Sur le flot de l'équation d'Euler à surface libre

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On the flow map of the Euler equation with free boundary

Ayman Rimah Said

Acknowledgment

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Résumé

L'équation d'Euler à surface libre décrit l'évolution de l'interface séparant l'air d'un fluide parfait irrotationnel. C'est un système de deux équations couplées : l'équation d'Euler à l'intérieur du domaine et une équation cinématique qui décrit les déformations du domaine. Les 4 travaux qui constituent le corps de cette thèse peuvent être divisés en trois sujets connectés au problème de Cauchy du système des water waves.

- Dans le prolongement des travaux de [3, 5, 6, 8], où les auteurs ont montré que le problème de Cauchy pour le système des water waves est bien posé et que le flot est continu sur des espaces de Sobolev suffisamment réguliers, nous montrons:
  - Dans [62] que le système des water waves avec ou sans tension de surface est quasi-linéaire au sens le plus fort du terme, c'est-à-dire que le flot n'est pas uniformément continu. De plus, dans le cas avec tension de surface, nous montrons que pour avoir une estimation de Lipschitz sur le flot, il faut au moins une perte de $\frac{1}{2}$ dérivés. Plus généralement, pour l'équation de Burgers avec terme dispersif de la forme $\partial_x|D|^{\alpha-1}, \alpha \in]1,2[$, nous montrons qu'il faut au moins une perte de $2-\alpha$ dérivés pour assurer un contrôle Lipschitz sur le flot.
  - Dans [63], nous montrons que les résultats obtenus dans [62] sont effectivement optimaux, c'est-à-dire que pour l'équation de Burgers avec un terme dispersif $\partial_x|D|^{\alpha-1}, \alpha \in]1,2[$ le flot est effectivement Lipschitz de H^s á $H^{s-2+\alpha}$ pour des données initiales périodiques de moyenne nulle. Pour le système des water waves avec tension de surface en deux dimensions d'espace, nous montrons qu'après re-normalisation, le flot est bien Lipschitz au prix d'une perte de $\frac{1}{2}$ dérivées.
- Afin de démontrer les résultats dans [63], nous avons développé une généralisation para-différentielle d'une transformation de jauge complexe de type Cole-Hopf introduite pour la première fois par T. Tao pour l'équation de Benjamin-Ono. Dans [64], nous l'utilisons pour améliorer les résultats connus sur une conjecture numérique dûe à Saut et Klein dans [47] sur l'équation de Burgers dispersive. Ce qui, à la connaissance de l'auteur, est la première fois que la transformation de jauge est mise en oeuvre à cette fin pour $\alpha \in]1,2[$.
- Afin de démontrer les différents résultats dans [62, 63, 64], nous avons étudié et affiné différents résultats connus en calcul paradifférentiel. Plus précisément,

dans [61], nous améliorons certaines estimations sur l'opérateur de paracomposition introduit par Alinhac, nous donnons une preuve du changement de variables dans le calcul paradifférentiel et enfin nous étudions comment le support du cut-off fréquentiel varie après la composition d'opérateurs para-différentiels.

Abstract

The Euler equation with free boundary, i.e the water waves system, describes the evolution of the interface between air and a perfect irrotational fluid. It is a system of two coupled equations: the Euler equation in the interior of the domain and a kinematic equation describing the deformation of the domain. The 4 works that constitute the body of this thesis can be divided into three connected subjects on the Cauchy problem of the water waves system.

- In the continuation of the works in [3, 5, 6, 8] where the Cauchy problem for the water waves system is shown to be well-posed and the flow map continuous on sufficiently regular Sobolev spaces we show:
  - In [62] that the water waves system with and without surface tension is quasi-linear in the strongest sense, i.e the flow map is not uniformly continuous. Moreover in the case with surface tension we show that in order to have a Lipschitz estimate on the flow map at least a loss of $\frac{1}{2}$ derivative is needed. More generally for the Burgers equation augmented by a dispersive term of the form $\partial_x |D|^{\alpha-1}, \alpha \in]1,2[$, we show that at least a loss of $2-\alpha$ derivative is needed to ensure Lipschitz control on the flow.
  - In [63] we show that the results obtained in [62] are indeed optimal, that is for the Burgers equation augmented with a dispersive term $\partial_x |D|^{\alpha-1}, \alpha \in]1,2[$ the flow map is indeed Lipschitz from H^s to $H^{s-2+\alpha}$ for periodic data with 0 mean value. For the water waves system with surface tension in two space dimension we show that after suitable re-normalization that the flow map is Lipschitz under $\frac{1}{2}$ loss of derivative.
- In order to prove the results in [63] we developed a paradifferential generalization of a complex Cole-Hopf type gauge transform first introduced by T. Tao for the Benjamin-Ono equation. In [64] we use this to improve upon known results on a numerical conjecture by Saut and Klein [47] on the dispersive Burgers equation, which to the author's knowledge is the first time the gauge transform was implemented to that end for $\alpha \in]1,2[$.
- In order to prove the different results in [62, 63, 64] we needed to study and refine different known results in paradifferential calculus. More precisely in [61], we improve some estimates on the paracomposition operator introduced by Alinhac, give a proof of the change of variables in paradifferential operators and finally study the frequency cut-off after composition of paradifferential operators.

Chapter 1

Introduction

Contents

At the core of this work is the study of the Cauchy problem of nonlinear evolution partial differential equations (PDE) with a strong choice of PDE coming from fluid mechanics. To situate the problem we consider a general evolution PDE of the form:

$$\partial_t u = F((\partial_x^\alpha u)_{|\alpha|\leq k}),\ u(0,\cdot) = u_0 \text{ in a Sobolev space } H^s, s \in \mathbb{R}. \tag{1.0.1}$$

Hadamard's well posedness conditions for an evolution PDE can be interpreted through the flow map:

- There exists a time $T > 0$ and a Sobolev space $s \in \mathbb{R}$, such that for all $0 \leq t \leq T$ and $u_0 \in H^s$, the flow map $u_0 \mapsto u(t,\cdot)$ is well defined and continuous from H^s to H^s.

Several immediate questions follow this definition i) how to prove well posedness? ii) is $T = +\infty$? iii) What is the lowest possible s for the problem to be well posed? iv) What is the flow map's regularity?

To answer question i) one has several methods to tackle the problem and that are usually guided by the "nature" of the PDE. We consider the following preliminary general definitions commonly found in the literature that we will revisit in section 1.1.

- Equation (1.0.1) is said to be linear if F is linear in all of its variables.
- It is quasi-linear if F is linear in the highest order derivatives of u, i.e the PDE can be written as:

$$\partial_t u = \sum_{|\alpha|=k} a_\alpha((\partial_x^\beta u)_{0\leq|\beta|\leq k-1})\partial_x^k u + a_0(\partial_x^\beta u)_{0\leq|\beta|\leq k-1}.$$

- And it is semi-linear if moreover $(a_\alpha)_{|\alpha|=k}$ does not depend on u.

For linear constant coefficient equations one looks to a Green function in order to solve the Cauchy problem. For semi-linear equations a Picard iteration scheme is expected to work and for quasi-linear equations more nonlinear techniques are used such that compactness argument and an energy method. Unfortunately this broad definition fails as shown in the next section and a revised definition through the flow map is given. The starting point of this thesis was to show that the water waves system with and without surface tension is quasi-linear.

Points i) and ii) are usually interconnected through conservation laws and form some of the hardest and deepest questions in PDE today. To answer one of those two questions, one usually has to understand the fine properties of the non linearity F in (1.0.1). One of the goals of this thesis is to show that one can extract information on the nonlinearity by studying the regularity properties of the flow map, i.e the study of point iv), in a simpler setting compared to i) and ii) in the sense that we can work with $T < +\infty$ and $s > 1 + \frac{d}{2}$, which we did for the water waves system and the dispersive Burgers equation. Moreover we show that the technique developed to study question iv) for the dispersive Burgers equation, i.e a complex para-differential generalization of the Cole-Hopf Gauge transform, can be used to give information on questions i) and ii).

The main techniques used in this thesis are at the interface of microlocal analysis, more specifically para-differential calculus, and fluid mechanics to try and give answers to question i),ii) and iv) for the water waves system and related models. All of the results presented here come from the author's works:

[61] A. R. Said: *On Paracomposition and change of variables in Paradifferential operators*, arXiv preprint, arXiv:2002.02943.

[62] A. R. Said: *A geometric proof of the Quasi-linearity of the Water-Waves system and the incompressible Euler equations*, arXiv preprint, arXiv:2002.02940.

[63] A. R. Said: *Regularity results on the flow map of periodic dispersive Burgers type equations and the Gravity-Capillary equations*, arXiv preprint, arXiv:2103.03576.

[64] A. R. Said: *On the Cauchy problem of dispersive Burgers type equations*, arXiv preprint, arXiv:2103.03588.

1.1 Semi-linearity and quasi-linearity of evolution PDE

By the previous definitions of semi-linearity and quasi-linearity the following equations (KPI) and (KPII) are semi-linear by the count of the derivatives, indeed they are given by:

$$(u_t + uu_x + u_{xxx})_x + u_{yy} = 0, \tag{KPI}$$
$$(u_t + uu_x + u_{xxx})_x - u_{yy} = 0. \tag{KPII}$$

Bourgain showed in [21] that (KPII) can be solved by an iteration scheme and that the flow map is regular. But Molinet, Saut and Tzvetkov showed in [55] that the flow map associated to (KPI) cannot be C^2 and that it cannot be solved by a Picard iteration scheme. They introduce the following definitions to quasi-linearity and semi-linearity [55]:

- A partial differential equation is said to be semi-linear if its flow map is regular (at least C^1).
- A partial differential equation is said to be quasi-linear if its flow map is not C^1.

This definition is more attractive in the sense that a claim of the form "equation (1.0.1) is semi/quasi-linear" gives a concrete information on the Cauchy problem and henceforth will be used.

A well known example of a quasi-linear equation is the Burgers equation:

$$\partial_t u + u\partial_x u = 0 \text{ on } \mathbb{R},$$

where it is known that the flow map fails to be uniformly continuous, giving the equation its quasi-linear nature, as for example shown in [65].

1.2 A model problem: the dispersive Burgers equation

An important class of equations that arises in the study of asymptotic models of the water waves equations is Burgers type equation with a dispersive term, for example the Benjamin-Ono equation:

$$\partial_t u + u\partial_x u + H\partial_x^2 u = 0 \text{ on } \mathbb{R}, \tag{BO}$$

and Korteweg-de Vries equation:

$$\partial_t u + u\partial_x u + \partial_x^3 u = 0 \text{ on } \mathbb{R}. \tag{KdV}$$

More generally, In [3] and [6], Alazard, Burq, and Zuily perform a paralinearization and symmetrization of the the water waves system that takes the form:

$$\partial_t u + T_{V(u)}.\nabla u + iT_\gamma u = f,$$

where γ is an elliptic symbol of order $\frac{3}{2}$ in the case with surface tension and $\frac{1}{2}$ in the case without, and $T.$ is the operation of para-multiplication that will be introduced below. For $s > 1 + \frac{1}{2}$ a good one dimensional "toy" model is the dispersive Burgers equation:

$$\partial_t u + u\partial_x u + \partial_x |D|^{\alpha-1} u = 0, \;\; \alpha \in \mathbb{R}^+, \;\; |D| = \mathrm{Op}(|\xi|). \tag{1.2.1}$$

Remark 1.2.1. *Notice that the* (BO) *equation and* (KdV) *equation are the special cases for* $\alpha = 2$ *and* $\alpha = 3$ *respectively.*

It was shown in [49], that the flow map associated to the Benjamin-Ono equation on $H^s(\mathbb{R}), s > \frac{3}{2}$ fails to be uniformly continuous. The proof relies heavily on the dimension, the structure of the equation and on some interactions between small and high frequencies thus it does not generalize to the case of the torus $\mathbb{T}$. More generally in [65], it is shown that the flow map fails to be C^2 (thus the equations are unsolvable by a Picard fixed point scheme) for equations of the form:

$$\partial_t u + u\partial_x u + \omega(D)\partial_x u = 0, \text{ with } |\omega(\xi)| \leq |\xi|^\gamma, \gamma < 2.$$

Here the proof relies heavily on the Duhamel formula, on the explicit solvability of the linear part using the Fourier transform and again on some interactions between small and high frequencies thus it does not generalize to the case of $\mathbb{T}$.

In [65], for the KdV equation, using Strichartz type dispersive estimates the Cauchy problem is solved by a Picard fixed point scheme and thus the flow map is regular, showing a change in nature for the problem. This shows that an interesting phenomena happens where the dispersive term can dominate the nonlinearity. On $\mathbb{R}$, the previous examples show that this change of regime happens for a dispersive term of order 3. Thus the result obtained in [65] is optimal in $d = 1$.

In this thesis we improve these results in several directions:

1. we prove the strongest result possible by proving that the flow is not uniformly continuous,
2. for $\alpha \in [0, 2[$ and $\epsilon > 0$ we prove that the flow cannot be C^1 from $H^s(\mathbb{D})$ to $C^0([0, T], H^{s-1+(\alpha-1)^+ +\epsilon}(\mathbb{D})))$, where $\mathbb{D} = \mathbb{T}$ or $\mathbb{R}$.
3. For those negative type results we give a robust geometric proof that will be translated to the more complex water waves system with and without tension and do so in any dimension.
4. For $\alpha \in]1, 2[$ we show that the flow map is indeed Lipschitz from $H^s_0(\mathbb{T}), s > 1 + \frac{2-\alpha}{\alpha-1} + \frac{1}{2}$ to $C^0([0, T], H_0^{s-2+\alpha}(\mathbb{T})))$, where $H^s_0(\mathbb{T})$ is the Sobolev space of function with 0 mean value.

Remark 1.2.2. *Point* 1 *is enough to deduce the quasi-linearity of the system but the drawback of this test of quasi-linearity is that it does not show the effect of the dispersive term. The natural question was then to ask if one can see the effect of the dispersive term by analyzing more precisely the regularity of the flow map.*

For this we can start by noticing that independently of α *the flow map is Lipschitz from bounded sets of* $H^s(\mathbb{D})$ *to* $C^0([0, T], H^{s-1}(\mathbb{D}))$ *and ask: can the space* $H^{s-1}(\mathbb{D})$ *be replaced by* $H^{s-\mu}(\mathbb{D})$ *with* $\mu < 1$ *depending on* α*? The answer to this question is exactly the goal of points* 2 *and* 3 *where it is shown that the optimal* μ *is* $2 - \alpha$.

1.2.1 The flow map regularity of the Dispersive Burgers equation

Points 1 and 2 are given in the following Theorem from [62].

Theorem 1.2.1 (from [62]). *Consider three real numbers $\alpha \in [0,2[$, $s \in]2+\frac{1}{2}, +\infty[$, $r > 0$ and $u_0 \in H^s(\mathbb{D})$. Then there exists $C_s > 0$ such that for $0 < T < \frac{C_s}{r+\|\partial_x u_0\|_{L^\infty(\mathbb{D})}}$ and all v_0 in the ball $B(u_0, r) \subset H^s(\mathbb{D})$ there exists a unique $v \in C([0,T], H^s(\mathbb{D}))$ solving the Cauchy problem:*

$$\begin{cases} \partial_t v + v\partial_x v + \partial_x |D|^{\alpha-1} v = 0 \\ v(0,\cdot) = v_0(\cdot), \end{cases} \tag{1.2.2}$$

where,

$$|D| = \mathrm{Op}(\xi).$$

Moreover, for all $\delta > 0$ and all of $\mu \in [0,s]$, we have:

$$\forall t \in [0,T], \ \|v(t)\|_{H^\mu(\mathbb{D})} \leq e^{C_\mu \|\partial_x v\|_{L^1([0,T], L^\infty(\mathbb{D}))}} \|v_0\|_{H^\mu(\mathbb{D})}. \tag{1.2.3}$$

Taking $v_0 \in B(u_0, r)$, and assuming moreover that $u_0 \in H^{s+1}(\mathbb{D})$ then:

$$\forall t \in [0,T], \|(u-v)(t)\|_{H^s(\mathbb{D})} \leq e^{C_s(\|\partial_x(u,v)\|_{L^1([0,t], L^\infty(\mathbb{D}))} + C_s t \|u_0\|_{H^{s+1}(\mathbb{D})})} \|u_0 - v_0\|_{H^s(\mathbb{D})}. \tag{1.2.4}$$

The lack of regularity of the flow map is given by the following results on balls centered at the origin for all $R > 0$, there exists a constant $C_s >$ such that for $T < \frac{1}{C_s R}$, the flow map:

$$\begin{aligned} B(0,R) &\to C([0,T], H^s(\mathbb{D})) \\ v_0 &\mapsto v \end{aligned}$$

is not uniformly continuous.

In addition, for all $\epsilon' > 0$ the flow map:

$$\begin{aligned} B(0,R) &\to C([0,T], H^{s-1+(\alpha-1)^+ + \epsilon'}(\mathbb{D})) \\ v_0 &\mapsto v \end{aligned}$$

is not C^1.

Remark 1.2.3. *We shall prove a stronger result (see Theorem 3.2.1) showing that for a dispersive perturbation of order $\alpha < 2$, the non-linear transport term dominates the flow's evolution locally and this happens independently of the dimension. This limited regularity of the flow implies that the Cauchy problem cannot be solved by a Picard fixed point scheme and thus those equations are quasi-linear. Another example of application is the Whitham equation on $\mathbb{R}$:*

$$\begin{cases} \partial_t u + u\partial_x u - Lu_x = 0, \\ Lf(x) = \int e^{ix\cdot\xi} p(x,\xi) \hat{f}(\xi) d\xi. \end{cases}$$

We prove that the latter is quasi-linear for $p \in S^\alpha, \alpha < 1$ and such that $\mathrm{Im}(p) \in S^0$ (See (2.2.1) for the definition of the symbol classes).

Looking to the literature to assess the optimality of the previous result, first in [65] the equation (1.2.1) is actually shown to be quasi-linear for $\alpha \in [0,3[$ and becomes semi-linear for $\alpha = 3$, *i.e* the Korteweg-de Vries equation, when $\mathbb{D} = \mathbb{R}$ suggesting that our results are sub-optimal. Then when $\mathbb{D} = \mathbb{T}$, in [56], for the case $\alpha = 2$ and the Benjamin-Ono equation, the flow map is shown to be Lipschitz (and even has analytic regularity) on bounded sets of H^s_0 the (Sobolev spaces of functions with mean value 0). Which suggests that our results could be optimal but with a subtlety in the low frequencies.

The aim of the next theorem is to prove that the results obtained in [62] are optimal on the torus while clarifying in all of those cases the effect brought on by the low frequencies.

Remark 1.2.4. *In Appendix 5.2.3, we look to the problem on $\mathbb{R}$ and use the same Gauge transform to show that the lack of regularity obtained in [65] for $\alpha \geq 2$ is essentially due to the lack of control of the L^1 norm in Sobolev spaces.*

Theorem 1.2.2 (from [63])**.** *Consider three real numbers $\alpha \in [1, +\infty[$, $s \in]1 + \frac{2-\alpha}{\alpha-1} + \frac{1}{2}, +\infty[$, $r > 0$ and $u_0 \in H^s_0(\mathbb{T}; \mathbb{R})$. Then the flow map associated to the Cauchy problem* (1.2.2)*:*

$$\begin{aligned} B(u_0, r) \cap H^s_0(\mathbb{T}; \mathbb{R}) &\rightarrow C([0,T], H_0^{s-(2-\alpha)^+}(\mathbb{T}; \mathbb{R})) \\ v_0 &\mapsto v \end{aligned}$$

is Lipschitz.

Several remarks are in order.

Remark 1.2.5. *1. As a corollary of Theorem 1.2.2 we prove in Section 5.1.2 the following.*

Corollary 1.2.1. *Consider three real numbers $\alpha \in [0, +\infty[$, $s \in]1 + \frac{2-\alpha}{\alpha-1} + \frac{1}{2}, +\infty[$, $r > 0$ and $u_0 \in H^s(\mathbb{T}; \mathbb{R})$.*

- *Then the flow map associated to the Cauchy problem* (1.2.2)*:*

$$\begin{aligned} B(u_0, r) &\rightarrow C([0,T], H^s(\mathbb{T}; \mathbb{R})) \\ v_0 &\mapsto v \end{aligned}$$

  is continuous but not uniformly continuous.

- *For all $\epsilon > 0$ the flow map:*

$$\begin{aligned} B(u_0, r) &\rightarrow C([0,T], H^{s-1+\epsilon}(\mathbb{T}; \mathbb{R})) \\ v_0 &\mapsto v \end{aligned}$$

  is not C^1.

2. The case $\alpha = \frac{3}{2}$ is closely related to the system obtained after reduction and para-linearization of the periodic water Waves system in dimension 1 obtained in [3] Proposition 3.3 by T. Alazard, N. Burq and C. Zuily, which we will treat in the next section on the water waves system.

3. The case $\alpha = 2$ is the Benjamin-Ono equation on the circle was obtained by Molinet in [56]. Though Molinet's result extends to the Cauchy problem on $L^2(\mathbb{T})$. Also in [56] only the flow map regularity for data with 0 mean value was studied.

1.2.2 Improved energy estimates of the paradifferential Dispersive Burgers equation

In order to prove Theorem 1.2.2 we developed a paradifferential generalization of a complex Cole-Hopf gauge transform that was first introduced by T. Tao in [72] in order to improve on the threshold of well-posedness for the Benjamin-Ono equation. Thus it was natural to ask if this can also be done for the dispersive Burgers equation with this new type of gauge of transform in hand. More precisely we will look at the well-posedness of the paralinearised "weak" dispersive perturbations of the Burgers equation:

$$\partial_t u + \partial_x[T_u u] - T_{\frac{\partial_x u}{2}} u + \partial_x |D|^{\alpha-1} u = 0, \; u_0 \in H^s, \tag{1.2.5}$$

which is derived from the "weak" dispersive perturbations of the Burgers equation:

$$\partial_t u + u\partial_x u + \partial_x |D|^{\alpha-1} u = 0, \text{ where } \alpha \in]1,2] \text{ and } |D| = \mathrm{Op}(|\xi|). \tag{1.2.6}$$

For $\alpha = 2$ in (1.2.6), we have the usual Benjamin-Ono equation, for $\alpha = 3$ we have the KdV equation, for $\alpha = \frac{1}{2}$ and $\alpha = \frac{3}{2}$ respectively this equation is a toy model for the system obtained by paralinearization and symetrization of water waves system with and without surface tension [7, 1, 8, 3, 5].

The Cauchy problem associated to (1.2.6) has been extensively studied in the literature. For a comprehensive and complete overview of those equations and their links to other problems coming from mechanical fluids and dispersive non linear equations in physics we refer to J-C. Saut's [65, 66].

For $\alpha \geq 2$ the Cauchy problem is now very well understood. A first approach is to use smoothing effects and refined Strichartz estimates see [60, 48]. A second approach is to use time dependent frequency localized spaces as in [42]. A third approach, first introduced by Tao in [72] is to use a gauge transform to eliminate the worst interaction terms, see also [23, 58] for the Benjamin-Ono equation and [33] for $\alpha \in]2,3[$, which to the author's knowledge is the only time the gauge transform was used to improve upon the local well-posedness of dispersive Burgers equation in the fractional dispersion case.

Except for the first approach, in the words of [69], those techniques face serious technical difficulties for $\alpha < 2$. The goal of this section is to show that using the gauge transform introduced in [63], the last approach can be still carried for $1 \leq \alpha < 2$ and it gives $H^s(\mathbb{D})$ under the estimates of $\left\| (1 + \|u\|_{L^\infty_x}) \|u\|_{W^{2-\alpha,\infty}_x} \right\|_{L^1_t}$, improving upon the known hyperbolic factor $\|\partial_x u\|_{L^1_t L^\infty_x}$. To the author's best knowledge this is the first time a gauge transform technique was carried out to improve upon the local well-posedness of the weakly dispersive Burgers equation.

We will also show that for $2 \leq \alpha \leq 3$, this gauge transform can be efficiently used to completely conjugate the paralinearized dispersive Burgers equation to the linear dispersive equation modulo a regular, i.e C^∞, remainder under control of $\|u\|_{L^\infty_t C^{2-\alpha}_*}$. Again to the author's best knowledge this is the first time such a transformation is carried out outside the integrable cases, i.e $\alpha = 2$ and $\alpha = 3$. For those cases, i.e the Benjamin Ono and the KdV equations, suitable Birkhoff coordinates were constructed to diagonalize the infinite dimension Hamiltonian, for this we refer to the pioneering works of Gérard, Kappeler and Topalov [29, 30, 43].

The following quantities are conserved by the flow associated to (1.2.2):

$$\|u(t)\|_{L^2} = \|u_0\|_{L^2}\,, \text{ and,} \tag{1.2.7}$$

$$H(u) = \int_{\mathbb{R}} \left|D^{\frac{\alpha-1}{2}} u\right|^2 (t,x)dx + \frac{1}{3}\int_{\mathbb{R}} u^3(t,x)dx = H(u_0). \tag{1.2.8}$$

By the the Sobolev embedding $H^{\frac{1}{6}} \hookrightarrow L^3$, $H(u)$ is well defined for $\alpha \geq 1 + \frac{1}{3}$. Moreover (1.2.6) is invariant under the scaling transformation:

$$u_\lambda = \lambda^{\alpha-1} u(\lambda^\alpha t, \lambda x),$$

for any positive λ. We have $\|u_\lambda(t,\cdot)\|_{\dot{H}^s} = \lambda^{\alpha+s-\frac{3}{2}} \|u(\lambda^\alpha t,\cdot)\|_{\dot{H}^s}$, thus the critical index corresponding to (1.2.6) is $s_c = \frac{3}{2} - \alpha$. In particular, (1.2.6) is L^2 critical for $\alpha = \frac{3}{2}$.

In the "low" dispersion case, i.e $\alpha \leq 2$ a complete numerical study was carried out by Klein and Saut in [47]. They conjectured among other things the following.

Conjecture 1.2.1.

1. *For $\alpha \leq 1$, sufficiently large solutions blow up in finite time and do so through a wave breaking scenario, i.e $\lim_{t\to T^*} \|u(t)\|_{L^\infty}$ stays bounded while $\|\partial_x u(t)\|_{L^\infty} \to +\infty$.*

2. *For $\alpha > 1$ we have global in time existence for small initial data.*

3. *For $1 < \alpha \leq \frac{3}{2}$, sufficiently large solutions blow up in finite time and do so through a "dispersive" blow up scenario, i.e $\|u(t)\|_{L^\infty_x} \to +\infty$.*

4. *For $\alpha > \frac{3}{2}$, solutions exist globally in time.*

In [24] and [38] blow up is proven for $\alpha < 1$ and in [39, 40] it is shown that for $\alpha < \frac{2}{3}$ the only possible blow up scenario is a wave breaking one, a simpler proof can be found in [68].

In [52] using Strichartz estimate well posedness is proved for $s > \frac{3}{2} - \frac{3(\alpha-1)}{8}$ and $\alpha > 1$, proving that even for very low dispersion the threshold of well posedness can be improved which again contrasts with the Burgers equation ($\alpha = 1$) where the equation is shown to be ill-posed for $s = \frac{3}{2}$ in [52].

This was improved in [59] using an adapted version of the I-method and refined Strichartz estimates in co-normal Bourgain type spaces. They proved well posedness for $s > \frac{3}{2} - \frac{5(\alpha-1)}{4}$ and $\alpha > 1$, thus proving by the conservation of $H(u)$ and scaling, global well posedness for $\alpha > 1 + \frac{6}{7}$. Which was the first result proving global existing results for $\alpha < 2$.

Finally for $\alpha \geq 2$ the Cauchy problem is much better understood. For the Benjamin Ono equation on $\mathbb{R}$, to the author's knowledge the best known result is L^2 global well-posedness derived in [42]. Recently Patrick Gerard, Thomas Kappeler and Peter Topalov proved in [29] global well-posedness for the periodic Benjamin-Ono equation all the way down to $s > -\frac{1}{2}$ and ill-posedness for $s < -\frac{1}{2} = s_c$, the critical Sobolev exponent. For $\alpha \in]2,3[$ on the real line, the best known local well posedness result is for $\alpha \geq \frac{3}{4}(2-\alpha)$ under a low frequency condition given in [32] and in L^2 without the low frequency condition in [33]. For the KdV equation, for both the periodic and real line cases the Cauchy problem is globally well posed on

$H^{-1}(\mathbb{R})$ as shown in [43, 46], which is the best possible well posedness result, i.e the KdV equation is ill-posed for $s < -1$ as shown in [57].

The remarkable well-posedness results for $\alpha = \{2, 3\}$ uses the integrability of the Benjamin-Ono equation and the KdV equation and the construction of Birkhoff coordinates and thus cannot be extended to the case $\alpha \neq \{2, 3\}$.

Remark 1.2.6. *It is interesting to compare (1.2.6) to the "fractal" Burgers equation, i.e the Burgers equation with a dissipative term:*

$$\partial_t u + u\partial_x u + (-\Delta)^{\frac{\alpha}{2}} u = 0, \ \ \alpha \geq 0. \tag{1.2.9}$$

For $\alpha = 2$, (1.2.9) is the usual Hopf equation. The local and global Cauchy problem associated to (1.2.9) is very well understood, we refer to [45] for a complete solution to the problem. For $\alpha < 1$, sufficiently large solutions of (1.2.9) blow up in finite time and that through a wave breaking mechanism. For $\alpha \geq 1$ solutions exist globally in time.

This contrasts with (1.2.6) in several directions:

- *the change of local/global well posedness is conjectured to happen at $\alpha = \frac{3}{2}$ and not 1,*
- *the conjectured existence of a new nonlinear blow up regime for $\alpha \in]1, \frac{3}{2}]$,*
- *the drastic difference of the global Cauchy problem between (1.2.6) and (1.2.9) for $\alpha = 1$.*

In this section we will look more closely to the equation:

$$\partial_t u + \partial_x [T_u u] - T_{\frac{\partial_x u}{2}} u + \partial_x |D|^{\alpha-1} u = 0, \ \ u_0 \in H^s, \tag{1.2.10}$$

where T is a paraproduct defined in section 1.4.2.1 below.

Remark 1.2.7. *The modifications made to pass from (1.2.6) to (1.2.10) are the following:*

- *We replaced the product $\frac{u^2}{2}$ by the paraproduct $T_u u$ and added the term $-T_{\frac{\partial_x u}{2}} u$ to ensure that $\partial_x [T_u u] - T_{\frac{\partial_x u}{2}} u$ is skew symmetric for the L^2 scalar product.*
- *We dropped the remainder terms*

$$\partial_x R(u, u) \text{ and } T_{\partial_x u} u.$$

The motivations to study the paralinearised version of the equations are the following:

1. *Equation (1.2.10) still contains the main "bad" term $u_{low}\partial_x u_{high}$. As remarked in [72] and [23] this is the main term obstructing straightforward estimates in $X^{0^+, \frac{1}{2}^+}$ for $\alpha = 2$.*
2. *Indeed looking through the literature [23] and [42], the neglected terms can be treated in localized Besov-Bourgain type spaces in our threshold of regularity. By contrast the results on (1.2.10) are simpler to write because they can be completely described in the usual Sobolev spaces.*

In this thesis we opted to study the problem in two steps, first in [64] we treated on the paralinearised version of the equation (1.2.10)*, and do so in the simplest setting of Sobolev spaces. In a second step we will then study the full equation* (1.2.6) *in Bourgain type spaces in a future work.*

Now we give the first theorem of this section.

Theorem 1.2.3. *Consider two real numbers $\alpha \in]1,2[$, $s \in]1+\frac{1}{2},+\infty[$. Then for all $v_0 \in H^s(\mathbb{D})$ and all $r > 0$ there exists $C_s > 0$ such that for $0 < T < \frac{C_s}{r+\|\partial_x u_0\|_{L^\infty(\mathbb{D})}}$ and all u_0 in the ball $B(v_0, r) \subset H^s(\mathbb{D})$ there exists a unique $u \in C([0,T], H^s(\mathbb{D}))$ solving the Cauchy problem:*

$$\partial_t u + iT^{B',b}_{\sigma^{B,b}_{u\xi}+(\sigma^{B,b}_{u\xi})^*} u + \partial_x |D|^{\alpha-1} u = 0, \; u_0 \in H^s, \; b > 0, \tag{1.2.11}$$

where $B > 2$ is given by Theorem 6.1.1, $\sigma^{B,b}_{\cdot}$ is a cutoff defining paradifferential operators (cf Definition 2.2.9), a^* is given by

$$a^*(x,\xi) = \frac{1}{2\pi}\int_{\mathbb{D}\times\hat{\mathbb{D}}} e^{-iy\cdot\eta}\bar{a}(x-y,\xi-\eta)dyd\eta,$$

and $B' > B$ is the cutoff corresponding to the left hand side that includes $T^{B,b}_{\cdot}$ and its a adjoint.

The flow map $u_0 \mapsto u$ is continuous from $B(v_0, r)$ to $C([0,T], H^s(\mathbb{D}))$.

Moreover we have the estimate for $t \in [0,T]$:

$$\|u(t)\|_{H^s} \leq e^{C\left\|(1+\|u\|_{L^\infty_x})\|u\|_{W^{2-\alpha,\infty}_x}\right\|_{L^1([0,t])}} \|u_0\|_{H^s}, \; \textit{for } 1 < \alpha < 2. \tag{1.2.12}$$

Remark 1.2.8.

- *The a priori estimate* (1.2.12) *is not enough to improve upon the local well-posedness theory, indeed we need an extra estimate on the difference of two solutions. A straightforward computation shows that taking the difference of two solutions $u - v$ we get:*

$$\partial_t(u-v)+T_{i\xi|\xi|^{\alpha-1}}(u-v)+\underbrace{\partial_x[T_u(u-v)] - T_{\frac{\partial_x u}{2}}(u-v)}_{(1)} + \underbrace{\partial_x[T_{u-v}v] - T_{\frac{\partial_x u-v}{2}}v}_{(2)} = 0.$$

  Term (1) *can be treated using the gauge transform but term* (2) *is not a paradifferential operator in the variable $u - v$ and can not be treated in our current restricted paradifferential-Sobolev space setting. Indeed term* (2) *has the same structure as the residual terms we dropped to get equation* (1.2.10) *and has to be treated in the Besov-Bourgain space type spaces which is not done here.*

- *The analogue of estimate* (1.2.12) *is still valid for $\alpha \geq 2$. In the real line case the standard Strichartz estimates give for $\alpha = 2$, i.e the Benjamin-Ono equation, the analogue of the well-posedness result of Burq and Planchon [23] and for $\alpha \in [2,3]$ the analogue of [32].*

We turn to the conjugation theorem for $\alpha \in]2,3[$.

Theorem 1.2.4. *Consider two real numbers* $\alpha \in]2,3[$, $s \in]\frac{1}{2} + 2 - \alpha, +\infty[$. *Then there exist* $T > 0$ *and* $r > 0$ *such that for all* u_0 *in the ball* $B(0,r) \subset H^s(\mathbb{D})$ *there exists a unique* $u \in C([0,T], H^s(\mathbb{D}))$ *solving the Cauchy problem:*

$$\partial_t u + iT^{B,b}_{u\xi} u + \partial_x |D|^{\alpha-1} u = 0, \; u_0 \in H^s, \; b > 0, \tag{1.2.13}$$

where $B \geq 1 + \frac{1}{3}$ *is given by Theorem 6.1.1. The flow map* $v_0 \mapsto v$ *is continuous from* $B(0,r)$ *to* $C([0,T], H^s(\mathbb{D}))$.

Moreover there exists a symbol $p \in W^{\infty,\infty}_{\tau}([0,\tau], C^1\Gamma^0_1(\mathbb{D}))$, *in the symbol classes with limited regularity in the frequency variable defined in by (6.1.1), such that for its hyperbolic flow map* $(A^{p(\tau,\cdot)}_{\tau})_{\tau=1}$ *defined by (4.1.1) we have:*

$$\partial_t (A^{p(\tau,\cdot)}_{\tau})_{\tau=1} u + \partial_x |D|^{\alpha-1} (A^{p(\tau,\cdot)}_{\tau})_{\tau=1} u = R_\infty(u), \tag{1.2.14}$$

and there exists a non decreasing functions C_s, *such that* $R_\infty(u)$ *verifies for all* $\mu \in \mathbb{R}$:

$$\|R_\infty(u)\|_{H^\mu} \leq C_\mu(\|u\|_{L^\infty([0,T],C^{2-\alpha}_*(\mathbb{D}))}).$$

Remark 1.2.9.

- *The method used here can be pushed to prove that* p *is in* $C^k\Gamma^0_1(\mathbb{D})_{k\in\mathbb{N}}$ *for all* $k \in \mathbb{N}$, *with a smaller radius* $r_k > 0$ *in Theorem 1.2.4 but without a lower bound on* r_k. *We chose to present the computation showing* $p \in C^1\Gamma^0_1(\mathbb{D})$, *which is the minimal regularity required for the definition of* A^p_1.
- *For the KdV equation this gives local well-posedness in* $H^{-\frac{1}{2}}(\mathbb{D})$ *which is the analogue of the result proven in [27] which is well below the optimal well-posedness in* $H^{-1}(\mathbb{D})$, *which on* $\mathbb{R}$ *was recently proved in [46], and in [43] for the periodic case.*

1.3 The water waves system

We now give the main results on the water waves system with and without surface tension. We follow here the presentation in [3] and [6].

1.3.1 Assumptions on the domain

We consider a domain with free boundary, of the form:

$$\left\{(t,x,y) \in [0,T] \times \mathbb{R}^d \times \mathbb{R} : (x,y) \in \Omega_t\right\},$$

where Ω_t is the domain located between a free surface

$$\Sigma_t = \left\{(x,y) \in \mathbb{R}^d \times \mathbb{R} : y = \eta(t,x)\right\}$$

and a given (general) bottom denoted by $\Gamma = \partial\Omega_t \setminus \Sigma_t$. More precisely we assume that initially ($t = 0$) we have the hypothesis (H_t) given by:

- The domain Ω_t is the intersection of the half space, denoted by $\Omega_{1,t}$, located below the free surface Σ_t,

$$\Omega_{1,t} = \left\{(x,y) \in \mathbb{R}^d \times \mathbb{R} : y < \eta(t,x)\right\} \tag{H_t}$$

and an open set $\Omega_2 \subset \mathbb{R}^{d+1}$ such that Ω_2 contains a fixed strip around Σ_t, which means that there exists $h > 0$ such that,

$$\Big\{(x,y) \in \mathbb{R}^d \times \mathbb{R} : \eta(t,x) - h \leq y \leq \eta(t,x)\Big\} \subset \Omega_2. \tag{H_t}$$

We shall assume that the domain Ω_2 (and hence the domain $\Omega_t = \Omega_{1,t} \cap \Omega_2$) is connected.

1.3.2 The equations

We consider an incompressible inviscid liquid, having unit density. The equations of motion are given by the Euler system on the velocity field v:

$$\begin{cases} \partial_t v + v \cdot \nabla v + \nabla P = -g e_y \\ \operatorname{div} v = 0 \end{cases} \quad \text{in } \Omega_t, \tag{1.3.1}$$

where $-ge_y$ is the acceleration of gravity ($g > 0$) and where the pressure term P can be recovered from the velocity by solving an elliptic equation. The problem is then coupled with the boundary conditions:

$$\begin{cases} v \cdot n = 0 & \text{on } \Gamma, \\ \partial_t \eta = \sqrt{1 + |\nabla \eta|^2} v \cdot \nu & \text{on } \Sigma_t, \\ P = -\kappa H(\eta) & \text{on } \Sigma_t, \end{cases} \tag{1.3.2}$$

where n and ν are the exterior normals to the bottom Γ and the free surface Σ_t, κ is the surface tension and $H(\eta)$ is the mean curvature of the free surface:

$$H(\eta) = \operatorname{div}\left(\frac{\nabla \eta}{\sqrt{1 + |\nabla \eta|^2}}\right).$$

We take $\kappa = 1$ for the case with surface tension and $\kappa = 0$ in the case of gravity water waves (without surface tension). The first condition in (1.3.2) expresses the fact that the particles in contact with the rigid bottom remain in contact with it. As no hypothesis is made on the regularity of Γ, this condition is shown to make sense in a weak variational meaning due to the hypothesis H_t, for more details on this we refer to Section 2 in [3] and Section 3 in [6].

The fluid motion is supposed to be irrotational and Ω_t is supposed to be simply connected thus the velocity field v derives from some potential ϕ i.e $v = \nabla \phi$ and:

$$\begin{cases} \Delta \phi = 0 \text{ in } \Omega, \\ \partial_n \phi = 0 \text{ on } \Gamma. \end{cases}$$

The boundary condition on ϕ becomes:

$$\begin{cases} \partial_n \phi = 0 & \text{on } \Gamma, \\ \partial_t \eta = \partial_y \phi - \nabla \eta \cdot \nabla \phi & \text{on } \Sigma_t, \\ \partial_t \phi = -g\eta + \kappa H(\eta) - \frac{1}{2} \left|\nabla_{x,y} \phi\right|^2 & \text{on } \Sigma_t. \end{cases} \tag{1.3.3}$$

Following Zakharov [75] and Craig-Sulem [28] we reduce the analysis to a system on the free surface Σ_t. If ψ is defined by

$$\psi(t,x) = \phi(t,x,\eta(t,x)),$$

then ϕ is the unique variational solution of

$$\Delta\phi = 0 \text{ in } \Omega_t,\ \ \phi_{|y=\eta} = \psi,\ \ \partial_n\phi = 0 \text{ on } \Gamma.$$

Define the Dirichlet-Neumann operator by

$$\begin{aligned}(G(\eta)\psi)(t,x) &= \sqrt{1+|\nabla\eta|^2}\partial_n\phi_{|y=\eta} \\ &= (\partial_y\phi)(t,x,\eta(t,x)) - \nabla\eta(t,x)\cdot(\nabla\phi)(t,x,\eta(t,x)).\end{aligned}$$

For the case with rough bottom we refer to [9], [3] and [6] for the well posedness of the variational problem and the Dirichlet-Neumann operator. Now (η,ψ) (see for example [28]) solves:

$$\begin{aligned}&\partial_t\eta = G(\eta)\psi, \\ &\partial_t\psi = -g\eta + \kappa H(\eta) + \frac{1}{2}\left|\nabla\psi\right|^2 + \frac{1}{2}\frac{\nabla\eta\cdot\nabla\psi + G(\eta)\psi}{1+|\nabla\eta|^2}.\end{aligned} \tag{1.3.4}$$

1.3.3 Gravity water waves: Pressure and Taylor Coefficients

Here we give a quick review of the ideas in [5]. Recall that by definition for gravity water waves we work with $\kappa = 0$ and we define the Taylor coefficient

$$a(t,x) = -(\partial_y P)(t,x,\eta(t,x)).$$

The stability of the waves is dictated by the Taylor sign condition, which is the assumption that there exists a positive constant c such that

$$a(t,x) \geq c > 0. \tag{1.3.5}$$

In [6] this condition is needed in the proof of the well posedness of the Cauchy problem and it is shown to be locally propagated by the flow.

Now we will show how to define P from the Zakharov formulation. Let R be the variational solution of

$$\Delta R = 0 \text{ in } \Omega_t,\quad R_{|y=\eta} = g\eta + \frac{1}{2}\left|\nabla_{x,y}\phi\right|^2_{|y=\eta}.$$

We define the pressure P in the domain Ω by

$$P(t,x,y) = R(t,x,y) - gy - \frac{1}{2}\left|\nabla_{x,y}\phi(t,x,y)\right|^2.$$

In [5] Alazard, Burq, and Zuily show that to a solution:

$$(\eta,\psi) \in C([0,T], H^{s+\frac{1}{2}}(\mathbb{R}^d)\times H^{s+\frac{1}{2}}(\mathbb{R}^d)),$$

for $s > \frac{d}{2} + \frac{1}{2}$ of the Zakharov/Craig-Sulem system (1.3.4) corresponds a unique solution v of the Euler system, in the case $\kappa = 0$.

1.3.4 Quasi-linearity of the water Wave system

In [3] and [6], Alazard, Burq, and Zuily perform a paralinearization and symmetrization of the the water waves system that takes the form:

$$\partial_t u + T_V.\nabla u + iT_\gamma u = f,$$

where γ is of order $\frac{3}{2}$ in the case with surface tension and $\frac{1}{2}$ in the case without. The terms V and γ verify the conditions required by Theorem 3.2.1 and thus the paralinearization of the water-waves system are quasi-linear in the considered thresholds of regularity. From this we will deduce the following two theorems.

First in the case of water waves with surface tension, i.e $\kappa = 1$, where the well-posedness of the Cauchy problem is proved in [3] we complete it by the following.

Theorem 1.3.1. *Fix the dimension $d \geq 1$ and consider two real numbers $r > 0$, $s \in]2 + \frac{d}{2}, +\infty[$ and $(\eta_0, \psi_0) \in H^{s+\frac{1}{2}}(\mathbb{R}^d) \times H^s(\mathbb{R}^d)$ such that*

$$\forall (\eta_0', \psi_0') \in B((\eta_0, \psi_0), r) \subset H^{s+\frac{1}{2}}(\mathbb{R}^d) \times H^s(\mathbb{R}^d)$$

the assumption $H_{t=0}$ is satisfied. Then there exists $T > 0$ such that the Cauchy problem (1.3.4) with initial data $(\eta_0', \psi_0') \in B((\eta_0, \psi_0), r)$ has a unique solution

$$(\eta', \psi') \in C^0([0, T]; H^{s+\frac{1}{2}}(\mathbb{R}^d) \times H^s(\mathbb{R}^d))$$

and such that the assumption H_t is satisfied for $t \in [0, T]$.

Moreover $\forall R > 0$ the flow map:

$$\begin{aligned} B(0, R) &\to C([0, T], H^{s+\frac{1}{2}}(\mathbb{R}^d) \times H^s(\mathbb{R}^d)) \\ (\eta_0', \psi_0') &\mapsto (\eta', \psi') \end{aligned}$$

is not uniformly continuous.

We show that at least a loss of $\frac{1}{2}$ derivative is necessary to have Lipschitz control over the flow map, i.e for all $\epsilon' > 0$ the flow map

$$\begin{aligned} B(0, R) &\to C([0, T], H^{s+\epsilon'}(\mathbb{R}^d) \times H^{s-\frac{1}{2}+\epsilon'}(\mathbb{R}^d)) \\ (\eta_0', \psi_0') &\mapsto (\eta', \psi') \end{aligned}$$

is not C^1.

Remark 1.3.1. *It is worth noticing that a previous result was obtained on the regularity of the flow map for the two dimensional gravity-capillary water waves (i.e with surface tension) in [26] its proved that the flow is not C^3 with respect to initial data $(\eta_0, \psi_0) \in H^{s+\frac{1}{2}}(\mathbb{R}^2) \times H^s(\mathbb{R}^2)$ for $s < 3$.*

This result is in contrast with our result which holds for $s > 3$ and this can indeed be seen in the fact that in [26] the lack of regularity of the flow is shown to be primarily due to the influence of surface tension. Though in our work the lack of regularity of the flow is shown to be due to the hydrodynamic term (the non-linear transport term).

Now we turn to gravity water waves, i.e $\kappa = 0$ where the well posedness of the Cauchy problem is proved in [6]. It is well known that the vertical and horizontal traces of the velocity on the free boundary play an important role in the well posedness of the Cauchy problem and are given by:

$$\begin{aligned} B &= (\partial_y \phi)_{|y=\eta} = \frac{\nabla\eta \cdot \nabla\psi + G(\eta)\psi}{1+|\nabla\eta|^2}, \\ V &= (\nabla_x \phi)_{|y=\eta} = \nabla\psi - B\nabla\eta. \end{aligned} \tag{1.3.6}$$

Theorem 1.3.2. *Fix the dimension $d \geq 1$ and consider two real numbers $r > 0$, $s \in]2+\frac{d}{2}, +\infty[$*[1] *and $(\eta_0, \psi_0) \in H^{s+\frac{1}{2}}(\mathbb{R}^d) \times H^{s+\frac{1}{2}}(\mathbb{R}^d)$ and consider*

$$(\eta_0', \psi_0') \in B((\eta_0, \psi_0), r) \subset H^{s+\frac{1}{2}}(\mathbb{R}^d) \times H^{s+\frac{1}{2}}(\mathbb{R}^d)$$

such that we have:

1. $V_0' \in H^s(\mathbb{R}^d), \quad B_0' \in H^s(\mathbb{R}^d)$,
2. $H_{t=0}$ *is satisfied,*
3. *there exists a positive constant c such that, $\forall x \in \mathbb{R}^d, a_0'(x) \geq c > 0$ (see (1.3.5)).*

We denote the set of such (η_0', ψ_0') by I_r. Then there exists $T > 0$ such that the Cauchy problem (1.3.4) with initial data (η_0', ψ_0') has a unique solution

$$(\eta', \psi') \in C^0([0,T]; H^{s+\frac{1}{2}}(\mathbb{R}^d) \times H^{s+\frac{1}{2}}(\mathbb{R}^d))$$

such that for $t \in [0,T]$ the assumption H_t is satisfied, $\forall x \in \mathbb{R}^d, a'(t,x) \geq \frac{c}{2}$ and

$$(V', B') \in C^0([0,T]; H^s(\mathbb{R}^d) \times H^s(\mathbb{R}^d)).$$

Moreover $\forall R > 0$, the flow map:

$$\begin{aligned} I_r &\to C([0,T], H^{s+\frac{1}{2}}(\mathbb{R}^d) \times H^{s+\frac{1}{2}}(\mathbb{R}^d)) \\ (\eta_0', \psi_0') &\mapsto (\eta', \psi') \end{aligned}$$

is not uniformly continuous.
Considering a weaker control norm we get: For all $\epsilon' > 0$, the flow map:

$$\begin{aligned} I_r &\to C([0,T], H^{s-\frac{1}{2}+\epsilon'}(\mathbb{R}^d) \times H^{s-\frac{1}{2}+\epsilon'}(\mathbb{R}^d)) \\ (\eta_0', \psi_0') &\mapsto (\eta', \psi') \end{aligned}$$

is not C^1.

Remark 1.3.2. *The previous results for the water waves on $\mathbb{R}^d$ extend to $\mathbb{T}^d$.*

[1] Here we are slightly above the threshold of well-posedness of $1 + \frac{d}{2}$ proved in [6].

1.3.5 Exact regularity of the flow map of the 2d Gravity Capillary equation

We consider the case when $d=1$ and η, ψ are 2π-periodic in the space variable x. Then Theorem 1.3.1 is completed by the following:

Theorem 1.3.3. *Consider two real numbers $r>0$, $s\in]3+\frac{1}{2},+\infty[$ and $(\eta_0,\psi_0)\in H^{s+\frac{1}{2}}(\mathbb{T})\times H^s(\mathbb{T})$ such that,*

$$\forall(\eta_0',\psi_0')\in B((\eta_0,\psi_0),r)\subset H^{s+\frac{1}{2}}(\mathbb{T})\times H^s(\mathbb{T})$$

the assumption $H_{t=0}$ is satisfied. Define (η,ψ) and (η',ψ') as the solutions to the Cauchy problem (1.3.4) on $[0,T]$, $T>0$. Define the following change of variables:

$$\begin{aligned}\chi(t,x)&=\int_0^x\frac{1}{\sqrt{1+(\partial_x\eta(t,y))^2}}dy-\int_0^t\int_\Sigma\left[\frac{1}{1+(\partial_x\eta(t,y))^2}+\partial_x\phi\right]d\Sigma \qquad (1.3.7)\\ &=\int_0^x\sqrt{1+(\partial_x\eta(t,y))^2}dy-\int_0^t\int_0^{2\pi}\left[\frac{1}{1+(\partial_x\eta(t,y))^2}+\partial_x\phi\right]\sqrt{1+(\partial_x\eta)^2}dy,\end{aligned}$$

and χ' is defined analogously from (η',ψ').

Then for r sufficiently small and $t\in[0,T]$ we have:

$$\begin{aligned}&\left\|(\eta,\psi)^*-(\eta',\psi')^{*'}(t,\cdot)\right\|_{H^s\times H^{s-\frac{1}{2}}}\\ &\leq C(\left\|(\eta_0,\psi_0,\eta_0',\psi_0')\right\|_{H^{s+\frac{1}{2}}\times H^s})\left\|(\eta_0,\psi_0)^*-(\eta_0',\psi_0')^{*'}\right\|_{H^s\times H^{s-\frac{1}{2}}}, \qquad (1.3.8)\end{aligned}$$

where * and $^{*'}$ are the paracomposition by χ and χ', which we recall it's definition in 2.4.

Remark 1.3.3. *The time integral in the re-normalization 1.3.7 is to insure that the mean value of the transport term vanishes. This re-normalization is needed here to compensate the non-linearity in the dispersive term of order $\frac{3}{2}$ here.*

1.4 Paradifferential calculus

The paradifferential approach in microlocal analysis was introduced by Bony [15, 16, 18, 19], see also Meyer [54], Hörmander [36], Metivier [53], Alinhac [10, 11], Taylor [73, 74], Bahouri, Chemin and Danchin [12].

1.4.1 Pseudodifferential operators

First let us introduce broadly speaking pseudodifferential operators, following the symbolic approach (see for example Hörmander [35, 36]), the goal is to mimic the construction around regular differential operators:

$$a(x,\xi)=\sum_{|\alpha|\leq m}a_\alpha(x)(i\xi)^\alpha,\ a_\alpha\in C_b^\infty.$$

Taking an operator T on $\mathscr{S}'$, it is said to be a pseudodifferential operator if the there exists a regular function (i.e C^∞) a, called the symbol of T, such that:

$$T(e^{ix\xi})=a(x,\xi)e^{ix\xi},\ \forall\alpha,\beta,\left|\partial_x^\alpha\partial_\xi^\beta a(x,\xi)\right|\leq C_{\alpha,\beta}(1+|\xi|)^{m-\beta}. \qquad (1.4.1)$$

To a symbol a the process that associates to it an operator $\mathrm{Op}(a)$ is called quantization, there are several possible quantizations which are analogous to (1.4.1), as for example for the Weil quantization, see for example [19] for more details on the possible choices of quantization. In this thesis we will only work with Bony's quantization (1.4.1).

Pseudodifferential calculus is the understanding of the different properties of $\mathrm{Op}(a)$ such as products, adjoints, continuity on different functional spaces etc.

1.4.2 Paradifferential operators

The initial goal of paradifferential calculus was to study the propagation of singularities for nonlinear equations and thus one needed a more flexible notion than pseudodifferential calculus that includes symbols with limited regularity in the space variable.

1.4.2.1 Paraproducts

The key observation by Bony is the following, using the Fourier inversion formula for the product of two functions:

$$
\begin{aligned}
f(x)g(x) &= \frac{1}{(2\pi)^{2d}} \int\int_{\mathbb{R}^d\times\mathbb{R}^d} e^{ix\cdot(\xi_1+\xi_2)}\mathscr{F}(f)(\xi_1)\mathscr{F}(g)(\xi_2)d\xi_1 d\xi_2 && (1.4.2)\\
&= \frac{1}{(2\pi)^{2d}} \int\int_{|\xi_1+\xi_2|\sim|\xi_2|} e^{ix\cdot(\xi_1+\xi_2)}\mathscr{F}(f)(\xi_1)\mathscr{F}(g)(\xi_2)d\xi_1 d\xi_2 && (1.4.3)\\
&+ \frac{1}{(2\pi)^{2d}} \int\int_{|\xi_1+\xi_2|\sim|\xi_1|} e^{ix\cdot(\xi_1+\xi_2)}\mathscr{F}(f)(\xi_1)\mathscr{F}(g)(\xi_2)d\xi_1 d\xi_2 \\
&+ \frac{1}{(2\pi)^{2d}} \int\int_{|\xi_1|\sim|\xi_2|} e^{ix\cdot(\xi_1+\xi_2)}\mathscr{F}(f)(\xi_1)\mathscr{F}(g)(\xi_2)d\xi_1 d\xi_2 \\
&= T_f g + T_g f + R(f,g), && (1.4.4)
\end{aligned}
$$

Then Bony proved that for $f \in L^\infty$, T_f is continuous from H^s to H^s for all $s \in \mathbb{R}$ which for is not the case of fg indeed for $g \in H^s$ and s large for fg to be in H^s it is necessary that $f \in H^s$.

The operator $T_f g$ is called the paraproduct of f and g and can be interpreted as follows. Loosely speaking, the term $T_f g$ takes into play high frequencies of g compared to those of f and demands more regularity in $g \in H^s$ than $f \in L^\infty$ thus the term $T_f g$ bears the "singularities" brought on by g in the product fg. Symmetrically $T_g f$ bears the "singularities" brought on by f in the product fg. The second key observation by Bony is that the remainder R is a smoother function ($H^{2s-\frac{d}{2}}$) for $s > 0$.

A heuristic interpretation of Paraproduct

A very useful and instructive heuristic to the notion of paraproducts was given by Shnirelman in [70]. For the sake of this discussion let us pretend that ∂_x is left-invertible with a choice of ∂_x^{-1} that acts continuously from H^s to H^{s+1}.

Another way to define the paraproduct of two functions $f, g \in H^s$ with s sufficiently large is: we differentiate fg k times, using the Leibniz formula, and then restore the function fg by the k-th power of ∂_x^{-1}:

$$\begin{aligned} fg &= \partial_x^{-k}\partial_x^k(fg) \\ &= \partial_x^{-k}\big(g\partial_x^k f + k\partial_x g\partial_x^{k-1}f + \cdots + k\partial_x f\partial_x^{k-1}g + g\partial_x^k f\big) \\ &= T_g f + T_f g + R, \end{aligned}$$

where,

$$T_g f = \partial_x^{-k}\big(g\partial_x^k f\big), \quad T_f g = \partial_x^{-k}\big(f\partial_x^k g\big),$$

and R is the sum of all remaining terms.

The key observation is that if $s > \frac{1}{2} + k$, then $g \mapsto T_f g$ is a continuous operator in H^s for $f \in H^{s-k}$. The remainder R is a continuous bilinear operator from H^s to H^{s+1}.

The operator $T_f g$ is then called the paraproduct of g and f and can be interpreted as follows. Again the term $T_f g$ takes into play high frequencies of g compared to those of f and demands more regularity in $g \in H^s$ than $f \in H^{s-k}$ thus the term $T_f g$ bears the "singularities" brought on by g in the product fg. Symmetrically $T_g f$ bears the "singularities" brought on by f in the product fg and the remainder R is a smoother function (H^{s+1}) and does not contribute to the main singularities of the product.

1.4.2.2 Cutoff functions

To formalize the computations above, we introduce a cut-off functions. Let ψ be a regular function with support bounded away from $(\eta, 0)$ and $(-\eta, \eta)$ at infinity. Then the terms in (1.4.3) and (1.4.4) can be formalized by writing:

$$T_f g = \frac{1}{(2\pi)^d}\int\int_{\mathbb{R}^d\times\mathbb{R}^d} \psi(\xi_1,\xi_2)e^{ix\cdot(\xi_1+\xi_2)}\mathscr{F}(f)(\xi_1)\mathscr{F}(g)(\xi_2)d\xi_1 d\xi_2,$$

$$T_g f = \frac{1}{(2\pi)^d}\int\int_{\mathbb{R}^d\times\mathbb{R}^d} \psi(\xi_2,\xi_1)e^{ix\cdot(\xi_1+\xi_2)}\mathscr{F}(f)(\xi_1)\mathscr{F}(g)(\xi_2)d\xi_1 d\xi_2,$$

$$R(f,g) = \frac{1}{(2\pi)^d}\int\int_{\mathbb{R}^d\times\mathbb{R}^d} (1-\psi(\xi_1,\xi_2)-\psi(\xi_2,\xi_1))e^{ix\cdot(\xi_1+\xi_2)}\mathscr{F}(f)(\xi_1)\mathscr{F}(g)(\xi_2)d\xi_1 d\xi_2.$$

More generally, given a symbol a and a cut-off ψ we define the paradifferential operator:

$$T_a^\psi = \mathrm{Op}(\sigma_a^\psi), \text{ where } \mathscr{F}_x\sigma_a^\psi(\xi,\eta) = \psi(\xi,\eta)\mathscr{F}_x a(\xi,\eta),$$

and $\mathscr{F}_x$ is the Fourier transform with respect to the spacial variable x. Paradifferential calculus is the understanding of the different properties of T_a^ψ such as products, adjoints, continuity on different functional spaces etc.

The main two types of regularization through cutoffs, $(\psi_H^B)_{B>2}$ found in the literature are defined by Hörmander in [36]:

$$\psi_H^B(\eta,\xi) = 0 \text{ when } \begin{cases} |\eta| > B(|\xi|+1), \\ |\xi| > B(|\eta+\xi|+1), \end{cases} \text{ and } \psi_H^B(\eta,\xi) = 1 \text{ when } |\xi| > B(|\eta|+1), \tag{1.4.5}$$

and $(\psi_M^\epsilon)_{\epsilon<1}$ defined by Métivier in [53]:

$$\psi_H^B(\eta,\xi) = 0 \text{ when } |\eta| \geq \epsilon(|\xi|+1), \text{ and } \psi_H^B(\eta,\xi) = 1 \text{ when } |\eta| \leq \frac{\epsilon}{2}(|\xi|+1). \tag{1.4.6}$$

The figures 1.1 and 1.2 illustrate the choice of cutoff functions in the plane (ξ, η) when $d = 1$.

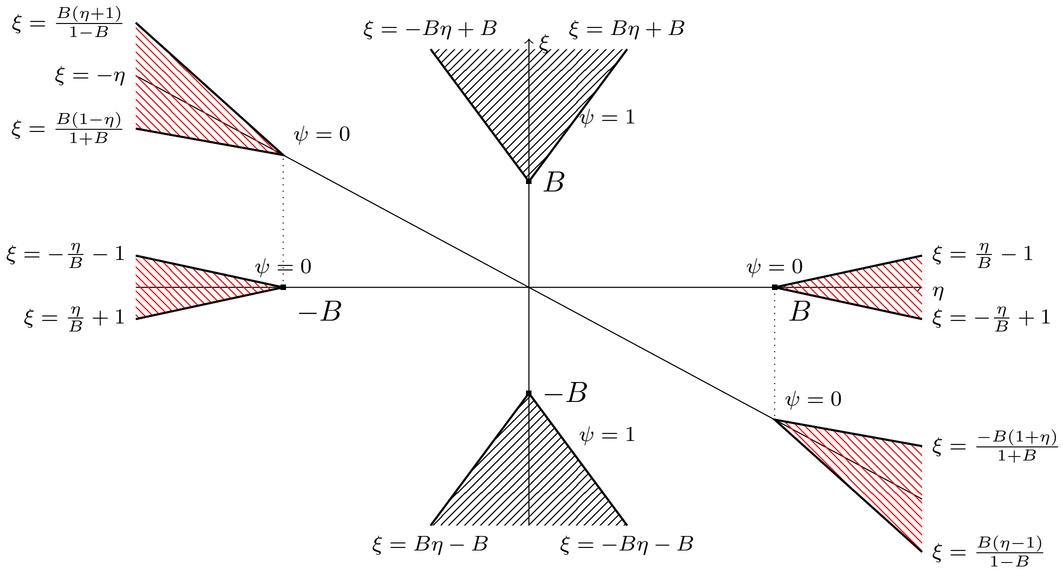

Figure 1.1: Hörmander's choice of cut-off function $(\psi_H^B)_{B>2}$.

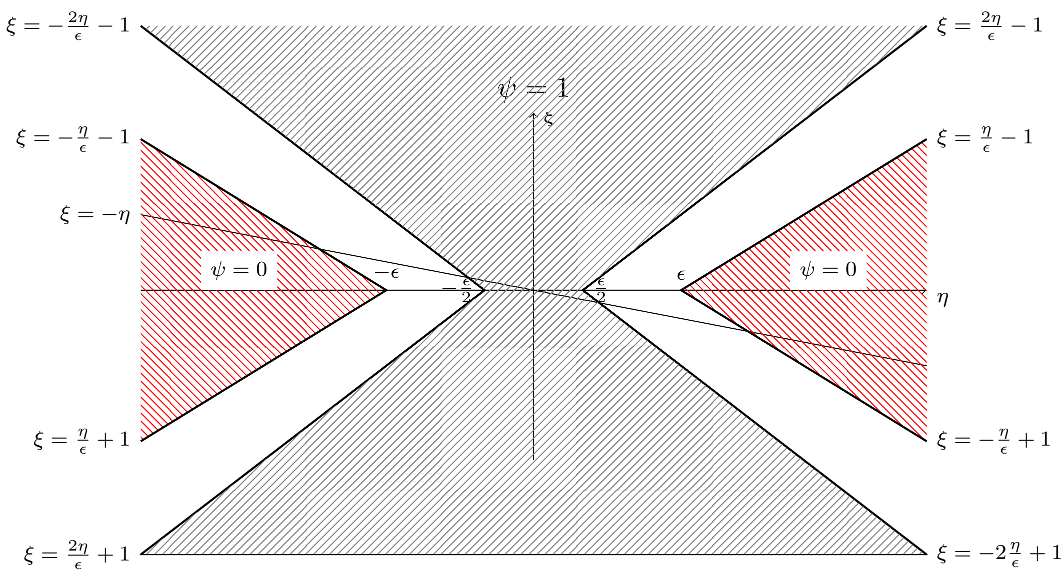

Figure 1.2: Métivier choice of cut-off function $(\psi_M^\epsilon)_{\epsilon<1}$.

1.4.2.3 Cutoff of the composition of paradifferential operators

In [63] we study the Baker-Campbell-Hausdorff type formula for commutation and conjugation of paradifferential operators by the flow map associated to a hyperbolic paradifferential equation. For this we need to study what happens to the frequency cutoff when composing two paradifferential operators.

The effect of the composition on the support of cutoff is seen by the following:

$$\sigma_a^{\psi_M^\epsilon} \circ \sigma_a^{\psi_M^\epsilon} = \sigma^{\psi_M^{2\epsilon+\epsilon^2}}_{\sigma_a^{\psi_M^\epsilon}\otimes\sigma_a^{\psi_M^\epsilon}} \text{ and } \sigma_a^{\psi_H^B} \circ \sigma_a^{\psi_H^B} = \sigma^{\psi_H^{\frac{B^2}{2B+1}}}_{\sigma_a^{\psi_H^B}\otimes\sigma_a^{\psi_H^B}}.$$

where $\sigma_a^\psi \otimes \sigma_a^\psi$ is given by:

$$\sigma_a^\psi \otimes \sigma_a^\psi(x,\xi) = (2\pi)^{-d} \int_{\mathbb{D}^d\times\hat{\mathbb{D}}^d} e^{i(x-y).(\xi-\eta)} \sigma_a^\psi \otimes (x,\eta)\sigma_a^\psi \otimes (y,\xi) dy d\eta.$$

Thus the composition of two paradifferential operators with cutoffs $(\psi_H^B)_{B>2}$ and $(\psi_M^\epsilon)_{\epsilon<1}$ are still paradifferential operators but with worse cutoffs $(\psi_H^{\frac{B^2}{2B+1}})_{B>2}$ and $\psi_M^{2\epsilon+\epsilon^2}$. This is not a problem when considering a finite number of composition of paradifferential operators but it becomes crucial if one for example needs to understand the limit of the series, for example $\sum \frac{(T_a)^k}{k!}$. The $k-$th order term of such a sum has the cutoffs with parameters $\sim 2^k\epsilon$ and $\sim \frac{B}{2^k}$ which are no longer paradifferential operators when the conditions $\frac{B}{2^k} > 2$ and $2^k\epsilon > 1$ are no longer verified.

To remedy this we introduce a special class of cutoffs $(\psi^{B,b})_{B>1,b>0}$, which is included in a modified version of the Hörmader class of cutoffs:

$$\psi^{B,b}(\eta,\xi) = 0 \text{ when } |\xi| < B|\eta| + b, \text{ and } \psi^{B,b}(\eta,\xi) = 1 \text{ when } |\xi| > B|\eta| + b + 1. \tag{1.4.7}$$

In the plane (η,ξ), when $d = 1$, this is illustrated by the following figure:

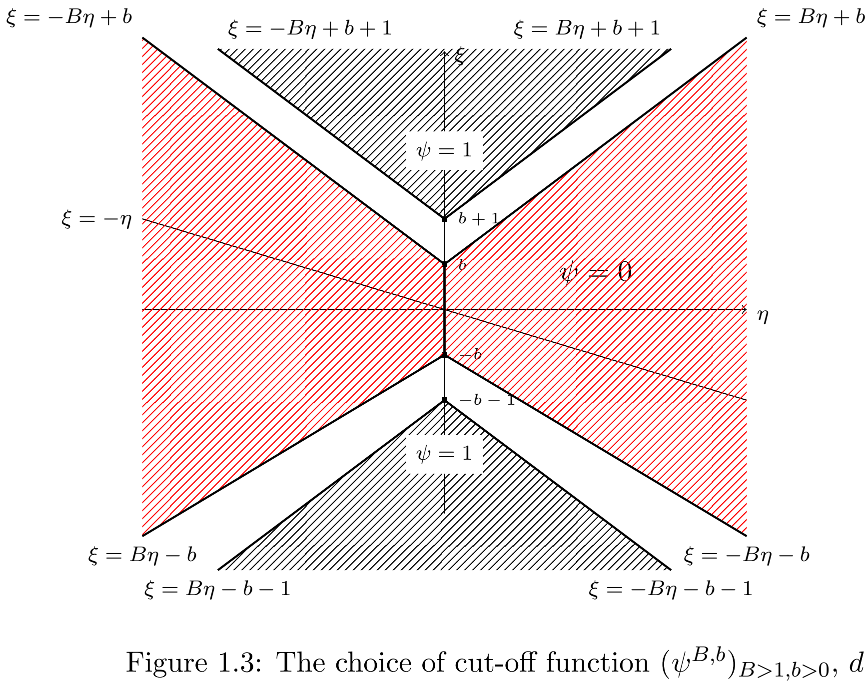

Figure 1.3: The choice of cut-off function $(\psi^{B,b})_{B>1,b>0}$, $d = 1$.

We then prove the following sharp estimate on the support of those cutoffs:

$$\sigma_a^{\psi^{B,b}} \circ \sigma_a^{\psi^{B,b}} = \sigma^{\psi^{\frac{B^2}{2B-1},b}}_{\sigma_a^{\psi^{B,b}}\otimes\sigma_a^{\psi^{B,b}}}.$$

Thus the composition of two paradifferential operators with this new class of cutoffs is still a paradifferential operator but with a slightly worse cutoff $\psi^{\frac{B^2}{2B-1},b}$. The gain from the following new cutoff family can be illustrated in the following, if

$$f(B) = \frac{B^2}{2B-1},$$

then f composed k times with itself converges to 1 when k goes to $+\infty$ for $B > 1$.

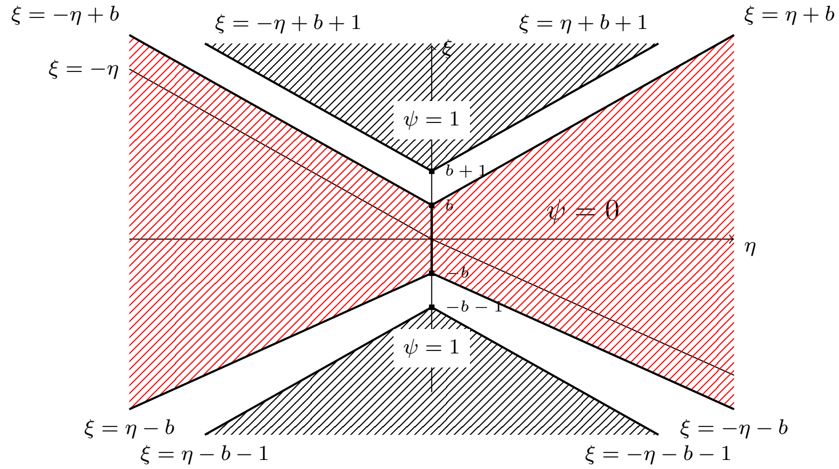

Figure 1.4: The cut-off function $(\psi^{1,b})_{B>1,b>0}$.

The regularization by the cutoff $\sigma^{\psi^{1,b}}_{\cdot}$ posses all of the same properties as $(\psi^B_H)_{B>2}$ and $(\psi^\epsilon_M)_{\epsilon<1}$ except their action on functions whose spectrum is located outside a ball:

$$\operatorname{supp} \mathscr{F}_x(f) \subset B(0,R) \Rightarrow \operatorname{supp} \mathscr{F}_x(\sigma_a^{\psi^{1,b}} f) \subset B(0, 2R - b),$$

$$\operatorname{supp} \mathscr{F}_x(f) \subset \{|\xi| \geq R\} \Rightarrow \operatorname{supp} \mathscr{F}_x(\sigma_a^{\psi^{1,b}} f) \subset \{|\xi| \geq 1\}\,.$$

The paradifferential operators with this cutoff, $T_a^{\psi^{1,b}}$, will be shown to verify all of the standard calculus properties but their continuity will be reduced to spaces $H^s, s > 0$. Thus in comparison with the other cutoffs, $(\psi^B_H)_{B>2}$, $(\psi^\epsilon_M)_{\epsilon<1}$ and $(\psi^{B,b})_{B>1,b>0})$, we only lose the continuity for $s \leq 0$. An interesting application to this new cutoff is following.

Corollary 1.4.1. *Consider four real numbers $\beta \leq 0$, $s > 0$, $B > 1$, $b > 0$, and symbol $a \in \Gamma^\beta_0$. Then $T_a^{\psi^{B,b}} \in \mathscr{L}(H^s)$ which is a Banach algebra, thus $e^{T_a^{\psi^{B,b}}} \in \mathscr{L}(H^s)$ is well defined. Moreover there exists a symbol $\exp(a) \in \Gamma^\beta_0$ such that:*

$$e^{T_a^{\psi^{B,b}}} = Id - T_1 + T^{\psi^{1,b}}_{\exp(a)} \text{ in } \mathscr{L}(H^s).$$

1.4.2.4 Paracomposition

In [10] Alinhac constructed an operator, called the paracomposition operator such that given a $\rho > 0$ and $C^{1+\rho}$ diffeomorphism $\chi : \Omega_1 \to \Omega_2$ between two open subsets of $\mathbb{R}^d$, then the paracomposition by χ is an operator $\chi^* : \mathfrak{D}'(\Omega_2) \to \mathfrak{D}'(\Omega_1)$ having analogous properties to the usual composition $u \to u \circ \chi$ but with limited dependency on the regularity of χ as for classical paradifferential operators i.e the paraproduct T_a is well defined from $H^s \to H^s$, for all s for a merely in L^∞.

Alinhac's construction was motivated by questions that arose from the study of non linear PDEs for example: the study of the transport of a distribution's wave front by a diffeomorphism with low regularity as in the works of E. Leichtnam in [51], the study of the singularities of solutions to semi-linear hyperbolic evolution problems and the characteristic surfaces of the associated operators (here having low regularity), the main reference being Bony's work on the subject ([16],[15],[17],[18]). More recently in [3] and [6], the Paracomposition appears naturally as the "good variable"[2] after a low regularity change of variable in treating the Cauchy problem for the Water Waves system with rough data. It will also appear in our proof of the quasi-linearity of the Water Waves system [62].

Finally the construction of χ^* gives a complete linearization formula to the composition of two functions (with one being a diffeomorphism) generalizing the classic para-linearization Theorem by Bony [16] in a low regularity case.

Bony showed that for $u \in C^\infty$ and $\chi \in H^s_{loc}, s > \frac{d}{2}$ (without the diffeomorphism hypothesis):

$$u \circ \chi = T_{u'(\chi)}\chi + \text{ remainder},$$

and Alinhac showed for $u \in C^\sigma_{loc}, \sigma > 1$ and $\chi \in C^{1+\rho}, \rho > 0$ a diffeomorphism:

$$u \circ \chi = \chi^* u + T_{u'(\chi)}\chi + \text{ remainder}. \tag{1.4.8}$$

Another fundamental result obtained by Alinhac is that the operator χ^* benefits from symbolic calculus properties, that is, it conjugates paradifferential operators. Given T_h a paradifferential operator, Alinhac proved a result in the form:

$$\chi^* T_h u = T_{h^*}\chi^* u + \text{ remainder},$$

where h^* is the pulled back symbol in the case of diffeomorphisms.

The main result in this thesis on paracomposition generalizes Bony's and Alinhac's work by:

- dropping the diffeomorphism hypothesis with a new operator $\chi^\star : \mathfrak{D}'(\Omega_2) \to \mathfrak{D}'(\Omega_1)$. $\chi^\star$ will coincide with Alinhac's operator χ^* modulo a regular remainder in the case of diffeomorphisms.
- Giving estimates in global spaces instead of local spaces.

We will then show that $\chi^\star$ benefits of symbolic calculus properties, for that we will start by discussing the pull-back of pseudodifferential and paradifferential operators by χ which then become Fourier integral operators. In this discussion we show that those Fourier Integral Operators obtained by pull-back are pseudodifferential or paradifferential operators if and only if they are pulled-back by a diffeomorphism i.e a change of variable. We also give a proof to the change of variables in paradifferential operators as we could not find a reference in the literature.

[2]The so called good unknown of Alinhac.

A heuristic interpretation of Paracomposition

We again work with $f \in H^s$ and $g \in C^s$ with s large and consider the composition of two functions $f \circ g$ which bears the singularities of both f and g, and our goal is to separate them. We proceed as before by differentiating $f \circ g$ k times, using the Faá di Bruno's formula, and then restore the function fg by the k-th power of ∂_x^{-1}:

$$\begin{aligned} f \circ g &= \partial_x^{-k} \partial_x^k (f \circ g) \\ &= \partial_x^{-k} \big((\partial_x^k f \circ g) \cdot (\partial_x g)^k + \cdots + (\partial_x f \circ g) \cdot \partial_x^k g \big) \\ &= g^* f + T_{\partial_x f \circ g} g + R, \end{aligned}$$

where,

$$g^* f = \partial_x^{-k} \big((\partial_x^k f \circ g) \cdot (\partial_x g)^k \big) \text{ is the paracomposition of } f \text{ by } g$$

and R is the sum of all remaining terms.

Again the key observation is that if $s > \frac{1}{2} + k$, then $f \mapsto g^* f$ is a continuous operator in H^s for $g \in C^{s-k}$. Thus this term bears essentially the singularities of f in $f \circ g$. As before $T_{\partial_x f \circ g} g$ bears essentially the singularities of g in $f \circ g$. The remainder R is a continuous bilinear operator from H^s to H^{s+1}. Thus we have separated the singularities of the composition $f \circ g$.

A heuristic interpretation of change of variable in Paradifferential operators

From what we have seen previously it seems likely that the adequate change of variables for paradifferential operators is one that comes from commuting with the paracomposition by a diffeomorphism. We carry on the previous computation with the trivial paradifferential operator $\partial_x \sim T_{i\xi}$ and we suppose moreover that g is a diffeomorphism.

$$\begin{aligned} g^* \partial_x f &= \partial_x^{-k} \big((\partial_x^{k+1} f \circ g) \cdot (\partial_x g)^k \big) \\ &= \partial_x^{-k} \big(\partial_x^k [\partial_x^{-k} (\partial_x^{k+1} f \circ g) \cdot (\partial_x g)^{k+1}] \cdot (\partial_x g)^{-1} \big) \\ &\sim T_{(\partial_x g)^{-1}} T_{i\xi} g^* f, \end{aligned}$$

and we notice that $(\partial_x g)^{-1} i\xi = (\partial_x)^*$ is the usual pull-back formula for pseudodifferential symbols by a diffeomorphism g, giving us the desired symbolic calculus rules.

1.4.3 An application of the paracomposition operator

Now that we have global spaces estimates on the paracomposition operator and the paralinearization formula (1.4.8), let us give some examples of applications to the study of composition in Sobolev spaces with limited regularity. The literature on this problem is rich and we focus on two recent articles treating this subject [13] and [41], in which the authors study composition in Sobolev spaces and the geometry of diffeomorphisms groups on manifolds. We will limit the discussion here to the Euclidean space in which the tools presented here significantly improve upon the results from [13] and [41]. First in [41] the composition estimates are proven on $H^n(\mathbb{R}^d) \times D^s(\mathbb{R}^d)$ with $n \in \mathbb{N}$, $s > 1 + \frac{d}{2}$ an integer and

$$D^s(\mathbb{R}^d) = \left\{ \psi - id \in H^s(\mathbb{R}^d), \psi \text{ is a diffeomorphism} \right\}.$$

Here we generalize this to n, s real number and from the paralinearization formula (1.4.8) it is justified to work in the class $D^s(\mathbb{R}^d)$ which appears naturally but it admits several generalization. The simplest one is for example using Zygmund spaces. We also clarify the need of the diffeomorphism hypothesis. More precisely we have the following,

Corollary 1.4.2. *Consider two real numbers $s \in \mathbb{R}$, $\rho \in R_+^* \setminus \mathbb{N}$, and take $\phi \in H^s(\mathbb{R}^d)$ and consider $\chi \in W^{1+\rho,\infty}_{loc}(\mathbb{R}^d)$ a diffeomorphism such that $D\chi \in W^{\rho,\infty}(\mathbb{R}^d)$. Then $\phi \circ \chi \in H^{min(s,\rho)}(\mathbb{R}^d)$.*

The result we have is even stronger indeed it's a Kato-Ponce like decomposition of the different terms that appear in the H^s estimates of composition, for example keeping the notations of the previous Corollary and taking $\psi \in D^s(\mathbb{R}^d)$ we can have estimates of the form:

$$\|\phi \circ \psi\|_{H^s} \le \|D\psi\|_{L^\infty} \|\psi\|_{H^s} + \|D\phi\|_{L^\infty} \|\psi - Id\|_{H^s}.$$

So if we were only working with Sobolev spaces more sophisticated versions of the previous inequality give,

Corollary 1.4.3. *Consider a real number $s > 1 + \frac{d}{2}$, and take $\phi \in H^s(\mathbb{R}^d)$ and consider $\chi \in W^{1+s-\frac{d}{2},\infty}_{loc}(\mathbb{R}^d)$ a diffeomorphism such that $D\chi \in W^{1,\infty}(\mathbb{R}^d)$ and $D^2\chi \in H^{s-2}(\mathbb{R}^d)$. Then $\phi \circ \chi \in H^s(\mathbb{R}^d)$.*

Secondly in [13] to prove the well posedness of EPDIFF equation the authors treat the case of change of variables in pseudodifferential operator with a diffeomorphism with limited regularity. The results obtained were restricted to skew-symmetric operators with compact support and a diffeomorphism in the class $D^s(\mathbb{R}^d)$. Our analysis of such a change of variable shows that the correct category in which the pull back of a pseudodifferential operator by a diffeomorphism with limited regularity is that of paradifferential operators. Thus here the more general case of symbols with limited regularity is treated, the pseudodifferential symbols being the the case where the symbols are regular, the ellipticity and symmetry hypothesis dropped and the need of diffeomorphisms when making the change of variable justified. More precisely we have

Corollary 1.4.4. *Consider a real number r, $A \in S^r(\mathbb{R}^d \times \mathbb{R}^d)$ and $\chi \in W^{1+s-\frac{d}{2},\infty}_{loc}(\mathbb{R}^d)$ a diffeomorphism such that $D\chi \in W^{1,\infty}(\mathbb{R}^d)$ and $D^2\chi \in H^{s-2}(\mathbb{R}^d)$. Then the pull back A^* of A by χ defined as*

$$u \in \mathscr{S}, A^* u = [A(u \circ \chi)] \circ \chi^{-1},$$

is extended to a linear bounded operator from $H^s(\mathbb{R}^d)$ to $H^{s-r}(\mathbb{R}^d)$.

1.5 Sketch of the different strategies on the model problem

In this section we will give the sketch of the proofs for the main Theorems 1.2.1, 1.2.2, 1.2.3 and 1.2.4 on the dispersive Burgers equation.

1.5.1 Sketch of the proof of Theorem 1.2.1

The point of start is to adapt the classic proof of the quasi-linearity of the Burgers equation, presented to me in a personal note of C. Zuily [76], that we will recall here.

1.5.1.1 Quasi-linearity of the Burgers equation

The result of quasi-linearity of the Burgers equation is that the flow map taken point-wise in time fails to be uniformly continuous. Such a result is obtained by constructing two families of solutions u and v from some initial data $u^0$and v^0 depending on parameters λ and ϵ such that

$$\lim_{\substack{\lambda\to+\infty\\ \epsilon\to 0}} \left\|u^0 - v^0\right\|_{H^s} = 0 \text{ and } \left\|(u-v)(t,\cdot)\right\|_{H^s} \geq c > 0, \text{ with } t > 0.$$

To show how to construct such families we start by recalling the usual geometric construction of the graph of a function $u(t,\cdot)$ solution to the Burgers equation with initial data u^0. Put

$$\chi(t,x) = x + tu^0(x)$$

the characteristic flow associated to the problem, which is a diffeomorphism in the x variable. Then,

$$u(t,\cdot) = u^0 \circ \chi^{-1}(t,x).$$

The action of χ^{-1} on the graph of u^0 is given by the Figure 1.5 below that also shows the shock formation phenomena.
Then u^0 and v^0 are chosen as a high frequency compactly supported ansatz depending on (λ,ϵ):

$$u^0(x) = \lambda^{\frac{1}{2}-s}\omega(\lambda x), \quad v^0(x) = u^0(x) + \epsilon\omega(x), \quad \text{with} \quad \omega \in C_0^\infty,$$

where ϵ represents a change in the initial speed of transport, and (ϵ,λ) verify:

- $\epsilon \to 0$ insuring that the difference in the H^s norm of the sequences of initial data goes to 0.
- $\lambda \to +\infty$ is the usual ansatz parameter hypothesis.
- $\epsilon\lambda \to +\infty$ insuring that the change of transport speed is enough to have disjoint supports at positive time.

Now if we put χ and $\tilde{\chi}$ to be the characteristic flows associated to the solutions u^0 and v^0 then:

$$\begin{aligned}(u-v)(t,x) &= u^0(\chi^{-1}(t,x)) - v^0(\tilde{\chi}^{-1}(t,x)) \\ &= u^0(\chi^{-1}(t,x)) - u^0(\tilde{\chi}^{-1}(t,x)) + O_{H^s}(\epsilon).\end{aligned}$$

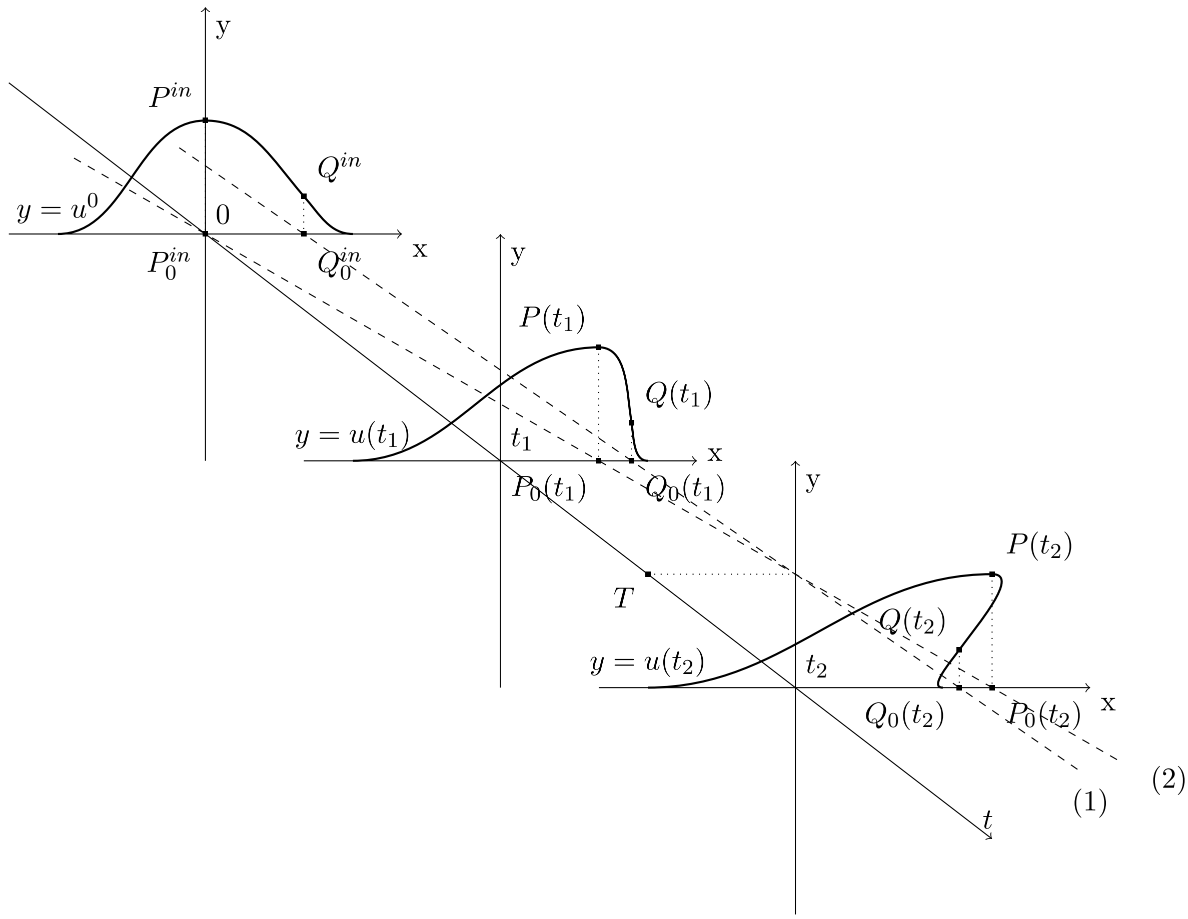

Figure 1.5: The lines (1) and (2) are the characteristic curves from Q_0^{in} and P_0^{in}. T is the time of formation of the shock wave.

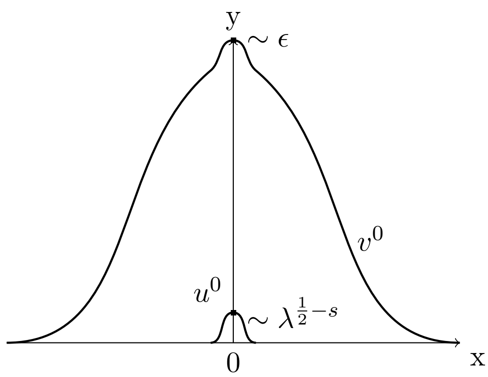

Figure 1.6: Graph of the ansatz.

Then using the compactly supported property of u^0 and the change of speed we prove that $u^0(\chi^{-1}(t,x))$ and $u^0(\tilde{\chi}^{-1}(t,x))$ have disjoint supports which is illustrated by Figure 1.7. We then prove that $\left\|u^0(\chi^{-1}(t,x))\right\|_{H^s} \geq c > 0$ which finishes the proof of the non uniform continuity of the flow map. For the control in a weaker norm, that is the flow map cannot be $C^1(H^s(\mathbb{D}), C^0([0,T], H^{s-1+\epsilon}(\mathbb{D})))$, we get it

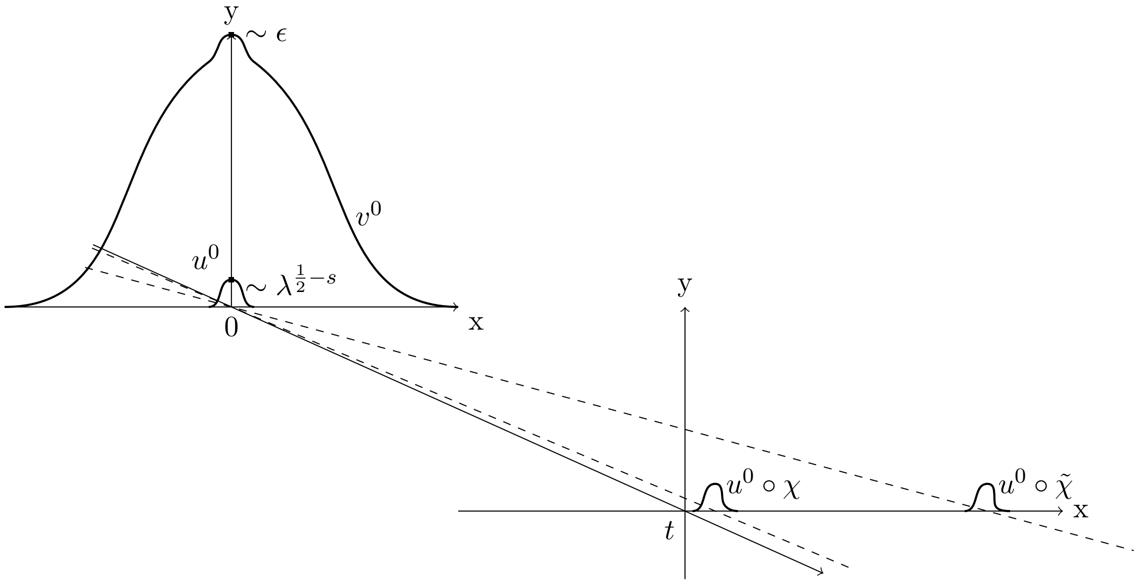

Figure 1.7: Transport of the ansatz.

from the estimate $\left\|u^0(\chi(t,x)^{-1})\right\|_{H^{s-\mu}} \geq c\lambda^{-\mu}$.

1.5.1.2 Quasi-linearity of problem (1.2.2)

Now if we adapt the proof to our current problem (1.2.2) we get:

$$\begin{aligned}(u-v)(t,x) &= f(t,\chi(t,x)^{-1}) - g(t,\tilde{\chi}(t,x)^{-1}) \\ &= f(t,\chi(t,x)^{-1}) - f(t,\tilde{\chi}(t,x)^{-1}) + O_{H^s}\big(\epsilon + t^2\epsilon\lambda^\alpha\big),\end{aligned}$$

where f and g are solutions to

$$\partial_t f + (\partial_x |D|^{\alpha-1})^* f = 0 \tag{1.5.1}$$

$$\partial_t g + \widetilde{(\partial_x |D|^{\alpha-1})}^* g = 0 \tag{1.5.2}$$

and $(\cdot)^*$ and $\widetilde{(\cdot)}^*$ are the change of variables by the characteristic flows defined for a symbol a by

$$\mathrm{Op}(a)^*(u\circ\chi) = (\mathrm{Op}(a)u)\circ\chi \text{ i.e } \mathrm{Op}(a)^*(u) = (\mathrm{Op}(a)[u\circ\chi^{-1}])\circ\chi,$$

and analogously for $\widetilde{(\cdot)}^*$. The pulled back equations (1.5.1) and (1.5.2) are shown to be well posed in Appendix 3.5.

The first immediate problem we face is the extra term $t^2\epsilon\lambda^\alpha$ which diverges. To remedy this problem, we give up wanting to control of the flow map punctually with a fixed time t and use a conveniently chosen sequence of small time (τ) to control $\tau^2\epsilon\lambda^\alpha$:

$$\tau \to 0 \text{ and } \lambda\epsilon\tau \to +\infty.$$

The second, deeper problem we face is that we lose control over the support of the solution. Indeed (1.5.1) and (1.5.2) are obtained by pull-back of the linear equation

$$\partial_t w + \partial_x |D|^{\alpha-1} w = 0 \tag{1.5.3}$$

which is a non local-dispersive equation that is expected to disperse the support of the solution and the L^∞ norm. This phenomena is thus expected to oppose the phenomena illustrated by the previous Figures (1.5) and (1.6) and indeed does so for the KdV equation on $\mathbb{R}$.

To remedy this, the idea is not to use u^0 and v^0 as initial data but by profiting of the time reversibility [3] of the equations use the backward in time solutions u^1 and v^1 defined by:

$$\begin{cases} \omega \text{ solution of } (1.5.3), \\ \omega(\tau,\cdot) = u^0, \\ \omega(0,\cdot) = u^1, \end{cases} \qquad \begin{cases} \omega' \text{ solution of } (1.5.3), \\ \omega'(\tau,\cdot) = v^0, \\ \omega'(0,\cdot) = v^1. \end{cases}$$

This gives us:

$$(u-v)(\tau,x) = u^0(\chi^{-1}(t,x))) - u^0(\tilde{\chi}^{-1}(t,x)) + O_{H^s}\big(\epsilon + \tau^2\epsilon\lambda^\alpha + \tau^2\lambda^{\alpha-1}\big).$$

We then prove that this gives the desired result, in the threshold $\alpha \in [0,2[$, by proving analogously to the Burgers equation: $\left\|u^0(\chi^{-1}(t,x))\right\|_{H^s} \geq c > 0$ and then using the compactly supported property of u^0 and the change of speed we prove that $u^0(\chi^{-1}(t,x)))$ and $u^0(\tilde{\chi}^{-1}(t,x))$ have disjoint supports.

1.5.2 Sketch of the proof of Theorem 1.2.2

For Theorem 1.2.2, we first work on H^s_0 and the main idea is to conjugate (1.2.2) to a semi-linear dispersive equation of the form:

$$\partial_t w + |D|^{\alpha-1}\partial_x w = Ru,$$

where R is continuous from H^s to itself. For the viscous Burgers equation such a result is obtained by the Cole-Hopf transformation that reduces the problem to a one dimensional heat equation. In [72], Tao used a complex version of the Cole-Hopf transformation to reduce the problem on the Benjamin-Ono equation to a one dimensional Schrödinger type equation, this idea was extensively used to lower the regularity needed for the well-posedness of the Cauchy problem as in Molinet's work in [56]. A generalized pseudodifferential form of this transformation was used in [7] to reduce the one dimensional water waves system to a a one dimensional semi-linear Schrödinger type system.

Formally if we follow the same lines of those previous papers, the transformation we will have to use is a pseudodifferential transformation of the form:

$$\begin{cases} w = \mathrm{Op}(a)u, \\ a = e^{\frac{1}{i\alpha}\xi|\xi|^{1-\alpha}U}, \end{cases} \tag{1.5.4}$$

where U is a real valued periodic primitive of u (that is $\partial_x U = u$), that exists because u has mean value 0.

[3] This idea fundamentally depends on the local reversibility in time of the linearised equations and thus fails for the fractional Burgers equation.

The main problem is that such an operator belongs to a Hörmander symbol class of the form $S^0_{\alpha-1,2-\alpha}$, which for $\alpha=\frac{3}{2}$ becomes $S^0_{\frac{1}{2},\frac{1}{2}}$ which is a "bad" symbol class with no general symbolic calculus rules. Thus we have to treat this transformation with care.

The idea here is inspired by the particular form of the formal computation, we express the desired operator as the time one of a flow map associated to a hyperbolic equation, i.e $a=A_1$ where $(A_\tau)_{\tau\in\mathbb{R}}$ is defined as the group generated by the paradifferential operator iT_p where p is a real valued symbol of order smaller than 1. This is inspired by previous results of Alazard, Baldi and P.Gérard [2].

Take a different operator T_b. The main new idea is to apply a Baker-Campbell-Hausdorff formula. Formally this allows one to express $A_\tau T_b A_{-\tau}$ as a series of successive Lie derivatives $[iT_p,\cdots,[iT_p,T_b]\cdots]$. The same kind of computations go for $[A_\tau,T_b]$. The convergence of such a series is a non trivial problem, equivalent to solving a linear ODE in the Fréchet space of paradifferential symbol classes $\Gamma^m_{+\infty}$ defined in Appendix 2.2.3. Such an ODE is not generally well posed and to solve such a problem one usually has to look at a Nash-Moser type scheme. Though in our case we have an explicit ODE that can be solved locally with loss of derivative thus inspired by Hörmander's [37] and Beals in [14], we prove the existence of a symbol b^τ such that $A_\tau T_b A_{-\tau}=T_{b^\tau}$, moreover b^τ is shown to have the asymptotic expansion given by the Baker-Campbell-Hausdorff formula. The use of paradifferential operators is key here, as in Hörmander's [37], because the continuity of paradifferential operators given by Theorem 2.2.3 insures that we do not need to control an infinite number of semi-norms as would have been the case for pseudodifferential operators.

Finally the transformation defined in this way helps us to reduce the transport term of order 1 to a term of order $2-\alpha$ which is enough for our problem.

Passing from H^s_0 to H^s we use the following gauge transform:

$$\tilde{u}(t,x)=u(t,x-t\fint u_0)-\fint u_0,\ \text{where}\ \frac{1}{|\mathbb{T}|}\int_{\mathbb{T}}=\fint,$$

which we prove is continuous on H^s but not uniformly continuous and C^1 only from H^s to H^{s-1}.

For the Gravity-Capillary equation the problem is more delicate. Indeed the model problems we studied are for the system after paralinearization and symmetrization. First the change of variable from the original system to the paralinearized and symmetrized one is known to be Lipschitz on H^s for $s>2+\frac{1}{2}$. Thus the problem is indeed reduced to the study of the flow map regularity of an equation of the form

$$\partial_t u+T_V\partial_x u+iT_\gamma u=f.$$

In the same spirit as [7, 1] we perform a para-change of variable, i.e we para-compose with χ defined by (1.3.7), to get:

$$\partial_t[u^*]+T_W\partial_x u^*+iT_{|\xi|^{\frac{3}{2}}}u^*=f,\ \text{with}\ \int_{\mathbb{T}}W=0.$$

We then proceed exactly as for Equation (1.2.2) (with the 0 mean value hypothesis insured by the choice of χ).

Remark 1.5.1. • *Transformation* (1.5.4)*, in which we use a primitive of the solution is called a gauge transform in the literature.*

- *As for the Cole-Hopf transformation, this gauge transform* (1.5.4) *is one dimensional and no suitable generalization exists for water waves system in higher dimension.*

- *The same type of transformation can be iterated and get at the step of order k a remainder of order $k+1-k\alpha$ which is acceptable for k sufficiently large as $\alpha > 1$ but the price to pay is that we have to assume that $s > 1 + \frac{1}{\alpha-1}$.*

1.5.3 Sketch of the proofs of Theorems 1.2.3 and 1.2.4

The starting point is to prove the key a priori estimate (1.2.12). Let us explain how to do so on the L^2 level first, passing from this to the H^s estimates adds a (significantly) more technical step to the proof.

To get the L^2 estimate, we are looking for an operator A such that:

$$\partial_t u + T_{i\xi}T_u u - T_{\frac{\partial_x u}{2}} u + T_{i\xi|\xi|^{\alpha-1}} u = 0 \Rightarrow \partial_t u + A^{-1}T_{i\xi|\xi|^{\alpha-1}} A u = R_{2-\alpha} + R_\infty,$$

where A is a unitary operator modulo, at least, an α-regularizing operator, $R_{2-\alpha}$ is of order $2-\alpha$ with $\mathrm{Re}(R_{2-\alpha})$ of order 0.

To find A we follow our construction in section 4.1 and define $A = A_1^p$, where A_τ^p is the flow of a hyperbolic paradifferential equation of the form:

$$\partial_\tau A_\tau^p h_0 - iT_p A_\tau^p h_0 = 0, \ A_0^p h_0 = h_0.$$

Using our result on the Baker-Campbell-Hausdorff formula for this type of flow proved in section 4.1 we are looking for p such that:

$$[A_1^p, T_{i\xi|\xi|^{\alpha-1}}] = A_1^p\big(T_{i\xi}T_u - T_{\frac{\partial_x u}{2}}\big) + R.$$

In the previous section, choosing $p = \frac{\xi|\xi|^{1-\alpha}}{\alpha}U$, where U is a primitive of u, was enough to have R as an $\alpha-1$ regularizing operator when $s > \frac{3}{2}$ which gave us the desired result on the flow map regularity at this threshold. At our threshold of regularity where we only control the $L^2([0,T], W_x^{2-\alpha})$ norm of u this would give R as an operator of order 1, that is there is no apparent gain that comes from this transformation.

To remedy this the idea is to construct p implicitly. Indeed using the stability of paradifferential operators by commutation with A_1^p proved in section 4.1, we write:

$$[A_1^p, T_{i\xi|\xi|^{\alpha-1}}] = A_1^p T_{i^c[\xi|\xi|^{\alpha-1}]_1^p}.$$

It is here where the two cases $\alpha < 2$ and $\alpha \geq 2$ have to be treated differently. Indeed, on one hand by Proposition 4.1.3, $i^c[\xi\,|\xi|^{\alpha-1}]_1^p$ belongs to a symbol class of the form $L_*^\infty S_{\min(\alpha-1,1),2-\alpha}$. And on the other hand $u\partial_x + \frac{\partial_x u}{2} \in \Gamma_0^1 = L_*^\infty S_{1,0}$, thus for $\alpha < 2$ there is no hope to solve:

$$i^c[\xi\,|\xi|^{\alpha-1}]_1^p = u\partial_x + \frac{\partial_x u}{2},$$

and for $\alpha \geq 2$ it seems possible through a local implicit function type theorem.

The case $\alpha < 2$: We solve the problem approximately, using the ellipticity of $\xi\,|\xi|^{\alpha-1}$, we show that we can fully solve the first term in the Baker-Campbell-Hausdorff expansion of $i^c[\xi\,|\xi|^{\alpha-1}]_1^p$, i.e we solve:

$$[ip, i\xi\,|\xi|^{\alpha-1}] = u\partial_x + \frac{\partial_x u}{2}.$$

This amounts to right inverting a linear operator in the Fréchet space of paradifferential operators. The problem is first reduced to a standard linear inversion in the scale of Banach spaces defining the Fréchet space of paradifferential operators. Then using an explicit approximate parametrix, given by the usual Cole-Hopf choice of gauge transformation, a careful choice of cut-off functions studied in [61] and symbolic calculus we show that a Neumann series can be carried out to correct the right parametrix into a right inverse in one Banach space in the scale. We then use a bootstrap argument to propagate the regularity to the hole scale of Banach spaces and thus the Fréchet space of paradifferential operators.

Getting back to the equation,

- We carefully compute the cutoffs used in our paradifferential operators in order to exactly control the remainder terms.
- We use the fact that $u\partial_x + \frac{\partial_x u}{2}$ and $i\xi\,|\xi|^{\alpha-1}$ are L^2 skew-symmetric to ensure that p can be chosen L^2 self-symmetric.

In doing so we construct an L^2 unitary operator $A = A_1^p$ such that:

$$[A_1^p, T_{i\xi|\xi|^{\alpha-1}}] = A_1^p\big(T_{i\xi}T_u - T_{\frac{\partial_x u}{2}}\big) + \underbrace{\int_0^1 A_{1-r}^p[T_{ip}, T_{u\partial_x + \frac{\partial_x u}{2}}]A_{r-1}^p dr}_{(1)}.$$

Crudely at our threshold of regularity using symbolic calculus (1) seems to be of order $1 + 2 - \alpha$ which seems worse that what we had before. The key cancellation is that using the identity:

$$[B, C]^* = [C^*, B^*],$$

we see that (1) is actually L^2 skew-adjoint, which gives the conservation of the L^2 norm.

At our level of regularity $s > 2 - \alpha + \frac{1}{2}$ this key cancellation does not hold for higher Sobolev estimates, except in the case of the Benjamin-Ono equation, i.e $\alpha = 2$, indeed in that case:

$$p(x, \xi) = \frac{\mathrm{Op}(\frac{1}{D})P_{\geq b}(D)u}{2} \text{ for } \xi \geq 0,$$

and,

$$\left[T_{ip}^{B',b}, iT^{B',b}_{\sigma_{u\xi}^{B,b} + (\sigma_{u\xi}^{B,b})^*}\right] = T_{iu^2} \text{ for } \xi \geq 0,$$

this "exceptional" algebraic cancellation in the commutator is due to the fact that $\partial_\xi p = 0$ which does not occur for fractional $\alpha \neq 2$. This difficulty was noted in [33] and the proposed solution was to use a gauge transform with indeed $\partial_\xi p = 0$ to eliminate only the lowest frequency terms in $u\partial_x u$, i.e $P_0(D)u\partial_x u$, and treat the remainder terms in some carefully modified Bourgain type spaces.

Inspired by this idea we will indeed use a choice of p such that $\partial_\xi p = 0$, which in the paradifferential setting developed here will amount to a simple approximation of the symbol p by step functions in the frequency variable ξ.

The case $\alpha \geq 2$: Using the ellipticity of $\xi |\xi|^{\alpha-1}$ and the paradifferential setting constructed we show that:

$$p \mapsto {}^{c}[\xi |\xi|^{\alpha-1}]_1^p$$

is indeed locally surjective around 0, which is the key technical result we prove in Theorem 6.1.2. This is a non trivial problem, equivalent to solving a nonlinear ODE in the Fréchet space of paradifferential symbols. Such an ODE is not generally well posed and to solve such a problem one usually has to look at a Nash-Moser type scheme[4]. In our case the choice of paradifferential setting, inspired by by Hörmander's [37], is shown to be stable by the gauge transformation in Proposition 4.1.4. Thus we show that the problem can be reduced to a standard implicit function theorem combined with bootstrap argument in order to insure propagation of regularity. The bootstrap trick is the analogue of the one used in the Picard fixed point theorem depending on a parameter.

Thus we get,

$$A_1^p \partial_t u + T_{i\xi|\xi|^{\alpha-1}} A_1^p u = R_\infty.$$

We would like to fully conjugate the quasi-linear equation (1.2.10) to a semi-linear equation, i.e:

$$\partial_t u + T_{i\xi} T_u u - T_{\frac{\partial_x u}{2}} u + T_{i\xi|\xi|^{\alpha-1}} u = 0 \Rightarrow \partial_t \tilde{A} u + T_{i\xi|\xi|^{\alpha-1}} \tilde{A} u = R,$$

Thus compared to first gauge transform we have to treat the term $\partial_t A$. At our threshold of regularity $\partial_t A$ is still of order 1 thus we don't have a gain on the order of the operator. The key idea is to iterate the gauge transform to eliminate those time derivatives, i.e construct $(A_1^{p_n})$ that eliminate the terms $(\partial_t A_1^{p_{n-1}})$ at each step. We then use Theorem 4.1.1 and the geometric decrease in the norms at each step to prove the convergence:

$$\lim_{n \to +\infty} \prod_{k=1}^{n} A_1^{p_n} = \tilde{A}_1^{\tilde{p}} = \tilde{A}.$$

Remark 1.5.2. *The continuity of operators of type $T_{e^{i\tau p}}$ on Zygmund space with loss of derivatives were studied by E. Stein [71] and by G. Bourdaud in [20]. In section 4.1 we need explicit estimates taking into play the exact symbol semi-norms. For this we give a complete study the continuity of paradifferential operators defined by symbols in these type of "exotic" symbol classes in Appendix 4.2. Our proofs follow the same lines and methods presented in [71, 74, 53].*

[4] Such a scheme can indeed be carried out here thanks to the tame estimates in Remark 4.1.4 but we show that this can be avoided here.

Chapter 2

On paracomposition and paradifferential operators

This chapter reviews basic ideas from microlocal analysis, which can be found in [35], [36], [73], [11] and [53], fixes the notations for latter chapters and contains the main results on para-differential calculus from article [61].

Contents

2.1 Notations and functional analysis

We present the definitions of the functional spaces that will be used.
We will use the usual definitions and standard notations for the regular functions C^k, C^k_0 for those with compact support, the distribution space $\mathscr{D}'$,$\mathscr{E}'$ for those with compact support, $\mathscr{D}'^k$,$\mathscr{E}'^k$ for distributions of order k, Lebesgue spaces (L^p), Sobolev spaces ($H^s, W^{p,q}$) and the Schwartz class $\mathscr{S}$ and it's dual $\mathscr{S}'$. All of those spaces are equipped with their standard topologies. We also use the *Landau notation* $O_{\| \ \|}(X)$.

Notation 2.1.1. *We will use $\mathbb{D}$ to denote $\mathbb{T}$ or $\mathbb{R}$ and $\hat{\mathbb{D}}$ to denote their duals that is $\mathbb{Z}$ in the case of $\mathbb{T}$ and $\mathbb{R}$ in the case of $\mathbb{R}$. For concision an integral on on $\mathbb{Z}^d$ i.e $\int_{\mathbb{Z}^d}$ should be understood as $\sum_{\mathbb{Z}}^{d}$. A function a is said to be in $C^\infty(\mathbb{T}^d \times \mathbb{Z}^d)$ if* *for every* $\xi \in \mathbb{Z}$ $a(\cdot,\xi) \in C^\infty(\mathbb{T}^d)$*. For* $\xi \in \mathbb{Z}^d$ *and* $i \in \{1,\cdots,d\}$*,* ∂_{ξ_i} *should be understood as the partial forward difference operator, i.e*

$$\partial_{\xi_i} a(\xi_1,\cdots,\xi_i,\cdots,\xi_d) = a(\xi_1,\cdots,\xi_i+1,\cdots,\xi_d) - a(\xi_1,\cdots,\xi_i,\cdots,\xi_d),\ \xi \in \mathbb{Z}^d.$$

We recall the following simple identities for the Fourier transform on the Torus:

$$\begin{cases} \mathscr{F}_{\mathbb{T}^d}(\partial_x^\alpha f)(\xi) = \xi^\alpha \mathscr{F}_{\mathbb{T}^d}(f)(\xi), \xi \in \mathbb{Z}^d, \\ \mathscr{F}_{\mathbb{T}^d}((e^{-2i\pi x}-1)^\alpha f)(\xi) = \xi^\alpha \mathscr{F}_{\mathbb{T}^d}(f)(\xi), \xi \in \mathbb{Z}^d,\ x \in \mathbb{T}^d. \end{cases}$$

Remark 2.1.1. *Henceforth in order to have compact statements including both cases of functions defined on* $\mathbb{T}$ *and* $\mathbb{R}$*, we will use the following abuse of notation* $\mathscr{S}(\mathbb{D}^d) = \mathscr{S}(\mathbb{R}^d)$*, when* $\mathbb{D} = \mathbb{R}$ *and* $\mathscr{S}(\mathbb{D}^d) = C^\infty(\mathbb{T}^d)$*, when* $\mathbb{D} = \mathbb{T}$*).*

Definition 2.1.1 (Littlewood-Paley decomposition)**.** *Pick* $P_0 \in C_0^\infty(\mathbb{R}^d)$ *so that,*

$$P_0(\xi) = 1 \text{ for } |\xi| < 1 \text{ and } 0 \text{ for } |\xi| > 2.$$

We define a dyadic decomposition of unity by:

$$\text{for } k \geq 1,\ P_{\leq k}(\xi) = \Phi_0(2^{-k}\xi),\ P_k(\xi) = P_{\leq k}(\xi) - P_{\leq k-1}(\xi).$$

Thus,

$$P_{\leq k}(\xi) = \sum_{0 \leq j \leq k} P_j(\xi) \text{ and } 1 = \sum_{j=0}^{\infty} P_j(\xi).$$

Introduce the operator acting on $\mathscr{S}'(\mathbb{R}^d)$*:*

$$P_{\leq k} u = \mathscr{F}^{-1}(P_{\leq k}(\xi)u) \text{ and } u_k = \mathscr{F}^{-1}(P_k(\xi)u).$$

Thus,

$$u = \sum_k u_k.$$

Finally put $\{k \geq 1, C_k = \operatorname{supp}\ P_k\}$ *the set of rings associated to this decomposition.*

Remark 2.1.2. *An interesting property of the Littlewood-Paley decomposition is that even if the decomposed function is merely a distribution the terms of the decomposition are regular, indeed they all have compact spectrum and thus are entire functions. On classical functions spaces this regularization effect can be "measured" by the following inequalities due to Bernstein.*

Proposition 2.1.1 (Bernstein's inequalities)**.** *Suppose that* $a \in L^p(\mathbb{R}^d)$ *has its spectrum contained in the ball* $\{|\xi| \leq \lambda\}$*. Then* $a \in C^\infty$ *and for all* $\alpha \in \mathbb{N}^d$ *and* $1 \leq p \leq q \leq +\infty$*, there is* $C_{\alpha,p,q}$ *(independent of* λ*) such that*

$$\|\partial_x^\alpha a\|_{L^q} \leq C_{\alpha,p,q} \lambda^{|\alpha| + \frac{d}{p} - \frac{d}{q}} \|a\|_{L^p}.$$

In particular,

$$\|\partial_x^\alpha a\|_{L^q} \leq C_\alpha \lambda^{|\alpha|} \|a\|_{L^p},\ \text{and for } p=2,\ p=\infty$$

$$\|a\|_{L^\infty} \leq C\lambda^{\frac{d}{2}} \|a\|_{L^2}.$$

If moreover a has it's spectrum is in $\{0 < \mu \leq |\xi| \leq \lambda\}$ *then:*

$$C_{\alpha,q}^{-1} \mu^{|\alpha|} \|a\|_{L^q} \leq \|\partial_x^\alpha a\|_{L^q} \leq C_{\alpha,q} \lambda^{|\alpha|} \|a\|_{L^q}.$$

Proposition 2.1.2. *For all $\mu > 0$, there is a constant C such that for all $\lambda > 0$ and for all $\alpha \in W^{\mu,\infty}$ with spectrum contained in $\{|\xi| \geq \lambda\}$. one has the following estimate:*

$$\|a\|_{L^\infty} \leq C\lambda^{-\mu} \|a\|_{W^{\mu,\infty}} .$$

Definition 2.1.2 (Singular support). *$f \in \mathscr{S}'(\mathbb{R}^d)$ is said to be C^∞ in a neighborhood of x, if there exists a neighborhood ω of x such that for all $\psi \in C_0^\infty(\omega)$ we have $\psi f \in C^\infty(\mathbb{R}^d)$.*
The singular support of a distribution f, sing supp f*, is defined as the complementary of such points and is clearly closed.*

Definition 2.1.3 (Zygmund spaces on $\mathbb{R}^d$). *For $r \in \mathbb{R}$ we define the space:*

$$C_*^r(\mathbb{R}^d) \subset \mathscr{S}'(\mathbb{R}^d), \text{ by } C_*^r(\mathbb{R}^d) = \left\{ u \in \mathscr{S}'(\mathbb{R}^d), \|u\|_r = \sup_q 2^{qr} \|u_q\|_\infty < \infty \right\}$$

equipped with its canonical topology giving it a Banach space structure.
It's a classical result that for $r \notin \mathbb{N}$, $C_*^r(\mathbb{R}^d) = W^{r,\infty}(\mathbb{R}^d)$ the classic Hölder spaces.
We define the local spaces:

$$C_{*,loc}^r(\mathbb{R}^d) = \left\{ u \in \mathscr{S}'(\mathbb{R}^d), \forall \psi \in C_0^\infty(\mathbb{R}^d), \psi u \in C_*^r(\mathbb{R}^d) \right\} .$$

Proposition 2.1.3. *Let B be a ball with center 0. There exists a constant C such that for all $r > 0$ and for all $(u_q)_{q\in\mathbb{N}} \in \mathscr{S}'(\mathbb{R}^d)$ verifying:*

$$\forall q, \operatorname{supp} \hat{u_q} \subset 2^q B \text{ and } (2^{qr} \|u_q\|_\infty)_{q\in\mathbb{N}} \text{ is bounded}$$

$$\text{then}, u = \sum_q u_q \in C_*^r(\mathbb{R}^d) \text{ and } \|u\|_r \leq \frac{C}{1 - 2^{-r}} sup_{q\in\mathbb{N}} 2^{qr} \|u_q\|_\infty .$$

For the definition of spaces in open subsets of $\mathbb{R}^d$ we follow the presentation of [25]. Let Ω be an open subset of $\mathbb{R}^d$.

Definition 2.1.4 (Zygmund spaces on Ω). *For $r \in \mathbb{R}$ we define the space:*

$$C_*^r(\Omega) \subset \mathscr{D}'(\Omega), \text{ by } C_*^r(\Omega) = \left\{ u \in \mathscr{D}'(\Omega), u = U_{|\Omega} \text{ for some } U \in C_*^r(\mathbb{R}^d) \right\}$$

equipped with its canonical topology i.e

$$\|u\|_{C_*^r(\Omega)} = \inf_{\substack{U \in C_*^r(\mathbb{R}^d) \\ U_{|\Omega} = u}} \|U\|_{C_*^r(\mathbb{R}^d)}$$

giving it a Banach space structure.
We define the local spaces:

$$C_{*,loc}^r(\Omega) = \left\{ u \in \mathscr{D}'(\Omega), \forall \psi \in C_0^\infty(\Omega), \psi u \in C_*^r(\Omega) \right\} .$$

Definition 2.1.5 (Sobolev spaces on $\mathbb{R}^d$). *It is also a classical result that for $s \in \mathbb{R}$:*

$$H^s(\mathbb{R}^d) = \left\{ u \in \mathscr{S}'(\mathbb{R}^d), |u|_s = \left(\sum_q 2^{2qs} \|u_q\|_{L^2}{}^2 \right)^{\frac{1}{2}} < \infty \right\}$$

with the right hand side equipped with its canonical topology giving it a Hilbert space structure and $|\ |_s$ *is equivalent to the usual norm on* $\|\ \|_{H^s}$ *.*
We define the local spaces:

$$H^s_{loc}(\mathbb{R}^d) = \left\{ u \in \mathscr{S}'(\mathbb{R}^d), \forall \psi \in C_0^\infty(\mathbb{R}^d), \psi u \in H^s(\mathbb{R}^d) \right\}.$$

Proposition 2.1.4. *Let* B *be a ball with center 0. There exists a constant* C *such that for all* $s > 0$ *and for all* $(u_q)_{\in\mathbb{N}} \in \mathscr{S}'(\mathbb{R}^d)$ *verifying:*

$$\forall q, \operatorname{supp} \hat{u_q} \subset 2^q B \text{ and } (2^{qs} \|u_q\|_{L^2})_{q\in\mathbb{N}} \text{ is in } L^2(\mathbb{N})$$

$$\text{then, } u = \sum_q u_q \in H^s(\mathbb{R}^d) \text{ and } |u|_s \leq \frac{C}{1-2^{-s}} \left(\sum_q 2^{2qs} \|u_q\|_{L^2}{}^2 \right)^{\frac{1}{2}}.$$

Remark 2.1.3. *The previous definition and properties of the Littlewood-Paley decomposition, Zygmund spaces and Sobolev spaces carries out naturally to* $\mathbb{T}^d$*.*

Definition 2.1.6 (Sobolev spaces on Ω)**.** *For* $s \in \mathbb{R}$ *we define the space*

$$H^s(\Omega) \subset \mathscr{D}'(\Omega), \text{ by } H^s(\Omega) = \left\{ u \in \mathscr{D}'(\Omega), u = U_{|\Omega} \text{ for some } U \in H^s(\mathbb{R}^d) \right\},$$

equipped with its canonical topology i.e,

$$\|u\|_{H^s(\Omega)} = \inf_{\substack{U \in H^s(\mathbb{R}^d) \\ U_{|\Omega} = u}} \|U\|_{H^s(\mathbb{R}^d)},$$

giving it a Hilbert space structure[1]*.*
We define the local spaces:

$$H^s_{loc}(\Omega) = \left\{ u \in \mathscr{D}'(\Omega), \forall \psi \in C_0^\infty(\Omega), \psi u \in H^s(\Omega) \right\}.$$

Remark 2.1.4. *This definition of the functions in an open subset might not seem as the most natural, in fact there are different ways(intrinsically, extrinsically, by interpolation etc...) to define* $H^s(\Omega)$ *and when no regularity assumption is put on* Ω *and they don't necessarily match. In [25] they show that when* Ω *has Lipschitz regularity all the different definitions of* $H^s(\Omega)$ *coincide.*

We recall the usual nonlinear estimates in Sobolev spaces:

- If $u_j \in H^{s_j}(\mathbb{R}^d), j = 1, 2,$ and $s_1 + s_2 > 0$ then $u_1 u_2 \in H^{s_0}(\mathbb{R}^d)$ and if

$$s_0 \leq s_j, j = 1, 2 \text{ and } s_0 \leq s_1 + s_2 - \frac{d}{2},$$

$$\text{then } \|u_1 u_2\|_{H^{s_0}} \leq K \|u_1\|_{H^{s_1}} \|u_2\|_{H^{s_2}},$$

  where the last inequality is strict if s_1 or s_2 or $-s_0$ is equal to $\frac{d}{2}$.

[1] This is not immediate from the definition but is a consequence of the fact that $H^s(\Omega)$ can be seen as a quotient of $H^s(\mathbb{R}^d)$ by a closed subset, for a full presentation see [25].

- For all C^∞ function F vanishing at the origin, if $u \in H^s(\mathbb{R}^d)$ with $s > \frac{d}{2}$, then,

$$\|F(u)\|_{H^s} \leq C(\|u\|_{H^s}),$$

for some non decreasing function C depending only on F.

Finally we present a classic result for operator estimates by Y.Meyer [54]:

Lemma 2.1.1 (Meyer multipliers)**.** *Let $\delta \in \mathbb{R}$, and suppose we have a sequence:*

$$m_p \in C^\infty,\ \forall k \in \mathbb{N},\ \sum_{|\alpha|=k} \|\partial^\alpha m_p\|_\infty \leq C_k 2^{p(k+\delta)}.$$

The mapping $M : u \mapsto \sum m_p u_p = Mu$ maps H^s to $H^{s-\delta}$ and C^r_* to $C^{r-\delta}_*$ for all $s, r > \delta$, with operators norms depending only on the C_k for $k \leq E(s-\delta)+1$ or $k \leq E(r-\delta)+1$.

Here we recall the usual Kato-Ponce [44] commutator estimates:

Proposition 2.1.5. *Consider $s > 0$ and $f, g \in H^s$ then*

$$\|[\langle \mathrm{D}\rangle^s, f]g\|_{L^2} \leq C(\|f\|_{W^{1,\infty}} \|g\|_{H^{s-1}} + \|f\|_{H^s} \|g\|_{L^\infty}).$$

2.2 Notions of microlocal analysis

In this paragraph we start by reviewing classic notations and results about pseudodifferential calculus, Fourier integral operators and paradifferential calculus, which can be found in [35], [36], [73], [11] and [53] as an accessible presentation to the theories and from which we follow the presentation. Moreover we complete this by our study of the support of the composition of two paradifferential operators.

2.2.1 Pseudodifferential Calculus

We introduce here the basic definitions and symbolic calculus results. We first introduce the classes of regular symbols.

Definition 2.2.1. *Given $m \in \mathbb{R}, 0 \leq \rho \leq 1$ and $0 \leq \sigma \leq 1$ we denote the symbol class $S^m_{\rho,\sigma}(\mathbb{D}^d \times \hat{\mathbb{D}}^d)$ the set of all $a \in C^\infty(\mathbb{D}^d \times \hat{\mathbb{D}}^d)$ such that for all multi-orders α, β we have the estimate:*

$$\left|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)\right| \leq C_{\alpha,\beta}(1+|\xi|)^{m-\rho\beta+\sigma\alpha}.$$

$S^m_{\rho,\sigma}(\mathbb{D}^d \times \hat{\mathbb{D}}^d)$ is a Fréchet space with the topology defined by the family of semi-norms:

$$M^m_{\alpha,\beta}(a) = \sup_{i\leq\alpha, j\leq\beta} \sup_{\mathbb{D}^d\times\hat{\mathbb{D}}^d} \left|\partial_x^i \partial_\xi^j a(x,\xi)(1+|\xi|)^{\rho j-m-\sigma i}\right|.$$

Set

$$S^m(\mathbb{D}^d \times \hat{\mathbb{D}}^d) = S^m_{1,0}(\mathbb{D}^d \times \hat{\mathbb{D}}^d),$$

$$S^{-\infty}(\mathbb{D}^d \times \hat{\mathbb{D}}^d) = \bigcap_{m\in\mathbb{R}} S^m(\mathbb{D}^d \times \hat{\mathbb{D}}^d) \text{ and } S^{+\infty}(\mathbb{D}^d \times \hat{\mathbb{D}}^d) = \bigcup_{m\in\mathbb{R}} S^m(\mathbb{D}^d \times \hat{\mathbb{D}}^d)$$

equipped with their canonically induced topology.

Given a symbol $a \in S^m(\mathbb{D}^d \times \hat{\mathbb{D}}^d)$, we define the pseudodifferential operator:

$$\mathrm{Op}(a)u(x) = a(x, D)u(x) = (2\pi)^{-n} \int_{\hat{\mathbb{D}}^d} e^{ix.\xi} a(x, \xi)\hat{u}(\xi)d\xi.$$

For $u \in \mathscr{S}(\mathbb{D}^d)$ we have

$$\begin{aligned} \mathrm{Op}(a)u(x) &= (2\pi)^{-d} \int_{\hat{\mathbb{D}}^d} e^{ix.\xi} a(x, \xi)\hat{u}(\xi)d\xi \\ &= (2\pi)^{-d} \int_{\hat{\mathbb{D}}^d} e^{ix.\xi} a(x, \xi) \int_{\mathbb{D}^d} e^{-iy.\xi} u(y)dyd\xi \\ &= \int_{\hat{\mathbb{D}}^d} \left((2\pi)^{-n} \int_{\mathbb{D}^d} e^{i(x-y)\cdot\xi} a(x, \xi)d\xi \right) u(y)dy. \end{aligned}$$

Thus giving us the following Proposition.

Proposition 2.2.1. *For $a \in S^m(\mathbb{D}^d \times \hat{\mathbb{D}}^d)$, $\mathrm{Op}(a)$ has a kernel K defined by*

$$K(x, y) = (2\pi)^{-d} \int_{\hat{\mathbb{D}}^d} e^{i(x-y)\cdot\xi} a(x, \xi)d\xi = (2\pi)^{-n} \mathscr{F}_\xi a(x, y - x). \tag{2.2.1}$$

Which can be inverted to give:

$$\begin{aligned} a(x, \xi) = \mathscr{F}_{y\to\xi} K(x, x - y) &= \int_{\mathbb{D}^d} e^{-iy.\xi} K(x, x - y)dy \\ &= (-1)^d e^{-ix.\xi} \int_{\mathbb{D}^d} e^{iy.\xi} K(x, y)dy. \end{aligned} \tag{2.2.2}$$

Definition 2.2.2. *Let $m \in \mathbb{R}$, an operator T is said to be of order m if, and only if, for all $\mu \in \mathbb{R}$, it is bounded from $H^\mu(\mathbb{D}^d)$ to $H^{\mu-m}(\mathbb{D}^d)$.*

Theorem 2.2.1. *If $a \in S^m(\mathbb{D}^d \times \hat{\mathbb{D}}^d)$, then $a(x, D)$ is an operator of order m. Moreover we have the norm estimate:*

$$\|a(x, D)\|_{H^\mu \to H^{\mu-m}} \le C M^m_{\mu, m+d/2+1}(a).$$

We will now present the main results in symbolic calculus associated to pseudodifferential operators.

Theorem 2.2.2. *Let $m, m' \in \mathbb{R}^d$, $a \in S^m(\mathbb{D}^d \times \hat{\mathbb{D}}^d)$ and $b \in S^{m'}(\mathbb{D}^d \times \hat{\mathbb{D}}^d)$.*

- *Composition: Then $\mathrm{Op}(a) \circ \mathrm{Op}(b)$ is a pseudodifferential operator of order $m + m'$ with symbol $a \otimes b$ defined by:*

  $$a \otimes b(x, \xi) = (2\pi)^{-d} \int_{\mathbb{D}^d \times \hat{\mathbb{D}}^d} e^{i(x-y).(\xi-\eta)} a(x, \eta)b(y, \xi)dyd\eta.$$

  Moreover,

  $$\mathrm{Op}(a)\circ\mathrm{Op}(b)(x, \xi) - \mathrm{Op}(\sum_{|\alpha|<k} \frac{1}{i^{|\alpha|}\alpha!} (\partial_\xi^\alpha a(x, \xi))(\partial_x^\alpha b(x, \xi))) \text{ is of order } m + m' - k$$

  for all $k \in \mathbb{N}$.

- *Adjoint: The adjoint operator of* $\mathrm{Op}(a)$*, that will note* $\mathrm{Op}(a)^t$ *to avoid confusion with the pullback operator defined in this work, is a pseudodifferential operator of order* m *with symbol* a^t *defined by:*

$$a^t(x,\xi) = (2\pi)^{-d} \int_{\mathbb{D}^d \times \hat{\mathbb{D}}^d} e^{-iy.\eta} \bar{a}(x-y, \xi-\eta) dy d\eta$$

Moreover,

$$\mathrm{Op}(a^t)(x,\xi) - \mathrm{Op}(\sum_{|\alpha|<k} \frac{1}{i^{|\alpha|}\alpha!}(\partial_\xi^\alpha \partial_x^\alpha \bar{a}(x,\xi))) \text{ is of order } m-k$$

for all $k \in \mathbb{N}$.

Definition 2.2.3. *Let* $(a_j) \in S^{m_j}(\mathbb{D}^d \times \hat{\mathbb{D}}^d)$ *be a series of symbols with* $(m_j) \in \mathbb{R}^d$ *decreasing to* $-\infty$*. We say that* $a \in S^{m_0}(\mathbb{D}^d \times \hat{\mathbb{D}}^d)$ *is the asymptotic sum of* (a_j) *if*

$$\forall k \in \mathbb{N}, a - \sum_{j=0}^{k} a_j \in S^{m_{k+1}}(\mathbb{D}^d).$$

We denote $a \sim \sum a_j$

Remark 2.2.1. *We can now write simply:*

$$a \otimes b \sim \sum_{|\alpha|} \frac{1}{i^{|\alpha|}\alpha!}(\partial_\xi^\alpha a(x,\xi))(\partial_x^\alpha b(x,\xi))$$

and

$$a^t \sim \sum_{|\alpha|} \frac{1}{i^{|\alpha|}\alpha!}(\partial_\xi^\alpha \partial_x^\alpha \bar{a}(x,\xi)).$$

Proposition 2.2.2 (Pseudo-local property)**.** *Let* $a \in S^m(\mathbb{D}^d \times \hat{\mathbb{D}}^d)$ *and let* K *be its kernel. Then* K *is* C^∞ *for* $x \neq y$*. In particular, for all* $u \in \mathscr{S}'$*:*

$$\text{sing supp}\, a(x,D)u \subset \text{sing supp}\, u$$

Proof. Let $x \neq y$, $\psi, \theta \in C_0^\infty(\mathbb{R}^d)$,$\psi = 1$ near x, $\theta = 1$ near y and $\operatorname{supp}\psi \cap \operatorname{supp}\theta = \emptyset$. Then $\tilde{K}(x,y) = \psi(x)K(x,y)\theta(y)$ is the kernel of the operator $\psi a \theta$. By Theorem 2.2.2 , $\psi a \theta \sim 0$ thus is of order $-\infty$ which finishes the proof.

□

Let Ω be an open subset of $\mathbb{R}^d$. We will now define the notion of local symbols and operators in an open set.

Definition 2.2.4 (Local operators and symbols)**.** *We define* $S^m(\Omega \times \mathbb{R}^d)$ *to be the set of* $a \in C^\infty(\Omega \times \mathbb{R}^d)$ *such that for all multi-orders* α, β *we have the estimate:*

$$\left|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)\right| \leq C_{\alpha,\beta}(1+|\xi|)^{m-\rho\beta+\sigma\alpha}.$$

$S^m(\Omega^d \times \mathbb{R}^d)$ *is a Fréchet space with the topology defined by the family of semi-norms:*

$$M^m_{\alpha,\beta}(a) = \sup_{i \leq \alpha, j \leq \beta} \sup_{\Omega \times \mathbb{R}^d} \left|\partial_x^i \partial_\xi^j a(x,\xi)(1+|\xi|)^{\rho j - m - \sigma i}\right|.$$

We define the local spaces:

$$S^m_{loc}(\Omega\times\mathbb{R}^d)=\left\{a\in C^\infty(\Omega\times\mathbb{R}^d),\forall\psi\in C_0^\infty(\Omega),\psi a\in S^m(\Omega\times\mathbb{R}^d)\right\},$$

equipped with its canonical topology giving it a Fréchet space structure.

If $a\in S^m(\Omega\times\mathbb{R}^d)$ or $S^m_{loc}(\Omega\times\mathbb{R}^d)$, the usual formula

$$Au(x)=a(x,D)u(x)=(2\pi)^{-d}\int_{\mathbb{R}^d}e^{ix.\xi}a(x,\xi)\hat{u}(\xi)d\xi$$

defines an operator respectively from $\mathscr{S}'(\mathbb{R}^d)$,$\mathscr{E}'(\Omega)$ to $\mathscr{D}'(\Omega)$, which can be restricted to an operator $\mathscr{E}'(\Omega)\to\mathscr{D}'(\Omega)$ and $C_0^\infty(\Omega)\to C^\infty(\Omega)$.
The link between such operators and the operators obtained by cut-off from global operators is given by the following Proposition:

Proposition 2.2.3. *Let $A:v\to C^\infty(\Omega)$ be a continuous linear operator such that for all $\psi,\theta\in C_0^\infty(\Omega)$, $\psi A\theta\in\mathrm{Op}(S^m)$. Then there exists $a'\in S^m(\Omega\times\mathbb{R}^d)$ with A=a'(x,D)+R, where R is an operator with kernel in $C^\infty(\Omega\times\Omega)$.*

Proof. Let $(\psi_j)\in C_0^\infty(\Omega)$ be a partition of unity locally finite over Ω. Put $\psi_jA\psi_k=A_{jk}\in\mathrm{Op}(S^m)$ then

$$Au=\sum_{j,k}\psi_jA\psi_k=\sum_{\substack{j,k\\ \operatorname{supp}\psi_j\cap\psi_k\neq\emptyset}}A_{jk}+\sum_{\substack{j,k\\ \operatorname{supp}\psi_j\cap\psi_k=\emptyset}}A_{jk}.$$

Then

$$a'=\sum_{\substack{j,k\\ \operatorname{supp}\psi_j\cap\psi_k\neq\emptyset}}A_{jk}\in S^m(\Omega\times\mathbb{R}^d)$$

because for $\forall\psi\in C_0^\infty(\Omega)$, $\psi a'$ is a finite sum by definition of a partition of unity locally finite.
The remainder has a kernel:

$$\sum_{\substack{j,k,\\ \operatorname{supp}\psi_j\cap\psi_k=\emptyset}}\psi_j(x)K(x,y)\psi_k(y)\in C^\infty(\Omega\times\Omega)$$

by the pseudo-local property, Proposition 2.2.2 . □

We see from the previous definition that there is subtlety with the support of the functions if one want for example to define A^t. The following class of local operators clarifies that problem:

Definition 2.2.5 (Properly supported operators)**.** *A continuous linear operator $A:C_0^\infty(\Omega)\to C^\infty(\Omega)$ is said to be properly supported if, for any compact subset $K\subset\Omega$, there exists a compact subset $K'\subset\Omega$ with:*

$$\operatorname{supp}u\subset K\Longrightarrow\operatorname{supp}Au\subset K'\ \textit{and}\ u=0\ \textit{on}\ K'\Longrightarrow Au=0\ \textit{on}\ K$$

We see that such an operator maps C_0^∞ to C_0^∞ and for example A^t can be extended in a standard way to an operator from $\mathscr{D}'(\Omega)$ to itself.

Proposition 2.2.4. *Let $A = a(x, D)$ where $a \in S^m_{loc}(\Omega \times \mathbb{R}^d)$. There exists an operator R with kernel in $C^\infty(\Omega \times \Omega)$ such that $A+R$ is properly supported.*

Proof. This is the same proof as Proposition 2.2.3 because

$$\sum_{\substack{j,k, \\ \operatorname{supp} \psi_j \cap \psi_k = \emptyset}} A_{jk}$$

is properly supported. □

Remark 2.2.2. *The previous Proposition tells us that for local regularity considerations we can essentially work with properly supported operators for local symbols (modulo a C^∞ kernel) and by Proposition 2.2.3 we can do the same for operators obtained by cut-off.*

2.2.2 Fourier Integral Operators

Here we will give basic definitions and results as presented in part 1 of Hörmander's [35].
We wish to define operators of the form :

$$\begin{aligned} A_\omega u(x) &= \int e^{iS(x,\xi)} a(x,\xi) \hat{u}(\xi) d\xi \\ &= \int e^{i(S(x,\xi) - y.\xi)} a(x,\xi) u(y) dy \\ &= \int e^{i\omega(x,y,\xi)} a(x,\xi) u(y) dy d\xi \end{aligned} \tag{2.2.3}$$

where u is a regular function, a is a symbol and ω is a given function defining the operator A. We can clearly see that for example $\omega = 0$ the integral in not defined for symbols with $m \geq -d$, we thus start by the following definition of suitable phase functions:

Definition 2.2.6. *Let $\omega(x, y, \xi)$ be a $C^\infty(\Omega \times \Omega \times \mathbb{R}^d)$ map which is positively homogeneous of degree one with respect to ξ. Put:*

$$R_\omega = \left\{ (x,y) \in \Omega \times \Omega, \forall \xi \in \mathbb{R}^d \setminus \{0\}, \omega(x,y,\xi) \text{ has no critical point} \right\}^{2},$$

and its compliment C_ω, which is the projection on $\Omega \times \Omega$ of the conic set (with respect to ξ) of:

$$C = \left\{ (x,y,\xi) \in \Omega \times \Omega \times \mathbb{R}^d \setminus \{0\}, D\omega_\xi(x,y,\xi) = 0 \right\}.$$

- *Then ω is called a phase function on $R_\omega \times \mathbb{R}^d$.*
- *ω is called a non-degenerate phase function if at any point in C, the differentials $D(\frac{\partial \omega}{\partial \xi_j}), j = 1, ..., d,$ are linearly independent.*
- *ω is called an operator phase function on $R_\omega \times \mathbb{R}^d$ if for each fixed x (or y) it has no critical point (y, ξ) $(or (x, \xi))$ with $\xi \neq 0$.*

[2] R_ω is clearly open.

- *For $U \subset \Omega$ define $C_\omega U = \{x, (x,y) \in C_\omega \text{ for some } y \in U\}$.*

The main example here are pseudodifferential operators with $\omega(x,y,\xi) = (x-y).\xi$, in that case C_ω is equal to the diagonal $\{(x,x), x \in \Omega\}$, and we see that all of the previous definitions naturally apply in this case.
The following Proposition will give a definition to the weak form of (2.2.3) :

$$< A_\omega u, v >=< op_\omega(a)u, v >= \int e^{i\omega(x,y,\xi)} a(x,y,\xi)u(y)v(x)dxdyd\xi,\ u,v \in C_0^\infty(\Omega). \tag{2.2.4}$$

Proposition 2.2.5. *Take a symbol $a \in S^m_{\rho,\sigma}(\Omega \times \Omega \times \mathbb{R}^d)$,$\rho > 0, \sigma < 1$, and a phase function ω on $\Omega \times \Omega \times \mathbb{R}^d$ (i.e $R_\omega = \Omega \times \Omega$). Then:*

1. *The oscillatory integral (2.2.4) exists and is a continuous bilinear form for the C_0^k topologies on u,v if*

   $$m - k\rho < -N,\ m - k(1-\sigma) < -N.$$

   Thus we obtain a continuous linear map A from $C_0^k(\Omega)$ to $\mathscr{D}'^k(\Omega)$ which has a distribution kernel $K_\omega \in \mathscr{D}'^k(\Omega \times \Omega)$ given by the oscillatory integral

   $$K_\omega(u) = \int e^{i\omega(x,y,\xi)} a(x,y,\xi)u(x,y)dxdyd\xi,\ u \in C_0^\infty(\Omega \times \Omega).$$

2. *If ω has no critical point (y,ξ) for each fixed x, then (2.2.3) is defined as an oscillatory integral and we obtain a continuous map $A : C_0^k(\Omega) \to C(\Omega)$. By differentiation under the integral sign it follows that A is also continuous map from $C_0^k(\Omega)$ to $C^j(\Omega)$ if*

   $$m - k\rho < -N - j,\ m - k(1-\sigma) < -N - j.$$

3. *If ω has no critical point (x,ξ) for each fixed y, then the adjoint of A is defined and has the properties listed in point 2, so A is a continuous map of $\mathscr{E}'^j(\Omega)$ into $\mathscr{D}'^k(\Omega)$. In particular A defines a continuous map from $\mathscr{E}'(\Omega)$ into $\mathscr{D}'(\Omega)$.*

4. *The oscillatory integral:*

   $$K_\omega(x,y) = \int e^{i\omega(x,y,\xi)} a(x,\xi)d\xi \text{ defines a } C^\infty(\Omega \times \Omega = R_\omega) \text{ map},$$

   it follows that A is an integral operator with C^∞ kernel, so A is a continuous map of $\mathscr{E}'(\Omega)$ to $C^\infty(\Omega)$.

5. *We have the generalization of the pseudo-local property:*

   $$\text{sing supp } op_\omega(a)u = C_\omega \,\text{sing supp}\, u.$$

When ω is an operator phase function it verifies all the previous properties.

Proposition 2.2.6. *Let $\omega(x, y, \xi)$ be a $C^\infty(\Omega \times \Omega \times \mathbb{R}^d)$ map which is positively homogeneous of degree one with respect to ξ and a be a symbol in $S^m_{\rho,\sigma}(\Omega\times\Omega\times\mathbb{R}^d)$, $\rho > \sigma$ and that either ω is linear or that $\rho + \sigma = 1$. Suppose that a vanishes of infinite order on C then we have the same results as in the previous Proposition with m replaced by $m - \rho + \sigma$.*
If a just vanishes on C then we can find $b \in S^{m-\delta+\rho}_{\rho,\sigma}(\Omega \times \Omega \times \mathbb{R}^d)$ such that we have the formal equality $op_\omega(a)u = op_\omega(b)u$.

As Hörmander summed up, when ω is non degenerate the singularities of the distribution $u \to op_\omega(a)u$ only depend on the Taylor expansion of a on the set C.

The following Proposition, taken from part 2 of [35], gives the natural link between pseudodifferential operators and Fourier Integral operators defined by the phase function $\omega(x, y, \xi) = (x - y).\xi$.

Proposition 2.2.7. *Consider a real number m and a symbol $c \in S^m(\Omega \times \Omega \times \mathbb{R}^d)$, then:*

$$a(x, \xi) = \int_{\Omega\times\mathbb{R}^d} c(x, y, \eta)e^{i(x-y).(\eta-\xi)}dyd\eta \in S^m(\Omega \times \mathbb{R}^d)$$

and we have:

$$\forall u \in C_0^\infty(\Omega), op_{(x-y).\xi}(c)u = \mathrm{Op}(a)u = (2\pi)^{-d}\int_{\mathbb{R}^d} e^{ix.\xi}a(x, \xi)\hat{u}(\xi)d\xi.$$

Moreover the asymptotic expansion of a is given by:

$$\forall N \in \mathbb{N}, a(x, \xi) - \sum_{|\alpha|<N} \frac{1}{i^{|\alpha|}\alpha!}\partial^\alpha_\xi\partial^\alpha_y c(x, y, \xi)_{|y=x} \in S^{m-N}(\Omega \times \mathbb{R}^d).$$

Remark 2.2.3. *In the previous setting c is often called an amplitude.*

We will not give the proof of these Propositions here but we will present the fundamental Lemma behind those results and the idea behind it. The main problem is to define oscillatory integrals of the form:

$$\int e^{i\omega(x,\xi)}a(x, \xi)u(x)dxd\xi,\ u \in C_0^\infty(\Omega),$$

We start by remarking that the integral is absolutely convergent if a is of order $m < -N$.

Lemma 2.2.1. *If ω has no critical point (x, ξ) with $\xi \neq 0$, then one can find a first order differential operator*

$$L = \sum_j h_j\frac{\partial}{\xi_j} + \tilde{h}_j\frac{\partial}{x_j} + c$$

with $h_j \in S^0(\Omega \times \mathbb{R}^d)$ and $\tilde{h}_j, c \in S^{-1}(\Omega \times \mathbb{R}^d)$ such that $L^te^{i\omega} = e^{i\omega}$.
L is a continuous map from $S^m_{\rho,\sigma}(\Omega \times \Omega \times \mathbb{R}^d)$ to $S^{m-\epsilon}_{\rho,\sigma}(\Omega \times \Omega \times \mathbb{R}^d)$ where $\epsilon = min(\rho, 1 - \sigma)$.

Taking a symbol a of order m we compute:

$$\int e^{i\omega(x,\xi)} a(x,\xi)u(x)dxd\xi = \int e^{i\omega(x,\xi)} La(x,\xi)u(x)dxd\xi$$
$$= \int e^{i\omega(x,\xi)} L^k a(x,\xi)u(x)dxd\xi,$$

under the hypothesis $\rho > 0$ and $\sigma < 1$ we have $\epsilon > 0$ and $L^k a \in S^{m-k\epsilon}_{\rho,\sigma}(\Omega \times \Omega \times \mathbb{R}^d)$, taking $m - k\epsilon < -N$ and applying the previous remark we see that the integral is then well defined.

2.2.3 Paradifferential Calculus

We start by the definition of symbols with limited spatial regularity. Let $\mathscr{W} \subset \mathscr{S}'$ be a Banach space.

Definition 2.2.7. *Given $\rho \geq 0$ and $m \in \mathbb{R}$, $\Gamma^m_{\mathscr{W}}(\mathbb{D}^d)$ denotes the space of locally bounded functions $a(x,\xi)$ on $\mathbb{D}^d \times (\hat{\mathbb{D}}^d \setminus 0)$, which are C^∞ with respect to ξ for $\xi \neq 0$ and such that, for all $\alpha \in \mathbb{N}^d$ and for all $\xi \neq 0$, the function $x \mapsto \partial^\alpha_\xi a(x,\xi)$ belongs to $\mathscr{W}$ and there exists a constant C_α such that for all $\epsilon > 0$:*

$$\forall |\xi| > \epsilon, \left\| \partial^\alpha_\xi a(.,\xi) \right\|_{\mathscr{W}} \leq C_{\alpha,\epsilon}(1+|\xi|)^{m-|\alpha|}. \tag{2.2.5}$$

The spaces $\Gamma^m_{\mathscr{W}}(\mathbb{D}^d)$ are equipped with their natural Fréchet topology induced by the semi-norms defined by the best constants in (2.2.5) .

We will essentially work with $\mathscr{W} = W^{\rho,\infty}$ and write $\Gamma^m_{\mathscr{W}} = \Gamma^m_\rho$.

For quantitative estimates we introduce as in [53]:

Definition 2.2.8. *For $m \in \mathbb{R}$ and $a \in \Gamma^m_{\mathscr{W}}(\mathbb{D})$, we set*

$$M^m_{\mathscr{W}}(a;n) = \sup_{|\alpha|\leq n} \sup_{|\xi| \geq \frac{1}{2}} \left\| (1+|\xi|)^{m-|\alpha|} \partial^\alpha_\xi a(.,\xi) \right\|_{\mathscr{W}}, \text{ for } n \in \mathbb{N}.$$

For $\mathscr{W} = W^{\rho,\infty}, \rho \geq 0$, we write:

$$\Gamma^m_{W^{\rho,\infty}}(\mathbb{D}) = \Gamma^m_\rho(\mathbb{D}) \text{ and } M^m_\rho(a) = M^m_{W^{\rho,\infty}}(a; 1 + \lfloor \frac{d}{2} \rfloor).$$

Moreover we introduce the following spaces equipped with their natural Fréchet space structure:

$$C^\infty_b(\mathbb{D}) = \cap_{\rho\geq 0} W^{\rho,\infty},\ \Gamma^m_\infty(\mathbb{D}) = \cap_{\rho \geq 0}\Gamma^m_\rho(\mathbb{D}),\ \Gamma^{-\infty}_\rho(\mathbb{D}) = \cap_{m\in\mathbb{R}}\Gamma^m_\rho(\mathbb{D}) \text{ and,}$$
$$\Gamma^{-\infty}_\infty(\mathbb{D}) = \cap_{\rho\geq 0} \cap_{m\in\mathbb{R}} \Gamma^m_\rho(\mathbb{D}).$$

As presented in [36, 53] the idea to define paradifferential operators is to regularize the symbols by a cutoff ψ, for a paradifferential symbol $a \in \Gamma^{m'}_\rho(\mathbb{D}^d)$ we will then associate a symbol $\sigma^\psi_a \in S^m_{1,1}(\mathbb{D}^d \times \hat{\mathbb{D}}^d)$. All of the results presented above were for the class $S^m_{1,0} \subset S^m_{1,1}$ and don't generalize to $S^m_{1,1}$, even the L^2 continuity. Looking more closely to $a \in S^m_{1,1}$ in [36], Hörmander shows that the essential problems that occur are localized in the frequency regions $(\eta, 0)$ and $(-\eta,\eta)$ of $\mathscr{F}_x(a)$. Thus the idea in paradifferential calculus is regularization by a cutoff in the frequency domain with support bounded away from $(\eta,0)$ and $(-\eta,\eta)$ at infinity. Then σ^ψ_a will have this extra spectral localization property that will give them the desired properties as in $S^m_{1,0}$.

Definition-Proposition 2.2.1. *Take $m \in \mathbb{R}$, $\Sigma^m_{\mathscr{W}}(\mathbb{D}^d)$ denotes the subclass of symbols $\sigma \in \Gamma^m_{\mathscr{W}}(\mathbb{D}^d)$ which satisfy the following spectral condition:*

$$\exists B > 1, b > 0,\ \mathscr{F}_x\sigma(\eta, \xi) = 0 \text{ for } |\xi| < B|\eta| + b. \tag{2.2.6}$$

${}^{lim}\Sigma^m_{\mathscr{W}}(\mathbb{D}^d)$ denotes the subclass of symbols $\sigma \in \Gamma^m_{\mathscr{W}}(\mathbb{D}^d)$ which satisfy the following spectral condition:

$$\exists b > 0,\ \mathscr{F}_x\sigma(\eta, \xi) = 0 \text{ for } |\xi| < |\eta| + b. \tag{2.2.7}$$

When $\mathscr{W} = W^{r,\infty}(\mathbb{D}^d)$ we write $\Sigma^m_{\mathscr{W}}(\mathbb{D}^d) = \Sigma^m_r(\mathbb{D}^d)$ and ${}^{lim}\Sigma^m_{\mathscr{W}}(\mathbb{D}^d) = {}^{lim}\Sigma^m_r(\mathbb{D}^d)$.

$\mathscr{W} \subset L^\infty(\mathbb{D}^d) \Rightarrow \Gamma^m_{\mathscr{W}}(\mathbb{D}^d) \subset \Gamma^m_0(\mathbb{D}^d),\ \Sigma^m_{\mathscr{W}}(\mathbb{D}^d) \subset \Sigma^m_0(\mathbb{D}^d),\ {}^{lim}\Sigma^m_{\mathscr{W}}(\mathbb{D}^d) \subset {}^{lim}\Sigma^m_0(\mathbb{D}^d)$.

Moreover, by the Bernstein inequalities (2.1.1):

$$\Sigma^m_0(\mathbb{D}^d), {}^{lim}\Sigma^m_0(\mathbb{D}^d) \subset S^m_{1,1}(\mathbb{D}^d).$$

More generally, the spectral condition implies that symbols in $\Sigma^m_{\mathscr{W}}(\mathbb{D}^d)$, ${}^{lim}\Sigma^m_0(\mathbb{D}^d)$ are smooth in x too.

Remark 2.2.4. *The interesting fact now is $\Sigma^m_0(\mathbb{D}^d)$ is shown to still enjoy all of the symbolic calculus and continuity properties announced above for $S^m_{1,0}(\mathbb{D}^d)$. For ${}^{lim}\Sigma^m_0(\mathbb{D}^d)$ it will still enjoy the symbolic calculus but it's continuity properties will be restricted to $H^s, s > 0$ and $C^\rho_*, \rho > 0$.*

Definition-Proposition 2.2.2. *Consider four strictly positive real numbers $b, (B_i)_{1\le i\le 3}$ verifying:*

$$B_1B_3 > 1 \text{ and } B_3B_2 > B_2 + B_2. \tag{2.2.8}$$

Consider ψ a C^∞ function such that:

1. $$\psi(\eta, \xi) = 0 \text{ when } |\eta| > B_1|\xi| + b \text{ or } |\xi| > B_2|\eta + \xi| + b,$$
$$\text{and } \psi(\eta, \xi) = 1 \text{ when } |\xi| > B_3|\eta| + b,$$

2. *for all $(\alpha, \beta) \in \mathbb{N}^d \times \mathbb{N}^d$, there is C_{α_β}, with $C_{0,0} \le 1$, such that:*

$$\forall (\xi, \eta) : \left|\partial^\alpha_\xi \partial^\beta_\eta \psi(\xi, \eta)\right| \le C_{\alpha,\beta}(1 + |\xi|)^{-|\alpha|-|\beta|}. \tag{2.2.9}$$

Such a ψ is called an admissible cut-off function for any positive $b, (B_i)_{i\in\{1,2,3\}}$ verifying (2.2.8) .

The cutoffs defined in the introduction (with the extra gross hypothesis (2.2.9)), $(\psi^B_H)_{B>2}$ by (1.4.5), (ψ^ϵ_M) by (1.4.6) and $(\psi^{B,b})_{B>1,b>0}$ by (1.4.7) are all admissible cutoff functions. $(\psi^{1,b})_{b>0}$ will be called limit cutoff functions.

Figure (2.2.3) illustrate the condition of admissible cutoff functions in the plane (ξ, η) when $d = 1$.

Definition-Proposition 2.2.3. *(Regularization of a symbol) Take $m \in \mathbb{R}$, $a \in \Gamma^m_{\mathscr{W}}$, ψ an admissible cut-off function and a limit cutoff function $\psi^{1,b}$. Define σ_a^{ψ} and $\sigma_a^{\psi^{1,b}}$:*

$$\mathscr{F}_x \sigma_a^{\psi}(\xi,\eta) = \psi(\xi,\eta)\mathscr{F}_x a(\xi,\eta) \text{ and } \mathscr{F}_x \sigma_a^{\psi^{1,b}}(\xi,\eta) = \psi^{1,b}(\xi,\eta)\mathscr{F}_x a(\xi,\eta).$$

Then $\sigma_a^{\psi} \in \Sigma^m_{\mathscr{W}}(\mathbb{D}^d)$ and $\sigma_a^{\psi^{1,b}} \in {}^{lim}\Sigma^m_{\mathscr{W}}(\mathbb{D}^d)$.

When $\mathscr{W} = W^{r,\infty}(\Omega)$ we have the following properties:

1. *This association is bounded:*

$$M^m_r(\sigma_a^{\psi}) \leq CM^m_r(a) \text{ and } M^m_r(\sigma_a^{\psi^{1,b}}) \leq CM^m_r(a).$$

2. *We have $a - \sigma_a^{\psi} \in \Gamma^m_0$ and $a - \sigma_a^{\psi^{1,b}} \in \Gamma^m_0$, moreover:*

$$M^{m-r}_0(\sigma_a^{\psi} - a) \leq CM^m_r(a) \text{ and } M^{m-r}_0(\sigma_a^{\psi^{1,b}} - a) \leq CM^m_r(a).$$

In particular, if ψ_1 and ψ_2 are two admissible cut-off functions then the difference $\sigma_a^{\psi_1} - \sigma_a^{\psi_2}$ belongs to Σ^{m-r}_0 and:

$$M^{m-r}_0(\sigma_a^{\psi_1} - \sigma_a^{\psi_2}) \leq CM^m_r(a).$$

Respectively we have $\sigma_a^{\psi^{1,b_1}} - \sigma_a^{\psi^{1,b_2}}$ belongs to ${}^{lim}\Sigma^{m-r}_0$ and:

$$M^{m-r}_0(\sigma_a^{\psi^{1,b_1}} - \sigma_a^{\psi^{1,b_2}}) \leq CM^m_r(a).$$

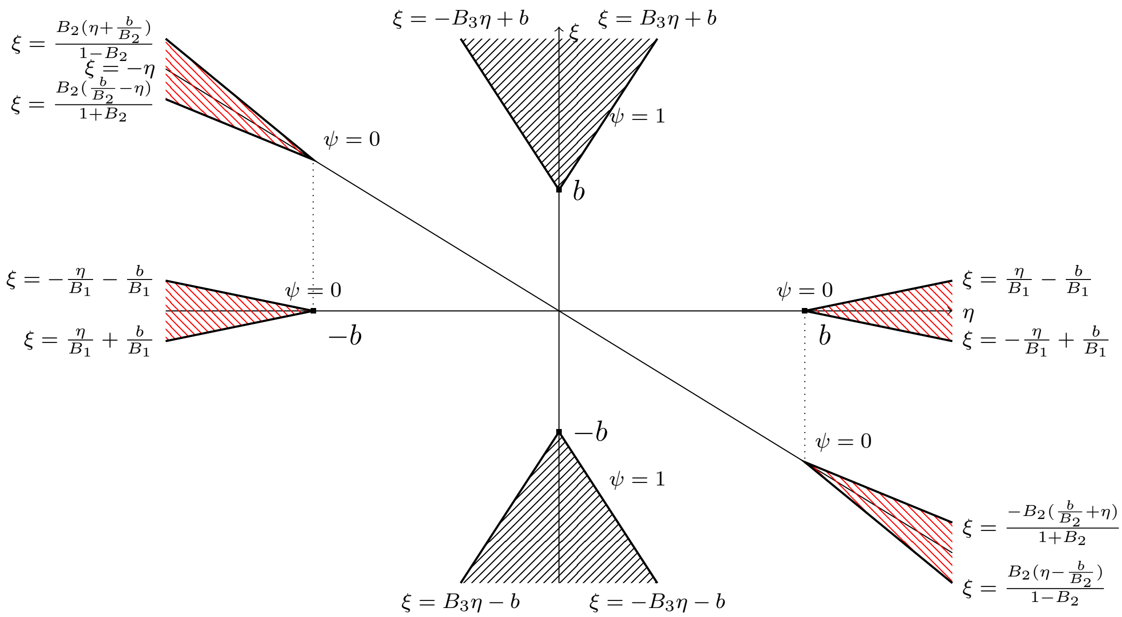

Figure 2.1: Admissible cut-off functions.

Now we list a couple of important calculus properties to the association $a \mapsto \sigma_a^{\psi}$ and $a \mapsto \sigma_a^{\psi^{1,b}}$. For the following Proposition we fix a choice of an admissible cutoff function ψ and $\psi^{1,b}$.

Proposition 2.2.8. • *For $m \in \mathbb{R}, r \geq 0, \alpha \in \mathbb{N}^d$ of length $|\alpha| \leq r$ and $a \in \Gamma_r^m$:*

$$\partial_x^\alpha \sigma_a^\psi = \sigma_{\partial_x^\alpha a}^\psi \in \Sigma_0^m, \quad \partial_x^\alpha \sigma_a^{\psi^{1,b}} = \sigma_{\partial_x^\alpha a}^{\psi^{1,b}} \in {}^{lim}\Sigma_0^m.$$

- *For $m \in \mathbb{R}, r \geq 0$ and $\alpha \in \mathbb{N}^d$ of length $|\alpha| \geq r$ the mappings $a \mapsto \partial_x^\alpha \sigma_a^\psi$ and $a \mapsto \partial_x^\alpha \sigma_a^{\psi^{1,b}}$ are bounded from Γ_r^m to $\Sigma_0^{m+|\alpha|-r}$ and ${}^{lim}\Sigma_0^{m+|\alpha|-r}$ respectively, more precisely:*

$$M_0^{m+|\alpha|-r}(\partial_x^\alpha \sigma_a^\psi, \partial_x^\alpha \sigma_a^{\psi^{1,b}}) \leq M_r^m(a).$$

- *For $m \in \mathbb{R}, r \geq 0, \beta \in \mathbb{N}^d$ and $a \in \Gamma_r^m$*

$$\partial_\xi^\beta \sigma_a^\psi - \sigma_{\partial_\xi^\beta a}^\psi \in \Sigma_0^{m-|\beta|-r}, \quad \partial_\xi^\beta \sigma_a^{\psi^{1,b}} - \sigma_{\partial_\xi^\beta a}^{\psi^{1,b}} \in {}^{lim}\Sigma_0^{m-|\beta|-r}$$

From [53] we give an approximation of symbols in $\Sigma_0^m(\mathbb{D}^d)$ and ${}^{\lim}\Sigma_0^m(\mathbb{D}^d)$ by symbols in the Schwartz class.

Lemma 2.2.2. *For all $\sigma \in \Sigma_0^m$($\sigma \in {}^{\lim}\Sigma_0^m$ respectively), there is a sequence of symbols $\sigma_n \in \mathscr{S}(\mathbb{D}^d \times \hat{\mathbb{D}}^d)$ such that*

1. *the family $\{\sigma_n\}$ is bounded in $S_{1,1}^m$,*
2. *the σ_n satisfy the spectral condition (2.2.6) ((2.2.7) respectively) for some $B > 1, b > 0$ independent of n,*
3. *$\sigma_n \to \sigma$ on compact subsets of $\mathbb{D}^d \times \hat{\mathbb{D}}^d$.*

The main difference between Σ_0^m and ${}^{\lim}\Sigma_0^m$ is captured in their actions on the spectrum of functions. First we give a general result for symbols in $S_{1,1}^m$ from [53].

Proposition 2.2.9. *Consider a real number m, $p \in S_{1,1}^m(\mathbb{D}^d \times \hat{\mathbb{D}}^d)$ and $u \in \mathscr{S}(\mathbb{D}^d)$ then the spectrum of $\operatorname{Op}(p)u$ is contained in the closure of the set:*

$$\{\xi + \eta, \xi \in \operatorname{supp} \mathscr{F}u, (\eta, \xi) \in \operatorname{supp} \mathscr{F}_x p\}.$$

This implies the following property for operators verifying the spectral conditions (2.2.6) and (2.2.7) .

Lemma 2.2.3. *Consider a real number m, $p \in \Sigma_0^m(\mathbb{D}^d), q \in {}^{\lim}\Sigma_0^m(\mathbb{D}^d)$ with parameter $B > 1, b > 0$ and $u \in \mathscr{S}(\mathbb{D}^d)$.*

- *For $R >> b$, if $\operatorname{supp} \mathscr{F}u \subset \{|\xi| \leq R\}$, then:*

$$\operatorname{supp} \mathscr{F} \operatorname{Op}(p)u \subset \left\{ |\xi| \leq (1 + \frac{1}{B})R - \frac{b}{B} \right\}, \tag{2.2.10}$$

$$\text{and } \operatorname{supp} \mathscr{F} \operatorname{Op}(q)u \subset \left\{ |\xi| \leq 2R - \frac{b}{B} \right\}. \tag{2.2.11}$$

- *For $R >> b$, if $\operatorname{supp} \mathscr{F}u \subset \{|\xi| \geq R\}$, then:*

$$\operatorname{supp} \mathscr{F} \operatorname{Op}(p)u \subset \left\{ |\xi| \geq (1 - \frac{1}{B})R + \frac{b}{B} \right\}, \tag{2.2.12}$$

$$\text{and } \operatorname{supp} \mathscr{F} \operatorname{Op}(q)u \subset \left\{ |\xi| \geq \frac{b}{B} \right\}. \tag{2.2.13}$$

Remark 2.2.5. *It is exactly the lack of lower control* (2.2.13) *that reduces the continuity properties of* ${}^{\lim}\Sigma_0^m(\mathbb{D}^d)$ *to* $H^s, s > 0$.

The key new result on the control of spectrum of composition of paradifferential is illustrated in the following.

Proposition 2.2.10. *Fix a real number* m.

1. *If* $p, q \in \Sigma_0^m(\mathbb{R}^d) \cap \mathscr{S}(\mathbb{D}^d \times \hat{\mathbb{D}}^d)$ *with parameters* $B > 1, b > 0$ *then*

$$p \otimes q(x,\xi) = (2\pi)^{-d} \int_{\mathbb{D}^d \times \hat{\mathbb{D}}^d} e^{i(x-y).(\xi-\eta)} p(x,\eta) q(y,\xi) dy d\eta,$$

verifies the spectral condition(2.2.19) *with parameters* $\frac{B^2}{2B-1} > 1, b > 0$.

2. *If* $p, q \in {}^{lim}\Sigma_0^m(\mathbb{R}^d) \cap \mathscr{S}(\mathbb{D}^d \times \hat{\mathbb{D}}^d)$ *with parameter* $b > 0$ *then*

$$p \otimes q(x,\xi) = (2\pi)^{-d} \int_{\mathbb{D}^d \times \hat{\mathbb{D}}^d} e^{i(x-y).(\xi-\eta)} p(x,\eta) q(y,\xi) dy d\eta,$$

verifies the spectral condition(2.2.7) *with parameter* $b > 0$.

Proof. We compute:

$$\begin{aligned}
&\mathscr{F}_x(p \otimes q)(\eta,\xi) = (2\pi)^{-d} \int e^{i(x-y).(\xi-\eta_1)} e^{-ix.\eta} p(x,\eta_1) q(y,\xi) dy d\eta_1 dx \\
&= (2\pi)^{-3d} \int e^{i(x-y).(\xi-\eta_1)} e^{-ix.\eta} e^{ix.\eta_2} e^{iy.\eta_3} \mathscr{F}_x(p)(\eta_2,\eta_1) \mathscr{F}_x(q)(\eta_3,\xi) dx dy d\eta_1 d\eta_2 d\eta_3 \\
&= (2\pi)^{-d} \int_{\substack{B|\tilde{\eta}|+b \le |\xi| \\ B|\eta-\tilde{\eta}|+b \le |\xi-\tilde{\eta}|}} \mathscr{F}_x(p)(\eta-\tilde{\eta},\xi-\tilde{\eta}) \mathscr{F}_x(q)(\tilde{\eta},\xi) d\tilde{\eta},
\end{aligned}$$

where we used $\mathscr{F}_x(e^{ix.\xi})(\eta) = (2\pi)^d \delta_0(\eta - \xi)$.

The goal is to investigate if one can find η and $\tilde{\eta}$ such that:

$$B|\eta| + b > \xi \tag{2.2.14}$$
$$B|\tilde{\eta}| + b \le |\xi| \tag{2.2.15}$$
$$B|\eta - \tilde{\eta}| + b \le |\xi - \tilde{\eta}| \tag{2.2.16}$$

and in that case find an upper bound on $|\eta|$. In order to have the largest $|\eta|$ possible by (2.2.16) and the triangle inequality, $\tilde{\eta}$ needs to be on the straight line (η,ξ).

Thus the problem is reduced to $d = 1$. The strategy is to investigate what happens in the different regions of the plane (η,ξ).

First for $\eta, \xi \ge 0$, we have:

$$B|\tilde{\eta}| + b \le |\xi| < B|\eta| + b \Rightarrow \begin{cases} |\tilde{\eta}| < \xi \\ |\tilde{\eta}| < \eta \end{cases} \Rightarrow \begin{cases} |\xi - \tilde{\eta}| = \xi - \tilde{\eta} \\ |\eta - \tilde{\eta}| = \eta - \tilde{\eta} \end{cases}.$$

Thus by (2.2.16):

$$B\eta - B\tilde{\eta} + b \le \xi - \tilde{\eta} \Rightarrow B\eta - \xi + b \le (B-1)\tilde{\eta},$$

if $B=1$ we get

$$\eta + b \leq \xi,$$

which is the desired result. If not, $B>1$, we have:

$$\frac{B}{B-1}\eta - \frac{\xi}{B-1} + \frac{b}{B-1} \leq \tilde{\eta},$$

thus by (2.2.15):

$$\frac{B^2}{B-1}\eta - \frac{B}{B-1}\xi + b\frac{B}{B-1} \leq \xi - b \Rightarrow \frac{B^2}{B-1}\eta + b\frac{2B-1}{B-1} \leq \frac{2B-1}{B-1}\xi,$$

which give the desired upper bound:

$$\frac{B^2}{2B-1}\eta + b \leq \xi.$$

For $\eta \leq 0, \xi \geq 0$ we see that:

$$\begin{aligned}
\mathscr{F}_x(p\otimes q)(\eta,\xi) &= \overline{\mathscr{F}_x(\mathrm{Re}(p\otimes q))(-\eta,\xi)} + i\overline{\mathscr{F}_x(\mathrm{Im}(p\otimes q))(-\eta,\xi)} \\
&= \overline{\mathscr{F}_x(\mathrm{Re}(p)\otimes \mathrm{Re}(q))(-\eta,\xi)} - \overline{\mathscr{F}_x(\mathrm{Im}(p)\otimes \mathrm{Im}(q))(-\eta,\xi)} \\
&+ i\overline{\mathscr{F}_x(\mathrm{Im}(p)\otimes \mathrm{Re}(q))(-\eta,\xi)} + i\overline{\mathscr{F}_x(\mathrm{Re}(p)\otimes \mathrm{Im}(q))(-\eta,\xi)},
\end{aligned}$$

where,

$$p(x,\xi) = \mathrm{Re}(p)(x,\xi) + i\,\mathrm{Im}(p)(x,\xi),\ \ q(x,\xi) = \mathrm{Re}(q)(x,\xi) + i\,\mathrm{Im}(q)(x,\xi).$$

By linearity $\mathrm{Re}(p), \mathrm{Im}(p), \mathrm{Re}(q)$ and $\mathrm{Im}(q)$ verify the spectral condition with parameters B, b, thus we deduce from the previous case:

$$-\frac{B^2}{2B-1}\eta + b \leq \xi \iff \frac{B^2}{2B-1}|\eta| + b \leq \xi.$$

For $\eta, \xi \leq 0$ we have:

$$B|\tilde{\eta}| + b \leq |\xi| < B|\eta| + b \Rightarrow \begin{cases} |\tilde{\eta}| < -\xi \\ |\tilde{\eta}| < -\eta \end{cases} \Rightarrow \begin{cases} |\xi - \tilde{\eta}| = \tilde{\eta} - \xi \\ |\eta - \tilde{\eta}| = \tilde{\eta} - \eta \end{cases}.$$

Thus by (2.2.16):

$$B\tilde{\eta} - B\eta + b \leq \tilde{\eta} - \xi \Rightarrow -B\eta + \xi + b \leq -(B-1)\tilde{\eta},$$

if $B=1$ we get:

$$-\eta + b \leq -\xi,$$

which is the desired result. If not, $B>1$, we have:

$$-\frac{B}{B-1}\eta + \frac{\xi}{B-1} + \frac{b}{B-1} \leq -\tilde{\eta},$$

thus by (2.2.15):

$$-\frac{B^2}{B-1}\eta + \frac{B}{B-1}\xi + b\frac{B}{B-1} \leq -\xi - b \Rightarrow -\frac{B^2}{B-1}\eta + b\frac{2B-1}{B-1} \leq -\frac{2B-1}{B-1}\xi,$$

which give the desired upper bound:

$$-\frac{B^2}{2B-1}\eta + b \leq -\xi.$$

For $\eta \geq 0, \xi \leq 0$ the result is again deduced from the identity:

$$\mathscr{F}_x(p \otimes q)(\eta, \xi) = \overline{\mathscr{F}_x(\mathrm{Re}(p \otimes q))(-\eta, \xi)} + i\overline{\mathscr{F}_x(\mathrm{Im}(p \otimes q))(-\eta, \xi)}.$$

□

Remark 2.2.6. *The proof verbatim generalizes to the case where we would have taken p, q verifying the spectral condition (2.2.7) with two different parameters B, B' and b, we would have found that $p \otimes q$ verifies the spectral condition with parameter $\frac{BB'}{B+B'-1}$ and b.*

Corollary 2.2.1. *Take $m, m' \in \mathbb{R}$, and $\rho > 0$, $a \in \Gamma^m_\rho(\mathbb{D}^d)$ and $b \in \Gamma^{m'}_\rho(\mathbb{D}^d)$. Consider an admissible cut-off function $\psi^{B,b}$ with $B > 1, b >$ defined by (1.4.7) and a limit cut-off function $\psi^{1,b}$. Then we have:*

$$\mathrm{Op}\left(\sigma_a^{\psi^{B,b}}\right) \circ \mathrm{Op}\left(\sigma_b^{\psi^{B,b}}\right) = \mathrm{Op}\left(\sigma^{\psi^{\frac{B^2}{2B-1},b}}_{\sigma_a^{\psi^{B,b}} \otimes \sigma_b^{\psi^{B,b}}}\right),$$

$$\mathrm{Op}\left(\sigma_a^{\psi^{1,b}}\right) \circ \mathrm{Op}\left(\sigma_b^{\psi^{1,b}}\right) = \mathrm{Op}\left(\sigma^{\psi^{1,b}}_{\sigma_a^{\psi^{1,b}} \otimes \sigma_b^{\psi^{1,b}}}\right).$$

Definition 2.2.9. *Consider a real numbers $m \in \mathbb{R}$, a symbol $a \in \Gamma^m_{\mathscr{W}}$ and an admissible cutoff function ψ define the paradifferential operator T_a by:*

$$\widehat{T_a u}(\xi) = (2\pi)^{-d} \int_{\hat{\mathbb{D}}^d} \psi(\xi - \eta, \eta)\hat{a}(\xi - \eta, \eta)\hat{u}(\eta) d\eta,$$

where $\hat{a}(\eta, \xi) = \int e^{-ix.\eta} a(x, \xi) dx$ is the Fourier transform of a with respect to the first variable. For a limit cut-off $\psi^{1,b}$ we define:

$$\widehat{T_a^{lim} u}(\xi) = (2\pi)^{-d} \int_{\hat{\mathbb{D}}^d} \psi^{1,b}(\xi - \eta, \eta)\hat{a}(\xi - \eta, \eta)\hat{u}(\eta) d\eta.$$

The connection between two different choices of cut-offs is the following:

$$\forall a \in \Gamma^m_\rho, (B, B', b, b') \in [1, +\infty[^2 \times]0, +\infty[^2,\ \sigma_a^{\psi^{B,b}} - \sigma_a^{\psi^{B',b'}} \in \Gamma^{m-\rho}_0. \tag{2.2.17}$$

The first main features of paradifferential operators is their continuity given by the following Theorems.

Theorem 2.2.3. *Take $m \in \mathbb{R}$. If $a \in \Gamma^m_0(\mathbb{D}^d)$, then T_a is of order m. Moreover, for all $\mu \in \mathbb{R}$ there exists a constant K such that:*

$$\|T_a\|_{H^\mu \to H^{\mu-m}} \leq K M_0^m(a), \text{ and,}$$

$$\|T_a\|_{W^{\mu,\infty} \to W^{\mu-m,\infty}} \leq K M_0^m(a), \mu \notin \mathbb{N}.$$

By Theorem 4.3.1 of [53] and Bernstein inequalities we get.

Theorem 2.2.4. *Take $m \in \mathbb{R}$ and $a \in \Gamma_0^m(\mathbb{D}^d)$, then for all $\mu > 0$ there exists a constant K such that:*

$$\left\| T_a^{lim} \right\|_{H^\mu \to H^{\mu-m}} \leq K M_0^m(a).$$

$$\left\| T_a^{lim} \right\|_{W^{\mu,\infty} \to W^{\mu-m,\infty}} \leq K M_0^m(a), \mu \notin \mathbb{N}.$$

The symbolic calculus for paradifferential operators is their continuity given by the following Theorem from [53].

Theorem 2.2.5. *Take $m, m' \in \mathbb{R}$, and $\rho > 0$, $a \in \Gamma_\rho^m(\mathbb{D}^d)$ and $b \in \Gamma_\rho^{m'}(\mathbb{D}^d)$.*

- *Composition: Then $T_a T_b$ is a paradifferential operator of order $m + m'$ and $T_a T_b - T_{a\#b}$ is of order $m + m' - \rho$ where $a\#b$ is defined by:*

$$a\#b = \sum_{|\alpha|<\rho} \frac{1}{i^{|\alpha|}\alpha!} \partial_\xi^\alpha a \partial_x^\alpha b$$

Moreover, for all $\mu \in \mathbb{R}$ there exists a constant K such that

$$\|T_a T_b - T_{a\#b}\|_{H^\mu \to H^{\mu-m-m'+\rho}} \leq K(M_\rho^m(a) M_0^{m'}(b) + M_\rho^m(a) M_0^{m'}(b)).$$

- *Adjoint: The adjoint operator of T_a, that we will note T_a^t to again avoid confusion with the pull back operator defined in this work, is a paradifferential operator of order m with symbol a^t defined by:*

$$a^t = \sum_{|\alpha|<\rho} \frac{1}{i^{|\alpha|}\alpha!} \partial_\xi^\alpha \partial_x^\alpha \bar{a} \tag{2.2.18}$$

Moreover, for all $\mu \in \mathbb{R}$ there exists a constant K such that:

$$\left\|T_a^t - T_{a^t}\right\|_{H^\mu \to H^{\mu-m+\rho}} \leq K M_\rho^m(a).$$

Combining Theorem 2.2.5 with Corollary 2.2.1 we get the following more precise Theorem on composition of paradifferential operators.

Theorem 2.2.6. *Take $m, m' \in \mathbb{R}$, and $\rho > 0$, $a \in \Gamma_\rho^m(\mathbb{D}^d)$ and $b \in \Gamma_\rho^{m'}(\mathbb{D}^d)$. Then there exists $r \in \Gamma_0^{m+m'-\rho}(\mathbb{D}^d)$ such that:*

$$M_0^{m+m'-\rho}(r) \leq K(M_\rho^m(a) M_0^{m'}(b) + M_\rho^m(a) M_0^{m'}(b)),$$

and we have

$$T_a^{\psi^{B,b}} T_b^{\psi^{B,b}} - T_{a\#b}^{\psi^{\frac{B^2}{2B-1},b}} = T_r^{\psi^{\frac{B^2}{2B-1},b}},$$

$$T_a^{lim} T_b^{lim} - T_{a\#b}^{lim} = T_r^{lim}.$$

If $a = a(x)$ is a function of x only, the paradifferential operator T_a is called a para-product. With a good choice of (B, b) in the definition of the cut-off function with respect to our choice of the dyadic decomposition of unity in the Littlewood-Paley decomposition we get that when $a = a(x)$, T_a takes the usual form:

$$T_a u = \sum_{k=1}^{\infty} \Phi_{k-1} a u_k.$$

It follows from Theorem 2.2.5 and the Sobolev embeddings that:

- If $a \in H^{\alpha}(\mathbb{D}^d)$ and $b \in H^{\beta}(\mathbb{D}^d)$ with $\alpha, \beta > \frac{d}{2}$, then

$$T_a T_b - T_{ab} \text{ is of order } -\left(min\{\alpha,\beta\} - \frac{d}{2}\right).$$

- If $a \in H^{\alpha}(\mathbb{D}^d)$ with $\alpha > \frac{d}{2}$, then

$$T_a^t - T_{a^t} \text{ is of order } -\left(\alpha - \frac{d}{2}\right).$$

An important feature of para-products is that they are well defined for function $a = a(x)$ which are not L^{∞} but merely in some Sobolev spaces H^r with $r < \frac{d}{2}$.

Proposition 2.2.11. *Take $m > 0$. If $a \in H^{\frac{d}{2}-m}(\mathbb{D}^d)$ and $u \in H^{\mu}(\mathbb{D}^d)$ then,*

$$T_a u \in H^{\mu-m}(\mathbb{D}^d), \text{ and } \quad \|T_a u\|_{H^{\mu-m}} \leq K \|a\|_{H^{\frac{d}{2}-m}} \|u\|_{H^{\mu}}.$$

A main feature of para-products is the existence of para-linearization Theorems which allow us to replace nonlinear expressions by paradifferential expressions, at the price of error terms which are smoother than the main terms.

Theorem 2.2.7. *Let $\alpha, \beta \in \mathbb{R}$ be such that $\alpha, \beta > \frac{d}{2}$, then*

- *Bony's Linearization Theorem: for all C^{∞} function F, if $a \in H^{\alpha}(\mathbb{D}^d)$ then*

$$F(a) - F(0) - T_{F'(a)} a \in H^{2\alpha - \frac{d}{2}}(\mathbb{D}^d).$$

- *If $a \in H^{\alpha}(\mathbb{D}^d)$ and $b \in H^{\beta}(\mathbb{D}^d)$, then $ab - T_a b - T_b a \in H^{\alpha+\beta-\frac{d}{2}}(\mathbb{D}^d)$. Moreover there exists a positive constant K independent of a and b such that:*

$$\|ab - T_a b - T_b a\|_{H^{\alpha+\beta-\frac{d}{2}}} \leq K \|a\|_{H^{\alpha}} \|b\|_{H^{\beta}}.$$

2.2.3.1 Link between Fourier Integral Operators and paradifferential operators

In order to give the link between Paradifferential operators and Fourier Integral Operators we start by defining the space of amplitudes for Paradifferential operators.

Definition-Proposition 2.2.4. *Take $m \in \mathbb{R}$, $A^m_{\mathscr{W}}(\mathbb{R}^d)$ denotes the subclass of symbols $c \in \Gamma^m(\mathscr{W} \times \mathscr{W} \times \mathbb{R}^d)$ which satisfy the following spectral condition*

$$\exists B > 1, b > 0, \mathscr{F}_{x,y} c(\xi, \zeta, \eta) = 0 \text{ for } B|\xi - \zeta| + b > |\eta| \text{ or } B|\zeta| + b > |\eta|. \quad (2.2.19)$$

When $\mathscr{W} = W^{r,\infty}(\Omega)$ we write $A^m_{\mathscr{W}}(\mathbb{R}^d) = A^m_r(\mathbb{R}^d)$.
By the Bernstein inequalities (2.1.1), $A^m_0(\mathbb{R}^d) \subset S^m_{1,1}(\mathbb{R}^d)$. More generally, the spectral condition implies that symbols in $A^m_{\mathscr{W}}(\mathbb{R}^d)$ are smooth in x, y too.

Proposition 2.2.12. *Consider two real numbers $m \in \mathbb{R}$, $r \in \mathbb{R}_+$ and an amplitude $c \in A^m_r(\mathbb{R}^d)$, then:*

$$\sigma(x,\xi) = \int_{\Omega \times \mathbb{R}^d} c(x,y,\eta) e^{i(x-y).(\eta-\xi)} dy d\eta \in \Sigma^m_r(\mathbb{R}^d)$$

and we have:

$$\forall u \in C_0^\infty(\Omega), op_{(x-y).\xi}(c)u = \mathrm{Op}(\sigma)u = (2\pi)^{-d}\int_{\mathbb{R}^d} e^{ix.\xi}\sigma(x,\xi)\hat{u}(\xi)d\xi.$$

Moreover the asymptotic expansion of a is given by:

$$\sigma(x,\xi) - \sum_{|\alpha|<N} \frac{1}{i^{|\alpha|}\alpha!}\partial_\xi^\alpha\partial_y^\alpha c(x,y,\xi)_{|y=x} \in \Sigma_{r-N}^{m-N}(\mathbb{R}^d).$$

Proof. First by Lemma 2.2.2 we can work with an amplitude c in $\mathscr{S}$. As $\mathscr{S} \subset S^m_{1,0}$ by Proposition 2.2.7 we have

$$\sigma(x,\xi) = \int_{\mathbb{R}^d\times\mathbb{R}^d} c(x,y,\eta)e^{i(x-y).(\eta-\xi)}dyd\eta \in \mathscr{S}.$$

Moreover writing

$$\mathscr{F}_x\sigma(\eta,\xi) = \int_{\mathbb{R}^d} \mathscr{F}_{x,y}c(\xi+\eta-\tilde{\eta},\tilde{\eta}-\xi,\tilde{\eta})d\tilde{\eta},$$

we see that if c verifies the spectral condition with parameters B, b then so does σ with parameter $B-1, b$ thus $\sigma \in \Sigma_r^m(\mathbb{R}^d)$. The asymptotic expansion comes from the one given in Proposition 2.2.7 combined by the symbolic calculus rules in Proposition 2.2.8. □

2.3 Pull-back of pseudo and para- differential operators

Let Ω, Ω' be two open subsets of $\mathbb{R}^d$. Henceforth we will note all variables in Ω' with a $'$ for clarity in the computations. Let $\chi : \Omega \to \Omega'$ be a C^∞ map, χ gives rise naturally to the pull back operation for functions and kernels:

$$\begin{array}{ll} C^\infty(\Omega') \to C^\infty(\Omega) & C^\infty(\Omega'\times\Omega') \to C^\infty(\Omega\times\Omega) \\ v \mapsto v\circ\chi = v^* & K(x',y') \mapsto K(\chi(x),\chi(y))|\det D\chi(y)| = K^*(x,y). \end{array}$$

This Pull back has the property:

$$\begin{aligned} K^*v^* &= \int_\Omega K(\chi(x),\chi(y))v(\chi(y))|\det D\chi(y)|dy \\ &= \int_{\Omega'} K(\chi(x),y')v(y')\#\chi^{-1}(y')dy' = (K(v\#\chi^{-1}))^*. \end{aligned} \tag{2.3.1}$$

Where $\#\chi^{-1} : \Omega' \to \bar{\mathbb{N}}$ is the function counting the number of pre-images and $v \in C_0^\infty(\Omega')$. We note that the the change of variables is well defined if and only if one of the two integrals is defined. If χ is a diffeomorphism we have the usual fonctorial property $K^*v^* = (Kv)^*$ which permits the definition of operators with kernels on manifolds.

The classic result on the change of variables in pseudo-differential operators is that for $A \in S^m_{loc}(\Omega'\times\mathbb{R}^d)$ properly supported with kernel K then the operator defined by K^* is a pseudo-differential operator A^* of order m on Ω which is also properly supported. Thus it can be seen as the stability of this sub-class of operators of kernels under the pull back by diffeomorphisms (modulo a C^∞ kernel as in Remark 2.2.2) and thus are well defined on manifolds by the same process. Before we start by presenting those classic results we will discuss why they are essentially optimal.

We start by computing for a pseudo-differential operator defined by $a \in S^m(\Omega' \times \mathbb{R}^d)$ with kernel K and $\chi : \Omega \to \Omega'$ a C^∞ map:

$$\begin{aligned} K^* u &= \int_\Omega K(\chi(x), \chi(y)) u(y) |\det D\chi(y)| dy \\ &= \int_{\Omega \times \Omega} (2\pi)^{-d} e^{i(\chi(x)-\chi(y)).\xi} a(\chi(x), \xi) u(y) |\det D\chi(y)| dy d\xi \end{aligned}$$

thus

$$K^* = op_{(\chi(x)-\chi(y)).\xi}(a(\chi(x), \xi)|\det D\chi(y)|)$$

with

$$a(\chi(x), \xi)|\det D\chi(y)| \in S^m(\Omega \times \Omega \times \mathbb{R}^d),$$

because and all the derivatives of χ are bounded. Put

$$\omega_\chi(x, y, \xi) = (\chi(x) - \chi(y)).\xi,$$

by the definitions on Fourier integral operators we have:

$$C_{\omega_\chi} = \left\{(x, y) \in \Omega^2, \chi(x) = \chi(y)\right\}.$$

We also see that w_χ is non degenerate on $\Omega \times \Omega$ if and only if χ is a local diffeomorphism. To sum up:

Proposition 2.3.1. *Take $a \in S^m(\Omega' \times \mathbb{R}^d)$ and $\chi \in C^\infty(\Omega, \Omega')$. Then the pull-back of* $\mathrm{Op}(a)$ *under χ is a Fourier Integral Operator with phase function w_χ and symbol $a(\chi(x), \xi)|\det D\chi(y)| \in S^m(\Omega \times \Omega \times \mathbb{R}^d)$. We have:*

$$C_{\omega_\chi} = \left\{(x, y) \in \Omega^2, \chi(x) = \chi(y)\right\}.$$

Moreover, w_χ is non-degenerate if and only of χ is a local diffeomorphism.

Now we ask the question if there exists a symbol a^* such that:

$$op_{\omega_\chi}(a(\chi(x), \xi)|\det D\chi(y)|) = \mathrm{Op}(a^*).$$

The classic result is that this is true if χ is a diffeomorphism. Now we precise that it's essentially optimal as it could be seen by the following two examples:

- The necessity of the injectivity of ξ: we take $\chi = |\ |$ which is a local diffeomorphism from $\mathbb{R} \setminus 0$ in to $\mathbb{R}_*^+$. We compute for $A = Id$ i.e $a = 1$:

  $$op_{\omega_\chi}(a(\chi(x), \xi)|\det D\chi(y)|)u = u(x) + u(-x),$$

  and the part $u(.) \mapsto u(-.)$ is not a pseudo-differential operator.

- The necessity of the local diffeomorphism hypothesis: we take $\chi = x^3$ which is a local diffeomorphism from $\mathbb{R} \setminus 0$ in to $\mathbb{R}$. We compute for $A = \frac{d}{dx}$ i.e $a = i\xi$:

$$op_{\omega_\chi}(a(\chi(x), \xi)|\det D\chi(y)|)u = \frac{u'(x)}{3x^2},$$

which is a pseudo-differential operator on $\mathbb{R} \setminus 0$ but cannot be extended to one on $\mathbb{R}$ with a regular symbol in 0. [3]

Now we present the classic results of change of variables in pseudo and para-differential operators under the hypothesis that χ is a diffeomorphism as they can be found in [10],[11] and [35].

Theorem 2.3.1. *Let $\chi : \Omega \to \Omega'$ be a C^∞ diffeomorphism and $A = a(x, D) \in S^m_{loc}(\Omega' \times \mathbb{R}^d)$ a properly supported pseudo-differential operator with kernel K. Then the operator A^* defined by K^* i.e:*

$$\forall u \in C_0^\infty(\Omega), A^* u = \int_\Omega K(\chi(x), \chi(y))u(y)|\det D\chi(y)|dy$$

is a properly supported pseudo-differential operator with symbol

$$a^*(x,\xi) = (-1)^d e^{-ix.\xi} \int_{\Omega\times\mathbb{R}^d} a(\chi(x), \eta)e^{i(\chi(x)-\chi(y)).\eta+iy.\xi}|\det D\chi(y)|dyd\eta \in S^m_{loc}(\Omega\times\mathbb{R}^d).$$

An expansion of a^* is given by:

$$a^*(x,\xi) \sim \sum_\alpha \frac{1}{\alpha!}\partial^\alpha a(\chi(x), D\chi^{-1}(\chi(x))^t\xi)P_\alpha(\chi(x), \xi), \tag{2.3.2}$$

where,

$$P_\alpha(x', \xi) = D^\alpha_{y'}(e^{i(\chi^{-1}(y')-\chi^{-1}(x')-D\chi^{-1}(x')(y'-x')).\xi})_{|y'=x'}$$

and P_α is polynomial in ξ of degree $\leq \frac{|\alpha|}{2}$, with $P_0 = 1, P_1 = 0$.

Remark 2.3.1. *This a classic result found commonly in the literature, And as in the Remark 2.2.4 an analogous result still holds in the class Σ^m_0 as will be shown in the proof of the next Theorem.*

For para-differential operators we have:

Theorem 2.3.2. *Let $\chi : \Omega \to \Omega'$ be a $W^{1+\rho,\infty}_{loc}$ diffeomorphism with $D\chi \in W^{\rho,\infty}$ and $\rho \geq 0$. Consider $a \in \Gamma^m_r(\mathbb{R}^d)$ a properly supported paradifferential operator. Then there exists a property supported $a^* \in \Gamma^m_{min(r,\rho)}(\mathbb{R}^d)$ defined by:*

$$(T_a u) \circ \chi = T_{a^*}(u \circ \chi) + (R\chi)u,$$

where $R \in \Gamma^m_r(\mathbb{R}^d)$ and $R\chi$ is a term depending essentially on χ and it's explicit formula is given in (2.4.4).

[3]In fact it can be treated in the more general frame of operators with singular symbols but this goes beyond the scope of this work.

Moreover a^* has the local expansion:

$$a^*(x,\xi) \sim \sum_{\substack{\alpha \\ |\alpha| \leq \lfloor min(r,\rho) \rfloor}} \frac{1}{\alpha!}\partial^\alpha a(\chi(x), D\chi^{-1}(\chi(x))^t \xi) P_\alpha(\chi(x),\xi), \tag{2.3.3}$$

where,

$$P_\alpha(x',\xi) = D^\alpha_{y'}(e^{i(\chi^{-1}(y')-\chi^{-1}(x')-D\chi^{-1}(x')(y'-x')).\xi})_{|y'=x'}$$

and P_α is polynomial in ξ of degree $\leq \frac{|\alpha|}{2}$, with $P_0 = 1, P_1 = 0$.

An analogous result still holds for para-differential operators modeled on the spaces $a \in C^r_*, r > 0$ and $\chi \in C^{1+\rho}_*$.
As we couldn't find a clear reference to this result in the literature, it is eluded to in [10][4], we give a simple proof of this Theorem.

Proof. Taking ψ a cut-off function with parameters $B > 1, b > 0$, and take $u \in C^\infty_0(\Omega)$ compute

$$\begin{aligned}(T_a(u \circ \chi^{-1})) \circ \chi &= op_{(\chi(x)-\chi(y)).\xi}(\sigma^\psi_a(\chi(x),\xi)\,|\det D\chi(y)|)u \\ &= \int_{\Omega\times\mathbb{R}^d} e^{i(\chi(x)-\chi(y))\cdot\xi}\sigma^\psi_a(\chi(x),\xi)\,|\det D\chi(y)|\,u(y)dyd\xi.\end{aligned}$$

As we remarked above the main contribution in this integral will come from $(x,y,\xi) \in C_{\omega_\chi}$ where we recall $\omega_\chi(x,y,\xi) = (\chi(x)-\chi(y))\cdot\xi$. To show this insert the smooth cut-off function $\theta(x,y)$ supported in a small neighborhood of the diagonal (x,x).

$$\begin{aligned}(T_a(u \circ \chi^{-1})) \circ \chi &= \int_{\Omega\times\mathbb{R}^d} e^{i(\chi(x)-\chi(y))\cdot\xi}\sigma^\psi_a(\chi(x),\xi)\,|\det D\chi(y)|\,u(y)dyd\xi \\ &= \int_{\Omega\times\mathbb{R}^d} e^{i(\chi(x)-\chi(y))\cdot\xi}\theta(x,y)\sigma^\psi_a(\chi(x),\xi)\,|\det D\chi(y)|\,u(y)dyd\xi \\ &+ \int_{\Omega\times\mathbb{R}^d} e^{i(\chi(x)-\chi(y))\cdot\xi}(1-\theta(x,y))\sigma^\psi_a(\chi(x),\xi)\,|\det D\chi(y)|\,u(y)dyd\xi\end{aligned}$$

Now ω_χ has no critical points on the support of $(1-\theta(x,y))$ and by integration by parts we have:

$$(T_a(u \circ \chi^{-1})) \circ \chi = \int_{\Omega\times\mathbb{R}^d} e^{i(\chi(x)-\chi(y))\cdot\xi}\theta(x,y)\sigma^\psi_a(\chi(x),\xi)\,|\det D\chi(y)|\,u(y)dyd\xi + Ru.$$

with $R \in \Gamma^{m-\min(r,\rho)}_0$. We now analyze when y is close to x. By the mean value Theorem, for y sufficiently close to x, there exists a invertible linear mapping $L_{x,y} \in W^{\rho,\infty}$ such that

$$\begin{cases}\chi(x)-\chi(y) = L_{x,y}\cdot(x-y) \\ L_{x,x} = D\chi(x).\end{cases}$$

[4] part 3.3 point h, which can be found in pages 114-115.

Thus we get,

$$
\begin{aligned}
&(T_a(u \circ \chi^{-1})) \circ \chi \\
&= \int_{\Omega \times \mathbb{R}^d} e^{i(\chi(x)-\chi(y))\cdot\xi} \theta(x,y) \sigma_a^{\psi}(\chi(x),\xi) \left|\det D\chi(y)\right| u(y) dy d\xi + Ru \\
&= \int_{\Omega \times \mathbb{R}^d} e^{i(x-y)\cdot\xi} \theta(x,y) \sigma_a^{\psi}(\chi(x), {L_{x,y}^t}^{-1}\xi) \left|\det D\chi(y)\right| \left|\det L_{x,y}^{-1}\right| u(y) dy d\xi + Ru.
\end{aligned}
$$

We get an operator with an amplitude

$$
c(x,y,\xi) = \theta(x,y)\sigma_a^{\psi}(\chi(x), {L_{x,y}^t}^{-1}\xi) \left|\det D\chi(y)\right| \left|\det L_{x,y}^{-1}\right| \in \Gamma_\rho^m(\mathbb{R}^d).
$$

In the frequency domain this amplitude depends on terms coming from $\sigma_a^{\psi}(\chi(x), {L_{x,y}^t}^{-1}\xi)$, $\left|\det D\chi(y)\right|$ and $\left|\det L_{x,y}^{-1}\right|$. Putting all of the high frequency terms depending on χ and χ^{-1} in a term $R\chi$ by defining $\tilde{\psi}$ as the cut-off function in both of the variables $(x-y, y)$ with parameters:

$$
c = \min(1, \sup D\chi^{-1}, \sup D\chi),\ \tilde{B} = cB,\ \tilde{b} = b.
$$

Thus by Proposition 2.2.3:

$$
\tilde{c}(x,y,\xi) = \tilde{\psi}(D,\cdot)c \in \Sigma^m_{min(r,\rho)}(\mathbb{R}^d),
$$

with,

$$
c = \tilde{c} + R\chi + R'
$$

and $R' \in \Gamma_0^{m-min(r,\rho)}$.
The result then follows from Proposition 2.2.12. □

2.4 Paracomposition

2.4.1 Main results for paracomposition on $\mathbb{R}^d$

We start by a formal computation, as in [73], using the Littlewood-Paley decomposition and two functions u and χ:

$$
\begin{aligned}
u \circ \chi &= \sum_{k \geq 0} u(\Phi_{k+1}\chi) - u(\Phi_k \chi) = \sum_{j,k} u_j(\Phi_{k+1}\chi) - u_j(\Phi_k\chi) \\
&= \sum_{j<k} u_j(\Phi_{k+1}\chi) - u_j(\Phi_k\chi) + \sum_{j \geq k} u_j(\Phi_{k+1}\chi) - u_j(\Phi_k\chi) \\
&= \underbrace{\sum_{k \geq 1} \Phi_{k-1}u(\Phi_k\chi) - \Phi_{k-1}u(\Phi_{k-1}\chi)}_{1} + \underbrace{\sum_{k \geq 0} u_k(\Phi_k\chi)}_{2}.
\end{aligned} \tag{2.4.1}
$$

Remark 2.4.1. *Heuristically the term 1 has frequencies of u smaller than that of χ and as in classical paradifferential results will depend mainly on the regularity of χ. This is indeed the main term in Bony's para-linearization Theorem modulo a more regular remainder:*

$$
\begin{aligned}
(1) &= \sum_{k\geq 1}\left(\int_0^1 \Phi_{k-1}u'(\tau\Phi_k\chi+(1-\tau)\Phi_{k-1})\chi d\tau\right)\phi_k\chi \\
&= \underbrace{\sum_{k\geq 1}\Phi_{k-1}(u'\circ\chi)(\phi_k\chi)}_{T_{u'\circ\chi}\chi} \qquad (2.4.2)\\
&+\underbrace{\sum_{k\geq 1}\left(\int_0^1 \Phi_{k-1}u'(\tau\Phi_k\chi+(1-\tau)\Phi_{k-1}\chi)-\Phi_{k-1}(u'\circ\chi)d\tau\right)\phi_k\chi}_{R_0}.
\end{aligned}
$$

Same as term 1, heuristically term 2 will essentially depend on the regularity of u, with a remainder depending on χ and u that is more regular when it's well defined. Thus (2) will naturally give rise to the paracomposition operator. To better understand it, let us suppose just for the next computation that χ is linear and invertible:

$$
\begin{aligned}
(2) &= \sum_{k\geq 0}\int_{\mathbb{R}^d}\phi_k(\xi)\hat{u}(\xi)e^{i\Phi_k\chi(x).\xi}d\xi \\
&= \sum_{k\geq 0}\int_{\mathbb{R}^d}\phi_k(\Phi_k\chi^{-t}\xi)\hat{u}(\Phi_k\chi^{-t}\xi)e^{ix.\xi}|\Phi_k\chi^{-t}(\xi)|d\xi
\end{aligned}
$$

Thus we essentially have to look at how $\Phi_k\chi^{-t}$ modifies the frequencies and thus how it modifies the rings in the Littlewood-Paley decomposition.
Put $\left\{k\geq 1, C'_k = \text{supp }\ \phi_k(\Phi_k\chi^{-t}.)\right\}$, we have:

$$
C'_k \approx \bigcup_{k-N'\leq l\leq k+N} C_l,
$$

where N and N' are such that $2^N > sup_{k,\mathbb{R}^d}|\Phi_k\chi'|$ and $2^{N'} > sup_{k,\mathbb{R}^d}|\Phi_k\chi'|^{-1}$ and the natural para-composition operator in this case is obtained by cutting the frequencies according to C'_k, this is exactly the "lemme de recoupe" in Alinhac's work.

Now we define N as in the previous remark and compute:

$$
\begin{aligned}
(2) &= \underbrace{\sum_{k\geq 0}\sum_{\substack{l\geq 0\\ l\leq k+N}}\phi_l(D)(u_k\circ\chi)}_{\chi^\star u} \qquad (2.4.3)\\
&+\underbrace{\sum_{k\geq 0}\sum_{\substack{l\geq 0\\ l\leq k+N}}\phi_l(D)[u_k\circ\Phi_k\chi-u_k\circ\chi]}_{R_1}+\underbrace{\sum_k(Id-\Phi_{k+N})(D)u_k\circ\Phi_k\chi}_{R_2}.
\end{aligned}
$$

Theorem 2.4.1. *Let $\chi : \mathbb{R}^d \to \mathbb{R}^d$ be a $C^{1+\rho}_{*,loc}$ map with $D\chi \in C^\rho_*$ and $\rho > 0$* [5]*. Then for all $\sigma, s \in \mathbb{R}^*_+$ the following maps are continuous:*

$$C^\sigma_*(\mathbb{R}^d) \to C^\sigma_*(\mathbb{R}^d) \qquad C^\sigma_{*,loc}(\mathbb{R}^d) \to C^\sigma_{*,loc}(\mathbb{R}^d)$$
$$u \mapsto \chi^\star u = \sum_{k\geq 0} \sum_{\substack{l\geq 0 \\ l\leq k+N}} \phi_l(D)u_k \circ \chi \qquad u \mapsto \chi^\star u = \sum_{k\geq 0} \sum_{\substack{l\geq 0 \\ l\leq k+N}} \phi_l(D)u_k \circ \chi.$$

If moreover χ is a diffeomorphism then we have the Sobolev estimates:

$$H^s(\mathbb{R}^d) \to H^s(\mathbb{R}^d) \qquad H^s_{loc}(\mathbb{R}^d) \to H^s_{loc}(\mathbb{R}^d)$$
$$u \mapsto \chi^\star u = \sum_{k\geq 0} \sum_{\substack{l\geq 0 \\ l\leq k+N}} \phi_l(D)u_k \circ \chi \qquad u \mapsto \chi^\star u = \sum_{k\geq 0} \sum_{\substack{l\geq 0 \\ l\leq k+N}} \phi_l(D)u_k \circ \chi.$$

Taking $\tilde\chi : \mathbb{R}^d \to \mathbb{R}^d$ a $C^{1+\tilde\rho}_{*,loc}$ map with $D\tilde\chi \in C^{\tilde\rho}_*$ and $\tilde\rho > 0$, then the previous operation has the natural fonctorial property:

$$\forall u \in C^\sigma_*(\mathbb{R}^d) \cup C^\sigma_{*,loc}(\mathbb{R}^d), \chi^\star\tilde\chi^\star u = (\chi \circ \tilde\chi)^\star u + Ru.$$

$$\text{with } R,\ R : C^\sigma_*(\mathbb{R}^d) \to C^{\sigma+min(\rho,\tilde\rho)}_*(\mathbb{R}^d),\ R : C^\sigma_{*,loc}(\mathbb{R}^d) \to C^{\sigma+min(\rho,\tilde\rho)}_{*,loc}(\mathbb{R}^d),$$

and if χ and $\tilde\chi$ are diffeomorphisms:

$$R : H^s(\mathbb{R}^d) \to H^{s+min(\rho,\tilde\rho)}(\mathbb{R}^d), \qquad R : H^s_{loc}(\mathbb{R}^d) \to H^{s+min(\rho,\tilde\rho)}_{loc}(\mathbb{R}^d).$$

Remark 2.4.2. *It's natural that the Sobolev estimates only hold when χ is a diffeomorphism because for example even the usual composition operation $u \mapsto u\circ\chi$ is not necessarily continuous on L^p spaces, $p < \infty$. An extra hypothesis that appears in the literature is χ is a local diffeomorphism with all of it's local inverses uniformly bounded in $\dot W^{1,\infty}$.*

Theorem 2.4.2. *Let u be a $W^{1,\infty}(\mathbb{R}^d)$ map and χ be a $C^{1+\rho}_{*,loc}$ map with $D\chi \in C^\rho_*$ and $\rho > 0$. Then:*

$$u \circ \chi(x) = \chi^\star u(x) + T_{u'\circ\chi}\chi(x) + R_0(x) + R_1(x) + R_2(x)$$

where the paracomposition given in the previous Theorem verifies the estimates:

$$\forall\sigma > 0, \|\chi^\star u(x)\|_\sigma \leq C(\|D\chi\|_\infty)\,\|u(x)\|_\sigma\,,$$

$$u' \circ \chi \in \Gamma^0_{W^{0,\infty}(\mathbb{R}^d)}(\mathbb{R}^d) \quad for\ u\ Lipchitz,$$

and the remainders verify the estimates:

- *In Zygmund Spaces, for $\sigma > 0$:*

$$\|R_0\|_{1+\rho+min(1+\rho,\sigma)} \leq C\,\|D\chi\|_\rho\,\|u\|_{1+\sigma}$$

$$for\ i \in \{1,2\}\,, \|R_i\|_{1+\rho+\sigma} \leq C(\|D\chi\|_\infty)\,\|D\chi\|_\rho\,\|u\|_{1+\sigma}\,.$$

[5]Clearly when there is no diffeomorphism hypothesis on χ we can choose $\chi : \mathbb{R}^d \to \mathbb{R}^{d'}$ with $d \neq d'$ and have the same results but for clarity we chose to present the same dimensions in this presentation.

- *In Sobolev Spaces, for $s > \frac{d}{2}$ we get the following estimates*
  - *without the diffeomorphism hypothesis:*

$$\|R_0\|_{H^{1+\rho+min(1+\rho,s-\frac{d}{2})}} \leq C \|D\chi\|_\rho \|u\|_{H^{1+s}}$$
$$\|R_1\|_{H^{1+\rho+s}} \leq C(\|D\chi\|_\infty) \|D\chi\|_\rho \|u\|_{H^{1+s}} .$$

  - *Suppose moreover that χ is a diffeomorphism:*

$$\|R_2\|_{H^{1+\rho+s}} \leq C(\|D\chi\|_\infty , \|D\chi^{-1}\|_\infty) \|D\chi\|_\rho \|u\|_{H^{1+s}} .$$

The same estimates hold in the local spaces.

As Alinhac remarked in [10], a particular case of the previous Theorem is Bony para-linearization Theorem but with the extra hypothesis of diffeomorphism, here it's is a full generalization because we dropped the diffeomorphism hypothesis. We find Bony's para-linearization Theorem when $\sigma = +\infty$, in this case only the term $T_{u'\circ\chi}\chi(x)$ appears and $\chi^\star u(x)$ is a part of the remainder. If on the other hand, $\chi \in C^\infty$, the term $T_{u'\circ\chi}\chi(x)$ becomes a part of the remainder and the paracomposition $\chi^\star u(x)$ coincides with the usual composition modulo a regularizing operator. Thus Theorem 2.4.2 appears as a linearization Theorem of $u \circ \chi$ as the sum of two terms, one depending mainly on the regularity of u (and "less" of χ) and the other depending mainly on the regularity of χ (and "less" of u).

Remark 2.4.3. *The simplest example for the paracomposition operator is when $\chi(x) = Ax$ is a linear operator and in that case we see that if N is chosen sufficiently large in the definition:*

$$u(Ax) = (Ax)^* u, \ \text{and } T_{u'(Ax)} Ax = 0.$$

Remark 2.4.4. *The proof of Theorem 2.4.2 tell us that the if in the sum defining $\chi^\star$ we choose a different $N' \geq N$ then the operator is modified by a ρ regularizing operator.*

Theorem 2.4.3. *Consider $a \in \Gamma^m_\beta(\mathbb{R}^d)$, with $\beta \geq 0$, $\chi : \mathbb{R}^d \to \mathbb{R}^d$ a $C^{1+\rho}_{*,loc}$ map with $D\chi \in C^\rho_*$, $\rho > 0$ and $1 + \rho \notin \mathbb{N}$. Then there exists $q \in \Gamma^{m-\beta}_0(\mathbb{R}^d)$ such that we have the following formal symbolic calculus rule:*

$$\chi^\star T_a u = op_{\omega_\chi}\left(\sigma_a(\chi(x), \xi)\frac{|\det D\chi(y)|}{\#\chi^{-1}(\chi(y))}\right)\chi^\star u + op_{\omega_\chi}\left(\sigma_q(\chi(x), \xi)\frac{|\det D\chi(y)|}{\#\chi^{-1}(\chi(y))}\right)\chi^\star u.$$

To join Alinhac's work, the following Proposition makes the link between his definition of the paracomposition operator in the case of a diffeomorphism and the one given here.

Theorem 2.4.4. *Let u be $W^{1,\infty}(\mathbb{R}^d)$ map and χ be a $C^{1+\rho}_{*,loc}$ diffeomorphism with $D\chi \in C^\rho_*$ and $\rho > 0$. Consider $\tilde{N}$ such that $2^{\tilde{N}} > sup_{k,\mathbb{R}^d} |\Phi_k \chi'|^{-1}$ and $2^{\tilde{N}} > sup_{k,\mathbb{R}^d} |\Phi_k \chi'|$. Put Alinhac's paracomposition operator:*

$$\chi^* u = \sum_{k\geq 1} \sum_{\substack{l\geq 0 \\ |l-k|\leq \tilde{N}}} \phi_l(D) u_k \circ \chi$$

$$\text{then: } \chi^* u = \chi^\star u + R_3,$$

Where the remainder verifies:

- *In Zygmund Spaces, for $\sigma > 0$:*

$$\|R_3\|_{1+\rho+\sigma} \leq C(\|D\chi^{-1}\|_\infty, \|D\chi\|_\infty) \|D\chi\|_\rho \|u\|_{1+\sigma}.$$

- *In Sobolev Spaces, for $s > \frac{d}{2}$:*

$$\|R_3\|_{H^{1+\rho+s}} \leq C(\|D\chi^{-1}\|_\infty, \|D\chi\|_\infty) \|D\chi\|_\rho \|u\|_{H^{1+s}}.$$

The same estimates hold in the local spaces.
Take $a \in \Gamma^m_\beta(\mathbb{R}^d)$ and q as in Theorem 2.4.3 then:

$$\chi^\star T_a u = T_{a^*}\chi^\star u + T_{q^*}\chi^\star u$$
$$\chi^* T_a u = T_{a^*}\chi^* u + T_{q'^*}\chi^* u \text{ with } q' \in \Gamma_0^{m-\beta}(\mathbb{R}^d).$$

Remark 2.4.5. *As in remark 2.4.4, the proof of Theorem 2.4.4 tell us that the if in the sum defining χ^* we choose a different $\tilde{N}' \geq \tilde{N}$ then the operator is modified by a ρ regularizing operator.*

Remark 2.4.6. *As a corollary of Theorem 2.4.4 we get that in Theorem 2.3.2:*

$$R\chi = T_{(T_a u)' \circ \chi}\chi - T_{a^*} T_{u' \circ \chi}\chi. \tag{2.4.4}$$

Remark 2.4.7. *All of the result of this section extend naturally to the functions and operators defined on the Torus.*

2.4.2 Proofs

We will give the proof for the estimates in global spaces, for local spaces it is sufficient to see that the given estimates hold under the hypothesis that all the functions used have a compact support and to pass to local spaces estimates it is sufficient to multiply by functions in C_0^∞ which don't modify the estimates given (we don't make any boundary estimates).

Proof of Theorem 2.4.1 and 2.4.2

Take $\chi : \mathbb{R}^d \to \mathbb{R}^d$ be a $C_*^{1+\rho}$ map with $\rho > 0$ put $\mathrm{B} = \mathrm{B}(0, N+1)$.
We start by the Zygmund spaces estimates (thus we don't suppose that χ is a diffeomorphism):

$$\|\Phi_{k+N} u_k \circ \chi\|_\infty \leq C \|u_k\|_\infty \leq 2^{-k\sigma} \|u\|_\sigma$$

and supp $\Phi_{k+N} u_k \circ \chi \subset 2^k\mathrm{B}$.
Thus by Proposition 2.1.3, for $\sigma > 0$:

$$\chi^\star u \in C_*^\sigma(\mathbb{R}^d) \text{ and } \|\chi^\star u\|_\sigma \leq \frac{C(N)}{1 - 2^{-\sigma}} \|u\|_\sigma.$$

For Sobolev estimates we suppose that χ is a diffeomorphism and by the change of variables formula we have for $s > 0$:

$$\|\Phi_{k+N} u_k \circ \chi\|_{L^2} \leq C(\|D\chi^{-1}\|_\infty) \|u_k\|_{L^2} \leq C(\|D\chi^{-1}\|_\infty) 2^{-ks} \|u\|_{H^s}$$

and supp $\Phi_{k+N} u_k \circ \chi \subset 2^k \mathrm{B}$.
Thus by Proposition 2.1.3, for $\sigma > 0$:

$$\chi^\star u \in H^s(\mathbb{R}^d) \text{ and } \|\chi^\star u\|_{H^s} \leq \frac{C(N, \|D\chi^{-1}\|_\infty)}{1 - 2^{-s}} \|u\|_{H^s}.$$

Now we compute the estimates on the remainders in the linearization formula.

$$R_0 = \sum_{k \geq 1} \left(\int_0^1 \Phi_{k-1} u'(\tau \Phi_k \chi + (1-\tau)\Phi_{k-1}\chi) - \Phi_{k-1}(u' \circ \chi) d\tau \right) \phi_k \chi = \sum_k r_k^0 \chi_k$$

$$\begin{aligned} r_k^0 &= \int_0^1 \Phi_{k-1} u'(\tau \Phi_k \chi + (1-\tau)\Phi_{k-1}\chi) - \Phi_{k-1}(u' \circ \chi) d\tau \\ &= \int_0^1 \int_0^1 \Phi_{k-1} u''(t\tau(\Phi_k \chi - \Phi_{k-1}\chi) \\ &+ t(\Phi_{k-1}\chi - \chi) - \chi) dt [\tau(\Phi_k \chi - \chi) + (1-\tau)(\Phi_{k-1}\chi - \chi)] d\tau. \end{aligned}$$

Thus if $\sigma \neq 1$:

$$\|r_k^0\|_\infty \leq C 2^{k(-\sigma-\rho)}$$

And if $\sigma = 1$:

$$\|r_k^0\|_\infty \leq C k 2^{k(-1-\rho)} \leq C 2^{-k},$$

Which sums up in $\|r_k^0\|_\infty \leq C 2^{-min(1+\rho,\sigma)k}$. By the same computations we have analogous estimates on $\|\partial^\alpha r_k^0\|$ and clearly $r_k^0 \in C^\infty$ which gives the desired estimates on R_0 by Lemma 2.1.1 and the fact that $r_0^0 = 0$, both in the Sobolev et Zygmund cases without the diffeomorphism hypothesis.

$$\begin{aligned} R_1 &= \sum_{k \geq 0} \phi_{k+N}(D)[u_k \circ \Phi_k \chi - u_k \circ \chi] \\ &= \sum_{k \geq 0} \phi_{k+N}(D)[(\int_0^1 u_k'(t\Phi_k + (1-t)\chi) dt)(\Phi_k \chi - \chi)] \\ &= \sum_{k \geq 0} \phi_{k+N}(D)[r_k^1(\Phi_k \chi - \chi)]. \end{aligned}$$

We have:

$$\|r_k^1\|_\infty \leq C 2^{-k\sigma}$$

combining this with Propositions 2.1.3, 2.1.4 and the fact that $r_0^1 = 0$ we get the desired estimates again in both in the Sobolev et Zygmund cases without the diffeomorphism hypothesis.
The proof of the estimates on R_2 relies on oscillatory integral techniques that come from Lemma 2.2.1 . For the sake of completion we will give the explicit computations without directly using the Lemma.

$$R_2(x) = \sum_k (Id - \Phi_{k+N})(D) u_k \circ \Phi_k \chi(x).$$

We will prove that for $j \geq k+N+1, \nu \geq \rho > 0$, we have:

$$\|\phi_j(D)u_k \circ \Phi_k \chi\|_\infty \leq C_\nu(\|D\chi\|_\rho)2^{-j\nu}2^{k(\nu-\rho)}\|u_k\|_\infty \tag{2.4.5}$$

which will be sufficient to give the Zygmund estimates on R_2 because we will have:

$$\begin{aligned}
\|\phi_j(D)R_2\|_\infty &\leq \sum_{\substack{k\geq 0\\ k\leq N-j+1}} \|\phi_j(D)u_k \circ \Phi_k\chi\|_\infty \\
&\leq \sum_{\substack{k\geq 0\\ k\leq N-j+1}} C_\nu(\|D\chi\|_\rho)2^{-j\nu}2^{k(\nu-\rho)}\|u_k\|_\infty \\
&\leq \sum_{\substack{k\geq 0\\ k\leq N-j+1}} C_\nu(\|D\chi\|_\rho)2^{-j\nu}2^{k(\nu-\rho)}\|u_k\|_\infty \\
&\leq \sum_{\substack{k\geq 0\\ k\leq N-j+1}} C_\nu(\|D\chi\|_\rho)2^{-j\nu}2^{k(\nu-\rho-\sigma-1)}\|u\|_{1+\sigma},
\end{aligned}$$

Taking $\nu > 1+\rho+\sigma$ we dominate the last expression by:

$$C_\nu(\|D\chi\|_\rho)2^{-j(\rho+\sigma+1)}\|u\|_{1+\sigma}$$

which gives the desired Zygmund estimate.

For the Sobolev estimates we will prove that:

$$\|\phi_j(D)u_k \circ \Phi_k\chi\|_2 \leq C_\nu(\|D\chi\|_\rho)2^{-j\nu}2^{k(\nu-\rho)}\|u_k \circ \Phi_k\chi\|_2, \tag{2.4.6}$$

which then necessitates the diffeomorphism hypothesis on χ to have:

$$\|\phi_j(D)u_k \circ \Phi_k\chi\|_2 \leq C_\nu(\|D\chi\|_\rho, \|D\chi^{-1}\|_\infty)2^{-j\nu}2^{k(\nu-\rho)}\|u_k\|_2,$$

And the desired estimates follow exactly as in the Zygmund case.
Now we prove (2.4.5) and (2.4.6),to make the desired estimates we will put in a test function $f \in C_b^\infty$ as it's usually done with oscillatory integral estimates:

$$\phi_j(D)fu_k \circ \Phi_k\chi(x) = \int e^{i(x-y).\xi}\phi_j(\xi)\phi_k(\eta)f(y)\hat{u_k}(\eta)e^{i\Phi_k\chi(y).\eta}d\eta dy d\xi \tag{2.4.7}$$

Set

$$\omega_k(y,\eta,\xi) = \Phi_k\chi(y).\eta - y.\xi,$$
$$L_k(y,\eta,\xi,\partial_y) = \frac{\Phi_k\chi'(y)^t.\eta - y.\xi}{i\left|\Phi_k\chi'(y)^t.\eta - y.\xi\right|^2}.\nabla_y.$$

Given the definition of N we have:

$$\left|\Phi_k\chi'(y)^t.\eta - y.\xi\right| \geq C(|\eta| + |\xi|) \text{ on supp } \phi_j(\xi)\phi_k(\eta),$$

Thus L_k is well defined and regular, moreover $L_k e^{i\omega_k} = e^{i\omega_k}$. Integrating by parts in (2.4.6):

$$\phi_j(D)fu_k \circ \Phi_k\chi(x) = \int e^{ix.\xi}\phi_j(\xi)\phi_k(\eta)\hat{u_k}(\eta)e^{i\omega_k}(L_k^t)^\nu f(y)d\eta dy d\xi.$$

Note that $(L_k^t)^\nu f$ is homogeneous with degree $-\nu$ in (η, ξ), and smooth on the support of $\phi_j(\xi)\phi_k(\eta)$. Also

$$\left|(L_k^t)^\nu f(y)\right| \le C(\|f\|_\nu, \|D\chi\|_\rho)2^{\nu-\sigma} \text{ on } |\xi|^2 + |\eta|^2 = 1. \tag{2.4.8}$$

Next on a box containing supp $\phi_j(\xi)\phi_k(\eta)$, write

$$(L_k^t)^\nu f(y) = \sum_{(\alpha,\beta)\in\Lambda} a_{k\nu\alpha\beta}(y)e^{i\alpha.\xi+i\beta.\eta} = 2^{-j\nu} \sum_{(\alpha,\beta)\in\Lambda} a_{k\nu\alpha\beta}(y)e^{i2^{-j}\alpha.\xi+i2^{-j}\beta.\eta},$$

where Λ is an appropriate lattice and

$$\sum_{(\alpha,\beta)\in\Lambda} \|a_{k\nu\alpha\beta}\|_\infty \le C(\|f\|_\nu, \|D\chi\|_\rho)2^{\nu-\sigma}. \tag{2.4.9}$$

So (2.4.7) becomes for $j \ge 1$:

$$\begin{aligned}
&\phi_j(D)fu_k \circ \Phi_k\chi(x) &(2.4.10)\\
&= 2^{-j\nu} \sum_{(\alpha,\beta)\in\Lambda} \int e^{ix.\xi}\phi_j(\xi)\phi_k(\eta)\hat{u_k}(\eta)e^{iw_k}a_{k\nu\alpha\beta}(y)e^{i2^{-j}\alpha.\xi+i2^{-j}\beta.\eta}d\eta dy d\xi\\
&= 2^{-j\nu} \sum_{(\alpha,\beta)\in\Lambda} \int e^{i(x-y).\xi}\phi_j(\xi)u_k(\Phi_k\chi(y)+2^{-j}\beta)a_{k\nu\alpha\beta}(y)e^{i2^{-j}\alpha.\xi}dy d\xi\\
&= 2^{-j\nu} \sum_{(\alpha,\beta)\in\Lambda} \int e^{i(x-y).\xi}2^{jn}\hat{\phi}_1(2^j(x-y)+\alpha)u_k(\Phi_k\chi(y)+2^{-j}\beta)a_{k\nu\alpha\beta}(y)dy.\\
&= 2^{-j\nu} \sum_{(\alpha,\beta)\in\Lambda} (a_{k\nu\alpha\beta}\cdot u_k(\Phi_k\chi+2^{-j}\beta)) * g_\alpha(x), &(2.4.11)
\end{aligned}$$

Where $g_\alpha(x) = 2^{jn}\hat{\phi}_1(2^jx+\alpha)$ thus

$$\|g_\alpha\|_{L^1} = 2^{jn}\int \left|\hat{\phi}_1(2^jx+\alpha)\right| dx = \left\|\hat{\phi}_1\right\|_{L^1}. \tag{2.4.12}$$

For $j = 0$ we have an analog inequality.
Using the classic Young and Hölder inequalities combined with (2.4.9), (2.4.12) and taking $f \to 1$ gives us (2.4.5) and (2.4.6). This concludes the proof.

Proof of Theorem 2.4.3

Take $a \in \Gamma_\beta^m(\mathbb{R}^d)$, with $\beta \ge 0$ and $\chi : \mathbb{R}^d \to \mathbb{R}^d$ a $C_*^{1+\rho}$ map with $\rho > 0$. We compute:

$$\chi^\star T_a u = \sum_{k\ge 0} \Phi_{k+N}[(T_a u)_k \circ \chi], \tag{2.4.13}$$

Note that $(T_a u)_k$ can been as $T_{\phi_k}T_a u$ and seeing this a modification of the cut-off function by Proposition 2.2.3 we get:

$$(T_a u)_k = T_{\phi_k}T_a u = T_a T_{\phi_k} u + T_{q^k}u, \text{ with } q^k \in \Gamma_0^{m-\beta}(\mathbb{R}^d).$$

Put $q = \sum q^k$ then (2.4.13) becomes:

$$\chi^\star T_a u = \sum_{k\ge 0} \Phi_{k+N}[(T_a u_k) \circ \chi] + \sum_{k\ge 0} \Phi_{k+N}[(T_{q^k}u_k) \circ \chi].$$

And the formal discussion and computations in part 2.3 give the desired result.

Proof of Theorem 2.4.4

The only thing left to prove is the estimate on R_3.

$$R_3 = \sum_k \underbrace{\Phi_{k-\tilde{N}}(D)u_k \circ \Phi_k\chi(x)}_{1} + \phi_N(D)u_k \circ \chi$$

$\phi_N(D)u_k \circ \chi$ is C^∞ so we only have to the estimate to the first term on the left hand side. Estimating 1 is exactly as (2.4.7) but with ϕ_j substituted by $\Phi_{k-\tilde{N}}$. The core of the estimation relies on the fact that L_k should be well defined and regular on $\operatorname{supp} \Phi_{k-\tilde{N}}(\xi)\phi_k(\eta)$ which is the case give our choice of $\tilde{N}$ and the fact $k \geq 1$. We also have the estimate:

$$\left|\Phi_k\chi'(y)^t.\eta - y.\xi\right| \geq C(|\eta| + |\xi|) \text{ on } \operatorname{supp} \Phi_{k-\tilde{N}}(\xi)\phi_k(\eta).$$

The proof than exactly follows as for R_2.

2.4.3 Main results for paracomposition on open subsets

The previous definition of the operator $\chi^\star$ on functions defined on $\mathbb{R}^d$ relied heavily on the Littelwood-Paley theory which doesn't make it immediately extendable to the open domain case. In [10], Alinhac was able to define such an operator profiting from the continuity of χ^* on the local function spaces and a partition of unity on the open domains. More precisely consider (V_i, Θ_i) a partition of unity locally finite of Ω' then:

$$u \circ \chi = \sum_i \Theta_i u \circ \chi$$

where $\Theta_i u$ is seen as a function of $\mathbb{R}^n$ with the natural extension by 0. In order to have the same natural extension for χ,

$$\chi^{-1}(\operatorname{supp} \Theta_i)$$

needs to be compact we thus have to suppose that χ is a proper map[6]. Under this hypothesis consider $\zeta_i \in C_0^\infty(\Omega)$ such that $\zeta_i = 1$ on $\chi^{-1}(\operatorname{supp} \Theta_i)$:

$$u \circ \chi = \sum_i \zeta_i \Theta_i u \circ \zeta_i \chi, \tag{2.4.14}$$

where $\zeta_i\chi$ is seen as a function of $\mathbb{R}^n$ with the natural extension by 0.

Theorem 2.4.5. *Let $\chi : \Omega \to \Omega'$ be a $C^{1+\rho}_{*,loc}$ proper map with $D\chi \in C^\rho_*$ and $\rho > 0$. Consider (V_i, Θ_i) a partition of unity locally finite of Ω' and ζ_i the associated functions as previously. Then for all $\sigma, s \in \mathbb{R}^*_+$ the following maps are continuous:*

$$\begin{aligned} C^\sigma_*(\Omega') &\to C^\sigma_*(\Omega) & \qquad C^\sigma_{*,loc}(\Omega') &\to C^\sigma_{*,loc}(\Omega) \\ u &\mapsto \chi^\star u = \sum_i \zeta_i \cdot (\zeta_i\chi)^\star \Theta_i u & \qquad u &\mapsto \chi^\star u = \sum_i \zeta_i \cdot (\zeta_i\chi)^\star \Theta_i u, \end{aligned}$$

[6] Note that this extra hypothesis is needed for the methods used to work and is not intrinsic to the problem. Also this hypothesis is immediately verified in the diffeomorphism case treated by Alinhac.

if moreover χ is a diffeomorphism then we have the Sobolev estimates:

$$H^s(\Omega') \to H^s(\Omega) \qquad\qquad H^s_{loc}(\Omega') \to H^s_{loc}(\Omega)$$
$$u \mapsto \chi^\star u = \sum_i \zeta_i \cdot (\zeta_i\chi)^\star \Theta_i u \qquad\qquad u \mapsto \chi^\star u = \sum_i \zeta_i \cdot (\zeta_i\chi)^\star \Theta_i u,$$

where $\Theta_i u$ and $\zeta_i\chi$ are treated as functions on $\mathbb{R}^d$. And in the sum defining each $(\zeta_i\chi)^\star$ a choice

$$N_i, 2^{N_i} \geq \sup_{\operatorname{supp}\Theta_i} \chi'$$

is made by the definition in section 2.4.1, but by remark 2.4.4 in order to simplify the computations we can take the same

$$N \geq N_i, 2^N \geq \sup_\Omega \chi'$$

uniformly for all the operators and this modifies the definition by a ρ regularizing operator.
Making a different choice $(V_i', \Theta_i', \zeta_i')$, which gives a different operator $\chi_1^\star$ then

$$\forall u \in C^\sigma_*(\mathbb{R}^d) \cup C^\sigma_{*,loc}(\mathbb{R}^d), \chi^\star u = \chi_1^\star u + R'u.$$

with $R'u \in C^\infty$.

Consider $\tilde{\chi} : \Omega' \to \Omega''$ a a $C^{1+\tilde{\rho}}_{*,loc}$ proper map with $D\tilde{\chi} \in C^{\tilde{\rho}}_*$ with $\tilde{\rho} > 0$, then the previous operation has the natural fonctorial property:

$$\forall u \in C^\sigma_*(\Omega'') \cup C^\sigma_{*,loc}(\Omega''), \chi^\star\tilde{\chi}^\star u = (\chi \circ \tilde{\chi})^\star u + \tilde{R}u.$$

$$\text{with } \tilde{R},\ \tilde{R} : C^\sigma_*(\Omega'') \to C^{\sigma+min(\rho,\tilde{\rho})}_*(\Omega),\ \tilde{R} : C^\sigma_{*,loc}(\Omega'') \to C^{\sigma+min(\rho,\tilde{\rho})}_{*,loc}(\Omega),$$

and if χ and $\tilde{\chi}$ are diffeomorphisms:

$$\tilde{R} : H^s(\Omega'') \to H^{s+min(\rho,\tilde{\rho})}(\Omega),\ \tilde{R} : H^s_{loc}(\Omega'') \to H^{s+min(\rho,\tilde{\rho})}_{loc}(\Omega).$$

Theorem 2.4.6. *Let u be a $W^{1,\infty}(\Omega)$ map and χ be a be a $C^{1+\rho}_{*,loc}$ proper map with $D\chi \in C^\rho_*$ and $\rho > 0$. Then:*

$$u \circ \chi(x) = \chi^\star u(x) + T_{u'\circ\chi}\chi(x) + R_0(x) + R_1(x) + R_2(x)$$

where the paracomposition given in the previous Theorem verifies the estimates:

$$\forall \sigma > 0, \|\chi^\star u(x)\|_\sigma \leq C(\|D\chi\|_\infty) \|u(x)\|_\sigma,$$
$$u' \circ \chi \in \Gamma^0_{W^{0,\infty}(\Omega)}(\mathbb{R}^d) \text{ for } u \text{ Lipchitz.}$$

The remainders are given by:

$$R_0 = \sum_i \sum_{k\geq 1} \zeta_i \bigg(\int_0^1 \Phi_{k-1}\Theta_i u'(\tau\Phi_k\zeta_i\chi + (1-\tau)\Phi_{k-1}\zeta_i\chi) - \Phi_{k-1}(\Theta_i u' \circ \zeta_i\chi) d\tau \bigg) \phi_k\zeta_i\chi,$$

$$R_1 = \sum_i \sum_{k\geq 0} \sum_{\substack{l\geq 0 \\ l\leq k+N}} \zeta_i(\phi_l(D)[\Theta_i u_k \circ \Phi_k\zeta_i\chi - \Theta_i u_k \circ \zeta_i\chi]),$$

$$R_2 = \sum_i \sum_k \zeta_i((Id - \Phi_{k+N})(D)\Theta_i u_k \circ \Phi_k\zeta_i\chi),$$

and the remainders verify the estimates:

- *In Zygmund Spaces, for $\sigma > 0$:*

$$\|R_0\|_{1+\rho+min(1+\rho,\sigma)} \leq C \|D\chi\|_\rho \|u\|_{1+\sigma}$$

$$for\ i \in \{1,2\}, \|R_i\|_{1+\rho+\sigma} \leq C(\|D\chi\|_\infty) \|D\chi\|_\rho \|u\|_{1+\sigma}.$$

- *In Sobolev Spaces, for $s > \frac{d}{2}$ we get the following estimates*
  - *without the diffeomorphism hypothesis:*

$$\|R_0\|_{H^{1+\rho+min(1+\rho,s-\frac{d}{2})}} \leq C \|D\chi\|_\rho \|u\|_{H^{1+s}}$$

$$\|R_1\|_{H^{1+\rho+s}} \leq C(\|D\chi\|_\infty) \|D\chi\|_\rho \|u\|_{H^{1+s}}.$$

  - *Suppose moreover that χ is a diffeomorphism:*

$$\|R_2\|_{H^{1+\rho+s}} \leq C(\|D\chi\|_\infty, \left\|D\chi^{-1}\right\|_\infty) \|D\chi\|_\rho \|u\|_{H^{1+s}}.$$

The same estimates hold in the local spaces.

Theorem 2.4.7. *Consider $a \in \Gamma^m_\beta(\mathbb{R}^d)$, with $\beta \geq 0$ and $\chi : \Omega \to \Omega'$ a $C^{1+\rho}_{*,loc}$ proper map with $D\chi \in C^\rho_*$, $\rho > 0$ and $1+\rho \notin \mathbb{N}$. Then there exists $q \in \Gamma^{m-\beta}_0(\mathbb{R}^d)$ such that we have the following formal symbolic calculus rule:*

$$\chi^\star T_a u = op_{\omega_\chi}\left(\sigma_a(\chi(x),\xi)\frac{|\det D\chi(y)|}{\#\chi^{-1}(\chi(y))}\right)\chi^\star u + op_{\omega_\chi}\left(\sigma_q(\chi(x),\xi)\frac{|\det D\chi(y)|}{\#\chi^{-1}(\chi(y))}\right)\chi^\star u.$$

Again to join Alinhac's work:

Theorem 2.4.8. *Let u be $W^{1,\infty}(\Omega)$ map and χ be a $W^{1,\infty}$ diffeomorphism, a $C^{1+\rho}_{*,loc}$ proper map with $D\chi \in C^\rho_*$ and $\rho > 0$. Again, consider (V_i, Θ_i) a partition of unity locally finite of Ω' and ζ_i the associated functions as previously. Put Alinhac's paracomposition operator:*

$$\chi^* u = \sum_i \zeta_i(\zeta_i\chi)^*\Theta_i u \ then:$$

$$\chi^* u = \chi^\star u + R_3,$$

Where the remainder verifies:

- *In Zygmund Spaces, for $\sigma > 0$:*

$$\|R_3\|_{1+\rho+\sigma} \leq C(\left\|D\chi^{-1}\right\|_\infty, \|D\chi\|_\infty) \|D\chi\|_\rho \|u\|_{1+\sigma}.$$

- *In Sobolev Spaces, for $s > \frac{d}{2}$:*

$$\|R_3\|_{H^{1+\rho+s}} \leq C(\left\|D\chi^{-1}\right\|_\infty, \|D\chi\|_\infty) \|D\chi\|_\rho \|u\|_{H^{1+s}}.$$

The same estimates hold in the local spaces.
Consider $a \in \Gamma^m_\beta(\mathbb{R}^d)$ and q as in Theorem 2.4.3 then:

$$\chi^\star T_a u = T_{a^*}\chi^\star u + T_{q^*}\chi^\star u$$
$$\chi^* T_a u = T_{a^*}\chi^* u + T_{q'^*}\chi^* u \ with\ q' \in \Gamma^{m-\beta}_0(\mathbb{R}^d).$$

Again we have the same "independence" of the definition of the operator χ^* (modulo a more regular term) with respect to the arbitrary choices made, more precisely, making a different choice $(V_i', \Theta_i', \zeta_i')$ which gives a different operator χ_1^* then

$$\forall u \in C_*^\sigma(\mathbb{R}^d) \cup C_{*,loc}^\sigma(\mathbb{R}^d), \chi^* u = \chi_1^* u + R'u.$$

with $R'u \in C^\infty$.

2.4.4 Proof

All of the estimates given come directly for the Theorems of section 2.4.1. The linearization formulas come from Equation (2.4.14) and the linearization Theorems in section 2.4.1. The only thing left to prove is the independency result with respect to the choice of (V_i, Θ_i, ζ_i). We start by the following Lemma:

Lemma 2.4.1. *Let $(\Theta, \zeta, \tilde{\zeta}) \in C_0^\infty(\Omega')$ be such that $\zeta = 1$ on $\chi^{-1}(\operatorname{supp}\Theta)$ and $\tilde{\zeta} = 1$ on $\operatorname{supp}\zeta$ then:*

$$\sum_{k\geq 0} \zeta \Phi_{k+N}(D)[(\Theta u)_k \circ \zeta\chi] = \sum_{k\geq 0} \tilde{\zeta} \Phi_{k+N}(D)[(\Theta u)_k \circ \tilde{\zeta}\chi] + F, \ F \in C^\infty$$

Proof. Take $\Theta' \in C_0^\infty(\Omega')$ such that $\Theta' \circ \chi = 0$ on supp ζ and $\Theta' \circ \chi = 1$ on supp $\tilde{\zeta} - \zeta$ and compute:

$$\begin{aligned}
&\sum_{k\geq 0} \zeta \Phi_{k+N}(D)[(\Theta u)_k \circ \zeta\chi] \\
&= \sum_{k\geq 0} \tilde{\zeta} \Phi_{k+N}(D)[(\Theta u)_k \circ \tilde{\zeta}\chi] + \sum_{k\geq 0} (\zeta - \tilde{\zeta}) \Phi_{k+N}(D)[(\Theta u)_k \circ \tilde{\zeta}\chi] \\
&= \sum_{k\geq 0} \tilde{\zeta} \Phi_{k+N}(D)[(\Theta u)_k \circ \tilde{\zeta}\chi] + \underbrace{\sum_{k\geq 0} (\zeta - \tilde{\zeta}) \Phi_{k+N}(D)[(\Theta'(\Theta u)_k) \circ \tilde{\zeta}\chi]}_{F}.
\end{aligned}$$

And we have by integration by parts, $\forall l \in \mathbb{N}$:

$$\Theta'(\Theta u)_k = 2^{-kl} \int \frac{e^{i(x'-y')\xi}}{i(x'-y')^l} \Theta'(x)\Theta(y)\phi_1(2^{-k}\xi)u(y)dyd\xi,$$

$$\text{thus, } \left\|\Theta'(\Theta u)_k\right\|_\infty \leq C_l 2^{-k(l-n)}, \text{ and } F \in C^\infty.$$

□

Given (i,j) such that $\operatorname{supp}\Theta_i \cap \operatorname{supp}\Theta_j' \neq \emptyset$ we define $\tilde{\zeta}_{i,j} \in C_0^\infty(\Omega)$ such that $\tilde{\zeta}_{i,j} = 1$ on $\operatorname{supp}\zeta_i \cup \operatorname{supp}\zeta_j'$.

$$\begin{aligned}
\chi^\star u &= \sum_i \zeta_i \cdot (\zeta_i\chi)^\star \Theta_i u = \sum_{k\geq 0} \sum_{i,j} \zeta_i \Phi_{k+N}(D)[(\Theta_i\Theta_j' u)_k \circ \zeta_i\chi] \\
&= \sum_{k\geq 0} \sum_{i,j} \tilde{\zeta}_{i,j} \Phi_{k+N}(D)[(\Theta_i\Theta_j' u)_k \circ \tilde{\zeta}_{i,j}\chi] + F, \ F \in C^\infty \\
&= \sum_{k\geq 0} \sum_{i,j} \zeta_j' \Phi_{k+N}(D)[(\Theta_i\Theta_j' u)_k \circ \zeta_j'\chi] + F + F', \ F' \in C^\infty \\
&= \chi_1^\star u + F + F',
\end{aligned}$$

which gives the desired result and ends the proof.

Chapter 3

On the quasi-linearity of the Water Waves System

In this section we give the proofs of Theorems 1.2.1, 1.3.1 and 1.3.2, which are the main the results from [62].

Contents

3.1 Study of the model equation

In this section we give a full proof of Theorem 1.2.1.

3.1.1 Prerequisites on the Cauchy Problems

For a real number $\alpha \in [0, 2[$, we consider the Cauchy problem[1]:

$$\begin{cases} \partial_t u + u\partial_x u + \partial_x |D|^{\alpha-1} u = 0 \\ u(0, \cdot) = u_0(\cdot) \in H^s(\mathbb{D}), \ s > \frac{3}{2}, \end{cases} \tag{3.1.1}$$

[1] Recall that $D = \mathrm{Op}(|\xi|)$.

It is well known that the problem is well posed in Sobolev spaces, this can be summarized in the following Theorem:

Theorem 3.1.1. *Consider two real numbers, $s \in]\frac{3}{2}, +\infty[$ and $r > 0$. Fix $u_0 \in H^s(\mathbb{D})$. Then there exists $C_s > 0$ such that for $0 < T < \frac{C_s}{r + \|\partial_x u_0\|_{L^\infty(\mathbb{D})}}$ and all $v_0 \in B(u_0, r) \subset H^s(\mathbb{D})$, the problem (3.1.1) with initial data v_0 has a unique solution $v \in C^0([0,T], H^s(\mathbb{D}))$, the map $v_0 \mapsto v$ is continuous from $B(u_0, r)$ to $C^0([0,T], H^s(\mathbb{D}))$ and maps real functions into real functions. Moreover we have the estimates for all $\delta > 0$ and all of $\mu \in [0, s]$, we have:*

$$\forall t \in [0,T], \ \|v(t)\|_{H^\mu(\mathbb{D})} \leq e^{C_\mu \|\partial_x v\|_{L^1([0,T], L^\infty(\mathbb{D}))}} \|v_0\|_{H^\mu(\mathbb{D})}. \tag{3.1.2}$$

Taking $v_0 \in B(u_0, r)$, and assuming moreover that $u_0 \in H^{s+1}(\mathbb{D})$ then:

$$\forall t \in [0,T], \|(u-v)(t)\|_{H^s(\mathbb{D})} \leq e^{C_s(\|\partial_x(u,v)\|_{L^1([0,t], L^\infty(\mathbb{D}))} + C_s t \|u_0\|_{H^{s+1}(\mathbb{D})})} \|u_0 - v_0\|_{H^s(\mathbb{D})}. \tag{3.1.3}$$

We will also need to remark that fixing the initial data at 0 is an arbitrary choice, that is all of the previous conclusions hold for the Cauchy problem defined for $t_0 \leq T$:

$$\begin{cases} \partial_t v + v\partial_x v + \partial_x |D|^{\alpha-1} v = 0 \\ v(t_0, \cdot) = v_0(\cdot) \in H^s(\mathbb{D}), \ s > \frac{3}{2}. \end{cases} \tag{3.1.4}$$

Remark 3.1.1. *Note that the previous Theorem holds for the Cauchy problem associated to the Burgers equation:*

$$\begin{cases} \partial_t u + u\partial_x u = 0 \\ u(0, \cdot) = u_0(\cdot) \in H^s(\mathbb{D}), \ s > \frac{3}{2}, \end{cases} \tag{3.1.5}$$

Though we have some extra estimates in Hölder type spaces:

$$\forall 0 \leq k < s - \frac{1}{2}, \|u(t)\|_{W^{k,\infty}(\mathbb{D})} \leq C_k \|u_0\|_{W^{k,\infty}(\mathbb{D})}, \tag{3.1.6}$$

Taking two different solution u, v, assuming moreover $u_0 \in H^{s+1}(\mathbb{D})$ then we have:

$$\forall 1 \leq k < s - \frac{1}{2}, \|(u-v)(t)\|_{W^{k,\infty}(\mathbb{D})} \leq \|u_0 - v_0\|_{W^{k,\infty}(\mathbb{D})} e^{C_k \int_0^t \|u(s)\|_{W^{k+1,\infty}(\mathbb{D})} ds}.$$

Remark 3.1.2. *We compute the change of scale for the evolution PDE (3.1.1):*

$$u_0 \mapsto \lambda^{\alpha-1} u_0(\lambda x)$$

gives the solution

$$\lambda^{\alpha-1} u(\lambda^\alpha t, \lambda x).$$

Thus giving the critical scaling in Sobolev spaces: $s_c = 1 + \frac{1}{2} - \alpha$, thus we prove quasi-linearity in the subcritical regime of the problem.

Notation 3.1.1. *In order not to be confused with the pull-back symbol, henceforth the conjugate of a symbol a will be written as $a^\top$.*

As the linearized equation is a hyperbolic pseudo-differential equation we recall the result on the Cauchy problem associated to this type of equations:

Theorem 3.1.2. *Consider $(a_t)_{t\in\mathbb{R}}$ a family of symbols in $S^\beta(\mathbb{D}^d)$,$\beta \in \mathbb{R}$, such that $t \mapsto a_t$ is continuous and bounded from $\mathbb{R}$ to $S^\beta(\mathbb{D}^d)$ and such that $\mathrm{Re}(a_t) = \frac{a_t + a_t^\top}{2}$ is bounded in $S^0(\mathbb{D}^d)$, and take $T > 0$. Then for all $s \in \mathbb{R}$, $u_0 \in H^s(\mathbb{D}^d)$ and $f \in C^0([0,T]; H^s(\mathbb{D}^d))$ the Cauchy problem:*

$$\begin{cases} \partial_t u + \mathrm{Op}(a)u = f \\ \forall x \in \mathbb{D}^d, u(0,x) = u_0(x) \end{cases} \tag{3.1.7}$$

has a unique solution $u \in C^0([0,T]; H^s(\mathbb{D}^d)) \cap C^1([0,T]; H^{s-\beta}(\mathbb{D}^d))$ which verifies the estimates:

$$\|u(t)\|_{H^s(\mathbb{D}^d)} \le e^{Ct} \|u_0\|_{H^s(\mathbb{D}^d)} + 2\int_0^t e^{C(t-t')} \left\|f(t')\right\|_{H^s(\mathbb{D}^d)} dt',$$

where C depends on a finite symbol semi-norm of $\mathrm{Re}(a_t)$. We will also need to remark that fixing the initial data at 0 is an arbitrary choice. More precisely, $\forall 0 \le t_0 \le T$ and all data $u_0 \in H^s(\mathbb{D}^d)$ the Cauchy problem:

$$\begin{cases} \partial_t u + \mathrm{Op}(a)u = f \\ \forall x \in \mathbb{D}^d, u(t_0,x) = u_0(x) \end{cases} \tag{3.1.8}$$

has a unique solution $u \in C^0([0,T]; H^s(\mathbb{D}^d)) \cap C^1([0,T]; H^{s-\beta}(\mathbb{D}^d))$ which verifies the estimate:

$$\|u(t)\|_{H^s(\mathbb{D}^d)} \le e^{C|t-t_0|} \|u_0\|_{H^s(\mathbb{D}^d)} + 2\left|\int_{t_0}^t e^{C(t-t')} \left\|f(t')\right\|_{H^s(\mathbb{D}^d)} dt'\right|.$$

3.1.2 Proof of Theorem 1.2.1

To prove the theorem we will show that there exists a positive constant C and two sequences $(u^\lambda_{\epsilon,\tau})$ and $(v^\lambda_{\epsilon,\tau})$ solutions of (1.2.2) on $[0,1]$ such that for every $t \in [0,1]$,

$$\sup_{\lambda,\epsilon,\tau} \left\|u^\lambda_{\epsilon,\tau}\right\|_{L^\infty([0,1],H^s(\mathbb{D}))} + \left\|v^\lambda_{\epsilon,\tau}\right\|_{L^\infty([0,1],H^s(\mathbb{D}))} \le C,$$

$(u^\lambda_{\epsilon,\tau})$ and $(v^\lambda_{\epsilon,\tau})$ satisfy initially

$$\lim_{\substack{\lambda\to+\infty \\ \epsilon,\tau\to 0}} \left\|u^\lambda_{\epsilon,\tau}(0,\cdot) - v^\lambda_{\epsilon,\tau}(0,\cdot)\right\|_{H^s(\mathbb{D})} = 0,$$

but,

$$\liminf_{\substack{\lambda\to+\infty \\ \epsilon,\tau\to 0}} \left\|u^\lambda_{\epsilon,\tau} - v^\lambda_{\epsilon,\tau}\right\|_{L^\infty([0,1],H^s(\mathbb{D}))} \ge c > 0.$$

Considering a weaker control norm we want to get, for all $\delta > 0$,

$$\liminf_{\substack{\lambda\to+\infty \\ \epsilon,\tau\to 0}} \frac{\left\|u^\lambda_{\epsilon,\tau} - v^\lambda_{\epsilon,\tau}\right\|_{L^\infty([0,1],H^{s-1+(\alpha-1)^+ +\delta}(\mathbb{D}))}}{\left\|u^\lambda_{\epsilon,\tau}(0,\cdot) - v^\lambda_{\epsilon,\tau}(0,\cdot)\right\|_{H^s(\mathbb{D})}} = +\infty.$$

3.1.2.1 Definition of the Ansatz

- For $\mathbb{D} = \mathbb{R}$, take $\omega \in C_0^\infty(\mathbb{R}), \omega(x) = 1$ if $|x| \leq \frac{1}{2}, \omega(x) = 0$ if $|x| \geq 1$.
- For $\mathbb{D} = \mathbb{T}$, we see functions on $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ as 2π periodic function on $\mathbb{R}$ and we take $\omega \in C_0^\infty(\mathbb{T})$ as the periodic extension of the function defined above.

Let (λ, ϵ) be two positive real sequences such that

$$\lambda \to +\infty, \ \epsilon \to 0, \ \lambda\epsilon \to +\infty. \tag{3.1.9}$$

Put

- for $\mathbb{D} = \mathbb{R}$,
$$u^0(x) = \lambda^{\frac{1}{2}-s}\omega(\lambda x), \quad v^0(x) = u^0(x) + \epsilon\omega(x),$$
- for $\mathbb{D} = \mathbb{T}$, u^0 and v^0 as the periodic extensions of the functions defined above.

Take $t_0 > 0$ smaller than a harmless constant which will be fixed later, and $(\tau), 0 < \tau \leq t_0$ and $\tau \to 0$.
Now let l, l' be the solutions to the Cauchy problem on $[0, t_0]$:

$$\begin{cases} \partial_t l + \partial_x |D|^{\alpha-1} l = 0, \\ l(\tau, \cdot) = u^0, \end{cases} \qquad \begin{cases} \partial_t l' + \partial_x |D|^{\alpha-1} l' = 0, \\ l'(\tau, \cdot) = v^0. \end{cases}$$

Put $u^1(x) = l(0, x)$ and define analogously $v^1(x) = l'(0, x)$.

Define u and v as the solution given by Theorem 3.1.1with initial data u^1 and v^1 on the intervals $[0, T]$ and $[0, T']$. Taking $0 < \delta < s - \frac{3}{2}$, u^0 and v^0 are uniformly bounded in $H^{\frac{3}{2}+\delta}(\mathbb{D})$ when $\lambda \to +\infty$ and thus by Theorem 3.1.2, u^1 and v^1 are also uniformly bounded in $H^{\frac{3}{2}+\delta}(\mathbb{D})$ and thus by the Sobolev injection Theorem they are bounded in $\dot{W}^{1,\infty}(\mathbb{D})$. Thus we can take a uniform $0 < T$ on which all the solutions are well defined and we take $0 < t_0 \leq T$ [2].

3.1.2.2 Change of variables by transport

Put

$$\begin{cases} \frac{d}{dt}\chi(t, s, x) = u(t, \chi(t, s, x)) \\ \chi(s, s, x) = x \end{cases},$$

and define analogously $\tilde{\chi}$ from v.

We recall that from the Cauchy-Lipschitz Theorem we have as u^0 and v^0 are $H^{+\infty}(\mathbb{D})$ functions, then u^1, v^1 are $H^{+\infty}(\mathbb{D})$ and u and v are $H^{+\infty}(\mathbb{D})$ with respect to the x variable thus $\chi, \tilde{\chi} \in C^1([0, T]^2, C^\infty)$. And they are both diffeomorphisms in the x variable.

[2] Heuristically, if the existence time of the solution with initial data ω is $[0, T]$ then the existence time of the solution with initial data u_0 is $\sim T\lambda^{s-\frac{3}{2}}$ which tends to infinity with λ, thus we are "dilating" the time scale of the problem with initial data ω and "zooming" for short time and in the $\dot{H}^s(\mathbb{D})$ norm. In this part of the evolution, we prove that the Burgers transport term is more important and gives this quasi-linear character to the PDE.

By the estimate (3.1.2) u and v are uniformly bounded in $\dot{W}^{1,\infty}(\mathbb{D})$ because their Sobolev norms are dominated by those of u^1 and v^1 thus by those of u^0 and v^0 by Theorem 3.1.2. By classic manipulations of ODEs we get the estimates:

$$\begin{cases} \exists C > 0, \forall t', t \leq t_0, \forall x, C^{-1} \leq |\partial_x \chi(t, t', x)| \leq C \\ \forall 2 \leq k < \lfloor s - \frac{1}{2} \rfloor, \left\| \partial_x^k \chi(t, t', x) \right\|_{L^\infty} \leq C \left\| u \right\|_{W^{k,\infty}}. \end{cases} \tag{3.1.10}$$

Analogous estimates hold for $\tilde{\chi}$ using v.
The classic transport computation reads:

$$\begin{cases} \partial_t(u(t, \chi(t, 0, x))) = (\partial_t u)(t, \chi(t, 0, x)) + \partial_t(\chi(t, 0, x))(\partial_x u)(t, \chi(t, 0, x)) \\ \qquad\qquad\qquad\qquad = -(\partial_x |D|^{\alpha-1} u)(t, \chi(t, 0, x)) \\ \qquad\qquad\qquad\qquad = -(\partial_x |D|^{\alpha-1})^*(u(t, \chi(t, 0, x))), \\ u(0, \chi(0, 0, x)) = u(0, x) = u^1(x). \end{cases}$$

where $(\cdot)^*$ is the change of variables by $\chi(t, 0, x)$ as presented in Theorem 2.3.1. Thus if we put f the solution to the following Cauchy problem, which is well posed by Appendix 3.5:

$$\begin{cases} \partial_t f + (\partial_x |D|^{\alpha-1})^* f = 0 \\ \forall x \in \mathbb{D}, f(0, x) = u^1(x) \end{cases} \tag{3.1.11}$$

we get:

$$u(t, \chi(t, 0, x)) = f(t, x) \Leftrightarrow u(t, x) = f(t, \chi(0, t, x)). \tag{3.1.12}$$

Analogously, if we put g the solution to the well posed Cauchy problem,

$$\begin{cases} \partial_t g + \widetilde{(\partial_x |D|^{\alpha-1})}^* g = 0 \\ \forall x \in \mathbb{D}, g(0, x) = v^1(x) \end{cases} \tag{3.1.13}$$

where $\widetilde{(\cdot)}^*$ is the change of variables by $\tilde{\chi}(t, 0, x)$, we get

$$v(t, x) = g(t, \tilde{\chi}(0, t, x)) \Leftrightarrow v(t, \tilde{\chi}(t, 0, x)) = g(t, x). \tag{3.1.14}$$

Returning to the ODEs defining χ and $\tilde{\chi}$, for a generic initial time $0 \leq t' \leq t_0$ we get:

$$\begin{cases} \chi(t, t', x) = x + \int_{t'}^t f(s, x) ds, \\ \tilde{\chi}(t, t', x) = x + \int_{t'}^t g(s, x) ds, \end{cases} \tag{3.1.15}$$

Proposition 3.1.1. *There exists $C > 0$ independent of $(\tau, \epsilon, \lambda)$ such that: $\forall h \in H^s(\mathbb{D}), \forall (t, t') \leq t_0$,*

$$C^{-1} \left\| h \right\|_{H^s} \leq \left\| h \circ \chi(t, t', x) \right\|_{H^s} \leq C \left\| h \right\|_{H^s},$$

$$C^{-1} \left\| h \right\|_{H^s} \leq \left\| h \circ \tilde{\chi}(t, t', x) \right\|_{H^s} \leq C \left\| h \right\|_{H^s}.$$

Proof. We will start by proving the upper bound for the estimate on the composition with χ. As u is bounded in $(\tau, \epsilon, \lambda)$ on $C([0,T], H^s(\mathbb{D}))$ then there exists a unique solution $H \in C([0,T], H^s(\mathbb{D}))$ to

$$\begin{cases} \partial_t H + u\partial_x H = 0, \\ H(t,x) = h(x), \end{cases}$$

and H is bounded in $(\tau, \epsilon, \lambda)$ on $C([0,T], H^s(\mathbb{D}))$. The desired bound come from the fact that we have the explicit formula for H:

$$H(t', x) = h \circ \chi(t, t', x).$$

Now to get the lower bound it suffices to write by the upper bound computations:

$$\begin{aligned} \|h\|_{H^s} &= \left\| h \circ \chi(t, t', x) \circ \chi(t', t, x) \right\|_{H^s} \\ &\leq C \left\| h \circ \chi(t, t', x) \right\|_{H^s}. \end{aligned}$$

We get analogously the estimates on the composition with $\tilde{\chi}$. □

3.1.2.3 Key Lemma and proof of the Theorem

Lemma 3.1.1. *Take $\epsilon' > 0$ sufficiently small, as $0 \leq \alpha < 2$ we can find a sequence $(\tau, \epsilon, \lambda)$ such that:*

$$\begin{cases} \tau \to 0, \\ \epsilon \to 0, \\ \lambda \to +\infty, \end{cases} \qquad \begin{cases} \tau\lambda^{(\alpha-1)^+} \to 0, \\ \epsilon^{-1}\lambda^{-1+(\alpha-1)^+ + \epsilon'} \to +\infty, \\ \lambda\epsilon\tau \to +\infty, \\ \lambda^\alpha \epsilon \tau^2 \to 0. \end{cases} \tag{3.1.16}$$

Then there exists $c > 0$ such that:

1. *For $\nu \geq 0$ and $\forall(\tau, \epsilon, \lambda), \left\| u^0 \circ \chi(0, \tau, x) - u^0 \circ \tilde{\chi}(0, \tau, x) \right\|_{H^{s-\nu}} > c\lambda^{-\nu}$.*
2. *For $\nu \geq 0$*

$$\begin{aligned} u(\tau, x) - v(\tau, x) &= u^0 \circ \chi(0, \tau, x) - u^0 \circ \tilde{\chi}(0, \tau, x) \\ &\quad + O_{H^{s-\nu}}\big(\epsilon + (\tau\lambda^{(\alpha-1)^+} + \tau\lambda^{\alpha-(s-\nu)})\lambda^{-\nu} + \tau^2\epsilon\lambda^{\alpha-\nu}\big). \end{aligned} \tag{3.1.17}$$

We will now show that this Lemma implies the Theorem 1.2.1. We have by combining the estimates (1) and (2) for $\nu = s$:

$$\forall(\tau, \epsilon, \lambda), \|u(\tau, x) - v(\tau, x)\|_{H^s} > \frac{c}{2} > 0 \text{ thus } \sup_{\tau,\epsilon,\lambda} \|u(\tau, x) - v(\tau, x)\|_{H^s} > \frac{c}{2} > 0.$$

Also by Theorem 3.1.2:

$$\exists C > 0, \left\| u^1(x) - v^1(x) \right\|_{H^s} \leq C\epsilon, \text{ thus } \left\| u^1(x) - v^1(x) \right\|_{H^s} \to 0,$$

which gives the non uniform continuity in the desired norms.
Now for the control in a weaker norm we write:

$$\frac{\|u(\tau, x) - v(\tau, x)\|_{H^{s-1+(\alpha-1)^+ + \epsilon'}}}{\|u^1(x) - v^1(x)\|_{H^s}} \geq c\epsilon^{-1}\lambda^{-1+(\alpha-1)^+ + \epsilon'} \to +\infty,$$

which gives the desired result.

3.1.2.4 Proof of point 1 of Lemma 3.1.1

We first prove that there exists $c > 0$ such that $\left\|u^0 \circ \chi(0,\tau,x)\right\|_{H^{s-\nu}} > c\lambda^{-\nu}$, indeed by Proposition 3.1.1 and change of variable:

$$\left\|u^0 \circ \chi(0,\tau,x)\right\|_{H^{s-\nu}} \geq C^{-1} \left\|u^0\right\|_{H^{s-\nu}} \geq C^{-1}\lambda^{-\nu} \left\|\omega\right\|_{H^{s-\nu}}. \tag{3.1.18}$$

Now we will show that $u^0 \circ \chi(0,\tau,x)$ and $u^0 \circ \tilde{\chi}(0,\tau,x)$ have disjoint supports which will suffice to conclude given (3.1.18). Put $y = \chi(0,\tau,x)$, thus $x = \chi(\tau,0,y)$. On the support of $u^0 \circ \chi(0,\tau,x)$ we have:

- If $\mathbb{D} = \mathbb{R}$, $\lambda |y| \leq 1$.
- If $\mathbb{D} = \mathbb{T}$, $\forall k \in \mathbb{N}, 2\pi k - 1 \leq \lambda |y| \leq 2\pi k + 1$.

We compute by the Taylor formula, since $x = \tilde{\chi}(\tau,0,y)$:

$$\begin{aligned}
\tilde{\chi}(0,\tau,x) &= \tilde{\chi}(0,\tau,\tilde{\chi}(\tau,0,y)) \\
&\quad + (\chi(\tau,0,y) - \tilde{\chi}(\tau,0,y)) \int_0^1 \partial_y \tilde{\chi}(0,\tau,r\chi(\tau,0,y) + (1-r)\tilde{\chi}(\tau,0,y))dr \\
&= y + (\chi(\tau,0,y) - \tilde{\chi}(\tau,0,y)) \int_0^1 \partial_y \tilde{\chi}(0,\tau,r\chi(\tau,0,y) + (1-r)\tilde{\chi}(\tau,0,y))dr.
\end{aligned} \tag{3.1.19}$$

First by (3.1.15),

$$\begin{aligned}
&\partial_y \tilde{\chi}(0,\tau,r\chi(\tau,0,y) + (1-r)\tilde{\chi}(\tau,0,y)) \\
&\qquad = 1 + \int_0^\tau \partial_y [g(t, r\chi(\tau,0,y) + (1-r)\tilde{\chi}(\tau,0,y))]dt.
\end{aligned}$$

Thus by estimates of Theorem 3.5.1, taking $0 < \delta < s - \frac{3}{2}$[3]:

$$\begin{aligned}
\partial_y \tilde{\chi}(0,\tau,r\chi(\tau,0,y) + (1-r)\tilde{\chi}(\tau,0,y)) &= 1 + O_{L^\infty}\left(\tau[1 + \left\|v^1\right\|_{H^{\frac{3}{2}+\delta}} + \left\|u^1\right\|_{H^{\frac{3}{2}+\delta}}]\right) \\
&= 1 + O_{L^\infty}(\tau),
\end{aligned}$$

Which gives

$$\int_0^1 \partial_y \tilde{\chi}(0,\tau,r\chi(\tau,0,y) + (1-r)\tilde{\chi}(\tau,0,y))dr = 1 + O_{L^\infty}(\tau). \tag{3.1.20}$$

Now we estimate $\chi(\tau,0,y) - \tilde{\chi}(\tau,0,y)$, by (3.1.15) :

$$\chi(\tau,0,y) - \tilde{\chi}(\tau,0,y) = \int_0^\tau f(t,y) - g(t,y)dt. \tag{3.1.21}$$

Taking $0 < \delta < s - \frac{3}{2}$, by estimates of Theorem 3.5.1:

$$\begin{aligned}
f(t,y) = f(0,y) + \int_0^t \partial_t f(r,y)dr &= u^1(y) + tO_{L^\infty}(\left\|u^1\right\|_{H^{\frac{1}{2}+\alpha+\delta}}) \\
&= u^1(y) + O_{L^\infty}(t\epsilon).
\end{aligned}$$

[3] Recall the notation $O_{\| \ \|}$ in 2.1.

Analogously we get:
$$g(t,y) = v^1(y) + O_{L^\infty}(t\epsilon).$$

Consider μ the solution to the Cauchy problem:
$$\begin{cases} \partial_t \mu + \partial_x |D|^{\alpha-1} \mu = 0 \\ \forall y \in \mathbb{D}, \mu(\tau, y) = \omega(y). \end{cases} \tag{3.1.22}$$

By definition:
$$\begin{aligned} u^1(y) - v^1(y) = -\epsilon\mu(0,y) &= -\epsilon\omega(y) + \epsilon \int_\tau^0 \partial_t \mu(t,y)dt \\ &= -\epsilon\omega(y) + O_{L^\infty}(\epsilon\tau). \end{aligned}$$

Thus,
$$\chi(\tau,0,y) - \tilde{\chi}(\tau,0,y) = -\epsilon\tau\omega(y) + O_{L^\infty}(\tau^2\epsilon),$$
and finally we get in (3.1.19),
$$\tilde{\chi}(\tau,0,x) - y = -\epsilon\tau\omega(y) + O_{L^\infty}(\tau^2\epsilon).$$

We get for $x \in \text{supp } u^0 \circ \chi(0,\tau,\cdot)$:

- For $\mathbb{D} = \mathbb{R}$:
$$\lambda |\tilde{\chi}(0,\tau,x)| \geq \tau\epsilon\lambda - 1 + o_{L^\infty}(\tau\epsilon\lambda) \geq 2,$$
by hypothesis $\tau\epsilon\lambda \to +\infty$, which gives the desired result.
- For $\mathbb{D} = \mathbb{T}$ given an adequate choice of τ, ϵ and λ:
$$2n\pi + 1 \leq \lambda |\tilde{\chi}(0,\tau,x)| \leq 2(n+1)\pi - 1,$$
which again gives the desired result.

3.1.2.5 Proof of point 2 of Lemma 3.1.1

We start by writing:
$$\begin{aligned} u(t,x) - v(t,x) &= f(t,\chi(0,t,x)) - g(t,\tilde{\chi}(0,t,x)) \\ &= \underbrace{f(t,\chi(0,t,x)) - f(t,\tilde{\chi}(0,t,x))}_{(1)} + (f-g)(t,\tilde{\chi}(0,t,x)). \end{aligned}$$

Term (1) resembles the main term in the usual transport estimates we used in point 1 of the Lemma [4] but with a main difference of f being some dispersed data and not compactly supported. The main trick here was to construct from u^0, v^0 the defocused data in the past u^1, v^1 and use this as the initial data for f and g.

$$\begin{aligned} u(\tau,x) - v(\tau,x) = \; & u^0 \circ \chi(0,\tau,x) - u^0 \circ \tilde{\chi}(0,\tau,x) \\ & + (f-u^0)(\tau,\chi(0,\tau,x)) - (f-u^0)(\tau,\tilde{\chi}(0,t,x)) + (f-g)(\tau,\tilde{\chi}(0,\tau,x)). \end{aligned}$$

[4] Like the ones used in proving the quasi-linearity of the Burgers equation.

The idea is then to see that by definition of l: $l(\tau,x)=u^0(x)$ and we get:

$$\begin{aligned}u(\tau,x)-v(\tau,x)&=u^0\circ\chi(0,\tau,x)-u^0\circ\tilde{\chi}(0,\tau,x)\\&+\underbrace{(f-l)(\tau,\chi(0,\tau,x))-(f-l)(\tau,\tilde{\chi}(0,t,x))}_{(1)}+\underbrace{(f-g)(\tau,\tilde{\chi}(0,\tau,x))}_{2}.\end{aligned}$$

We start by estimating (1), by Proposition 3.1.1:

$$\|(f-l)(\tau,\chi(0,\tau,x))\|_{H^s}\le C\|(f-l)(\tau,\cdot)\|_{H^s}.$$

Now $f-l$ solves:

$$\begin{cases}\partial_t(f-l)+\partial_x|D|^{\alpha-1}(f-l)=(\partial_x|D|^{\alpha-1}-\partial_x|D|^{\alpha-1*})f\\ \forall x\in\mathbb{D},(f-l)(0,x)=0.\end{cases}\tag{3.1.23}$$

Thus we have the estimates:

$$\begin{aligned}\|f-l(\tau,\cdot)\|_{H^\nu}&\le C\left\|(\partial_x|D|^{\alpha-1}-\partial_x|D|^{\alpha-1*})f\right\|_{L^1([0,\tau],H^\nu)}\\&\le C\tau\left\|(\partial_x|D|^{\alpha-1}-\partial_x|D|^{\alpha-1*})f\right\|_{L^\infty([0,\tau],H^\nu)}\end{aligned}$$

By Theorem 2.3.1 and the Kato-Ponce commutator estimates (2.1.5),

$$\begin{aligned}\|f-l(\tau,\cdot)\|_{H^\nu}&\le C\tau\|(Id-D\chi(0,t,\chi(t,0,x)))\|_{L^\infty}\|f\|_{L^\infty([0,\tau],H^{\nu+\alpha})}\\&+C\tau\|Id-D\chi(0,t,\chi(t,0,x))\|_{L^\infty([0,\tau],W^{\nu,\infty})}\|f\|_{L^\infty([0,\tau],H^\alpha)}\\&+C\tau\|Id-D\chi(0,t,\chi(t,0,x))\|_{L^\infty([0,\tau],W^{1,\infty})}\|f\|_{L^\infty([0,\tau],H^{\nu+\alpha-1})}.\end{aligned}$$

Using Theorem 3.5.3 and applying the Sobolev embedding Theorem with $\delta>0$ and $\delta<s-\alpha-\frac{1}{2}$, we get:

$$\|f-l(\tau,\cdot)\|_{H^\nu}\le C(\tau\lambda^{(\alpha-1)^+}+\tau\lambda^{\alpha-\nu})\lambda^{\nu-s}.\tag{3.1.24}$$

Thus we get

$$\|(f-l)(\tau,\chi(0,\tau,x))\|_{H^\nu}\le C(\tau\lambda^{(\alpha-1)^+}+\tau\lambda^{\alpha-\nu})\lambda^{\nu-s}.$$

Analogously we get

$$\|(f-l)(\tau,\tilde{\chi}(0,\tau,x))\|_{H^\nu}\le C(\tau\lambda^{(\alpha-1)^+}+\tau\lambda^{\alpha-\nu})\lambda^{\nu-s},$$

which gives

$$\|(1)\|_{H^\nu}\le C(\tau\lambda^{(\alpha-1)^+}+\tau\lambda^{\alpha-\nu})\lambda^{\nu-s}.\tag{3.1.25}$$

Now we estimate (2) in the same manner, by Proposition 3.1.1:

$$\|(f-g)(\tau,\tilde{\chi}(0,\tau,x))\|_{H^\nu}\le\|(f-g)(\tau,\cdot)\|_{H^\nu}$$

$f-g$ solve:

$$\begin{cases}\partial_t(f-g)+H\partial_x|D|^{\alpha-1*}(f-g)+(\partial_x|D|^{\alpha-1*}-\partial_x\widetilde{|D|^{\alpha-1}}^{*})g=0\\ \forall x\in\mathbb{D},(f-g)(0,x)=(u^1-v^1)(x).\end{cases}\tag{3.1.26}$$

By Theorem 2.3.1 and the Kato-Ponce commutator estimates (2.1.5),

$$\begin{aligned}
&\|f - g(\tau,\cdot)\|_{H^\nu} \\
&\leq C\|u_1 - v_1\|_{H^\nu} \\
&+ C\tau \|(D\tilde{\chi}(0,t,\tilde{\chi}(t,0,x)) - D\chi(0,t,\chi(t,0,x)))\|_{L^\infty} \|g\|_{L^\infty([0,\tau],H^{\nu+\alpha})} \\
&+ C\tau \|D\tilde{\chi}(0,t,\tilde{\chi}(t,0,x)) - D\chi(0,t,\chi(t,0,x))\|_{L^\infty([0,\tau],W^{\nu,\infty})} \|g\|_{L^\infty([0,\tau],H^{\alpha})} \\
&+ C\tau \|D\tilde{\chi}(0,t,\tilde{\chi}(t,0,x)) - D\chi(0,t,\chi(t,0,x))\|_{L^\infty([0,\tau],W^{1,\infty})} \|g\|_{L^\infty([0,\tau],H^{\nu+\alpha-1})}.
\end{aligned}$$

Using Theorem 3.5.3 and taking $0 < \delta < s - \alpha - \frac{3}{2}$:

$$\|f - g(\tau,\cdot)\|_{H^\nu} \leq C(\epsilon + \tau^2\epsilon\lambda^\alpha\lambda^{\nu-s} + \epsilon\tau + \tau\lambda^{(\alpha-1)^+}\lambda^{\nu-s}),$$

which gives

$$\|(2)\|_{H^\nu} \leq C(\epsilon + \tau^2\epsilon\lambda^\alpha\lambda^{\nu-s} + \epsilon\tau + \tau\lambda^{(\alpha-1)^+}\lambda^{\nu-s}), \tag{3.1.27}$$

finishing the proof of Lemma 3.1.1 and Theorem 1.2.1.

3.2 A technical generalization

The techniques used in the previous proof will be generalized but with some care in the estimates due to the non linearity we add to the dispersive term. This extra "complication" is crucial for our application to the Water Waves system.

Theorem 3.2.1. *Consider five real numbers $\alpha \in [0,2[$, $s \in]2 + \frac{d}{2}, +\infty[$ and $T > 0$ and $(\beta, k) \in \mathbb{R}^+$ verifying:*

$$\begin{cases} k \geq 1, \quad \beta \leq \alpha, \\ \beta < (k+1)\alpha - 2k + 1. \end{cases}$$

Consider a elliptic skew symmetric[5] *C^1 symbol $a : [0,T] \times H^s(\mathbb{D}^d) \to \Gamma_1^\alpha(\mathbb{D}^d)$,*

$$\textit{i.e such that } \operatorname{Re}(a_t) = \frac{a_t + a_t^\top}{2} \textit{ is bounded in } \Gamma_1^0(\mathbb{D}^d),$$

$$\exists C > 0, \forall (t,u,x) \in [0,T] \times H^s(\mathbb{D}^d) \times \mathbb{D}^d, \forall \xi, |\xi| \geq \frac{1}{2}, |a(t,u,x,\xi)| \geq C|\xi|^\alpha.$$

Moreover we suppose the following bounds on the nonlinearity in a:

$$\forall \mu \in \mathbb{R}, \forall g \in H^\mu(\mathbb{D}^d), \left\|T_{D_u a} g + \overline{T_{D_u a} g}\right\|_{H^\mu} \leq M_0^\beta(D_u a + \overline{D_u a}) \|g\|_{H^{\mu+\beta}}. \tag{3.2.1}$$

$$\forall (t,u), M_1^\alpha(a) \leq C(1 + \|u\|_{W^{1,\infty}}), \; M_0^\beta(D_u a + \overline{D_u a}(u,\cdot)) \leq C\|u\|_{L^\infty}^{k-1}. \tag{3.2.2}$$

Consider a C^1 function $V(t,x,u) : [0,T] \times \mathbb{D}^d \times H^s(\mathbb{D}^d) \to H^s(\mathbb{D}^d;\mathbb{R}^d)$ and a function $F \in L^\infty\big([0,T], W^{1,\infty}\big(H^s(\mathbb{D}^d), H^s(\mathbb{D}^d)\big)\big)$.

[5] Recall the notation $a^\top$ for the adjoint of an operator a.

*Suppose that the following hypothesis **H1** is verified, there exists $\omega \in C_c^\infty(\mathbb{D}^d)$ supported in $B(0,1)$ such that*

$$\forall (t,x) \in [0,T] \times \operatorname{supp} \omega, C_x^{-1} t \leq \left| \int_0^t D_u V(s,x,0)[\omega(x)] ds \right| \leq C_x t. \tag{H1}$$

for a constant $C_x > 0$ when $\omega(x) \neq 0$.

Fix $u_0 \in H^s(\mathbb{D}^d)$ and take $r > 0$, then there exists a constant $C_s > 0$ such that for

$$T' < \frac{C_s}{r + M_0^\alpha((a, D_x a)(0,\cdot,u_0)) + M_0^\beta(D_u + \overline{D_u} a(0,\cdot,u_0)) + \|(V, D_{x,u}V)(0,\cdot,u_0\|_{L^\infty} + \|\partial_x u_0\|_{L^\infty}} \tag{3.2.3}$$

and $T' < T$ then for all v_0 in the ball $B(u_0, r) \subset H^s(\mathbb{D}^d)$ the Cauchy problem:

$$\begin{cases} \partial_t v + T_{V(t,x,v)} \cdot \nabla v + T_{a(t,v)} v = F(t,v) \\ v(0,\cdot) = v_0(\cdot), \end{cases} \tag{3.2.4}$$

has a unique solution $v \in C([0,T'], H^s(\mathbb{D}^d))$. Moreover $\forall R > 0$, the flow map:

$$\begin{aligned} B(0,R) &\to C([0,T_R], H^s(\mathbb{D}^d)) \\ v_0 &\mapsto v \end{aligned}$$

is not uniformly continuous, where T_R is chosen sufficiently small for the flow map to be defined on $C([0,T_R], H^s(\mathbb{D}^d)$ uniformly on $B(0,R)$.

Considering a weaker control norm we get, for all $\epsilon > 0$ the flow map:

$$\begin{aligned} B(0,R) &\to C([0,T_R], H^{s-1+(\alpha-1)^+ + \epsilon}(\mathbb{D}^d)) \\ v_0 &\mapsto v \end{aligned}$$

is not C^1.

In the proof of quasi-linearity of the water waves systems we will need the following slight generalization to systems given by the following corollary.

Corollary 3.2.1. *Consider a positive integer $n \geq 1$ and five real numbers $\alpha \in [0,2[$, $s \in]2 + \frac{d}{2}, +\infty[$, $T > 0$ and $(\beta, k) \in \mathbb{R}^+$ verifying:*

$$\begin{cases} k \geq 1, \quad \beta \leq \alpha, \\ \beta < (k+1)\alpha - 2k + 1. \end{cases}$$

Consider a C^1 skew symmetric elliptic symbol $a : [0,T] \times H^s(\mathbb{D}^d; \mathbb{R}^n) \to \Gamma_1^\alpha(\mathbb{D}^d; M_n(\mathbb{R}))$.

Moreover we suppose the following bounds on the nonlinearity in a:

$$\forall \mu \in \mathbb{R}, \forall g \in H^\mu(\mathbb{D}^d), \left\| T_{D_u a} g + \overline{T_{D_u a} g} \right\|_{H^\mu} \leq M_0^\beta(D_u a + \overline{D_u a}) \|g\|_{H^{\mu+\beta}}. \tag{3.2.5}$$

$$\forall (t,u), M_1^\alpha(a) \leq C(1 + \|u\|_{W^{1,\infty}}), \ M_0^\beta(D_u a + \overline{D_u a}(u,\cdot)) \leq C \|u\|_{L^\infty}^{k-1}. \tag{3.2.6}$$

Consider a C^1 function $V(t,x,u) : [0,T] \times \mathbb{D}^d \times H^s(\mathbb{D}^d; \mathbb{C}^n) \to H^s(\mathbb{D}^d; \mathbb{R}^n)$ and a function $F \in L^\infty\big([0,T], W^{1,\infty}\big(H^s(\mathbb{D}^d; \mathbb{R}^n), H^s(\mathbb{D}^d; \mathbb{R}^n)\big)\big)$.

Suppose that the following hypothesis ***H1*** *is verified, there exists* $\omega \in C_c^\infty(\mathbb{D}^d;\mathbb{R}^n)$ *supported in* $B(0,1)$ *such that*

$$\forall (t,x) \in [0,T] \times \operatorname{supp}\omega, C_x^{-1}t \leq \left| \int_0^t D_u V(s,x,0)[\omega(x)]ds \right| \leq C_x t, \qquad (\mathbf{H1})$$

for a constant $C_x > 0$ *when* $\omega(x) \neq 0$.

Fix $u_0 \in H^s(\mathbb{D}^d;\mathbb{R}^n)$ *and take* $r > 0$, *then there exists* $T' > 0$ *verifying analogous conditions as in Theorem 3.2.2 such that for all* v_0 *in the ball* $B(u_0,r) \subset H^s(\mathbb{D}^d;\mathbb{R}^n)$ *the Cauchy problem:*

$$\begin{cases} \forall i \in [1,..,n],\ \partial_t v_i + V(t,x,v)\cdot\nabla v_i + (T_{a(t,v)}v)_i = F_i(t,v), \\ v(0,\cdot) = v_0(\cdot), \end{cases}$$

has a unique solution $v \in C([0,T'],H^s(\mathbb{D}^d;\mathbb{R}^n))$. *Moreover* $\forall R > 0$, *the flow map:*

$$\begin{aligned} B(0,R) &\to C([0,T_R],H^s(\mathbb{D}^d;\mathbb{R}^n)) \\ v_0 &\mapsto v \end{aligned}$$

is not uniformly continuous, where T_R *is chosen sufficiently small for the flow map to be defined on* $C([0,T_R],H^s(\mathbb{D}^d)$ *uniformly on* $B(0,R)$.

Considering a weaker control norm we get, for all $\epsilon' > 0$ *the flow map:*

$$\begin{aligned} B(0,R) &\to C([0,T_R],H^{s-1+(\alpha-1)^+ +\epsilon'}(\mathbb{D}^d;\mathbb{R}^n)) \\ v_0 &\mapsto v \end{aligned}$$

is not C^1.

3.2.1 Prerequisites on the Cauchy problem

We consider the Cauchy problem associated to Theorem 3.2.1:

$$\begin{cases} \partial_t u + T_{V(t,x,u)}\cdot\nabla u + T_a u = F(t,u) \\ u(0,\cdot) = u_0(\cdot) \in H^s(\mathbb{D}^d),\ \ s > 1 + \frac{d}{2}, \end{cases} \tag{3.2.7}$$

Theorem 3.2.2. *Consider* $0 \leq \alpha < 2$, $T > 0$, *an elliptic* C^1 *symbol* $a : [0,T] \times H^s(\mathbb{D}^d) \to \Gamma_1^\alpha(\mathbb{D}^d)$ *skew symmetric*

$$\textit{i.e such that}\ \ \operatorname{Re}(a_t) = \frac{a_t + a_t^\top}{2}\ \textit{is bounded in}\ \Gamma_1^0(\mathbb{D}^d).$$

Moreover we suppose the following bounds on the nonlinearity in a:

$$\forall \mu \in \mathbb{R}, \forall g \in H^\mu(\mathbb{D}^d), \left\| T_{D_u a}g + \overline{T_{D_u a}g} \right\|_{H^\mu} \leq M_0^\beta(D_u a + \overline{D_u a}) \|g\|_{H^{\mu+\beta}}. \tag{3.2.8}$$

Consider a C^1 *function* $V(t,x,u) : [0,T] \times \mathbb{D}^d \times H^s(\mathbb{D}^d) \to H^s(\mathbb{D}^d;\mathbb{R}^d)$ *and a function* $F \in L^\infty\big([0,T], W^{1,\infty}\big(H^s(\mathbb{D}^d), H^s(\mathbb{D}^d)\big)\big)$.

Consider $s > 1 + \dfrac{d}{2}$, $r > 0$ *and* $u_0 \in H^s(\mathbb{D}^d)$ *such that for all* $v_0 \in B(u_0,r)$, *and* T' *verifying* (3.2.3), *the problem* (3.2.7) *with initial data* v_0 *has a unique solution* $v \in C^0([0,T'],H^s(\mathbb{D}^d))$ *and the map* $v_0 \mapsto v$ *is continuous from* $B(u_0,r)$ *to* $C^0([0,T'],H^s(\mathbb{D}^d))$. *Moreover we have the estimates:*

$$\forall 0 \leq \mu \leq s, \forall t \in [0,T'], \|v(t)\|_{H^\mu(\mathbb{D}^d)} \leq e^{C_\mu(a,V)\|\partial_x v\|_{L^1([0,T'],L^\infty(\mathbb{D}^d))}} \|v_0\|_{H^\mu(\mathbb{D}^d)}. \tag{3.2.9}$$

where $C_\mu(a,V)$ *verifies:*

$$C_\mu(a,V) \le C_\mu\big(M_0^\alpha((a,D_xa)(0,\cdot,u_0)) + M_0^\beta(D_u+\overline{D_u}a(0,\cdot,u_0)) \\ + \|(V,D_{x,u}V)(0,\cdot,u_0\|_{L^\infty} + \|\partial_x u_0\|_{L^\infty}\big).$$

Taking two different solutions v, v'*, assuming moreover* $v_0 \in H^{s+\beta}(\mathbb{D}^d)$ *then we have for* $t \in [0,T']$*:*

$$\left\|(v-v')(t)\right\|_{H^s(\mathbb{D}^d)} \le e^{C_s(a,V)(\|\partial_x(v,v')\|_{L^1([0,T'],L^\infty(\mathbb{D}^d))}+t\|v_0\|_{H^{s+\beta}(\mathbb{D}^d)})}\left\|v_0-v_0'\right\|_{H^s(\mathbb{D}^d)}. \tag{3.2.10}$$

We will also work with linear hyperbolic paradifferential equations and we summarize the properties needed in the following Theorem:

Theorem 3.2.3. *Consider* $(a_t)_{t\in\mathbb{R}}$ *a family of symbols in* $\Gamma_1^\beta(\mathbb{D}^d)$ *with* $\beta \in \mathbb{R}$*, such that* $t \mapsto a_t$ *is continuous and bounded from* $\mathbb{R}$ *to* $\Gamma_1^\beta(\mathbb{D}^d)$ *and such that* $\mathrm{Re}(a_t) = \frac{a_t+a_t^\top}{2}$ *is bounded in* $\Gamma_1^0(\mathbb{D}^d)$*, and take* $T>0$*. Then for all initial data* $u_0 \in H^s(\mathbb{D}^d)$*, and* $f \in C^0([0,T];H^s(\mathbb{D}^d))$ *the Cauchy problem:*

$$\begin{cases} \partial_t u + T_a u = f \\ \forall x \in \mathbb{D}^d, u(0,x) = u_0(x) \end{cases} \tag{3.2.11}$$

has a unique solution $u \in C^0([0,T];H^s(\mathbb{D}^d)) \cap C^1([0,T];H^{s-\beta}(\mathbb{D}^d))$ *which verifies the estimates:*

$$\|u(t)\|_{H^s(\mathbb{D}^d)} \le e^{Ct}\|u_0\|_{H^s(\mathbb{D}^d)} + 2\int_0^t e^{C(t-t')}\left\|f(t')\right\|_{H^s(\mathbb{D}^d)}dt',$$

where C *depends on a finite symbol semi-norm* $M_1^0(\mathrm{Re}(a_t))$*.*

Remark 3.2.1. *We will also need to remark that fixing the initial data at 0 is an arbitrary choice. More precisely,* $\forall 0 \le t_0 \le T$ *and all data* $u_0 \in H^s(\mathbb{D}^d)$ *the Cauchy problem:*

$$\begin{cases} \partial_t u + T_a u = f \\ \forall x \in \mathbb{D}^d, u(t_0,x) = u_0(x) \end{cases} \tag{3.2.12}$$

has a unique solution $u \in C^0([0,T];H^s(\mathbb{D}^d)) \cap C^1([0,T];H^{s-\beta}(\mathbb{D}^d))$ *which verifies the estimate:*

$$\|u(t)\|_{H^s(\mathbb{D}^d)} \le e^{C|t-t_0|}\|u_0\|_{H^s(\mathbb{D}^d)} + 2\left|\int_{t_0}^t e^{C(t-t')}\left\|f(t')\right\|_{H^s(\mathbb{D}^d)}dt'\right|.$$

3.2.2 Proof of Theorem 3.2.1

As for Theorem 1.2.1, for the proof we will show that there exists a positive constant C and two sequences $(u_{\epsilon,\tau}^\lambda)$ and $(v_{\epsilon,\tau}^\lambda)$ solutions of (3.2.4) on $[0,1]$ such that for every $t \in [0,1]$,

$$\sup_{\lambda,\epsilon,\tau}\left\|u_{\epsilon,\tau}^\lambda\right\|_{L^\infty([0,1],H^s(\mathbb{D}^d))} + \left\|v_{\epsilon,\tau}^\lambda\right\|_{L^\infty([0,1],H^s(\mathbb{D}^d))} \le C,$$

$(u^\lambda_{\epsilon,\tau})$ and $(v^\lambda_{\epsilon,\tau})$ satisfy initially

$$\lim_{\substack{\lambda\to+\infty \\ \epsilon,\tau\to 0}} \left\| u^\lambda_{\epsilon,\tau}(0,\cdot) - v^\lambda_{\epsilon,\tau}(0,\cdot) \right\|_{H^s(\mathbb{D}^d)} = 0,$$

but,

$$\liminf_{\substack{\lambda\to+\infty \\ \epsilon,\tau\to 0}} \left\| u^\lambda_{\epsilon,\tau} - v^\lambda_{\epsilon,\tau} \right\|_{L^\infty([0,1],H^s(\mathbb{D}^d))} \geq c > 0.$$

Considering a weaker control norm we want to get, for all $\epsilon' > 0$,

$$\liminf_{\substack{\lambda\to+\infty \\ \epsilon,\tau\to 0}} \frac{\left\| u^\lambda_{\epsilon,\tau} - v^\lambda_{\epsilon,\tau} \right\|_{L^\infty([0,1],H^{s-1+(\alpha-1)^+ + \epsilon'}(\mathbb{D}^d))}}{\left\| u^\lambda_{\epsilon,\tau}(0,\cdot) - v^\lambda_{\epsilon,\tau}(0,\cdot) \right\|_{H^s(\mathbb{D}^d)}} = +\infty.$$

3.2.2.1 Definition of the Ansatz

Let (λ, ϵ) be two positive real sequences such that

$$\lambda \to +\infty, \;\; \epsilon \to 0, \;\; \lambda\epsilon \to +\infty, \tag{3.2.13}$$

and put

- on $\mathbb{R}^d$,
$$u^0(x) = \lambda^{\frac{d}{2}-s}\omega(\lambda x), \;\; v^0(x) = u^0(x) + \epsilon\omega(x),$$
- on $\mathbb{T}^d$, u^0 and v^0 as the periodic extensions of the functions defined above.

Take $t_0 > 0$ smaller than a harmless constant which will be fixed later, and (τ), $0 < \tau \leq t_0$.
Now let l be the solutions to the Cauchy problem on $[0, t_0]$:

$$\begin{cases} \partial_t l + T_{a(t,l)} l = F(t,l) \\ \forall x \in \mathbb{D}^d, l(\tau,x) = u^0(x). \end{cases} \tag{3.2.14}$$

Put $u^1(x) = l(0,x)$ and define l' to be the solutions to the Cauchy problem on $[0, t_0]$:

$$\begin{cases} \partial_t l' + T_{a(t,l)} l' = F(t,l') \\ \forall x \in \mathbb{D}^d, l(\tau,x) = v^0(x). \end{cases} \tag{3.2.15}$$

and put $v^1 = l(0,x)$.

Remark 3.2.2. *It's important to notice that we use the same term $T_{a(t,l)}$ in (3.2.14) and (3.2.15) and thus (l,l') have Lipschitz dependence on the data (u^0, v^0).*

Define u and v as the solution given by Theorem 3.2.2 with initial data u^1 and v^1 on the intervals $[0,T]$, $[0,T']$. Taking $0 < \delta < s - 1 - \frac{d}{2}$, u^0 and v^0 are uniformly bounded in $H^{1+\frac{d}{2}+\delta}(\mathbb{D}^d)$ and thus by Theorem 3.2.3, u^1 and v^1 are also uniformly bounded in $H^{1+\frac{d}{2}+\delta}(\mathbb{D}^d)$ and thus by the Sobolev injection Theorems they are bounded in $\dot{W}^{1,\infty}(\mathbb{D}^d)$. Thus we can take a uniform $0 < T$ on which all the solutions are well defined and we take $0 < t_0 \leq T$.

3.2.2.2 Change of variables by transport

Put

$$\begin{cases} \frac{d}{dt}\chi(t,s,x) = V\big(t,\chi(t,s,x),u(t,\chi(t,s,x))\big), \\ \chi(s,s,x) = x, \end{cases}$$

and define analogously $\tilde{\chi}$ from v. We recall that from the Cauchy-Lipschitz Theorem as u^0 and v^0 are $H^{+\infty}(\mathbb{D}^d)$ functions, then u^1, v^1 are $H^{+\infty}$ and u and v are $H^{+\infty}(\mathbb{D}^d)$ with respect to the x variable thus $\chi, \tilde{\chi} \in C^1([0,T]^2, C^\infty(\mathbb{D}^d))$. And they are both diffeomorphisms in the x variable.

By the estimate (3.1.2) u and v are uniformly bounded in $\dot{W}^{1,\infty}(\mathbb{D}^d)$ say by $M > 0$ and their Sobolev norms are dominated by those of u^1 and v^1 thus by those of u^0 and v^0 by Theorem 3.2.3. By classic manipulations of ODEs we get the estimates:

$$\begin{cases} \exists C>0, \forall (t',t) \in [0,t_0], \forall x \in \mathbb{D}^d, C^{-1} \leq |D\chi(t,t',x)| \leq C \\ \forall 2 \leq k < \lfloor s - \frac{d}{2} \rfloor, \left\| D^k\chi(t,t',x) \right\|_{L^\infty} \leq C \left\| u \right\|_{W^{k,\infty}} \end{cases} \tag{3.2.16}$$

Analogous estimates hold for $\tilde{\chi}$ using v.
Now we compute the analogue of the classic transport computation but with the paracomposition operator which reads:

$$\begin{aligned} \partial_t(\chi(t,0,x)^*u(t,x)) &= \chi(t,0,x)^*\partial_t u + T_{\partial_t\chi(t,0,x)} \cdot \chi(t,0,x)^*\nabla u(t,x) + R(t,u) \\ &= -\chi(t,0,x)^*(T_{a(t,u)}u)(t,x) + \chi(t,0,x)^*F(t,u) + R(t,u) \\ &= -T_{a(t,u)^*}\chi(t,0,x)^*u(t,x) + \chi(t,0,x)^*F(t,u) + R(t,u) + R'(t,u), \\ \chi(0,0,x)^*u(0,x) &= u(0,x) = u^1(x). \end{aligned}$$

where $(\cdot)^*$ is the change of variables by $\chi(t,0,x)$ as presented in Theorem 2.4.2. We can assemble the terms R, R' and F in a new term F' verifying the same hypothesis as F, thus without loss of generality henceforth we will keep the generic notation F for all the terms verifying the same hypothesis.
Thus if we put f the solution to the Cauchy problem, which is well posed by Appendix 3.5:

$$\begin{cases} \partial_t f + T_{a(t,u)^*} f = \chi(t,0,x)^*F(t,u) \\ \forall x \in \mathbb{D}^d, f(0,x) = u^1(x) \end{cases} \tag{3.2.17}$$

we get:

$$\chi(t,0,x)^*u(t,x) = f(t,x). \tag{3.2.18}$$

Analogously, if we put g the solution to the well posed Cauchy problem,

$$\begin{cases} \partial_t g + T_{\widetilde{a(t,v)}^*} g = \tilde{\chi}(t,0,x)^*F(t,v) \\ \forall x \in \mathbb{D}^d, g(0,x) = v^1(x) \end{cases} \tag{3.2.19}$$

where $\widetilde{(\cdot)}^*$ is the change of variables by $\tilde{\chi}(t,0,x)$, we get

$$\tilde{\chi}(t,0,x)^*v(t,x) = g(t,x). \tag{3.2.20}$$

Returning to the ODEs defining χ and $\tilde{\chi}$ we get:

$$\begin{cases} \chi(t,t',x) = x + \int_{t'}^{t} V(s,\chi(s,t',x), f(s,x)))ds, \\ \tilde{\chi}(t,t',x) = x + \int_{t'}^{t} V(s,\tilde{\chi}(s,t',x), g(s,x))ds. \end{cases} \tag{3.2.21}$$

Proposition 3.2.1. *There exists $C > 0$ independent of $(\tau, \epsilon, \lambda)$ such that:* $\forall h \in H^s(\mathbb{D}^d), \forall (t,t') \le t_0$,

$$C^{-1}\|h\|_{H^s} \le \left\|h \circ \chi(t,t',x)\right\|_{H^s} \le C\|h\|_{H^s},$$

$$C^{-1}\|h\|_{H^s} \le \left\|h \circ \tilde{\chi}(t,t',x)\right\|_{H^s} \le C\|h\|_{H^s}.$$

Proof. We will start by proving the upper bound for the estimate on the composition with χ. As u is bounded in $(\tau, \epsilon, \lambda)$ on $C([0,T], H^s(\mathbb{D}^d))$ then there exists a unique solution $H \in C([0,T], H^s(\mathbb{D}))$ to

$$\begin{cases} \partial_s H(s,x) + V(s,x,u)\cdot\nabla H(s,x) = 0 \\ H(t,x) = h(x) \end{cases}$$

and H is bounded in $(\tau, \epsilon, \lambda)$ on $C([0,T], H^s(\mathbb{D}^d))$. The desired bounds come from the fact that we have the explicit formula for H:

$$H(t',x) = h \circ \chi(t,t',x).$$

Now to get the lower bound it suffices to write by the upper bound computations:

$$\begin{aligned} \|h\|_{H^s} &= \left\|h \circ \chi(t,t',x) \circ \chi(t',t,x)\right\|_{H^s} \\ &\le C\left\|h \circ \chi(t,t',x)\right\|_{H^s}. \end{aligned}$$

We get analogously the estimates on the composition with $\tilde{\chi}$. □

3.2.2.3 Key Lemma and proof of the Theorem

Lemma 3.2.1. *As $0 \le \alpha < 2$ we can find a sequence $(\tau, \epsilon, \lambda)$ such that for all $\epsilon' > 0$ sufficiently small:*

$$\begin{cases} \tau \to 0, \\ \tau = \lambda^{1-\alpha},\ \text{for } \alpha > 1, \\ \tau\epsilon^k\lambda^\beta \to 0, \end{cases} \qquad \begin{cases} \tau^2\epsilon\lambda^\alpha \to 0 \\ \epsilon^{-1}\lambda^{-1+(\alpha-1)^+-\epsilon'} \to +\infty, \\ \tau\lambda\epsilon \to +\infty. \end{cases} \tag{3.2.22}$$

Then there exists $c > 0$ such that:

1. $\forall (\tau, \epsilon, \lambda, \nu), \left\|u^0 \circ \chi(0,\tau,x) - u^0 \circ \tilde{\chi}(0,\tau,x)\right\|_{H^{s-\nu}} > c\lambda^{-\nu}$.
2. *For δ such that $0 < \delta < s - 1 - \frac{d}{2}$:*

$$u(\tau,x) - v(\tau,x) = u^0\circ\chi(0,\tau,x) - u^0\circ\tilde{\chi}(0,\tau,x) + O_{H^{s-\nu}}\big(\epsilon + [\tau^2\lambda^{\alpha-1} + \tau^2\epsilon\lambda^\alpha + \tau\epsilon^k\lambda^\beta]\lambda^{-\nu}\big).$$

We will now show that this Lemma implies the Theorem. We have by combining the estimates (1) and (2) for $\nu = s$:

$$\forall(\tau,\epsilon,\lambda), \|u(\tau,x)-v(\tau,x)\|_{H^s} > \frac{c}{2} > 0 \text{ thus } \sup_{\tau,\epsilon,\lambda}\|u(\tau,x)-v(\tau,x)\|_{H^s} > \frac{c}{2} > 0$$

Also by Theorem 3.2.3 and Remark 3.2.2:

$$\exists C>0, \left\|u^1(x)-v^1(x)\right\|_{H^s} \leq C\epsilon \text{ thus } \left\|u^1(x)-v^1(x)\right\|_{H^s} \to 0,$$

which gives the non uniform continuity in the desired norms. Now for the control in a weaker norm we write:

$$\frac{\|u(\tau,x)-v(\tau,x)\|_{H^{s-1+(\alpha-1)^+-\epsilon'}}}{\|u^1(x)-v^1(x)\|_{H^s}} \geq c\epsilon^{-1}\lambda^{-1+(\alpha-1)^+-\epsilon'} \to +\infty,$$

which gives the desired result.

3.2.2.4 Proof of point 1 of Lemma 3.2.1

We first prove that $\exists c>0$ such that

$$\left\|u^0 \circ \chi(0,\tau,x)\right\|_{H^s} > c\lambda^{-\nu},$$

indeed by Proposition 3.2.1 and change of variable:

$$\left\|u^0 \circ \chi(0,\tau,x)\right\|_{H^{s-\nu}} \geq C^{-1}\left\|u^0\right\|_{H^{s-\nu}} \geq C^{-1}\lambda^{-\nu}\|\omega\|_{H^{s-\nu}}. \tag{3.2.23}$$

Now we will show that $u^0 \circ \chi(0,\tau,x)$ and $u^0 \circ \tilde{\chi}(0,\tau,x)$ have disjoint supports which will suffice to conclude given (3.2.23). Put $y = \chi(0,\tau,x)$, thus $x = \chi(\tau,0,y)$. On the support of $u^0 \circ \chi(0,\tau,x)$ we have:

- If $\mathbb{D}^d = \mathbb{R}^d$, $\lambda|y| \leq 1$.
- If $\mathbb{D}^d = \mathbb{T}^d$, $\forall k\in\mathbb{N}, 2\pi k - 1 \leq \lambda|y| \leq 2\pi k + 1$.

We then compute by the Taylor formula:

$$\begin{aligned}\tilde{\chi}(0,\tau,x) &= \tilde{\chi}(0,\tau,\tilde{\chi}(\tau,0,y)) \\ &\quad + \int_0^1 D_y\tilde{\chi}(0,\tau,r\chi(\tau,0,y)+(1-r)\tilde{\chi}(\tau,0,y))dr[\chi(\tau,0,y)-\tilde{\chi}(\tau,0,y)] \\ &= y + \int_0^1 D_y\tilde{\chi}(0,\tau,r\chi(\tau,0,y)+(1-r)\tilde{\chi}(\tau,0,y))dr[\chi(\tau,0,y)-\tilde{\chi}(\tau,0,y)].\end{aligned} \tag{3.2.24}$$

First,

$$\begin{aligned}&D_y\tilde{\chi}(0,\tau,r\chi(\tau,0,y)+(1-r)\tilde{\chi}(\tau,0,y)) \\ &= Id + \int_0^\tau D_y[V(t,\tilde{\chi}(0,\tau,r\chi(\tau,0,y)+(1-r)\tilde{\chi}(\tau,0,y)),g(t,r\chi(\tau,0,y)+(1-r)\tilde{\chi}(\tau,0,y)))]dt.\end{aligned}$$

Thus by estimates of Theorem 3.5.1taking $0 < \delta < s - \frac{d}{2} - 1$:

$$D_y\tilde{\chi}(0,\tau,r\chi(\tau,0,y)+(1-r)\tilde{\chi}(\tau,0,y)) = Id + O_{L^\infty}(\tau(\|v^1\|_{H^{1+\frac{d}{2}+\delta}} + \|u^1\|_{H^{1+\frac{d}{2}+\delta}}))$$
$$= Id + O_{L^\infty}(\tau),$$

which gives

$$\int_0^1 D_y\tilde{\chi}(0,\tau,r\chi(\tau,0,y)+(1-r)\tilde{\chi}(\tau,0,y))dr = Id + O_{L^\infty}(\tau). \tag{3.2.25}$$

Now we estimate $\chi(\tau,0,y)-\tilde{\chi}(\tau,0,y)$, by (3.2.21) :

$$\chi(\tau,0,y)-\tilde{\chi}(\tau,0,y) = \int_0^\tau V(t,\chi(t,0,y),f(t,y)) - V(t,\tilde{\chi}(t,0,y),g(t,y))dt \tag{3.2.26}$$

$$= \int_0^\tau\int_0^1 D_uV(t,\chi(t,0,y),rf(t,y)+(1-r)g(t,y))[f(t,y)-g(t,y)]dtdr$$
$$+ \int_0^\tau\int_0^1 D_xV(t,r\chi(t,0,y)+(1-r)\tilde{\chi}(t,0,y),g(t,y))[\chi(t,0,y)-\tilde{\chi}(t,0,y)]dtdr.$$

Taking $0 < \delta < s - \alpha - \frac{d}{2}$, by estimates of Theorem 3.5.1:

$$f(t,y) = f(0,y) + \int_0^t \partial_t f(r,y)dr$$
$$= u^1(y) + O_{L^\infty}(t(\|u^1\|_{H^{\frac{d}{2}+\alpha+\delta}})) = u^1(y) + O_{L^\infty}(t\epsilon).$$

Analogously we get:

$$g(t,y) = v^1(y) + O_{L^\infty}(t\epsilon).$$

Now $(u^1 - v^1)(y) = (l-l')(0,y)$ is the evaluation of the solution of the following Cauchy problem at $t = 0$:

$$\begin{cases} \partial_t(l-l') + T_{a(t,l)}(l-l') = F(t,l) - F(t,l') \\ \forall y \in \mathbb{D}^d, (l-l')(\tau,y) = -\epsilon\omega(y). \end{cases} \tag{3.2.27}$$

Thus by estimates of Theorem 3.5.1:

$$u^1(y) - v^1(y) = (l-l')(0,y) = -\epsilon\omega(y) + \int_\tau^0 \partial_t(l-l')(t,y)dt$$
$$= -\epsilon\omega(y) + O_{L^\infty}(\tau(\|v^1\|_{H^{\alpha+\frac{d}{2}+\delta}} + \|u^1\|_{H^{\alpha+\frac{d}{2}+\delta}}))$$
$$= -\epsilon\omega(y) + O_{L^\infty}(\tau\epsilon).$$

Thus,

$$\chi(\tau,0,y)-\tilde{\chi}(\tau,0,y)$$
$$= -\epsilon[\int_0^\tau\int_0^1 D_uV(t,\chi(t,0,y),rf(t,y)+(1-r)g(t,y))dtdr][\omega(y)] + O_{L^\infty}(\tau^2\epsilon)$$
$$+ \int_0^\tau\int_0^1 D_xV(t,r\chi(t,0,y)+(1-r)\tilde{\chi}(t,0,y),g(t,y))dr[\underbrace{\chi(t,0,y)-\tilde{\chi}(t,0,y)}_{*}]dt.$$

Iterating the computation in (*):

$$\begin{aligned}
&\chi(\tau,0,y)-\tilde{\chi}(\tau,0,y)\\
&=-\epsilon\Big[\int_0^\tau\int_0^1 D_uV(t,\chi(t,0,y),rf(t,y)+(1-r)g(t,y))dtdr\Big][\omega(y)]+O_{L^\infty}(\tau^2\epsilon)\\
&=-\epsilon\Big[\int_0^\tau D_uV(t,y,0)dt\Big][\omega(y)]+O_{L^\infty}(\tau^2\epsilon+\epsilon^2\tau).
\end{aligned}$$

and finally we get in (3.2.24),

$$\begin{aligned}
&\tilde{\chi}(\tau,0,x)-y\\
&=-\epsilon\Big[\int_0^\tau D_uV(t,y,0)dt\Big][\omega(y)]+O_{L^\infty}(\tau^2\epsilon+\epsilon^2\tau).
\end{aligned}$$

We get for $x\in$ supp $u^0\circ\chi(0,\tau,\cdot)$:

- For $\mathbb{D}^d=\mathbb{R}^d$:

$$\begin{aligned}
&\lambda\,|\tilde{\chi}(0,\tau,x)|\\
&\geq\epsilon\lambda\left|\int_0^\tau\int_0^1 D_uV(t,y,0)drdt[\omega(y)]\right|-1+o_{L^\infty}(\tau\lambda\epsilon)\\
&\geq 2,
\end{aligned}$$

  which gives the desired result.

- For $\mathbb{D}^d=\mathbb{T}^d$ given an adequate choice of τ,ϵ and λ we get:

$$2n\pi+1\leq\lambda\,|\tilde{\chi}(0,\tau,x)|\leq 2(n+1)\pi-1,$$

  which again gives the desired result.

3.2.2.5 Proof of point 2 of Lemma 3.2.1

We start by writing:

$$u(t,x)-v(t,x)=\chi(0,t,x)^*f(t,x)-\tilde{\chi}(0,t,x)^*g(t,x)+R(f)-R(g)$$

where R is a regularizing operator of order 2,

$$\begin{aligned}
u(t,x)-v(t,x)=&\underbrace{\chi(0,t,x)^*f(t,x)-\tilde{\chi}(0,t,x)^*f(t,x)}_{(1)}\\
&+\tilde{\chi}(0,t,x)^*(f-g)(t,x)+R(f)-R(g).
\end{aligned}$$

Term (1) resembles the main term in the usual transport estimates we used in point 1 of the Lemma [6] but with a main difference is f being some dispersed data and not compactly supported and the use of the paracomposition operator. Again, the

[6] Like the ones used in proving the quasi-linearity of the Burgers equation.

main trick here was to construct from u^0, v^0 the defocused data in the past u^1, v^1 and use this as the initial data for f and g.

$$\begin{aligned}
u(\tau,x)-v(\tau,x) &= u^0\circ\chi(0,\tau,x)-u^0\circ\tilde{\chi}(0,\tau,x)\\
&+T_{(u^0)'\circ\chi(0,\tau,x)}\chi(0,\tau,x)-T_{(u^0)'\circ\tilde{\chi}(0,\tau,x)}\tilde{\chi}(0,\tau,x)\\
&+\chi(0,\tau,x)^*(f-u^0)(\tau,x)-\tilde{\chi}(0,t,x)^*(f-u^0)(\tau,x)\\
&+\tilde{\chi}(0,\tau,x)^*(f-g)(\tau,x)+R(f)-R(g).\\
&=u^0\circ\chi(0,\tau,x)-u^0\circ\tilde{\chi}(0,\tau,x)\\
&+\underbrace{\chi(0,\tau,x)^*(f-l)(\tau,x)-\tilde{\chi}(0,t,x)^*(f-l)(\tau,x)}_{(1)}\\
&+\underbrace{\tilde{\chi}(0,\tau,x)^*(f-g)(\tau,x)}_{(2)}+R(f)-R(g)\\
&+\underbrace{T_{(u^0)'\circ\chi(0,\tau,x)}\chi(0,\tau,x)-T_{(u^0)'\circ\tilde{\chi}(0,\tau,x)}\tilde{\chi}(0,\tau,x)}_{(3)},
\end{aligned}$$

where l is defined by (3.2.14) and R was modified to contain other regularizing operators of order 2 that appear by symbolic calculus rules. The easiest part to estimate is the remainder one because of the gain of derivatives and Theorem (3.2.2):

$$\|R(f)-R(g)\|_{H^s}\leq C\epsilon.$$

We turn to estimating (1), by Theorem 2.4.2:

$$\|\chi(0,\tau,x)^*(f-l)(\tau,x)\|_{H^s}\leq C\,\|(f-l)(\tau,\cdot)\|_{H^s}.$$

Now $f-l$ solves:

$$\begin{cases}\partial_t(f-l)+T_{a(t,l)}(f-l)=(T_{a(t,l)}-T_{a(t,u)^*})f-F(t,l)+\chi(t,0,x)^*F(t,u)F(t,f)\\ \forall x\in\mathbb{D}^d,(f-l)(0,x)=0.\end{cases} \tag{3.2.28}$$

Writing

$$\chi(t,0,x)^*F(t,u)-F(t,l)=G_1(f-l)$$

where G_1 is a continuous linear operator on H^s we get the estimates by Theorem 3.5.1:

$$\begin{aligned}
&\|f-l(\tau,\cdot)\|_{H^\nu}\\
&\leq C[\left\|(T_{a(\tau,l)}-T_{a(t,l)^*})f\right\|_{L^1([0,\tau],H^\nu)}+\left\|(T_{a(\tau,l)^*}-T_{a(\tau,u)^*})f\right\|_{L^1([0,\tau],H^\nu)}]\\
&\leq C[\tau^2\left\|Id-D\chi^{-1}\right\|_{L^\infty}\|f\|_{H^{\nu+\alpha}}+\tau\|u-l\|_{L^\infty}\|f\|_{H^{\nu+\alpha}}].
\end{aligned}$$

As $s>2+\frac{d}{2}$,

$$\|f-l(\tau,\cdot)\|_{H^\nu}\leq C[\tau^2\lambda^{\alpha-1}+\tau^2\lambda^{\alpha-1}]\lambda^{\nu-s},$$

which gives

$$\|\chi(0,\tau,x)^*(f-l)(\tau,x)\|_{H^\nu}\leq C\tau\lambda^{\alpha-1}\lambda^{\nu-s},$$

and

$$\|\tilde{\chi}(0,\tau,x)^*(f-l)(\tau,x)\|_{H^\nu}\leq C\tau\lambda^{\alpha-1}\lambda^{\nu-s}.$$

Thus we finally get

$$\|(1)\|_{H^\nu} \leq C\tau\lambda^{\alpha-1}\lambda^{\nu-s}. \tag{3.2.29}$$

Now we estimate (2) and (3) in the same manner, by Theorem 2.4.2:

$$\|\tilde{\chi}(0,\tau,x)^*(f-g)(\tau,x)\|_{H^\nu} \leq C\,\|(f-g)(\tau,\cdot)\|_{H^\nu}\,,$$

And as $s > 2 + \frac{d}{2}$ and by (3.2.21):

$$\left\|T_{(u^0)'\circ\chi(0,\tau,x)}\chi(0,\tau,x) - T_{(u^0)'\circ\tilde{\chi}(0,\tau,x)}\tilde{\chi}(0,\tau,x)\right\| \leq C\,\|(f-g)(\tau,\cdot)\|_{H^\nu}\,.$$

Now $f-g$ solve:

$$\begin{cases} \partial_t(f-g) + T_{a(t,u)^*}(f-g) - (T_{a(t,v)^*} - T_{\widetilde{a(t,v)}^*})g \\ -\chi(t,0,x)^*F(t,u) + \tilde{\chi}(t,0,x)^*F(t,v) = (T_{a(t,u)^*} - T_{a(t,v)^*})g \\ \forall x \in \mathbb{D}^d, (f-g)(0,x) = (u^1-v^1)(x). \end{cases} \tag{3.2.30}$$

Here will need to be more careful as the nonlinearity in the dispersive term can be more "harmful" than the transport term when $\alpha \geq 1$, which was not there in the treatment of the model problem. More precisely we write:

$$\chi(t,0,x)^*F(t,u) - F(t,l) = G_2(f-g),$$

where G_2 is a continuous linear operator on H^s and we get by Theorem 3.2.2:

$$\begin{aligned} &\|f-g(\tau,\cdot)\|_{H^\nu} \\ &\leq C\left[\left\|(T_{a(t,v)^*} - T_{\widetilde{a(t,v)}^*})g\right\|_{L^1([0,\tau],H^\nu)} + \left\|(T_{a(t,u)^*} - T_{a(t,v)^*})g\right\|_{L^1([0,\tau],H^\nu)} + \epsilon\right] \\ &\leq C[\tau\left\|D\chi^{-1} - D\tilde{\chi}^{-1}\right\|_{L^\infty}\|g\|_{H^{\nu+\alpha}} + \tau\,\|f-g\|_{L^\infty}\|(f,g)\|^k_{L^\infty}\|g\|_{H^{\nu+\beta}} + \epsilon] \\ &\leq C[\tau^2\epsilon\lambda^\alpha\lambda^{\nu-s} + \tau\epsilon^k\lambda^\beta\lambda^{\nu-s} + \epsilon], \end{aligned}$$

which gives

$$\|(2)\|_{H^\nu} \leq C(\tau^2\epsilon\lambda^\alpha\lambda^{\nu-s} + \tau\epsilon^k\lambda^\beta\lambda^{\nu-s} + \epsilon), \tag{3.2.31}$$

and

$$\|(3)\|_{H^\nu} \leq C(\tau^2\epsilon\lambda^\alpha\lambda^{\nu-s} + \tau\epsilon^k\lambda^\beta\lambda^{\nu-s} + \epsilon), \tag{3.2.32}$$

finishing the proof of Lemma 3.2.1 and Theorem 3.2.1.

Remark 3.2.3. *For the application to the water waves system we need to remark that the restriction on β comes from the $T_{a(t,u)^*} - T_{a(t,v)^*}g$. Naively estimating this term we see that in the case of water waves with surface tension we are working in the limit case $\beta = \alpha = \frac{3}{2}$ and $k = 2$ which is barely missed by Theorem 3.2.1. We will will show that this can be avoided in the special case of water waves system by carefully choosing the ansatz. For this we notice that one slightly modify the last two estimates and we write:*

$$\|(2)\|_{H^\nu} \leq C(\tau^2\epsilon\lambda^\alpha\lambda^{\nu-s} + \tau^2\epsilon^k\lambda^\beta\lambda^{\nu-s} + \epsilon + \left\|(T_{a(t,u_0)^*} - T_{a(t,v_0)^*})g\right\|_{L^1([0,\tau],H^\nu)}), \tag{3.2.33}$$

and

$$\|(3)\|_{H^\nu} \leq C(\tau^2\epsilon\lambda^\alpha\lambda^{\nu-s} + \tau^2\epsilon^k\lambda^\beta\lambda^{\nu-s} + \epsilon + \left\|(T_{a(t,u_0)^*} - T_{a(t,v_0)^*})g\right\|_{L^1([0,\tau],H^\nu)}). \tag{3.2.34}$$

3.3 Quasi-linearity of the Water-Waves system with surface tension

In this section we always have $\kappa = 1$.

3.3.1 Prerequisites from the Cauchy problem

We start by recalling the apriori estimates given by Proposition 5.2 of [3]. We keep the notations of Theorem 1.3.1.

Proposition 3.3.1. *(From [3]) Let $d \geq 1$ be the dimension and consider a real number $s > 2 + \frac{d}{2}$. Then there exists a non decreasing function C such that, for all $T \in]0,1]$ and all solution (η, ψ) of (1.3.4) such that*

$$(\eta, \psi) \in C^0([0,T]; H^{s+\frac{1}{2}}(\mathbb{R}^d) \times H^s(\mathbb{R}^d)) \text{ and } H_t \text{ is verified for } t \in [0,T],$$

we have

$$\|(\eta, \psi)\|_{L^\infty(0,T;H^{s+\frac{1}{2}} \times H^s)} \leq C((\eta_0, \psi_0)_{H^{s+\frac{1}{2}} \times H^s}) + TC(\|(\eta, \psi)\|_{L^\infty(0,T;H^{s+\frac{1}{2}} \times H^s)}).$$

The proof will rely on the para-linearised and symmetrized version of (1.3.4) given by Proposition 4.8 and corollary 4.9 of [3]. Before we recall this, for clarity as in [3] we introduce a special class of operators $\Sigma^m \subset \Gamma_0^m$ given by:

Definition 3.3.1. *(From [3]) Given $m \in \mathbb{R}$, Σ^m denotes the class of symbols a of the form*

$$a = a^{(m)} + a^{(m-1)}$$

with

$$a^{(m)} = F(\nabla \eta(t,x), \xi)$$
$$a^{(m-1)} = \sum_{|k|=2} G_\alpha(\nabla \eta(t,x), \xi) \partial_x^k \eta(t,x),$$

such that

1. *T_a maps real valued functions to real-valued functions;*
2. *F is of class C^∞ real valued function of $(\zeta, \xi) \in \mathbb{R}^d \times (\mathbb{R}^d \setminus 0)$, homogeneous of order m in ξ; and such that there exists a continuous function $K = K(\zeta) > 0$ such that*
$$F(\zeta, \xi) \geq K(\zeta) |\xi|^m,$$
for all $(\zeta, \xi) \in \mathbb{R}^d \times (\mathbb{R}^d \setminus 0)$;
3. *G_α is a C^∞ complex valued function of $(\zeta, \xi) \in \mathbb{R}^d \times (\mathbb{R}^d \setminus 0)$, homogeneous of order $m-1$ in ξ.*

Σ^m enjoys all the usual symbolic calculus properties modulo acceptable remainders that we define by the following:

Definition-Notation 3.3.1. *(From [3]) Let $m \in \mathbb{R}$ and consider two families of operators of order m,*

$$\{A(t) : t \in [0,T]\}, \quad \{B(t) : t \in [0,T]\}.$$

We shall say that $A \sim B$ if $A - B$ is of order $m - \frac{3}{2}$ and satisfies the following estimate: for all $\mu \in \mathbb{R}$, there exists a continuous function C such that for all $t \in [0,T]$,

$$\|A(t) - B(t)\|_{H^\mu \to H^{\mu - m + \frac{3}{2}}} \le C(\|\eta(t)\|_{H^{s+\frac{1}{2}}}).$$

In the next Proposition we recall the different symbols that appear in the para-linearisation and symmetrization of the equations.

Proposition 3.3.2. *(From [3])We work under the hypothesis of Proposition 3.3.1. Put*

$$\lambda = \lambda^{(1)} + \lambda^{(0)}, \ l = l^{(2)} + l^{(1)} \ \textit{with},$$

$$\begin{cases} \lambda^{(1)} = \sqrt{(1 + |\nabla\eta|^2)\,|\xi|^2 - (\nabla\eta \cdot \xi)^2}, \\ \lambda^{(0)} = \frac{1+|\nabla\eta|^2}{2\lambda^{(1)}} \left\{ div\left(\alpha^{(1)}\nabla\eta\right) + i\partial_\xi \lambda^{(1)} \cdot \nabla\alpha^{(1)} \right\}, \\ \alpha^{(1)} = \frac{1}{\sqrt{1+|\nabla\eta|^2}} \left(\lambda^{(1)} + i\nabla\eta\cdot\xi\right). \end{cases} \tag{3.3.1}$$

$$\begin{cases} l^{(2)} = (1 + |\nabla\eta|^2)^{-\frac{1}{2}}\left(|\xi|^2 - \frac{(\nabla\eta\cdot\xi)^2}{1+|\nabla\eta|^2}\right), \\ l^{(1)} = -\frac{i}{2}(\partial_x \cdot \partial_\xi) l^{(2)}. \end{cases} \tag{3.3.2}$$

Now let $q \in \Sigma^0, p \in \Sigma^{\frac{1}{2}}, \gamma \in \Sigma^{\frac{3}{2}}$ be defined by

$$q = (1 + |\nabla\eta|^2)^{-\frac{1}{2}},$$
$$p = (1 + |\nabla\eta|^2)^{-\frac{5}{4}}\sqrt{\lambda^{(1)}} + p^{(-\frac{1}{2})},$$
$$\gamma = \sqrt{l^{(2)}\lambda^{(1)}} + \sqrt{\frac{l^{(2)}}{\lambda^{(1)}}}\frac{Re\lambda^{(0)}}{2} - \frac{i}{2}(\partial_\xi \cdot \partial_x)\sqrt{l^{(2)}\lambda^{(1)}},$$
$$p^{(-\frac{1}{2})} = \frac{1}{\gamma^{(\frac{3}{2})}}\left\{ q l^{(1)} - \gamma^{(\frac{1}{2})} p^{(\frac{1}{2})} + i\partial_\xi \gamma^{(\frac{3}{2})} \cdot \partial_x p^{(\frac{1}{2})} \right\}.$$

Then

$$T_q T_\lambda \sim T_\gamma T_q, \ T_q T_l \sim T_\gamma T_p, \ T_\gamma \sim (T_\gamma)^\top.$$

Now we can write the para-linearization and symmetrization of the equations (1.3.4) after a change of variable:

Corollary 3.3.1. *(From [3])Under the hypothesis of Proposition 3.3.1, introduce the unknowns*

$$U = \psi - T_B\eta^{7}, \ \Phi_1 = T_p\eta \ \textit{and} \ \Phi_2 = T_q U,$$

where we recall,

$$\begin{cases} B = (\partial_y\phi)_{|y=\eta} = \frac{\nabla\eta\cdot\nabla\psi + G(\eta)\psi}{1+|\nabla\eta|^2}, \\ V = (\nabla_x\phi)_{|y=\eta} = \nabla\psi - B\nabla\eta. \end{cases}$$

[7] U is commonly called the "good" unknown of Alinhac.

Then $\Phi_1, \Phi_2 \in C^0([0,T]; H^s(\mathbb{R}^d))$ *and*

$$\begin{cases} \partial_t \Phi_1 + T_V \cdot \nabla \Phi_1 - T_\gamma \Phi_2 = f_1, \\ \partial_t \Phi_2 + T_V \cdot \nabla \Phi_2 + T_\gamma \Phi_1 = f_2, \end{cases} \tag{3.3.3}$$

with $f_1, f_2 \in L^\infty(0,T; H^s(\mathbb{R}^d))$, *and* f_1, f_2 *have* C^1 *dependence on* (U, θ) *verifying:*

$$\|(f_1, f_2)\|_{L^\infty(0,T;H^s(\mathbb{R}^d))} \leq C(\|(\eta, \psi)\|_{L^\infty(0,T;H^{s+\frac{1}{2}} \times H^s(\mathbb{R}^d))}).$$

3.3.2 Proof of Theorem 1.3.1

Corollary 3.3.1 shows that the para-linearization and symmetrization of the equations (1.3.4) are of the form of the equations treated in Theorem 3.2.1. The goal of the proof is thus to mainly show that the previous change of unknowns preserves the quasi-linear structure of the equations. This we will be proved but with a slightly different change of unknowns that will satisfy the same type of equations.

3.3.2.1 Reducing the problem around 0

Fix $T > 0$, $r > 0$ as in the proof of Theorem 3.2.1, given the local nature of the result we see that we can work on balls with radius r small. Henceforth we will be working on $\mathrm{B}(0,r) \subset C^0([0,T]; H^{s+\frac{1}{2}}(\mathbb{R}^d) \times H^s(\mathbb{R}^d))$ and without loss of generality we suppose that H_t is always verified on $[0,T]$ on that set.

3.3.2.2 New change of unknowns

Lemma 3.3.1. *Under the hypothesis of Proposition 3.3.1, fix* $\epsilon > 0$ *and introduce the unknowns*

$$U = \psi - T_B \eta, \;\; \tilde{\Phi}_1 = [T_p + \epsilon(I - T_1)]\eta \text{ and } \tilde{\Phi}_2 = [T_q + \epsilon(I - T_1)]U.$$

Then $\tilde{\Phi}_1, \tilde{\Phi}_2 \in C^0([0,T]; H^s(\mathbb{R}^d))$ *and*

$$\begin{cases} \partial_t \tilde{\Phi}_1 + T_V \cdot \nabla \tilde{\Phi}_1 - T_\gamma \tilde{\Phi}_2 = \tilde{f}_1, \\ \partial_t \tilde{\Phi}_2 + T_V \cdot \nabla \tilde{\Phi}_2 + T_\gamma \tilde{\Phi}_1 = \tilde{f}_2, \end{cases} \tag{3.3.4}$$

with $\tilde{f}_1, \tilde{f}_2 \in L^\infty(0,T; H^s(\mathbb{R}^d))$, *and* $\tilde{f}_1, \tilde{f}_2$ *have* C^1 *dependence on* (U, θ) *verifying:*

$$\left\|(\tilde{f}_1, \tilde{f}_2)\right\|_{L^\infty(0,T;H^s(\mathbb{R}^d))} \leq C(\|(\eta, \psi)\|_{L^\infty(0,T;H^{s+\frac{1}{2}} \times H^s(\mathbb{R}^d))}).$$

Proof. The Lemma simply follows from the fact that $I - T_1$ is a regularizing operator. □

3.3.2.3 The new change of unknowns locally preserves the structure of the equations:

To apply Theorem 3.2.1 we simply note that $DV(0,0)(h,k) = \nabla h$. Thus proof of Theorem 1.3.1 in the threshold $s > 2 + \frac{d}{2}$ will then follow from Theorem 3.2.1 combined with Lemma 3.3.1 and the following Lemma.

Lemma 3.3.2. *Let $d \geq 1$ and $s > 2 + \frac{d}{2}$. There exists $r, \epsilon > 0$ such that:*

$$\begin{aligned}\tilde{\Phi} : B(0,r) &\to C^0([0,T]; H^s(\mathbb{R}^d))\\ (\eta,\psi) &\mapsto (\tilde{\Phi}_1, \tilde{\Phi}_2)\end{aligned}$$

is a C^∞ diffeomorphism upon it's image and $\tilde{\Phi}(0)$=0.

Proof.

$$\tilde{\Phi}(\eta,\psi) = \underbrace{\begin{pmatrix} T_p + \epsilon(I - T_1) & 0 \\ 0 & T_q + \epsilon(I - T_1)\end{pmatrix}}_{(1)} \underbrace{\begin{pmatrix} I & 0 \\ -T_B & I\end{pmatrix}}_{(2)} \begin{pmatrix}\eta \\ \psi\end{pmatrix}$$

(2) being clearly a diffeomorphism we will concentrate on (1).
First we see that for r small enough $T_q+\epsilon(I-T_1)$ is a perturbation of the $T_1+\epsilon(I-T_1)$, indeed by symbolic calculus rules:

$$\begin{aligned}\|T_q + \epsilon(I - T_1) - T_1 - \epsilon(I - T_1)\|_{\mathcal{L}(H^s)} &= \|T_q - T_1\|_{\mathcal{L}(H^s)}\\ &\leq M_0^0(q-1)\\ &\leq C(\|\eta\|_{W^{1,\infty}}) \|\eta\|_{W^{1,\infty}}\\ &\leq C(\|\eta\|_{H^s}) \|\eta\|_{H^s}\end{aligned}$$

which gives the desired result.
Now we turn to $T_p + \epsilon(I - T_I)$. First notice that for $\epsilon > 0$:

$$T_{|\xi|^{\frac{1}{2}}} + \epsilon(I - T_I) : C^0([0,T]; H^{s+\frac{1}{2}}(\mathbb{R}^d)) \to C^0([0,T]; H^s(\mathbb{R}^d))$$

is a C^∞ diffeomorphism. And now we see that $T_p + \epsilon(I - T_I)$ is a perturbation of $T_{|\xi|^{\frac{1}{2}}} + \epsilon(I - T_I)$ indeed by symbolic calculus rules:

$$\begin{aligned}\left\| T_p - T_{|\xi|^{\frac{1}{2}}} \right\|_{\mathcal{L}(H^{s+\frac{1}{2}}, H^s)} &\leq C(\|\eta\|_{W^{1,\infty}}) \|\eta\|_{W^{1,\infty}}\\ &\leq C(\|\eta\|_{H^s}) \|\eta\|_{H^s}\end{aligned}$$

$\square$

Now to conclude the proof of Theorem 1.3.1, we want to apply Corollary 3.2.1 but as remarked previously we find ourselves in the limit case $\beta = \alpha = \frac{3}{2}$ and $k = 2$ which is not apriori covered by the Corollary. The key observation is that we have:

$$V(\eta,\psi) = \nabla\psi - B\nabla\eta, \ \gamma = \gamma(\eta),$$

and that $V(0,\psi)$ and $|\xi|^{\frac{3}{2}} = \gamma(0)$ do verify the hypothesis of Corollary 3.2.1 which is sufficient in order to apply the Corollary by Remark 3.2.3. Thus by Lemma 3.3.2, the equations (3.3.4) verify the hypothesis of Corollary 3.2.1 in the threshold $s > 2 + \frac{d}{2}$ with the choice $\eta_0 = 0$ thus we have two sequences:

$$\begin{cases} \exists(\tilde{\Phi}_1^0, \tilde{\Phi}_2^0) \in C^0([0,T]; H^s(\mathbb{R}^d)) \ \text{ solution of } (3.3.4),\\ \exists(\tilde{\Phi}_1^1, \tilde{\Phi}_2^1) \in C^0([0,T]; H^s(\mathbb{R}^d)) \ \text{ solution of } (3.3.4),\end{cases}$$

such that $\exists c > 0$,

$$\begin{cases} \left\| (\tilde{\Phi}_1^0, \tilde{\Phi}_2^0)(0,\cdot) - (\tilde{\Phi}_1^1, \tilde{\Phi}_2^1)(0,\cdot) \right\|_{H^s} = \left\| (0, \tilde{\Phi}_2^0)(0,\cdot) - (0, \tilde{\Phi}_2^1)(0,\cdot) \right\|_{H^s} \to 0, \\ \left\| (\tilde{\Phi}_1^0, \tilde{\Phi}_2^0) - (\tilde{\Phi}_1^1, \tilde{\Phi}_2^1) \right\|_{L^\infty([0,T],H^s)} > c. \end{cases}$$

Now putting $(\eta^0, \psi^0) = \tilde{\Phi}^{-1}(\tilde{\Phi}_1^0, \tilde{\Phi}_2^0)$ and $(\eta^1, \psi^1) = \tilde{\Phi}^{-1}(\tilde{\Phi}_1^1, \tilde{\Phi}_2^1)$ we get from Lemmas 3.3.1 and 3.3.2:

$$\begin{cases} (\eta^0, \psi^0) \in C^0([0,T]; H^{s+\frac{1}{2}}(\mathbb{R}^d) \times H^s(\mathbb{R}^d)) \text{ is a solution of (1.3.4)}, \\ (\eta^1, \psi^1) \in C^0([0,T]; H^{s+\frac{1}{2}}(\mathbb{R}^d) \times H^s(\mathbb{R}^d)) \text{ is a solution of (1.3.4)}, \end{cases}$$

such that

$$\begin{cases} \left\| (\eta^0, \psi^0)(0,\cdot) - (\eta^1, \psi^1)(0,\cdot) \right\|_{H^{s+\frac{1}{2}} \times H^s} = \left\| (0, \psi^0)(0,\cdot) - (0, \psi^1)(0,\cdot) \right\|_{H^{s+\frac{1}{2}} \times H^s} \to 0, \\ \left\| (\eta^0, \psi^0) - (\eta^1, \psi^1) \right\|_{L^\infty([0,T],H^{s+\frac{1}{2}} \times H^s)} > c. \end{cases}$$

thus giving us the desired result. As the change of unknowns is a diffeomorphism (thus is Lipschitz) we get analogously the result on the control in weaker norms.

3.4 Quasi-Linearity of the Gravity Water Waves

In this section we always have $\kappa = 0$. The proof will follow as in the previous section but with some extra care, taking into account the lower regularity framework.

3.4.1 Prerequisites from the Cauchy problem

We start by recalling the apriori estimates given by Proposition 4.1 of [6], we keep the notations of Theorem 1.3.2.

Proposition 3.4.1. *(From [6]) Let $d \geq 1$ be the dimension and consider a real number $s > 1 + \frac{d}{2}$. Then there exists a non decreasing function C such that, for all $T \in]0,1]$ and all solution (η, ψ) of (1.3.4) such that:*

$$\begin{cases} (\eta, \psi) \in C^0([0,T]; H^{s+\frac{1}{2}}(\mathbb{R}^d) \times H^{s+\frac{1}{2}}(\mathbb{R}^d)), \\ H_t \text{ is verified for } t \in [0,T], \\ \exists c_0 > 0, \forall t \in [0,T], a(t,x) \geq c_0, \end{cases}$$

we have [8]

$$\begin{aligned} &\|(\eta, \psi, V, B)\|_{L^\infty([0,T];H^{s+\frac{1}{2}} \times H^{s+\frac{1}{2}} \times H^s \times H^s)} \\ &\leq C(\|(\eta_0, \psi_0, V_0, B_0)\|_{H^{s+\frac{1}{2}} \times H^{s+\frac{1}{2}} \times H^s \times H^s}) \\ &+ TC(\|(\eta, \psi, V, B)\|_{L^\infty(0,T;H^{s+\frac{1}{2}} \times H^{s+\frac{1}{2}} \times H^s \times H^s)}). \end{aligned}$$

[8] Recall B and V are defined by (1.3.6).

The proof will rely on the para-linearised and symmetrized version of (1.3.4) given by Proposition 4.8 and 4.10 of [6]. Given the low regularity threshold, η and thus Ω_t are in $W^{\frac{3}{2},\infty}(\mathbb{R}^d)$ for the gravity water waves by contrast to $W^{\frac{5}{2},\infty}(\mathbb{R}^d)$ frame work for the case with surface tension, the para-linearisation of (1.3.4) is done with the variables V and B. This will only add a technical level to our proof of quasi-linearity.

Proposition 3.4.2. *(From [6])Under the hypothesis of Proposition 3.4.1, suppose moreover that $\|(V_0, B_0)\|_{H^s \times H^s} < +\infty$ thus by Proposition (3.4.1)this regularity is propagated on $[0,T]$. Now introduce the unknowns*

$$\begin{cases} \zeta = \nabla\eta, \\ U = V + T_\zeta B, \end{cases} \quad \textit{where,} \quad \begin{cases} B = (\partial_y \phi)_{|y=\eta} = \frac{\nabla\eta\cdot\nabla\psi + G(\eta)\psi}{1+|\nabla\eta|^2}, \\ V = (\nabla_x \phi)_{|y=\eta} = \nabla\psi - B\nabla\eta. \end{cases}$$

Now define the symbols:

$$\begin{cases} \lambda = \sqrt{(1+|\nabla\eta|^2)\,|\xi|^2 - (\nabla\eta\cdot\xi)^2}, \\ \gamma = \sqrt{a\lambda}, \\ q = \sqrt{\frac{a}{\lambda}}. \end{cases}$$

Set $\theta = T_q\zeta$. Then $\theta, U \in C^0([0,T]; H^s(\mathbb{R}^d))$ and

$$\begin{cases} \partial_t U + T_V \cdot \nabla U + T_\gamma \theta = f_1, \\ \partial_t \theta + T_V \cdot \nabla\theta - T_\gamma U = f_2, \end{cases} \tag{3.4.1}$$

with $f_1, f_2 \in L^\infty(0,T; H^s(\mathbb{R}^d))$, and f_1, f_2 have C^1 dependence on (U,θ) verifying:

$$\|(f_1,f_2)\|_{L^\infty(0,T;H^s)} \leq C(\|(\eta,\psi,V,B)\|_{L^\infty(0,T;H^{s+\frac{1}{2}}\times H^{s+\frac{1}{2}}\times H^s\times H^s)})$$

3.4.2 Proof of Theorem 1.3.2

As in the proof of Theorem 1.3.1, Proposition 3.4.2 shows that the para-linearisation and symmetrisation of the Equations (1.3.4) are of the form of the equations treated in Theorem 3.2.1. Thus again, the goal of the proof is thus to mainly show that the previous change of unknowns preserves the quasi-linear structure of the equations. This we will be proved but with a slightly different change of unknowns that will satisfy the same type of equations but where we take into account the low frequencies. For concision we will omit the $(\mathbb{R}^d)$ when writing the functional spaces.

3.4.2.1 Reducing the problem around 0

Fix $T>0$, $r>0$ as in the proof of Theorem 3.2.1 and 1.3.1, given the local nature of the result we see that first we can work on balls centered at 0 with radius r small. Put

$$I_{s,T} = \left\{(\eta,\psi) \in C^0([0,T]; H^{s+\frac{1}{2}} \times H^{s+\frac{1}{2}}), (V,B) \in C^0([0,T]; H^s \times H^s), \exists c>0, a \geq c\right\},$$

$$I_{s,0} = \left\{(\eta_0,\psi_0) \in H^{s+\frac{1}{2}} \times H^{s+\frac{1}{2}}, (V_0,B_0) \in H^s \times H^s, \exists c>0, a \geq c\right\},$$

henceforth we will be working on $\mathrm{B}(0,r) \subset I_{s,T}$ and without loss of generality we suppose that H_t is always verified on $[0,T]$, on that set.

3.4.2.2 New change of unknowns

Lemma 3.4.1. *Consider $\epsilon > 0$ and $\omega \in C_0^\infty(\mathbb{R}^d)$ such that $\omega = 1$ on $B(0,1)$ and $\omega = 0$ on $\mathbb{R}^d \setminus B(0,2)$. Under the hypothesis of Proposition (3.4.2), introduce the unknowns*

$$\begin{cases} \tilde{\zeta} = (1-\omega(D))\nabla\eta, \\ \tilde{U} = (1-\omega(D))(V + T_\zeta B), \\ aux_1 = \omega(D)\psi, \\ aux_2 = \omega(D)\eta, \end{cases} \quad \textit{where,} \quad \begin{cases} B = (\partial_y\phi)_{|y=\eta} = \frac{\nabla\eta\cdot\nabla\psi + G(\eta)\psi}{1+|\nabla\eta|^2}, \\ V = (\nabla_x\phi)_{|y=\eta} = \nabla\psi - B\nabla\eta. \end{cases}$$

and set $\tilde{\theta} = T_q\tilde{\zeta} + \epsilon(I - T_1)$, where q is defined in Proposition (3.4.2).
Then $\tilde{\theta}, U, aux_1, aux_2 \in C^0([0,T]; H^s)$ and

$$\begin{cases} \partial_t\tilde{U} + T_V\cdot\nabla\tilde{U} + T_\gamma\tilde{\theta} = f_1', \\ \partial_t\tilde{\theta} + T_V\cdot\nabla\tilde{\theta} - T_\gamma\tilde{U} = f_2', \end{cases} \tag{3.4.2}$$

with $f_1', f_2' \in L^\infty(0,T;H^s)$, and f_1', f_2' have C^1 dependence on (U,θ) verifying:

$$\|(f_1', f_2')\|_{L^\infty(0,T;H^s)} \leq C(\|(\eta,\psi,V,B)\|_{L^\infty(0,T;H^{s+\frac{1}{2}}\times H^{s+\frac{1}{2}}\times H^s\times H^s)})$$

Proof. Again the lemma simply follows from the fact that $I - T_1$ and $\omega(D)$ are regularizing operators. ☐

3.4.2.3 Decomposing the change of variable:

Set

$$\begin{aligned} \Phi : I_{s,T} &\to C^0([0,T]; H^s) \\ (\eta,\psi) &\mapsto (\tilde{U}, \tilde{\theta}, aux_1, aux_2) \end{aligned} \qquad \begin{aligned} \Phi : I_{s,0} &\to H^s \\ (\eta,\psi) &\mapsto (\tilde{U}, \tilde{\theta}, aux_1, aux_2) \end{aligned}$$

The goal is to prove that Φ is locally invertible and then the proof will follow from Theorem 3.2.1.
We write $\Phi = \Phi_1 \circ \Phi_2$ with

$$\begin{aligned} \Phi_1 : I_{s,T} &\to C^0([0,T]; H^s \times H^{s-\frac{1}{2}} \times H^s \times H^s) \\ (\eta,\psi) &\mapsto (\tilde{U}, \tilde{\zeta}, aux_1, aux_2) \end{aligned}$$

and,

$$\begin{aligned} \Phi_2 : C^0([0,T]; H^s \times H^{s-\frac{1}{2}} \times H^s \times H^s) &\to C^0([0,T]; H^s) \\ (\tilde{U}, \tilde{\zeta}, aux_1, aux_2) &\mapsto (\tilde{U}, \tilde{\theta}, aux_1, aux_2) \end{aligned}$$

We define Φ_1 and Φ_2 analogously when Φ is defined on $I_{s,0}$.

Lemma 3.4.2. *There exists $r, r_1, \epsilon > 0$ such that:*

$$\Phi_1 : B(0,r) \cap I_{s,T} \to C^0([0,T]; H^s \times H^{s-\frac{1}{2}} \times H^s \times H^s)$$

is a C^∞ diffeomorphism upon it's image.

$$\Phi_2 : B(0,r_1) \cap C^0([0,T]; H^s \times H^{s-\frac{1}{2}} \times H^s \times H^s) \to C^0([0,T]; H^s)$$

is a C^∞ diffeomorphism upon it's image.
Analogous results hold when Φ is defined on $I_{s,0}$.

The proof of Theorem 1.3.2 follows as in the previous section from Corollary 3.2.1 and the previous Lemma combined with the fact that $\Phi_1(0) = 0$ thus we have

$$\mathrm{B}(0,r_1)\cap C^0([0,T];H^{+\infty}\times H^{+\infty}\times H^{+\infty}\times H^{+\infty}) \subset \Phi_1\Big(\mathrm{B}(0,r)\cap C^0([0,T];H^{s+\frac{1}{2}}\times H^{s+\frac{1}{2}})\Big).$$

Also $\Phi_2(0) = 0$ thus there exists r_2:

$$\mathrm{B}(0,r_2)\cap C^0([0,T];H^{+\infty}) \subset \Phi_2\Big(\mathrm{B}(0,r_1)\cap C^0([0,T];H^{+\infty}\times H^{+\infty}\times H^s\times H^s\Big).$$

We now turn to the proof of the lemma.

Proof. As all of the estimates used are pointwise in time thus the proof is the same for $I_{s,T}$ and $I_{s,0}$ and we only write the one for $I_{s,T}$. We start by Φ_1, first the part $\eta \mapsto (\tilde{\zeta}, aux_2)$ is invertible with inverse

$$\mathscr{F}[\Phi_1^{-1}(\tilde{\zeta},aux_2)](\xi) = \frac{1}{d}\sum_j (1-\omega(\xi))\frac{\mathscr{F}[\partial_i\tilde{\zeta}](\xi)}{i\xi_j} + \omega(\xi)\mathscr{F}[aux_2].$$

By the same argument $\psi \mapsto ((1-\omega(D))\nabla\psi, \omega(D)\psi)$ is invertible and we see that $(\tilde{U}, aux_1)$ is a perturbation of that map indeed:

$$\begin{aligned}\left\|((1-\omega(D))\nabla\psi,\omega(D)\psi)-(\tilde{U},aux_1)\right\|_{\mathcal{L}(H^{s+\frac{1}{2}},H^s)} &\leq C(\|B\|_{W^{\frac{1}{2},\infty}})\|\eta\|_{H^{s+\frac{1}{2}}}\\ &\leq C(\|(\eta,\psi)\|_{H^{s+\frac{1}{2}}})\|\eta\|_{H^{s+\frac{1}{2}}}\end{aligned}$$

thus for r small enough we get the desired result.
Now we turn to Φ_2. This operator is the identity on $\tilde{U}, aux_1, aux_2$ thus we only have to work on $\tilde{\theta}$. Put a_0 as the Taylor coefficient associated to the solution of the problem (0,0). Now notice that for $\epsilon > 0$:

$$T_{\sqrt{a_0}|\xi|^{-\frac{1}{2}}} + \epsilon(I - T_I) : C^0([0,T];H^{s-\frac{1}{2}}) \to C^0([0,T];H^s)$$

is a C^∞ diffeomorphism. And now we see that $T_q + \epsilon(I - T_1)$ is a perturbation of $T_{\sqrt{a_0}|\xi|^{-\frac{1}{2}}} + \epsilon(I - T_1)$ indeed by symbolic calculus rules:

$$\left\|T_q - T_{\sqrt{a_0}|\xi|^{-\frac{1}{2}}}\right\|_{\mathcal{L}(H^{s-\frac{1}{2}},H^s)} \leq C(\|\eta\|_{H^s})\|\eta\|_{H^s},$$

which gives the result by taking r small. $\square$

3.5 Appendix: Energy estimates and well-posedness of some pulled back hyperbolic equations

Theorem 3.5.1. *Let $T > 0$, $\chi \in W^{1,\infty}([0,T], W^{1,\infty}_{loc}(\mathbb{D}^d))$ with $D_x\chi \in L^{\infty}([0,T], L^\infty(\mathbb{D}^d))$ and consider $(a_t)_{t\in\mathbb{R}}$ a family of symbols in $\Gamma^\beta_1(\mathbb{D}^d)$ with $\beta \in \mathbb{R}$, such that $t \mapsto a_t$ is continuous and bounded from $\mathbb{R}$ to $\Gamma^\beta_1(\mathbb{D}^d)$ and such that $Re(a_t) = \frac{a_t + a_t^\top}{2}$ is bounded* *in* $\Gamma^0_1(\mathbb{D}^d)$. *Suppose moreover that* $\chi(t,\cdot)$ *is a diffeomorphism between open sets of* $\mathbb{D}^d$ *and that we have the bounds:*

$$\exists C>0, \forall t\leq T, \forall x,\ C^{-1}\leq |D_x\chi(t,x)|\leq C. \tag{3.5.1}$$

Put $(\cdot)^*$ *is the change of variables by* χ *as presented in Theorem 2.4.2.*
Then for all initial data $u_0\in H^s(\mathbb{D}^d)$ *and* $f\in C^0([0,T];H^s(\mathbb{D}^d))$ *the Cauchy problem:*

$$\begin{cases}\partial_t u + T_{a^*}u = f\\ \forall x\in\mathbb{D}^d, u(0,x)=u_0(x)\end{cases} \tag{3.5.2}$$

has a unique solution $u\in C^0([0,T];H^s(\mathbb{D}^d))\cap C^1([0,T];H^{s-\beta}(\mathbb{D}^d))$ *which verifies the estimates:*

$$\begin{aligned}\|u(t)\|_{H^s} &\leq e^{C(\|D_x\chi\|_{L^\infty L^\infty},\|D_x\chi^{-1}\|_{L^\infty L^\infty},M^0_0(Re(a)))t}\|u_0\|_{H^s} \\ &+2\int_0^t e^{C(\|D_x\chi\|_{L^\infty L^\infty},\|D_x\chi^{-1}\|_{L^\infty L^\infty},M^0_0(Re(a)))(t-t')}\left\|f(t')\right\|_{H^s}dt'.\end{aligned} \tag{3.5.3}$$

Again fixing the initial data at 0 is an arbitrary choice. More precisely, $\forall 0\leq t_0\leq T$ *and all data*
$u_0\in H^s(\mathbb{D}^d)$ *the Cauchy problem:*

$$\begin{cases}\partial_t u + T_{a^*}u = f\\ \forall x\in\mathbb{D}^d, u(t_0,x)=u_0(x)\end{cases} \tag{3.5.4}$$

has a unique solution $u\in C^0([0,T];H^s(\mathbb{D}^d))\cap C^1([0,T];H^{s-\beta}(\mathbb{D}^d))$ *which verifies the estimate:*

$$\begin{aligned}\|u(t)\|_{H^s} &\leq e^{C(\|D_x\chi\|_{L^\infty L^\infty},\|D_x\chi^{-1}\|_{L^\infty L^\infty},M^0_0(Re(a)))|t-t_0|}\|u_0\|_{H^s} \\ &+2\left|\int_{t_0}^t e^{C(\|D_x\chi\|_{L^\infty L^\infty},\|D_x\chi^{-1}\|_{L^\infty L^\infty},M^0_0(Re(a)))(t-t')}\left\|f(t')\right\|_{H^s}dt'\right|.\end{aligned}$$

Proof. The existence of a solution follows from standard compacities arguments after regularization given the apriori estimates (3.5.3). Also, the equation being linear those estimates give the uniqueness immediately. Thus we will only show the desired apriori estimates.
Put $\Gamma_s=\langle \mathrm{D}\rangle^s$, we will compute $\frac{d}{dt}({\Gamma_s}^*u,{\Gamma_s}^*u)_{L^2(\mathbb{D}^d,|D_x\chi(t,x)|dx)}$ in two different ways.

- **Method 1.** First notice that by Theorem 2.4.2

$${\Gamma_s}^*(x,\xi)\sim([D\chi^{-1}(t,\chi(t,x))]^t\xi)^s+R$$

  where R is of order $s-1$.
  Thus using the lower and upper bound on $|D\chi(t,x)|$ combined with upper bound on $\frac{d}{dt}|D\chi(t,x)|$ we have:

$$\begin{aligned}&C(\|D_x\chi^{-1}\|_{L^\infty L^\infty})\frac{d}{dt}[(\Gamma_s u,\Gamma_s u)_{L^2}]-C(\|D_x\chi\|_{L^\infty L^\infty})\|\Gamma_s u\|^2_{L^2}\\ &\leq\frac{d}{dt}({\Gamma_s}^*u,{\Gamma_s}^*u)_{L^2(|D_x\chi(t,x)|dx)}.\end{aligned}$$

- **Method 2.** Now we use the PDE,

$$\begin{aligned}
&\frac{d}{dt}({\Gamma_s}^* u, {\Gamma_s}^* u)_{L^2(|D_x\chi(t,x)|dx)} \\
&= 2Re((\partial_t {\Gamma_s}^* u, {\Gamma_s}^* u))_{L^2(|D\chi(t,x)|dx)}) + ({\Gamma_s}^* u, {\Gamma_s}^* u)_{L^2(\frac{d}{dt}|D\chi(t,x)|dx)} \\
&= -2Re(({\Gamma_s}^* T_{a^*} u, {\Gamma_s}^* u)_{L^2(|D\chi(t,x)|dx)}) + 2Re(({\Gamma_s}^* f, {\Gamma_s}^* u))_{L^2(|D\chi(t,x)|dx)} \\
&+ 2Re(([\partial_t {\Gamma_s}^*] u, {\Gamma_s}^* u)_{L^2(|D\chi(t,x)|dx)}) + ({\Gamma_s}^* u, {\Gamma_s}^* u)_{L^2(\frac{d}{dt}|D\chi(t,x)|dx)}
\end{aligned}$$

by change of variables,

$$\begin{aligned}
&\frac{d}{dt}({\Gamma_s}^* u, {\Gamma_s}^* u)_{L^2(|D_x\chi(t,x)|dx)} \\
&= -2Re(({\Gamma_s}^* T_{a^*} u \circ \chi^{-1}, {\Gamma_s}^* u \circ \chi^{-1})_{L^2}) + 2Re(({\Gamma_s}^* f, {\Gamma_s}^* u))_{L^2(|D\chi(t,x)|dx)} \\
&+ 2Re(([\partial_t {\Gamma_s}^*] u, {\Gamma_s}^* u)_{L^2(|D\chi(t,x)|dx)}) + ({\Gamma_s}^* u, {\Gamma_s}^* u)_{L^2(\frac{d}{dt}|D\chi(t,x)|dx)}.
\end{aligned}$$

Now notice that,

$$\operatorname{Re}\left(\int T_{D[{\Gamma_s}^* T_{a^*} u \overline{{\Gamma_s}^* u}]\circ\chi^{-1}} \chi^{-1} dx\right) = \int T_{D[\Gamma_s T_{\mathrm{Re}(a)} u \overline{\Gamma_s u}]\circ\chi^{-1}} \chi^{-1} dx + R,$$

where R verifies by Theorem 2.4.2:

$$|R| \leq C(\left\|D\chi^{-1}\right\|_{L^\infty L^\infty}) \left\|\Gamma_s u\right\|_{L^2}^2. \tag{3.5.5}$$

Thus by Theorem 2.4.2:

$$\begin{aligned}
&\frac{d}{dt}({\Gamma_s}^* u, {\Gamma_s}^* u)_{L^2(|D_x\chi(t,x)|dx)} \qquad (3.5.6)\\
&= -2(\Gamma_s T_{Re(a)}[(\chi^{-1})^* u], \Gamma_s[(\chi^{-1})^* u])_{L^2} + \operatorname{Re}\left(\int T_{D[{\Gamma_s}^* T_{a^*} u \overline{{\Gamma_s}^* u}]\circ\chi^{-1}} \chi^{-1} dx\right) + R \\
&+ 2Re(({\Gamma_s}^* f, {\Gamma_s}^* u)_{L^2(|D\chi(t,x)|dx)}) + 2Re(([\partial_t {\Gamma_s}^*] u, {\Gamma_s}^* u)_{L^2(|D\chi(t,x)|dx)}) \\
&+ ({\Gamma_s}^* u, {\Gamma_s}^* u)_{L^2(\frac{d}{dt}|D\chi(t,x)|dx)}, \\
&= -2(\Gamma_s T_{Re(a)}[(\chi^{-1})^* u], \Gamma_s[(\chi^{-1})^* u])_{L^2} + \int T_{D[\Gamma_s T_{\mathrm{Re}(a)} u \overline{\Gamma_s u}]\circ\chi^{-1}} \chi^{-1} dx + R \\
&+ 2Re(({\Gamma_s}^* f, {\Gamma_s}^* u)_{L^2(|D\chi(t,x)|dx)}) + 2Re(([\partial_t {\Gamma_s}^*] u, {\Gamma_s}^* u)_{L^2(|D\chi(t,x)|dx)}) \\
&+ ({\Gamma_s}^* u, {\Gamma_s}^* u)_{L^2(\frac{d}{dt}|D\chi(t,x)|dx)}.
\end{aligned}$$

Now we have

$$\left|\int T_{D[\Gamma_s T_{\mathrm{Re}(a)} u \overline{\Gamma_s u}]\circ\chi^{-1}} \chi^{-1} dx\right| \leq C(\left\|D\chi^{-1}\right\|_{L^\infty}) \left\|\Gamma_s u\right\|_{L^2}^2. \tag{3.5.7}$$

By the upper bound on $\left|D\chi^{-1}(t,x)\right|$:

$$\begin{aligned}
&(\Gamma_s T_{Re(a)}[(\chi^{-1})^* u], \Gamma_s[(\chi^{-1})^* u])_{L^2(\mathbb{D}^d)} \\
&\leq M_0^0(Re(a)) C(\left\|D_x\chi^{-1}\right\|_{L^\infty L^\infty}) \left\|\Gamma_s u\right\|_{L^2}^2.
\end{aligned} \tag{3.5.8}$$

Now by the upper bound on $\frac{d}{dt}|D\chi(t,x)|$ and $\frac{d}{dt}\left|D\chi^{-1}(t,x)\right|$ we have:

$$({\Gamma_s}^* u, {\Gamma_s}^* u)_{L^2(\frac{d}{dt}|D\chi(t,x)|dx)} \leq C(\|D_x\chi\|_{L^\infty L^\infty}) \left\|{\Gamma_s}^* u\right\|_{L^2}^2$$

Now using the upper bound on $|D\chi(t,x)|$:

$$({\Gamma_s}^* u, {\Gamma_s}^* u)_{L^2(\frac{d}{dt}|D\chi(t,x)|dx)} \leq C(\|D_x\chi\|_{L^\infty L^\infty}, \left\|D_x\chi^{-1}\right\|_{L^\infty L^\infty}) \|\Gamma_s u\|_{L^2}^2 . \tag{3.5.9}$$

Analogously we get:

$$({\Gamma_s}^* f, {\Gamma_s}^* u)_{L^2(|D\chi(t,x)|dx)} \leq C(\|D_x\chi\|_{L^\infty L^\infty(\mathbb{D}^d)}, \left\|D_x\chi^{-1}\right\|_{L^\infty L^\infty}) \|\Gamma_s u\|_{L^2} \|\Gamma_s f\|_{L^2} , \tag{3.5.10}$$

$$([\partial_t {\Gamma_s}^*] u, {\Gamma_s}^* u)_{L^2(|D\chi(t,x)|dx)} \leq C(\|D_x\chi\|_{L^\infty L^\infty}, \left\|D_x\chi^{-1}\right\|_{L^\infty L^\infty}) \|\Gamma_s u\|_{L^2}^2 . \tag{3.5.11}$$

Thus finally we get by combining (3.5.6), (3.5.5), (3.5.7), (3.5.8), (3.5.9), (3.5.10) and (3.5.11):

$$(\Gamma_s T_{Re(a)}[(\chi^{-1})^* u], \Gamma_s[(\chi^{-1})^* u])_{L^2(\mathbb{D}^d)} \leq C(\|D_x\chi\|_{L^\infty L^\infty}, \left\|D_x\chi^{-1}\right\|_{L^\infty L^\infty}) \|\Gamma_s u\|_{L^2}^2 \tag{3.5.12}$$

To conclude we combine the computations from both methods and get:

$$\begin{aligned} \frac{d}{dt}[(\Gamma_s u, \Gamma_s u)_{L^2}] &\leq C(\|D_x\chi\|_{L^\infty L^\infty}, \left\|D_x\chi^{-1}\right\|_{L^\infty L^\infty}, M_0^0(Re(a))) \|\Gamma_s u\|_{L^2}^2 \\ &\quad + C(\|D_x\chi\|_{L^\infty L^\infty}, \left\|D_x\chi^{-1}\right\|_{L^\infty L^\infty}) \|\Gamma_s u\|_{L^2} \|\Gamma_s f\|_{L^2} . \end{aligned}$$

The result then follows from the Gronwall Lemma. □

We see that the proof depends essentially on symbolic calculus rules and those still clearly hold in the case of pseudodifferential operators as presented in section 2.2.

Theorem 3.5.2. *Let $T > 0$, $\chi \in W^{1,\infty}([0,T], C^\infty(\mathbb{D}^d))$ such that $D_x\chi \in C_b^\infty(\mathbb{D}^d)$ and consider $(a_t)_{t\in\mathbb{R}}$ a family of symbols in $S^\beta(\mathbb{D}^d)$ with $\beta \in \mathbb{R}$, such that $t \mapsto a_t$ is continuous and bounded from $\mathbb{R}$ to $S^\beta(\mathbb{D}^d)$ and such that $Re(a_t) = \frac{a_t + a_t^\top}{2}$ is bounded in $S^0(\mathbb{D}^d)$. Suppose moreover that $\chi(t,\cdot)$ is a diffeomorphism between open sets of $\mathbb{D}^d$ and that we have the bounds:*

$$\exists C > 0, \forall t \leq T, \forall x, C^{-1} \leq |D_x\chi(t,x)| \leq C. \tag{3.5.13}$$

Put $(\cdot)^*$ is the change of variables by χ as presented in Theorem 2.3.1.
Then for all initial data $u_0 \in H^s(\mathbb{D}^d)$ and $f \in C^0([0,T]; H^s(\mathbb{D}^d))$ the Cauchy problem:

$$\begin{cases} \partial_t u + \mathrm{Op}(a^*) u = f \\ \forall x \in \mathbb{D}^d, u(0,x) = u_0(x) \end{cases} \tag{3.5.14}$$

has a unique solution $u \in C^0([0,T]; H^s(\mathbb{D}^d)) \cap C^1([0,T]; H^{s-\beta}(\mathbb{D}^d))$ which verifies the estimates:

$$\begin{aligned} \|u(t)\|_{H^s} &\leq e^{C(\|D_x\chi\|_{L^\infty L^\infty}, \left\|D_x\chi^{-1}\right\|_{L^\infty L^\infty})t} \|u_0\|_{H^s} \\ &\quad + 2\int_0^t e^{C(\|D_x\chi\|_{L^\infty L^\infty}, \left\|D_x\chi^{-1}\right\|_{L^\infty L^\infty})(t-t')} \left\|f(t')\right\|_{H^s} dt', \end{aligned} \tag{3.5.15}$$

where C depends also on a finite symbol semi-norm of $Re(a_t)$. Again fixing the initial data at 0 is an arbitrary choice. More precisely, $\forall 0 \leq t_0 \leq T$ and all data $u_0 \in H^s(\mathbb{D}^d)$ the Cauchy problem:

$$\begin{cases} \partial_t u + \mathrm{Op}(a^*)u = f \\ \forall x \in \mathbb{D}^d, u(t_0, x) = u_0(x) \end{cases} \tag{3.5.16}$$

has a unique solution $u \in C^0([0,T]; H^s(\mathbb{D}^d)) \cap C^1([0,T]; H^{s-\beta}(\mathbb{D}^d))$ which verifies the estimate:

$$\begin{aligned} \|u(t)\|_{H^s} &\leq e^{C(\|D_x\chi\|_{L^\infty L^\infty}, \|D_x\chi^{-1}\|_{L^\infty L^\infty})|t-t_0|} \|u_0\|_{H^s} \\ &+ 2\left| \int_{t_0}^t e^{C(\|D_x\chi\|_{L^\infty L^\infty}, \|D_x\chi^{-1}\|_{L^\infty L^\infty})(t-t')} \|f(t')\|_{H^s} dt' \right|. \end{aligned}$$

We finally show a general regularizing effect due to integration in time.

Theorem 3.5.3. *Consider $(a_t)_{t\in\mathbb{R}}$ a family of symbols in $S^\beta(\mathbb{D}^d)$ with $\beta \in \mathbb{R}$, such that $t \mapsto a_t$ is continuous and bounded from $\mathbb{R}$ to $S^\beta(\mathbb{D}^d)$ and such that $\mathrm{Re}(a_t) = \frac{a_t + a_t^\top}{2}$ is bounded in $S^0(\mathbb{D}^d)$, and take $T > 0$. Then for all initial data $u_0 \in H^s(\mathbb{D}^d)$, and $f \in C^0([0,T]; H^s(\mathbb{D}^d))$ the Cauchy problem:*

$$\begin{cases} \partial_t u + op(a)u = f \\ \forall x \in \mathbb{D}^d, u(0, x) = u_0(x) \end{cases} \tag{3.5.17}$$

has a unique solution $u \in C^0([0,T]; H^s(\mathbb{D}^d)) \cap C^1([0,T]; H^{s-\beta}(\mathbb{D}^d))$ which verifies the estimates:

$$\|u(t)\|_{H^s(\mathbb{D}^d)} \leq e^{Ct} \|u_0\|_{H^s(\mathbb{D}^d)} + 2\int_0^t e^{C(t-t')} \|f(t')\|_{H^s(\mathbb{D}^d)} dt',$$

where C depends on a finite symbol semi-norm $M_1^0(\mathrm{Re}(a_t))$.
Suppose moreover that a is elliptic that is:

$$\forall (x,\xi) \in \mathbb{R}^{2d}, |a(x,\xi)| \geq C\langle\xi\rangle^\beta.$$

Then $\forall t \in [0,T]$:

$$\left\| \int_0^t u(s,\cdot)ds \right\|_{H^s} \leq C(\|u_0\|_{H^{s-1}} + \|u_0\|_{H^{s-\beta}} + \|f\|_{L^\infty([0,T],H^{s-1})} + \|f\|_{L^\infty([0,T],H^{s-\beta})}).$$

Proof. We start by writing:

$$\partial_t u + \mathrm{Op}(a)u = f$$

we then apply $\mathrm{Op}(a^{-1})$:

$$\mathrm{Op}(a^{-1})\partial_t u + u = \mathrm{Op}(a^{-1})f + Ru$$

with $\mathrm{R} \in S^{-1}(\mathbb{D}^d)$,

$$\partial_t \mathrm{Op}(a^{-1})u + u = \mathrm{Op}(a^{-1})f + Ru + \mathrm{Op}(\partial_t a^{-1})u = \mathrm{Op}(a^{-1})f + Ru + \mathrm{Op}(\frac{\partial_t a}{a^2})u.$$

the proof then follows by integration in time and the usual elliptic estimates. □

Chapter 4

On the Baker-Campbell-Hausdorff formula for hyperbolic paradifferential flows

In this section we give a complete symbolic calculus study of the flow of para-differential hyperbolic equations and their conjugation and commutation with para-differential operators. The results presented here are a combination of results obtained in [63] and [64].

Contents

4.1 Baker-Campbell-Hausdorff formula: composition and commutator estimates

We will start by giving the propositions defining the operators used in the gauge transforms and the symbolic calculus associated to them.

Notation 4.1.1. *We will essentially compute the conjugation and commutation of operators with a flow map which naturally bring into play Lie derivatives i.e commutators, thus we introduce the following notation for commutation between operators:*

$$\mathfrak{L}_a^0 b = b,\ \mathfrak{L}_a b = [a,b] = a \circ b - b \circ a,\ \mathfrak{L}_a^2 b = [a,[a,b]],\ \mathfrak{L}_a^k b = \underbrace{[a,[\cdots,[a,}_{k\ times} b]]\cdots].$$

In the following propositions the variable $t \in [0,T]$ is the generic time variable that appears in all throughout this thesis and a new variable $\tau \in \mathbb{R}$ will be used and they should not be confused.

We also need to recall the definition of the adjoint of a paradifferential symbol p that we write p^* and is given by:

$$p^*(x,\xi) = \frac{1}{2\pi}\int_{\mathbb{D}\times\hat{\mathbb{D}}} e^{-iy.\eta}\bar{p}(x-y,\xi-\eta)dyd\eta$$

We start with the proposition defining the Flow map and its standard properties.

Proposition 4.1.1. *Consider two real numbers $\delta \le 1$, $s \in \mathbb{R}$ and a symbol $p \in \Gamma_0^\delta(\mathbb{D})$ such that:*

$$\operatorname{Im}(p) = \frac{p-p^*}{2i} \in \Gamma_0^{\tilde{\delta}}(\mathbb{D}), \ \textit{with } \tilde{\delta} \le 0,$$

$$[\langle \mathrm{D}\rangle^s, \sigma_p^\psi] \in \Gamma_0^0(\mathbb{D}), \ \textit{for all } s \in \mathbb{R}, \ \textit{with} \ :$$

$$\begin{aligned} [\langle \mathrm{D}\rangle^s, \sigma_p^\psi] &= \langle \mathrm{D}\rangle^s \otimes \sigma_p^\psi - \sigma_p^\psi \otimes \langle \mathrm{D}\rangle^s \\ &= (2\pi)^{-d}\int_{\mathbb{D}^d\times\hat{\mathbb{D}}^d} e^{i(x-y).(\xi-\eta)}[\langle\eta\rangle^s\sigma_p^\psi(y,\xi) - \sigma_p^\psi(x,\eta)\langle\xi\rangle^s]dyd\eta. \end{aligned}$$

The following linear hyperbolic equation is well posed on $\mathbb{R}$:

$$\begin{cases} \partial_\tau h - iT_p h = 0, \\ h(0,\cdot) = h_0(\cdot) \in H^s. \end{cases} \tag{4.1.1}$$

For $\tau \in \mathbb{R}$, define A_τ^p as the flow map associated to (4.1.1) i.e,

$$\begin{aligned} A_\tau^p : &H^s(\mathbb{D}) \to H^s(\mathbb{D}) \\ &h_0 \mapsto h(\tau,\cdot). \end{aligned} \tag{4.1.2}$$

Then for $\tau \in \mathbb{R}$ we have,

1. *$A_\tau^p \in \mathscr{L}(H^s(\mathbb{D}))$ and,*

$$\|A_\tau^p\|_{H^s\to H^s} \le e^{C|\tau|(M_0^0(\operatorname{Im}(p))+M_0^0([\langle \mathrm{D}\rangle^s,\sigma_p^\psi]))}.$$

2. $$iT_p \circ A_\tau^p = A_\tau^p \circ iT_p, \ A_{\tau+\tau'}^p = A_\tau^p A_{\tau'}^p.$$

3. *A_τ^p is invertible and,*

$$(A_\tau^p)^{-1} = A_{-\tau}^p.$$

   Moreover the L^2 adjoint of A_τ^p verifies:

$$(A_\tau^p)^* = A_{-\tau}^{(T_p)^*} = A_{-\tau}^p + R,$$

   where R is a $\tilde{\delta}$ regularizing operator and $A_\tau^{(T_p)^*}$ is the flow generated by the Cauchy problem:

$$\begin{cases} \partial_\tau h - i(T_p)^* h = 0, \\ h(0,\cdot) = h_0(\cdot) \in H^s(\mathbb{D}). \end{cases} \tag{4.1.3}$$

4. *Taking a different symbol $\tilde{p}$ verifying the same hypothesis as p we have:*

$$\begin{aligned}&\left\|[A^p_\tau - A^{\tilde{p}}_\tau]h_0\right\|_{H^s}\\ &\leq C\,|\tau|\, e^{C|\tau|(M^0_0(\mathrm{Im}(p),\mathrm{Im}(\tilde{p}))+M^0_0([\langle \mathrm{D}\rangle^s,\sigma^\psi_p],[\langle \mathrm{D}\rangle^s,\sigma^\psi_{\tilde{p}}]))} M^\delta_0(p-\tilde{p})\,\|h_0\|_{H^{s+\delta}}\,.\end{aligned} \tag{4.1.4}$$

Proof. Points $(1),(2),(3)$ are simple consequences of the hyperbolicity and well posedness of the Cauchy problem (4.1.1). Point (4) comes by writing:

$$\partial_\tau[A^p_\tau - A^{\tilde{p}}_\tau]h_0 - iT_p[A^p_\tau - A^{\tilde{p}}_\tau]h_0 = iT_{p-\tilde{p}}A^{\tilde{p}}_\tau h_0,$$

and making the usual energy estimate. □

Remark 4.1.1. *In this proposition the hypothesis $\delta \leq 1$ is superfluous, for example this proposition extends with no problem to the linear Schrödinger flow where $\delta = 2$. We put it in the hypothesis because we wanted to put an emphasis on the type of symbols we will use in the gauge transform. Indeed we will be working symbols $p \in \Gamma^\delta_1(\mathbb{D})$ with p real valued for which we have $\tilde{\delta} \leq \delta - 1$ by symbolic calculus and the inequality:*

$$M^{\delta-1}_0(\mathrm{Im}(p),[\langle \mathrm{D}\rangle^s,\sigma^\psi_p]) \leq CM^{\delta-1}_0(\partial_\xi\partial_x p),\ \tilde{\delta} \leq 0.$$

This can even be relaxed to the minimal hypothesis $p \in \Gamma^\delta_\delta$.

Later on we will need to study the continuity of A^p_τ on Hölder/Zygmund spaces. This a non trivial result, indeed hyperbolic flows are not in general continuous on L^p spaces for $p \neq 2$, as it is for example the case for the Schrödinger equation, or equations of the form $\partial_t h + i\,|D|^\alpha\, h = 0, \alpha \neq 1$ that are not continuous on Zygmund spaces as shown in the Appendix of [8]. To study the continuity of A^p_τ on Hölder/Zygmund we start by studying its symbol. First we recall the following Lemma we proved in [63] adapting the classical result by Beals on pseudodifferential operators in [14] to the limited regularity setting.

Lemma 4.1.1. *Consider an operator A continuous from $\mathscr{S}(\mathbb{D})$ to $\mathscr{S}'(\mathbb{D})$ and let $a \in \mathscr{S}'(\mathbb{D}\times\hat{\mathbb{D}})$ the unique symbol associated to A (cf [19] for the uniqueness), i.e, let K be the kernel associated to A then:*

$$u, v \in \mathscr{S}(\mathbb{D}), (Au, v) = K(u \otimes v), \quad a(x,\xi) = \mathscr{F}_{y\to\xi}K(x, x-y).$$

- *If A is continuous from H^m to L^2, with $m \in \mathbb{R}$, and $[\frac{1}{i}\frac{d}{dx}, A]$ is continuous from $H^{m+\delta}$ to L^2 with $\delta < 1$, then $(1+|\xi|)^{-m}a(x,\xi) \in L^\infty_{x,\xi}(\mathbb{D}\times\hat{\mathbb{D}})$ and we have the estimate:*

$$\left\|(1+|\xi|)^{-m}a\right\|_{L^\infty_{x,\xi}} \leq C_m\left[\|A\|_{H^m\to L^2} + \left\|\left[\frac{1}{i}\frac{d}{dx}, A\right]\right\|_{H^{m+\delta}\to L^2}\right]. \tag{4.1.5}$$

- *If A is continuous from H^m to L^2, with $m \in \mathbb{R}$, and $[ix, A]$ is continuous from from $H^{m-\rho}$ to L^2 with $\rho \geq 0$, then $(1+|\xi|)^{-m}a(x,\xi) \in L^\infty_{x,\xi}(\mathbb{D}\times\hat{\mathbb{D}})$ and we have the estimate:*

$$\left\|(1+|\xi|)^{-m}a\right\|_{L^\infty_{x,\xi}} \leq C_m[\|A\|_{H^m\to L^2} + \|[ix, A]\|_{H^{m-\rho}\to L^2}]. \tag{4.1.6}$$

Proof. First without loss of generality through a standard mollification argument we work with $a \in \mathscr{S}(\mathbb{D}\times\hat{\mathbb{D}})$. We study the cases on $\mathbb{T}$ and $\mathbb{R}$ separately.

Operators defined on $\mathbb{T}$ The first key observation is the following:

$$(x,\xi) \in \mathbb{T}\times\mathbb{Z},\ e^{-ix\cdot\xi}Ae^{ix\cdot\xi} = a(x,\xi), \tag{4.1.7}$$

which one can right as $e^{ix\cdot\xi} \in L^2_x(\mathbb{T})$. Thus taking L^2 norms in x we get:

$$\|a(\cdot,\xi)\|_{L^2_x} \le \|A\|_{H^m\to L^2}\left\|e^{ix\cdot\xi}\right\|_{H^m_x} \le C_m\|A\|_{H^m\to L^2}(1+|\xi|)^m. \tag{4.1.8}$$

Now to get the analogue of (4.1.8) but in the ξ variable we observe that the continuity hypothesis reads for $(u,v)\in\mathscr{S}$:

$$\left|\int_{\mathbb{T}\times\mathbb{Z}} e^{ix\cdot\xi}a(x,\xi)(1+|\xi|)^{-m}\mathscr{F}(v)(\xi)u(x)dxd\xi\right| \le \|A\|_{H^m\to L^2}\|\mathscr{F}(v)\|_{L^2_\xi}\|u\|_{L^2_x},$$

which we rewrite as:

$$\left|\int_{\mathbb{Z}}\mathscr{F}(v)(\xi)\left[\int_{\mathbb{T}} e^{ix\cdot\xi}(1+|\xi|)^{-m}a(x,\xi)u(x)dx\right]d\xi\right| \le \|A\|_{H^m\to L^2}\|\mathscr{F}(v)\|_{L^2_\xi}\|u\|_{L^2_x},$$

Now we treat a as an operator with roles of (x,ξ) exchanged. Choosing,

$$u(x) = \mathscr{F}^{-1}(e^{ix\cdot\xi}),$$

we get analogously to (4.1.8):

$$\left\|(1+|\xi|)^{-m}a(x,\xi)\right\|_{L^\infty_x L^2_\xi} \le C_m\|A\|_{H^m\to L^2}. \tag{4.1.9}$$

The second key observation is:

$$e^{-ix\cdot\xi}\left[\frac{1}{i}\frac{d}{dx}, A\right]e^{ix\cdot\xi} = \partial_x a(x,\xi), \quad e^{-ix\cdot\xi}[ix,A]e^{ix\cdot\xi} = \partial_\xi a(x,\xi). \tag{4.1.10}$$

Iterating the previous computations we get:

$$\|\partial_x a(\cdot,\xi)\|_{L^2_x} \le C_m(1+|\xi|)^{m+\delta}\left\|\left[\frac{1}{i}\frac{d}{dx}, A\right]\right\|_{H^{m+\delta}\to L^2}, \tag{4.1.11}$$

and as $\rho \ge 0$:

$$\left\|\partial_\xi[(1+|\xi|)^{-m}a(x,\xi)]\right\|_{L^\infty_x L^2_\xi} \le C_m[\|A\|_{H^m\to L^2} + \|[ix,A]\|_{H^{m-\rho}\to L^2}]. \tag{4.1.12}$$

By the Sobolev embedding (4.1.9) and (4.1.12) give the desired result (4.1.6) on the torus.

To get (4.1.5) we introduce:

$$b(x,\xi,\xi_0) = a\left(\frac{x}{(1+|\xi_0|)^\delta}, (1+|\xi_0|)^\delta\xi\right).$$

As $\delta < 1$ we have that $(1+|\xi|)^\delta \sim (1+|\xi_0|)^\delta$ for $|\xi-\xi_0| \le c(1+|\xi_0|)^\delta$ for some fixed $c>0$. Considering b as a function of (x,ξ) on $\mathbb{T}\times \mathrm{B}(\xi_0,c)$, inequalities (4.1.8) and (4.1.11) give uniform L^2_x and H^1_x estimates thus by the Sobolev embedding we get:

$$\|b(x,\xi)\|_{L^\infty_x} \le C_m\left[\|A\|_{H^m\to L^2} + \left\|\left[\frac{1}{i}\frac{d}{dx}, A\right]\right\|_{H^{m+\delta}\to L^2}\right](1+|\xi|)^m, \tag{4.1.13}$$

which transferred back to a give the desired result (4.1.5) on the torus.

Operators defined on $\mathbb{R}$ The main problem we face on $\mathbb{R}$ when adapting the previous proof is we can no longer use $e^{ix\cdot\xi}$ as a test function as it no longer belongs to $L^2(\mathbb{R})$. One way to get over this was given by Beals in [**?**], we choose g in $\mathscr{S}(\mathbb{R})$ such that $g(0) = 1$, $\mathscr{F}(g)$ is supported in $\{|\xi| \leq 1\}$ and $g(x) = g(-x)$. Let $g_x(y) = g(y - x)$ and compute for $u \in \mathscr{S}$:

$$u(x) = u(x)g_x(x) = \int_{\mathbb{R}\times\mathbb{R}} e^{i(x-y)\cdot\xi} g_x(y)u(y)dyd\xi = \int_{\mathbb{R}\times\mathbb{R}} e^{-iy\cdot\xi} e^{ix} g_y(x)u(y)dyd\xi.$$

We now compute an analogue of (4.1.7):

$$\begin{aligned}
Au(x) &= \int_{\mathbb{R}\times\mathbb{R}} e^{-iy\cdot\xi} A\big(e^{ix} g_y\big)(x)u(y)dyd\xi \\
&= \int_{\mathbb{R}\times\mathbb{R}} e^{i(x-y)\cdot\xi} a_0(x, y, \xi)u(y)dyd\xi \\
&= \int_{\mathbb{R}} e^{ix\cdot\xi} a(x, \xi)\mathscr{F}(u)(\xi)d\xi,
\end{aligned}$$

where,

$$a_0(x, y, \xi) = e^{-ix\cdot\xi} A\big(e^{ix\cdot\xi} g_y\big)(x),$$

and,

$$a(x, \xi) = \int_{\mathbb{R}\times\mathbb{R}} e^{i(x-y)\cdot(\eta-\xi)} a_0(x, y, \eta)d\eta dy.$$

Applying the same arguments as in the periodic case we get:

$$\left\|(1 + |\xi|)^{-m}\partial_y^k a_0\right\|_{L^\infty_{x,y,\xi}} \leq C_{m,k}\left[\|A\|_{H^m\to L^2} + \left\|\left[\frac{1}{i}\frac{d}{dx}, A\right]\right\|_{H^{m+\delta}\to L^2}\right], k \in \mathbb{N} \tag{4.1.14}$$

and,

$$\left\|(1 + |\xi|)^{-m}\partial_y^k a_0\right\|_{L^\infty_{x,y,\xi}} \leq C_{m,k}[\|A\|_{H^m\to L^2} + \|[ix, A]\|_{H^{m-\rho}\to L^2}], k \in \mathbb{N}. \tag{4.1.15}$$

Thus to conclude the proof we need to transfer the information on the amplitude a_0 to the symbol a which is a simple application of Oscillatory Integrals. Indeed it suffices to write:

$$\begin{aligned}
a(x, \xi) &= \int_{\mathbb{R}\times\mathbb{R}} e^{i(x-y)\cdot(\eta-\xi)} a_0(x, y, \eta)d\eta dy \\
&= \int_{\mathbb{R}\times\mathbb{R}} \frac{1}{\langle\xi - \eta\rangle^2}(I - \Delta_y)e^{i(x-y)\cdot(\eta-\xi)} a_0(x, y, \eta)d\eta dy \\
&= \int_{\mathbb{R}\times\mathbb{R}} \frac{1}{\langle\xi - \eta\rangle^2} e^{i(x-y)\cdot(\eta-\xi)}(I - \Delta_y)^* a_0(x, y, \eta)d\eta dy\Bigg],
\end{aligned}$$

which gives the desired result as $a_0(x, y, \eta)$ and all of it's y derivatives are y integrable. ☐

Remark 4.1.2. *It is worth noting that if instead of Sobolev estimates we had Hölder continuity estimates on A, then combined with* (4.1.7) *it gives directly the analogue of estimates* (4.1.5) *and* (4.1.6).

We can now define the limited-regularity symbol classes to which A^p_τ belongs.

Definition 4.1.1. *Consider $s \in \mathbb{R}_+$, for $0 \leq \delta, \rho < 1$, we say:*

$$p \in W^{s,\infty}S^m_{\rho,\delta}(\mathbb{D}) \iff \begin{cases} \left|D^k_\xi p(x,\xi)\right| \leq C_k \langle\xi\rangle^{m-\rho k} \\ \left\|D^k_\xi p(\cdot,\xi)\right\|_{W^{s,\infty}} \leq C_k \langle\xi\rangle^{m-\rho k+s\delta} \end{cases}, (x,\xi) \in \mathbb{D}\times\tilde{\mathbb{D}},\ k \geq 0. \tag{4.1.16}$$

The best constant in (4.1.16) defines a seminorm denoted by ${}^{\rho,\delta}M^m_s(\cdot;k), k \in \mathbb{N}$ where k is the number of derivatives we make on the frequency variable ξ, we also define the seminorm ${}^{\rho,\delta}M^m_s = {}^{\rho,\delta}M^m_s(\cdot;1)$. We define analogously $W^{s,\infty}S^m_{\rho,\delta}(\mathbb{D}^* \times \tilde{\mathbb{D}})$.

Motivated by Lemma 4.1.1 we introduce the following family of seminorms:

$${}^{\rho,\delta}H^m_0(p;k) = \sum_{j=0}^{k} \left\|\mathfrak{L}^j_{ix} p\right\|_{H^m \to H^{j\rho}},$$

$${}^{\rho,\delta}H^m_n(p;k) = \sum_{l=0}^{n}\sum_{j=0}^{k} \left\|\mathfrak{L}^j_{ix}\mathfrak{L}^l_{\frac{1}{i}\frac{d}{dx}} p\right\|_{H^m \to H^{j\rho - l\delta}},\ n \in \mathbb{N}, n \leq \rho,$$

and if $s \notin \mathbb{N}$:

$${}^{\rho,\delta}H^m_s(p;k) = H^m_{\lfloor s\rfloor}(p;k) + \sup_{n\in\mathbb{N}} 2^{n(s-\lfloor s\rfloor)} \sum_{j=0}^{k} \left\|\mathfrak{L}^j_{ix}\mathfrak{L}^{\lfloor s\rfloor}_{\frac{1}{i}\frac{d}{dx}}[P_n(D)p]\right\|_{H^m \to H^{j\rho - l\delta}},$$

where $P_n(D)$ is applied to p in the x variable.

Then ${}^{\rho,\delta}H^m_s(p;k)_{k\in\mathbb{N}}$ induces an equivalent Fréchet topology to ${}^{\rho,\delta}M^m_s(\cdot;k)_{k\in\mathbb{N}}$ on $W^{s,\infty}S^m_{\rho,\delta}$.

In Stein's [71], such symbols are called "exotic", their continuity on different L^p spaces is completely studied but only in the regular case and without explicit estimates depending on the semi-norms. To make such estimates explicit we have given a full proof of such continuity Theorems in Appendix 4.2.

Proposition 4.1.2. *Consider two real numbers $\delta < 1$, $\rho \geq 0$ and a symbol $p \in \Gamma^\delta_\rho(\mathbb{D})$ such that:*

$$\mathrm{Im}(p) = \frac{p - p^*}{2i} \in \Gamma^{\tilde{\delta}}_0(\mathbb{D}),\ \textit{with } \tilde{\delta} \leq 0,$$

$$[\langle \mathrm{D}\rangle^s, \sigma^\psi_p] \in \Gamma^0_0(\mathbb{D}),\ \textit{for all } s \in \mathbb{R}.$$

Let $A^p_\tau, \tau \in \mathbb{R}$ be the flow map defined by Proposition 4.1.1, then there exists a symbol $e^{i\tau p}_\otimes \in W^{\rho,\infty}S^0_{1-\delta,\delta}(\mathbb{D}^* \times \tilde{\mathbb{D}})$ such that:

$$A^p_\tau = T^{lim}_{e^{i\tau p}_\otimes} + A^p_\tau(Id - T_1). \tag{4.1.17}$$

Moreover we have the identities:

$$\begin{cases} \partial_\tau[T^{lim}_{e^{i\tau p}_\otimes} h_0] = iT_p T^{lim}_{e^{i\tau p}_\otimes} h_0, \\ T^{lim}_{e^{i\tau p}_\otimes} h_0 \ _{|\tau=0} = T_1 h_0. \end{cases} \textit{for } h_0 \in H^s(\mathbb{D}), s \in \mathbb{R}. \tag{4.1.18}$$

$$T^{lim}_{e^{i\tau p}_\otimes} = T_{e^{i\tau p}} + \int_0^\tau A^p_{\tau-s}\big(T_{ip}T_{e^{isp}} - T_{ipe^{isp}}\big)ds. \tag{4.1.19}$$

Proof. The idea is that morally $T^{lim}_{e_{\otimes}^{i\tau p}}$ should be defined by the asymptotic series:

$$T^{lim}_{e_{\otimes}^{i\tau p}} \sim \sum \frac{i^k \tau^k}{k!}(T_p)^k = \sum \frac{i^k \tau^k}{k!} T_{\overset{k}{\otimes} p}, \tag{4.1.20}$$

where $\overset{k}{\otimes} p$ is defined by Theorem 2.2.6. To make this series converge we introduce a mollification with the Gaussian function $\phi_\epsilon(D)$ with symbol:

$$\phi_\epsilon(\xi) = \frac{1}{\epsilon\sqrt{2\pi}} e^{-\frac{\xi^2}{2\epsilon^2}}, \epsilon > 0. \tag{4.1.21}$$

Other than the standard properties of mollifiers, we have the following properties:

- For $h \in H^s(\mathbb{D})$, $\phi_\epsilon(D)h$ is real analytic.
- The moments of the Gaussian can be explicitly computed by, for $k \in n$:

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |\xi|^k e^{-\frac{\xi^2}{2}} d\xi = \frac{2^{\frac{k}{2}} \Gamma(\frac{k+1}{2})}{\sqrt{\pi}} = (k-1)!! \begin{cases} 1 \text{ if } k \text{ is even} \\ \sqrt{\frac{2}{\pi}} \text{ if } k \text{ is odd} \end{cases}.$$

- From the moments of the Gaussian we deduce that, for $h \in H^s(\mathbb{D})$ and $k \in n$:

$$\left\| \partial_x^k \phi_\epsilon(D) h \right\|_{H^s} \leq C_k \epsilon^{-k} \|h\|_{H^s},$$

  and C_k verifies for all $K > 0$, $K^k C_k = o(k!)$.

Indeed this last control on C_k is what motivates the choice of Gaussian mollifiers as we have the factor $\frac{1}{k!}$ in (4.1.20) which will essentially compensate for one the losses in the mollification scheme.

By the symbolic calculus rules in Theorem 2.2.6 we then have that there exists a symbol ${}^{\epsilon}e_{\otimes}^{i\tau p} \in C^{\rho}_{*} S^0_{1-\delta,\delta}(\mathbb{D}^* \times \tilde{\mathbb{D}})$ such that:

$$\sum_{k=0}^{+\infty} \frac{i^k \tau^k}{k!}(T_p)^k \phi_\epsilon(D) = \sum_{k=0}^{+\infty} \frac{i^k \tau^k}{k!} T_{\overset{k}{\otimes} p} \phi_\epsilon(D) = T^{lim}_{{}^{\epsilon}e_{\otimes}^{i\tau p}}, \tag{4.1.22}$$

$${}^{1-\delta,\delta}M^0_\rho({}^{\epsilon}e_{\otimes}^{i\tau p}) \leq \sum_{k=0}^{+\infty} C^k C_k \frac{|\tau|^k}{k!} \epsilon^{-k\delta} M^\delta_\rho(p)^k, \tag{4.1.23}$$

where C_k verifies for all $K > 0$, $K^k C_k = o(k!)$.

Now in order to pass to the limit in ϵ we need to get uniform estimates on ${}^{1-\delta,\delta}H^m_s({}^{\epsilon}e_{\otimes}^{i\tau p}; k)_{k\in\mathbb{N}}$. To do so we see that:

$$\begin{cases} \partial_\tau [T^{lim}_{{}^{\epsilon}e_{\otimes}^{i\tau p}} h_0] = i T_p T^{lim}_{{}^{\epsilon}e_{\otimes}^{i\tau p}} h_0, \\ T^{lim}_{e_{\otimes}^{i\tau p}} h_0 \;|_{\tau=0} = \phi_\epsilon(D) T_1 h_0, \end{cases} \quad \text{for } h_0 \in H^s(\mathbb{D}), s \in \mathbb{R}. \tag{4.1.24}$$

Thus a standard energy estimate combined with the commutation identities

$$\begin{cases} [\frac{1}{i}\frac{d}{dx}, A^p_\tau] = \int_0^\tau A^p_{\tau-r} [\frac{1}{i}\frac{d}{dx}, T_{ip}] A^p_r, \\ [ix, A^p_\tau] = \int_0^\tau A^p_{\tau-r} [ix, T_{ip}] A^p_r, \end{cases}$$

and Lemma 4.1.1 we get:

$$ {}^{1-\delta,\delta}H^0_n({}^{\epsilon}e_{\otimes}^{i\tau p};k)_{k\in\mathbb{N}} \leq e^{C_{k,n}|\tau|(M_0^0(\mathrm{Im}(p))+M_0^0([\langle \mathrm{D}\rangle^{-k(1-\delta)+n\delta},\sigma_p^{\psi}]))}M_n^{\delta}(p;k), \ (k,n)\in\mathbb{N}. $$

In order to get higher Zygmund estimates we see that by getting back to the sum (4.1.22), we have:

$$ \begin{cases} \partial_\tau[T^{lim}_{P_{\leq k}(D)[{}^{\epsilon}e_{\otimes}^{i\tau p}]}h_0] = iT_{[P_{\leq k}(D)p]}T^{lim}_{[P_{\leq k+1}(D){}^{\epsilon}e_{\otimes}^{i\tau p}]}h_0, \\ T^{lim}_{[P_{\leq k}(D)e_{\otimes}^{i\tau p}]}h_0 \ |_{\tau=0} = \phi_\epsilon(D)h_0. \end{cases} \quad \text{for } h_0\in H^s(\mathbb{D}), s\in\mathbb{R}. \tag{4.1.25} $$

Again commuting with ix and $\frac{1}{i}\frac{d}{dx}$ and Proposition 2.1.3 we get:

$$ {}^{1-\delta,\delta}H^0_s({}^{\epsilon}e_{\otimes}^{i\tau p};k)_{k\in\mathbb{N}} \leq e^{C_{k,n}|\tau|(M_0^0(\mathrm{Im}(p))+M_0^0([\langle \mathrm{D}\rangle^{-k(1-\delta)+s\delta},\sigma_p^{\psi}]))}M_s^{\delta}(p;k), \ (k,n)\in\mathbb{N}. $$

Thus we can pass to the limit in when $\epsilon\to 0$ and get the desired result.

Finally for identity (4.1.18) we pass to the limit in (4.1.24). Identity (4.1.19) comes from the following computation. Fix an $h_0\in H^s, s\in\mathbb{R}$, then $[A^p_\tau - T_{e^{i\tau p}}]h_0$ solves:

$$ \begin{cases} \partial_\tau\big([A^p_\tau - T_{e^{i\tau p}}]h_0\big) - iT_p\big([A^p_\tau - T_{e^{i\tau p}}]h_0\big) = \big(T_{ip}T_{e^{i\tau p}} - T_{ipe^{i\tau p}}\big)h_0, \\ \big([A^p_\tau - T_{e^{i\tau p}}]h_0\big)(0,\cdot) = (Id - T_1)h_0(\cdot), \end{cases} \tag{4.1.26} $$

which by definition of A^p_τ gives (4.1.19). $\square$

Combining the previous Proposition with Theorem 4.2.2 we get the following:

Corollary 4.1.1. *Consider two real numbers $\delta<1$, $\rho\geq 0$ and a symbol $p\in\Gamma^\delta_\rho(\mathbb{D})$ such that:*

$$ \mathrm{Im}(p) = \frac{p-p^*}{2i} \in \Gamma^{\tilde{\delta}}_0(\mathbb{D}), \ \textit{with } \tilde{\delta}\leq 0, $$
$$ [\langle \mathrm{D}\rangle^s, \sigma_p^{\psi}] \in \Gamma^0_0(\mathbb{D}), \ \textit{for all } s\in\mathbb{R}. $$

Then $T^{lim}_{e_{\otimes}^{i\tau p}}$ is continuous from C^s_* to $C^{s-\frac{\delta}{2}}_*$ and from $W^{s+(\frac{1}{2}-\frac{1}{p})\delta,p}$ to $W^{s,p}$ for $s>0$. Moreover we have the estimate:

$$ \left\| T^{lim}_{e_{\otimes}^{i\tau p}} \right\|_{W^{s+(\frac{1}{2}-\frac{1}{p})\delta,p}\to W^{s,p}} \leq K \ {}^{1-\delta,\delta}M^0_0(e_{\otimes}^{i\tau p};1), \ \textit{and,} $$

$$ \left\| T^{lim}_{e_{\otimes}^{i\tau p}} \right\|_{C_*^{s+\frac{1}{2}\delta}\to C^s_*} \leq K \ {}^{1-\delta,\delta}M^0_0(e_{\otimes}^{i\tau p};1). $$

An improvement one could make on Proposition 4.1.2 is on the frequency localization and limit cutoff.

Proposition 4.1.3. *Consider two real numbers $\delta<1$, $\rho\geq 0$ and a symbol $p\in\Gamma^\delta_\rho(\mathbb{D})$ such that:*

$$ \mathrm{Im}(p) = \frac{p-p^*}{2i} \in \Gamma^{\tilde{\delta}}_0(\mathbb{D}), \ \textit{with } \tilde{\delta}\leq 0, $$
$$ [\langle \mathrm{D}\rangle^s, \sigma_p^{\psi}] \in \Gamma^0_0(\mathbb{D}), \ \textit{for all } s\in\mathbb{R}. $$

Let $A^p_\tau, \tau \in \mathbb{R}$ be the flow map defined by Proposition 4.1.1 with the choice of cutoff $T^{B,n}_{ip}$ with $B \geq 2, b \geq 1$ and $e^{i\tau p}_{\otimes} \in \Gamma^1 S^0_{1-\delta,\delta}(\mathbb{R}^* \times \mathbb{R})$ the symbol given by Proposition 4.1.2.

$$A^p_\tau = T^{2,b}_{e^{i\tau p}_{\otimes}} + R_\infty, \text{ and,} \tag{4.1.27}$$

there exists a constant C such that R_∞, verifies,

$$\forall u \in \mathscr{S}', \ \operatorname{supp} \mathscr{F}(R_\infty u) \subset B(0, C).$$

Proof. As $B \geq 2$, $i(T^{B,b}_p)^*$ is still a paradifferential operator, thus there exists $(e^{i\tau p}_{\otimes})^*$ such that:

$$(A^p_\tau)^* = A^{(T^{B,b}_p)^*}_\tau = T^{lim}_{(e^{i\tau p}_{\otimes})^*} + A^{(T^{B,b}_p)^*}_\tau (Id - (T^{B,b}_1)^*),$$

On the other hand conjugating (4.1.17) we have:

$$(A^p_\tau)^* = \left(T^{lim}_{e^{i\tau p}_{\otimes}}\right)^* + (A^p_\tau)^*(Id - (T^{B,b}_1)^*).$$

Now if one calls a^* the symbol of the adjoint of an operator $\operatorname{Op}(a)$, then we have the following identify:

$$\mathscr{F}_x(a^*)(\eta - \xi, \xi) = \overline{\mathscr{F}_x(a)(\xi - \eta, \xi)}. \tag{4.1.28}$$

Thus writing:

$$\mathscr{F}_x\left(T^{lim}_{(e^{i\tau p}_{\otimes})^*} + A^{(T^{B,b}_p)^*}_\tau (Id - (T^{B,b}_1)^*)\right)(\eta, \xi) = \mathscr{F}_x\left(T^{lim}_{(e^{i\tau p}_{\otimes})^*} + A^{(T^{B,b}_p)^*}_\tau (Id - (T^{B,b}_1)^*)\right)(\eta, \xi),$$

and applying identity (4.1.28) we get the desired result in the zones $\xi \geq 0, \eta \leq 0$ and $\xi \leq 0, \eta \geq 0$. For the other two zones we see that:

$$\mathscr{F}_x(a)(\eta, \xi) = \overline{\mathscr{F}_x(a)(-\eta, \xi)},$$

and applying the previous result to $\bar{p}$ we obtain the desired result by symmetry. □

The key commutation and conjugation result is given by the following proposition.

Proposition 4.1.4. *Consider two real numbers $\delta < 1$, $\rho \geq 0$ and a symbol $p \in \Gamma^\delta_\rho(\mathbb{D})$ such that:*

$$\operatorname{Im}(p) = \frac{p - p^*}{2i} \in \Gamma^{\tilde{\delta}}_0(\mathbb{D}), \text{ with } \tilde{\delta} \leq 0,$$
$$[\langle \mathrm{D} \rangle^s, \sigma^\psi_p] \in \Gamma^0_0(\mathbb{D}), \text{ for all } s \in \mathbb{R}.$$

Let $A^p_\tau, \tau \in \mathbb{R}$ be the flow map defined by Proposition 4.1.1 and take a symbol $b \in \Gamma^\beta_\rho(\mathbb{D}), \beta \in \mathbb{R}$ then we have:

5. *For $\rho \geq 1$, there exists $b^p_\tau \in W^{\rho,\infty} S^\beta_{1-\delta,\delta}(\mathbb{D})$ such that:*

$$A^p_\tau \circ T_b \circ A^p_{-\tau} = T^{lim}_{b^p_\tau}. \tag{4.1.29}$$

Moreover we have the estimates:

$$\left\| T^{lim}_{b^p_\tau} - \sum_{k=0}^{\lceil \rho - 1 \rceil} \frac{\tau^k}{k!} \mathfrak{L}^k_{iT_p} T_b \right\|_{H^s \to H^{s-\beta-\lceil \rho \rceil \delta + \rho}} \leq C_\rho M^\beta_\rho(b) M^\delta_\rho(p)^{\lceil \rho \rceil}, \tag{4.1.30}$$

$$
{}^{1-\delta,\delta}H^{\beta}_{\rho}(b^p_\tau;k)
\leq e^{C_{\rho,k}|\tau|(M^0_0(\mathrm{Im}(p))+M^0_0([\langle \mathrm{D}\rangle^{-k(1-\delta)+\rho\delta},\sigma^{\psi}_p]))}[H^\beta_\rho(b;k)+H^\beta_\rho(b;k)H^\delta_\rho(p;k)],\ k\in\mathbb{N},
\tag{4.1.31}
$$

where $C_{\rho,k}$ is a constant depending only on ρ and k.

6. *There exists ${}^c b^p_\tau \in W^{\rho-1,\infty}S^{\beta+\delta-1}_{1-\delta,\delta}(\mathbb{D})$ such that:*

$$
[A^p_\tau, T_b] = A^p_\tau T^{lim}_{{}^c b^p_\tau} \iff T^{lim}_{{}^c b^p_\tau} = T_b - T^{lim}_{b^p_{-\tau}}.
\tag{4.1.32}
$$

Moreover we have the estimates:

$$
\left\| T^{lim}_{{}^c b^p_\tau} - \sum_{k=1}^{\lceil \rho-1\rceil}(-1)^{k-1}\frac{\tau^k}{k!}\mathfrak{L}^k_{iT_p}T_b \right\|_{H^s\to H^{s-\beta-\lceil\rho\rceil\delta+\rho}} \leq C_\rho M^\beta_\rho(b)M^\delta_\rho(p)^{\lceil\rho\rceil},
\tag{4.1.33}
$$

$$
{}^{1-\delta,\delta}H^{\beta+\delta-1}_{\rho-1}({}^c b^p_\tau;k)
\leq e^{C_{\rho,k}|\tau|(M^0_0(\mathrm{Im}(p))+M^0_0([\langle \mathrm{D}\rangle^{-k(1-\delta)+\rho\delta},\sigma^{\psi}_p]))}[H^\beta_\rho(b;k)+H^\beta_\rho(b;k)H^\delta_\rho(p;k)], k\in\mathbb{N},
\tag{4.1.34}
$$

where $C_{\rho,k}$ is a constant depending only on ρ and k.

The link between b^p_τ and ${}^c b^p_\tau$ is given by the following:

$$
T^{lim}_{{}^c b^p_\tau} = \int_0^\tau A^p_r T_{\mathfrak{L}_{ip}b}A^p_{-r}dr = \int_0^\tau T^{lim}_{(\mathfrak{L}_{ip}b)^p_r}dr,
$$

where $\mathfrak{L}_{ip}b$ is the paradifferential symbol associated to $\mathfrak{L}_{iT_p}T_b$ by Theorem 2.2.6. Moreover by Proposition 4.1.2 we have the following more precise frequency cut-off:

$$
A^p_\tau \circ T^{\psi^{B,R}}_b \circ A^p_{-\tau} = T^{\psi^{2\star B\star 2,R}}_{b^p_\tau} + R_\infty,
\tag{4.1.35}
$$

where A^p_τ is defined by the choice of cut-off $T^{\psi^{B',R}}_p$, $B'<1$, $B\star B' = \frac{BB'}{B+B'-1}$ and,

$$
\begin{aligned}
R^\infty = &- T^{\psi^{2,R}}_{e^{ip}}T^{\psi^{B,R}}_b A^p_{-1}(Id - T^{\psi^{B',R}}_1) - A^p_1(Id-T^{\psi^{B',R}}_1)T^{\psi^{B,R}}_p T^{\psi^{2,R}}_{e^{-ip}}\\
&- A^p_1(Id-T^{\psi^{B',R}}_1)T^{\psi^{B,R}}_b A^p_{-1}(Id-T^{\psi^{B',R}}_1).
\end{aligned}
$$

Remark 4.1.3. • *It is important to notice that the main result of this proposition is the factorization of the A^p_τ terms in (4.1.29) and (4.1.32) where the right hand sides contain symbols in the usual classes modulo a more regular remainder. This was not a priori the case of the left hand sides containing A^p_τ. In other words we prove the stability of Γ^m_ρ under the conjugation by A^p_τ.*

This is crucial when studying the regularity of the flow map for:

$$
s > 1 + \frac{2-\alpha}{\alpha-1} + \frac{1}{2}.
$$

Indeed if p depends on a parameter λ, $D_\lambda A^p_\tau \circ T_b \circ A^p_{-\tau}$ is a priori an operator of order $\beta+\delta$ by (4.1.4), but $D_\lambda T^{lim}_{b^p_\tau}$ is shown in Proposition 4.1.6 to be an operator of order β.

- *In the language of pseudodifferential operators, $T_{b^p_\tau}$ is the asymptotic sum of the series $(\frac{\tau^k}{k!}\mathfrak{L}^k_{iT_p}T_b)$ i.e the Baker-Campbell-Hausdorff formal series. Though $T_{b^p_\tau}$ is not necessarily equal to this sum, for this sum need not converge.*
- *By the right hand side we have the continuity of $T^{lim}_{b^p_\tau}$ on H^s for $s \in \mathbb{R}$ and not only $s > 0$ which is an improvement on (2.2.4).*

Proof. The structure of the proof is as follows:

(I) We will give a proof of the estimate (4.1.30) assuming b^p_τ exists.

(II) We will prove the existence of $b^p_\tau \in W^{\rho,\infty}S^\beta_{1-\delta,\delta}(\mathbb{D})$ which is the subtle part of the proof.

(III) Finally we will deduce point (6) from point (5).

Point (I) For point (5) we compute,

$$\partial_\tau[A^p_\tau \circ T_b \circ A^p_{-\tau}] = iT_p \circ A^p_\tau \circ T_b \circ A^p_{-\tau} - A^p_\tau \circ T_b \circ iT_p \circ A^p_{-\tau}$$

Using (2),

$$\begin{aligned}\partial_\tau[A^p_\tau \circ T_b \circ A^p_{-\tau}] &= A^p_\tau \circ iT_p \circ T_b \circ A^p_{-\tau} - A^p_\tau \circ T_b \circ iT_p \circ A^p_{-\tau} \\ &= A^p_\tau[iT_p, T_b]A^p_{-\tau}.\end{aligned} \tag{4.1.36}$$

As $A_0 = Id$, integrating on $[0, \tau]$ we get:

$$A^p_\tau \circ T_b \circ A^p_{-\tau} = T_b + \int_0^\tau \underbrace{A^p_r[iT_p, T_b]A^p_{-r}}_{*}dr.$$

Iterating the computation in $*$ we get for $n \in \mathbb{N}^*$,

$$A^p_\tau \circ T_b \circ A^p_{-\tau} = \sum_{k=0}^n \frac{\tau^k}{k!}\mathfrak{L}^k_{iT_p}T_b + \int_0^\tau \frac{(\tau-r)^n}{n!}A^p_r\mathfrak{L}^{n+1}_{iT_p}T_bA^p_{-r}dr. \tag{4.1.37}$$

Now the key point is the continuity of paradifferential operators given by Theorem 2.2.3 and the symbolic calculus rules given by Theorem 2.2.6. By Lemma 4.1.2:

$$\left\|\mathfrak{L}^{\lceil\rho\rceil}_{iT_p}T_b\right\|_{H^s\to H^{s-\beta-\lceil\rho\rceil\delta+\rho}} \leq C_\rho M^\beta_\rho(b)M^\delta_\rho(p)^{\lceil\rho\rceil}, \text{ for } \rho \in \mathbb{R}_+, \tag{4.1.38}$$

where $\lceil\rho\rceil$ is the upper integer part of ρ.

Thus applying point (1) combined with (4.1.38) we get (4.1.30).

Point (II) The constant C_ρ in (4.1.38) is estimated "brutally" by Lemma 4.1.2: $O(2^\rho \times \lceil\rho\rceil!)$, thus even though one has a $\frac{1}{\lceil\rho\rceil!}$ in (4.1.37) the convergence result is non trivial. To get past this let us express explicitly the difficulty in the problem. Rearranging the terms in (4.1.36) we see that:

$$\partial_\tau[A^p_\tau \circ T_b \circ A^p_{-\tau}] = A^p_\tau[iT_p, T_b]A^p_{-\tau} = [iT_p, A^p_\tau T_b A^p_{-\tau}], \tag{4.1.39}$$

thus we have to solve the following the Cauchy problem in $\mathscr{L}(H^s(\mathbb{D}), H^{s-\beta}(\mathbb{D}))$:

$$\begin{cases} \partial_\tau f(\tau) = [iT_p, f(\tau)] \in \Gamma^\beta_{\rho-1}(\mathbb{D}), \\ f(0) = T_b \in \Gamma^\beta_\rho(\mathbb{D}). \end{cases} \tag{4.1.40}$$

This amounts to the non trivial problem of solving a linear ODE in the Fréchet space $\Gamma^\beta_{+\infty}(\mathbb{D})$, indeed such a problem need not have a solution in general, and even if it does, it need not be unique. To solve such a problem one usually has to look at a Nash-Moser type scheme, though in our case we have an explicit ODE that can be solved with a series and a loss of derivative. Thus inspired by Hörmander's [37], we remark another key estimate given by the continuity of paradifferential operators given by Theorem 2.2.3 and the symbolic calculus rules given by Theorem 2.2.6 (cf Lemma 4.1.2):

$$\left\| \mathfrak{L}^{\lceil \rho \rceil}_{iT_p} T_b \right\|_{H^s \to H^{s-\beta-\lceil \rho \rceil \delta}} \leq C^{\lceil \rho \rceil} M^\beta_0(b) M^\delta_0(p)^{\lceil \rho \rceil}, \text{ for } \rho \in \mathbb{R}_+. \tag{4.1.41}$$

This means that if we can compensate the loss of $\beta + \lceil \rho \rceil \delta$ derivatives with a cost negligible in comparison to $\lceil \rho \rceil !$, we would have a convergent series in (4.1.37). A first approach would be to interpolate (4.1.38) and (4.1.41) which gives:

$$\begin{aligned} \left\| \mathfrak{L}^{\lceil \rho \rceil}_{iT_p} T_b \right\|_{H^s \to H^{s-\beta}} \leq & \; C^{\frac{\rho - \lceil \rho \rceil \delta}{\rho} \lceil \rho \rceil} M^\beta_0(b)^{\frac{\rho - \lceil \rho \rceil \delta}{\rho}} M^\delta_0(p)^{\frac{\rho - \lceil \rho \rceil \delta}{\rho} \lceil \rho \rceil} \\ & \times C^{(\lceil \rho \rceil + 1) \frac{\lceil \rho \rceil \delta}{\rho}} 2^{\lceil \rho \rceil \frac{\lceil \rho \rceil \delta}{\rho}} \lceil \rho \rceil !^{\frac{\lceil \rho \rceil \delta}{\rho}} M^\beta_\rho(b)^{\frac{\lceil \rho \rceil \delta}{\rho}} M^\delta_\rho(p)^{\lceil \rho \rceil \frac{\lceil \rho \rceil \delta}{\rho}}, \end{aligned} \tag{4.1.42}$$

This indeed solves the cost $\lceil \rho \rceil !$ of (4.1.38) but depends on $M^\cdot_\rho$ norms of b and p. An idea to control those norms in a cost negligible in comparison to $\lceil \rho \rceil !$ would be to mollify p and b using an analytic mollifier, this might work but we found it better to mollify differently.

For this we introduce a mollification with the Gaussian function $\phi_\epsilon(D)$ with symbol:

$$\phi_\epsilon(\xi) = \frac{1}{\epsilon\sqrt{2\pi}} e^{-\frac{\xi^2}{2\epsilon^2}}, \epsilon > 0. \tag{4.1.43}$$

Other than the standard properties of mollifiers, we have the following properties:

- For $h \in H^s(\mathbb{D})$, $\phi_\epsilon(D)h$ is real analytic.
- The moments of the Gaussian can be explicitly computed by, for $k \in n$:

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |\xi|^k e^{-\frac{\xi^2}{2}} d\xi = \frac{2^{\frac{k}{2}} \Gamma(\frac{k+1}{2})}{\sqrt{\pi}} = (k-1)!! \begin{cases} 1 \text{ if } k \text{ is even} \\ \sqrt{\frac{2}{\pi}} \text{ if } k \text{ is odd} \end{cases}.$$

- From the moments of the Gaussian we deduce that, for $h \in H^s(\mathbb{D})$ and $k \in n$:

$$\left\| \partial^k_x \phi_\epsilon(D) h \right\|_{H^s} \leq C_k \epsilon^{-k} \|h\|_{H^s},$$

  and C_k verifies for all $K > 0$, $K^k C_k = o(k!)$.

Now by the symbolic calculus rules given by Theorem 2.2.6, for $\epsilon > 0$, there exists ${}^{\epsilon}b^p_\tau \in \Gamma^0_\rho(\mathbb{D})$ such that:

$$\sum_{k=0}^{+\infty} \frac{\tau^k}{k!} \mathfrak{L}^k_{iT_p} T_b \phi_\epsilon(D) = T^{lim}_{{}^{\epsilon}b^p_\tau}, \text{ with,}$$

$$M^0_\rho({}^{\epsilon}b^p_\tau) \leq \sum_{k=0}^{+\infty} C^k C_k \frac{|\tau|^k}{k!} \epsilon^{-k\delta-\beta} M^\beta_\rho(b) M^\delta_\rho(p)^k. \tag{4.1.44}$$

In order to pass to the limit in ϵ we will express ${}^{\epsilon}b^p_\tau$ differently, for all $\epsilon > 0$, $\sum_{k=0}^{n} \frac{\tau^k}{k!} \mathfrak{L}^k_{iT_p} T_b \phi_\epsilon(D)$ converges in $\mathscr{L}(H^s(\mathbb{D}))$, thus by unicity of the limit:

$$A^p_\tau \circ T_b \circ A^p_{-\tau} \phi_\epsilon(D) = \sum_{k=0}^{+\infty} \frac{\tau^k}{k!} \mathfrak{L}^k_{iT_p} T_b \phi_\epsilon(D). \tag{4.1.45}$$

Thus,

$$A^p_\tau \circ T_b \circ A^p_{-\tau} \phi_\epsilon(D) = T^{lim}_{{}^{\epsilon}b^p_\tau}. \tag{4.1.46}$$

Now we estimate the ${}^{\delta-1}H^\beta_\rho(\cdot;k)_{k\in\mathbb{N}}$ norms of ${}^{\epsilon}b^p_\tau$. To do so we need, in the word of Hörmander [36], a result which interpolates between information on the norm a of an operator and bounds for the derivatives of it's symbol. This was exactly the goal of Lemma 4.1.1.

By commuting $\frac{1}{i}\frac{d}{dx}$ and ix with (4.1.46) we get:

$$\begin{aligned}[\frac{1}{i}\frac{d}{dx}, T^{lim}_{{}^{\epsilon}b^p_\tau}] =& [\frac{1}{i}\frac{d}{dx}, A^p_\tau] \circ T_b \circ A^p_{-\tau}\phi_\epsilon(D) + A^p_\tau \circ [\frac{1}{i}\frac{d}{dx}, T_b] \circ A^p_{-\tau}\phi_\epsilon(D) \\ &+ A^p_\tau \circ T_b[\frac{1}{i}\frac{d}{dx}, \circ A^p_{-\tau}]\phi_\epsilon(D) + A^p_\tau \circ T_b \circ A^p_{-\tau}[\frac{1}{i}\frac{d}{dx}, \phi_\epsilon(D)].\end{aligned}$$

and,

$$\begin{aligned}[ix, T^{lim}_{{}^{\epsilon}b^p_\tau}] =& [ix, A^p_\tau] \circ T_b \circ A^p_{-\tau}\phi_\epsilon(D) + A^p_\tau \circ [ix, T_b] \circ A^p_{-\tau}\phi_\epsilon(D) \\ &+ A^p_\tau \circ T_b[ix, \circ A^p_{-\tau}]\phi_\epsilon(D) + A^p_\tau \circ T_b \circ A^p_{-\tau}[ix, \phi_\epsilon(D)].\end{aligned}$$

To estimate $[\frac{1}{i}\frac{d}{dx}, A^p_\tau]$ and $[ix, A^p_\tau]$ we get back to (4.1.1) and see that:

$$\begin{cases} [\frac{1}{i}\frac{d}{dx}, A^p_\tau] = \int_0^\tau A^p_{\tau-r}[\frac{1}{i}\frac{d}{dx}, T_{ip}]A^p_r, \\ [ix, A^p_\tau] = \int_0^\tau A^p_{\tau-r}[ix, T_{ip}]A^p_r. \end{cases} \tag{4.1.47}$$

Thus by iteration, the continuity of A^p_τ and Lemma 4.1.1 we get for $(k,n) \in \mathbb{N}$:

$${}^{1-\delta,\delta}H^\beta_n(b^p_\tau;k) \leq e^{C_{k,n}|\tau|(M^0_0(\mathrm{Im}(p))+M^0_0([\langle \mathrm{D}\rangle^{-k(1-\delta)+n\delta},\sigma^\psi_p]))}[H^\beta_n(b;k)+H^\beta_n(b;k)H^\delta_n(p;k)].$$

Thus we can pass to the limit in ϵ in (4.1.46), there exist $b^p_\tau \in W^{0,\infty}S^\beta_{1-\delta,0}$ such that:

$$A^p_\tau \circ T_b \circ A^p_{-\tau} = T^{lim}_{b^p_\tau}. \tag{4.1.48}$$

Now to get ${}^{1-\delta,\delta}H^{\beta}_{\rho}(b^p_\tau;k), \rho\notin\mathbb{N}$ estimates we will use the Littelwood Paley decomposition. Indeed recalling $(P_{\leq k})$ as the Littlewood Paley projectors defined in 2.1.1 we have by Lemma 2.2.3:

$$T^{lim}_a P_{\leq k} = P_{\leq k+1} T^{lim}_{(P_{\leq k+1}(D)a)} P_{\leq k},$$

where $\mathscr{F}(P_{\leq k+1}(D)a) = P_{\leq k+1}(\eta)\mathscr{F}(a)(\eta,\xi)$. Thus going back to the sum (4.1.45) we find:

$$A^{(P_{\leq k+1}(D)p}_{\tau} \circ T_{(P_{\leq k+1}(D)b} \circ A^{(P_{\leq k+1}(D)p}_{-\tau} P_{\leq k} = P_{\leq k+1} T^{lim}_{(P_{\leq k+1}(D)b^p_\tau} P_{\leq k}$$

thus we get by commuting with ix, $\frac{1}{i}\frac{d}{dx}$ and estimating as previously:

$${}^{1-\delta,\delta}H^{\beta}_{0}(P_{\leq k+1}(D)\mathfrak{L}^{\lfloor\rho\rfloor}_{\frac{1}{i}\frac{d}{dx}} b^p_\tau;k) \leq e^{C_k|\tau|(M^0_0(\mathrm{Im}(p))+M^0_0([\langle \mathrm{D}\rangle^{-k(1-\delta)},\sigma^\psi_p]))} \times (*)$$

$$\text{where } (*) = H^{\beta}_{n}(P_{\leq k+1}(D)\mathfrak{L}^{\lfloor\rho\rfloor}_{\frac{1}{i}\frac{d}{dx}} b;k) + H^{\beta}_{n}(P_{\leq k+1}(D)\mathfrak{L}^{\lfloor\rho\rfloor}_{\frac{1}{i}\frac{d}{dx}} b;k) H^{\delta}_{n}(\mathfrak{L}^{\lfloor\rho\rfloor}_{\frac{1}{i}\frac{d}{dx}} p;k).$$

Thus by Proposition 2.1.3 characterizing Zygmund spaces and Definition 4.1.1 we get fpr $(k,n)\in\mathbb{N}$:

$$\begin{aligned}&{}^{1-\delta,\delta}H^{\beta}_{\rho}(b^p_\tau;k)\\ &\leq e^{C_{k,\rho}|\tau|(M^0_0(\mathrm{Im}(p))+M^0_0([\langle \mathrm{D}\rangle^{-k(1-\delta)+\rho\delta},\sigma^\psi_p]))}[H^{\beta}_{\rho}(b;k)+H^{\beta}_{\rho}(b;k)H^{\delta}_{\rho}(p;k)].\end{aligned}$$

which gives (4.1.31) for $\rho\notin\mathbb{N}$.

Point (III) For point (6) we compute:

$$\partial_\tau[A^p_\tau, T_b] = [iT_p\circ A^p_\tau, T_b] = iT_p[A^p_\tau, T_b] + [iT_p, T_b]A^p_\tau.$$

Thus by definition of A^p_τ as the flow map we get the following Duhamel formula,

$$\begin{aligned}[A^p_\tau, T_b] &= \int_0^\tau A^p_{\tau-r}[iT_p, T_b]A^p_r dr,\\ &= A_\tau \int_0^\tau \underbrace{A^p_{-r}[iT_p, T_b]A^p_r(u)dr}_{\star}.\end{aligned}$$

Applying point (5) to $\star$ we get:

$$\begin{aligned}[A^p_\tau, T_b] = &A^p_\tau\sum_{k=1}^{n}(-1)^{k-1}\frac{\tau^k}{k!}\mathfrak{L}^k_{iT_p}T_b\\ &+(-1)^n A^p_\tau\int_0^\tau\frac{(\tau-r)^n}{n!}A^p_{-r}\mathfrak{L}^{n+1}_{iT_p}T_bA^p_r dr.\end{aligned}$$

Again applying point (1) combined with (4.1.38) we get (4.1.33).

To get (4.1.32) we inject (4.1.29) in $\star$, which concludes the proof. □

Lemma 4.1.2. *Consider two real numbers δ,β, $\rho\geq 0$, and two symbols $p\in\Gamma^\delta_\rho(\mathbb{D})$ and $b\in\Gamma^\beta_\rho(\mathbb{D})$ then the exists a constant $C>0$ such that:*

$$\left\|\mathfrak{L}^{\lceil\rho\rceil}_{T_p}T_b\right\|_{H^s\to H^{s-\beta-\lceil\rho\rceil\delta+\rho}} \leq C^{\lceil\rho\rceil+1}2^{\lceil\rho\rceil}\lceil\rho\rceil! M^\beta_\rho(b)M^\delta_\rho(p)^{\lceil\rho\rceil},\ \textit{for } \rho\in\mathbb{R}_+, \quad (4.1.49)$$

$$\left\|\mathfrak{L}^{\lceil\rho\rceil}_{T_p}T_b\right\|_{H^s\to H^{s-\beta-\lceil\rho\rceil\delta}} \leq C^{\lceil\rho\rceil}M^\beta_0(b)M^\delta_0(p)^{\lceil\rho\rceil},\ \textit{for } \rho\in\mathbb{R}_+. \quad (4.1.50)$$

Proof. For (4.1.50), we notice that $\mathfrak{L}^{\lceil\rho\rceil}_{T_p}T_b$ contains $2^{\lceil\rho\rceil}$ terms of the form:

$$T_p\circ\cdots\circ\underbrace{T_b}_{\text{position } i}\circ\cdots\circ T_p,\ i\in[0,2^{\lceil\rho\rceil}],$$

Now by the continuity of paradifferential operators given in Theorem 2.2.3 we have:

$$\|T_p\circ\cdots\circ T_b\circ\cdots\circ T_p\|_{H^s\to H^{s-\beta-\lceil\rho\rceil\delta}}\leq K^{\lceil\rho\rceil}M_0^\beta(b)M_0^\delta(p)^{\lceil\rho\rceil},$$

which gives (4.1.50).

For (4.1.49), we start by the case $k\in\mathbb{N}^*$ is a an integer. We first notice that again by Theorem 2.2.3 we have:

$$\begin{cases}\mathfrak{L}^1_{T_p}T_b\in\Gamma^{\beta+\delta-1}_{k-1},\\ M^{\beta+\delta-1}_{k-1}(\mathfrak{L}^1_{T_p}T_b)\leq CM_k^\beta(b)M_k^\delta(p).\end{cases}$$

Thus iterating this formula we get:

$$\begin{cases}\mathfrak{L}^k_{T_p}T_b\in\Gamma^{\beta+k\delta-k}_{0},\\ M^{\beta+\delta-k}_{0}(\mathfrak{L}^k_{T_p}T_b)\leq C_kM_k^\beta(b)\prod_{i\geq1}^{k}M_i^\delta(p),\end{cases}$$

and C_k verifies:

$$C_k=2kC_{k-1}\Rightarrow C_k=C2^kk!,$$

Thus giving the result in the case k integer. For $\rho\geq0$, it suffice to see that for $\rho\leq1$ again by Theorem 2.2.3 we have:

$$\begin{cases}\mathfrak{L}^1_{T_p}T_b\in\Gamma^{\beta+\delta-\rho}_{0},\\ M^{\beta+\delta-\rho}_{0}(\mathfrak{L}^1_{T_p}T_b)\leq CM_\rho^\beta(b)M_\rho^\delta(p),\end{cases}$$

which concludes the proof. □

Remark 4.1.4. *We would like to note that in the special case $\delta=0$ we have the refined tame estimates for $k\in\mathbb{N}$:*

$$M_0^\beta(\partial_\xi^k b_\tau^p;0)\leq\sum_{j=0}^{k}\sum_{l=0}^{j}\binom{k}{j}\binom{j}{l}M_0^0(\partial_\xi^{k-j}\otimes e^{i\tau p};0)M_0^\beta(\partial_\xi^{j-l}b_\tau^p;0)M_0^0(\partial_\xi^l\otimes e^{-i\tau p};0).\tag{4.1.51}$$

We won't explicitly use the tameness in our proof as we avoid using a Nash-Moser type scheme but it's worth noting that implicitly it is this condition that ensures that the constructions in Section 6.1 converge, for more details on the necessity of this condition we refer to the following complete and instructive article by Hamilton [31].

Proof. This is the consequences of the Leibniz formula combined with the computation of $[ix,b_\tau^p]$:

$$[ix,A_\tau^pT_bA_{-\tau}^p]=[ix,A_\tau^p]T_bA_{-\tau}^p+A_\tau^p[ix,T_b]A_{-\tau}^p+A_\tau^pT_b[ix,A_{-\tau}^p].$$

□

The different Gateaux derivatives of the operators defined above are given by the following propositions.

Proposition 4.1.5. *Consider two real numbers $\delta < 1$, $\rho \geq 0$, two symbols $p, p' \in \Gamma^{\delta}_{\rho}(\mathbb{D})$ such that,*

$$\operatorname{Im}(p) = \frac{p - p^*}{2i} \in \Gamma^{\tilde{\delta}}_{0}(\mathbb{R}), \operatorname{Im}(p') = \frac{p' - p'^*}{2i} \in \Gamma^{\tilde{\delta}}_{0}(\mathbb{R}), \ \tilde{\delta} \leq 0,$$

$$\big([\langle \mathrm{D} \rangle^s, \sigma^{\psi}_{p}], [\langle \mathrm{D} \rangle^s, \sigma^{\psi}_{p'}]\big) \in \Gamma^{0}_{0}(\mathbb{D}), \ \textit{for all } s \in \mathbb{R}.$$

Let $A^{p}_{\tau}, A^{p'}_{\tau}, \tau \in \mathbb{R}$ be the flow maps defined by Proposition 4.1.1, then for $\tau \in \mathbb{R}$ we have:

$$A^{p}_{\tau} - A^{p'}_{\tau} = \int_0^{\tau} A^{p}_{\tau - r} T_{ip' - p} A^{p'}_{r} dr. \tag{4.1.52}$$

Another way to express this is with the Gateaux derivative of $p \mapsto A^{p}_{\tau}$ on the Fréchet space $\Gamma^{\delta}_{\rho}(\mathbb{D})$ is given by:

$$D_p A^{p}_{\tau}(h) = \int_0^{\tau} A^{p}_{\tau - r} T_{ih} A^{p}_{r} dr. \tag{4.1.53}$$

Moreover consider an open interval $I \subset \mathbb{R}$, and a symbols $p \in C^1(I, \Gamma^{\delta}_{\rho}(\mathbb{D}))$ such that for all $z \in I$:

$$\operatorname{Im}(p(z)) = \frac{p - p^*}{2i} \in \Gamma^{\tilde{\delta}}_{0}(\mathbb{R}),$$

$$[\langle \mathrm{D} \rangle^s, \sigma^{\psi}_{p}] \in \Gamma^{0}_{0}(\mathbb{D}), \ \textit{for all } s \in \mathbb{R}.$$

Let $A^{p}_{\tau}, \tau \in \mathbb{R}$ be the flow map defined by Proposition 4.1.1 then for $\tau \in \mathbb{R}, z \in I$ we have:

$$\partial_z A^{p}_{\tau} = \int_0^{\tau} A^{p}_{\tau - r} T_{i\partial_z p} A^{p}_{r} dr. \tag{4.1.54}$$

Proof. Fix $h_0 \in H^s, s \in \mathbb{R}$ then:

$$\partial_\tau [A^{p}_{\tau} h_0] - iT_p [A^{p}_{\tau} h_0] = 0 \Rightarrow \partial_\tau [\partial_z A^{p}_{\tau} h_0] - iT_p [\partial_z A^{p}_{\tau} h_0] - T_{i\partial_z p}[A^{p}_{\tau} h_0] = 0,$$

which gives (4.1.54) by the definition of A^{p}_{τ} and the Duhamel formula. The identities (4.1.52) and (4.1.53) are obtained in the same way. $\square$

Proposition 4.1.6. *Consider two real numbers $\delta < 1$, $\rho > 1$, $\rho \notin \mathbb{N}$, and two symbols $p, p' \in \Gamma^{\delta}_{\rho}(\mathbb{D})$ verifying:*

$$\operatorname{Im}(p) = \frac{p - p^*}{2i} \in \Gamma^{\tilde{\delta}}_{0}(\mathbb{R}), \operatorname{Im}(p') = \frac{p' - p'^*}{2i} \in \Gamma^{\tilde{\delta}}_{0}(\mathbb{R}), \ \tilde{\delta} \leq 0,$$

$$\big([\langle \mathrm{D} \rangle^s, \sigma^{\psi}_{p}], [\langle \mathrm{D} \rangle^s, \sigma^{\psi}_{p'}]\big) \in \Gamma^{0}_{0}(\mathbb{D}), \ \textit{for all } s \in \mathbb{R}.$$

Let $A^{p}_{\tau}, A^{p'}_{\tau}, \tau \in \mathbb{R}$ be the flow maps defined by Proposition 4.1.1 and take a symbol $b \in \Gamma^{\beta}_{\rho}(\mathbb{R})$ then for $\tau \in \mathbb{R}$ we have:

$$T^{lim}_{b^{p}_{\tau}} - T^{lim}_{b^{p'}_{\tau}} = \int_0^{\tau} A^{p}_{\tau - r} \mathscr{L}_{iT_{p-p'}} T^{lim}_{b^{p'}_{r}} A^{p}_{r - \tau} dr \tag{4.1.55}$$

$$= \int_0^{\tau} \mathscr{L}_{iT^{lim}_{p-(p')^{p}_{\tau - r}}} T^{lim}_{(b^{p'}_{r})^{p}_{\tau - r}} dr. \tag{4.1.56}$$

Another way to express this is with the Gateaux derivative of $p \mapsto T^{lim}_{b^p_\tau}$ on the Fréchet space $\Gamma^\delta_\rho(\mathbb{R})$ is given by:

$$D_p T^{lim}_{b^p_\tau}(h) = \int_0^\tau \mathscr{L}_{iT^{lim}_{h^p_{\tau-r}}} T^{lim}_{b^p_\tau} dr = \mathscr{L}_{i\int_0^\tau T^{lim}_{h^p_{\tau-r}} dr} T^{lim}_{b^p_\tau}. \tag{4.1.57}$$

Writing, $T^{lim}_{{}^c b^p_\tau} = T_b - T^{lim}_{b^p_{-\tau}}$, and, $T^{lim}_{{}^c b^{p'}_\tau} = T_b - T^{lim}_{b^{p'}_{-\tau}}$ we get:

$$T^{lim}_{{}^c b^p_\tau} - T^{lim}_{{}^c b^{p'}_\tau} = -\int_0^{-\tau} A^p_{-\tau-r} \mathscr{L}_{iT_{p-p'}} T^{lim}_{b^{p'}_r} A^p_{r+\tau} dr \tag{4.1.58}$$

$$= -\int_0^{-\tau} \mathscr{L}_{iT^{lim}_{p-(p')^p_{-\tau-r}}} T^{lim}_{(b^{p'}_r)^p_{-\tau-r}} dr. \tag{4.1.59}$$

$$D_p T^{lim}_{{}^c b^p_\tau}(h) = -\int_0^{-\tau} \mathscr{L}_{iT^{lim}_{h^p_{-\tau-r}}} T^{lim}_{b^p_{-\tau}} dr = -\mathscr{L}_{i\int_0^{-\tau} T^{lim}_{h^p_{-\tau-r}} dr} T^{lim}_{b^p_{-\tau}}. \tag{4.1.60}$$

Proof. From (4.1.39) and (4.1.40) we have:

$$\begin{cases} \partial_\tau [T^{lim}_{b^p_\tau} - T^{lim}_{b^{p'}_\tau}] = \mathscr{L}_{iT_p}(T^{lim}_{b^p_\tau} - T^{lim}_{b^{p'}_\tau}) + \mathscr{L}_{iT_{p-p'}} T^{lim}_{b^{p'}_\tau}, \\ T^{lim}_{b^p_0} - T^{lim}_{b^{p'}_0} = 0. \end{cases} \tag{4.1.61}$$

Thus the Duhamel formula gives (4.1.55) and (4.1.56). For the Gateaux derivative passing to the limit in (4.1.55) we have:

$$D_p T^{lim}_{b^p_\tau}(h) = \int_0^\tau A^p_{\tau-r} \mathscr{L}_{iT_h} T^{lim}_{b^p_r} A^p_{r-\tau} dr,$$

which gives (4.1.57). □

We now study the composition of two different flows.

Theorem 4.1.1. *Consider two real numbers $\delta < 1$, $\rho \geq 0$, two symbols $p, p' \in \Gamma^\delta_\rho(\mathbb{D})$ such that,*

$$\mathrm{Im}(p) = \frac{p - p^*}{2i} \in \Gamma^{\tilde\delta}_0(\mathbb{R}), \mathrm{Im}(p') = \frac{p' - p'^*}{2i} \in \Gamma^{\tilde\delta}_0(\mathbb{R}),\ \tilde\delta \leq 0,$$

$$\big([\langle \mathrm{D}\rangle^s, \sigma^\psi_p], [\langle \mathrm{D}\rangle^s, \sigma^\psi_{p'}]\big) \in \Gamma^0_0(\mathbb{D}),\ \textit{for all } s \in \mathbb{R}.$$

Then for $\tau \in \mathbb{R}$ we have:

$$A^p_\tau A^{p'}_\tau = \tilde{A}^{p+(p')^p_\tau}_\tau,$$

where $\tilde{A}^{p+(p')^p_\tau}_\tau$ is understood as the flow generated by $iT_p + iT^{lim}_{(p')^p_\tau}$.

Remark 4.1.5. *Strictly speaking we only presented flows that were generated by operators independent of the τ variable which is not the case of $\tilde{A}^{p+(p')^p_\tau}_\tau$. We did so to avoid burdening the presentation, one can see all the results of this section can in verbatim be generalized to operators with Lipschitz dependence on τ by the usual Cauchy-Lipschitz theorem.*

Moreover by Appendix 2.2.3 $T^{lim}_{(p')^p_\tau}$ enjoys all of the same properties as a paradifferential operator with the usual cut-off except the continuity for $s \leq 0$. We still recover the continuity for $s \leq 0$ by the continuity of A^p_τ for $s \leq 0$. We can also recover the continuity for $s \leq 0$ by Proposition 4.1.3.

Proof. Fix $h_0 \in H^s, s \in \mathbb{R}$ and compute:

$$\partial_\tau [A^p_\tau A^{p'}_\tau h_0] = -iT_p[A^p_\tau A^{p'}_\tau h_0] - i[A^p_\tau T_{p'} A^{p'}_\tau h_0],$$

thus by Proposition 4.1.4,

$$\partial_\tau [A^p_\tau A^{p'}_\tau h_0] = -i(T_p + T^{lim}_{(p')^p_\tau})[A^p_\tau A^{p'}_\tau h_0],$$

and $A^p_\tau A^{p'}_\tau h_0(0, \cdot) = h_0(\cdot)$ which gives the desired result. $\square$

Control of the differential of the gauge transform in high Sobolev regularity

Corollary 4.1.2. *Consider three real numbers $\alpha > 1$, $\beta < \alpha$ and $s \in \mathbb{R}$, two real valued symbols $a \in \Gamma^\alpha_{2+\frac{1+\beta-\alpha}{\alpha-1}}(\mathbb{D})$ and $b \in \Gamma^\beta_{1+\frac{1+\beta-\alpha}{\alpha-1}}(\mathbb{D})$. Suppose that there exists a real valued symbol $p \in \Gamma^{\beta+1-\alpha}_{2+\frac{1+\beta-\alpha}{\alpha-1}}(\mathbb{D})$ such that:*

$$b = -\partial_\xi p \partial_x a + \partial_x p \partial_\xi a. \tag{4.1.62}$$

Define $A^p_\tau(u)$ as the flow map generated by iT_p from Proposition 4.1.1. For $\tau \in \mathbb{R}$, Let,

$$R_\tau = \tau T_{ib} + \int_0^\tau A^p_{-s}[T_{ip}, T_{ia}]A^p_s ds, \tag{4.1.63}$$

and,

$$\tilde{R_\tau} = A^p_\tau R_\tau A^p_{-\tau} = \tau A^p_\tau i T_b A^p_{-\tau} + [A^p_\tau, T_{ia}]A^p_{-\tau}. \tag{4.1.64}$$

Then $R_\tau, \tilde{R_\tau} \in \mathscr{L}(H^{s+(\beta+1-\alpha)^+}(\mathbb{D}), H^s(\mathbb{D}))$ and

$$\left\|(R_\tau, \tilde{R_\tau})\right\|_{H^{s+(\beta+1-\alpha)^+} \to H^s} \leq C M^\alpha_{2+\frac{1+\beta-\alpha}{\alpha-1}}(a) M^\beta_{1+\frac{1+\beta-\alpha}{\alpha-1}}(b) M^{\beta+1-\alpha}_{2+\frac{1+\beta-\alpha}{\alpha-1}}(p).$$

Moreover taking three different symbols a', b' and p' and defining analogously $R'_\tau, \tilde{R}'_\tau$, we have for $h \in H^s$:

$$\begin{aligned} &\left\|[R_\tau - R'_\tau]h\right\|_{H^s} \\ &\leq C M^\alpha_{2+\frac{1+\beta-\alpha}{\alpha-1}}(a, a', a-a') M^\beta_{1+\frac{1+\beta-\alpha}{\alpha-1}}(b, b', b-b') M^{\beta+1-\alpha}_{2+\frac{1+\beta-\alpha}{\alpha-1}}(p, p', p-p') \\ &\quad \times \|h\|_{H^{s+(\beta+1-\alpha)^+}}, \end{aligned}$$

and,

$$\begin{aligned} &\left\|[\tilde{R}_\tau - \tilde{R}'_\tau]h\right\|_{H^s} \\ &\leq C M^\alpha_{2+\frac{1+\beta-\alpha}{\alpha-1}}(a, a', a-a') M^\beta_{1+\frac{1+\beta-\alpha}{\alpha-1}}(b, b', b-b') M^{\beta+1-\alpha}_{2+\frac{1+\beta-\alpha}{\alpha-1}}(p, p', p-p') \\ &\quad \times \|h\|_{H^{s+(\beta+1-\alpha)^+}}. \end{aligned}$$

Proof. First we notice that by definition $R_\tau, \tilde{R_\tau}$ are of order β and that we can write as p, a and b have a regularity of $2+\frac{2-\alpha}{\alpha-1}$ and $1+\frac{2-\alpha}{\alpha-1}$ respectively:

$$\begin{cases} R_\tau = T^{lim}_{r_\tau^{(\beta)}} + T^{lim}_{r_\tau^{(\beta+1-\alpha)}} + R_\tau^{(\alpha-1-\beta)^-}, \\ \tilde{R_\tau} = T^{lim}_{\tilde{r_\tau}^{(\beta)}} + T^{lim}_{\tilde{r_\tau}^{(\beta+1-\alpha)}} + \tilde{R_\tau}^{(\alpha-1-\beta)^-}, \end{cases}$$

where $r_\tau^{(\beta)}, r_\tau^{(\beta+1-\alpha)}, \tilde{r_\tau}^{(\beta)}, \tilde{r_\tau}^{(\beta+1-\alpha)}$ are operators in the usual paradifferential classes and thus there differential with respect to p do not generate the undesired loss of $1+\beta-\alpha$ derivative.

Now by Proposition 4.1.4:

$$T^{lim}_{r_\tau^{(\beta)}} = T^{lim}_{\tilde{r_\tau}^{(\beta)}} = T_{ib} + [T_{ip}, T_{ia}],$$

but the choice of p ensures by Theorem 2.2.6 that $T_{ib}+[T_{ip},T_{ia}]$ is of order $\beta+1-\alpha$, giving the desired result.

□

4.2 Appendix: Continuity of limited regularity paradifferential exotic symbols on L^p spaces

We start by giving the following analogue of Theorem 2.1.A of [74].

Theorem 4.2.1. *Consider four real numbers $r>0, m\in\mathbb{R}$ and $0\leq\delta,\rho\leq 1$, then for all $a(x,\xi)\in C^r_* S^m_{\rho,\delta}$ such that $a^*(x,\xi)\in C^r_* S^m_{\rho,\delta}$ where:*

$$a^*(x,\xi) = \frac{1}{2\pi}\int_{\mathbb{D}\times\hat{\mathbb{D}}} e^{-iy.\eta}\bar{a}(x-y,\xi-\eta)dyd\eta,$$

then,

$$\mathrm{Op}(a) : W^{s+m+(\frac{1}{2}-\frac{1}{p})(1-\rho),p} \to W^{s,p}, \text{ with } p\in[2,+\infty]$$

provided $0<s<r$. Furthermore, under these hypothesis,

$$\mathrm{Op}(a) : C_*^{s+m+\frac{1}{2}(1-\rho)} \to C^s_*.$$

Moreover there exists a constant K such that:

$$\|\mathrm{Op}(a)\|_{W^{s+m+(\frac{1}{2}-\frac{1}{p})(1-\rho),p}\to W^{s,p}} \leq K\ {}^*M^{m,r}_{\rho,\delta}(a;1), \text{ and,}$$

$$\|\mathrm{Op}(a)\|_{C_*^{s+m+\frac{1}{2}(1-\rho)}\to C^s_*} \leq K\ {}^*M^{m,r}_{\rho,\delta}(a;1).$$

Remark 4.2.1. *In higher dimension the factor $(\frac{1}{2}-\frac{1}{p})(1-\rho)$ should be adapted to $d(\frac{1}{2}-\frac{1}{p})(1-\rho)$ and the semi norm of order 1 in the ξ variable in the estimates should be adapted to $\lfloor\frac{d}{2}\rfloor+1$.*

If moreover $\delta<1$ then all of the previous results extend to L^2 continuity (i.e $s=0$), this results from an almost orthogonal decomposition combined with a TT^* argument as shown in Theorem 2, Section 2.5 of [71].

It's a result by Hörmander [36] that if $\delta<\rho$ or $\delta=\rho<1$ the hypothesis on a^* is automatically verified. This hypothesis is also shown to be necessary for $\rho=\delta=1$.

Proof. We first notice that it suffices to make H^s and C^s_* estimates as the L^p are obtained directly by interpolation.

The key estimate follows from the following adaptation of Lemma 4.3.2 of [53]:

Lemma 4.2.1. *There are constants C and C' such that, for all $\lambda > 0$ and $q \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ satisfying:*

$$\operatorname{supp} q \subset \mathbb{R}^d \times \{|\xi| \le \lambda\}, \quad M = \sup_{|\beta| \le \tilde{d}} \lambda^{\rho|\beta|} \left\| \partial_\xi^\beta q \right\|_{L^\infty} < \infty, \text{ with } \tilde{d} = \left\lfloor \frac{d}{2} \right\rfloor + 1.$$

Suppose moreover that q and it's derivatives in ξ are uniformly continuous on $\mathbb{R}^d \times \mathbb{R}^d$. Then the function,

$$Q(y) = \int e^{-iy\cdot\xi} q(y,\xi) d\xi,$$

satisfies:

$$\int (1 + |\lambda y|^2)^{\tilde{d}} |Q(y)|^2 dy \le C\lambda^d M^2 (1 + \lambda^{2(1-\rho)})^{\tilde{d}}, \tag{4.2.1}$$

and,

$$\|Q\|_{L^1(\mathbb{R}^d)} \le C' M (1 + \lambda^{2(1-\rho)})^{\frac{\tilde{d}}{2}}. \tag{4.2.2}$$

Proof of Lemma 4.2.1. For $|\alpha| \le \tilde{d}$ we have:

$$y^\alpha Q(y) = \int e^{-iy\cdot\xi} D_\xi^\alpha q(y,\xi) d\xi.$$

At this step we would like to apply Plancherel's theorem to deduce:

$$\int \left|y^{2\alpha}\right| |Q(y)|^2 dy \le C\lambda^{d-2\rho\alpha} M^2, \tag{4.2.3}$$

which is the argument given in [53]. As the application of the Plancherel's theorem does not seem immediate to us we opted to expand upon it to make it's application more immediate. We first notice that it suffices to prove (4.2.3) for $\alpha = 0$. To do so we introduce the function:

$$\tilde{Q}(x,y) = \int e^{-iy\cdot\xi} q(x,\xi) d\xi = \mathscr{F}_\xi q(x,y),$$

in this setting we can apply Plancherel's theorem to deduce:

$$\|Q\|_{L^\infty_x L^2_y} \le \|q\|_{L^\infty_x L^2_\xi}. \tag{4.2.4}$$

getting back to (4.2.3) we want to estimate $\left\|\tilde{Q}(y,y)\right\|^2_{L^2_y(\mathbb{R}^d)}$, to do so we estimate uniformly on cubes the norms $\left\|\tilde{Q}(y,y)\right\|^2_{L^2_y(C(y_0,R))}$. For this we define the set:

$$K(y_0,R,\epsilon) = \{(x,y), y \in C(y_0,R), |x_j - y_j| \le \epsilon, j \in [1,\cdots,d]\}.$$

Thus by the fundamental Theorem of calculus:

$$\frac{1}{c_d\epsilon^d} \int_{K(y_0,R,\epsilon)} \left|\tilde{Q}(x,y)\right|^2 dxdy \underset{\epsilon\to 0}{\longrightarrow} \int_{C(y_0,R)} \left|\tilde{Q}(y,y)\right|^2 dy$$

$$\left\|\tilde{Q}(y,y)\right\|^2_{L^2_y(C(y_0,R))} \leq C_{K(y_0,R,\epsilon)}\frac{1}{\epsilon^d}\left\|\tilde{Q}(x,y)\right\|^2_{L^2(K(R,\epsilon))}$$
$$\leq C_{K(R,\epsilon)}\left\|q\right\|^2_{L^\infty_x(C(0,\epsilon);L^2_\xi(C(0,R))},$$

where the constant $C_{K(y_0,R,\epsilon)}$ can be chosen uniformly in y_0 by the uniform continuity of q. Now to explicit the dependence of $C_{K(R,\epsilon)}$ on the different parameters, by Plancherel's theorem we have:

$$\frac{R^d}{c'_d\epsilon^d}\int_{\mathbb{R}^d}\left|\prod_{j=1}^d \operatorname{sinc}\left(2\xi_j R\right)e^{iy_0^j\xi_j} * q(x,\xi)\right|^2 dxd\xi = \frac{1}{c_d\epsilon^d}\int_{K(R,\epsilon)}\left|\tilde{Q}(x,y)\right|^2 dxdy$$
$$\underset{\epsilon\to 0}{\longrightarrow}\int_{C(0,R)}\left|\tilde{Q}(y,y)\right|^2 dy.$$

Thus we cover the diagonal (y,y) in $\mathbb{R}^d\times\mathbb{R}^d$ by compact sets $(K(y_0^i,R_i,\epsilon_i))_{i\in\mathbb{N}}$ where y_0^i, R_i and ϵ_i are chosen in a manner to ensure that the sum of the volume of the different intersections between elements of this cover is summable.

Thus we have:
$$\int\left|y^{2\alpha}\right|\left|Q(y)\right|^2 dy \leq C\lambda^{d-2\rho\alpha}M^2.$$

Multiplying by $\lambda^{2|\alpha|}$ and summing in α, implies (4.2.1). Since $\tilde{d} > \frac{d}{2}$, the second estimate (4.2.2) follows. □

Getting back to the proof of Theorem 4.2.1, we are working with $d = \tilde{d} = 1$ in Lemma 4.2.1.

We start by making a Littlewood-Paley type decomposition by writing:

$$a(x,\xi) = a(x,\xi)P_0(\xi) + \sum_{k=1}^{\infty} a(x,\xi)P_k(\xi) = a_0(x,\xi) + \sum_{k=1}^{\infty} a_k(x,\xi).$$

The proof will be divided in two paragraphs where we do the Sobolev and Zygmund estimates respectively.

Continuity in H^s

Lemma 4.2.2. *For $k \geq 0$, $\operatorname{Op}(a_k)$ maps to L^2 to H^∞. Moreover for all $\alpha \in \mathbb{N}$, there is C_α such that for all $a \in C^r_* S^m_{\rho,\delta}$, $k \geq 0$ and all $f \in L^2$:*

$$\left\|\partial_x^\alpha \operatorname{Op}(a_k)f\right\|_{L^2} \leq C_\alpha M^{m,\alpha}_{\rho,\delta}(a;1)\left\|f\right\|_{L^2} 2^{k(m+\alpha)} \tag{4.2.5}$$

Proof of Lemma 4.2.2. Since a_k is compactly supported in ξ, one sees that $\operatorname{Op}(a_k)f$ is given by the convergent integral:

$$\operatorname{Op}(a_k)f(x) = \int A_k(x,y)f(y)dy, \tag{4.2.6}$$

where the kernel $A_k(x,y)$ is given by the convergent integral:

$$\operatorname{Op}(a_k) = \frac{1}{2\pi}\int e^{i(x-y)\xi}a_k(x,\xi)d\xi. \tag{4.2.7}$$

Moreover on the support of a_k, $1+|\xi|\simeq 2^k$. Therefore Lemma 4.2.1 can be applied with $\lambda = 2^{k+1}$, implying that:

$$\int (1+2^{2k}|x-y|^2)\,|A_k(x,y)|^2\,dy \leq C2^{2km+k+2(1-\rho)k} M^{m,0}_{\rho,\delta}(a;1)^2. \tag{4.2.8}$$

Hence for $f\in \mathscr{S}(\mathbb{D})$, Cauchy-Schwartz inequality implies that:

$$|\mathrm{Op}(a_k)f(x)|^2 \leq C2^{2km+k+2(1-\rho)k} M^{m,0}_{\rho,\delta}(a;1)^2 \int \frac{2^{2km+k+2(1-\rho)k}\,|f(y)|^2}{(1+2^{2k}|x-y|^2)}dy. \tag{4.2.9}$$

The integral $2^k\int(1+2^{2k}|x-y|^2)^{-1}dx = C'$ is finite and independent of k. Thus:

$$\|\mathrm{Op}(a_k)f\|_{L^2} \leq C_\alpha M^{m,0}_{\rho,\delta}(a;1)\,\|f\|_{L^2}\,2^{k(m+(1-\rho))}. \tag{4.2.10}$$

In order to eliminate the extra factor $2^{k(1-\rho))}$ in (4.2.10), we use the fundamental TT^* trick, indeed writing $(a_k)^*$ as the formal symbol of the operator $(\mathrm{Op}(a_k))^*$, by the frequency localization we see that $(a_k)^*\in C^r_* S^m_{\rho,\delta}$. Thus $a_k(a_k)^*\in C^r_* S^{2m}_{\rho,\delta}$ and applying the previous estimate, as it's uniform in the choice of symbol, to $\mathrm{Op}(a_k)(\mathrm{Op}(a_k))^*$ we get:

$$\|\mathrm{Op}(a_k)(\mathrm{Op}(a_k))^* f\|_{L^2} \leq C_\alpha M^{m,0}_{\rho,\delta}(a;1)^2\,\|f\|_{L^2}\,2^{k(2m+(1-\rho))}, \tag{4.2.11}$$

thus by the standard TT^* lemma:

$$\|\mathrm{Op}(a_k)f\|_{L^2} \leq C_\alpha M^{m,0}_{\rho,\delta}(a;1)\,\|f\|_{L^2}\,2^{k(m+\frac{1}{2}(1-\rho))}. \tag{4.2.12}$$

Iterating this estimate we get (4.2.5) for $\alpha=0$. The symbol ∂_x^α is $(i\xi+\partial_x)^\alpha a_k(x,\xi)$, which gives the desired estimate for larger α. □

Now getting back to the continuity in H^s of $\mathrm{Op}(a)$ we write for $f\in\mathscr{S}(\mathbb{D})$ by the support localization of a_k:

$$\mathrm{Op}(a_k)f = \sum_{|j-k|\leq 3} \mathrm{Op}(a_k)P_j f,$$

Thus by Lemma 4.2.2,

$$\|\partial_x^\alpha \mathrm{Op}(a_k)f\|_{L^2} \leq C_\alpha M^{m,\alpha}_{\rho,\delta}(a;1)\sum_{|j-k|\leq 3}\|P_j f\|_{L^2}\,2^{k(m+\alpha)},$$

then by Definition 2.1.5,

$$\|\partial_x^\alpha \mathrm{Op}(a_k)f\|_{L^2} \leq C_\alpha M^{m,\alpha}_{\rho,\delta}(a;1)2^{k(\alpha-s)}\epsilon_k, \tag{4.2.13}$$

with,

$$\sum_k \epsilon_k^2 \leq \|f\|_{H^{s+m}}\,. \tag{4.2.14}$$

Now to conclude we recall the following Proposition from [53]:

Proposition 4.2.1 (Proposition 4.1.13 of [53]). *Let $0 < s$ and let n be an integer, $n > s$. There is a constant C such that, for all sequence $(f_k)_{k\geq 0} \in H^n(\mathbb{D}^d)$ satisfying for all $\alpha \in \mathbb{N}^d, |\alpha| \leq n$:*

$$\|\partial_x^\alpha f_k\|_{L^2(\mathbb{D}^d)} \leq 2^{k(|\alpha|-s)}\epsilon_k, \ \textit{with } (\epsilon_k) \in l^2, \tag{4.2.15}$$

the sum $f = \sum f_k$ belongs to $H^s(\mathbb{D}^d)$ and,

$$\|f\|^2_{H^s(\mathbb{D}^d)} \leq C \sum_{k=0}^{\infty} \epsilon_k^2. \tag{4.2.16}$$

Applying Proposition 4.2.1 to $\mathrm{Op}(a_k)f$ we get the desired Sobolev continuity and the desired estimate.

Continuity in C^s_* The proof follows the same lines as previously, indeed applying (4.2.2) to (4.2.6) we get the following Lemma.

Lemma 4.2.3. *For $k \geq 0$, $\mathrm{Op}(a_k)$ maps to L^∞ to $W^{\infty,\infty}$. Moreover for all $\alpha \in \mathbb{N}$, there is C_α such that for all $a \in C^r_* S^m_{\rho,\delta}$, $k \geq 0$ and all $f \in L^\infty$:*

$$\|\partial_x^\alpha \mathrm{Op}(a_k)f\|_{L^\infty} \leq C_\alpha M^{m,\alpha}_{\rho,\delta}(a;1)\, \|f\|_{L^\infty}\, 2^{k(m+\alpha+\frac{1}{2})}. \tag{4.2.17}$$

Again we have:

$$\|\partial_x^\alpha \mathrm{Op}(a_k)f\|_{L^\infty} \leq C_\alpha M^{m,\alpha}_{\rho,\delta}(a;1)2^{k(\alpha-s)}\, \|f\|_{C_*^{s+m+\frac{1}{2}}}, \tag{4.2.18}$$

which gives the desired result and estimate. $\square$

Theorem 4.2.2. *Consider four real numbers $r > 0, m \in \mathbb{R}$ and $0 \leq \delta, \rho \leq 1$, then for all $a(x,\xi) \in \Gamma^0 S^m_{\rho,\delta}$,*

$$T_a : W^{s+m+(\frac{1}{2}-\frac{1}{p})(1-\rho),p} \to W^{s,p}, \ \textit{with } p \in [2,+\infty], \quad s \in \mathbb{R},$$

$$T_a^{lim} : W^{s+m+(\frac{1}{2}-\frac{1}{p})(1-\rho),p} \to W^{s,p}, \ \textit{with } p \in [2,+\infty], \quad s > 0.$$

Furthermore, under these hypothesis,

$$T_a : C_*^{s+m+\frac{1}{2}(1-\rho)} \to C^s_*, \quad s \in \mathbb{R},$$

$$T_a^{lim} : C_*^{s+m+\frac{1}{2}(1-\rho)} \to C^s_*, \quad s > 0.$$

Moreover there exists a constant K such that:

$$\|T_a\|_{W^{s+m+(\frac{1}{2}-\frac{1}{p})(1-\rho),p}\to W^{s,p}} \leq K\ M^{m,0}_{\rho,\delta}(a;1), \ \textit{and,}$$

$$\|T_a\|_{C_*^{s+m+\frac{1}{2}(1-\rho)}\to C^s_*} \leq K\ M^{m,0}_{\rho,\delta}(a;1),$$

$$\left\|T_a^{lim}\right\|_{W^{s+m+(\frac{1}{2}-\frac{1}{p})(1-\rho),p}\to W^{s,p}} \leq K\ M^{m,0}_{\rho,\delta}(a;1), \ \textit{and,}$$

$$\left\|T_a^{lim}\right\|_{C_*^{s+m+\frac{1}{2}(1-\rho)}\to C^s_*} \leq K\ M^{m,0}_{\rho,\delta}(a;1),$$

Proof. This simply follows from the spectral localization property of paradifferential operators, indeed taking $f \in \mathscr{S}$, then $\mathrm{Op}(\sigma_{ak})f$ is supported in a ring C_k where $|\xi| \sim 2^k$, which is not necessarily the case for $\mathrm{Op}(a_k)f$. The spectral localization property also ensures that the adjoint operator verifies the hypothesis of Theorem 4.2.1. Thus rewriting estimates (4.2.13) and (4.2.18) with $\alpha = 0$ then by definition of Sobolev spaces and Zygmund spaces using the Littlewood-Paley decomposition we get:

$$\|T_a\|_{W^{s+m+(\frac{1}{2}-\frac{1}{p})(1-\rho),p} \to W^{s,p}} \leq K\ {}^*M^{m,s}_{\rho,\delta}(\sigma_a;2), \text{ and,}$$
$$\|T_a\|_{C_*^{s+m+\frac{1}{2}(1-\rho)} \to C^s_*} \leq K\ {}^*M^{m,s}_{\rho,\delta}(\sigma_a;2), \quad s \in \mathbb{R},$$

which gives the desired result by the Bernstein inequalities.

The proof for T^{lim}_a follows exactly the same lines with the sole difference being that $\mathrm{Op}({\sigma^{lim}_a}_k)f$ is supported in the ball $\mathrm{B}(0, C2^k)$ and not a ring C_k, which explains the restrictions to $s > 0$ by Propositions 2.1.3 and 2.1.4. $\square$

Chapter 5

Exact flow map regularity of the weakly dispersive Burgers type equation and the Gravity Capillary equation

In this section we give the proofs of Theorems 1.2.2 and 1.3.3 and Corollary 1.2.1, which are the main the results from [63].

Contents

5.1 Study of the model problems

5.1.1 Proof of Theorem 1.2.2, the estimates on H_0^s

We keep the notations of Theorem 1.2.2, fixing $u_0 \in H_0^s(\mathbb{T};\mathbb{R})$ and $r > 0$ and taking:

$$v_0, w_0 \in \mathrm{B}(u_0, r) \subset H_0^s(\mathbb{T};\mathbb{R}).$$

As the mean value is conserved by the flow of (1.2.2) we consider the solutions $u, v, w \in C^0([0,T]; H_0^s(\mathbb{T};\mathbb{R}))$ to (1.2.2) with initial data u_0, v_0, w_0 and on a uniform small interval $[0,T]$.

The main goal of the proof is to show the following estimate:

$$\|v(t,\cdot) - w(t,\cdot)\|_{H^{s-(2-\alpha)^+}} \leq C(\|(v_0,w_0)\|_{H^s}) \|v_0 - w_0\|_{H^{s-(2-\alpha)^+}}, \tag{5.1.1}$$

with the following tame control,

$$C(\|(v_0,w_0)\|_{H^s}) \leq C'(\|(v_0,w_0)\|_{H^{s-(2-\alpha)^+}})[\|(v_0,w_0)\|_{H^s}+1], \tag{5.1.2}$$

where C and C' are non decreasing positive functions.

The final simplification we make in this paragraph is that given the well-posedness of the Cauchy problem in H^s, and the density of $H^{+\infty}$ in H^s, it suffice to prove (5.1.1) for $v_0, w_0 \in H^{+\infty}$, which henceforth we will suppose.

We start by applying the paralinearization Theorem 2.2.6 to the term $u\partial_x v$ to get:

$$\begin{cases} \partial_t v + T_{vi\xi} v + T_{i|\xi|^{\alpha-1}\xi} v = R_1(v)v, \\ v(0,\cdot) = v_0(\cdot), \end{cases} \tag{5.1.3}$$

where R_1 verifies as $s > 1 + \frac{1}{2}$:

$$\begin{aligned} &\|R_1(v)\|_{H^{s-(2-\alpha)^+} \to H^{s-(2-\alpha)^+}} \leq C(\|v\|_{W^{1,\infty}}) \leq C(\|v\|_{H^s}), \\ &\|[R_1(v)-R_1(w)]v\|_{H^{s-(2-\alpha)^+} \to H^{s-(2-\alpha)^+}} \leq C\|v-w\|_{H^{s-(2-\alpha)^+}} \|v\|_{H^s}, \end{aligned}$$

where C verifies 5.1.2. Now we reduce $H^{s-(2-\alpha)^+}$ estimates to L^2 ones by defining $f_1 = \langle \mathrm{D} \rangle^{s-(2-\alpha)^+} v$. Commuting $\langle \mathrm{D} \rangle^{s-(2-\alpha)^+}$ with (5.1.3), using the symbolic calculus rules of Theorem 2.2.6, we get that:

$$\begin{cases} \partial_t f_1 + T_{vi\xi} f_1 + T_{i|\xi|^{\alpha-1}\xi} f_1 = R_1(v) f_1 \\ f_1(0,\cdot) = \langle \mathrm{D} \rangle^{s-(2-\alpha)^+} v_0(\cdot), \end{cases} \tag{5.1.4}$$

where R_1 was modified to include terms verifying the same estimate i.e:

$$\begin{aligned} \|R_1(v)\|_{L^2 \to L^2} &\leq C(\|v\|_{H^s}), \\ \|[R_1(v)-R_1(w)]f_1\|_{L^2} &\leq C\|v-w\|_{H^{s-(2-\alpha)^+}} \|f_1\|_{H^{(2-\alpha)^+}}. \end{aligned}$$

We define analogously $g_1 = \langle \mathrm{D} \rangle^{s-(2-\alpha)^+} w$ and notice that by definition:

$$\|f_1 - g_1\|_{L^2} = \|v-w\|_{H^{s-(2-\alpha)^+}},$$

thus the problem is reduced to getting L^2 estimates on $f_1 - g_1$.

Here we explain the scheme of the proof using estimates we proved in Section 4.1.

5.1.1.1 Gauge transform and Energy estimate

The goal of this section is to find an operator A_v such that

$$\partial_t [A_v f_1] + A_v T_{i|\xi|^{\alpha-1}\xi} f_1 + A_v T_{vi\xi} f_1 + [A_v, T_{i|\xi|^{\alpha-1}\xi}] f_1 = (\partial_t A_v) f_1 + A_v R_1(f_1) f_1,$$

and $A_v T_{vi\xi} + [A_v, T_{i|\xi|^{\alpha-1}\xi}]$ is a hyperbolic operator of order $(2-\alpha)^+ < 1$.
If we define $V = \partial_x^{-1} v$ which is the periodic zero mean value primitive of v, then

$$\hat{V}(0) = 0 \text{ and } \hat{V}(\xi) = \frac{\hat{v}(\xi)}{i\xi}, \text{ for } \xi \in \mathbb{Z}^*,$$

and we define analogously W from w. Then a formal computation shows that one can choose $A_v = T_{e^{\frac{1}{i\alpha}\xi|\xi|^{1-\alpha}V}} \in S^0_{\alpha-1,2-\alpha}(\mathbb{T}\times\mathbb{Z})$ which is a symbol class with no general symbolic calculus rules. Here we will define A_v differently [1].

A_v is defined as the time one of the flow map generated by:

$$p_v = \frac{1}{\alpha}\xi\,|\xi|^{1-\alpha}\,V \in \Gamma^{2-\alpha}_2(\mathbb{T}),$$

which is well defined by Proposition 4.1.1. We define analogously A_w and p_w from w. Now introduce:

$$f_2 = A_v f_1, \quad g_2 = A_w g_1. \tag{5.1.5}$$

The study of the symbolic calculus associated to this very specific form of symbols is given by Proposition 4.1.1 and the change of variable (5.1.5) is Lipschitz from L^2 to L^2 but under $H^{(2-\alpha)^+}$ control on (f_2, g_2). Indeed we write:

$$\begin{aligned}\|f_2 - g_2\|_{L^2} &= \|A_v f_1 - A_w g_1\|_{L^2}\\ &\le \|A_v[f_1 - g_1]\|_{L^2} + \|(A_v - A_w)g_2\|_{L^2}.\end{aligned}$$

Applying estimate (1) of Proposition 4.1.1 and estimate (4.1.4):

$$\begin{aligned}\|f_2 - g_2\|_{L^2} &\le C(\|v\|_{H^s})\,\|f_1 - g_1\|_{L^2} + \|V - W\|_{L^\infty}\,\|g_2\|_{H^{(2-\alpha)^+}}\\ &\le C(\|v\|_{H^s})\,\|f_1 - g_1\|_{L^2} + \|v - w\|_{H^{s-(2-\alpha)^+}}\,\|g_2\|_{H^{(2-\alpha)^+}}\\ &\le C(\|(v,w)\|_{H^s})\,\|f_1 - g_1\|_{L^2},\end{aligned}$$

where C verifies the estimate (5.1.2). As A_v^{-1} and A_w^{-1} are the time -1 generated by the flow map p_v, p_w respectively which is well defined by Proposition 4.1.1. We get by symmetry:

$$\|f_1 - g_1\|_{L^2} \le C(\|(v,w)\|_{H^s})\,\|f_2 - g_2\|_{L^2},$$

thus,

$$C^{-1}(\|(v,w)\|_{H^s})\,\|f_1 - g_1\|_{L^2} \le \|f_2 - g_2\|_{L^2} \le C(\|(v,w)\|_{H^s})\,\|f_1 - g_1\|_{L^2},$$

and the problem is reduced to getting L^2 estimates on $f_2 - g_2$.

To get the equations on f_2 and g_2 we commute A_v and A_w with (5.1.4), we make the computations for f_2, those for g_2 are obtained by symmetry:

$$A_v\partial_t f_1 + A_v T_{vi\xi} f_1 + A_v T_{i|\xi|^{\alpha-1}\xi} f_1 = A_v R_1(v) f_1, \text{ thus,}$$

$$\partial_t\big(A_v f_1\big) + T_{i|\xi|^{\alpha-1}\xi} A_v f_1 + (A_v T_{vi\xi} - [T_{i|\xi|^{\alpha-1}\xi}, A_v]) f_1 + [A_v, \partial_t] f_1 = A_v R_1(v) f_1.$$

By definition of p_v and Proposition 4.1.5 we have:

$$\partial_\xi(\xi\,|\xi|^{\alpha-1})\partial_x p_v = v\xi \text{ and } [A_v, \partial_t] = -A_v \int_0^1 A^p_{-r} T_{i\partial_t p_v} A^p_r\, dr.$$

[1] Similar ideas were used in Appendix C of [7] to get estimates on a change of variable operator which are still in the usual symbol classes $S^m_{1,0}$, the difficulty here being that we are no longer in those symbol classes.

Thus by Corollary 4.1.2 we have:

$$\partial_t f_2 + T_{i|\xi|^{\alpha-1}\xi} f_2 = R_2(v) f_2 + A_v R_1(v) A_v^{-1} f_2, \tag{5.1.6}$$

where R_2 and $A_v R_1(v) A_v^{-1}$ verify:

$$\begin{aligned} \|\mathrm{Re}(R_2(v))\|_{L^2 \to L^2} &\le C(\|v\|_{H^s}), \\ \|[R_2(v) - R_2(w)]g_2\|_{L^2} &\le C \|v-w\|_{H^{s-(2-\alpha)^+}} \|g_2\|_{H^{(2-\alpha)^+}}, \\ \left\|[A_v R_1(v) A_v^{-1} - A_w R_1(w) A_w^{-1}]g_2\right\|_{L^2} &\le C \|v-w\|_{H^{s-(2-\alpha)^+}} \|g_2\|_{H^{(2-\alpha)^+}}. \end{aligned}$$

We get analogously on g_2,

$$\partial_t g_2 + T_{i|\xi|^{\alpha-1}\xi} g_2 = R_2(w) g_2 + A_w R_1(w) A_w^{-1} g_2. \tag{5.1.7}$$

Taking the difference between (5.1.6) and (5.1.7) yields:

$$\partial_t (f_2 - g_2) + T_{i|\xi|^{\alpha-1}\xi}(f_2 - g_2) = [R_2(w) - R_2(v) - A_v R_1(v) A_v^{-1} + A_w R_1(w) A_w^{-1}] g_2.$$

Thus the usual energy estimate combined with the Gronwall lemma on $f_2 - g_2$ gives for $0 \le t \le T$:

$$\begin{aligned} \|f_2 - g_2\|_{L^2} &\le C(\|(u_0, v_0)\|_{H^s}, \|(f_2, g_2)(0,\cdot)\|_{H^{(2-\alpha)^+}}) \|(f_2 - g_2)(0,\cdot)\|_{L^2} \\ &\le C(\|(u_0, v_0)\|_{H^s}) \|(f_2 - g_2)(0,\cdot)\|_{L^2}, \end{aligned}$$

with C verifying (5.1.2), which concludes the proof.

5.1.2 Proof of Corollary 1.2.1, the estimates on H^s

The starting point is noticing that the mean value is preserved by (1.2.2) and by doing the change of unknowns:

$$\begin{cases} \tilde{u}(t,x) = u(t, x - t⨍ u_0) - ⨍ u_0 \\ \tilde{v}(t,x) = v(t, x - t⨍ v_0) - ⨍ v_0 \end{cases}, \tag{5.1.8}$$

where $⨍ u_0 = \frac{1}{2\pi}\int_{\mathbb{T}} u_0$ is the mean value, we can reduce the Cauchy problem for general data to ones with 0 mean value by verifying that $\tilde{u}, \tilde{v} \in H_0^s$ still solve (1.2.2). Thus the main goal is to prove that the change of variable (5.1.8) is not regular. More precisely we will show that there exists a positive constant C and two sequences (u_ϵ^λ) and (v_ϵ^λ) solutions of 1.2.2 in $C^0([0,1], H^s(\mathbb{T}))$ such that for every $t \le T$, where T is a uniform small time,

$$\sup_{\lambda,\epsilon} \left\|u_\epsilon^\lambda\right\|_{H^s(\mathbb{T})(t,\cdot)} + \left\|v_\epsilon^\lambda(t,\cdot)\right\|_{H^s(\mathbb{T})} \le C,$$

$(u_{\epsilon,\tau}^\lambda)$ and $(v_{\epsilon,\tau}^\lambda)$ satisfy initially:

$$\lim_{\substack{\lambda\to+\infty\\ \epsilon\to 0}} \left\|u_\epsilon^\lambda(0,\cdot) - v_\epsilon^\lambda(0,\cdot)\right\|_{H^s(\mathbb{T})} = 0,$$

but,

$$\liminf_{\substack{\lambda\to+\infty\\ \epsilon\to 0}} \left\|u_\epsilon^\lambda(t,\cdot) - v_\epsilon^\lambda(t,\cdot)\right\|_{H^s(\mathbb{T})} \ge c > 0.$$

Which proves the non uniform continuity. Considering a weaker control norm we want to get, for all $\delta > 0$ and for $t > 0$:

$$\liminf_{\substack{\lambda\to+\infty\\ \epsilon\to 0}} \frac{\left\|u_\epsilon^\lambda(t,\cdot) - v_\epsilon^\lambda(t,\cdot)\right\|_{H^{s-1+\delta}(\mathbb{T})}}{\left\|u_\epsilon^\lambda(0,\cdot) - v_\epsilon^\lambda(0,\cdot)\right\|_{H^s(\mathbb{T})}} = +\infty.$$

5.1.2.1 Definition of the Ansatz

Take $\omega \in C_0^\infty(\mathbb{T})$ such that for $x \in [0, 2\pi]$:

$$\omega(x) = 1 \text{ if } |x| \leq \frac{1}{2},\ \omega(x) = 0 \text{ if } |x| \geq 1.$$

Let (λ, ϵ) be two positive real sequences such that:

$$\lambda \to +\infty,\ \epsilon \to 0,\ \lambda\epsilon \to +\infty. \tag{5.1.9}$$

Put for $x \in [0, 2\pi]$,

$$u^0(x) = \lambda^{\frac{1}{2}-s}\omega(\lambda x),\ v^0(x) = u^0(x) + \epsilon\omega(x),$$

and extend u^0 and v^0 periodically. The main trick here will be to use the time reversibility of equation (1.2.2) by defining $\tilde{u}, \tilde{v}$ as the solution of (1.2.2) with data fixed at time $t > 0$ given by

$$\begin{cases} \tilde{u}(t,x) = u_0 - \fint u_0 \\ \tilde{v}(t,x) = v_0 - \fint v_0 \end{cases}, \tag{5.1.10}$$

where $t \leq t_0$ is chosen small enough for the equations to be well-posed. Finally, define u and v by (5.1.8).

5.1.2.2 Main estimates

First the estimates at time 0, for $0 \leq \nu \leq s$:

$$\|u(0,x) - v(0,x)\|_{H^\nu} = \left\|\tilde{u}(0,x) - \tilde{v}(0,x) + \fint u_0 - \fint v_0\right\|_{H^\nu}$$

By the estimate (5.1.1), the tame control (5.1.2) and the Cauchy-Schwartz inequality,

$$\begin{aligned} \|u(0,x) - v(0,x)\|_{H^\nu} &\leq C(\|(u_0, v_0)\|_{H^{\nu+(2-\alpha)^+}}) \|u_0 - v_0\|_{H^\nu} \\ &\leq C[1 + \lambda^{\nu-s+(2-\alpha)^+}]\epsilon. \end{aligned} \tag{5.1.11}$$

Now the estimates at a fixed time $t > 0$, by construction:

$$\begin{aligned} \|u(t,x) - v(t,x)\|_{H^\nu} &= \left\|u_0(x + t\fint u_0) - v_0(x + t\fint v_0)\right\|_{H^\nu} \\ &= \left\|u_0(x + t\fint u_0) - u_0(x + t\fint v_0)\right\|_{H^\nu} + O_{H^\nu}(\epsilon) \end{aligned}$$

Now by hypothesis $\lambda\epsilon \to +\infty$ and $t\fint \omega > 0$, thus $u_0(\cdot + t\fint u_0)$ and $u_0(\cdot + t\fint v_0)$ have disjoint supports, thus

$$\begin{aligned} \|u(t,x) - v(t,x)\|_{H^\nu} &= \left\|u_0(x + t\fint u_0)\right\|_{H^\nu} + \left\|u_0(x + t\fint v_0)\right\|_{H^\nu} + O_{H^\nu}(\epsilon) \\ &= C\lambda^{\nu-s} + O_{H^\nu}(\epsilon). \end{aligned} \tag{5.1.12}$$

Now to conclude the proof we differentiate the cases:

- in the case of non uniform continuity we take ϵ such that $\epsilon\lambda^{(2-\alpha)^+} \to 0$ and apply the previous estimates with $\nu = s$.
- In the case of non Lipschitz control we take ϵ such that $\lambda^{-1+\delta}\epsilon^{-1} \to +\infty$ and apply the previous estimates with $\nu = s - 1 + \delta$.

5.2 Flow map regularity for the periodic Gravity Capillary equation

5.2.1 Prerequisites from the Cauchy problem

We start by recalling the apriori estimates given by Proposition 5.2 of [3] combined with the results of [8]. We keep the notations of Theorem 1.3.3.

Proposition 5.2.1. *(From [3] and [8]) Consider a real number $s > 2 + \frac{1}{2}$. Then there exists a non decreasing function C such that, for all $T \in]0, 1]$ and all solution (η, ψ) of (1.3.4) such that:*

$$(\eta, \psi) \in C^0([0, T]; H^{s+\frac{1}{2}}(\mathbb{T}) \times H^s(\mathbb{T})) \text{ and } (H_t) \text{ is verified for } t \in [0, T],$$

we have:

$$\|(\eta, \psi)\|_{L^\infty(0,T;H^{s+\frac{1}{2}} \times H^s)} \leq C((\eta_0, \psi_0)_{H^{s+\frac{1}{2}} \times H^s}) + TC(\|(\eta, \psi)\|_{L^\infty(0,T;H^{s+\frac{1}{2}} \times H^s)}).$$

The proof will rely on the para-linearised and symmetrized version of (1.3.4) given by Proposition 4.8 and corollary 4.9 of [3] which are valid on $\mathbb{T}$ as shown in [8]. Before we recall this, for clarity as in [3], we introduce a special class of operators $\Sigma^m \subset \Gamma^m_0$ given by:

Definition 5.2.1. *(From [3]) Given $m \in \mathbb{R}$, Σ^m denotes the class of symbols a of the form*

$$a = a^{(m)} + a^{(m-1)},$$

with,

$$a^{(m)} = F(\partial_x \eta(t, x), \xi),$$
$$a^{(m-1)} = \sum_{|k|=2} G_\alpha(\partial_x \eta(t, x), \xi) \partial_x^k \eta(t, x),$$

such that

1. *T_a maps real valued functions to real-valued functions;*
2. *F is of class C^∞ real valued function of $(\zeta, \xi) \in \mathbb{R} \times (\mathbb{Z} \setminus 0)$, homogeneous of order m in ξ; and such that there exists a continuous function $K = K(\zeta) > 0$ such that*
$$F(\zeta, \xi) \geq K(\zeta) |\xi|^m,$$
for all $(\zeta, \xi) \in \mathbb{R} \times (\mathbb{Z} \setminus 0)$;
3. *G_α is a C^∞ complex valued function of $(\zeta, \xi) \in \mathbb{R} \times (\mathbb{Z} \setminus 0)$, homogeneous of order $m - 1$ in ξ.*

Σ^m enjoys all the usual symbolic calculus properties modulo acceptable remainders that we define by the following:

Definition-Notation 5.2.1. *(From [3]) Let $m \in \mathbb{R}$ and consider two families of operators of order m,*

$$\{A(t) : t \in [0, T]\}, \quad \{B(t) : t \in [0, T]\}.$$

We shall say that $A \sim B$ if $A - B$ is of order $m - \frac{3}{2}$ and satisfies the following estimate: for all $\mu \in \mathbb{R}$, there exists a continuous function C such that for all $t \in [0, T]$,

$$\|A(t) - B(t)\|_{H^\mu \to H^{\mu - m + \frac{3}{2}}} \le C(\|\eta(t)\|_{H^{s+\frac{1}{2}}}).$$

In the next Proposition we recall the different symbols that appear in the paralinearization and symmetrization of the equations.

Proposition 5.2.2. *(From [3])We work under the hypothesis of Proposition 5.2.1. Put*

$$\lambda = \lambda^{(1)} + \lambda^{(0)},\ l = l^{(2)} + l^{(1)} \text{ with,}$$

$$\begin{cases} \lambda^{(1)} = |\xi|, \\ \lambda^{(0)} = \frac{1+|\partial_x \eta|^2}{2|\xi|} \left\{ \partial_x \left(\alpha^{(1)} \partial_x \eta \right) + i \frac{\xi}{|\xi|} \partial_x \alpha^{(1)} \right\}, \\ \alpha^{(1)} = \frac{1}{\sqrt{1+|\partial_x \eta|^2}} \left(|\xi| + i \partial_x \eta \xi \right). \end{cases} \tag{5.2.1}$$

$$\begin{cases} l^{(2)} = (1 + |\partial_x \eta|^2)^{-\frac{3}{2}} \xi^2, \\ l^{(1)} = -\frac{i}{2} (\partial_x \cdot \partial_\xi) l^{(2)}. \end{cases} \tag{5.2.2}$$

Now let $q \in \Sigma^0, p \in \Sigma^{\frac{1}{2}}, \gamma \in \Sigma^{\frac{3}{2}}$ be defined by

$$q = (1 + |\partial_x \eta|^2)^{-\frac{1}{2}},$$
$$p = (1 + |\partial_x \eta|^2)^{-\frac{5}{4}} |\xi|^{\frac{1}{2}} + p^{(-\frac{1}{2})},$$
$$\gamma = \sqrt{l^{(2)} \lambda^{(1)}} + \sqrt{\frac{l^{(2)}}{\lambda^{(1)}}} \frac{\operatorname{Re} \lambda^{(0)}}{2} - \frac{i}{2} (\partial_\xi \cdot \partial_x) \sqrt{l^{(2)} \lambda^{(1)}},$$
$$p^{(-\frac{1}{2})} = \frac{1}{\gamma^{(\frac{3}{2})}} \left\{ q l^{(1)} - \gamma^{(\frac{1}{2})} p^{(\frac{1}{2})} + i \partial_\xi \gamma^{(\frac{3}{2})} \cdot \partial_x p^{(\frac{1}{2})} \right\}.$$

Then

$$T_q T_\lambda \sim T_\gamma T_q,\ T_q T_l \sim T_\gamma T_p,\ T_\gamma \sim (T_\gamma)^*,$$

where $(T_\gamma)^*$ is the adjoint of T_γ.

Now we can write the paralinearization and symmetrization of the equations (1.3.4) after a change of variable:

Corollary 5.2.1. *(From [3]) Under the hypothesis of Proposition 5.2.1, introduce the unknowns[2]*

$$U = \psi - T_B \eta,\ \Phi_1 = T_p \eta \text{ and } \Phi_2 = T_q U,$$

where we recall,

$$\begin{cases} B = (\partial_y \phi)_{|y=\eta} = \frac{\partial_x \eta \cdot \partial_x \psi + G(\eta)\psi}{1+(\partial_x \eta)^2}, \\ V = (\partial_x \phi)_{|y=\eta} = \partial_x \psi - B \partial_x \eta. \end{cases}$$

[2] U is commonly called the "good" unknown of Alinhac. Introduced by Alazard-Metivier in [9], following earlier works by Lannes in [50].

Then $\Phi_1, \Phi_2 \in C^0([0,T]; H^s(\mathbb{T}))$ *and*

$$\begin{cases} \partial_t \Phi_1 + T_V \times \partial_x \Phi_1 - T_\gamma \Phi_2 = f_1, \\ \partial_t \Phi_2 + T_V \times \partial_x \Phi_2 + T_\gamma \Phi_1 = f_2, \end{cases} \tag{5.2.3}$$

with $f_1, f_2 \in L^\infty(0,T; H^s(\mathbb{T}))$, *and* f_1, f_2 *have* C^1 *dependence on* (Φ_1, Φ_2) *verifying:*

$$\|(f_1, f_2)\|_{L^\infty(0,T;H^s(\mathbb{T}))} \leq C(\|(\eta,\psi)\|_{L^\infty(0,T;H^{s+\frac{1}{2}}\times H^s(\mathbb{T}))}).$$

5.2.2 Proof of Theorem 1.3.3

Corollary 5.2.1 shows that the paralinearization and symmetrization of the equations (1.3.4) are of the form of the equations treated in Theorem 1.2.2, so the proof will follow the same lines but with more care in treating the non linearity in the dispersive term.

We keep the notations of Theorem 1.3.3, fixing $(\eta_0, \psi_0) \in H^{s+\frac{1}{2}}(\mathbb{T}) \times H^s(\mathbb{T})$ and $r > 0$. We begin by taking $(\tilde{\eta}_0, \tilde{\psi}_0) \in \mathrm{B}((\eta_0, \psi_0), r) \subset H^{s+\frac{1}{2}}(\mathbb{T}) \times H^s(\mathbb{T})$ and consider the solutions $(\eta, \psi), (\tilde{\eta}, \tilde{\psi}) \in C^0([0,T]; H^{s+\frac{1}{2}}(\mathbb{T}) \times H^s(\mathbb{T}))$ to (1.3.4) with initial data $(\eta_0, \psi_0), (\tilde{\eta}_0, \tilde{\psi}_0)$, on a uniform small interval $[0,T]$ where the hypothesis (H_t) is also supposed to be verified. Define the following change of variables:

$$\begin{aligned} \chi(t,x) &= \int_0^x \frac{1}{\sqrt{1+(\partial_x \eta(t,y))^2}} dy - \int_0^t \int_\Sigma \left[\frac{1}{1+(\partial_x \eta(t,y))^2} + \partial_x \phi\right] d\Sigma \\ &= \int_0^x \sqrt{1+(\partial_x \eta(t,y))^2} dy - \int_0^t \int_0^{2\pi} \left[\frac{1}{1+(\partial_x \eta(t,y))^2} + \partial_x \phi\right] \sqrt{1+(\partial_x \eta)^2} dy, \end{aligned} \tag{5.2.4}$$

and $\tilde{\chi}$ is defined analogously from $(\tilde{\eta}, \tilde{\psi})$.

The main goal of the proof is to show the following estimate:

$$\begin{aligned} &\left\|(\eta,\psi)^*(t,\cdot) - (\tilde{\eta},\tilde{\psi})^{\tilde{*}}(t,\cdot)\right\|_{H^s \times H^{s-\frac{1}{2}}} \\ &\leq C\left(\left\|(\eta_0, \psi_0, \tilde{\eta}_0, \tilde{\psi}_0)\right\|_{H^{s+\frac{1}{2}}\times H^s}\right) \left\|(\eta_0,\psi_0)^* - (\tilde{\eta}_0, \tilde{\psi}_0)^{\tilde{*}}\right\|_{H^s\times H^{s-\frac{1}{2}}}, \end{aligned} \tag{5.2.5}$$

where * and $^{\tilde{*}}$ are the paracomposition operators defined by χ and $\tilde{\chi}$ respectively. We recall that the definition of the paracomposition operator is given in Section 2.4.

Put $\Phi = (\Phi_1, \Phi_2)$ the unknowns obtained from (η, ψ) after paralinearization and symmetrization of the equations as in Corollary 5.2.1. Define analogously $\tilde{\Phi} = (\tilde{\phi_1}, \tilde{\phi_2})$ from $(\tilde{\eta}, \tilde{\psi})$. Let us notice that, in order to prove (5.2.5), it suffice to get estimates on $\Phi - \tilde{\Phi}$. Indeed by the ellipticity of the symbols p and q combined with the immediate L^2 estimates(as $s > 2 + \frac{1}{2}$) we have:

$$\begin{aligned} &\left\|(\eta,\psi)^*(t,\cdot) - (\tilde{\eta},\tilde{\psi})^{\tilde{*}}(t,\cdot)\right\|_{H^s \times H^{s-\frac{1}{2}}} \\ &\leq C\left(\left\|(\eta_0, \psi_0, \tilde{\eta}_0, \tilde{\psi}_0)\right\|_{H^{s+\frac{1}{2}}\times H^s}\right) \left\|\Phi^*(t,\cdot) - \tilde{\Phi}^{\tilde{*}}(t,\cdot)\right\|_{H^{s-\frac{1}{2}}\times H^{s-\frac{1}{2}}}, \end{aligned} \tag{5.2.6}$$

$$\begin{aligned} &\left\|\Phi^*(t,\cdot) - \tilde{\Phi}^{\tilde{*}}(t,\cdot)\right\|_{H^{s-\frac{1}{2}}\times H^{s-\frac{1}{2}}} \\ &\leq C\left(\left\|(\eta_0, \psi_0, \tilde{\eta}_0, \tilde{\psi}_0)\right\|_{H^{s+\frac{1}{2}}\times H^s}\right) \left\|(\eta,\psi)^*(t,\cdot) - (\tilde{\eta},\tilde{\psi})^{\tilde{*}}(t,\cdot)\right\|_{H^s\times H^{s-\frac{1}{2}}}. \end{aligned} \tag{5.2.7}$$

5.2.2.1 Gauge transform

Again, as $s > 2+\frac{1}{2}$ we have an immediate L^2 estimates on $\Phi - \tilde{\Phi}$, thus we only need to get $\dot{H}^{s-\frac{1}{2}} \times \dot{H}^{s-\frac{1}{2}}$ estimates. Let us start by by writing $\Phi = \Phi_1 + i\Phi_2$ in equation (5.2.3):

$$\partial_t \Phi + T_V \cdot \partial_x \Phi + iT_\gamma \Phi = R_1(\Phi)\Phi, \tag{5.2.8}$$

Where R_1 verifies

$$\begin{cases} \|R_1(\Phi)\|_{H^{s-\frac{1}{2}} \to H^{s-\frac{1}{2}}} \leq C\Big(\left\|(\eta_0, \psi_0, \tilde{\eta}_0, \tilde{\psi}_0)\right\|_{H^{s+\frac{1}{2}} \times H^s} \Big), \\ \left\|[R_1(\Phi) - R_1(\tilde{\Phi})]\tilde{\Phi}\right\|_{H^{s-\frac{1}{2}}} \leq C\Big(\left\|(\eta_0, \psi_0, \tilde{\eta}_0, \tilde{\psi}_0)\right\|_{H^{s+\frac{1}{2}} \times H^s} \Big) \left\|\Phi - \tilde{\Phi}\right\|_{H^{s-\frac{1}{2}}}. \end{cases}$$

The next step is to perform the change of variable by χ, by Theorem 2.4.2 we get:

$$\partial_t \Phi^* + T_W \cdot \partial_x \Phi + iT_{|\xi|^{\frac{3}{2}}} \Phi^* = R_1'(\Phi^*)\Phi^*, \tag{5.2.9}$$

where R_1 verifies

$$\left\|[R_1'(\Phi^*) - R_1(\tilde{\Phi}^{\tilde{*}})]\tilde{\Phi}^{\tilde{*}}\right\|_{H^{s-\frac{1}{2}}} \leq C\Big(\left\|(\eta_0, \psi_0, \tilde{\eta}_0, \tilde{\psi}_0)\right\|_{H^{s+\frac{1}{2}} \times H^s} \Big) \left\|\Phi^* - \tilde{\Phi}^{\tilde{*}}\right\|_{H^{s-\frac{1}{2}}}.$$

We get the same equation on $\tilde{\Phi}^{\tilde{*}}$ by symmetry.

Introduce the following gauge transform A_Φ as the time one of the flow map defined by Proposition 4.1.1 with

$$p_\Phi = \frac{2}{3} |\xi|^{\frac{1}{2}} \partial_x^{-1} W \in \Gamma_2^{2-\alpha}(\mathbb{T}),$$

and put,

$$\theta = A_\Phi \Phi^*. \tag{5.2.10}$$

We define analogously $A_{\tilde{\Phi}}$ and $\tilde{\theta}$ from $\tilde{\Phi}^{\tilde{*}}$. From Proposition 4.1.1 the change of variable (5.2.10) is Lipschitz from $H^{s-\frac{1}{2}}$ to $H^{s-\frac{1}{2}}$ but under H^s control on $(\Phi, \tilde{\Phi})$ which is equivalent by Theorem 2.2.6 to a control on $\left\|(\eta_0, \psi_0, \tilde{\eta}_0, \tilde{\psi}_0)\right\|_{H^{s+\frac{1}{2}} \times H^s}$. We have:

$$\left\|\Phi^*(t,\cdot) - \tilde{\Phi}^{\tilde{*}}(t,\cdot)\right\|_{\dot{H}^{s-\frac{1}{2}}} \leq C\Big(\left\|(\eta_0, \psi_0, \tilde{\eta}_0, \tilde{\psi}_0)\right\|_{H^{s+\frac{1}{2}} \times H^s} \Big) \left\|\theta(t,\cdot) - \tilde{\theta}(t,\cdot)\right\|_{\dot{H}^{s-\frac{1}{2}}}, \tag{5.2.11}$$

$$\left\|\theta(t,\cdot) - \tilde{\theta}(t,\cdot)\right\|_{\dot{H}^{s-\frac{1}{2}}} \leq C\Big(\left\|(\eta_0, \psi_0, \tilde{\eta}_0, \tilde{\psi}_0)\right\|_{H^{s+\frac{1}{2}} \times H^s} \Big) \left\|\Phi^*(t,\cdot) - \tilde{\Phi}^{\tilde{*}}(t,\cdot)\right\|_{\dot{H}^{s-\frac{1}{2}}}. \tag{5.2.12}$$

To get the equations on θ and $\tilde{\theta}$ we commute A_Φ and $A_{\tilde{\Phi}}$ with (5.2.9), we make the computations for θ, those for $\tilde{\theta}$ are obtained by symmetry:

$$A_\Phi \partial_t \Phi^* + A_\Phi T_W \cdot \partial_x \Phi^* + iA_\Phi T_{|\xi|}^{\frac{3}{2}} \Phi = A_\Phi R_1'(\Phi^*)\Phi^*$$

$$\partial_t A_\Phi \Phi^* + iT_{|\xi|}^{\frac{3}{2}} A_\Phi \Phi^* + (A_\Phi T_W \partial_x - [iT_{|\xi|}^{\frac{3}{2}}, A_\Phi])\Phi^* - (\partial_t A_\Phi)\Phi^* = A_\Phi R_1'(\Phi^*)\Phi^*$$

By definition of p_Φ and Proposition 4.1.5 we have:

$$\partial_\xi(|\xi|^{\frac{3}{2}})\partial_x p_\Phi = W\xi \text{ and } \partial_t A_\Phi = A_\Phi \int_0^1 A_{-r}^{p_\Phi} T_{i\partial_t p_\Phi} A_r^{p_\Phi} dr.$$

thus by Corollary 4.1.2 we get:

$$\partial_t \theta + iT_{|\xi|}^{\frac{3}{2}}\theta = R_2(\theta)\theta + A_\Phi R_1(\Phi^*)A_\Phi^{-1}\Phi^*, \tag{5.2.13}$$

where R_2 and $A_\Phi R_1(\Phi)A_\Phi^{-1}$, as $s > 3 + \frac{1}{2}$, verify:

$$\begin{aligned}
&\|\mathrm{Re}(R_2(\theta))\|_{H^{s-\frac{1}{2}} \to H^{s-\frac{1}{2}}} \leq C\left(\left\|(\eta_0, \psi_0, \tilde{\eta}_0, \tilde{\psi}_0)\right\|_{H^{s+\frac{1}{2}} \times H^s}\right),\\
&\left\|[R_2(\theta) - R_2(\tilde{\theta})]\tilde{\theta}\right\|_{H^{s-\frac{1}{2}}} \leq C\left(\left\|(\eta_0, \psi_0, \tilde{\eta}_0, \tilde{\psi}_0)\right\|_{H^{s+\frac{1}{2}} \times H^s}\right)\left\|\theta(t,\cdot) - \tilde{\theta}(t,\cdot)\right\|_{H^{s-\frac{1}{2}}},
\end{aligned}$$

and,

$$\begin{aligned}
&\left\|[A_\Phi R_1(\Phi)A_\Phi^{-1} - A_{\tilde{\Phi}} R_1(\tilde{\Phi})A_{\tilde{\Phi}}^{-1}]\tilde{\theta}\right\|_{H^{s-\frac{1}{2}}}\\
&\leq C\left(\left\|(\eta_0, \psi_0, \tilde{\eta}_0, \tilde{\psi}_0)\right\|_{H^{s+\frac{1}{2}} \times H^s}\right)\left\|\theta(t,\cdot) - \tilde{\theta}(t,\cdot)\right\|_{H^{s-\frac{1}{2}}}.
\end{aligned}$$

Thus we have succeeded to eliminate the term $T_V \cdot \partial_x$ of order 1 in (5.2.9) and got a term of order $\frac{1}{2}$. The result then follows with a standard energy estimate.

5.2.3 Appendix: Gauge transform on $\mathbb{R}$

Closely inspecting the two problems in [65] and [56], we saw that the lack of regularity obtained in [65] for $\alpha \geq 2$ is essentially due to the lack of control of the L^1 norm in Sobolev spaces. To show this we start give a simple, albeit an artificial example:

Theorem 5.2.1. *Consider two real numbers $s \in]1+\frac{1}{2}, +\infty[$, $r > 0$ and $u_0 \in H^s(\mathbb{R})$. Then there exists $T > 0$ such that for all v_0 in the ball $B(u_0, r) \subset H^s(\mathbb{R})$ there exists a unique $v \in C([0,T], H^s(\mathbb{R}))$ solving the Cauchy problem:*

$$\begin{cases} \partial_t v + \mathrm{Re}(v)\partial_x v + i\partial_x^2 v = 0, \\ v(0,\cdot) = v_0(\cdot), \end{cases} \tag{5.2.14}$$

Moreover we have the estimates:

$$\forall 0 \leq \mu \leq s, \|v(t)\|_{H^\mu(\mathbb{R})} \leq C_\mu \|v_0\|_{H^\mu(\mathbb{R})}. \tag{5.2.15}$$

Taking two different solutions u, v such that $u - v \in L^1(\mathbb{R})$, then:

$$\begin{aligned}
\|(u-v)(t)\|_{H^s(\mathbb{R})} &\leq C(\|u_0\|_{H^s(\mathbb{R})})\|u_0 - v_0\|_{H^s(\mathbb{R})}\\
&\quad + C(\|u_0\|_{H^s(\mathbb{R})})[\|u_0 - v_0\|_{L^1(\mathbb{R})} + \|(u-v)(t)\|_{L^1(\mathbb{R})}].
\end{aligned} \tag{5.2.16}$$

Proof. This is the simplest theorem to prove as the transformation is straightforward and the symbols used are in the usual Hörmander symbol classes $S^m_{1,0}$. Given the well posedness of the Cauchy problem in H^s, and the density of $\mathscr{S}$ in H^s, it suffice to prove the result for $u_0, v_0 \in \mathscr{S}(\mathbb{R})$ which henceforth we suppose. Define u, v as the solution to (5.2.14) with initial data u_0, v_0 on $[0, T]$.

The first step we reduce H^s estimates to L^2 ones by defining $f_1 = \langle \mathrm{D} \rangle^s u$. Commuting $\langle \mathrm{D} \rangle^s$ with (5.2.14), by the symbolic calculus rules in Appendix 2.2.3 we get the PDE on f_1:

$$\begin{cases} \partial_t f_1 + \mathrm{Re}(u)\partial_x f_1 + i\partial_x^2 f_1 = R_1(f_1)f_1, \\ f_1(0, \cdot) = \langle \mathrm{D} \rangle^s u_0(\cdot), \end{cases} \tag{5.2.17}$$

where R_1 verifies

$$\|R_1(f_1)\|_{L^2 \to L^2} \leq C(\|u\|_{H^s}), \|\partial_{f_1} R_1(f_1)\|_{L^2 \to L^2} \leq C(\|u\|_{H^s}),$$

We define analogously g_1 from v and notice that by definition:

$$\|f_1 - g_1\|_{L^2} = \|u - v\|_{H^s}.$$

thus the problem is reduced to getting L^2 estimates on $f_1 - g_1$.

Then we introduce $F(t, x) = \int_0^x \mathrm{Re}(u)(t, y)dy \in C^\infty(\mathbb{R})$ and make the following change of variable:

$$f_2 = e^{-\frac{i}{2}F} f_1.$$

Analogously define G and g_2 from v. As remarked in [72], F, G do not necessarily decay at infinity but we still have $e^{-\frac{i}{2}F}, e^{-\frac{i}{2}G} \in S^m_{1,0}$. Indeed because $\partial_x F = \mathrm{Re}(u) \in H^{+\infty}$ and $\partial_x G = \mathrm{Re}(v) \in H^{+\infty}$. Now to get Lipschitz control we have

$$\left\| [e^{-\frac{i}{2}G} - e^{-\frac{i}{2}F}] f_1 \right\|_{L^2} \leq \|G - F\|_{L^\infty} \|f_1\|_{L^2} \leq \|v - u\|_{L^1} \|f_1\|_{L^2},$$
$$\left\| [e^{\frac{i}{2}G} - e^{\frac{i}{2}F}] f_2 \right\|_{L^2} \leq \|G - F\|_{L^\infty} \|f_2\|_{L^2} \leq \|v - u\|_{L^1} \|f_2\|_{L^2},$$

thus,

$$\begin{cases} \|f_2 - g_2\|_{L^2} \leq C[\|f_1 - g_1\|_{L^2} + \|v - u\|_{L^1} \|f_1\|_{L^2}], \\ \|f_1 - g_1\|_{L^2} \leq C[\|f_2 - g_2\|_{L^2} + \|v - u\|_{L^1} \|f_2\|_{L^2}], \end{cases}$$

and the problem is reduced to getting L^2 estimates on $f_2 - g_2$. Commuting $e^{\frac{i}{2}F}$ with (5.2.17) yields:

$$\begin{cases} \partial_t f_2 + i\partial_x^2 f_2 = R_2(f_2)f_2, \\ f_2(0, \cdot) = e^{\frac{i}{2}F_0} \langle \mathrm{D} \rangle^s u_0(\cdot), \end{cases} \tag{5.2.18}$$

where R_2 verifies

$$\|R_2(f_2)\|_{L^2 \to L^2} \leq C(\|u\|_{H^s}), \|\partial_{f_2} R_2(f_2)\|_{L^2 \to L^2} \leq C(\|u\|_{H^s}).$$

Analogously we get on g_2

$$\begin{cases} \partial_t g_2 + i\partial_x^2 g_2 = R_2(g_2)g_2 \\ g_2(0, \cdot) = e^{\frac{i}{2}G_0} \langle \mathrm{D} \rangle^s v_0(\cdot). \end{cases} \tag{5.2.19}$$

Now the usual energy estimate on $f_2 - g_2$ combined the Gronwall lemma on $f_2 - g_2$ gives for $0 \leq t \leq T$:

$$\|f_2 - g_2\|_{L^2} \leq C(\|(u, v)\|_{H^s}) \|(f_2 - g_2)(0, \cdot)\|_{L^2}$$

As $s > 1 + \frac{1}{2}$, by the Sobolev embedding Theorem:

$$\begin{aligned}
\|f_2 - g_2\|_{L^2} &\leq C(\|(u_0, v_0)\|_{H^s}) \left\| e^{\frac{i}{2}F_0} \langle \mathrm{D} \rangle^s u_0 - e^{\frac{i}{2}G_0} \langle \mathrm{D} \rangle^s v_0 \right\|_{L^2} \\
&\leq C(\|(u_0, v_0)\|_{H^s})[\|u_0 - v_0\|_{H^s} + \|u_0 - v_0\|_{L^1}],
\end{aligned}$$

which concludes the proof. □

Chapter 6

On the threshold of well posedness of the weakly dispersive Burgers type equation

In this section we give the proofs of Theorems 1.2.3 and 1.2.4 which are the main results from [64].

Contents

6.1 Implicit construction of symbols

In this section we give the different theorems permitting the construction of the gauge transforms used in the subsequent sections. The following two Theorems should be compared to Corollary 4.1.2. Indeed in both cases we are looking to solve a problem of the form

$$A^p_1 T_b A^p_{-1} = T_a + R, \tag{6.1.1}$$

where the symbols a and b are given, the unknown is the symbol p, A^p_1 is the time one of the flow generated by p defined in Section 4.1 and R is an acceptable remainder depending essentially on the threshold of regularity in which we are working.

In Corollary 4.1.2 an approximate solution to this problem (6.1.1) was given in the high regularity setting which permitted immediate control on the remainder R. The goal of this section is to improve upon on this by showing that we can control efficiently the remainder, i.e with no extra regularity assumption than the minimal ones required for the definition of a, b, p and A^p_1, in two directions. First for the linearised version of (6.1.1), i.e:

$$\mathfrak{L}_{ip}T_b = T_a. \tag{6.1.2}$$

Second when solving the full problem (6.1.1) in the case where the linearised problem (6.1.2) can be solved with p of negative in order.

Theorem 6.1.1. *Consider two real numbers $\alpha \geq 1$, $\beta \in \mathbb{R}$ and a symbol $a \in \Gamma_0^\beta(\mathbb{D})$. Then there exists $B > 1$ and a symbol $p \in \Gamma_1^{\beta+1-\alpha}(\mathbb{D})$ such that,*

$$\sigma_p^{B,b} \otimes \sigma_{\xi|\xi|^{\alpha-1}}^{B,b} - \sigma_{\xi|\xi|^{\alpha-1}}^{B,b} \otimes \sigma_p^{B,b} = \sigma_a^{B,b}, \tag{6.1.3}$$

where $\otimes$ is the symbol product defined formally by:

$$\mathrm{Op}(p) \circ \mathrm{Op}(q) = \mathrm{Op}(p \otimes q), \text{ where}$$
$$p \otimes q(x,\xi) = \frac{1}{2\pi}\int_{\mathbb{D}\times\hat{\mathbb{D}}} e^{i(x-y).(\xi-\eta)}p(x,\eta)q(y,\xi)dyd\eta.$$

Moreover we have the estimates:

$$M_0^{\beta+1-\alpha}(\partial_x\sigma_p^{B,b};0) \leq \frac{M_0^\beta(\sigma_a^{B,b};0)}{B[1-(1-\frac{1}{B})^\alpha]}, \tag{6.1.4}$$

$$\begin{aligned} M_0^{\beta-\alpha}(\partial_\xi\partial_x\sigma_p^{B,b};0) \leq &\frac{M_0^{\beta-1}(\partial_\xi\sigma_a^{B,b};0)}{B[1-(1-\frac{1}{B})^\alpha]} \\ &+ \alpha\frac{(1+\frac{1}{B})^{\alpha-1}-1}{1-(1-\frac{1}{B})^\alpha}M_0^{\beta+1-\alpha}(\partial_x\sigma_p^{B,b};0). \end{aligned} \tag{6.1.5}$$

Remark 6.1.1. *The choice of the same cut-off parameters in the right hand side and left hand side of (6.1.3) is not immediate, indeed by the general rule of composition of paradifferential operators given in Proposition 2.2.6, the cut-off on the left hand side is given by $B \star B = \frac{B^2}{2B-1} > B$. But in the special case where one of the operators is a Fourier multiplier we have this refined property where the cut-off on the left hand side is indeed given by B.*

Proof. First the case $\alpha = 1$ has the immediate solution with the choice of p as the primitive of $\sigma_a^{B,b}$ in the x variable. Henceforth we suppose $\alpha > 1$.

We start by defining the scale of Banach spaces that define the Fréchet space of Paradifferential operators.

Definition 6.1.1. *Given $m \in \mathbb{R}$, $k \in \mathbb{N}$ and $\mathscr{W} \subset \mathscr{S}'$ a Banach space. Define $C^k\Gamma_{\mathscr{W}}^m(\mathbb{D}\times\hat{\mathbb{D}}\setminus B(0,R))$ as the space of locally bounded functions $a(x,\xi)$ defined on $\mathbb{D}\times\hat{\mathbb{D}}\setminus B(0,R)$, which are C^k with respect to ξ and such that, for all $j \leq k$ and for all $\xi \geq R$, the function $x \mapsto \partial_\xi^\alpha a(x,\xi)$ belongs to $\mathscr{W}$ and there exists a constant C_k such that:*

$$\forall |\xi| \geq R, j \leq k, \left\|\partial_\xi^j a(.,\xi)\right\|_{\mathscr{W}} \leq C_k(1+|\xi|)^{m-|\alpha|}. \tag{6.1.6}$$

The space $C^k\Gamma_{\mathscr{W}}^m(\mathbb{D}\times\hat{\mathbb{D}}\setminus B(0,R))$ is equipped with it's natural Banach space topology by the best constant C_k. When $\mathscr{W} = W^{\rho,\infty}$, the best constant is the seminorm $M_\rho^\beta(\cdot;k)$.

We define:

$$\psi^{B,b}\Big(\Gamma^m_{\mathscr{W}}(\mathbb{D})\Big)=\Big\{\sigma^{B,b}_p, p\in\Gamma^m_{\mathscr{W}}(\mathbb{D})\Big\},$$

$$\psi^{B,b}\Big(C^k\Gamma^m_{\mathscr{W}}(\mathbb{D}\times\hat{\mathbb{D}}\setminus \mathrm{B}(0,R))\Big)=\Big\{\sigma^{B,b}_p, p\in C^k\Gamma^m_{\mathscr{W}}(\mathbb{D}\times\hat{\mathbb{D}}\setminus \mathrm{B}(0,R))\Big\},$$

equipped with their natural Fréchet and Banach topologies induced by the continuity of the map $p\mapsto\sigma^{B,b}_p$. Now we reinterpret Theorem 6.1.1 by introducing the linear operator:

$$\begin{aligned}L :&C^k\Gamma^{\beta+1-\alpha}_1(\mathbb{D}\times\hat{\mathbb{D}}\setminus \mathrm{B}(0,b))\to\psi^{B,b}\Big(C^k\Gamma^{\beta}_0(\mathbb{D}\times\hat{\mathbb{D}}\setminus \mathrm{B}(0,b))\Big)\\ &p\mapsto\sigma^{B,b}_p\otimes\sigma^{B,b}_{\xi|\xi|^{\alpha-1}}-\sigma^{B,b}_{\xi|\xi|^{\alpha-1}}\otimes\sigma^{B,b}_p.\end{aligned}$$

The proof will then proceed in two steps, we first prove that L has a right inverse on the Banach space $C^0\Gamma^{\beta+1-\alpha}_1(\mathbb{R}\setminus \mathrm{B}(0,b))$ and then we prove propagation of regularity in the frequency variable ξ for solutions of the equation (6.1.3).

To construct a right inverse for L the key idea here is simply that a right hand parametrix is given by the standard Cole-Hopf gauge transform:

$$\begin{aligned}E_{approx} :&C^k\Gamma^{\beta}_0(\mathbb{D}\times\hat{\mathbb{D}}\setminus \mathrm{B}(0,b))\to\psi^{B,b}\Big(C^k\Gamma^{\beta+1-\alpha}_0(\mathbb{D}\times\hat{\mathbb{D}}\setminus \mathrm{B}(0,b))\Big)\\ &a(x,\xi)\to\frac{|\xi|^{1-\alpha}}{i\alpha}\operatorname{Op}(\frac{1}{D})[\sigma^{B,b}_a(\cdot,\xi)](x,\xi),\end{aligned}$$

where,

$$\mathscr{F}_x(\operatorname{Op}(\frac{1}{D}[\sigma^{B,b}_a(\cdot,\xi)])(\eta)=\frac{1}{i\eta}\mathscr{F}_x(\sigma^{B,b}_a(x,\xi))(\eta),$$

which is well defined as $P_0(D)\operatorname{Op}(\frac{1}{D})[\sigma^{B,b}_a(\cdot,\xi)]=0$ where $P_0(D)$ is the Littelwood-Paley projector defined in Section 2.1. We then compute:

$$L\circ E_{approx}=\sigma^{B,b}_{\cdot}(Id-r),\ \text{ where:}$$

$$\begin{aligned}r :&C^{\beta+1-\alpha}\Gamma^m_0(\mathbb{D}\times\hat{\mathbb{D}}\setminus \mathrm{B}(0,b))\to\psi^{B,b}\Big(C^k\Gamma^{\beta+1-\alpha}_0(\mathbb{D}\times\hat{\mathbb{D}}\setminus \mathrm{B}(0,b))\Big)\\ &a\mapsto\frac{\alpha(\alpha-1)}{4\pi}\int_0^1\int_{\mathbb{D}\times\hat{\mathbb{D}}}e^{i(x-y)\eta}\sigma^{B,b}_{\xi|\xi|^{\alpha-2}}(\xi+t\eta)\psi^{B,b}(\eta,\xi)\sigma^{B,b}_{\frac{1}{i\alpha}\partial_x a|\xi|^{1-\alpha}}(y,\xi)dyd\eta dt.\end{aligned}$$

Let us remark that the remainder can also be written as:

$$r(a)(x,\xi)=\frac{\alpha-1}{2}\underbrace{\operatorname{Op}_x\Big(\int_0^1\sigma^{B,b}_{\xi|\xi|^{\alpha-2}}(\xi+t\eta)\psi^{B,b}(\eta,\xi)dt\Big)}_{(*)}[\sigma^{B,b}_{\frac{1}{i}\partial_x a|\xi|^{1-\alpha}}](x,\xi),$$

where $(*)$ is seen a Fourier multiplier in the x variable for ξ fixed. Thus estimating the semi-norms of $r(a)$ using the continuity of Fourier multipliers combined with the Bernstein inequalities we get we get by the frequency localization of Paradifferential operators:

$$M^{\beta+1-\alpha}_0(r(a);0)\leq\frac{C}{B}M^{\beta+1-\alpha}_0(\sigma^{B,b}_a;0).$$

Thus for B sufficiently large L has a right inverse on $\psi^{B,b}\Big(C^0\Gamma_1^{\beta+1-\alpha}(\mathbb{R}\setminus \mathrm{B}(0,b))\Big)$ given by the Neumann series:

$$E=\sum_{k=0}^{+\infty} E_{approx} r^k.$$

Thus we have constructed a $p\in C^0\Gamma_1^{\beta+1-\alpha}(\mathbb{R}\setminus \mathrm{B}(0,b))$ such that:

$$\sigma_p^{B,b}\otimes\sigma_{\xi|\xi|^{\alpha-1}}^{B,b}-\sigma_{\xi|\xi|^{\alpha-1}}^{B,b}\otimes\sigma_p^{B,b}=\sigma_a^{B,b}.$$

Now we want to prove that $p\in C^k\Gamma_1^{\beta+1-\alpha}(\mathbb{R}\setminus \mathrm{B}(0,b))$ for all k. We start by the following computation that comes from commuting with ix:

$$\sigma_{\partial_\xi p}^{B,b}\otimes\sigma_{\xi|\xi|^{\alpha-1}}^{B,b}-\sigma_{\xi|\xi|^{\alpha-1}}^{B,b}\otimes\sigma_{\partial_\xi p}^{B,b}=\sigma_{\partial_\xi a}^{B,b}+\alpha[\sigma_{|\xi|^{\alpha-1}}^{B,b}\otimes\sigma_p^{B,b}-\sigma_p^{B,b}\otimes\sigma_{|\xi|^{\alpha-1}}^{B,b}]. \tag{6.1.7}$$

Now to get the desired bound we use the following Lemma.

Lemma 6.1.1. *let $p\in\mathscr{S}'(\mathbb{D}\times\hat{\mathbb{D}})$ be a symbol such that for some cut-off parameters B,b we have:*

$$\sigma_p^{B,b}\otimes\sigma_{\xi|\xi|^{\alpha-1}}^{B,b}-\sigma_{\xi|\xi|^{\alpha-1}}^{B,b}\otimes\sigma_p^{B,b}=\sigma_a^{B,b},$$

for some $a\in C^0\Gamma_0^{\beta}(\mathbb{R}\setminus B(0,b))$ wit $\alpha>0$ and $\beta\in\mathbb{R}$.

Then $\partial_x\sigma_p^{B,b}\in C^0\Gamma_0^{\beta+1-\alpha}(\mathbb{R}\setminus B(0,b))$ and moreover we have the estimate:

$$M_0^{\beta+1-\alpha}(\partial_x\sigma_p^{B,b};0)\le\frac{M_0^{\beta}(\sigma_a^{B,b};0)}{B[1-(1-\frac{1}{B})^{\alpha}]}.$$

Proof of Lemma 6.1.1. Without loss of generality, we suppose $p\in\mathscr{S}$ as the result can be deduced by a standard density argument. We rewrite the identity verified by p as follows:

$$\frac{1}{2\pi}\int_0^1\int_{\mathbb{D}\times\hat{\mathbb{D}}} e^{i(x-y)\eta}\sigma_{|\xi|^{\alpha-1}}^{B,b}(\xi+t\eta)\psi^{B,b}(\eta,\xi)\sigma_{\partial_x p}^{B,b}(y,\xi)dyd\eta dt=\frac{i}{\alpha}\sigma_a^{B,b},$$

thus,

$$\underbrace{\mathrm{Op}_x\left(\int_0^1\sigma_{|\xi|^{\alpha-1}}^{B,b}(\xi+t\eta)\psi^{B,b}(\eta,\xi)dt\right)}_{(*)}[\sigma_{\partial_x p}^{B,b}](x,\xi)=\frac{i}{\alpha}\sigma_a^{B,b},$$

where $(*)$ is an elliptic Fourier multiplier in the x variable for ξ fixed with the bound:

$$\frac{B}{\alpha}[1-(1-\frac{1}{B})^{\alpha}]\,|\xi|^{\alpha-1}=\int_0^t\left(1-\frac{t}{B}\right)^{\alpha-1}dt\,|\xi|^{\alpha-1}\le\int_0^1\sigma_{|\xi|^{\alpha-1}}^{B,b}(\xi+t\eta)\psi^{B,b}(\eta,\xi)dt$$

which gives the desired result. □

Getting back to the proof of Theorem 6.1.1 and applying Lemma 6.1.1 to (6.1.7) we get $p\in C^1\Gamma_1^{\beta+1-\alpha}(\mathbb{R}\setminus \mathrm{B}(0,b))$ and iterating (6.1.7) we get $p\in C^k\Gamma_1^{\beta+1-\alpha}(\mathbb{R}\setminus \mathrm{B}(0,b))$ for all k. □

A corollary of Lemma 6.1.1 is the following uniqueness result.

Corollary 6.1.1. *let $p \in \mathscr{S}'(\mathbb{D} \times \hat{\mathbb{D}})$ be a symbol such that for some cut-off parameters B, b we have:*

$$\sigma_p^{B,b} \otimes \sigma_{\xi|\xi|^{\alpha-1}}^{B,b} - \sigma_{\xi|\xi|^{\alpha-1}}^{B,b} \otimes \sigma_p^{B,b} = 0.$$

Then $\sigma_p^{B,b}$ is a Fourier multiplier, i.e there exits a Fourier multiplier m such that:

$$\sigma_p^{B,b}(x,\xi) = \sigma_m^{B,b}(x,\xi) = \psi^{B,b}(0,\xi)m(\xi).$$

We now give the main Theorem that permits the construction of the gauge transform.

Theorem 6.1.2. *Consider two real numbers $\alpha \geq 1$, $\beta \leq \alpha - 1$ and a symbol $a \in \Gamma_0^\beta(\mathbb{R})$. Then there exists $\epsilon > 0, B \geq \frac{4}{3}$ such that for,*

$$M_0^\beta(a;0) \leq \epsilon,$$

there exists a symbol p such that $p \in \Gamma_1^{\beta+1-\alpha}(\mathbb{R})$ and:

$$\sigma_{\xi|\xi|^{\alpha-1}}^{B,b} - \sigma_{\otimes e^{ip}}^{2,b} \otimes \sigma_{\xi|\xi|^{\alpha-1}}^{B,b} \otimes \sigma_{\otimes e^{-ip}}^{2,b} = \sigma_a^{B,b}. \tag{6.1.8}$$

Moreover we have the estimates:

$$M_0^{\beta+1-\alpha}(\partial_x \sigma_p^{B,b};0) \leq \frac{1+C\epsilon}{\alpha} M_0^\beta(\sigma_a^{B,b};0), \tag{6.1.9}$$

$$\begin{aligned} M_0^{\beta-\alpha}(\partial_\xi\partial_x \sigma_p^{B,b};0) \leq (1 + CM_0^\beta(\partial_\xi \sigma_a^{B,b};0)) \times \Bigg[&\frac{M_0^{\beta-1}(\partial_\xi \sigma_a^{B,b};0)}{B[1-(1-\frac{1}{B})^\alpha]} \\ &+ \alpha \frac{(1+\frac{1}{B})^{\alpha-1}-1}{1-(1-\frac{1}{B})^\alpha} M_0^{\beta+1-\alpha}(\partial_x \sigma_p^{B,b};0) \Bigg]. \end{aligned} \tag{6.1.10}$$

Remark 6.1.2. *We note that $\beta + 1 - \alpha \leq 0$ thus the hypothesis on $\mathrm{Im}(p)$ is automatically verified.*

Proof. The proof should in spirit amount to a Nash-Moser scheme as we are looking to prove an implicit function type of result on the Fréchet space of paradifferential symbols. In our case the problem is simpler due to the following key observation, for $k \geq 0$ the underlining map is well defined on the Banach spaces in the scale defining the Fréchet space of paradifferential operators:

$$\begin{aligned} F :& C^k\Gamma_1^{\beta+1-\alpha}(\mathbb{D} \times \hat{\mathbb{D}} \setminus \mathrm{B}(0,b)) \to \psi^{\frac{4}{3},b}\Big(C^k\Gamma_0^\beta(\mathbb{D} \times \hat{\mathbb{D}} \setminus \mathrm{B}(0,b))\Big) \\ &p \mapsto \sigma_{\xi|\xi|^{\alpha-1}}^{B,b} - \sigma_{\otimes e^{ip}}^{2,b} \otimes \sigma_{\xi|\xi|^{\alpha-1}}^{B,b} \otimes \sigma_{\otimes e^{-ip}}^{2,b} \\ F(p) =& \sigma^{\frac{4}{3},b}_{\int_0^1 \sigma_{e^{irp}}^{2,b} \otimes [\sigma_p^{B,b} \otimes \sigma_{\xi|\xi|^{\alpha-1}}^{B,b} - \sigma_{\xi|\xi|^{\alpha-1}}^{B,b} \otimes \sigma_p^{B,b}] \otimes \sigma_{\otimes e^{-irp}}^{2,b} dr}. \end{aligned}$$

The proof will again proceed in two steps, we first prove that F has a right inverse on the Banach space $C^0\Gamma_1^{\beta+1-\alpha}(\mathbb{R} \setminus \mathrm{B}(0,b))$ and then we prove propagation of regularity in the frequency variable ξ for solutions of the equation (6.1.3).

Noticing that $F(0) = 0$, the goal is thus to prove the local surjectivity of F around the origin. Now that we reduced the problem to Banach spaces, by the inverse function theorem it suffice to find a right inverse to the differential of F at 0. Computing the differential at 0 we get:

$$D_0F(h) = \sigma_h^{B,b} \otimes \sigma_{\xi|\xi|^{\alpha-1}}^{B,b} - \sigma_{\xi|\xi|^{\alpha-1}}^{B,b} \otimes \sigma_h^{B,b} = L(h),$$

Thus by Theorem 6.1.1 and the local inversion Theorem in Banach spaces we get the desired local surjectivity.

Now we turn to propagation of regularity in the ξ variable, we fix:

$$p \in C^0\Gamma_1^{\beta+1-\alpha}(\mathbb{R} \setminus \mathrm{B}(0,b)) \text{ and } a \in \Gamma_0^{\beta}(\mathbb{R} \setminus \mathrm{B}(0,b)).$$

To make all of the computations rigorous we suppose $p \in \mathscr{S}$ and the desired result is obtained by a density argument. The computation behind the propagation of regularity is the following analogue of (6.1.7). We start from:

$$\sigma_{\xi|\xi|^{\alpha-1}}^{B,b} - \sigma_{\otimes e^{ip}}^{B,b} \otimes \sigma_{\xi|\xi|^{\alpha-1}}^{B,b} \otimes \sigma_{\otimes e^{-ip}}^{B,b} = \sigma_a^{B,b},$$

commuting with ix we get:

$$\sigma_{\alpha|\xi|^{\alpha-1}}^{B,b} - \sigma_{\partial_\xi a}^{B,b} = \sigma_{\otimes e^{ip}}^{B,b} \otimes \left[\sigma_{\otimes e^{-ip}}^{B,b} \otimes ix \otimes \sigma_{\otimes e^{ip}}^{B,b}, \sigma_{\xi|\xi|^{\alpha-1}}^{B,b}\right]_\otimes \otimes \sigma_{\otimes e^{-ip}}^{B,b},$$

where $[a,b]_\otimes = a \otimes b - b \otimes a$. Using the Duhamel formula combined with Proposition 4.1.2:

$$\sigma_{\otimes e^{-ip}}^{B,b} \otimes ix \otimes \sigma_{\otimes e^{ip}}^{B,b} = ix + \int_0^1 \sigma_{\otimes e^{-irp}}^{B,b} \otimes \sigma_{\partial_\xi p}^{B,b} \otimes \sigma_{\otimes e^{irp}}^{B,b} dr.$$

Thus,

$$\begin{aligned}\sigma_{\otimes e^{-ip}}^{B,b} \otimes \sigma_{\alpha|\xi|^{\alpha-1}}^{B,b} \otimes \sigma_{\otimes e^{ip}}^{B,b} - \sigma_{\alpha|\xi|^{\alpha-1}}^{B,b} - \sigma_{\otimes e^{-ip}}^{B,b} \otimes \sigma_{\partial_\xi a}^{B,b} \otimes \sigma_{\otimes e^{ip}}^{B,b} \\ = \Big[\int_0^1 \sigma_{\otimes e^{-irp}}^{B,b} \otimes \sigma_{\partial_\xi p}^{B,b} \otimes \sigma_{\otimes e^{irp}}^{B,b} dr, \sigma_{\xi|\xi|^{\alpha-1}}^{B,b}\Big]_\otimes. \quad (6.1.11)\end{aligned}$$

Applying Lemma 6.1.1 we get that:

$$\partial_x\left[\int_0^1 \sigma_{\otimes e^{-irp}}^{B,b} \otimes \sigma_{\partial_\xi p}^{B,b} \otimes \sigma_{\otimes e^{irp}}^{B,b} dr\right] \in C^0\Gamma_0^{\beta+1-\alpha}(\mathbb{R} \setminus \mathrm{B}(0,b)),$$

and by the frequency localization of paradifferential operators and the Bernstein inequalities:

$$\int_0^1 \sigma_{\otimes e^{-irp}}^{B,b} \otimes \sigma_{\partial_\xi p}^{B,b} \otimes \sigma_{\otimes e^{irp}}^{B,b} dr \in C^0\Gamma_1^{\beta+1-\alpha}(\mathbb{R} \setminus \mathrm{B}(0,b)).$$

Getting back to the definition of $\otimes e^{-irp}$ as $\beta + 1 - \alpha \leq 0$:

$$\sigma_{\otimes e^{irp}}^{lim,b} = \sum_{k=0}^{\infty} \frac{i^k r^k}{k!} \otimes^k \sigma_p^{B,b},$$

thus,

$$\sigma_{\otimes e^{irp}}^{lim,b} = 1 + O_{M_0^0(\cdot;0)}(\epsilon),$$

which gives:

$$\sigma_{\partial_\xi p}^{B,b} \in C^0\Gamma_1^{\beta+1-\alpha}(\mathbb{R} \setminus \mathrm{B}(0,b)).$$

We get the desired result by iteration. $\square$

6.2 Sobolev estimate on the weakly dispersive Burgers equation

The goal of this section is to prove Theorem 1.2.3. First it suffice to make the estimate (1.2.12) for $u_0 \in C_0^\infty$ and deduce the general result by density. We take u a solution to the Cauchy problem (1.2.11).

By Theorem 6.1.1 there exists $p \in \Gamma_1^{2-\alpha}(\mathbb{R})$ such that:

$$[T_{ip}^{B',b}, T_{i\xi|\xi|^{\alpha-1}}^{B',b}] = -iT^{B',b}_{\sigma_{u\xi}^{B,b}+(\sigma_{u\xi}^{B,b})^*}, \tag{6.2.1}$$

where $B' > B$ is the cut-off corresponding to the left hand side that includes $T^{B,b.}$ and it's a adjoint.

Now by Corollary 6.1.1 as $(T_{\xi|\xi|^{\alpha-1}}^{B',b})^* = T_{\xi|\xi|^{\alpha-1}}^{B',b}$ and the left hand side being L^2 skew-adjoint we get:

$$\left(T_p^{B',b}\right)^* = T_p^{B',b} \text{ in } L^2, \tag{6.2.2}$$

Henceforth $R_\infty(u)$ will designate a generic infinitely regularizing operator on on Sobolev spaces, i.e $R_\infty(u) : H^\mu \to H^{\mu'}$ for all $\mu, \mu' \in \mathbb{R}$ with the estimate:

$$\|R_\infty(u)\|_{H^\mu \to H^{\mu'}} \leq C_{\mu,\mu',\mu''}(\|u\|_{H^{\mu''}}), \mu'' \in \mathbb{R}.$$

Consider $(A_r^p)_{r\in\mathbb{R}}$ defined by Proposition 4.1.1 with the choice of cutoff in the paradifferential operator given by B', b. Now by construction we have:

$$A_1^p \partial_t u + T^{B',b}_{i\xi|\xi|^{\alpha-1}} A_1^p u + \frac{1}{2}\left(\int_0^1 A_{1-r}^p \left[T_{ip}^{B',b}, iT^{B',b}_{\sigma_{u\xi}^{B,b}+(\sigma_{u\xi}^{B,b})^*}\right] A_{r-1}^p dr\right) A_1^p u = R^\infty(u). \tag{6.2.3}$$

The key cancellation here is that even though $[T_{ip}^{B',b}, iT^{B',b}_{\sigma_{u\xi}^{B,b}+(\sigma_{u\xi}^{B,b})^*}] \in \Gamma_0^{3-\alpha}$ we have:

$$\left(\int_0^1 A_{1-r}^p \left[T_{ip}^{B',b}, iT^{B',b}_{\sigma_{u\xi}^{B,b}+(\sigma_{u\xi}^{B,b})^*}\right] A_{r-1}^p dr\right)^* = -\int_0^1 A_{1-r}^p \left[T_{ip}^{B',b}, iT^{B',b}_{\sigma_{u\xi}^{B,b}+(\sigma_{u\xi}^{B,b})^*}\right] A_{r-1}^p dr.$$

This gives conservation of the L^2 norm but it presents no concrete gain as the Cauchy problem (1.2.11) conserves the L^2 norm by a straightforward computation but we find it to be a good self check when making the computations.

At our level of regularity $s > 2 - \alpha + \frac{1}{2}$ this key cancellation does not hold for higher Sobolev estimates (except in the "exceptional" case of the Benjamin-Ono equation as noted in the introduction). To remedy this, inspired by [33], we preform an approximation of the symbol p given in (6.2.1) by step functions in the frequency variable ξ and a Littlewood-Paley decomposition in x.

To get higher Sobolev estimates the goal is estimate the L^2 norm of the k-th dyadic frequency shell of u, for this we start by rewriting (1.2.11) using p defined by (6.2.1):

$$\partial_t u - [T_{ip}^{B',b}, T_{i\xi|\xi|^{\alpha-1}}^{B',b}]u + T_{i\xi|\xi|^{\alpha-1}}^{B',b} u = R_\infty, \ u_0 \in H^s, \ b > 0. \tag{6.2.4}$$

We consider $C_k = [w_1 2^k, w_2 2^k]$ with $w_1 < 1 < w_2$ the k-th dyadic shell, i.e the support of $P_k(D)$. We fix the increasing sequence:

$$a_j = w_1 2^k + j^{\frac{1}{2-\alpha}} 2^{\frac{(1-\alpha)k}{2-\alpha}}, a_{-j} = a_j \text{ for } 0 \le j \le \left\lfloor (w_2 - w_1)^{\frac{1}{2-\alpha}} 2^k \right\rfloor = j_k,$$

and define the intervals:

$$\begin{cases} I_j = [\frac{2a_{j-1}+a_j}{3}, \frac{2a_j + a_{j+1}}{3}], \text{ for } |j| \le j_k, \\ I_{j_k} = [j_k, w_2 2^k], I_{-j_k} = [-w_2 2^k, -j_k]. \end{cases}$$

For $|j| \le j_k$ consider smooth functions $\phi_j : \mathbb{R} \to [0,1]$ supported in I_j and $\phi_j = 0$ for $j \notin [-j_k - 1, j_k + 1]$ such that:

$$\sum_{j\in\mathbb{Z}} \phi_j = 1, \ \left|\partial_\xi^n \phi_j(\xi)\right| \le C_n |\xi|^{-n}, n \in \mathbb{N}.$$

We now set,

$$p_j^l(x) = P_l(D)\left(\frac{1}{|I_j|}\int_{I_j} p(x,\xi)d\xi\right), |j| \le j_k, l \le \left\lfloor w_2 2^k \right\rfloor = l_k.$$

We compose (6.2.4) by the following:

$$\sum_{j=0}^{j_k} \phi_j \prod_{l=1}^{l_k} A_1^{p_j^l} \partial_t u - \sum_{j=0}^{j_k} \phi_j \prod_{l=1}^{l_k} A_1^{p_j^l} [T_{ip}^{B',b}, T_{i\xi|\xi|^{\alpha-1}}^{B',b}] + \underbrace{\sum_{j=0}^{j_k} \phi_j \prod_{l=1}^{l_k} A_1^{p_j^l} T_{i\xi|\xi|^{\alpha-1}}}_{(*)} u = R_\infty, \tag{6.2.5}$$

and we compute the commutator in $(*)$:

$$\begin{aligned} &\left[\prod_{l=1}^{l_k} A_1^{p_j^l}, T_{i\xi|\xi|^{\alpha-1}}^{B',b}\right] \\ &= \prod_{l=1}^{l_k} A_1^{p_j^l} \sum_{l=1}^{l_k} [T_{ip_j^l}^{B',b}, T_{i\xi|\xi|^{\alpha-1}}^{B',b}] + \prod_{l=1}^{l_k} A_1^{p_j^l} \sum_{l=1}^{l_k} \int_0^1 A_r^{p_j^l} \mathfrak{L}_{ip_j^l}^2 T_{i\xi|\xi|^{\alpha-1}}^{B',b} A_{-r}^{p_j^l} dr \\ &+ \sum_{l=1}^{l_k} \sum_{m=1}^{l-1} \prod_{n=l}^{l_k} A_1^{p_j^n} \mathfrak{L}_{A_1^{p_j^l}} \mathfrak{L}_{A_1^{p_j^m}} T_{i\xi|\xi|^{\alpha-1}}^{B',b} \prod_{n=1}^{l-1} A_1^{p_j^n}. \end{aligned} \tag{6.2.6}$$

Getting back to (6.2.5) we get:

$$\begin{aligned} &\underbrace{\sum_{j=0}^{j_k} \phi_j \prod_{l=1}^{l_k} A_1^{p_j^l} \partial_t u}_{(1)} + \underbrace{T_{i\xi|\xi|^{\alpha-1}} \sum_{j=0}^{j_k} \phi_j \prod_{l=1}^{l_k} A_1^{p_j^l} u}_{(2)} \\ &= \underbrace{\sum_{j=0}^{j_k} \phi_j \left[\prod_{l=1}^{l_k} A_1^{p_j^l} [T_{i(\tilde{\phi}_j p - \sum_{l=1}^{l_k} p_j^l)}^{B',b}, T_{i\xi|\xi|^{\alpha-1}}^{B',b}] u\right.}_{(3)} - \underbrace{\left.\prod_{l=1}^{l_k} A_1^{p_j^l} \sum_{l=1}^{l_k} \int_0^1 A_r^{p_j^l} \mathfrak{L}_{ip_j^l}^2 T_{i\xi|\xi|^{\alpha-1}}^{B',b} A_{-r}^{p_j^l} dr u\right]}_{(4)} \\ &\underbrace{- \sum_{j=0}^{j_k} \phi_j \sum_{l=1}^{l_k} \sum_{m=1}^{l-1} \prod_{n=l}^{l_k} A_1^{p_j^n} \mathfrak{L}_{A_1^{p_j^l}} \mathfrak{L}_{A_1^{p_j^m}} T_{i\xi|\xi|^{\alpha-1}}^{B',b} \prod_{n=1}^{l-1} A_1^{p_j^n} u}_{(5)} + R_\infty, \end{aligned} \tag{6.2.7}$$

where $\tilde{\phi_j}$ is a bump function with a slightly larger support than ϕ_j.

We now compute the L^2 scalar product of (6.2.7) with $\sum_{j=0}^{j_k} \phi_j \prod_{l=1}^{l_k} A_1^{p_j^l} u$. Given the complexity of equation (6.2.7) we will treat each term from (1) to (5) separately.

For term (1):

$$\begin{aligned}
&\operatorname{Re}\big((1), \sum_{j=0}^{j_k} \phi_j \prod_{l=1}^{l_k} A_1^{p_j^l} u)_{L^2}\big) \\
&= \sum_{j=0}^{j_k} \operatorname{Re}\big((\phi_j \prod_{l=1}^{l_k} A_1^{p_j^l} \partial_t u, \phi_j \prod_{l=1}^{l_k} A_1^{p_j^l} u)_{L^2}\big) + 2 \operatorname{Re}\big((\phi_{j-1} \prod_{l=1}^{l_k} A_1^{p_{j-1}^l} \partial_t u, \phi_j \prod_{l=1}^{l_k} A_1^{p_j^l} u)_{L^2}\big) \\
&+ 2 \sum_{j=0}^{j_k} \operatorname{Re}\big((\phi_{j+1} \prod_{l=1}^{l_k} A_1^{p_{j+1}^l} \partial_t u, \phi_j \prod_{l=1}^{l_k} A_1^{p_j^l} u)_{L^2}\big),
\end{aligned}$$

by the L^2 skew symmetry of p (6.2.2):

$$\begin{aligned}
&\operatorname{Re}\big((1), \sum_{j=0}^{j_k} \phi_j \prod_{l=1}^{l_k} A_1^{p_j^l} u)_{L^2}\big) = 5 \sum_{j=0}^{j_k} \operatorname{Re}\big((\tilde{\phi_j} \partial_t u, \tilde{\phi_j} u)_{L^2}\big) + R_\infty(u) \\
&+ \underbrace{2 \sum_{j=0}^{j_k} \operatorname{Re}\big(([\phi_{j+1} \prod_{l=1}^{l_k} A_1^{p_{j+1}^l} + \phi_{j-1} \prod_{l=1}^{l_k} A_1^{p_{j-1}^l} - 2\phi_j \prod_{l=1}^{l_k} A_1^{p_j^l}] \partial_t u, \phi_j \prod_{l=1}^{l_k} A_1^{p_j^l} u)_{L^2}\big)}_{R_1(u)}.
\end{aligned} \tag{6.2.8}$$

For $R_1(u)$ we see that the terms,

$$\begin{cases} \phi_{j+1}(D)\phi_j(D)[\phi_{j+1}(D)A^{p_{j+1}^l} - \phi_j(D)A_1^{p_j^l}] \\ \phi_{j-1}(D)\phi_j(D)[\phi_{j-1}(D)A^{p_{j-1}^l} - \phi_j(D)A_1^{p_j^l}] \end{cases}$$

can be seen as residual terms coming from a change of cut-off in the definition of Paradifferential operators. Thus by (2.2.17) and the choice of a_j we get:

$$\begin{cases} \phi_{j+1}(D)\phi_j(D)[\phi_{j+1}(D)A^{p_{j+1}^l} - \phi_j(D)A_1^{p_j^l}] \\ \phi_{j-1}(D)\phi_j(D)[\phi_{j-1}(D)A^{p_{j-1}^l} - \phi_j(D)A_1^{p_j^l}] \end{cases} \quad \text{are of order } 2 - 2\alpha,$$

thus, crudely estimating the time derivative by $2^{k\alpha}$ we get the estimate on $R_1(u)$:

$$|R_1(u)| \leq C \sum_{j=0}^{j_k} \sum_{l=0}^{j} e^{C \sum_l \|p_j^l\|_{L_x^\infty}} 2^{j(2-\alpha)} \left\| p_j^l \right\|_{L_x^\infty} \left\| \tilde{\phi_j} u \right\|_{L^2}^2 \tag{6.2.9}$$

For term (2), we have immediately by skew symmetry:

$$\operatorname{Re}\big((2), \sum_{j=0}^{j_k} \phi_j \prod_{l=1}^{l_k} A_1^{p_j^l} u)_{L^2}\big) = 0. \tag{6.2.10}$$

For term (3), (4) and (5) we have the estimates:

$$\left\|\mathrm{Re}([T^{B',b}_{i(\tilde{\phi_j p - \sum_{l=1}^{l_k} p_j^l})}, T^{B',b}_{i\xi|\xi|^{\alpha-1}}])\right\|_{L^2\to L^2} \leq C2^{j(2-\alpha)} \|P_{\leq j}(D)u\|_{L^\infty_x}, \tag{6.2.11}$$

$$\left\|\mathrm{Re}(\mathfrak{L}^2_{ip_j^l} T^{B',b}_{i\xi|\xi|^{\alpha-1}})\right\|_{L^2\to L^2} \leq C2^{j(2-\alpha)} \|P_l(D)u\|^2_{L^\infty_x} \tag{6.2.12}$$

$$\left\|\mathrm{Re}(\mathfrak{L}_{A_1^{p_j^l}} \mathfrak{L}_{A_1^{p_j^m}} T^{B',b}_{i\xi|\xi|^{\alpha-1}})\right\|_{L^2\to L^2} \leq e^{C\|p_j^l\|_{L^\infty_x} + \|p_j^m\|_{L^\infty_x}} 2^{k(2-\alpha)} \|P_l(D)u\|_{L^\infty_x} \|P_m(D)u\|_{L^\infty_x}. \tag{6.2.13}$$

Finally combining estimates (6.2.9),(6.2.10),(6.2.11),(6.2.12) and (6.2.13) we get:

$$\frac{d}{dt}\|P_k(D)u\|^2_{L^2} \leq C(1 + \left\|\tilde{P}_{\leq k}(D)u\right\|_{L^\infty})2^{k(2-\alpha)} \left\|\tilde{P}_{\leq k}(D)u\right\|_{L^\infty} \left\|\tilde{P}_k(D)u\right\|^2_{L^2},$$

where $\tilde{P}_k(D)$ is defined by a bump function with a slightly larger support than the one defining $P_k(D)$. This gives the desired result by the Littlewood-Paley decomposition, the Gronwall lemma and a standard frequency envelopes argument (see [72]).

6.3 Complete gauge transform for the dispersive Burgers equation

In contrast to the previous section in the first step to conjugating (1.2.13) we make the choice, by Theorem 6.1.1, $p_1 \in \Gamma^0_1(\mathbb{R})$ such that:

$$\sigma^{2,b}_{\otimes e^{ip_1}} \otimes \sigma^{B,b}_{\xi|\xi|^{\alpha-1}} - \sigma^{B,b}_{\xi|\xi|^{\alpha-1}} \otimes \sigma^{2,b}_{\otimes e^{-ip_1}} = \sigma^{B,b}_{iu\xi}, \tag{6.3.1}$$

$$\mathrm{Im}(p^1) = \frac{p^1 - p^{1*}}{2i} \in \Gamma^0_0(\mathbb{R}),\ M^0_0(\mathrm{Im}(p)) \leq CM^0_0(\partial_x p^1),$$

$$M^0_0(\partial_x p^1(t,\cdot);0) \leq \frac{1+C\epsilon}{\alpha} M^{\alpha-1}_0(\sigma^{B,b}_{iu\xi};0), \tag{6.3.2}$$

$$M^{-1}_0(\partial_\xi\partial_x p^1;0) \leq (1 + CM^{\alpha-2}_0(u;0)) \times \left[\frac{M^{\alpha-2}_0(u;0)}{\alpha} + (\alpha-1)M^0_0(\partial_x p^1;0)\right]. \tag{6.3.3}$$

where we implicitly used the Bernstein inequalities to see the injection:

$$u \in C^{2-\alpha}_* \Rightarrow \sigma^{B,b}_{iu\xi} \in \Gamma^{\alpha-1}_0.$$

Define $u^1 = A^{p^1}_1 u$ and commute with (1.2.13) we get by Proposition 4.1.5:

$$\partial_t u^1 + T^{\psi^{B,b}}_{i\xi|\xi|^{\alpha-1}} u^1 - T^{2,b}_{i\int_0^1 (\partial_t p^1)^{p^1}_r dr} u^1 = R^{1,\infty},\ u^1(0,\cdot) = A^{p^1}_1 u_0(\cdot). \tag{6.3.4}$$

Remark 6.3.1. *For the proof of well-posedness we also need to see that for 2 different solutions u, v we have by the proof of Theorem 6.1.2:*

$$M_0^0([p^1(u)-p^1(v)](t,\cdot);1) \le e^{C\|(u,v)\|_{C_*^{2-\alpha}}} \left\| \mathrm{Op}(\frac{1}{D})P_{\geq b}[u-v](t,\cdot)\right\|_{C_*^{2-\alpha}}.$$

Let us assess what we obtained from this first gauge transform we eliminated the term:

$$\sigma^{B,b}_{iu\xi} \in \Gamma^{\alpha-1}_0(\mathbb{R})$$

which generates an operator of order $\alpha-1$ and obtained $i\int_0^1 (\partial_t p^1)_r^{p^1}\, dr$ which under mere $C_*^{2-\alpha}$ control on u is still in $\Gamma_0^{\alpha-1}(\mathbb{R})$ which is still of order $\alpha-1$. Thus apriori in the low regularity setting in which we are working, we do not have a gain on the order of the remainder. The key idea is then to iterate this gauge transform, prove that the remainder goes to 0 and that the successive application of this gauge transform converges.

Applying successively Theorem 6.1.2 we construct a series of symbols $p^j \in \Gamma^0_1(\mathbb{R}), j\geq 2$ such that:

$$\sigma^{2,b}_{\otimes e^{ip_j}} \otimes \sigma^{B,b}_{\xi|\xi|^{\alpha-1}} - \sigma^{B,b}_{\xi|\xi|^{\alpha-1}} \otimes \sigma^{2,b}_{\otimes e^{-ip_j}} = \sigma^{2,b}_{\int_0^1 (\partial_t p^{j-1})_r^{p^{j-1}} dr}, \tag{6.3.5}$$

$$\mathrm{Im}(p^j) = \frac{p^j - p^{j*}}{2i} \in \Gamma^0_0(\mathbb{R}),\ M_0^0(\mathrm{Im}(p^j)) \le CM_0^0(\partial_x p^j),$$

$$\begin{aligned} M_0^0(\partial_x p^j(t,\cdot);0) \le{}& \frac{1 + Ce^{CM_0^0(p^{j-1}(t,\cdot);0)} M_0^0(\partial_t p^{j-1}|\xi|^{1-\alpha}(t,\cdot);0)}{\alpha} \\ &\times e^{CM_0^0(p^{j-1}(t,\cdot);0)} M_0^0(\partial_t p^{j-1}|\xi|^{1-\alpha}(t,\cdot);0). \end{aligned} \tag{6.3.6}$$

$$M_0^{-1}(\partial_\xi\partial_x p^j;0) \le (1 + C\times(*))\times\left[\frac{(*)}{\alpha}\right] + C(\alpha-1)e^{CM_0^0(p^{j-1}(t,\cdot);0)}M_0^0(\partial_x p^j;0)$$

where,

$$\begin{aligned}(*) = e^{CM_0^0(p^{j-1}(t,\cdot);0)}[&CM_0^0(\partial_t p^{j-1}|\xi|^{1-\alpha}(t,\cdot);0)M_0^{-1}(\partial_\xi p^{j-1}(t,\cdot);0) \\ &+ M_0^{-1}(\partial_\xi\partial_t p^{j-1}|\xi|^{1-\alpha}(t,\cdot);0)]\end{aligned} \tag{6.3.7}$$

From this we deduce that for $\|u\|_{C_*^{2-\alpha}} \le r$ with r sufficiently small we have:

$$M_0^0(\partial_x p^j(t,\cdot);0) \le C\left(\frac{1+Cr}{\alpha}\right)^j \|u(t,\cdot)\|_{C_*^{2-\alpha}}, \tag{6.3.8}$$

$$M_0^{-1}(\partial_\xi\partial_x p^j(t,\cdot);0) \le C\left(\frac{1+Cr}{\alpha}\right)^j \|u(t,\cdot)\|_{C_*^{2-\alpha}}. \tag{6.3.9}$$

Consider $A_1^{p^j}$ defined by Proposition 4.1.1 with the choice of cutoff in the paradifferential operator given by B, b and define $u^j = A_1^{p^j}u^{j-1}$, by construction we have:

$$\partial_t u^j + T_{i\xi|\xi|^{\alpha-1}}^{\psi^{B,b}} u^j - T^{\psi^{2,b}}_{i\int_0^1 (\partial_t p^j)_r^{p^j} dr} = R^{j,\infty},\ u^j(0,\cdot) = A_1^{p^j}u^{j-1}(0,\cdot), \tag{6.3.10}$$

where,

$$R^{j,\infty} = r_\infty^j + A_1^{p^j} R^{j-1,\infty},$$

where r_∞^j is generated by the different cut-off errors generated by the difference of $A_1^{p^j}$ and $T_{\otimes e^{ip^j}}$.

We now need to study $\prod_{j=1}^{k} A_1^{p^j}$, for this we define for $\tau \in \mathbb{R}$:

$$\tilde{p^1}(\tau, t, x) = p^1(t, x),\ \tilde{p^k}(\tau, t, x) = p^{k+1}(t, x) + (p^k(t, x))_\tau^{p^{k+1}(t,x)}, \tag{6.3.11}$$

thus by Proposition 4.1.1:

$$\prod_{j=1}^{k} A_1^{p_j} = [A_\tau^{\tilde{p^k}(\tau,t,x)}]_{\tau=1}. \tag{6.3.12}$$

We then have the estimates:

$$\sup_{|\tau|\leq 1} M_0^0(\widetilde{p^{k+1}}(t,\cdot);0) \leq M_0^0(p^j(t,\cdot);0) + e^{CM_0^0(p^j(t,\cdot);0)} \sup_{|\tau|\leq 1} M_0^0(\tilde{p^k}(t,\cdot);0), \tag{6.3.13}$$

$$\begin{aligned}&\sup_{|\tau|\leq 1} M_0^{-1}(\partial_\xi \widetilde{p^{k+1}}(t,\cdot);0) \leq M_0^{-1}(\partial_\xi p^j(t,\cdot);0)\\ &+ e^{CM_0^0(p^j(t,\cdot);0)}\left[\sup_{|\tau|\leq 1} M_0^{-1}(\partial_\xi \tilde{p^k}(t,\cdot);0) + CM_0^{-1}(\partial_\xi p^j(t,\cdot);0)\sup_{|\tau|\leq 1} M_0^0(\tilde{p^k}(t,\cdot);0)\right],\end{aligned} \tag{6.3.14}$$

Thus $(\tilde{p^k})_{k\in\mathbb{N}}$ converges for the seminorm $L^\infty_{|\tau|\leq 1} M_0^0(\cdot, 1)$, the iteration of this argument shows that $(\tilde{p^k})_{k\in\mathbb{N}}$ converges for all of the seminorm $W^{m,\infty}_{|\tau|\leq 1} M_1^0(\cdot, 1), m \in \mathbb{N}^*$, thus there exits $\tilde{p}$ such that:

$$\tilde{p} \in W^{\infty,\infty}([-1,1]; C^1\Gamma_1^0(\mathbb{D})),\quad W_\tau^{m,\infty} M_1^0(\tilde{p}(t,\cdot)) \leq C \left\|u(t,\cdot)\right\|_{C_*^{2-\alpha}},$$

$$\operatorname{Im}(\tilde{p}) = \frac{\tilde{p} - \tilde{p}^*}{2i} \in \Gamma_0^0(\mathbb{D}) \text{ and } M_0^0(\operatorname{Im}(\tilde{p})) \leq CM_0^0(\partial_x \tilde{p})$$

Thus passing to the limit in (6.3.10) we get:

$$\partial_t\left[(A_\tau^{\tilde{p}(\tau,\cdot)})_{\tau=1} u\right] + T^{\psi^{B,b}}_{i\xi|\xi|^{\alpha-1}}\left[(A_\tau^{\tilde{p}(\tau,\cdot)})_{\tau=1} u\right] = R^{\infty,\infty}(u), \tag{6.3.15}$$

where there exists a non decreasing functions C_s, such that $R^{\infty,\infty}(u)$ verifies for all $\mu \in \mathbb{R}$:

$$\left\|R^{\infty,\infty}(u)\right\|_{H^\mu} \leq C_\mu(\|u\|_{L^\infty([0,T],C_*^{2-\alpha}(\mathbb{D}))}),$$

which is the desired conjugation result.

Finally the proof of well-posedness follows from a standard energy estimate on (6.3.15) combined with Proposition 4.1.1 and the Lipschtiz estimate:

$$M_0^0([\tilde{p}(u) - \tilde{p}(v)](t,\cdot);1) \leq e^{C\|(u,v)\|_{C_*^{2-\alpha}}} \left\|\operatorname{Op}(\frac{1}{D}) P_{\geq b}[u-v](t,\cdot)\right\|_{C_*^{2-\alpha}}.$$

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ÉCOLE DOCTORALE
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Titre: Sur le flow de l'équation d'Euler à surface libre

Mots clés: Flot, équation d'Euler, regularité, équation dispersive, équation de Burgers, analyse microlocale, paracomposition, calcul paradifferentiel, transformation de jauge de Cole-Hopf, formule de Baker-Campbell-Hausdorff.

Résumé: L'équation d'Euler à surface libre décrit l'évolution de l'interface séparant l'air d'un fluide parfait irrotationnel. C'est un système de deux équations couplées : l'équation d'Euler à l'intérieur du domaine et une équation cinématique qui décrit les déformations du domaine. Les 4 travaux qui constituent le corps de cette thèse peuvent être divisés en trois sujets connectés au problème de Cauchy du système des water waves.

- Dans le prolongement des travaux de [2, 4, 5, 7], où les auteurs ont montré que le problème de Cauchy pour le système des water waves est bien posé et que le flot est continu sur des espaces de Sobolev suffisamment réguliers, nous montrons :

  - Dans [60] que le système des water waves avec ou sans tension de surface est quasi-linéaire au sens le plus fort du terme, c'est-à-dire que le flot n'est pas uniformément continu. De plus, dans le cas avec tension de surface, nous montrons que pour avoir une estimation de Lipschitz sur le flot, il faut au moins une perte de $\frac{1}{2}$ dérivés. Plus généralement, pour l'équation de Burgers avec terme dispersif de la forme $\partial_x|D|^{\alpha-1}, \alpha \in]1,2[$, nous montrons qu'il faut au moins une perte de $2-\alpha$ dérivés pour assurer un contrôle Lipschitz sur le flot.

  - Dans [61], nous montrons que les résultats obtenus dans [60] sont effectivement optimaux, c'est-à-dire que pour l'équation de Burgers avec un terme dispersif $\partial_x|D|^{\alpha-1}, \alpha \in]1,2[$ le flot est effectivement Lipschitz de H^s á $H^{s-2+\alpha}$ pour des données initiales périodiques de moyenne nulle. Pour le système des water waves avec tension de surface en deux dimensions d'espace, nous montrons qu'après re-normalisation, le flot est bien Lipschitz au prix d'une perte de $\frac{1}{2}$ dérivées.

- Afin de démontrer les résultats dans [61], nous avons développé une généralisation para-différentielle d'une transformation de jauge complexe de type Cole-Hopf introduite pour la première fois par T. Tao pour l'équation de Benjamin-Ono. Dans [62], nous l'utilisons pour améliorer les résultats connus sur une conjecture numérique dûe à Saut et Klein dans [45] sur l'équation de Burgers dispersive. Ce qui, à la connaissance de l'auteur, est la première fois que la transformation de jauge est mise en oeuvre à cette fin pour $\alpha \in]1,2[$.

- Afin de démontrer les différents résultats dans [60, 61, 62], nous avons étudié et affiné différents résultats connus en calcul paradifférentiel. Plus précisément, dans [59], nous améliorons certaines estimations sur l'opérateur de paracomposition introduit par Alinhac, nous donnons une preuve du changement de variables dans le calcul paradifférentiel et enfin nous étudions comment le support du cut-off fréquentiel varie après la composition d'opérateurs para-différentiels.

Title: On the flow map of the Euler equation with free boundary

Keywords: flow map, Euler equation, regularity, dispersive equations, microlocal analysis, paracomposition, paradifferential calculus, Cole-Hopf Gauge transform, Baker-Campbell-Hausdorff formula

Abstract: The Euler equation with free boundary, i.e the water waves system, describes the evolution of the interface between air and a perfect irrotational fluid. It is a system of two coupled equations: the Euler equation in the interior of the domain and a kinematic equation describing the deformation of the domain. The 4 works that constitute the body of this thesis can be divided into three connected subjects on the Cauchy problem of the water waves system.

- In the continuation of the works in [2, 4, 5, 7] where the Cauchy problem for the water waves system is shown to be well-posed and the ow map continuous on sufficiently regular Sobolev spaces we show:
  - In [60] that the water waves system with and without surface tension is quasi-linear in the strongest sense, i.e the flow map is not uniformly continuous. Moreover in the case with surface tension we show that in order to have Lipschitz estimate on the flow map at least a loss of $\frac{1}{2}$ derivative. More generally for the Burgers equation augmented by a dispersive term of the form $\partial_x|D|^{\alpha-1}, \alpha \in]1,2[$, we show that at least a loss of $2-\alpha$ derivative is needed to ensure Lipschitz control on the flow.
  - In [61] we show that the results obtained in [60] are indeed optimal, that is for the Burgers equation augmented with a dispersive term $\partial_x|D|^{\alpha-1}, \alpha \in]1,2[$ the flow map is indeed Lipschitz from H^s to $H^{s-2+\alpha}$ for periodic data with 0 mean value. For the water waves system with surface tension in two space dimension we show that after suitable renormalization that the flow map is Lipschitz under $\frac{1}{2}$ loss of derivative.
- In order to prove the results in [61] we developed a paradifferential generalization of a complex Cole-Hopf type gauge transform first introduced by T. Tao for the Benjamin-Ono equation. In [62] we use this to improve upon known results on a numerical conjecture by Saut and Klein [45] on the dispersive Burgers equation, which to the author's knowledge is the first time the gauge transform was implemented to that for $\alpha \in]1,2[$.
- In order to prove the different results in [60, 61, 62] we needed to study and refine different know results in paradifferential calculus. More precisely in [59], we improve some estimates on the paracomposition operator introduced by Alinhac, give a proof of the change of variables in paradifferential operators and finally study the frequency cut-off after composition of paradifferential operators.

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