdot)-(\tilde\eta,\tilde\psi)^{\tilde *}(t,\cdot)\right\|_{H^s\times H^{s-\frac12}}.\end{aligned} \tag{3.7}$$

3.2.1. *Gauge transform.* Again, as $s>2+\frac12$ we have an immediate L^2 estimates on $\Phi-\tilde\Phi$, thus we only need to get $\dot H^{s-\frac12}\times\dot H^{s-\frac12}$ estimates. Let us start by by writing $\Phi=\Phi_1+i\Phi_2$ in equation (3.3):

$$\partial_t\Phi+T_V\cdot\partial_x\Phi+iT_\gamma\Phi=R_1(\Phi)\Phi, \tag{3.8}$$

Where R_1 verifies

$$\begin{cases}\|R_1(\Phi)\|_{H^{s-\frac12}\to H^{s-\frac12}}\le C\Big(\left\|(\eta_0,\psi_0,\tilde\eta_0,\tilde\psi_0)\right\|_{H^{s+\frac12}\times H^s}\Big),\\ \left\|[R_1(\Phi)-R_1(\tilde\Phi)]\tilde\Phi\right\|_{H^{s-\frac12}}\le C\Big(\left\|(\eta_0,\psi_0,\tilde\eta_0,\tilde\psi_0)\right\|_{H^{s+\frac12}\times H^s}\Big)\left\|\Phi-\tilde\Phi\right\|_{H^{s-\frac12}}.\end{cases}$$

The next step is to preform the change of variable by χ, by Theorem A.6 we get:

$$\partial_t\Phi^*+T_W\cdot\partial_x\Phi+iT_{|\xi|^{\frac32}}\Phi^*=R_1'(\Phi^*)\Phi^*,\ \text{with} \tag{3.9}$$

Where R_1 verifies

$$\left\|[R_1'(\Phi^*)-R_1(\tilde\Phi^{\tilde *})]\tilde\Phi^{\tilde *}\right\|_{H^{s-\frac12}}\le C\Big(\left\|(\eta_0,\psi_0,\tilde\eta_0,\tilde\psi_0)\right\|_{H^{s+\frac12}\times H^s}\Big)\left\|\Phi^*-\tilde\Phi^{\tilde *}\right\|_{H^{s-\frac12}}.$$

We get the same equation on $\tilde\Phi^{\tilde *}$ by symmetry.

Introduce the following gauge transform A_Φ as the time one of the flow map defined by Propositions 4.1 with

$$p_\Phi = \frac{2}{3} |\xi|^{\frac{1}{2}} \partial_x^{-1} W \in \Gamma_2^{2-\alpha}(\mathbb{T}),$$

and put,

$$\theta = A_\Phi \Phi^*. \tag{3.10}$$

We define analogously $A_{\tilde{\Phi}}$ and $\tilde{\theta}$ from $\tilde{\Phi}^{\tilde{*}}$. From Theorem Propositions 4.1 the change of variable (3.10) is Lipschitz from $H^{s-\frac{1}{2}}$ to $H^{s-\frac{1}{2}}$ but under H^s control on $(\Phi, \tilde{\Phi})$ which is equivalent by Theorem A.3 to a control on $\left\|(\eta_0, \psi_0, \tilde{\eta}_0, \tilde{\psi}_0)\right\|_{H^{s+\frac{1}{2}} \times H^s}$. We have:

$$\left\|\Phi^*(t,\cdot) - \tilde{\Phi}^{\tilde{*}}(t,\cdot)\right\|_{\dot{H}^{s-\frac{1}{2}}} \leq C\left(\left\|(\eta_0, \psi_0, \tilde{\eta}_0, \tilde{\psi}_0)\right\|_{H^{s+\frac{1}{2}} \times H^s} \right) \left\|\theta(t,\cdot) - \tilde{\theta}(t,\cdot)\right\|_{\dot{H}^{s-\frac{1}{2}}}, \tag{3.11}$$

$$\left\|\theta(t,\cdot) - \tilde{\theta}(t,\cdot)\right\|_{\dot{H}^{s-\frac{1}{2}}} \leq C\left(\left\|(\eta_0, \psi_0, \tilde{\eta}_0, \tilde{\psi}_0)\right\|_{H^{s+\frac{1}{2}} \times H^s} \right) \left\|\Phi^*(t,\cdot) - \tilde{\Phi}^{\tilde{*}}(t,\cdot)\right\|_{\dot{H}^{s-\frac{1}{2}}}. \tag{3.12}$$

To get the equations on θ and $\tilde{\theta}$ we commute A_Φ and $A_{\tilde{\Phi}}$ with (3.9), we make the computations for θ, those for $\tilde{\theta}$ are obtained by symmetry:

$$A_\Phi \partial_t \Phi^* + A_\Phi T_W \cdot \partial_x \Phi^* + i A_\Phi T_{|\xi|}^{\frac{3}{2}} \Phi = A_\Phi R_1'(\Phi^*)\Phi^*$$

$$\partial_t A_\Phi \Phi^* + i T_{|\xi|}^{\frac{3}{2}} A_\Phi \Phi^* + (A_\Phi T_W \partial_x - [i T_{|\xi|}^{\frac{3}{2}}, A_\Phi])\Phi^* - (\partial_t A_\Phi)\Phi^* = A_\Phi R_1'(\Phi^*)\Phi^*$$

By definition of p_Φ and Proposition 4.4 we have:

$$\partial_\xi(|\xi|^{\frac{3}{2}})\partial_x p_\Phi = W\xi \text{ and } \partial_t A_\Phi = A_\Phi \int_0^1 A_{-r}^{p_\Phi} T_{i\partial_t p_\Phi} A_r^{p_\Phi} dr.$$

thus by Corollary 4.2 we get:

$$\partial_t \theta + i T_{|\xi|}^{\frac{3}{2}} \theta = R_2(\theta)\theta + A_\Phi R_1(\Phi^*) A_\Phi^{-1} \Phi^*, \tag{3.13}$$

where R_2 and $A_\Phi R_1(\Phi) A_\Phi^{-1}$, as $s > 3 + \frac{1}{2}$, verify:

$$\|\mathrm{Re}(R_2(\theta))\|_{H^{s-\frac{1}{2}} \to H^{s-\frac{1}{2}}} \leq C\left(\left\|(\eta_0, \psi_0, \tilde{\eta}_0, \tilde{\psi}_0)\right\|_{H^{s+\frac{1}{2}} \times H^s} \right),$$

$$\left\|[R_2(\theta) - R_2(\tilde{\theta})]\tilde{\theta}\right\|_{H^{s-\frac{1}{2}}} \leq C\left(\left\|(\eta_0, \psi_0, \tilde{\eta}_0, \tilde{\psi}_0)\right\|_{H^{s+\frac{1}{2}} \times H^s} \right) \left\|\theta(t,\cdot) - \tilde{\theta}(t,\cdot)\right\|_{H^{s-\frac{1}{2}}},$$

and,

$$\begin{aligned} &\left\|[A_\Phi R_1(\Phi) A_\Phi^{-1} - A_{\tilde{\Phi}} R_1(\tilde{\Phi}) A_{\tilde{\Phi}}^{-1}]\tilde{\theta}\right\|_{H^{s-\frac{1}{2}}} \\ &\leq C\left(\left\|(\eta_0, \psi_0, \tilde{\eta}_0, \tilde{\psi}_0)\right\|_{H^{s+\frac{1}{2}} \times H^s} \right) \left\|\theta(t,\cdot) - \tilde{\theta}(t,\cdot)\right\|_{H^{s-\frac{1}{2}}}. \end{aligned}$$

Thus we have succeeded to eliminate the term $T_V \cdot \partial_x$ of order 1 in (3.9) and got a term of order $\frac{1}{2}$. The result then follows with a standard energy estimate.

4. Baker-Campbell-Hausdorff formula: composition and commutator estimates

We will start by giving the propositions defining the operators used in the gauge transforms and the symbolic calculus associated to them. From those propositions we will deduce the direct estimates used in in Sections 2.1.1 and 3.2.1.

Notation 4.1. *To compute the conjugation and commutation of operators with a flow map, we introduce Lie derivatives, i.e commutators. More precisely we introduce the following notations for commutation between operators:*

$$\mathfrak{L}_a^0 b = b,\ \mathfrak{L}_a b = [a, b] = a \circ b - b \circ a,\ \mathfrak{L}_a^2 b = [a, [a, b]],\ \mathfrak{L}_a^k b = \underbrace{[a, [\cdots, [a,}_{k \text{ times}} b]] \cdots].$$

In the following proposition the variable $t \in [0, T]$ is the generic time variable that appeared in the previous section and a new variable $\tau \in \mathbb{R}$ will be used and they should not be confused.

We start with the proposition defining the Flow map and its standard properties.

Proposition 4.1. *Consider two real numbers $\delta < 1$, $s \in \mathbb{R}$ and a real valued symbol $p \in \Gamma_1^\delta(\mathbb{D})$. The following linear hyperbolic equation is globally well-posed:*

$$\begin{cases} \partial_\tau h - iT_p h = 0, \\ h(0, \cdot) = h_0(\cdot) \in H^s(\mathbb{D}). \end{cases} \tag{4.1}$$

For $\tau \in \mathbb{R}$, define A_τ^p as the flow map associated to 4.1 i.e:

$$\begin{aligned} A_\tau^p : &H^s(\mathbb{D}) \to H^s(\mathbb{D}) \\ &h_0 \mapsto h(\tau, \cdot). \end{aligned} \tag{4.2}$$

Then for $\tau \in \mathbb{R}$ we have,

(1) $A_\tau^p \in \mathscr{L}(H^s(\mathbb{D}))$ *and*
$$\|A_\tau^p\|_{H^s \to H^s} \leq e^{C|\tau| M_1^\delta(p)}.$$

(2)
$$iT_p \circ A_\tau^p = A_\tau^p \circ iT_p,\ A_{\tau+\tau'}^p = A_\tau^p A_{\tau'}^p.$$

(3) A_τ^p *is invertible and,*
$$(A_\tau^p)^{-1} = A_{-\tau}^p.$$
Moreover,
$$(A_\tau^p)^* = A_{-\tau}^{(T_p)^*} = A_{-\tau}^p + R,$$
where R is a $\delta - 1$ regularizing operator and $A_\tau^{(T_p)^*}$ is the flow generated by the Cauchy problem:
$$\begin{cases} \partial_\tau h - i(T_p)^* h = 0, \\ h(0, \cdot) = h_0(\cdot) \in H^s(\mathbb{D}). \end{cases} \tag{4.3}$$

(4) Taking a real valued symbol $\tilde{p} \in \Gamma_1^\delta(\mathbb{D})$ we have:
$$\left\|[A_\tau^p - A_\tau^{\tilde{p}}]h_0\right\|_{H^s} \leq C\,|\tau|\, e^{C|\tau| M_1^\delta(p,\tilde{p})} M_0^\delta(p - \tilde{p})\, \|h_0\|_{H^{s+\delta}}\,. \tag{4.4}$$

Proof. Points $(1), (2), (3)$ are simple consequences of the hyperbolicity and well posedness of the Cauchy problem (4.1). Point (4) comes by writing:

$$\partial_\tau [A_\tau^p - A_\tau^{\tilde{p}}]h_0 - iT_p[A_\tau^p - A_\tau^{\tilde{p}}]h_0 = iT_{p-\tilde{p}} A_\tau^{\tilde{p}} h_0,$$

and making the usual energy estimate. □

Remark 4.1. *The hypothesis $p \in \Gamma_1^\delta$ can be relaxed to $p \in \Gamma_\delta^\delta$, which is the minimal hypothesis to ensure well posedness in Sobolev spaces of (4.1).*

At the moment the only bounds we obtained on A^p_τ is the continuity bounds on Sobolev spaces, in order to study it's symbol and the symbol of conjugated operators we need to transfer those continuity bounds to estimates of symbols seminorms. This was first done by Beals in [9] for pseudodifferential operators in the class $S^m_{\rho,\rho}$ with $\rho < 1, m \in \mathbb{R}$. The following Lemma gives explicitly the key estimate adapted from [9] given by (4.5), and we give one new estimate (4.6) that can then be directly applied to the paradifferential setting.

Lemma 4.1. *Consider an operator A continuous from $\mathscr{S}(\mathbb{D})$ to $\mathscr{S}'(\mathbb{D})$ and let $a \in \mathscr{S}'(\mathbb{D} \times \hat{\mathbb{D}})$ the unique symbol associated to A (cf [10] for the uniqueness), i.e, let K be the kernel associated to A then:*

$$u, v \in \mathscr{S}(\mathbb{D}), (Au, v) = K(u \otimes v), \quad a(x,\xi) = \mathscr{F}_{y\to\xi} K(x, x-y).$$

- *If A is continuous from H^m to L^2, with $m \in \mathbb{R}$, and $[\frac{1}{i}\frac{d}{dx}, A]$ is continuous from $H^{m+\delta}$ to L^2 with $\delta < 1$, then $(1+|\xi|)^{-m} a(x,\xi) \in L^\infty_{x,\xi}(\mathbb{D}\times\hat{\mathbb{D}})$ and we have the estimate:*

$$\left\|(1+|\xi|)^{-m} a\right\|_{L^\infty_{x,\xi}} \leq C_m \left[\|A\|_{H^m\to L^2} + \left\| \left[\frac{1}{i}\frac{d}{dx}, A\right] \right\|_{H^{m+\delta}\to L^2} \right]. \tag{4.5}$$

- *If A is continuous from H^m to L^2, with $m \in \mathbb{R}$, and $[ix, A]$ is continuous from from $H^{m-\rho}$ to L^2 with $\rho \geq 0$, then $(1+|\xi|)^{-m} a(x,\xi) \in L^\infty_{x,\xi}(\mathbb{D}\times\hat{\mathbb{D}})$ and we have the estimate:*

$$\left\|(1+|\xi|)^{-m} a\right\|_{L^\infty_{x,\xi}} \leq C_m [\|A\|_{H^m\to L^2} + \|[ix, A]\|_{H^{m-\rho}\to L^2}]. \tag{4.6}$$

Proof. First without loss of generality through a standard mollification argument we work with $a \in \mathscr{S}(\mathbb{D}\times\hat{\mathbb{D}})$. We study the cases on $\mathbb{T}$ and $\mathbb{R}$ separately.
Operators defined on $\mathbb{T}$. The first key observation is the following:

$$(x,\xi) \in \mathbb{T}\times\mathbb{Z}, \; e^{-ix\cdot\xi} A e^{ix\cdot\xi} = a(x,\xi), \tag{4.7}$$

which one can right as $e^{ix\cdot\xi} \in L^2_x(\mathbb{T})$. Thus taking L^2 norms in x we get:

$$\|a(\cdot,\xi)\|_{L^2_x} \leq \|A\|_{H^m\to L^2} \left\|e^{ix\cdot\xi}\right\|_{H^m_x} \leq C_m \|A\|_{H^m\to L^2} (1+|\xi|)^m. \tag{4.8}$$

Now to get the analogue of (4.8) but in the ξ variable we observe that the continuity hypothesis reads for $(u,v) \in \mathscr{S}$:

$$\left| \int_{\mathbb{T}\times\mathbb{Z}} e^{ix\cdot\xi} a(x,\xi)(1+|\xi|)^{-m} \mathscr{F}(v)(\xi) u(x) dx d\xi \right| \leq \|A\|_{H^m\to L^2} \|\mathscr{F}(v)\|_{L^2_\xi} \|u\|_{L^2_x},$$

which we rewrite as:

$$\left| \int_{\mathbb{Z}} \mathscr{F}(v)(\xi) \left[\int_{\mathbb{T}} e^{ix\cdot\xi} (1+|\xi|)^{-m} a(x,\xi) u(x) dx \right] d\xi \right| \leq \|A\|_{H^m\to L^2} \|\mathscr{F}(v)\|_{L^2_\xi} \|u\|_{L^2_x},$$

Now we treat a as an operator with roles of (x,ξ) exchanged. Choosing,

$$u(x) = \mathscr{F}^{-1}(e^{ix\cdot\xi}),$$

we get analogously to (4.8):

$$\left\|(1+|\xi|)^{-m} a(x,\xi)\right\|_{L^\infty_x L^2_\xi} \leq C_m \|A\|_{H^m\to L^2}. \tag{4.9}$$

The second key observation is:

$$e^{-ix\cdot\xi} \left[\frac{1}{i}\frac{d}{dx}, A\right] e^{ix\cdot\xi} = \partial_x a(x,\xi), \quad e^{-ix\cdot\xi}[ix, A] e^{ix\cdot\xi} = \partial_\xi a(x,\xi). \tag{4.10}$$

Iterating the previous computations we get:

$$\|\partial_x a(\cdot,\xi)\|_{L^2_x} \leq C_m (1+|\xi|)^{m+\delta} \left\| \left[\frac{1}{i}\frac{d}{dx}, A\right] \right\|_{H^{m+\delta}\to L^2}, \tag{4.11}$$

and as $\rho \geq 0$:

$$\left\|\partial_\xi[(1+|\xi|)^{-m}a(x,\xi)]\right\|_{L^\infty_x L^2_\xi} \leq C_m[\|A\|_{H^m\to L^2} + \|[ix,A]\|_{H^{m-\rho}\to L^2}]. \tag{4.12}$$

By the Sobolev embedding (4.9) and (4.12) give the desired result (4.6) on the torus.

To get (4.5) we introduce:

$$b(x,\xi,\xi_0) = a\left(\frac{x}{(1+|\xi_0|)^\delta}, (1+|\xi_0|)^\delta \xi\right).$$

As $\delta < 1$ we have that $(1+|\xi|)^\delta \sim (1+|\xi_0|)^\delta$ for $|\xi-\xi_0| \leq c(1+|\xi_0|)^\delta$ for some fixed $c>0$. Considering b as a function of (x,ξ) on $\mathbb{T}\times \mathrm{B}(\xi_0,c)$, inequalities (4.8) and (4.11) give uniform L^2_x and H^1_x estimates thus by the Sobolev embedding we get:

$$\|b(x,\xi)\|_{L^\infty_x} \leq C_m\left[\|A\|_{H^m\to L^2} + \left\|\left[\frac{1}{i}\frac{d}{dx}, A\right]\right\|_{H^{m+\delta}\to L^2}\right](1+|\xi|)^m, \tag{4.13}$$

which transferred back to a give the desired result (4.5) on the torus.

Operators defined on $\mathbb{R}$. The main problem we face on $\mathbb{R}$ when adapting the previous proof is we can no longer use $e^{ix\cdot\xi}$ as a test function as it no longer belongs to $L^2(\mathbb{R})$. One way to get over this was given by Beals in [9], we choose g in $\mathscr{S}(\mathbb{R})$ such that $g(0)=1$, $\mathscr{F}(g)$ is supported in $\{|\xi|\leq 1\}$ and $g(x)=g(-x)$. Let $g_x(y) = g(y-x)$ and compute for $u \in \mathscr{S}$:

$$u(x) = u(x)g_x(x) = \int_{\mathbb{R}\times\mathbb{R}} e^{i(x-y)\cdot\xi} g_x(y)u(y)dyd\xi = \int_{\mathbb{R}\times\mathbb{R}} e^{-iy\cdot\xi}e^{ix}g_y(x)u(y)dyd\xi.$$

We now compute an analogue of (4.7):

$$\begin{aligned} Au(x) &= \int_{\mathbb{R}\times\mathbb{R}} e^{-iy\cdot\xi} A\big(e^{ix}g_y\big)(x)u(y)dyd\xi \\ &= \int_{\mathbb{R}\times\mathbb{R}} e^{i(x-y)\cdot\xi} a_0(x,y,\xi)u(y)dyd\xi \\ &= \int_{\mathbb{R}} e^{ix\cdot\xi} a(x,\xi)\mathscr{F}(u)(\xi)d\xi, \end{aligned}$$

where,

$$a_0(x,y,\xi) = e^{-ix\cdot\xi}A\big(e^{ix\cdot\xi}g_y\big)(x),$$

and,

$$a(x,\xi) = \int_{\mathbb{R}\times\mathbb{R}} e^{i(x-y)\cdot(\eta-\xi)} a_0(x,y,\eta)d\eta dy.$$

Applying the same arguments as in the periodic case we get:

$$\left\|(1+|\xi|)^{-m}\partial_y^k a_0\right\|_{L^\infty_{x,y,\xi}} \leq C_{m,k}\left[\|A\|_{H^m\to L^2} + \left\|\left[\frac{1}{i}\frac{d}{dx}, A\right]\right\|_{H^{m+\delta}\to L^2}\right], k\in\mathbb{N} \tag{4.14}$$

and,

$$\left\|(1+|\xi|)^{-m}\partial_y^k a_0\right\|_{L^\infty_{x,y,\xi}} \leq C_{m,k}[\|A\|_{H^m\to L^2} + \|[ix,A]\|_{H^{m-\rho}\to L^2}], k\in\mathbb{N}. \tag{4.15}$$

Thus to conclude the proof we need to transfer the information on the amplitude a_0 to the symbol a which is a simple application of Oscillatory Integrals. Indeed it suffices to write:

$$\begin{aligned} a(x,\xi) &= \int_{\mathbb{R}\times\mathbb{R}} e^{i(x-y)\cdot(\eta-\xi)} a_0(x,y,\eta)d\eta dy \\ &= \int_{\mathbb{R}\times\mathbb{R}} \frac{1}{\langle\xi-\eta\rangle^2}(I-\Delta_y)e^{i(x-y)\cdot(\eta-\xi)} a_0(x,y,\eta)d\eta dy \\ &= \int_{\mathbb{R}\times\mathbb{R}} \frac{1}{\langle\xi-\eta\rangle^2} e^{i(x-y)\cdot(\eta-\xi)}(I-\Delta_y)^* a_0(x,y,\eta)d\eta dy \Big], \end{aligned}$$

which gives the desired result as $a_0(x,y,\eta)$ and all of it's y derivatives are y integrable. □

Remark 4.2. *It is worth noting that if instead of Sobolev estimates we had Hölder continuity estimates on A, then combined with (4.7) it gives directly the analogue of estimates (4.5) and (4.6).*

Analogously to Beals characterization of pseudodifferential through the continuity of the successive commutators $\mathrm{Op}(x)$, $\frac{1}{i}\frac{d}{dx}$ with A on Sobolev spaces, we give the following characterization of Paradifferential operators through estimate (4.6).

Corollary 4.1. *Consider two real numbers m and $\rho \geq 0$ and the spaces of paradifferential symbols $\Gamma^m_\rho(\mathbb{D})$ equipped with the topology induced by the seminorms $M^m_\rho(\cdot;k)$ for $k \in \mathbb{N}$ defined in A.5 giving it a Fréchet space structure.*

For $p \in \Gamma^m_\rho(\mathbb{D})$ we introduce the following family of seminorms:

$$H^m_0(p;k) = \sum_{j=0}^{k} \left\| \mathfrak{L}^j_{ix} T_p \right\|_{H^m\to H^j},$$

$$H^m_n(p;k) = \sum_{l=0}^{n}\sum_{j=0}^{k} \left\| \mathfrak{L}^j_{ix}\mathfrak{L}^l_{\frac{1}{i}\frac{d}{dx}} T_p \right\|_{H^m\to H^j}, \ n\in\mathbb{N}, n\leq\rho,$$

and if $\rho \notin \mathbb{N}$:

$$H^m_\rho(p;k) = H^m_{\lfloor\rho\rfloor}(p;k) + \sup_{n\in\mathbb{N}} 2^{n(\rho-\lfloor\rho\rfloor)} \sum_{j=0}^{k} \left\| \mathfrak{L}^j_{ix}\mathfrak{L}^{\lfloor\rho\rfloor}_{\frac{1}{i}\frac{d}{dx}} T_p P_{\leq n}(D) \right\|_{H^m\to H^j}.$$

Then $H^m_\rho(p;k)_{k\in\mathbb{N}}$ induces an equivalent Fréchet topology to $M^m_\rho(\cdot;k)_{k\in\mathbb{N}}$ on:

$$\psi^{B,b}\left(\Gamma^m_\rho(\mathbb{D})\right) = \left\{\sigma^{B,b}_p, p\in\Gamma^m_\rho(\mathbb{D})\right\}.$$

In order to give the key symbolic calculus results in Propositions 4.2 and 4.3 we need to introduce the paradifferential analogue of the Hörmander symbol class $S^0_{1-\delta,\delta}$. For this we follow [26] and introduce the space of non regular symbols:

Definition-Proposition 4.1. *Consider $s \in \mathbb{R}_+$, for $0 \leq \delta, \rho < 1$, we say:*

$$p \in W^{s,\infty}S^m_{\rho,\delta}(\mathbb{D}) \iff \begin{cases} \left|D^k_\xi p(x,\xi)\right| \leq C_k\langle\xi\rangle^{m-\rho k} \\ \left\|D^k_\xi p(\cdot,\xi)\right\|_{W^{s,\infty}} \leq C_k\langle\xi\rangle^{m-\rho k+s\delta} \end{cases}, (x,\xi)\in\mathbb{D}\times\tilde{\mathbb{D}},\ k\geq 0. \tag{4.16}$$

The best constant in (4.16) defines a seminorm denoted by ${}^{\rho,\delta}M^m_s(\cdot;k), k\in\mathbb{N}$ where k is the number of derivatives we make on the frequency variable ξ, we also define the seminorm ${}^{\rho,\delta}M^m_s = {}^{\rho,\delta}M^m_s(\cdot;1)$. We define analogously $W^{s,\infty}S^m_{\rho,\delta}(\mathbb{D}^*\times\tilde{\mathbb{D}})$.

Analogously to Corollary 4.1 we introduce the following family of seminorms:

$${}^{\rho,\delta}H^m_0(p;k) = \sum_{j=0}^{k} \left\| \mathfrak{L}^j_{ix} p \right\|_{H^m\to H^{j\rho}},$$

$$ {}^{\rho,\delta}H^m_n(p;k) = \sum_{l=0}^{n}\sum_{j=0}^{k}\left\|\mathfrak{L}^j_{ix}\mathfrak{L}^l_{\frac{1}{i}\frac{d}{dx}}p\right\|_{H^m\to H^{j\rho-l\delta}},\ n\in\mathbb{N}, n\le\rho, $$

and if $s\notin\mathbb{N}$:

$$ {}^{\rho,\delta}H^m_s(p;k) = H^m_{\lfloor s\rfloor}(p;k) + \sup_{n\in\mathbb{N}} 2^{n(s-\lfloor s\rfloor)}\sum_{j=0}^{k}\left\|\mathfrak{L}^j_{ix}\mathfrak{L}^{\lfloor s\rfloor}_{\frac{1}{i}\frac{d}{dx}}[P_n(D)p]\right\|_{H^m\to H^{j\rho-l\delta}}, $$

where $P_n(D)$ is applied to p in the x variable.

Then ${}^{\rho,\delta}H^m_s(p;k)_{k\in\mathbb{N}}$ induces an equivalent Fréchet topology to ${}^{\rho,\delta}M^m_s(\cdot;k)_{k\in\mathbb{N}}$ on $W^{s,\infty}S^m_{\rho,\delta}$.

Proof. The L^2 continuity of such operators and control by their symbol seminorms is given in Appendix B of [22] and the reciprocal is given by Lemma 4.1. □

Remark 4.3. *We note that for the standard paradifferential symbols classes we have:*

$$ \Gamma^m_\rho = W^{\rho,\infty}S^m_{1,0}. $$

Now all of the ingredients are in place to give the key commutation and conjugation result.

Proposition 4.2. *Consider three real numbers $\delta<1$, $s\in\mathbb{R}$, $\rho\ge 1$, a real valued symbol $p\in\Gamma^\delta_\rho(\mathbb{D})$. Let $A^p_\tau, \tau\in\mathbb{R}$ be the flow map defined by Proposition 4.1 and take a symbol $b\in\Gamma^\beta_\rho(\mathbb{D}), \beta\in\mathbb{R}$ then we have:*

(5) There exists $b^p_\tau\in W^{\rho,\infty}S^\beta_{1-\delta,\delta}(\mathbb{D})$ such that:

$$ A^p_\tau\circ T_b\circ A^p_{-\tau} = T_{b^p_\tau}. \tag{4.17} $$

Moreover we have the estimates:

$$ \left\|T^{lim}_{b^p_\tau} - \sum_{k=0}^{\lceil\rho-1\rceil}\frac{\tau^k}{k!}\mathfrak{L}^k_{iT_p}T_b\right\|_{H^s\to H^{s-\beta-\lceil\rho\rceil\delta+\rho}} \le C_\rho M^\beta_\rho(b)M^\delta_\rho(p)^{\lceil\rho\rceil}, \tag{4.18} $$

$$ {}^{1-\delta,\delta}H^\beta_\rho(b^p_\tau;k) \le C_k(M^\delta_1(p))[H^\beta_\rho(b;k)+H^\beta_\rho(b;k)H^\delta_\rho(p;k)],\ \ k\in\mathbb{N}. \tag{4.19} $$

(6) There exists ${}^cb^p_\tau\in W^{\rho-1,\infty}S^{\beta+\delta-1}_{1-\delta,\delta}(\mathbb{D})$ such that:

$$ [A^p_\tau, T_b] = A^p_\tau T^{lim}_{{}^cb^p_\tau} \iff T^{lim}_{{}^cb^p_\tau} = T_b - T^{lim}_{b^p_{-\tau}}. \tag{4.20} $$

Moreover we have the estimates:

$$ \left\|T^{lim}_{{}^cb^p_\tau} - \sum_{k=1}^{\lceil\rho-1\rceil}(-1)^{k-1}\frac{\tau^k}{k!}\mathfrak{L}^k_{iT_p}T_b\right\|_{H^s\to H^{s-\beta-\lceil\rho\rceil\delta+\rho}} \le C_\rho M^\beta_\rho(b)M^\delta_\rho(p)^{\lceil\rho\rceil}, \tag{4.21} $$

$$ {}^{1-\delta,\delta}H^{\beta+\delta-1}_{\rho-1}({}^cb^p_\tau;k) \le C_k(M^\delta_1(p))[H^\beta_\rho(b;k)+H^\beta_\rho(b;k)H^\delta_\rho(p;k)], k\in\mathbb{N}. \tag{4.22} $$

Remark 4.4. • *It is important to notice that the main result of this proposition is the factorization of the A^p_τ terms in (4.17) and (4.20) where the right hand sides contain symbols in the usual classes modulo a more regular remainder. This was not a priori the case of the left hand sides containing A^p_τ. In other words we prove the stability of Γ^m_ρ under the conjugation by A^p_τ.*

This is crucial when studying the regularity of the flow map for:

$$ s > 1 + \frac{2-\alpha}{\alpha-1} + \frac{1}{2}. $$

Indeed if p depends on a parameter λ, $D_\lambda A^p_\tau \circ T_b \circ A^p_{-\tau}$ is a priori an operator of order $\beta+\delta$ by (4.4), but $D_\lambda T^{lim}_{b^p_\tau}$ is shown in proposition 4.5 to be an operator of order β.

- *In the language of pseudodifferential operators, $T_{b^p_\tau}$ is the asymptotic sum of the series $(\frac{\tau^k}{k!}\mathfrak{L}^k_{iT_p}T_b)$ i.e the Baker-Campbell-Hausdorff formal series. Though $T_{b^p_\tau}$ is not necessarily equal to this sum, for this sum need not converge.*
- *By the right hand side we have the continuity of $T^{lim}_{b^p_\tau}$ on H^s for $s\in\mathbb{R}$ and not only $s>0$ which is an improvement on A.2.*

Proof. The structure of the proof is as follows:

(I) We will give a proof of the estimate (4.18) assuming b^p_τ exists.

(II) We will prove the existence of $b^p_\tau W^{\rho,\infty}S^\beta_{1-\delta,\delta}(\mathbb{D})$ which is the subtle part of the proof.

(III) Finally we will deduce point (6) from point (5).

Point (I). For point (5) we compute,

$$\partial_\tau[A^p_\tau\circ T_b\circ A^p_{-\tau}] = iT_p\circ A^p_\tau\circ T_b\circ A^p_{-\tau} - A^p_\tau\circ T_b\circ iT_p\circ A^p_{-\tau}$$

Using (2),

$$\begin{aligned}\partial_\tau[A^p_\tau\circ T_b\circ A^p_{-\tau}] &= A^p_\tau\circ iT_p\circ T_b\circ A^p_{-\tau} - A^p_\tau\circ T_b\circ iT_p\circ A^p_{-\tau}\\ &= A^p_\tau[iT_p,T_b]A^p_{-\tau}.\end{aligned} \tag{4.23}$$

As $A_0=Id$, integrating on $[0,\tau]$ we get:

$$A^p_\tau\circ T_b\circ A^p_{-\tau} = T_b + \int_0^\tau \underbrace{A^p_r[iT_p,T_b]A^p_{-r}}_{*}\,dr.$$

Iterating the computation in $*$ we get for $n\in\mathbb{N}^*$,

$$A^p_\tau\circ T_b\circ A^p_{-\tau} = \sum_{k=0}^n \frac{\tau^k}{k!}\mathfrak{L}^k_{iT_p}T_b + \int_0^\tau \frac{(\tau-r)^n}{n!}A^p_r\mathfrak{L}^{n+1}_{iT_p}T_bA^p_{-r}dr. \tag{4.24}$$

Now the key point is the continuity of paradifferential operators given by Theorem A.1 and the symbolic calculus rules given by Theorem A.3. By Lemma 4.2:

$$\left\|\mathfrak{L}^{\lceil\rho\rceil}_{iT_p}T_b\right\|_{H^s\to H^{s-\beta-\lceil\rho\rceil\delta+\rho}} \le C_\rho M^\beta_\rho(b)M^\delta_\rho(p)^{\lceil\rho\rceil},\ \text{for } \rho\in\mathbb{R}_+, \tag{4.25}$$

where $\lceil\rho\rceil$ is the upper integer part of ρ.

Thus applying point (1) combined with (4.25) we get (4.18).

Point (II). The constant C_ρ in (4.25) is estimated "brutally" by Lemma 4.2: $O(2^\rho\times\lceil\rho\rceil!)$, thus even though one has a $\frac{1}{\lceil\rho\rceil!}$ in (4.24) the convergence result is non trivial. To get past this let us express explicitly the difficulty in the problem. Rearranging the terms in (4.23) we see that:

$$\partial_\tau[A^p_\tau\circ T_b\circ A^p_{-\tau}] = A^p_\tau[iT_p,T_b]A^p_{-\tau} = [iT_p, A^p_\tau T_bA^p_{-\tau}], \tag{4.26}$$

thus we have to solve the following the Cauchy problem in $\mathscr{L}(H^s(\mathbb{D}),H^{s-\beta}(\mathbb{D}))$:

$$\begin{cases}\partial_\tau f(\tau) = [iT_p, f(\tau)]\in\Gamma^\beta_{\rho-1}(\mathbb{D}),\\ f(0)=T_b\in\Gamma^\beta_\rho(\mathbb{D}).\end{cases} \tag{4.27}$$

This amounts to the non trivial problem of solving a linear ODE in the Fréchet space $\Gamma^\beta_{+\infty}(\mathbb{D})$, indeed such a problem need not have a solution in general, and even if it does, it need not be unique. To solve such a problem one usually has to look at a Nash-Moser type scheme, though in our case we have an explicit ODE that can be solved with a series and a loss of derivative. Thus inspired by Hörmander's [15], we remark another key estimate given by the continuity of paradifferential operators given by Theorem A.1 and the symbolic calculus rules given by Theorem A.3(cf Lemma 4.2):

$$\left\|\mathfrak{L}_{iT_p}^{\lceil\rho\rceil}T_b\right\|_{H^s\to H^{s-\beta-\lceil\rho\rceil\delta}} \leq C^{\lceil\rho\rceil}M_0^\beta(b)M_0^\delta(p)^{\lceil\rho\rceil},\ \text{for } \rho\in\mathbb{R}_+. \tag{4.28}$$

This means that if we can compensate the loss of $\beta+\lceil\rho\rceil\delta$ derivatives with a cost negligible in comparison to $\lceil\rho\rceil!$, we would have a convergent series in (4.24). A first approach would be to interpolate (4.25) and (4.28) which gives:

$$\begin{aligned}\left\|\mathfrak{L}_{iT_p}^{\lceil\rho\rceil}T_b\right\|_{H^s\to H^{s-\beta}} \leq\ & C^{\frac{\rho-\lceil\rho\rceil\delta}{\rho}\lceil\rho\rceil}M_0^\beta(b)^{\frac{\rho-\lceil\rho\rceil\delta}{\rho}}M_0^\delta(p)^{\frac{\rho-\lceil\rho\rceil\delta}{\rho}\lceil\rho\rceil} \\ & \times C^{(\lceil\rho\rceil+1)\frac{\lceil\rho\rceil\delta}{\rho}}2^{\lceil\rho\rceil\frac{\lceil\rho\rceil\delta}{\rho}}\lceil\rho\rceil!^{\frac{\lceil\rho\rceil\delta}{\rho}}M_\rho^\beta(b)^{\frac{\lceil\rho\rceil\delta}{\rho}}M_\rho^\delta(p)^{\lceil\rho\rceil\frac{\lceil\rho\rceil\delta}{\rho}},\end{aligned} \tag{4.29}$$

This indeed solves the cost $\lceil\rho\rceil!$ of (4.25) but depends on $M_\rho^\cdot$ norms of b and p. An idea to control those norms in a cost negligible in comparison to $\lceil\rho\rceil!$ would be to mollify p and b using an analytic mollifier, this might work but we found it better to mollify differently.

For this we introduce a mollification with the Gaussian function $\phi_\epsilon(D)$ with symbol:

$$\phi_\epsilon(\xi) = \frac{1}{\epsilon\sqrt{2\pi}}e^{-\frac{\xi^2}{2\epsilon^2}}, \epsilon>0. \tag{4.30}$$

Other than the standard properties of mollifiers, we have the following properties:

- For $h\in H^s(\mathbb{D})$, $\phi_\epsilon(D)h$ is real analytic.
- The moments of the Gaussian can be explicitly computed by, for $k\in n$:

$$\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}|\xi|^k e^{-\frac{\xi^2}{2}}d\xi = \frac{2^{\frac{k}{2}}\Gamma(\frac{k+1}{2})}{\sqrt{\pi}} = (k-1)!!\begin{cases}1 \text{ if } k \text{ is even}\\ \sqrt{\frac{2}{\pi}} \text{ if } k \text{ is odd}\end{cases}.$$

- From the moments of the Gaussian we deduce that, for $h\in H^s(\mathbb{D})$ and $k\in n$:

$$\left\|\partial_x^k\phi_\epsilon(D)h\right\|_{H^s}\leq C_k\epsilon^{-k}\|h\|_{H^s},$$

  and C_k verifies for all $K>0$, $K^kC_k=o(k!)$.

Now by the symbolic calculus rules given by Theorem A.3, for $\epsilon>0$, there exists ${}^\epsilon b_\tau^p\in\Gamma_\rho^0(\mathbb{D})$ such that:

$$\sum_{k=0}^{+\infty}\frac{\tau^k}{k!}\mathfrak{L}_{iT_p}^kT_b\phi_\epsilon(D) = T^{lim}_{{}^\epsilon b_\tau^p},\ \text{with},$$

$$M_\rho^0({}^\epsilon b_\tau^p)\leq\sum_{k=0}^{+\infty}C^kC_k\frac{|\tau|^k}{k!}\epsilon^{-k\delta-\beta}M_\rho^\beta(b)M_\rho^\delta(p)^k. \tag{4.31}$$

In order to pass to the limit in ϵ we will express ${}^\epsilon b_\tau^p$ differently, for all $\epsilon>0$, $\sum_{k=0}^{n}\frac{\tau^k}{k!}\mathfrak{L}_{iT_p}^kT_b\phi_\epsilon(D)$ converges in $\mathscr{L}(H^s(\mathbb{D}))$, thus by unicity of the limit:

$$A_\tau^p\circ T_b\circ A_{-\tau}^p\phi_\epsilon(D) = \sum_{k=0}^{+\infty}\frac{\tau^k}{k!}\mathfrak{L}_{iT_p}^kT_b\phi_\epsilon(D). \tag{4.32}$$

Thus,

$$A_\tau^p\circ T_b\circ A_{-\tau}^p\phi_\epsilon(D) = T^{lim}_{{}^\epsilon b_\tau^p}. \tag{4.33}$$

Now we estimate the ${}^{\delta-1}H_\rho^\beta(\cdot;k)_{k\in\mathbb{N}}$ norms of ${}^\epsilon b_\tau^p$. To do so we need, in the word of Hörmander [14], a result which interpolates between information on the norm a of an operator and bounds for the derivatives of it's symbol. This was exactly the goal of Lemma 4.1.

By commuting $\frac{1}{i}\frac{d}{dx}$ and ix with (4.33) we get:

$$\begin{aligned}[\frac{1}{i}\frac{d}{dx}, T^{lim}_{\epsilon b^p_\tau}] =& [\frac{1}{i}\frac{d}{dx}, A^p_\tau] \circ T_b \circ A^p_{-\tau}\phi_\epsilon(D) + A^p_\tau \circ [\frac{1}{i}\frac{d}{dx}, T_b] \circ A^p_{-\tau}\phi_\epsilon(D) \\ &+ A^p_\tau \circ T_b[\frac{1}{i}\frac{d}{dx}, \circ A^p_{-\tau}]\phi_\epsilon(D) + A^p_\tau \circ T_b \circ A^p_{-\tau}[\frac{1}{i}\frac{d}{dx}, \phi_\epsilon(D)].\end{aligned}$$

and,

$$\begin{aligned}[ix, T^{lim}_{\epsilon b^p_\tau}] =& [ix, A^p_\tau] \circ T_b \circ A^p_{-\tau}\phi_\epsilon(D) + A^p_\tau \circ [ix, T_b] \circ A^p_{-\tau}\phi_\epsilon(D) \\ &+ A^p_\tau \circ T_b[ix, \circ A^p_{-\tau}]\phi_\epsilon(D) + A^p_\tau \circ T_b \circ A^p_{-\tau}[ix, \phi_\epsilon(D)].\end{aligned}$$

To estimate $[\frac{1}{i}\frac{d}{dx}, A^p_\tau]$ and $[ix, A^p_\tau]$ we get back to (4.1) and see that:

$$\begin{cases}[\frac{1}{i}\frac{d}{dx}, A^p_\tau] = \int_0^\tau A^p_{\tau-r}[\frac{1}{i}\frac{d}{dx}, T_{ip}]A^p_r, \\ [ix, A^p_\tau] = \int_0^\tau A^p_{\tau-r}[ix, T_{ip}]A^p_r.\end{cases} \tag{4.34}$$

Thus by iteration, the continuity of A^p_τ and Lemma 4.1 we get:

$$^{1-\delta,\delta}H^\beta_n(b^p_\tau; k) \leq C_{n,k}(M^\delta_1(p))[H^\beta_n(b; k) + H^\beta_n(b; k)H^\delta_n(p; k)], \ (k, n) \in \mathbb{N}.$$

Thus we can pass to the limit in ϵ in (4.33), there exist $b^p_\tau \in W^{0,\infty}S^\beta_{1-\delta,0}$ such that:

$$A^p_\tau \circ T_b \circ A^p_{-\tau} = T^{lim}_{b^p_\tau}. \tag{4.35}$$

Now to get $^{1-\delta,\delta}H^\beta_\rho(b^p_\tau; k), \rho \notin \mathbb{N}$ estimates we will use the Littelwood Paley decomposition. Indeed recalling $(P_{\leq k})$ as the Littelwood Paley projectors defined in A.1 we have by Lemma A.1:

$$T^{lim}_a P_{\leq k} = P_{\leq k+1} T^{lim}_{(P_{\leq k+1}(D)a)} P_{\leq k},$$

where $\mathscr{F}(P_{\leq k+1}(D)a) = P_{\leq k+1}(\eta)\mathscr{F}(a)(\eta, \xi)$. Thus going back to the sum (4.32) we find:

$$A^{(P_{\leq k+1}(D)p}_\tau \circ T_{(P_{\leq k+1}(D)b} \circ A^{(P_{\leq k+1}(D)p}_{-\tau} P_{\leq k} = P_{\leq k+1} T^{lim}_{(P_{\leq k+1}(D)b^p_\tau} P_{\leq k}$$

thus we get by commuting with ix, $\frac{1}{i}\frac{d}{dx}$ and estimating as previously:

$$\begin{aligned}{}^{1-\delta,\delta}H^\beta(P_{\leq k+1}(D)\mathfrak{L}^{\lfloor\rho\rfloor}_{\frac{1}{i}\frac{d}{dx}} b^p_\tau; k) \leq& C_{n,k}(M^\delta_1(p))H^\beta_n(P_{\leq k+1}(D)\mathfrak{L}^{\lfloor\rho\rfloor}_{\frac{1}{i}\frac{d}{dx}} b; k) \\ &+ C_k(M^\delta_1(p))H^\beta_n(P_{\leq k+1}(D)\mathfrak{L}^{\lfloor\rho\rfloor}_{\frac{1}{i}\frac{d}{dx}} b; k)H^\delta_n(\mathfrak{L}^{\lfloor\rho\rfloor}_{\frac{1}{i}\frac{d}{dx}} p; k)\end{aligned}$$

Thus by Proposition A.3 characterizing Zygmund spaces and Definition 4.1 we get:

$$^{1-\delta,\delta}H^\beta_\rho(b^p_\tau; k) \leq C_{\rho,k}(M^\delta_1(p))[H^\beta_\rho(b; k) + H^\beta_\rho(b; k)H^\delta_\rho(p; k)], \ (k, n) \in \mathbb{N}.$$

which gives (4.19) for $\rho \notin \mathbb{N}$.

Point (III). For point (6) we compute:

$$\partial_\tau[A^p_\tau, T_b] = [iT_p \circ A^p_\tau, T_b] = iT_p[A^p_\tau, T_b] + [iT_p, T_b]A^p_\tau.$$

Thus by definition of A^p_τ as the flow map we get the following Duhamel formula,

$$\begin{aligned}[A^p_\tau, T_b] &= \int_0^\tau A^p_{\tau-r}[iT_p, T_b]A^p_r dr, \\ &= A_\tau \int_0^\tau \underbrace{A^p_{-r}[iT_p, T_b]A^p_r(u)dr}_{\star}.\end{aligned}$$

Applying point (5) to $\star$ we get:

$$[A^p_\tau, T_b] = A^p_\tau \sum_{k=1}^{n} (-1)^{k-1} \frac{\tau^k}{k!} \mathfrak{L}^k_{iT_p} T_b$$
$$+ (-1)^n A^p_\tau \int_0^\tau \frac{(\tau - r)^n}{n!} A^p_{-r} \mathfrak{L}^{n+1}_{iT_p} T_b A^p_r dr.$$

Again applying point (1) combined with (4.25) we get (4.21).

To get (4.20) we inject (4.17) in $\star$, which concludes the proof. $\square$

Lemma 4.2. *Consider two real numbers δ, β, $\rho \geq 0$, and two symbols $p \in \Gamma^\delta_\rho(\mathbb{D})$ and $b \in \Gamma^\beta_\rho(\mathbb{D})$ then the exists a constant $C > 0$ such that:*

$$\left\| \mathfrak{L}^{\lceil \rho \rceil}_{T_p} T_b \right\|_{H^s \to H^{s-\beta-\lceil \rho \rceil \delta + \rho}} \leq C^{\lceil \rho \rceil + 1} 2^{\lceil \rho \rceil} \lceil \rho \rceil ! M^\beta_\rho(b) M^\delta_\rho(p)^{\lceil \rho \rceil}, \text{ for } \rho \in \mathbb{R}_+, \tag{4.36}$$

$$\left\| \mathfrak{L}^{\lceil \rho \rceil}_{T_p} T_b \right\|_{H^s \to H^{s-\beta-\lceil \rho \rceil \delta}} \leq C^{\lceil \rho \rceil} M^\beta_0(b) M^\delta_0(p)^{\lceil \rho \rceil}, \text{ for } \rho \in \mathbb{R}_+. \tag{4.37}$$

Proof. For (4.37), we notice that $\mathfrak{L}^{\lceil \rho \rceil}_{T_p} T_b$ contains $2^{\lceil \rho \rceil}$ terms of the form:

$$T_p \circ \cdots \circ \underbrace{T_b}_{\text{position } i} \circ \cdots \circ T_p, \ i \in [0, 2^{\lceil \rho \rceil}],$$

Now by the continuity of paradifferential operators given in Theorem A.1 we have:

$$\| T_p \circ \cdots \circ T_b \circ \cdots \circ T_p \|_{H^s \to H^{s-\beta-\lceil \rho \rceil \delta}} \leq K^{\lceil \rho \rceil} M^\beta_0(b) M^\delta_0(p)^{\lceil \rho \rceil},$$

which gives (4.37).

For (4.36), we start by the case $k \in \mathbb{N}^*$ is a an integer. We first notice that again by Theorem A.1 we have:

$$\begin{cases} \mathfrak{L}^1_{T_p} T_b \in \Gamma^{\beta+\delta-1}_{k-1}, \\ M^{\beta+\delta-1}_{k-1}(\mathfrak{L}^1_{T_p} T_b) \leq C M^\beta_k(b) M^\delta_k(p). \end{cases}$$

Thus iterating this formula we get:

$$\begin{cases} \mathfrak{L}^k_{T_p} T_b \in \Gamma^{\beta+k\delta-k}_0, \\ M^{\beta+\delta-k}_0(\mathfrak{L}^k_{T_p} T_b) \leq C_k M^\beta_k(b) \prod_{i \geq 1}^{k} M^\delta_i(p), \end{cases}$$

and C_k verifies:

$$C_k = 2k C_{k-1} \Rightarrow C_k = C 2^k k!,$$

Thus giving the result in the case k integer. For $\rho \geq 0$, it suffice to see that for $\rho \leq 1$ again by Theorem A.1 we have:

$$\begin{cases} \mathfrak{L}^1_{T_p} T_b \in \Gamma^{\beta+\delta-\rho}_0, \\ M^{\beta+\delta-\rho}_0(\mathfrak{L}^1_{T_p} T_b) \leq C M^\beta_\rho(b) M^\delta_\rho(p), \end{cases}$$

which concludes the proof. $\square$

We give a result on the symbol of A^p_τ

Proposition 4.3. *Consider two real numbers $\delta < 1$, $\rho \geq 1$ and a real valued symbol $p \in \Gamma^\delta_\rho(\mathbb{D})$.*

Let $A^p_\tau, \tau \in \mathbb{R}$ be the flow map defined by Proposition 4.1, then there exists a symbol $e^{i\tau p}_\otimes \in W^{\rho,\infty} S^0_{1-\delta,\delta}(\mathbb{D}^* \times \tilde{\mathbb{D}})$ such that:

$$A^p_\tau = T^{lim}_{e^{i\tau p}_\otimes} + A^p_\tau(Id - T_1). \tag{4.38}$$

Moreover we have the identities:

$$\begin{cases}\partial_\tau [T^{lim}_{e_\otimes^{i\tau p}} h_0] = i T_p T^{lim}_{e_\otimes^{i\tau p}} h_0, \\ T^{lim}_{e_\otimes^{i\tau p}} h_0 \ |_{\tau=0} = T_1 h_0.\end{cases} \quad \text{for } h_0 \in H^s(\mathbb{D}), s \in \mathbb{R}. \tag{4.39}$$

$$T^{lim}_{e_\otimes^{i\tau p}} = T_{e^{i\tau p}} + \int_0^\tau A^p_{\tau - s}\big(T_{ip} T_{e^{isp}} - T_{ipe^{isp}}\big) ds. \tag{4.40}$$

Proof. The idea is that morally $T^{lim}_{e_\otimes^{i\tau p}}$ should be defined by the asymptotic series:

$$T^{lim}_{e_\otimes^{i\tau p}} \sim \sum \frac{i^k \tau^k}{k!} (T_p)^k = \sum \frac{i^k \tau^k}{k!} T_{\overset{k}{\otimes} p},$$

where $\overset{k}{\otimes} p$ is defined by Theorem A.3. To make this series converge we again introduce the Gaussian multiplier $\phi_\epsilon(D)$ defined by (4.30), as we still have the factor $\frac{1}{k!}$ as in Proposition 4.2.

By the symbolic calculus rules in Theorem A.3 we then have that there exists a symbol ${}^\epsilon e_\otimes^{i\tau p} \in C^\rho_* S^0_{1-\delta,\delta}(\mathbb{D}^* \times \tilde{\mathbb{D}})$ such that:

$$\sum_{k=0}^{+\infty} \frac{i^k \tau^k}{k!} (T_p)^k \phi_\epsilon(D) = \sum_{k=0}^{+\infty} \frac{i^k \tau^k}{k!} T_{\overset{k}{\otimes} p} \phi_\epsilon(D) = T^{lim}_{{}^\epsilon e_\otimes^{i\tau p}}, \tag{4.41}$$

$${}^{1-\delta,\delta} M^0_\rho({}^\epsilon e_\otimes^{i\tau p}) \leq \sum_{k=0}^{+\infty} C^k C_k \frac{|\tau|^k}{k!} \epsilon^{-k\delta} M^\delta_\rho(p)^k, \tag{4.42}$$

where C_k verifies for all $K > 0$, $K^k C_k = o(k!)$.

Now in order to pass to the limit in ϵ we need to get uniform estimates on ${}^{1-\delta,\delta} H^m_s({}^\epsilon e_\otimes^{i\tau p}; k)_{k \in \mathbb{N}}$. To do so we see that:

$$\begin{cases}\partial_\tau [T^{lim}_{{}^\epsilon e_\otimes^{i\tau p}} h_0] = i T_p T^{lim}_{{}^\epsilon e_\otimes^{i\tau p}} h_0, \\ T^{lim}_{e_\otimes^{i\tau p}} h_0 \ |_{\tau=0} = \phi_\epsilon(D) T_1 h_0,\end{cases} \quad \text{for } h_0 \in H^s(\mathbb{D}), s \in \mathbb{R}. \tag{4.43}$$

Thus a standard energy estimate combined with the commutation identities (4.34) and Lemma 4.1 we get:

$${}^{1-\delta,\delta} H^m_n({}^\epsilon e_\otimes^{i\tau p}; k)_{k \in \mathbb{N}} \leq C_{k,n}(M^\delta_1(p;k)) M^\delta_n(p;k), \ (k,n) \in \mathbb{N}.$$

In order to get higher Zygmund estimates we see that by getting back to the sum (4.41), we have:

$$\begin{cases}\partial_\tau [T^{lim}_{P_{\leq k}(D)[{}^\epsilon e_\otimes^{i\tau p}]} h_0] = i T_{[P_{\leq k}(D) p]} T^{lim}_{[P_{\leq k}(D) {}^\epsilon e_\otimes^{i\tau p}]} h_0, \\ T^{lim}_{[P_{\leq k}(D) e_\otimes^{i\tau p}]} h_0 \ |_{\tau=0} = \phi_\epsilon(D) h_0.\end{cases} \quad \text{for } h_0 \in H^s(\mathbb{D}), s \in \mathbb{R}. \tag{4.44}$$

As previously commuting with ix and $\frac{1}{i}\frac{d}{dx}$ and Proposition A.3 we get:

$${}^{1-\delta,\delta} H^m_s({}^\epsilon e_\otimes^{i\tau p}; k)_{k \in \mathbb{N}} \leq C_{k,n}(M^\delta_1(p;k)) M^\delta_s(p;k), \ (k,n) \in \mathbb{N}.$$

Thus we can pass to the limit in when $\epsilon \to 0$ and get the desired result.

Finally for identity (4.39) we pass to the limit in (4.43). Identity (4.40) comes from the following computation. Fix an $h_0 \in H^s, s \in \mathbb{R}$, then $[A^p_\tau - T_{e^{i\tau p}}] h_0$ solves:

$$\begin{cases}\partial_\tau \big([A^p_\tau - T_{e^{i\tau p}}] h_0\big) - i T_p \big([A^p_\tau - T_{e^{i\tau p}}] h_0\big) = \big(T_{ip} T_{e^{i\tau p}} - T_{ipe^{i\tau p}}\big) h_0, \\ \big([A^p_\tau - T_{e^{i\tau p}}] h_0\big)(0, \cdot) = (Id - T_1) h_0(\cdot),\end{cases} \tag{4.45}$$

which by definition of A^p_τ gives (4.40). □

We will now compute the different Gateaux derivatives of the operators defined above.

Proposition 4.4. *Consider two real numbers $\delta < 1$, $\rho \geq 1$, two real valued symbols $p, p' \in \Gamma^{\delta}_{\rho}(\mathbb{D})$. Let $A^p_\tau, A^{p'}_\tau, \tau \in \mathbb{R}$ be the flow maps defined by Proposition 4.1, then for $\tau \in \mathbb{R}$ we have:*

$$A^p_\tau - A^{p'}_\tau = \int_0^\tau A^p_{\tau-r} T_{ip'-p} A^{p'}_r dr. \tag{4.46}$$

Another way to see this is with the Gateaux derivative of $p \mapsto A^p_\tau$ on the Fréchet space $\Gamma^\delta_\rho(\mathbb{D})$ is given by:

$$D_p A^p_\tau(h) = \int_0^\tau A^p_{\tau-r} T_{ih} A^p_r dr. \tag{4.47}$$

Moreover consider an open interval $I \subset \mathbb{R}$, and a real valued symbols $p \in C^1(I, \Gamma^\delta_\rho(\mathbb{D}))$. Let $A^p_\tau, \tau \in \mathbb{R}$ be the flow map defined by Proposition 4.1 then for $\tau \in \mathbb{R}, z \in I$ we have:

$$\partial_z A^p_\tau = \int_0^\tau A^p_{\tau-r} T_{i\partial_z p} A^p_r dr. \tag{4.48}$$

Proof. Fix $h_0 \in H^s, s \in \mathbb{R}$ then:

$$\partial_\tau [A^p_\tau h_0] - iT_p[A^p_\tau h_0] = 0 \Rightarrow \partial_\tau[\partial_z A^p_\tau h_0] - iT_p[\partial_z A^p_\tau h_0] - T_{i\partial_z p}[A^p_\tau h_0] = 0,$$

which gives (4.48) by the definition of A^p_τ and the Duhamel formula. The identities (4.46) and (4.47) are obtained in the same way. □

Proposition 4.5. *Consider two real numbers $\delta < 1$, $\rho \geq 1$, two real valued symbols $p, p' \in \Gamma^\delta_\rho(\mathbb{D})$. Let $A^p_\tau, A^{p'}_\tau, \tau \in \mathbb{R}$ be the flow maps defined by Proposition 4.1 and take a symbol $b \in \Gamma^\beta_\rho(\mathbb{D})$ then for $\tau \in \mathbb{R}$ we have:*

$$T^{lim}_{b^p_\tau} - T^{lim}_{b^{p'}_\tau} = \int_0^\tau A^p_{\tau-r} \mathscr{L}_{iT_{p-p'}} T^{lim}_{b^{p'}_r} A^p_{r-\tau} dr \tag{4.49}$$

$$= \int_0^\tau \mathscr{L}_{iT^{lim}_{p-(p')^p_{\tau-r}}} T^{lim}_{(b^{p'}_r)^p_{\tau-r}} dr. \tag{4.50}$$

Another way to see this is with the Gateaux derivative of $p \mapsto T^{lim}_{b^p_\tau}$ on the Fréchet space $\Gamma^\delta_\rho(\mathbb{D})$ is given by:

$$D_p T^{lim}_{b^p_\tau}(h) = \int_0^\tau \mathscr{L}_{iT^{lim}_{h^p_{\tau-r}}} T^{lim}_{b^p_\tau} dr = \mathscr{L}_{i\int_0^\tau T^{lim}_{h^p_{\tau-r}} dr} T^{lim}_{b^p_\tau}. \tag{4.51}$$

Writing, $T^{lim}_{cb^p_\tau} = T_b - T^{lim}_{b^p_{-\tau}}$, and, $T^{lim}_{cb^{p'}_\tau} = T_b - T^{lim}_{b^{p'}_{-\tau}}$ we get:

$$T^{lim}_{cb^p_\tau} - T^{lim}_{cb^{p'}_\tau} = -\int_0^{-\tau} A^p_{-\tau-r} \mathscr{L}_{iT_{p-p'}} T^{lim}_{b^{p'}_r} A^p_{r+\tau} dr \tag{4.52}$$

$$= -\int_0^{-\tau} \mathscr{L}_{iT^{lim}_{p-(p')^p_{-\tau-r}}} T^{lim}_{(b^{p'}_r)^p_{-\tau-r}} dr. \tag{4.53}$$

$$D_p T^{lim}_{cb^p_\tau}(h) = -\int_0^{-\tau} \mathscr{L}_{iT^{lim}_{h^p_{-\tau-r}}} T^{lim}_{b^p_{-\tau}} dr = -\mathscr{L}_{i\int_0^{-\tau} T^{lim}_{h^p_{-\tau-r}} dr} T^{lim}_{b^p_{-\tau}}. \tag{4.54}$$

Proof. From (4.26) and (4.27) we have:

$$\begin{cases} \partial_\tau [T^{lim}_{b^p_\tau} - T^{lim}_{b^{p'}_\tau}] = \mathscr{L}_{iT_p}(T^{lim}_{b^p_\tau} - T^{lim}_{b^{p'}_\tau}) + \mathscr{L}_{iT_{p-p'}} T^{lim}_{b^{p'}_\tau}, \\ T^{lim}_{b^p_0} - T^{lim}_{b^{p'}_0} = 0. \end{cases} \tag{4.55}$$

Thus the Duhamel formula gives (4.49) and (4.50). For the Gateaux derivative passing to the limit in (4.49) we have:

$$D_p T^{lim}_{b^p_\tau}(h) = \int_0^\tau A^p_{\tau-r} \mathscr{L}_{iT_h} T^{lim}_{b^p_r} A^p_{r-\tau} dr,$$

which gives (4.51). $\square$

Corollary 4.2. *Consider three real numbers $\alpha > 1$, $\beta < \alpha$ and $s \in \mathbb{R}$, two real valued symbols $a \in \Gamma^{\alpha}_{2+\frac{1+\beta-\alpha}{\alpha-1}}(\mathbb{D})$ and $b \in \Gamma^{\beta}_{1+\frac{1+\beta-\alpha}{\alpha-1}}(\mathbb{D})$. Suppose that there exists a real valued symbol $p \in \Gamma^{\beta+1-\alpha}_{2+\frac{1+\beta-\alpha}{\alpha-1}}(\mathbb{D})$ such that:*

$$b = -\partial_\xi p \partial_x a + \partial_x p \partial_\xi a. \tag{4.56}$$

Define $A^p_\tau(u)$ as the flow map generated by iT_p from Proposition 4.1. For $\tau \in \mathbb{R}$, Let,

$$R_\tau = \tau T_{ib} + \int_0^\tau A^p_{-s}[T_{ip}, T_{ia}]A^p_s ds, \tag{4.57}$$

and,

$$\tilde{R_\tau} = A^p_\tau R_\tau A^p_{-\tau} = \tau A^p_\tau i T_b A^p_{-\tau} + [A^p_\tau, T_{ia}]A^p_{-\tau}. \tag{4.58}$$

Then $R_\tau, \tilde{R_\tau} \in \mathscr{L}(H^{s+(\beta+1-\alpha)^+}(\mathbb{D}), H^s(\mathbb{D}))$ and

$$\left\|(R_\tau, \tilde{R_\tau})\right\|_{H^{s+(\beta+1-\alpha)^+} \to H^s} \leq C M^{\alpha}_{2+\frac{1+\beta-\alpha}{\alpha-1}}(a) M^{\beta}_{1+\frac{1+\beta-\alpha}{\alpha-1}}(b) M^{\beta+1-\alpha}_{2+\frac{1+\beta-\alpha}{\alpha-1}}(p).$$

Moreover taking three different symbols a', b' and p' and defining analogously $R'_\tau, \tilde{R}'_\tau$, we have for $h \in H^s$:

$$\begin{aligned}
&\left\|[R_\tau - R'_\tau]h\right\|_{H^s} \\
&\leq C M^{\alpha}_{2+\frac{1+\beta-\alpha}{\alpha-1}}(a, a', a-a') M^{\beta}_{1+\frac{1+\beta-\alpha}{\alpha-1}}(b, b', b-b') M^{\beta+1-\alpha}_{2+\frac{1+\beta-\alpha}{\alpha-1}}(p, p', p-p') \\
&\quad \times \|h\|_{H^{s+(\beta+1-\alpha)^+}},
\end{aligned}$$

and,

$$\begin{aligned}
&\left\|[\tilde{R_\tau} - \tilde{R}'_\tau]h\right\|_{H^s} \\
&\leq C M^{\alpha}_{2+\frac{1+\beta-\alpha}{\alpha-1}}(a, a', a-a') M^{\beta}_{1+\frac{1+\beta-\alpha}{\alpha-1}}(b, b', b-b') M^{\beta+1-\alpha}_{2+\frac{1+\beta-\alpha}{\alpha-1}}(p, p', p-p') \\
&\quad \times \|h\|_{H^{s+(\beta+1-\alpha)^+}}.
\end{aligned}$$

Proof. First we notice that by definition $R_\tau, \tilde{R_\tau}$ are of order β and that we can write as p, a and b have a regularity of $2 + \frac{2-\alpha}{\alpha-1}$ and $1 + \frac{2-\alpha}{\alpha-1}$ respectively:

$$\begin{cases}
R_\tau = T^{lim}_{r^{(\beta)}_\tau} + T^{lim}_{r^{(\beta+1-\alpha)}_\tau} + R^{(\alpha-1-\beta)^-}_\tau, \\
\tilde{R_\tau} = T^{lim}_{\tilde{r_\tau}^{(\beta)}} + T^{lim}_{\tilde{r_\tau}^{(\beta+1-\alpha)}} + \tilde{R}^{(\alpha-1-\beta)^-}_\tau,
\end{cases}$$

where $r^{(\beta)}_\tau, r^{(\beta+1-\alpha)}_\tau, \tilde{r_\tau}^{(\beta)}, \tilde{r_\tau}^{(\beta+1-\alpha)}$ are operators in the usual paradifferential classes and thus there differential with respect to p do not generate the undesired loss of $1 + \beta - \alpha$ derivative.

Now by Proposition 4.2:

$$T^{lim}_{r^{(\beta)}_\tau} = T^{lim}_{\tilde{r_\tau}^{(\beta)}} = T_{ib} + [T_{ip}, T_{ia}],$$

but the choice of p ensures by Theorem A.3 that $T_{ib} + [T_{ip}, T_{ia}]$ is of order $\beta + 1 - \alpha$, giving the desired result. $\square$

Appendix A. Paradifferential Calculus

In this paragraph we review classic notations and results about paradifferential and pseudodifferential calculus that we need in this paper. We follow the presentations in [13], [14], [25], and [17] which give an accessible and complete presentation.

Notation A.1. *In the following presentation we will use the usual definitions and standard notations for the regular functions* C^k, C^k_b *for bounded ones and* C^k_0 *for those with compact support, the distribution space* $\mathscr{D}'$,$\mathscr{E}'$ *for those with compact support,* $\mathscr{D}'^k$,$\mathscr{E}'^k$ *for distributions of order* k*, Lebesgue spaces* (L^p)*, Sobolev spaces* ($H^s, W^{p,q}$) *and the Schwartz class* $\mathscr{S}$ *and it's dual* $\mathscr{S}'$*. All of those spaces are equipped with their standard topologies. We also use the Landau notation* $O_{\| \ \|}(X)$*.*

For the definition of the periodic symbol classes we will need the following definitions and notations.

Notation A.2. *We will use* $\mathbb{D}$ *to denote* $\mathbb{T}$ *or* $\mathbb{R}$ *and* $\hat{\mathbb{D}}$ *to denote their duals that is* $\mathbb{Z}$ *in the case of* $\mathbb{T}$ *and* $\mathbb{R}$ *in the case of* $\mathbb{R}$*. For concision an integral on on* $\mathbb{Z}$ *i.e* $\displaystyle\int_{\mathbb{Z}}$ *should be understood as* $\displaystyle\sum_{\mathbb{Z}}$*. A function* a *is said to be in* $C^\infty(\mathbb{T}\times\mathbb{Z})$ *if for every* $\xi\in\mathbb{Z}$, $a(\cdot,\xi)\in C^\infty(\mathbb{T})$*. For* $\xi\in\mathbb{Z}$, ∂_ξ *should be understood as the forward difference operator, i.e*

$$\partial_\xi a(\xi) = a(\xi+1) - a(\xi),\ \xi\in\mathbb{Z}.$$

We recall the following simple identities for the Fourier transform on the Torus:

$$\begin{cases} \mathscr{F}_{\mathbb{T}}(\partial_x^\alpha f)(\xi) = \xi^\alpha \mathscr{F}_{\mathbb{T}}(f)(\xi), \xi\in\mathbb{Z}, \\ \mathscr{F}_{\mathbb{T}}((e^{-2i\pi x}-1)^\alpha f)(\xi) = \xi^\alpha \mathscr{F}_{\mathbb{T}}(f)(\xi), \xi\in\mathbb{Z}. \end{cases}$$

A.1. **Littlewood-Paley Theory.**

Definition A.1 (Littlewood-Paley decomposition)**.** *Pick* $P_0\in C_0^\infty(\mathbb{R}^d)$ *so that:*

$$P_0(\xi) = 1 \text{ for } |\xi|<1 \text{ and } 0 \text{ for } |\xi|>2.$$

We define a dyadic decomposition of unity by:

$$\text{for } k\geq 1,\ P_{\leq k}(\xi) = \Phi_0(2^{-k}\xi),\ P_k(\xi) = P_{\leq k}(\xi) - P_{\leq k-1}(\xi).$$

Thus,

$$P_{\leq k}(\xi) = \sum_{0\leq j\leq k} P_j(\xi) \text{ and } 1 = \sum_{j=0}^{\infty} P_j(\xi).$$

Introduce the operator acting on $\mathscr{S}'(\mathbb{D}^d)$*:*

$$P_{\leq k}u = \mathscr{F}^{-1}(P_{\leq k}(\xi)u) \text{ and } u_k = \mathscr{F}^{-1}(P_k(\xi)u).$$

Thus,

$$u = \sum_k u_k.$$

Finally put $\{k\geq 1, C_k = \operatorname{supp}\ P_k\}$ *the set of rings associated to this decomposition.*

Remark A.1. *An interesting property of the Littlewood-Paley decomposition is that even if the decomposed function is merely a distribution the terms of the decomposition are regular, indeed they all have compact spectrum and thus are entire functions. On classical functions spaces this regularization effect can be "measured" by the following inequalities due to Bernstein.*

Proposition A.1 (Bernstein's inequalities). *Suppose that $a \in L^p(\mathbb{D}^d)$ has its spectrum contained in the ball $\{|\xi| \leq \lambda\}$.*

Then $a \in C^\infty$ and for all $\alpha \in \mathbb{N}^d$ and $1 \leq p \leq q \leq +\infty$, there is $C_{\alpha,p,q}$ (independent of λ) such that

$$\|\partial_x^\alpha a\|_{L^q} \leq C_{\alpha,p,q}\lambda^{|\alpha|+\frac{d}{p}-\frac{d}{q}} \|a\|_{L^p}.$$

In particular,

$$\|\partial_x^\alpha a\|_{L^q} \leq C_\alpha \lambda^{|\alpha|} \|a\|_{L^p}, \text{ and for } p=2,\ p=\infty$$

$$\|a\|_{L^\infty} \leq C\lambda^{\frac{d}{2}} \|a\|_{L^2}.$$

If moreover a has it's spectrum is in $\{0 < \mu \leq |\xi| \leq \lambda\}$ then:

$$C_{\alpha,q}^{-1}\mu^{|\alpha|} \|a\|_{L^q} \leq \|\partial_x^\alpha a\|_{L^q} \leq C_{\alpha,q}\lambda^{|\alpha|} \|a\|_{L^q}.$$

Proposition A.2. *For all $\mu > 0$, there is a constant C such that for all $\lambda > 0$ and for all $\alpha \in W^{\mu,\infty}$ with spectrum contained in $\{|\xi| \geq \lambda\}$. one has the following estimate:*

$$\|a\|_{L^\infty} \leq C\lambda^{-\mu} \|a\|_{W^{\mu,\infty}}.$$

Definition A.2 (Zygmund spaces on $\mathbb{D}^d$). *For $r \in \mathbb{R}$ we define the space:*

$$C_*^r(\mathbb{D}^d) \subset \mathscr{S}'(\mathbb{D}^d),\ C_*^r(\mathbb{D}^d) = \left\{u \in \mathscr{S}'(\mathbb{D}^d), \|u\|_r = \sup_q 2^{qr}\|u_q\|_\infty < \infty\right\}$$

equipped with its canonical topology giving it a Banach space structure.
It's a classical result that for $r \notin \mathbb{N}$, $C_*^r(\mathbb{D}^d) = W^{r,\infty}(\mathbb{D}^d)$ the classic Hölder spaces.

Proposition A.3. *Let B be a ball with center 0. There exists a constant C such that for all $r > 0$ and for all $(u_q)_{q\in\mathbb{N}} \in \mathscr{S}'(\mathbb{D}^d)$ verifying:*

$$\forall q, \operatorname{supp} \hat{u_q} \subset 2^q B \text{ and } (2^{qr}\|u_q\|_\infty)_{q\in\mathbb{N}} \text{ is bounded,}$$

$$\text{then}, u = \sum_q u_q \in C_*^r(\mathbb{D}^d) \text{ and } \|u\|_r \leq \frac{C}{1-2^{-r}} sup_{q\in\mathbb{N}} 2^{qr}\|u_q\|_\infty.$$

Definition A.3 (Sobolev spaces on $\mathbb{D}^d$). *It is also a classical result that for $s \in \mathbb{R}$:*

$$H^s(\mathbb{D}^d) = \left\{u \in \mathscr{S}'(\mathbb{D}^d), |u|_s = \left(\sum_q 2^{2qs}\|u_q\|_{L^2}{}^2\right)^{\frac{1}{2}} < \infty\right\}$$

with the right hand side equipped with its canonical topology giving it a Hilbert space structure and $|\ |_s$ is equivalent to the usual norm on $\|\ \|_{H^s}$.

Proposition A.4. *Let B be a ball with center 0. There exists a constant C such that for all $s > 0$ and for all $(u_q)_{\in\mathbb{N}} \in \mathscr{S}'(\mathbb{D}^d)$ verifying:*

$$\forall q,\ \operatorname{supp} \hat{u_q} \subset 2^q B \text{ and } (2^{qs}\|u_q\|_{L^2})_{q\in\mathbb{N}} \text{ is in } L^2(\mathbb{N}),$$

$$\text{then},\ u = \sum_q u_q \in H^s(\mathbb{D}^d) \text{ and } |u|_s \leq \frac{C}{1-2^{-s}}\left(\sum_q 2^{2qs}\|u_q\|_{L^2}{}^2\right)^{\frac{1}{2}}.$$

We recall the usual nonlinear estimates in Sobolev spaces:

- If $u_j \in H^{s_j}(\mathbb{D}^d), j = 1,2$, and $s_1 + s_2 > 0$ then $u_1u_2 \in H^{s_0}(\mathbb{D}^d)$ and if

$$s_0 \leq s_j, j = 1,2 \text{ and } s_0 \leq s_1 + s_2 - \frac{d}{2},$$

$$\text{then } \|u_1u_2\|_{H^{s_0}} \leq K\|u_1\|_{H^{s_1}}\|u_2\|_{H^{s_2}},$$

  where the last inequality is strict if s_1 or s_2 or $-s_0$ is equal to $\frac{d}{2}$.

- For all C^∞ function F vanishing at the origin, if $u \in H^s(\mathbb{D}^d)$ with $s > \frac{d}{2}$, then
$$\|F(u)\|_{H^s} \leq C(\|u\|_{H^s}),$$
for some non decreasing function C depending only on F.

A.2. **Paradifferential operators.** We start by the definition of symbols with limited spatial regularity. Let $\mathscr{W} \subset \mathscr{S}'$ be a Banach space.

Definition A.4. *Given $m \in \mathbb{R}$, $\Gamma^m_{\mathscr{W}}(\mathbb{D})$ denotes the space of locally bounded functions $a(x,\xi)$ on $\mathbb{D} \times (\hat{\mathbb{D}} \setminus 0)$, which are C^∞ with respect to ξ for $\xi \neq 0$ and such that, for all $\alpha \in \mathbb{N}^d$ and for all $\xi \neq 0$, the function $x \mapsto \partial_\xi^\alpha a(x,\xi)$ belongs to $\mathscr{W}$ and there exists a constant C_α such that, for all $\epsilon > 0$:*
$$\forall\, |\xi| > \epsilon, \left\|\partial_\xi^\alpha a(.,\xi)\right\|_{\mathscr{W}} \leq C_{\alpha,\epsilon}(1+|\xi|)^{m-|\alpha|}. \tag{A.1}$$
The spaces $\Gamma^m_{\mathscr{W}}(\mathbb{D})$ are equipped with their natural Fréchet topology induced by the semi-norms defined by the best constants in (A.1).

For quantitative estimates we introduce as in [17]:

Definition A.5. *For $m \in \mathbb{R}$ and $a \in \Gamma^m_{\mathscr{W}}(\mathbb{D})$, we set*
$$M^m_{\mathscr{W}}(a;n) = \sup_{|\alpha|\leq n} \sup_{|\xi|\geq \frac{1}{2}} \left\|(1+|\xi|)^{m-|\alpha|}\partial_\xi^\alpha a(.,\xi)\right\|_{\mathscr{W}}, \text{ for } n \in \mathbb{N}.$$
For $\mathscr{W} = W^{\rho,\infty}, \rho \geq 0$, we write:
$$\Gamma^m_{W^{\rho,\infty}}(\mathbb{D}) = \Gamma^m_\rho(\mathbb{D}) \text{ and } M^m_\rho(a) = M^m_{W^{\rho,\infty}}(a;1).$$
Moreover we introduce the following spaces equipped with their natural Fréchet space structure:
$$C_b^\infty(\mathbb{D}) = \cap_{\rho\geq 0} W^{\rho,\infty},\ \Gamma^m_\infty(\mathbb{D}) = \cap_{\rho\geq 0}\Gamma^m_\rho(\mathbb{D}),\ \Gamma^{-\infty}_\rho(\mathbb{D}) = \cap_{m\in\mathbb{R}}\Gamma^m_\rho(\mathbb{D}) \text{ and,}$$
$$\Gamma^{-\infty}_\infty(\mathbb{D}) = \cap_{\rho\geq 0} \cap_{m\in\mathbb{R}} \Gamma^m_\rho(\mathbb{D}).$$

Remark A.2. *In higher dimension the 1 in the definition of M^m_ρ should be replaced by $1 + \lfloor \frac{d}{2} \rfloor$.*

Definition A.6. *Define an admissible cutoff function as a function $\psi^{B,b} \in C^\infty$, $B > 1, b > 0$ that verifies:*

(1)
$$\psi^{B,b}(\eta,\xi) = 0 \text{ when } |\xi| < B|\eta| + b, \text{ and } \psi^{B,b}(\eta,\xi) = 1 \text{ when } |\xi| > B|\eta| + b + 1.$$

(2) for all $(\alpha,\beta) \in \mathbb{N}^d \times \mathbb{N}^d$, there is $C_{\alpha\beta}$, with $C_{0,0} \leq 1$, such that:
$$\forall(\xi,\eta) : \left|\partial_\xi^\alpha \partial_\eta^\beta \psi(\xi,\eta)\right| \leq C_{\alpha,\beta}(1+|\xi|)^{-|\alpha|-|\beta|}. \tag{A.2}$$
$(\psi^{1,b})_{B>1,b>0}$ will be called limit cutoff functions.

Definition A.7. *Consider a real numbers $m \in \mathbb{R}$, a symbol $a \in \Gamma^m_{\mathscr{W}}$ and an admissible cutoff function $\psi^{B,b}$ define the paradifferential operator T_a by:*
$$\widehat{T_a u}(\xi) = (2\pi)\int_{\hat{\mathbb{D}}} \psi^{B,b}(\xi-\eta,\eta)\hat{a}(\xi-\eta,\eta)\hat{u}(\eta)d\eta,$$
where $\hat{a}(\eta,\xi) = \int e^{-ix.\eta}a(x,\xi)dx$ is the Fourier transform of a with respect to the first variable. In the language of pseudodifferential operators:
$$T_a u = op(\sigma_a)u, \text{ where } \mathscr{F}_x\sigma_a(\xi,\eta) = \psi^{B,b}(\xi,\eta)\mathscr{F}_x a(\xi,\eta).$$
For a limit cutoff $\psi^{1,b}$ we define:
$$\widehat{T_a^{lim} u}(\xi) = (2\pi)\int_{\hat{\mathbb{D}}} \psi^{1,b}(\xi-\eta,\eta)\hat{a}(\xi-\eta,\eta)\hat{u}(\eta)d\eta,$$

and define analogously σ_a^{lim}*.*

An important property of paradifferential operators is their action on functions with localized spectrum.

Lemma A.1. *Consider two real numbers* $m \in \mathbb{R}$*,* $\rho \geq 0$*, a symbol* $a \in \Gamma_0^m(\mathbb{D})$*, an admissible cutoff function* $\psi^{B,b}$*, a limit cutoff function* $\psi^{1,b}$ *and* $u \in \mathscr{S}(\mathbb{D}^d)$*.*

- *For* $R >> b$*, if* $\operatorname{supp} \mathscr{F} u \subset \{|\xi| \leq R\}$*, then:*

$$\operatorname{supp} \mathscr{F} T_a u \subset \left\{ |\xi| \leq (1 + \frac{1}{B}) R - \frac{b}{B} \right\}, \tag{A.3}$$

$$\text{and } \operatorname{supp} \mathscr{F} T_a^{lim} u \subset \left\{ |\xi| \leq 2R - \frac{b}{B} \right\}. \tag{A.4}$$

- *For* $R >> b$*, if* $\operatorname{supp} \mathscr{F} u \subset \{|\xi| \geq R\}$*, then:*

$$\operatorname{supp} \mathscr{F} T_a u \subset \left\{ |\xi| \geq (1 - \frac{1}{B}) R + \frac{b}{B} \right\}, \tag{A.5}$$

$$\text{and } \operatorname{supp} \mathscr{F} T_a^{lim} u \subset \left\{ |\xi| \geq \frac{b}{B} \right\}. \tag{A.6}$$

The main features of symbolic calculus for paradifferential operators are given by the following Theorems taken from [17] and [20].

Theorem A.1. *Let* $m \in \mathbb{R}$*. if* $a \in \Gamma_0^m(\mathbb{D})$*, then* T_a *is of order* m*. Moreover, for all* $\mu \in \mathbb{R}$ *there exists a constant* K *such that:*

$$\|T_a\|_{H^\mu \to H^{\mu - m}} \leq K M_0^m(a), \text{ and,}$$

$$\|T_a\|_{W^{\mu,\infty} \to W^{\mu - m, \infty}} \leq K M_0^m(a), \mu \notin \mathbb{N}.$$

Theorem A.2. *Take* $m \in \mathbb{R}$ *and* $a \in \Gamma_0^m(\mathbb{R}^d)$*, then for all* $\mu > 0$ *there exists a constant* K *such that:*

$$\left\| T_a^{lim} \right\|_{H^\mu \to H^{\mu - m}} \leq K M_0^m(a).$$

$$\left\| T_a^{lim} \right\|_{W^{\mu,\infty} \to W^{\mu - m, \infty}} \leq K M_0^m(a), \mu \notin \mathbb{N}.$$

Theorem A.3. *Let* $m, m' \in \mathbb{R}$*, and* $\rho > 0$*,* $a \in \Gamma_\rho^m(\mathbb{D})$ *and* $b \in \Gamma_\rho^{m'}(\mathbb{D})$*.*

- *Composition: Then* $T_a T_b$ *is a paradifferential operator with symbol:*

$$a \otimes b \in \Gamma_\rho^{m+m'}(\mathbb{D}), \text{ more precisely,}$$

$$T_a^{\psi^{B,b}} T_b^{\psi^{B',b}} = T_{a \otimes b}^{\psi^{\frac{BB'}{B+B'-1},b}}.$$

Moreover $T_a T_b - T_{a\#b}$ *is of order* $m + m' - \rho$ *where* $a\#b$ *is defined by:*

$$a\#b = \sum_{|\alpha| < \rho} \frac{1}{i^{|\alpha|}\alpha!} \partial_\xi^\alpha a \partial_x^\alpha b,$$

and there exists $r \in \Gamma_0^{m+m'-\rho}(\mathbb{D})$ *such that:*

$$M_0^{m+m'-\rho}(r) \leq K(M_\rho^m(a) M_0^{m'}(b) + M_\rho^m(a) M_0^{m'}(b)),$$

and we have

$$T_a^{\psi^{B,b}} T_b^{\psi^{B',b}} - T_{a\#b}^{\psi^{\frac{BB'}{B+B'-1},b}} = T_r^{\psi^{\frac{BB'}{B+B'-1},b}},$$

$$T_a^{lim} T_b^{lim} - T_{a\#b}^{lim} = T_r^{lim}.$$

- *Adjoint: The adjoint operator of T_a, T_a^* is a paradifferential operator of order m with symbol a^* defined by:*

$$a^* = \sum_{|\alpha|<\rho} \frac{1}{i^{|\alpha|}\alpha!} \partial_\xi^\alpha \partial_x^\alpha \bar{a}.$$

Moreover, for all $\mu \in \mathbb{R}$ there exists a constant K such that

$$\|T_a^* - T_{a^*}\|_{H^\mu \to H^{\mu-m+\rho}} \leq K M_\rho^m(a).$$

If $a = a(x)$ is a function of x only, the paradifferential operator T_a is called a paraproduct. It follows from Theorem A.3 and the Sobolev embedding that:

- If $a \in H^\alpha(\mathbb{D})$ and $b \in H^\beta(\mathbb{D})$ with $\alpha, \beta > \frac{d}{2}$, then

$$T_a T_b - T_{ab} \text{ is of order } -\left(min\left\{\alpha, \beta\right\} - \frac{d}{2}\right).$$

- If $a \in H^\alpha(\mathbb{D})$ with $\alpha > \frac{d}{2}$, then

$$T_a^* - T_{a^*} \text{ is of order } -\left(\alpha - \frac{d}{2}\right).$$

- If $a \in W^{r,\infty}(\mathbb{D})$, $r \in \mathbb{N}$ then:

$$\|au - T_a u\|_{H^r} \leq C \|a\|_{W^{r,\infty}} \|u\|_{L^2}.$$

An important feature of paraproducts is that they are well defined for function $a = a(x)$ which are not L^∞ but merely in some Sobolev spaces H^r with $r < \frac{d}{2}$.

Proposition A.5. *Let $m > 0$. If $a \in H^{\frac{d}{2}-m}(\mathbb{D})$ and $u \in H^\mu(\mathbb{D})$ then $T_a u \in H^{\mu-m}(\mathbb{D})$. Moreover,*

$$\|T_a u\|_{H^{\mu-m}} \leq K \|a\|_{H^{\frac{d}{2}-m}} \|u\|_{H^\mu}$$

A main feature of paraproducts is the existence of paralinearisation Theorems which allow us to replace nonlinear expressions by paradifferential expressions, at the price of error terms which are smoother than the main terms.

Theorem A.4. *Let $\alpha, \beta \in \mathbb{R}$ be such that $\alpha, \beta > \frac{d}{2}$, then*

- *Bony's Linearization Theorem: For all C^∞ function F, if $a \in H^\alpha(\mathbb{D})$ then;*

$$F(a) - F(0) - T_{F'(a)} a \in H^{2\alpha - \frac{d}{2}}(\mathbb{D}).$$

- *If $a \in H^\alpha(\mathbb{D})$ and $b \in H^\beta(\mathbb{D})$, then $ab - T_a b - T_b a \in H^{\alpha+\beta-\frac{d}{2}}(\mathbb{D})$. Moreover there exists a positive constant K independent of a and b such that:*

$$\|ab - T_a b - T_b a\|_{H^{\alpha+\beta-\frac{d}{2}}} \leq K \|a\|_{H^\alpha} \|b\|_{H^\beta}.$$

A.3. **Paracomposition.** We recall the main properties of the paracomposition operator first introduced by S. Alinhac in [8] to treat low regularity change of variables. Here we present the results we reviewed and generalized in some cases in [20].

Theorem A.5. *Let $\chi : \mathbb{D}^d \to \mathbb{D}^d$ be a $W^{1+r,\infty}_{loc}$ diffeomorphism with $D\chi \in W^{r,\infty}$, $r > 0, r \notin \mathbb{N}$ and take $s \in \mathbb{R}$ then the following maps are continuous:*

$$H^s(\mathbb{D}^d) \to H^s(\mathbb{D}^d)$$
$$u \mapsto \chi^* u = \sum_{k \geq 0} \sum_{\substack{l \geq 0 \\ k-N \leq l \leq k+N}} P_l(D) u_k \circ \chi,$$

where $N \in \mathbb{N}$ is chosen such that $2^N > sup_{k,\mathbb{D}^d} |\Phi_k D\chi|^{-1}$ and $2^N > sup_{k,\mathbb{D}^d} |\Phi_k D\chi|$.

Taking $\tilde{\chi} : \mathbb{D}^d \to \mathbb{D}^d$ a $C^{1+\tilde{r}}$ diffeomorphism with $D\chi \in W^{\tilde{r},\infty}$ map with $\tilde{r} > 0$, then the previous operation has the natural fonctorial property:

$$\forall u \in H^s(\mathbb{D}^d), \chi^* \tilde{\chi}^* u = (\chi \circ \tilde{\chi})^* u + Ru,$$

$$\textit{with, } R : H^s(\mathbb{R}^d) \to H^{s+min(r,\tilde{r})}(\mathbb{R}^d) \textit{ continous.}$$

We now give the key paralinearization theorem taking into account the paracomposition operator.

Theorem A.6. *Let u be a $W^{1,\infty}(\mathbb{D}^d)$ map and $\chi : \mathbb{D}^d \to \mathbb{D}^d$ be a $W^{1+r,\infty}_{loc}$ diffeomorphism with $D\chi \in W^{r,\infty}$, $r > 0, r \notin \mathbb{N}$. Then:*

$$u \circ \chi(x) = \chi^* u(x) + T_{Du\circ\chi}\chi(x) + R_0(x) + R_1(x) + R_2(x)$$

where the paracomposition given in the previous Theorem verifies the estimates:

$$\forall s \in \mathbb{R}, \|\chi^* u(x)\|_{H^s} \le C(\|D\chi\|_\infty) \|u(x)\|_{H^s},$$

$$u' \circ \chi \in \Gamma^0_{W^{0,\infty}(\mathbb{D}^d)}(\mathbb{D}^d) \textit{ for } u \textit{ Lipchitz,}$$

and the remainders verify the estimates:

$$\|R_0\|_{H^{1+r+min(1+\rho,s-\frac{d}{2})}} \le C \|D\chi\|_r \|u\|_{H^{1+s}}$$
$$\|R_1\|_{H^{1+r+s}} \le C(\|D\chi\|_\infty) \|D\chi\|_r \|u\|_{H^{1+s}}.$$
$$\|R_2\|_{H^{1+r+s}} \le C(\|D\chi\|_\infty, \|D\chi^{-1}\|_\infty) \|D\chi\|_r \|u\|_{H^{1+s}}.$$

Finally the commutation between a paradifferential operator $a \in \Gamma^m_\beta(\mathbb{D}^d)$ and a paracomposition operator χ^* is given by the following

$$\chi^* T_a u = T_{a^*}\chi^* u + T_{q^*}\chi^* u \textit{ with } q \in \Gamma^{m-\beta}_0(\mathbb{D}^d),$$

where a^* has the local expansion:

$$a^*(x,\xi) \sim \sum_{\substack{\alpha \\ |\alpha| \le \lfloor min(r,\rho) \rfloor}} \frac{1}{\alpha!} \partial^\alpha a(\chi(x), D\chi^{-1}(\chi(x))^\top \xi) P_\alpha(\chi(x),\xi) \in \Gamma^m_{\min(r,\beta)}(\mathbb{D}^d), \tag{A.7}$$

where,

$$P_\alpha(x',\xi) = D^\alpha_{y'}(e^{i(\chi^{-1}(y') - \chi^{-1}(x') - D\chi^{-1}(x')(y'-x')).\xi})_{|y'=x'}$$

and P_α is polynomial in ξ of degree $\le \frac{|\alpha|}{2}$, with $P_0 = 1, P_1 = 0$.

Remark A.3. *The simplest example for the paracomposition operator is when $\chi(x) = Ax$ is a linear operator and in that case we see that if N is chosen sufficiently large in the definition:*

$$u(Ax) = (Ax)^* u, \textit{ and } T_{u'(Ax)} Ax = 0.$$

Appendix B. Gauge transform on $\mathbb{R}$

Closely inspecting the two problems in [23] and [19], we saw that the lack of regularity obtained in [23] for $\alpha \ge 2$ is essentially due to the lack of control of the L^1 norm in Sobolev spaces. To show this we start give a simple, albeit an artificial example:

Theorem B.1. *Consider two real numbers $s \in]1 + \frac{1}{2}, +\infty[$, $r > 0$ and $u_0 \in H^s(\mathbb{R})$. Then there exists $T > 0$ such that for all v_0 in the ball $B(u_0, r) \subset H^s(\mathbb{R})$ there exists a unique $v \in C([0,T], H^s(\mathbb{R}))$ solving the Cauchy problem:*

$$\begin{cases} \partial_t v + \mathrm{Re}(v)\partial_x v + i\partial_x^2 v = 0, \\ v(0,\cdot) = v_0(\cdot), \end{cases} \tag{B.1}$$

Moreover we have the estimates:

$$\forall 0 \leq \mu \leq s, \|v(t)\|_{H^\mu(\mathbb{R})} \leq C_\mu \|v_0\|_{H^\mu(\mathbb{R})}. \tag{B.2}$$

Taking two different solutions u, v such that $u - v \in L^1(\mathbb{R})$, then:

$$\begin{aligned}\|(u-v)(t)\|_{H^s(\mathbb{R})} &\leq C(\|u_0\|_{H^s(\mathbb{R})}) \|u_0 - v_0\|_{H^s(\mathbb{R})} \\ &\quad + C(\|u_0\|_{H^s(\mathbb{R})})[\|u_0 - v_0\|_{L^1(\mathbb{R})} + \|(u-v)(t)\|_{L^1(\mathbb{R})}].\end{aligned} \tag{B.3}$$

Proof. This is the simplest theorem to prove as the transformation is straightforward and the symbols used are in the usual Hörmander symbol classes $S^m_{1,0}$. Given the well posedness of the Cauchy problem in H^s, and the density of $\mathscr{S}$ in H^s, it suffice to prove the result for $u_0, v_0 \in \mathscr{S}(\mathbb{R})$ which henceforth we suppose. Define u, v as the solution to (B.1) with initial data u_0, v_0 on $[0, T]$.

The first step we reduce H^s estimates to L^2 ones by defining $f_1 = \langle \mathrm{D} \rangle^s u$. Commuting $\langle \mathrm{D} \rangle^s$ with (B.1), by the symbolic calculus rules in Appendix A we get the PDE on f_1:

$$\begin{cases} \partial_t f_1 + \mathrm{Re}(u)\partial_x f_1 + i\partial_x^2 f_1 = R_1(f_1) f_1, \\ f_1(0, \cdot) = \langle \mathrm{D} \rangle^s u_0(\cdot), \end{cases} \tag{B.4}$$

where R_1 verifies

$$\|R_1(f_1)\|_{L^2 \to L^2} \leq C(\|u\|_{H^s}), \|\partial_{f_1} R_1(f_1)\|_{L^2 \to L^2} \leq C(\|u\|_{H^s}),$$

We define analogously g_1 from v and notice that by definition:

$$\|f_1 - g_1\|_{L^2} = \|u - v\|_{H^s}.$$

thus the problem is reduced to getting L^2 estimates on $f_1 - g_1$.

Then we introduce $F(t,x) = \int_0^x \mathrm{Re}(u)(t,y)dy \in C^\infty(\mathbb{R})$ and make the following change of variable:

$$f_2 = e^{-\frac{i}{2}F} f_1.$$

Analogously define G and g_2 from v. As remarked in [24], F, G do not necessarily decay at infinity but we still have $e^{-\frac{i}{2}F}, e^{-\frac{i}{2}G} \in S^m_{1,0}$. Indeed because $\partial_x F = \mathrm{Re}(u) \in H^{+\infty}$ and $\partial_x G = \mathrm{Re}(v) \in H^{+\infty}$. Now to get Lipschitz control we have

$$\left\| [e^{-\frac{i}{2}G} - e^{-\frac{i}{2}F}] f_1 \right\|_{L^2} \leq \|G - F\|_{L^\infty} \|f_1\|_{L^2} \leq \|v - u\|_{L^1} \|f_1\|_{L^2},$$

$$\left\| [e^{\frac{i}{2}G} - e^{\frac{i}{2}F}] f_2 \right\|_{L^2} \leq \|G - F\|_{L^\infty} \|f_2\|_{L^2} \leq \|v - u\|_{L^1} \|f_2\|_{L^2},$$

thus,

$$\begin{cases} \|f_2 - g_2\|_{L^2} \leq C[\|f_1 - g_1\|_{L^2} + \|v - u\|_{L^1} \|f_1\|_{L^2}], \\ \|f_1 - g_1\|_{L^2} \leq C[\|f_2 - g_2\|_{L^2} + \|v - u\|_{L^1} \|f_2\|_{L^2}], \end{cases}$$

and the problem is reduced to getting L^2 estimates on $f_2 - g_2$. Commuting $e^{\frac{i}{2}F}$ with (B.4) yields:

$$\begin{cases} \partial_t f_2 + i\partial_x^2 f_2 = R_2(f_2) f_2, \\ f_2(0, \cdot) = e^{\frac{i}{2}F_0} \langle \mathrm{D} \rangle^s u_0(\cdot), \end{cases} \tag{B.5}$$

where R_2 verifies

$$\|R_2(f_2)\|_{L^2 \to L^2} \leq C(\|u\|_{H^s}), \|\partial_{f_2} R_2(f_2)\|_{L^2 \to L^2} \leq C(\|u\|_{H^s}).$$

Analogously we get on g_2

$$\begin{cases} \partial_t g_2 + i\partial_x^2 g_2 = R_2(g_2) g_2 \\ g_2(0, \cdot) = e^{\frac{i}{2}G_0} \langle \mathrm{D} \rangle^s v_0(\cdot). \end{cases} \tag{B.6}$$

Now the usual energy estimate on $f_2 - g_2$ combined the Gronwall lemma on $f_2 - g_2$ gives for $0 \leq t \leq T$:

$$\|f_2 - g_2\|_{L^2} \leq C(\|(u,v)\|_{H^s}) \|(f_2 - g_2)(0,\cdot)\|_{L^2}$$

As $s > 1 + \frac{1}{2}$, by the Sobolev embedding Theorem:

$$\begin{aligned} \|f_2 - g_2\|_{L^2} &\leq C(\|(u_0,v_0)\|_{H^s}) \left\| e^{\frac{i}{2}F_0} \langle \mathrm{D} \rangle^s u_0 - e^{\frac{i}{2}G_0} \langle \mathrm{D} \rangle^s v_0 \right\|_{L^2} \\ &\leq C(\|(u_0,v_0)\|_{H^s})[\|u_0 - v_0\|_{H^s} + \|u_0 - v_0\|_{L^1}], \end{aligned}$$

which concludes the proof. $\square$

References

[1] T. Alazard, P. Baldi,: *Gravity capillary standing water waves*,Arch. Ration. Mech. Anal., 217 (2015), no 3, 741-830.

[2] T. Alazard, P. Baldi, D. Han-Kwan: *Control for water waves*, J. Eur. Math. Soc., 20 (2018) 657-745.

[3] T. Alazard, N. Burq, C. Zuily,: *Cauchy theory for the gravity water waves system with non localized initial data*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 33 (2016), 337-395.

[4] T. Alazard, N. Burq, C. Zuily: *On the water waves equations with surface tension*, Duke Math. J. 158(3), 413-499 (2011).

[5] T. Alazard, N. Burq, C. Zuily: *The water-waves equations: from Zakharov to Euler*, Studies in Phase Space Analysis with Applications to PDEs. Progress in Nonlinear Differential Equations and Their Applications Volume 84, 2013, pp 1-20.

[6] T. Alazard, G. Metivier: *Paralinearization of the Dirichlet to Neumann operator, and regularity of diamond waves*, Comm. Partial Differential Equations, 34 (2009), no. 10-12, 1632-1704.

[7] Previous results of T. Alazard, P. Baldi, P. Gérard, Personal communication by T. Alazard.

[8] S. Alinhac *Paracomposition et operateurs paradifferentiels*, Communications in Partial Differential Equations, 1986,11:1, 87-121.

[9] R. Beals: *Characterization of pseudodifferential operators and applications*, Duke Math. J, Volume 44, Number 1 (1977), 45-57.

[10] JM. Bony: *On the Characterization of Pseudodifferential Operators (Old and New)*, Studies in Phase Space Analysis with Applications to PDEs. Progress in Nonlinear Differential Equations and Their Applications, vol 84. Birkhuser, New York, NY. https://doi.org/10.1007/978-1-4614-6348-1_2

[11] J. Bourgain: *Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations,* Geom. Funct.Anal.3(1993), 3: 209. https://doi.org/10.1007/BF01895688.

[12] W. Craig, C. Sulem: *Numerical simulation of gravity water waves*, J. Comput. Phys. 108(1), 73-83 (1993).

[13] L. Hörmander, *Fourier integral operators. I*, Acta Math. 127 (1971), 79-183.

[14] L. Hörmander: *Lectures on nonlinear hyperbolic differential equations*, Berlin ; New York : Springer, 1997.

[15] L. Hörmander, *The Nash-Moser theorem and paradifferential operators*, Analysis, et cetera, 429-449, Academic Press, Boston, MA, 1990.

[16] D. Lannes, *Well-posedness of the water waves equations*, J. Amer. Math. Soc., 18(3):605-654 (electronic), 2005.

[17] G. Metivier, *Para-differential calculus and applications to the Cauchy problem for non linear systems*, Ennio de Giorgi Math. res. Center Publ., Edizione della Normale, 2008.

[18] L. Molinet, J. C. Saut and N. Tzvetkov *Well-posedness and ill-posedness results for the Kadomtsev-Petviashvili-I equation*, Duke Math. J. Volume 115, Number 2 (2002), 353-384.

[19] L. Molinet, *Global Well-Posedness in L^2 for the Periodic Benjamin-Ono Equation*, American Journal of Mathematics, Johns Hopkins University Press, 2008, 130 (3), pp.635-683.

[20] A. R. Said: *On Paracompisition and change of variables in Paradifferential operators*, arXiv preprint, arXiv:2002.02943.

[21] A. R. Said: *A geometric proof of the Quasi-linearity of the Water-Waves system and the incompressible Euler equations*, arXiv preprint, arXiv:2002.02940.

[22] A. R. Said: *On the Cauchy problem of dispersive Burgers type equations*, arXiv preprint.

[23] J. C. Saut *Asymptotic Models for Surface and Internal waves*, 29 Brazilian Mathematical Colloquia, IMPA Mathematical Publications ,2013.

[24] T. Tao: *Global well-posedness of the Benjamin-Ono equation in $H^1(\mathbb{R})$, j. Hyperbolic Differ, Equ **1** (2004), 27-49.*

[25] M. E. Taylor, *Tools for PDE: Pseudodifferential Operators, Paradifferential Operators, and Layer Potentials*, American Mathematical Soc., 2007.

[26] M. E. Taylor, *Pseudodifferential Operators and Nonlinear PDE*, Brickhauser, Boston, 1991.

[27] V.E. Zakharov: *Stability of periodic waves of finite amplitude on the surface of a deep fluid*, J. Appl. Mech. Techn. Phys. 9(2), 190-194 (1968).