# REGULARITY RESULTS ON THE FLOW MAP OF PERIODIC DISPERSIVE BURGERS TYPE EQUATIONS AND THE GRAVITY-CAPILLARY EQUATIONS

AYMAN RIMAH SAID

ABSTRACT. In the first part of this paper we prove that the flow associated to the Burgers equation with a non local term of the form $D^{\alpha-1}\partial_x u$, $\alpha \in [1,+\infty[$ is Lipschitz from bounded sets of $H_0^s(\mathbb{T};\mathbb{R})$ to $C^0([0,T],H_0^{s-(2-\alpha)^+}(\mathbb{T};\mathbb{R}))$ for $T>0$ and $s>1+\frac{1}{2}$, where $H_0^s$ are the Sobolev spaces of function with 0 mean value, proving that the result obtained in [14] is optimal on the torus. We also prove that passing from $H_0^s(\mathbb{T})$ to $H^s(\mathbb{T})$ induces a loss of regularity on the flow map i.e it is not uniformly continuous from $H^s(\mathbb{T};\mathbb{R})$ to $C^0([0,T],H^s(\mathbb{T};\mathbb{R}))$ and is not Lipschitz from bounded sets of $H^s(\mathbb{T};\mathbb{R})$ to $C^0([0,T],H^{s-1+\epsilon}(\mathbb{T};\mathbb{R})), \epsilon>0$. We give an analogous example on $\mathbb{R}$ clarifying the need of the 0 mean value hypothesis. The proof relies on a pseudodifferential generalization of a complex Cole-Hopf gauge transformation introduced by T.Tao in [16] for the Benjamin-Ono equation.

In the second part of this paper we use a paralinearisation version of the previous method to prove that the flow of the one dimensional periodic Gravity Capillary equations is Lipschitz from bounded sets of $H^s$ verifying the symmetry hypothesis **S** to $C^0([0,T],H^{s-\frac{1}{2}})$ for $T>0$ and $s>\frac{5}{2}+\frac{1}{2}$ proving that the result obtained in [14] is optimal for the Water Waves system. We present for this a general scheme for the study of the flow map regularity in PDE, which is the main result of this paper.

***Keywords***— **Flow map, Regularity, Quasi-linear, nonlinear Burgers type dispersive equations, Water Waves system, Gravity-Capillary equations, Cole-Hopf Gauge transform.**

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## 1. INTRODUCTION

In our study of the quasi-linearity of the Water Wave system in [14] we studied the flow map regularity for some model non linear dispersive equations of the form:

$$\partial_t u + u\partial_x u + D^{\alpha-1}\partial_x u = 0 \text{ on } \mathbb{D}, \tag{1.1}$$

where $\mathbb{D}=\mathbb{R}$ or $\mathbb{T}$, $\alpha \in [0,2[$ and $D$ is the Fourier multiplier $|\xi|$. We proved that they are quasi-linear. We based our work on the following distinction between semi-linearity and quasi-linearity given in [12]:

- A partial differential equation is said to be semi-linear if its flow map is regular (at least $C^1$).
- A partial differential equation is said to be quasi-linear if its flow map is not Lipschitz.

PhD student at CMLA, Batiment Laplace 61, Avenue du President Wilson 94235 Cachan Cedex. email: aymanrimah@gmail.com.

More precisely we proved that:

- the flow map associated to (1.1) fails to be uniformly continuous from bounded sets of $H^s(\mathbb{D})$ to $C^0([0,T],H^s(\mathbb{D}))$ for $T>0$ and $s>2+\frac{1}{2}$.

The draw back of this test of quasi-linearity is that it does not show the effect of the dispersive term. The natural question was then to ask if one can see the effect of the dispersive term by analyzing more precisely the regularity of the flow map.

For this we can start by noticing that independently of $\alpha$ the flow is Lipschitz from bounded sets of $H^s(\mathbb{D})$ to $C^0([0,T],H^{s-1}(\mathbb{D}))$ and ask the natural question, can the space $H^{s-1}(\mathbb{D})$ can be improved to $H^{s-\mu}(\mathbb{D}))$ with $\mu<1$ and $\mu$ depending on $\alpha$. Again in [14] we proved that the best $\mu$ one can hope for is $\mu=1-(\alpha-1)^+$, more precisely we showed that

- the flow map cannot be Lipschitz from bounded sets of $H^s(\mathbb{D})$ to $C^0([0,T],H^{s-1+(\alpha-1)^++\epsilon}(\mathbb{D}))$ for $\epsilon>0$.

Looking to the literature to assess the optimality of the result, first in [15] the equation (1.1) is actually shown to be quasi-linear for $\alpha\in[0,3[$ and becomes semi-linear for $\alpha=3$ on $\mathbb{R}$ suggesting that our results are sub-optimal. Then on the circle, in [13], for the case $\alpha=2$ and the the Benjamin-Ono equation, the flow map is shown to be Lipschitz (and even has analytic regularity) on bounded sets of $H^s_0$ the Sobolev spaces of functions with mean value 0. Which suggests that our results could be optimal but with a subtlety in the low frequencies.

The aim of the current paper is to prove that the results obtained in [14] are optimal in several directions on $\mathbb{R}$ and on the torus separately and finally for the full periodic Water Waves system with surface tension, i.e the Gravity Capillary equation, while clarifying in all of those cases the effect brought on by the low frequencies.

### 1.1. **On the torus.**

We show that the flow map associated to (1.1) is Lipschitz from bounded sets of $H^s_0(\mathbb{T})$ to $C^0([0,T],H_0^{s-1+(\alpha-1)^+}(\mathbb{T}))$:

**Theorem 1.1.** *Consider three real numbers $\alpha\in[1,+\infty[$, $s\in]1+\frac{1}{2},+\infty[$, $r>0$ and $u_0\in H^s(\mathbb{T};\mathbb{R})$. Then there exists $T>0$ such that for all $v_0$ in the ball $B(u_0,r)\subset H^s(\mathbb{T};\mathbb{R})$ there exists a unique $v\in C([0,T],H^s(\mathbb{T};\mathbb{R}))$ solving the Cauchy problem:*

$$\begin{cases}\partial_t v+v\partial_x v+D^{\alpha-1}\partial_x v=0\\ v(0,\cdot)=v_0(\cdot)\end{cases}. \tag{1.2}$$

*Moreover we have the estimates:*

$$\forall 0\le\mu\le s, \|v(t)\|_{H^\mu(\mathbb{T})}\le C_\mu\|v_0\|_{H^\mu(\mathbb{T})}. \tag{1.3}$$

*Taking Two different solution $u,v$, such that $u_0\in H^{s+1}(\mathbb{T})$:*

$$\forall 0\le\mu\le s, \|(u-v)(t)\|_{H^\mu(\mathbb{T})}\le\|u_0-v_0\|_{H^\mu(\mathbb{T})}e^{C_\mu\int_0^t\|u(s)\|_{H^{\mu+1}(\mathbb{T})}ds}. \tag{1.4}$$

*From [14] for $s>2+\frac{1}{2}$ and $\alpha\in[0,2[$:*

- *the flow map:*
$$\begin{aligned}B(u_0,r)&\to C([0,T],H^s(\mathbb{T};\mathbb{R}))\\ v_0&\mapsto v\end{aligned}$$
*is continuous but not uniformly continuous.*
- *Considering a weaker control norm we get, for all $\epsilon>0$ the flow map:*
$$\begin{aligned}B(u_0,r)&\to C([0,T],H^{s-1+(\alpha-1)^++\epsilon}(\mathbb{T};\mathbb{R}))\\ v_0&\mapsto v\end{aligned}$$
*is not $C^1$.*

*We complete that by the following:*

(1) *We show that for $\alpha \in [1, +\infty[$, $s \in ]1 + \frac{1}{2}, +\infty[$, the flow map:*

$$\begin{aligned} B(u_0, r) \cap H_0^s(\mathbb{T}; \mathbb{R}) &\to C([0,T], H_0^{s-(2-\alpha)^+}(\mathbb{T}; \mathbb{R})) \\ v_0 &\mapsto v \end{aligned}$$

*is Lipschitz.*

(2) *From which we deduce for $\alpha \in [0, +\infty[$, $s \in ]1 + \frac{1}{2}, +\infty[$,*

- *the flow map:*

$$\begin{aligned} B(u_0, r) &\to C([0,T], H^s(\mathbb{T}; \mathbb{R})) \\ v_0 &\mapsto v \end{aligned}$$

  *is continuous but not uniformly continuous.*

- *For all $\epsilon > 0$ the flow map:*

$$\begin{aligned} B(u_0, r) &\to C([0,T], H^{s-1+\epsilon}(\mathbb{T}; \mathbb{R})) \\ v_0 &\mapsto v \end{aligned}$$

  *is not $C^1$.*

**Remark 1.1.** *The case $\alpha = \frac{3}{2}$ is closely related to the system obtained after reduction and para-linearization of the periodic Water Waves system in dimension 1 obtained in [4] Proposition 3.3 by T. Alazard, N. Burq and C. Zuily, which we will treat in the second part of this paper.*

*The case $\alpha = 2$ and the Benjamin-Ono equation on the circle was obtained by Molinet in [13]. Though Molinet's result extends to the Cauchy problem on $L^2(\mathbb{T})$ and only studied the flow map regularity for data with $0$ mean value.*

1.2. **On $\mathbb{R}$.** Closely inspecting the two problems in [15] and [13], we saw that the lack of regularity obtained in [15] for $\alpha \geq 2$ is essentially due to the lack of control of the $L^1$ norm in Sobolev spaces. To show this for this we start by a simple, albeit an artificial example:

**Theorem 1.2.** *Consider two real numbers $s \in ]1 + \frac{1}{2}, +\infty[$, $r > 0$ and $u_0 \in H^s(\mathbb{R})$. Then there exists $T > 0$ such that for all $v_0$ in the ball $B(u_0, r) \subset H^s(\mathbb{R})$ there exists a unique $v \in C([0,T], H^s(\mathbb{R}))$ solving the Cauchy problem:*

$$\begin{cases} \partial_t v + \mathrm{Re}(v)\partial_x v + i\partial_x^2 v = 0 \\ v(0, \cdot) = v_0(\cdot) \end{cases}. \tag{1.5}$$

*Moreover we have the estimates:*

$$\forall 0 \leq \mu \leq s, \|v(t)\|_{H^\mu(\mathbb{R})} \leq C_\mu \|v_0\|_{H^\mu(\mathbb{R})}. \tag{1.6}$$

*Taking two different solution $u, v$ such that $u - v \in L^1(\mathbb{R})$, then for all $0 \leq \mu \leq s$:*

$$\begin{aligned} \|(u-v)(t)\|_{H^\mu(\mathbb{R})} \leq C(\|u_0\|_{H^\mu(\mathbb{R})}) \|u_0 - v_0\|_{H^\mu(\mathbb{R})} \\ + C(\|u_0\|_{H^\mu(\mathbb{R})})[\|u_0 - v_0\|_{L^1(\mathbb{R})} + \|(u-v)(t)\|_{L^1(\mathbb{R})}]. \end{aligned} \tag{1.7}$$

1.3. **The periodic Gravity Capillary equation.** The second and main application is the water waves system with and without surface tension. We follow here the presentation in [4], [3] and [2].

1.3.1. *Assumptions on the domain.* We consider a domain with free boundary, of the form:

$$\{(t,x,y) \in [0,T] \times \mathbb{R} \times \mathbb{R} : (x,y) \in \Omega_t\},$$

where $\Omega_t$ is the domain located between a free surface

$$\Sigma_t = \{(x,y) \in \mathbb{R} \times \mathbb{R} : y = \eta(t,x)\}$$

and a given (general) bottom denoted by $\Gamma = \partial\Omega_t \setminus \Sigma_t$. More precisely we assume that initially ($t = 0$) we have the hypothesis $H_t$ given by:

- The domain $\Omega_t$ is the intersection of the half space, denoted by $\Omega_{1,t}$, located below the free surface $\Sigma_t$,

  $$\Omega_{1,t} = \{(x,y) \in \mathbb{R} \times \mathbb{R} : y < \eta(t,x)\}$$

  and an open set $\Omega_2 \subset \mathbb{R}^{1+1}$ such that $\Omega_2$ contains a fixed strip around $\Sigma_t$, which means that there exists $h > 0$ such that,

  $$\{(x,y) \in \mathbb{R} \times \mathbb{R} : \eta(t,x) - h \leq y \leq \eta(t,x)\} \subset \Omega_2.$$

  We shall assume that the domain $\Omega_2$ (and hence the domain $\Omega_t = \Omega_{1,t} \cap \Omega_2$) is connected.

1.3.2. *The equations.* We consider an incompressible inviscid liquid, having unit density. The equations of motion are given by the Euler system on the velocity field $v$:

$$\begin{cases} \partial_t v + v \cdot \nabla v + \nabla P = -g e_y \\ \operatorname{div} v = 0 \end{cases} \quad \text{in } \Omega_t, \tag{1.8}$$

where $-g e_y$ is the acceleration of gravity ($g > 0$) and where the pressure term $P$ can be recovered from the velocity by solving an elliptic equation. The problem is then coupled with the boundary conditions:

$$\begin{cases} v \cdot n = 0 & \text{on } \Gamma \\ \partial_t \eta = \sqrt{1 + |\nabla\eta|^2}\, v \cdot \nu & \text{on } \Sigma_t\,, \\ P = -\kappa H(\eta) & \text{on } \Sigma_t \end{cases} \tag{1.9}$$

where $n$ and $\nu$ are the exterior normals to the bottom $\Gamma$ and the free surface $\Sigma_t$, $\kappa$ is the surface tension and $H(\eta)$ is the mean curvature of the free surface:

$$H(\eta) = \operatorname{div}\left(\frac{\nabla\eta}{\sqrt{1 + |\nabla\eta|^2}}\right).$$

We are interested in the case with surface tension and take $\kappa = 1$. The first condition in (1.9) expresses in fact that the particles in contact with the rigid bottom remain in contact with it. As no hypothesis is made on the regularity of $\Gamma$, this condition is shown to make sense in a weak variational meaning due to the hypothesis $H_t$, for more details on this we refer to Section 2 in [4].

The fluid motion is supposed to be irrotational and $\Omega_t$ is supposed to be simply connected thus the velocity $v$ field derives from some potential $\phi$ i.e $v = \nabla\phi$ and:

$$\begin{cases} \Delta\phi = 0 \text{ in } \Omega, \\ \partial_n \phi = 0 \text{ on } \Gamma. \end{cases}$$

The boundary condition on $\phi$ becomes:

$$\begin{cases} \partial_n \phi = 0 & \text{on } \Gamma \\ \partial_t \eta = \partial_y \phi - \nabla\eta \cdot \nabla\phi & \text{on } \Sigma_t\,. \\ \partial_t \phi = -g\eta + H(\eta) - \frac{1}{2}\left|\nabla_{x,y}\phi\right|^2 & \text{on } \Sigma_t \end{cases} \tag{1.10}$$

Following Zakharov [18] and Craig-Sulem [8] we reduce the analysis to a system on the free surface $\Sigma_t$. If $\psi$ is defined by

$$\psi(t,x) = \phi(t,x,\eta(t,x)),$$

then $\phi$ is the unique variational solution of

$$\Delta\phi = 0 \text{ in } \Omega_t,\ \ \phi_{|y=\eta} = \psi,\ \ \partial_n\phi = 0 \text{ on } \Gamma.$$

Define the Dirichlet-Neumann operator by

$$\begin{aligned}(G(\eta)\psi)(t,x) &= \sqrt{1+|\nabla\eta|^2}\partial_n\phi_{|y=\eta}\\ &= (\partial_y\phi)(t,x,\eta(t,x)) - \nabla\eta(t,x)\cdot(\nabla\phi)(t,x,\eta(t,x)).\end{aligned}$$

For the case with rough bottom we refer to [6], [4] and [3] for the well posedness of the variational problem and the Dirichlet-Neumann operator. Now $(\eta,\psi)$ (see for example [8]) solves:

$$\begin{aligned}&\partial_t\eta = G(\eta)\psi,\\ &\partial_t\psi = -g\eta + H(\eta) + \frac{1}{2}|\nabla\psi|^2 + \frac{1}{2}\frac{\nabla\eta\cdot\nabla\psi + G(\eta)\psi}{1+|\nabla\eta|^2}.\end{aligned} \tag{1.11}$$

The system is completed with initial data

$$\eta(0,\cdot) = \eta_{in},\ \ \psi(0,\cdot) = \psi_{in}.$$

We consider the case when $\eta$, $\psi$ are $2\pi$-periodic in the space variable $x$. Moreover we make the following symmetry hypothesis:

$$\forall t \geq 0,\ \int_\Sigma \partial_x\phi d\Sigma = \int_0^{2\pi}\partial_x\phi(t,x,\eta(x))\sqrt{1+(\partial_x\eta)^2}dx = 0. \tag{$\mathbf{S}$}$$

This hypothesis is for example true for even initial data $\eta_{in}$, and $\psi_{in}$ as this property is propagated by the flow of the equation.

1.3.3. *Flow map regularity.* In [4] and [3], Alazard, Burq, and Zuily perform a paralinearization and symmetrization of the the water waves system that take the form:

$$\partial_t u + T_V.\nabla u + iT_\gamma u = f,$$

where $\gamma$ is elliptic of order $\frac{3}{2}$ which closely resembles the model problem we presented on $\mathbb{T}$ but with an extra non linearity in $\gamma$. The paralinearization and symmetrization of system was used to prove the well-posedness of the Cauchy problem in the optimal threshold $s > 2+\frac{1}{2}$ in which the velocity field $v$ is in $W^{1,\infty}$. We will complete this and our result [14] to give the precise regularity of the flow map:

**Theorem 1.3.** *From [3, 4], consider two real numbers $r > 0$, $s \in ]2+\frac{1}{2},+\infty[$ and $(\eta_0,\psi_0) \in H^{s+\frac{1}{2}}(\mathbb{T})\times H^s(\mathbb{T})$ such that*

$$\forall(\eta_0',\psi_0') \in B((\eta_0,\psi_0),r) \subset H^{s+\frac{1}{2}}(\mathbb{T})\times H^s(\mathbb{T})$$

*the assumption $H_{t=0}$ is satisfied. Then there exists $T > 0$ such that the Cauchy problem (1.11) with initial data $(\eta_0',\psi_0') \in B((\eta_0,\psi_0),r)$ has a unique solution*

$$(\eta',\psi') \in C^0([0,T]; H^{s+\frac{1}{2}}(\mathbb{T})\times H^s(\mathbb{T}))$$

*and such that the assumption $H_t$ is satisfied for $t \in [0,T]$.*

*Moreover from [14], for all $R > 0$ the flow map:*

$$\begin{aligned}B(0,R) &\rightarrow C([0,T], H^{s+\frac{1}{2}}(\mathbb{T})\times H^s(\mathbb{T}))\\ (\eta_0',\psi_0') &\mapsto (\eta',\psi')\end{aligned}$$

*is not uniformly continuous.*

*And we showed that at least a loss of $\frac{1}{2}$ derivative is necessary to have Lipschitz control over the flow map, i.e for all $\epsilon' > 0$ the flow map*

$$\begin{aligned}B(0,R) &\to C([0,T], H^{s+\epsilon'}(\mathbb{T}) \times H^{s-\frac{1}{2}+\epsilon'}(\mathbb{T}))\\ (\eta_0', \psi_0') &\mapsto (\eta', \psi')\end{aligned}$$

*is not Lipschitz.*

*Here we complete this by showing that it is sufficient under the symmetry Hypothesis* (**S**)*, i.e taking* $(\eta, \psi)$ *and* $(\eta', \psi')$ *in* $B((\eta, \psi), r)$ *verifying Hypothesis* (**S**) *then*

$$\begin{aligned}&\left\|(\eta, \psi) - (\eta', \psi')(t, \cdot)\right\|_{H^s \times H^{s-\frac{1}{2}}}\\ &\leq C(\left\|(\eta_0, \psi_0, \eta_0', \psi_0')\right\|_{H^{s+\frac{1}{2}} \times H^s}) \left\|(\eta_0, \psi_0) - (\eta_0', \psi_0')\right\|_{H^s \times H^{s-\frac{1}{2}}}. \end{aligned} \tag{1.12}$$

1.4. **Strategy of the proof.** For Theorem 1.1, we first work on $H_0^s$ and the main idea is to conjugate (1.1) to a semi-linear dispersive equation of the form:

$$\partial_t w + D^{\alpha-1}\partial_x w = Ru,$$

where R is continuous from $H^s$ to itself. For the viscous Burger equation such a result is obtained by the Cole-Hopf transformation that reduces the problem to a one dimensional heat equation. In [16], T.Tao used a complex version of the Cole-Hopf transformation to reduce the problem to a one dimensional Schrodinger equation, this idea was extensively used to lower the regularity needed for the well-posedness of the Cauchy problem as in Molinet's work in [13]. A generalized pseudodifferential form of this transformation was used in [1] to reduce the one dimensional water waves system to a a one dimensional semi-linear Schrodinger type system.

The transformation we use is a pseudodifferential transformation of the form:

$$\begin{cases} w = \mathrm{Op}(a)u, \\ a = e^{\frac{1}{i\alpha}\xi|\xi|^{1-\alpha}U}, \end{cases} \tag{1.13}$$

Where $U$ is a real valued periodic primitive of $u$ that exists because $u \in H_0^s$. Thus the hypothesis $u$ is real valued and $u$ has 0 mean value are crucial here. The main problem here is to notice that such an operator belongs to a Hörmander symbol class of the form $S^0_{\alpha-1,2-\alpha}$, which for $\alpha = \frac{3}{2}$ becomes $S^0_{\frac{1}{2},\frac{1}{2}}$ which is a "bad" symbol class with no general symbolic calculus rules. Thus we have to treat this transformation with care, for this we use the particular form of the transformation we have here to prove symbolic calculus rules. This transformation helps us reduce the transport term of order 1 to a term of order $2-\alpha$ which is enough for our problem.

Passing from $H_0^s$ to $H^s$ we use the following gauge transform

$$\tilde{u}(t,x) = u(t, x - t\fint u_0) - \fint u_0,$$

which we prove is continuous on $H^s$ but not uniformly continuous and only $C^1$ from $H^s$ to $H^{s-1}$.

For the Gravity-Capillary equation the problem is a bit more subtle. Indeed the model problems we study are for the paralinearized and symmetrized system but this change of variable is known to Lipschitz on $H^s$ for $s > 2 + \frac{1}{2}$. Thus the problem is reduced to the study of the flow map regularity of an equation of the form

$$\partial_t u + T_V.\nabla u + iT_\gamma u = f.$$

Then we preform a transformation in the same spirit as for Equation (1.1) but with a more complicated paradifferential symbol due to the non linearity in the dispersive term $\gamma$ and with the hypothesis of 0 mean value replaced here by the symmetry Hypothesis (**S**) which is intrinsic to the Water Waves system.

After this second transformation we get an equation of the form

$$\partial_t u + iT_\gamma u = R, \tag{1.14}$$

where $R$ contains harmless or hyperbolic terms of order at most $\frac{1}{2}$. The main problem here is that Equation (1.14) is a non linear equation with the naive estimate to control the flow map costing $\frac{3}{2}$ derivatives to bound which is a full derivative above the control we want to prove. The strategy here is inspired by the ODE method in which one can get an ODE on the differential of the flow map but here will get a PDE. Indeed if we write that

$$u(t,\cdot) = F(t,u_0),$$

where $F$ is the flow map of Equation (1.14) then we differentiate F in $u_0 \in H^s$, defining

$$\text{for } h_0 \in H^s, h = DF(t,u_0)h_0$$

we get:

$$\begin{cases} \partial_t h + iT_\gamma h + iT_{D\gamma h} u = Rh \\ h(0,\cdot) = h_0. \end{cases}$$

The problematic term here is $T_{D\gamma h}u$, which costs $\frac{3}{2}$ derivative in $u$ to control. To treat this term we use crucially the ellipticity of $\gamma$ and yet another change of variable to get the desired estimate. It's worth noting that this change of variable proves an even stronger and new result that is the Equation (1.14) is actually semi-linear.

**Remark 1.2.**
- *Transformation (1.13) in which we use a primitive of the solution is called a gauge transform in the literature.*
- *As for the Cole-Hopf transformation, this gauge transform (1.13) is essentially one dimensional.*
- *It's interesting to know that the same type of transformation can be iterated and get at the step of order $k$ a remainder of order $k+1-k\alpha$ which is acceptable for $k$ sufficiently large as $\alpha > 1$ but the price you pay is $s > 1 + \frac{1}{\alpha - 1}$.*
- *Finally we would like to remark that the strategy of differentiating on $H^s$ the flow map to get a PDE on it's differential and trying to get the best estimates possible seems to us to be a robust and useful method in the study of the flow map regularity in PDE.*

1.5. **Acknowledgement.** I would like to express my sincere gratitude to my thesis advisor Thomas Alazard.

## 2. Study of the model problems

2.1. **Proof of Theorem 1.2.** This is the simplest theorem to prove as the transformation is straightforward and the symbols used are in the usual Hörmander symbol classes $S^m_{1,0}$. Given the well posedness of the Cauchy problem in $H^s$, and the density of $\mathscr{S}$ in $H^s$, it suffice to prove the result for $u_0, v_0 \in \mathscr{S}(\mathbb{R})$ which henceforth we will be supposed and we define $u, v$ as the solution to (1.5) with initial data $u_0, v_0$ on $[0,T]$.

The first step we reduce $H^s$ estimates to $L^2$ ones by defining $f_1 = \langle \mathrm{D} \rangle^s u$. Commuting $\langle \mathrm{D} \rangle^s$ with (1.5), by the symbolic calculus rules in Appendix A.1 we get the the PDE on $f_1$:

$$\begin{cases} \partial_t f_1 + \mathrm{Re}(u)\partial_x f_1 + i\partial_x^2 f_1 = R_1(f_1)f_1 \\ f_1(0,\cdot) = \langle \mathrm{D} \rangle^s u_0(\cdot), \end{cases} \tag{2.1}$$

where $R_1$ verifies

$$\|R_1(f_1)\|_{L^2\to L^2} \le C(\|u\|_{W^{1,\infty}}, \|u\|_{H^s}), \|\partial_{f_1} R_1(f_1)\|_{L^2\to L^2} \le C(\|u\|_{W^{1,\infty}}, \|u\|_{H^s}),$$

We define analogously $g_2$ from $v$ and notice that there exists $C > 0$ such that:

$$C^{-1}\|f_1 - g_1\|_{L^2} \le \|u - v\|_{H^s} \le C\|f_1 - g_1\|_{L^2}$$

thus the problem is reduced to getting $L^2$ estimates on $f_1 - g_1$.

Then we introduce $F(t,x) = \int_0^x Re(u)(t,y)dy \in C^\infty(\mathbb{R})$ and make the following change of variable:

$$f_2 = e^{-\frac{i}{2}F} f_1.$$

Define analogously define $G$ and $g_2$ from v. As remarked in [16], $F, G$ do not necessarily decay at infinity but we still have $e^{-\frac{i}{2}F}, e^{-\frac{i}{2}G} \in S^m_{1,0}$ indeed because $\partial_x F = u \in H^{+\infty}$ and $\partial_x G = v \in H^{+\infty}$. Now to get Lipschitz control we have

$$\left\| e^{\frac{i}{2}(G-F)} u \right\|_{L^2} \le \|G - F\|_{L^\infty} \|u\|_{L^2} \le \|v - u\|_{L^1} \|u\|_{L^2},$$

thus,

$$\begin{cases} \|f_2 - g_2\|_{L^2} \le C[\|f_1 - g_1\|_{L^2} + \|v - u\|_{L^1}], \\ \|f_1 - g_1\|_{L^2} \le C[\|f_2 - g_2\|_{L^2} + \|v - u\|_{L^1}], \end{cases}$$

and the problem is reduced to getting $L^2$ estimates on $f_2 - g_2$. Commuting $e^{\frac{i}{2}F}$ with (2.1) we get again by the symbolic calculus rules in Appendix A.1:

$$\begin{cases} \partial_t f_2 + i\partial_x^2 f_2 = R_2(f_2) f_2 \\ f_2(0,\cdot) = e^{\frac{i}{2}F_0} \langle \mathrm{D} \rangle^s u_0(\cdot), \end{cases} \tag{2.2}$$

where $R_2$ verifies

$$\|R_2(f_2)\|_{L^2\to L^2} \le C(\|u\|_{W^{1,\infty}}, \|u\|_{H^s}), \|\partial_{f_2} R_2(f_2)\|_{L^2\to L^2} \le C(\|u\|_{W^{1,\infty}}, \|u\|_{H^s}).$$

Analogously we get on $g_2$

$$\begin{cases} \partial_t g_2 + i\partial_x^2 g_2 = R_2(g_2) g_2 \\ g_2(0,\cdot) = e^{\frac{i}{2}G_0} \langle \mathrm{D} \rangle^s v_0(\cdot). \end{cases} \tag{2.3}$$

Now the usual energy estimate combined the Gronwall lemma on $f_2 - g_2$ gives for $0 \le t \le T$:

$$\|f_2 - g_2\|_{L^2} \le C(\|(u,v)\|_{W^{1,\infty}}, \|(u,v)\|_{H^s}) \|(f_2 - g_2)(0,\cdot)\|_{L^2}$$

As $s > 1 + \frac{1}{2}$ and the Sobolev embedding Theorem

$$\begin{aligned} \|f_2 - g_2\|_{L^2} &\le C(\|(u_0,v_0)\|_{H^s}) \left\| e^{\frac{i}{2}F_0} \langle \mathrm{D} \rangle^s u_0 - e^{\frac{i}{2}G_0} \langle \mathrm{D} \rangle^s v_0 \right\|_{L^2} \\ &\le C(\|(u_0,v_0)\|_{H^s})[\|u_0 - v_0\|_{H^s} + \|u_0 - v_0\|_{L^1}], \end{aligned}$$

which concludes the proof.

Now we turn to the model on the torus.

### 2.2. Proof of Theorem 1.1, point 1, the estimates on $H^s_0$.

2.2.1. *Reduction of the problem to a ball around* 0. We keep the notations of Theorem 1.1, fixing $u_0 \in H^s_0(\mathbb{T};\mathbb{R})$ and $r > 0$ and begin by taking $v_0, w_0 \in \mathrm{B}(u_0, r) \subset H^s_0(\mathbb{T};\mathbb{R})$ and as the mean value is conserved by the flow of (1.2) we consider the solutions $u, v, w \in C^0([0,T]; H^s_0(\mathbb{T};\mathbb{R}))$ to (1.2) with initial datum $u_0, v_0, w_0$ and on a uniform small interval $[0,T]$.
The main goal of the proof is to show the following estimate:

$$\|v(t,\cdot) - w(t,\cdot)\|_{H^{s-(2-\alpha)^+}} \leq C(\|(v_0,w_0)\|_{H^s}) \|v_0 - w_0\|_{H^{s-(2-\alpha)^+}}, \tag{2.4}$$

with the following tame control on the constant,

$$C(\|(v_0,w_0)\|_{H^s}) \leq C(\|(v_0,w_0)\|_{H^{s-(2-\alpha)^+}})[\|(v_0,w_0)\|_{H^s} + 1]. \tag{2.5}$$

In a later step when we will make the gauge transform for the symbolic calculus to hold we will need to work with data in a small ball around. Without loss of generality by density we can suppose $u_0 \in \mathscr{S}$ we make the regular change of variable $v \mapsto v - u$ and $w \mapsto w - u$ which modifies the equation (1.2) to

$$\begin{cases} \partial_t(v-u) + (v-u)\partial_x(v-u) + D^{\alpha-1}\partial_x(v-u) = L(u)(u-v) \\ (v-u)(0,\cdot) = (v_0 - u_0)(\cdot) \end{cases}, \tag{2.6}$$

where $L$ is a linear operator depending on $u$. We get an analogous equation on $w-u$. Thus we will henceforth drop $u_0$ from our notations and work on $\mathrm{B}(0,r)$ with $r$ can be chose small due to the local nature of the result.

The final simplification we make in this paragraph is given the well posedness of the Cauchy problem in $H^s$, and the density of $\mathscr{S}$ in $H^s$, it suffice to prove (2.4) for $v_0, w_0 \in \mathscr{S}$ which henceforth we will suppose.

2.2.2. *Gauge transform and Energy estimate.* The proof will follow the same lines as for Theorem 1.2 but with the Gauge transform being a bit trickier.

So analogously we start by reducing $H^{s-(2-\alpha)^+}$ estimates to $L^2$ ones by defining $f_1 = \langle \mathrm{D}\rangle^{s-(2-\alpha)^+} v$. Commuting $\langle \mathrm{D}\rangle^{s-(2-\alpha)^+}$ with (1.5), by the symbolic calculus rules in Appendix A.1 we get the the PDE on $f_1$:

$$\begin{cases} \partial_t f_1 + v\partial_x f_1 + D^{\alpha-1}\partial_x f_1 = R_1(f_1)f_1 \\ f_1(0,\cdot) = \langle \mathrm{D}\rangle^s v_0(\cdot), \end{cases} \tag{2.7}$$

where $R_1$ verifies

$$\|R_1(f_1)\|_{L^2\to L^2} \leq C(\|v\|_{W^{1,\infty}}, \|v\|_{H^{s-(2-\alpha)^+}}),$$
$$\|\partial_{f_1} R_1(f_1)\|_{L^2\to L^2} \leq C(\|v\|_{W^{1,\infty}}, \|v\|_{H^{s-(2-\alpha)^+}}).$$

We define analogously $g_1$ from $w$ and notice that there exists $C > 0$ such that:

$$C^{-1}\|f_1 - g_1\|_{L^2} \leq \|v - w\|_{H^{s-(2-\alpha)^+}} \leq C\|f_1 - g_1\|_{L^2}$$

thus the problem is reduced to getting $L^2$ estimates on $f_1 - g_1$.

Now we define $V = \partial_x^{-1} v$ which is the periodic, zero mean value, primitive of u,

$$\hat{V}(0) = 0 \text{ and } \hat{V}(\xi) = \frac{\hat{v}(\xi)}{i\xi}, \text{ for } \xi \in \mathbb{Z}^*,$$

and we define analogously $W$ from $w$.
Now introduce the following gauge transform:

$$\begin{cases} a_v = e^{\frac{1}{i\alpha}\xi|\xi|^{1-\alpha}V} \in S^0_{\alpha-1,2-\alpha}(\mathbb{T}\times\mathbb{Z}), \\ a_w = e^{\frac{1}{i\alpha}\xi|\xi|^{1-\alpha}W} \in S^0_{\alpha-1,2-\alpha}(\mathbb{T}\times\mathbb{Z}), \end{cases}$$

$$\begin{cases} f_2 = \mathrm{Op}(a_v)f_1, \\ g_2 = \mathrm{Op}(a_w)g_1. \end{cases} \tag{2.8}$$

where the pseudodifferential symbol classes are defined in A.1. The study of the symbolic calculus associated to this very specific form of symbols is done in section 2.4 gives for $r$ sufficiently small the change of variable (2.8) is Lipschitz from $L^2$ to $L^2$ but under $H^{(2-\alpha)^+}$ control on $(f_2, g_2)$ i.e we have:

$$C^{-1}(\|(f_2,g_2)\|_{H^{(2-\alpha)^+}})\,\|f_1-g_1\|_{L^2} \le \|f_2-g_2\|_{L^2} \le C(\|(f_2,g_2)\|_{H^{(2-\alpha)^+}})\,\|f_1-g_1\|_{L^2}\,,$$

where $C$ and $C^{-1}$ verifies the tame estimate (2.5). Again by section 2.4, the transformation (2.8) is of order 0 thus

$$C^{-1}(\|(v,w)\|_{H^s})\,\|f_1-g_1\|_{L^2} \le \|f_2-g_2\|_{L^2} \le C(\|(v,w)\|_{H^s})\,\|f_1-g_1\|_{L^2}\,,$$

and the problem is reduced to getting $L^2$ estimates on $f_2 - g_2$.

To get the equations on $f_2$ and $g_2$ we commute $a_v$ and $a_w$ with (2.7), we make the computations for $f_2$, those for $g_2$ are obtained by symmetry:

$$\mathrm{Op}(a_v)\partial_t f_1 + \mathrm{Op}(a_v)\,\mathrm{Op}(vi\xi)f_1 + \mathrm{Op}(a_v)\,\mathrm{Op}(i\,|\xi|^{\alpha-1}\,\xi)f_1 = 0$$

$$\partial_t\,\mathrm{Op}(a_v)f_1 + \mathrm{Op}(i\,|\xi|^{\alpha-1}\,\xi)\,\mathrm{Op}(a_v)f_1 + (\mathrm{Op}(a_v)\,\mathrm{Op}(vi\xi) - [i\,|\xi|^{\alpha-1}\,\xi, \mathrm{Op}(a_v)])f_1 \\ - \mathrm{Op}(\partial_t a_v)f_1 = 0$$

By definition of $a_v$ we have:

$$\frac{1}{i}\partial_\xi(i\xi\,|\xi|^{\alpha-1})\partial_x a_v = a_v iv\xi.$$

Thus by Lemma 2.1 we get:

$$\partial_t f_2 + \mathrm{Op}(i\,|\xi|^{\alpha-1}\,\xi)f_2 + \mathrm{Op}(\frac{1}{i}\partial_\xi a_v\partial_x(iv\xi) - \frac{1}{2}\partial_x^2 a_v\partial_\xi^2(i\xi\,|\xi|^{\alpha-1}) - \partial_t a_v)f_1 = R_2(f_1)f_2,$$

$$\partial_t f_2 + \mathrm{Op}(i\,|\xi|^{\alpha-1}\,\xi)f_2 + \mathrm{Op}(c_v)f_2 = R_2(f_2)f_2 \tag{2.9}$$

where $R_2$ verifies the following estimates with constants verifying (2.5):

$$\|R_2(f_2)\|_{L^2\to L^2} \le C(\|v\|_{W^{1,\infty}}, \|v\|_{H^s}),$$
$$\|\partial_{f_2}R_2(f_2)\|_{L^2\to L^2} \le C(\|v\|_{W^{1,\infty}}, \|v\|_{H^s}),$$

and $c_v$ is defined by:

$$c_v(t,x,\xi) = \frac{\alpha-2}{\alpha i}\xi\,|\xi|^{1-\alpha}\,V\partial_x v + \frac{i\alpha(\alpha-1)}{2}\,|\xi|^{\alpha-2}\,\frac{\xi}{|\xi|}[\frac{i\xi\,|\xi|^{1-\alpha}}{\alpha}\partial_x v + (\frac{\xi\,|\xi|^{1-\alpha}\,v}{\alpha})^2] \\ + \frac{i}{\alpha}\xi\,|\xi|^{1-\alpha}\,\partial_t V.$$

We get analogously on $g_2$

$$\partial_t g_2 + \mathrm{Op}(i\,|\xi|^{\alpha-1}\,\xi)g_2 + \mathrm{Op}(c_w)g_2 = R_2(g_2)g_2. \tag{2.10}$$

Thus we have succeeded to eliminate the term $\mathrm{Op}(iv\xi)$ of order 1 in (2.7) and got a hyperbolic term of order $2-\alpha \le 1$. Now the usual energy estimate combined the Gronwall lemma on $f_2 - g_2$ gives for $0 \le t \le T$:

$$\|f_2-g_2\|_{L^2} \le C(\|(u_0,v_0)\|_{H^s}, \|(f_2,g_2)(0,\cdot)\|_{H^{(2-\alpha)^+}})\,\|(f_2-g_2)(0,\cdot)\|_{L^2} \\ \le C(\|(u_0,v_0)\|_{H^s})\,\|(f_2-g_2)(0,\cdot)\|_{L^2}\,,$$

with $C$ verifying (2.5), which concludes the proof.

2.3. **Proof of Theorem 1.1, point 2, the estimates on $H^s$.** The starting point is noticing that the mean value is preserved by (1.2) and by doing the change of unknowns:

$$\begin{cases} \tilde{u}(t,x) = u(t, x - t\fint u_0) - \fint u_0 \\ \tilde{v}(t,x) = v(t, x - t\fint v_0) - \fint v_0 \end{cases}, \tag{2.11}$$

where $\fint u_0 = \frac{1}{2\pi}\int_{\mathbb{T}} u_0$ is the mean value we van reduce the Cauchy problem for general data to ones with 0 mean value by verifying that $\tilde{u}, \tilde{v} \in H_0^s$ still solve (1.2). Thus the main goal is to prove that the change of variable (2.11) is not regular. The goal is to show that there exists a positive constant $C$ and two sequences $(u_\epsilon^\lambda)$ and $(v_\epsilon^\lambda)$ solutions of 1.2 in $C^0([0,1], H^s(\mathbb{T}))$ such that for every $t \leq T$, where $T$ is a uniform small time,

$$\sup_{\lambda,\epsilon} \left\| u_\epsilon^\lambda \right\|_{H^s(\mathbb{T})(t,\cdot)} + \left\| v_\epsilon^\lambda(t,\cdot) \right\|_{H^s(\mathbb{T})} \leq C,$$

$(u_{\epsilon,\tau}^\lambda)$ and $(v_{\epsilon,\tau}^\lambda)$ satisfy initially

$$\lim_{\substack{\lambda\to+\infty\\ \epsilon\to 0}} \left\| u_\epsilon^\lambda(0,\cdot) - v_\epsilon^\lambda(0,\cdot) \right\|_{H^s(\mathbb{T})} = 0,$$

but,

$$\liminf_{\substack{\lambda\to+\infty\\ \epsilon\to 0}} \left\| u_\epsilon^\lambda(t,\cdot) - v_\epsilon^\lambda(t,\cdot) \right\|_{H^s(\mathbb{T})} \geq c > 0.$$

Which proves the non uniform continuity. Considering a weaker control norm we want to get, for all $\delta > 0$ and for $t > 0$:

$$\liminf_{\substack{\lambda\to+\infty\\ \epsilon\to 0}} \frac{\| u_\epsilon^\lambda(t,\cdot) - v_\epsilon^\lambda(t,\cdot) \|_{H^{s-1+\delta}(\mathbb{T})}}{\| u_\epsilon^\lambda(0,\cdot) - v_\epsilon^\lambda(0,\cdot) \|_{H^s(\mathbb{T})}} = +\infty.$$

As the case $\alpha = 1$ is simply the Burgers equation for which the theorem is known to be true henceforth we will suppose $\alpha > 1$.

2.3.1. *Definition of the Ansatz.* Take $\omega \in C_0^\infty(\mathbb{T})$ such that for $x \in [0, 2\pi]$, $\omega(x) = 1$ if $|x| \leq \frac{1}{2}$, $\omega(x) = 0$ if $|x| \geq 1$. Let $(\lambda, \epsilon)$ be two positive real sequences such that:

$$\lambda \to +\infty, \ \epsilon \to 0, \ \lambda\epsilon \to +\infty. \tag{2.12}$$

Put for $x \in [0, 2\pi]$,

$$u^0(x) = \lambda^{\frac{1}{2}-s}\omega(\lambda x), \ v^0(x) = u^0(x) + \epsilon\omega(x),$$

and extend $u^0$ and $v^0$ periodically. The main trick here will be to use the time reversibility of equation (1.2) by defining $\tilde{u}, \tilde{v}$ as the solution of (1.2) with data fixed at time $t > 0$ given by

$$\begin{cases} \tilde{u}(t,x) = u_0 - \fint u_0 \\ \tilde{v}(t,x) = v_0 - \fint v_0 \end{cases}, \tag{2.13}$$

where $t \leq t_0$ is chosen small enough from the equations to be well posed. And define $u$ and $v$ by (2.11).

2.3.2. *Main estimates.* First the estimates at time 0, for $0 \leq \nu \leq s$:

$$\| u(0,x) - v(0,x) \|_{H^\nu} = \left\| \tilde{u}(0,x) - \tilde{v}(0,x) + \fint u_0 - \fint v_0 \right\|_{H^\nu}$$

By the esitmate (2.4), the tame control (2.5) and the Cauchy-Schwartz inequality,

$$\begin{aligned} \| u(0,x) - v(0,x) \|_{H^\nu} &\leq C(\|(u_0, v_0)\|_{H^{\nu+(2-\alpha)^+}}) \| u_0 - v_0 \|_{H^\nu} \\ &\leq C[1 + \lambda^{\nu-s+(2-\alpha)^+}]\epsilon. \end{aligned} \tag{2.14}$$

Now the estimates at a fixed time $t > 0$, by construction:

$$\begin{aligned}\|u(t,x) - v(t,x)\|_{H^\nu} &= \left\| u_0(x + t\fint u_0) - v_0(x + t\fint v_0) \right\|_{H^\nu} \\ &= \left\| u_0(x + t\fint u_0) - u_0(x + t\fint v_0) \right\|_{H^\nu} + O_{H^\nu}(\epsilon)\end{aligned}$$

We now remark that as by hypothesis $\lambda\epsilon \to +\infty$ and $t\fint \omega > 0$, $u_0(x + t\fint u_0)$ and $u_0(x + t\fint v_0)$ have disjoint supports, thus

$$\begin{aligned}\|u(t,x) - v(t,x)\|_{H^\nu} &= \left\| u_0(x + t\fint u_0) \right\|_{H^\nu} + \left\| u_0(x + t\fint v_0) \right\|_{H^\nu} + O_{H^\nu}(\epsilon) \\ &= C\lambda^{\nu - s} + O_{H^\nu}(\epsilon). \end{aligned} \tag{2.15}$$

Now to conclude the proof:

- in the case of non uniform continuity we take $\epsilon$ such that $\epsilon\lambda^{(2-\alpha)^+} \to 0$ and apply the previous estimates with $\nu = s$.
- In the case of non Lipschitz control we take $\epsilon$ such that $\lambda^{-1+\delta}\epsilon^{-1} \to +\infty$ and apply the previous estimates with $\nu = s - 1 + \delta$.

2.4. **Composition, commutator and inverse estimates.** Here we give the key calculus lemma that gives the estimates needed in section 2.2.

**Lemma 2.1.** *Consider three real numbers $m, m', m''$ such that $m < 1$ and three symbols $p \in S^m_{1,0}(\mathbb{T} \times \mathbb{Z})$, $q \in S^{m''}_{1,0}(\mathbb{T} \times \mathbb{Z})$ and $b \in S^{m'}_{1,0}(\mathbb{T} \times \mathbb{Z})$ respectively. Suppose moreover that $p$ is real valued and that*

$$\forall (x,\xi) \in \mathbb{T} \times \mathbb{Z}^*, |p(x,\xi)| < |\xi|, |\partial_x p(x,\xi)| < |\xi| \text{ and } |\partial_\xi p(x,\xi)| < 1. \tag{2.16}$$

*Define:*

$$a = qe^{ip} \in S^0_{1-m,m}(\mathbb{T} \times \mathbb{Z}).$$

(1) *Then $\mathrm{Op}(a) \circ \mathrm{Op}(b) \in S^{m'}_{1-m,m}(\mathbb{T} \times \mathbb{Z})$ is of order $m' + m''$ with symbol $a\#b$ defined by:*

$$a\#b(x,\xi) = (2\pi)^{-d} \int_{\mathbb{T}\times\mathbb{Z}} e^{i(x-y).(\xi-\eta)} a(x,\eta) b(y,\xi) dy d\eta$$

*Moreover,*

$$\mathrm{Op}(a) \circ \mathrm{Op}(b)(x,\xi) - \mathrm{Op}(\sum_{|\alpha|<k} \frac{1}{i^{|\alpha|}\alpha!} (\partial_\xi^\alpha a(x,\xi))(\partial_x^\alpha b(x,\xi))) \text{ is of order } m' + m'' - k(1-m)$$

*for all $k \in \mathbb{N}$. We have the norm estimate:*

$$\|\mathrm{Op}(a) \circ \mathrm{Op}(b)\|_{H^\mu \to H^{\mu - m' - m''}} \le C M^{m'}_{\mu, \frac{1}{2}+1}(b) M^{m''}_{\mu, \frac{1}{m}(\frac{1}{2}+1)}(a).$$

(2) *Also $\mathrm{Op}(b) \circ \mathrm{Op}(a) \in S^{m'}_{1-m,m}(\mathbb{T} \times \mathbb{Z})$ is of order $m' + m''$ with symbol $b\#a$ defined by:*

$$b\#a(x,\xi) = (2\pi)^{-d} \int_{\mathbb{T}\times\mathbb{Z}} e^{i(x-y).(\xi-\eta)} b(x,\eta) a(y,\xi) dy d\eta$$

*Moreover,*

$$\mathrm{Op}(b) \circ \mathrm{Op}(a)(x,\xi) - \mathrm{Op}(\sum_{|\alpha|<k} \frac{1}{i^{|\alpha|}\alpha!} (\partial_\xi^\alpha b(x,\xi))(\partial_x^\alpha a(x,\xi))) \text{ is of order } m' + m'' - k(1-m)$$

*for all $k \in \mathbb{N}$. We have the norm estimate:*

$$\|\mathrm{Op}(b) \circ \mathrm{Op}(a)\|_{H^\mu \to H^{\mu - m}} \le C M^{m'}_{\mu, \frac{1}{2}+1}(b) M^{m''}_{\mu, \frac{1}{m}(\frac{1}{2}+1)}(a).$$

**Corollary 2.1.** *Consider two real numbers $m, m'$ such that $m < 1$ and two symbols $p$ and $b$ in $S^m_{1,0}(\mathbb{T}\times\mathbb{Z})$ and $S^{m'}_{1,0}(\mathbb{T}\times\mathbb{Z})$ respectively. Suppose moreover that $p$ is real valued and that*

$$\forall(x,\xi)\in\mathbb{T}\times\mathbb{Z}^*, |p(x,\xi)| < |\xi|\,, |\partial_x p(x,\xi)| < |\xi| \text{ and } |\partial_\xi p(x,\xi)| < 1. \tag{2.17}$$

*Define:*

$$a = e^{ip} \in S^0_{1-m,m}(\mathbb{T}\times\mathbb{Z}).$$

- *Then $a(x,D)$ is an operator of order 0. Moreover we have the norm estimate:*
$$\|a(x,D)\|_{H^\mu\to H^\mu} \le CM^0_{\mu,\frac{1}{m}(\frac{1}{2}+1)}(a).$$
- *Also $\mathrm{Op}(a)\circ\mathrm{Op}(b) - \mathrm{Op}(b)\circ\mathrm{Op}(a)$ is of order $m'+m-1$. Moreover,*

$$\mathrm{Op}(a)\circ\mathrm{Op}(b)(x,\xi) - \mathrm{Op}(b)\circ\mathrm{Op}(a)$$
$$- \mathrm{Op}\Big(\sum_{|\alpha|<k}\frac{1}{i^{|\alpha|}\alpha!}(\partial^\alpha_\xi a(x,\xi))(\partial^\alpha_x b(x,\xi)) - \partial^\alpha_\xi b(x,\xi))(\partial^\alpha_x a(x,\xi))\Big)$$

*is of order $m' - k(1-m)$ for all $k\in\mathbb{N}$.*

*Proof of Lemma 2.1.* Here we cannot directly apply the symbolic calculus of the Appendix A.1 because we are computing the the composition and commutator estimates for symbols in two different classes $S^{1,0}_{m'}$ and $S^0_{1-m,m}$, moreover it's known that there is no symbolic calculus possible in the general class $S^0_{1-m,m}$, though we are working here with explicit symbols and will make use of their particular form and don't need the full generality of symbol classes.

First by the smallness hypothesis on $p$ in (2.16), $a\#b$ and $b\#a$ are well defined as oscillatory integrals with non degenerate phase functions. Let us notice that it suffice to prove the first step of the lemma:

$$\begin{cases} a\#b - ab - \frac{1}{i}\partial_\xi a\partial_x b \\ b\#a - ba - \frac{1}{i}\partial_\xi b\partial_x a \end{cases}, \text{ are of order } m'+m''-(1-m). \tag{2.18}$$

Indeed the the theorem then follows recursively with the Taylor formula.

Now the starting point is from a remark by Patrick Gerard [9] [1] and it's to see $a$ as the time one of a flow for an ODE and reducing the problem to getting commutator estimates with the ODE. More precisely define:

$$\begin{cases} a_\tau = qe^{i\tau p} \\ a_1 = a \end{cases}, \text{ and } \begin{cases} \partial_\tau a_\tau = ipa_\tau \\ a_0 = q \end{cases}. \tag{2.19}$$

To prove (2.18) we introduce respectively:

$$\begin{cases} R^1_\tau = a_\tau\#b - a_\tau b - \frac{1}{i}\partial_\xi a_\tau\partial_x b, \\ R^2_\tau = b\#a_\tau - ba_\tau - \frac{1}{i}\partial_\xi b\partial_x a_\tau, \end{cases} \tag{2.20}$$

Differentiating (2.20) with respect to $\tau$ we get:

$$\partial_\tau R^1_\tau(x,\xi) = ip(x,\xi)R^1_\tau(x,\xi) - \underbrace{a_\tau(x,\xi)\partial_\xi p(x,\xi)\partial_x b(x,\xi)}_{(1)}$$
$$\underbrace{+ (2\pi)^{-d}\int_{\mathbb{T}\times\mathbb{Z}} e^{i(x-y).(\xi-\eta)}a(x,\eta)b(y,\xi)i(p(x,\eta)-p(x,\xi))dyd\eta,}_{(2)} \tag{2.21}$$

[1]The same kind of ideas were used in Appendix C of [1] to get estimates on a change of variable operator.

$$\partial_\tau R^2_\tau = ipR^2_\tau - \underbrace{\partial_x ib(x,\xi)a_\tau(x,\xi)\partial_x p(x,\xi)}_{(3)}$$
$$+ \underbrace{(2\pi)^{-d}\int_{\mathbb{T}\times\mathbb{Z}} e^{i(x-y).(\xi-\eta)}b(x,\eta)a(y,\xi)i(p(y,\xi)-p(x,\xi))dyd\eta}_{(4)}. \quad (2.22)$$

Notice that as $m<1$, Equations (2.21) and (2.22) are hyperbolic and that the desired estimates hold for $\tau=0$ thus we only need to get an estimate (1), (2), (3) and (4).

We first start with $L^2$ estimates. For (1) and (2) we notice that $a_\tau$ is bounded from $H^{m'+m''}\to L^2$ and thus as $p,q,b\in S^m_{1,0}, S^{m''}_{1,0}$ and $S^{m'}_{1,0}$ in which we have symbolic calculus by Theorem A.2 we get the desired estimates on (1) and (3) in $L^2$. For (2) we write:

$$\begin{aligned}
&\int_{\mathbb{T}\times\mathbb{Z}} e^{i(x-y).(\xi-\eta)}a(x,\eta)b(y,\xi)i(p(x,\eta)-p(x,\xi))dyd\eta\\
&=\int_{\mathbb{T}\times\mathbb{Z}} e^{i(x-y).(\xi-\eta)}a(x,\eta)b(y,\xi)i(\eta-\xi)\left(\int_0^1 \partial_\xi p(x,r\eta+(1-r)\xi)dr\right)dyd\eta\\
&=\int_{\mathbb{T}\times\mathbb{Z}} \partial_y[e^{i(x-y).(\xi-\eta)}]a(x,\eta)b(y,\xi)\left(\int_0^1 \partial_\xi p(x,r\eta+(1-r)\xi)dr\right)dyd\eta\\
&=-\int_{\mathbb{T}\times\mathbb{Z}} e^{i(x-y).(\xi-\eta)}a(x,\eta)\partial_y b(y,\xi)\left(\int_0^1 \partial_\xi p(x,r\eta+(1-r)\xi)dr\right)dyd\eta,
\end{aligned}$$

and the $L^2$ estimates then follow. For (4) we write:

$$\begin{aligned}
&\int_{\mathbb{T}\times\mathbb{Z}} e^{i(x-y).(\xi-\eta)}b(x,\eta)a(y,\xi)i(p(y,\xi)-p(x,\xi))dyd\eta\\
&=\int_{\mathbb{T}\times\mathbb{Z}} e^{i(x-y).(\xi-\eta)}b(x,\eta)a(y,\xi)i(y-x)\left(\int_0^1 \partial_x p(ry+(1-r)x,\xi)dr\right)dyd\eta\\
&=\int_{\mathbb{T}\times\mathbb{Z}} \partial_\eta[e^{i(x-y).(\xi-\eta)}]b(x,\eta)a(y,\xi)i(y-x)\left(\int_0^1 \partial_x p(ry+(1-r)x,\xi)dr\right)dyd\eta\\
&=-\int_{\mathbb{T}\times\mathbb{Z}} e^{i(x-y).(\xi-\eta)}\partial_\eta b(x,\eta)a(y,\xi)i(y-x)\left(\int_0^1 \partial_x p(ry+(1-r)x,\xi)dr\right)dyd\eta,
\end{aligned}$$

and again the the $L^2$ estimates follow.

Finally to get the general $H^s$ estimates we notice that we got the $L^2$ estimates for all symbols $b\in S^{m'}_{1,0}, q\in S^{m''}_{1,0}$ with $m',m''\in\mathbb{R}$ and that it suffices to replace $b$ by $\langle \mathrm{D}\rangle^s b$ or $q$ by $\langle \mathrm{D}\rangle^s q$. □

**Lemma 2.2.** *Consider a real number $m<1$, a real valued symbol $p$ $S^m_{1,0}(\mathbb{T}\times\mathbb{Z})$ such that*

$$\forall(x,\xi)\in\mathbb{T}\times\mathbb{Z}^*, |p(x,\xi)|<|\xi|, |\partial_x p(x,\xi)|<|\xi| \text{ and } |\partial_\xi p(x,\xi)|<1, \quad (2.23)$$

*and define:*

$$a=e^{ip}\in S^0_{1-m,m}(\mathbb{T}\times\mathbb{Z}).$$

*Then* $\mathrm{Op}(a)$ *is invertible and*

$$\mathrm{Op}(a)^{-1}=\mathrm{Op}(e^{-ip}).$$

*Proof.* We start as previously with:

$$\begin{cases} a_\tau = e^{i\tau p}\\ a_1 = a\end{cases} \text{ and } \begin{cases}\partial_\tau a_\tau = ipa_\tau\\ a_0 = I\end{cases} \quad (2.24)$$

Using the fundamental relation for a flow $\phi$ of an ODE:

$$\phi_{\tau_2}^{\tau_1} \circ \phi_{\tau_1}^{\tau_0} = \phi_{\tau_2}^{\tau_0},$$

we get the desired result by writing:

$$\mathrm{Op}(a)^{-1} = \mathrm{Op}(a_{-1}).$$

□

## 3. Flow map regularity for the periodic Gravity Capillary equation

**3.1. Prerequisites from the Cauchy problem.** We start by recalling the apriori estimates given by Proposition 5.2 of [4] combined with the results of [3]. We keep the notations of Theorem 1.3.

**Proposition 3.1.** *(From [4] and [3]) Consider a real number $s > 2 + \frac{1}{2}$. Then there exists a non decreasing function $C$ such that, for all $T \in ]0,1]$ and all solution $(\eta, \psi)$ of (1.11) such that*

$$(\eta, \psi) \in C^0([0,T]; H^{s+\frac{1}{2}}(\mathbb{T}) \times H^s(\mathbb{T})) \text{ and } H_t \text{ is verified for } t \in [0,T],$$

*we have*

$$\|(\eta,\psi)\|_{L^\infty(0,T;H^{s+\frac{1}{2}}\times H^s)} \leq C((\eta_0,\psi_0)_{H^{s+\frac{1}{2}}\times H^s}) + TC(\|(\eta,\psi)\|_{L^\infty(0,T;H^{s+\frac{1}{2}}\times H^s)}).$$

The proof will rely on the para-linearised and symmetrized version of (1.11) given by Proposition 4.8 and corollary 4.9 of [4] which are valid on $\mathbb{T}$ as shown in [3]. Before we recall this, for clarity as in [4] we introduce a special class of operators $\Sigma^m \subset \Gamma_0^m$ given by:

**Definition 3.1.** *(From [4]) Given $m \in \mathbb{R}$, $\Sigma^m$ denotes the class of symbols $a$ of the form*

$$a = a^{(m)} + a^{(m-1)}$$

*with*

$$a^{(m)} = F(\partial_x \eta(t,x), \xi)$$
$$a^{(m-1)} = \sum_{|k|=2} G_\alpha(\partial_x \eta(t,x), \xi) \partial_x^k \eta(t,x),$$

*such that*

(1) *$T_a$ maps real valued functions to real-valued functions;*
(2) *$F$ is of class $C^\infty$ real valued function of $(\zeta, \xi) \in \mathbb{R} \times (\mathbb{Z} \setminus 0)$, homogeneous of order $m$ in $\xi$; and such that there exists a continuous function $K = K(\zeta) > 0$ such that*
$$F(\zeta, \xi) \geq K(\zeta) |\xi|^m,$$
*for all $(\zeta, \xi) \in \mathbb{R} \times (\mathbb{Z} \setminus 0)$;*
(3) *$G_\alpha$ is a $C^\infty$ complex valued function of $(\zeta, \xi) \in \mathbb{R} \times (\mathbb{Z} \setminus 0)$, homogeneous of order $m-1$ in $\xi$.*

$\Sigma^m$ enjoys all the usual symbolic calculus properties modulo acceptable reminders that we define by the following:

**Definition-Notation 3.1.** *(From [4]) Let $m \in \mathbb{R}$ and consider two families of operators of order $m$,*

$$\{A(t) : t \in [0,T]\}, \quad \{B(t) : t \in [0,T]\}.$$

*We shall say that $A \sim B$ if $A - B$ is of order $m - \frac{3}{2}$ and satisfies the following estimate: for all $\mu \in \mathbb{R}$, there exists a continuous function $C$ such that for all $t \in [0,T]$,*

$$\|A(t) - B(t)\|_{H^\mu \to H^{\mu - m + \frac{3}{2}}} \leq C(\|\eta(t)\|_{H^{s+\frac{1}{2}}}).$$

In the next Proposition we recall the different symbols that appear in the para-linearization and symmetrization of the equations.

**Proposition 3.2.** *(From [4])We work under the hypothesis of Proposition 3.1. Put*

$$\lambda = \lambda^{(1)} + \lambda^{(0)},\ l = l^{(2)} + l^{(1)} \text{ with,}$$

$$\begin{cases} \lambda^{(1)} = |\xi|\,, \\ \lambda^{(0)} = \frac{1+|\partial_x\eta|^2}{2|\xi|}\left\{div\Big(\alpha^{(1)}\partial_x\eta\Big) + i\frac{\xi}{|\xi|}\partial_x\alpha^{(1)}\right\}, \\ \alpha^{(1)} = \frac{1}{\sqrt{1+|\partial_x\eta|^2}}\Big(|\xi| + i\partial_x\eta\xi\Big). \end{cases} \tag{3.1}$$

$$\begin{cases} l^{(2)} = (1+|\partial_x\eta|^2)^{-\frac{3}{2}}\xi^2, \\ l^{(1)} = -\frac{i}{2}(\partial_x\cdot\partial_\xi)l^{(2)}. \end{cases} \tag{3.2}$$

*Now let* $q \in \Sigma^0, p \in \Sigma^{\frac{1}{2}}, \gamma \in \Sigma^{\frac{3}{2}}$ *be defined by*

$$q = (1+|\partial_x\eta|^2)^{-\frac{1}{2}},$$

$$p = (1+|\partial_x\eta|^2)^{-\frac{5}{4}}\,|\xi|^{\frac{1}{1}} + p^{(-\frac{1}{2})},$$

$$\gamma = \sqrt{l^{(2)}\lambda^{(1)}} + \sqrt{\frac{l^{(2)}}{\lambda^{(1)}}}\frac{Re\lambda^{(0)}}{2} - \frac{i}{2}(\partial_\xi\cdot\partial_x)\sqrt{l^{(2)}\lambda^{(1)}},$$

$$p^{(-\frac{1}{2})} = \frac{1}{\gamma^{(\frac{3}{2})}}\left\{ql^{(1)} - \gamma^{(\frac{1}{2})}p^{(\frac{1}{2})} + i\partial_\xi\gamma^{(\frac{3}{2})}\cdot\partial_x p^{(\frac{1}{2})}\right\}.$$

*Then*

$$T_qT_\lambda \sim T_\gamma T_q,\ T_qT_l \sim T_\gamma T_p,\ T_\gamma \sim (T_\gamma)^\top,$$

*where* $(T_\gamma)^\top$ *is the adjoint of* $T_\gamma$.

Now we can write the para-linearization and symmetrization of the equations (1.11) after a change of variable:

**Corollary 3.1.** *(From [4])Under the hypothesis of Proposition 3.1, introduce the unknowns*

$$U = \psi - T_B\eta^{2},\ \Phi_1 = T_p\eta \text{ and } \Phi_2 = T_qU,$$

*where we recall,*

$$\begin{cases} B = (\partial_y\phi)_{|y=\eta} = \frac{\partial_x\eta\cdot\partial_x\psi + G(\eta)\psi}{1+|\partial_x\eta|^2}, \\ V = (\nabla_x\phi)_{|y=\eta} = \partial_x\psi - B\partial_x\eta. \end{cases}$$

*Then* $\Phi_1, \Phi_2 \in C^0([0,T]; H^s(\mathbb{R}^d))$ *and*

$$\begin{cases} \partial_t\Phi_1 + T_V\cdot\partial_x\Phi_1 - T_\gamma\Phi_2 = f_1, \\ \partial_t\Phi_2 + T_V\cdot\partial_x\Phi_2 + T_\gamma\Phi_1 = f_2, \end{cases} \tag{3.3}$$

*with* $f_1, f_2 \in L^\infty(0,T; H^s(\mathbb{R}^d))$*, and* $f_1, f_2$ *have* $C^1$ *dependence on* $(\Phi_1, \Phi_2)$ *verifying:*

$$\|(f_1,f_2)\|_{L^\infty(0,T;H^s(\mathbb{T}))} \le C(\|(\eta,\psi)\|_{L^\infty(0,T;H^{s+\frac{1}{2}}\times H^s(\mathbb{T}))}).$$

[2] U is commonly called the "good" unknown of Alinhac.

3.2. **Proof of Theorem 1.3.** Corollary 3.1 shows that the para-linearization and symmetrization of the equations (1.11) are of the form of the equations treated in Theorem 1.1 so the proof will follow the same main lines but with more care in treating the non linearity in the dispersive term.

We keep the notations of Theorem 1.3, fixing $(\eta_0, \psi_0) \in H^{s+\frac{1}{2}}(\mathbb{T}) \times H^s(\mathbb{T})$ and $r > 0$ and begin by taking $(\tilde{\eta}_0, \tilde{\psi}_0) \in \mathrm{B}((\eta_0, \psi_0), r) \subset H^{s+\frac{1}{2}}(\mathbb{T}) \times H^s(\mathbb{T})$ and we consider the solutions $(\eta, \psi), (\tilde{\eta}, \tilde{\psi}) \in C^0([0,T]; H^{s+\frac{1}{2}}(\mathbb{T}) \times H^s(\mathbb{T}))$ to (1.2) with initial datum $(\eta_0, \psi_0), (\tilde{\eta}_0, \tilde{\psi}_0)$ and on a uniform small interval $[0,T]$ where the hypothesis $H_t$ is also supposed to be verified.
The main goal of the proof is to show the following estimate:

$$\begin{aligned}
&\left\|(\eta,\psi)(t,\cdot) - (\tilde{\eta},\tilde{\psi})(t,\cdot)\right\|_{H^s \times H^{s-\frac{1}{2}}} \\
&\leq C\left( \left\|(\eta_0, \psi_0, \tilde{\eta}_0, \tilde{\psi}_0)\right\|_{H^{s+\frac{1}{2}} \times H^s} \right) \left\|(\eta_0,\psi_0) - (\tilde{\eta}_0,\tilde{\psi}_0)\right\|_{H^s \times H^{s-\frac{1}{2}}}, \qquad (3.4)
\end{aligned}$$

when $(\eta, \psi)$ and $(\tilde{\eta}, \tilde{\psi})$ verify the symmetry hypothesis (**S**).

Put $\Phi = (\Phi_1, \Phi_2)$ the unknowns obtained from $(\eta, \psi)$ after paralinearization and symmetrization of the equations as in Corollary 3.1. Define analogously $\tilde{\Phi} = (\tilde{\phi_1}, \tilde{\phi_2})$ from $(\tilde{\eta}, \tilde{\psi})$. Let us notice that in order to prove 3.4 it suffice to get estimates on $\Phi - \tilde{\Phi}$, indeed by the ellipticity of the symbols $p$ and $q$ combined with the immediate $L^2$ estimates( as $s > 2 + \frac{1}{2}$) we have:

$$\begin{aligned}
&\left\|(\eta,\psi)(t,\cdot) - (\tilde{\eta},\tilde{\psi})(t,\cdot)\right\|_{H^s \times H^{s-\frac{1}{2}}} \\
&\leq C\left( \left\|(\eta_0, \psi_0, \tilde{\eta}_0, \tilde{\psi}_0)\right\|_{H^s \times H^{s-\frac{1}{2}}} \right) \left\|\Phi(t,\cdot) - \tilde{\Phi}(t,\cdot)\right\|_{H^{s-\frac{1}{2}} \times H^{s-\frac{1}{2}}}, \qquad (3.5) \\
&\left\|\Phi(t,\cdot) - \tilde{\Phi}(t,\cdot)\right\|_{H^{s-\frac{1}{2}} \times H^{s-\frac{1}{2}}} \\
&\leq C\left( \left\|(\eta_0, \psi_0, \tilde{\eta}_0, \tilde{\psi}_0)\right\|_{H^{s+\frac{1}{2}} \times H^s} \right) \left\|(\eta,\psi)(t,\cdot) - (\tilde{\eta},\tilde{\psi})(t,\cdot)\right\|_{H^s \times H^{s-\frac{1}{2}}}. \qquad (3.6)
\end{aligned}$$

3.2.1. *Gauge transform.* Again as $s > 2 + \frac{1}{2}$ we have an immediate $L^2$ estimates on $\Phi - \tilde{\Phi}$ thus we only need to get $\dot{H}^{s-\frac{1}{2}} \times \dot{H}^{s-\frac{1}{2}}$ estimates. Let us start by rewriting equation (3.3) by writing $\Phi = \Phi_1 + i\Phi_2$:

$$\partial_t \phi + T_V \cdot \partial_x \phi + i T_\gamma \phi = R_1, \qquad (3.7)$$

Where $R_1$ verifies

$$\begin{cases}
\|R_1(\phi)\|_{H^{s-\frac{1}{2}} \to H^{s-\frac{1}{2}}} \leq C\left( \left\|(\eta_0, \psi_0, \tilde{\eta}_0, \tilde{\psi}_0)\right\|_{H^s \times H^{s-\frac{1}{2}}} \right), \\
\|\partial_\phi R_1(\phi)\|_{H^{s-\frac{1}{2}} \to H^{s-\frac{1}{2}}} \leq C\left( \left\|(\eta_0, \psi_0, \tilde{\eta}_0, \tilde{\psi}_0)\right\|_{H^s \times H^{s-\frac{1}{2}}} \right).
\end{cases}$$

We get the same equation on $\tilde{\Phi}$ by symmetry.

Now we define $\Theta = \partial_x^{-1}[V(1+(\partial_x\eta)^2)^{-\frac{1}{2}}] = \partial_x^{-1}[\partial_x \phi(t,x,\eta(x))\sqrt{1+(\partial_x\eta)^2}]$ which exists and is periodic with zero mean value by hypothesis (**S**).
Now introduce the following gauge transform:

$$\begin{cases}
a_\Phi = e^{\frac{2}{i3}|\xi|^{\frac{1}{2}}\Theta} \in \bigcap_{0 \leq k \leq 2} \Gamma^{\frac{k}{2}}_{2-k,\frac{1}{2}}(\mathbb{T}), \\
\theta = T_{a_\Phi}\Phi,
\end{cases} \qquad (3.8)$$

where the symbol classes are defined in Section 3.3. We define analogously $a_{\tilde{\Phi}}$ and $\tilde{\theta}$ from $\tilde{\Phi}$. The generalization of the symbolic calculus in Section 2.4 to the paradifferential setting is done in 3.3. From Theorem 3.1 the change of variable

(3.8) is Lipschitz from $H^{s-\frac{1}{2}}$ to $H^{s-\frac{1}{2}}$ but under $H^s$ control on $(\Phi,\tilde{\Phi})$ which is equivalent by Theorem 3.1 to a control on $\left\|(\eta_0,\psi_0,\tilde{\eta}_0,\tilde{\psi}_0)\right\|_{H^{s+\frac{1}{2}}\times H^s}$ i.e we have:

$$\left\|\Phi(t,\cdot)-\tilde{\Phi}(t,\cdot)\right\|_{\dot{H}^{s-\frac{1}{2}}} \leq C\Big(\left\|(\eta_0,\psi_0,\tilde{\eta}_0,\tilde{\psi}_0)\right\|_{H^{s+\frac{1}{2}}\times H^s}\Big)\left\|\theta(t,\cdot)-\tilde{\theta}(t,\cdot)\right\|_{\dot{H}^{s-\frac{1}{2}}}, \tag{3.9}$$

$$\left\|\theta(t,\cdot)-\tilde{\theta}(t,\cdot)\right\|_{\dot{H}^{s-\frac{1}{2}}} \leq C\Big(\left\|(\eta_0,\psi_0,\tilde{\eta}_0,\tilde{\psi}_0)\right\|_{H^{s+\frac{1}{2}}\times H^s}\Big)\left\|\Phi(t,\cdot)-\tilde{\Phi}(t,\cdot)\right\|_{\dot{H}^{s-\frac{1}{2}}}. \tag{3.10}$$

To get the equations on $\theta$ and $\tilde{\theta}$ we commute $a_\Phi$ and $a_{\tilde{\Phi}}$ with (3.7), we make the computations for $\theta$, those for $\tilde{\theta}$ are obtained by symmetry:

$$T_{a_\Phi}\partial_t\Phi + T_{a_\Phi}T_V\cdot\partial_x\Phi + iT_{a_\Phi}T_\gamma\Phi = T_{a_\Phi}R_1$$
$$\partial_t T_{a_\Phi}\Phi + iT_\gamma T_{a_\Phi}\Phi + (T_{a_\Phi}T_V\partial_x - [iT_\gamma, T_{a_\Phi}])\Phi - T\partial_t a_\Phi\phi = T_{a_\Phi}R_1$$

By definition of $a_\Phi$ we have:

$$\frac{1}{i}[\partial_\xi(i\gamma)\partial_x a_\Phi - \partial_x(i\gamma)\partial_\xi a_\Phi] = a_\Phi iV\xi.$$

Thus by Lemma 3.1 we get:

$$\partial_t\theta + iT_\gamma\theta + T_{\frac{1}{i}\partial_\xi a_\Phi\partial_x(iV\xi)-\frac{1}{2}\partial_x^2 a_\Phi\partial_\xi^2(i\gamma)+\frac{1}{2}\partial_\xi^2 a_\Phi\partial_x^2(i\gamma)-\partial_t a_\Phi}\Phi = R_2(\theta)\theta,$$
$$\partial_t\theta + iT_\gamma\theta + T_{c_\phi}\theta = R_2(\theta)\theta \tag{3.11}$$

where $R_2$ verifies the following estimates:

$$\|R_2(\theta)\|_{H^{s-\frac{1}{2}}\to H^{s-\frac{1}{2}}} \leq C\Big(\left\|(\eta_0,\psi_0,\tilde{\eta}_0,\tilde{\psi}_0)\right\|_{H^{s+\frac{1}{2}}\times H^s}\Big),$$
$$\|\partial_\theta R_2(\theta)\|_{H^{s-\frac{1}{2}}\to H^{s-\frac{1}{2}}} \leq C\Big(\left\|(\eta_0,\psi_0,\tilde{\eta}_0,\tilde{\psi}_0)\right\|_{H^{s+\frac{1}{2}}\times H^s}\Big),$$

and $c_\phi$ is defined by:

$$\begin{aligned}c_\phi(t,x,\xi) &= \frac{2}{3i}|\xi|^{\frac{1}{2}}\Theta\partial_x V + \frac{2i}{3}|\xi|^{\frac{1}{2}}\partial_t\Theta \\ &- \frac{i}{2}\partial_\xi^2\gamma\Big(\frac{2}{i3}|\xi|^{\frac{1}{2}}[\partial_x V(1+(\partial_x\eta)^2)^{\frac{1}{2}} + \frac{V\partial_x\eta}{(1+(\partial_x\eta)^2)^{\frac{1}{2}}}] - \frac{4}{9}|\xi|V^2(1+(\partial_x\eta)^2)\Big) \\ &+ \frac{i}{2}\partial_x^2\gamma\Big([\frac{1}{i3}|\xi|^{-\frac{3}{2}}\xi\Theta]^2 - \frac{1}{i6}|\xi|^{-\frac{3}{2}}\Theta\Big).\end{aligned}$$

Thus we have succeeded to eliminate the term $T_V\cdot\partial_x$ of order 1 in (3.7) and got a hyperbolic term of order $\frac{1}{2}$. Here we need to be careful because of the non linearity in $\gamma$, indeed we cannot just simply make the usual energy estimate as in Section 2.2.2 because we will lose $\frac{3}{2}$ derivative. For this we will show in the next section a general scheme that can be used in the study of the flow map, where we differentiate the flow map with respect to the data, get a linear PDE on the differential and try and get the estimates from this new linear PDE.

3.2.2. *Estimate on the differential of the flow map.* Here we will give the $\dot{H}^{s-\frac{1}{2}}$ estimate needed on $\theta-\tilde{\theta}$, for this we will work by the mean value theorem on the differential of the flow map in $\theta(0,\cdot)$. If we write formally $\theta(t,\cdot)=F(t,\theta(0,\cdot))$ and $\delta(t,\cdot)=D_{\theta_0}F(t,\theta(0,\cdot))\delta_0$ for a certain $\delta_0\in \mathrm{B}(\theta(0,\cdot),r)\subset H^{s-\frac{1}{2}}(\mathbb{T})$. Then we get by differentiating in (3.11) we get the linear PDE in $\delta$ on $[0,T]$:

$$\begin{cases}\partial_t\delta + T_\gamma\delta = T_{\partial_x T_{p^{-1}}\delta}T_\Gamma\theta + R \\ \delta(0,\cdot)=\delta_0(\cdot)\end{cases} \tag{3.12}$$

where $\Gamma \in \Gamma_1^\alpha(\mathbb{D}^d)$ and $R \in C^0([0,T], H^{s-\frac{1}{2}}(\mathbb{T}))$ verifying

$$\begin{cases} M_1^\alpha(\Gamma) \leq C\Big( \left\|(\eta_0, \psi_0, \tilde{\eta}_0, \tilde{\psi}_0)\right\|_{W^{2,\infty}}\Big), \\ \forall t \in [0,T], \|R(t,\cdot)\|_{H^{s-\frac{1}{2}}} \leq C\Big( \left\|(\eta_0, \psi_0, \tilde{\eta}_0, \tilde{\psi}_0)\right\|_{H^{s+\frac{1}{2}}\times H^s}\Big). \end{cases} \tag{3.13}$$

The goal of this section is to prove the following estimate:

$$\begin{aligned} \forall t \in [0,T], \|\delta(t)\|_{\dot{H}^{s-\frac{1}{2}}} &\leq C(\left\|(\eta_0, \psi_0, \tilde{\eta}_0, \tilde{\psi}_0)\right\|_{H^{s+\frac{1}{2}}\times H^s}) \|\delta_0\|_{\dot{H}^{s-\frac{1}{2}}} \\ &+ 2\left|\int_{t_0}^t e^{C(\left\|(\eta_0,\psi_0,\tilde{\eta}_0,\tilde{\psi}_0)\right\|_{H^{s+\frac{1}{2}}\times H^s})(t-t')} \|R(t',\cdot)\|_{\dot{H}^{s-\frac{1}{2}}}\, dt'\right|, \end{aligned} \tag{3.14}$$

which by the mean value theorem concludes the proof of Theorem 1.3.

*Proof of Estimate 3.14.* We start by noticing that we have the immediate estimate

$$\begin{aligned} \forall t \in [0,T], \|\delta(t)\|_{\dot{H}^{s-\frac{1}{2}}} &\leq C(\left\|(\eta_0, \psi_0, \tilde{\eta}_0, \tilde{\psi}_0)\right\|_{H^{s+\frac{3}{2}}\times H^{s+1}}) \|\delta_0\|_{\dot{H}^{s-\frac{1}{2}}} \\ &+ 2\left|\int_{t_0}^t e^{C(\left\|(\eta_0,\psi_0,\tilde{\eta}_0,\tilde{\psi}_0)\right\|_{H^{s+\frac{3}{2}}\times H^{s+1}})(t-t')} \|R(t',\cdot)\|_{\dot{H}^{s-\frac{1}{2}}}\, dt'\right| \end{aligned} \tag{3.15}$$

The goal of this proof is thus to significantly improve this estimate on $\theta_0$ i.e on $(\eta_0, \psi_0, \tilde{\eta}_0, \tilde{\psi}_0)$.

We first reduce the $\dot{H}^{s-\frac{1}{2}}$ estimates to $\dot{H}^{s-1}$ estimates by defining $g = \partial_x T_{p^{-1}}\theta$ and $h = \partial_x T_{p^{-1}}\delta$, and by the ellipticity of $\partial_x T_{p^{-1}}$ it suffice to get the desired $\dot{H}^{s-1}$ estimates on $h$. Commuting $\partial_x T_{p^{-1}}$ with (3.11) and (3.12) we get the PDE on $g$ and $h$:

$$\begin{cases} \partial_t g + iT_\gamma g = F, \\ \partial_t h + T_\gamma h = T_h T_\Gamma g + f, \end{cases}$$

where $(F, f)$ verify

$$\forall t \in [0,T], \|(F,f)(t,\cdot)\|_{H^{s-1}} \leq C(\left\|(\eta_0, \psi_0, \tilde{\eta}_0, \tilde{\psi}_0)\right\|_{H^{s+\frac{1}{2}}\times H^s}).$$

Now the idea is to get an equation on $T_h T_\Gamma g$ where we "morally" use the ellipticity to write $T_{\gamma^{-1}}\partial_t$ as an operator of order 0 at the price of an acceptable remainder:

$$\partial_t g + T_\gamma g = F$$
$$T_h T_\Gamma T_{\gamma^{-1}}\partial_t g + T_h T_\Gamma g = T_\Gamma T_{\gamma^{-1}} F$$

as $s > 2 + \frac{d}{2}$,

$$T_h \partial_t\big(T_\Gamma T_{\gamma^{-1}} g\big) + T_h T_\Gamma g = T_h T_\Gamma T_{\gamma^{-1}} F$$

where $F$ was modified on each line to include terms which verify the same $H^{s-1}$ estimate. The "gain" is that $T_\Gamma T_{\gamma^{-1}}$ is of order 0 and that the "cost" of $\frac{3}{2}$ derivative in the spatial variable is put on $\partial_t$.

Getting back to h we have:

$$\partial_t h + T_\gamma h = -T_h \partial_t\big(T_\Gamma T_{\gamma^{-1}} g\big) + T_h T_\Gamma T_{\gamma^{-1}} F. \tag{3.16}$$

The key idea is that now we now how to solve $\partial_t h + h\partial_t\big(T_\Gamma T_{\gamma^{-1}} g\big) = 0$ explicitly so we make the change of unknowns:

$$u = e^{T_\Gamma T_{\gamma^{-1}} g} h. \tag{3.17}$$

As $s - 1 > 1 + \frac{d}{2}$ by the Sobolev embedding we have $g \in W^{1,\infty}$ and it's clear that $H^s$ estimates on $h$ are equivalent to ones on $u$ i.e for all $t \in [0,T]$:

$$C^{-1}(\|g_0\|_{H^{s-1}}) \|u\|_{H^{s-1}} \leq \|h\|_{H^{s-1}} \leq C(\|g_0\|_{H^{s-1}}) \|u\|_{H^{s-1}}. \tag{3.18}$$

Now we compute the $PDE$ verified by $u$:

$$\partial_t u + e^{T_\Gamma T_{\gamma^{-1}}g} T_{\tilde{\gamma}} e^{-T_\Gamma T_{\gamma^{-1}}g} u = T_{\partial_t g} u + e^{T_\Gamma T_{\gamma^{-1}}g} T_h T_\Gamma T_{\gamma^{-1}} F \\ + e^{T_\Gamma T_{\gamma^{-1}}g} f + e^{T_\Gamma T_{\gamma^{-1}}g} R \tag{3.19}$$

where $R$ verifies by A.5:

$$\|R\|_{H^{s-1}} \leq C \|g_0\|_{H^{s-1}} \|h\|_{H^{s-1}}. \tag{3.20}$$

Now we reduce the $H^{s-1}$ estimates on $u$ to $L^2$, first by noticing that $Te^{T_\Gamma T_{\gamma^{-1}}g} T_\gamma T e^{-T_\Gamma T_{\gamma^{-1}}g}$ is an elliptic paradifferential operator of a symbol we will call $\beta$. Then by making the change of variables $\phi = T_{|\beta|^{\frac{s-1}{\alpha}}} u$. By the ellipticity of $\beta$ and the immediate $L^2$ estimate on $u$, an $H^{s-1}$ estimate on $u$ is equivalent to an $L^2$ estimate on $\phi$ i.e

$$C^{-1}(\|g_0\|_{H^{s-1}}) \|u\|_{H^{s-1}} \leq \|\phi\|_{L^2} \leq C(\|g_0\|_{H^{s-1}}) \|u\|_{H^{s-1}}. \tag{3.21}$$

We then commute $T_{|\beta|^{\frac{s-1}{\alpha}}}$ with equation (3.19) and get by assembling the different terms verifying the same estimates as $R$ in one term:

$$\partial_t \phi + e^{T_\Gamma T_{\gamma^{-1}}g} T_\gamma e^{-T_\Gamma T_{\gamma^{-1}}g} \phi = T_{|\beta|^{\frac{s-1}{\alpha}}} e^{T_\Gamma T_{\gamma^{-1}}g} [f + T_h T_\lambda T_{\gamma^{-1}} F] + T_{|\beta|^{\frac{s-1}{\alpha}}} R. \tag{3.22}$$

Now to finally get the $L^2$ estimate on $\phi$ we need to do the classic energy estimate but with the adequate choice of basis in which $e^{T_\Gamma T_{\gamma^{-1}}g} T_\gamma e^{-T_\Gamma T_{\gamma^{-1}}g}$ is skew symmetric. For this the energy estimate is made on

$$\partial_t \|\phi\|^2_{L^2\left(e^{-2\,\mathrm{Re}(T_\Gamma T_{\gamma^{-1}}g)}dx\right)},$$

and the residual term

$$\|\phi\|^2_{L^2\left(\partial_t e^{-2\,\mathrm{Re}(T_\Gamma T_{\gamma^{-1}}g)}dx\right)}$$

is controlled as $s > 2 + \frac{d}{2}$. Combining this estimates and the Gronwall lemma we get

$$\|\phi\|_{L^2} \leq C(M_0^0(Re(\gamma)), \|g_0\|_{H^{s-1}}, \|F\|_{L^\infty([0,T],H^{s-1})}) \|\phi_0\|_{L^2} \\ + 2\left|\int_{t_0}^t e^{C(\|g_0\|_{H^{s-1}(\mathbb{D}^d)}, M_0^0(Re(\gamma)))(t-t')} \|f(t')\|_{H^{s-1}} dt'\right|,$$

which concludes the proof by (3.18) and (3.21). $\square$

3.3. **Composition, commutator and inverse estimates.** Here we give the key calculus lemmas that gives the estimates on the different remainder terms that appear in our computation and that do not come from the usual symbolic calculus rules on paradifferential operators. We start by the definition of a modified symbol class.

**Definition 3.2.** *Given $m \in \mathbb{R}$ and $0 \leq \rho \leq 1$, $\Gamma^m_{\mathscr{W},\rho}(\mathbb{T})$ denotes the space of locally bounded functions $a(x,\xi)$ on $\mathbb{T} \times \mathbb{Z}^*$, which are $C^\infty$ with respect to $\xi$ for $\xi \neq 0$ and such that, for all $\alpha \in \mathbb{N}^d$ and for all $\xi \neq 0$, the function $x \mapsto \partial_\xi^\alpha a(x,\xi)$ belongs to $\mathscr{W}$ and there exists a constant $C_\alpha$ such that,*

$$\forall |\xi| > \frac{1}{2}, \left\|\partial_\xi^\alpha a(.,\xi)\right\|_{\mathscr{W}} \leq C_\alpha (1+|\xi|)^{m-\rho|\alpha|}.$$

*Introduce the family of semi-norms $k \in \mathbb{N}$*

$$M^{m,k}_{\mathscr{W},\rho}(a) = \sup_{|\alpha| \leq k} \sup_{|\xi| \geq \frac{1}{2}} \left\|(1+|\xi|)^{m-\rho|\alpha|} \partial_\xi^\alpha a(.,\xi)\right\|_{\mathscr{W}},$$

*Then $\Gamma^m_{\mathscr{W},\rho}(\mathbb{T})$ equipped with the topology induced by the family $\left(M^{m,k}_{\mathscr{W},\rho}(a)\right)_{k\in\mathbb{N}}$ of semi-norms is a Fréchet space.*

Given a symbol $a$, define the paradifferential operator $T_a$ by

$$\widehat{T_a u}(\xi) = (2\pi)^{-d} \int_{\mathbb{Z}} \theta(\xi - \eta, \eta)\hat{a}(\xi - \eta, \eta)\psi(\eta)\hat{u}(\eta)d\eta,$$

where $\hat{a}(\eta, \xi) = \int e^{-ix.\eta} a(x, \xi)dx$ is the Fourier transform of $a$ with respect to the first variable; $\theta$ and $\psi$ are two fixed $C^\infty$ functions such that:

$$\psi(\eta) = 0 \text{ for } |\eta| \leq 1, \ \psi(\eta) = 1 \text{ for } |\eta| \geq 2,$$

and $\theta(\xi, \eta)$ is homogeneous of degree 0 and satisfies for $0 < \epsilon_1 < \epsilon_2$ small enough,

$$\theta(\xi, \eta) = 1 \text{ if } |\xi| \leq \epsilon_1 |\eta|, \ \theta(\xi, \eta) = 0 \text{ if } |\xi| \geq \epsilon_2 |\eta|.$$

For quantitative estimates we introduce:

**Definition 3.3.** *For $m \in \mathbb{R}$,$0 \leq \rho \leq 1$, $r \geq 0$ and $a \in \Gamma^m_{\mathscr{W},\rho}(\mathbb{T})$, we set*

$$M^m_{\mathscr{W},\rho}(a) = \sup_{|\alpha| \leq \frac{d}{2}+1+c} \sup_{|\xi| \geq \frac{1}{2}} \left\| (1 + |\xi|)^{m - \rho|\alpha|} \partial_\xi^\alpha a(., \xi) \right\|_{\mathscr{W}}, \ \textit{with } c > 0.$$

*We will essentially work with $\mathscr{W} = W^{r,\infty}$ and write $M^m_{W^{r,\infty},\rho}(a) = M^m_{r,\rho}(a)$ with $c = r$.*

**Remark 3.1.** *First remark that for $\rho = 1$ we get $\Gamma^m_{\mathscr{W}} = \Gamma^m_{\mathscr{W},1}$ the usual symbol class of Appendix A.2. The symbol class $\Gamma^m_{\mathscr{W},\rho}$ is modeled upon the pseudodifferential spaces $S^m\rho, \cdot$. Here we cannot directly apply the symbolic calculus of the Appendix, indeed it's known that there is no symbolic calculus possible in the general class $S^0_{1-\rho,\rho}$, though we are working here with explicit symbols and will make use of their particular form and don't need the full generality of symbol classes.*

**Theorem 3.1.** *Consider four real numbers $m, m', m''$ and $\rho > 0$ with $m < 1$, and three symbols $p \in \Gamma^m_\rho(\mathbb{T})$, $q \in \Gamma^{m''}_\rho(\mathbb{T})$ and $b \in \Gamma^{m'}_\rho(\mathbb{T})$. Suppose moreover that $p$ is real valued. Define:*

$$a = qe^{ip} \in \bigcap_{0 \leq k \leq \rho} \Gamma^{m''+km}_{\rho-k,1-m}(\mathbb{T}).$$

*In all of the following the paradifferential operators $T.$ are defined with a cut-off $\psi$ in Definition A.4 such that*

$$\psi(\eta) = 0 \textit{ for } |\eta| \leq C, \ \psi(\eta) = 1 \textit{ for } |\eta| \geq C + 1, \tag{3.23}$$

*where $C$ verifies,*

$$sup_{x,\xi} \left|p(x, \xi)\xi^{-1}\right| + \left|\partial_x p(x, \xi)\xi^{-1}\right| + |\partial_\xi p(x, \xi)| \leq C.$$

- *$T_a$ is an operator of order $m''$. Moreover we have the norm estimate:*

$$\|T_a\|_{H^\mu \to H^\mu} \leq CM^{m''}_{0,1-m}(a).$$

- $$T_a T_b \in \bigcap_{0 \leq k \leq \rho} \Gamma^{km+m'+m''}_{\rho-k,1-m}(\mathbb{T})$$

  *and $T_a T_b - T_{a\#b}$ is of order $m'' + m' - \rho(1 - m)$ where $a\#b$ is defined by:*

$$a\#b = \sum_{|\alpha| < \rho} \frac{1}{i^{|\alpha|}\alpha!} \partial_\xi^\alpha a \partial_x^\alpha b.$$

  *Moreover, for all $\mu \in \mathbb{R}$ there exists a constant $K$ such that*

$$\|T_a T_b - T_{a\#b}\|_{H^\mu \to H^{\mu - m'' - m' + \rho(1-m)}} \leq KM^m_\rho(p)M^{m''}_\rho(q)M^{m'}_\rho(b).$$

- $$T_b T_a \in \bigcap_{0\le k\le \rho} \Gamma^{km+m'+m''}_{\rho-k,1-m}(\mathbb{T})$$
  *and $T_bT_a - T_{b\#a}$ is of order $m''+m'-\rho(1-m)$ where $b\#a$ is defined by:*
  $$b\#a = \sum_{|\alpha|<\rho} \frac{1}{i^{|\alpha|}\alpha!}\partial_\xi^\alpha b \partial_x^\alpha a.$$
  *Moreover, for all $\mu\in\mathbb{R}$ there exists a constant $K$ such that*
  $$\|T_bT_a - T_{b\#a}\|_{H^\mu\to H^{\mu-m''-m'+\rho(1-m)}} \le K M^m_\rho(p) M^{m''}_\rho(q) M^{m'}_\rho(b).$$
- *$T_a$ has a left and right parametrix given by $a^{-1}=e^{-ip}$, i.e*
  $$T_aT_{a^{-1}} = T_{a^{-1}}T_a = T_1.$$

*Proof.* First by the cut-off hypothesis on $\phi$ in (3.23), $T_{a\#b}$ and $T_{b\#a}$ are well defined as oscillatory integrals with non degenerate phase functions.

The result then follows from Theorems 2.1, 2.2 and this lemma on approximation of symbols in $\Gamma^m_{\mathscr{W}} = \Gamma^m_{\mathscr{W},1}$ by symbols in the Schwartz class, which in turn is obtained by the usual mollification argument.

**Lemma 3.1.** *For all $a\in\Gamma^m_{\mathscr{W},\rho}(\mathbb{T})$, there is a sequence of symbols $\sigma_n\in\mathscr{S}(\mathbb{T}\times\mathbb{Z})$ such that*

*(1) the family $\{\sigma_n\}$ is bounded in $\Gamma^m_{\mathscr{W},\rho}(\mathbb{T})$,*

*(2) $\sigma_n\to a$ in $\Gamma^m_{\mathscr{W},\rho}(\mathbb{T})$.*

□

## Appendix A. Pseudodifferential and Paradifferential Calculus

In this paragraph we review classic notations and results about paradifferential and pseudodifferential calculus that we need in this paper. We follow the presentations in [10] , [17], and [11] which give an accessible and complete presentation.

**Notation A.1.** *In the following presentation we will use $\mathbb{D}$ to denote generically $\mathbb{T}$ or $\mathbb{R}$ and $\hat{\mathbb{D}}$ to denote their Pontryagin duals that is $\mathbb{Z}$ in the case of $\mathbb{T}$ and $\mathbb{R}$ in the case of $\mathbb{R}$.*

*We will use the usual definitions and standard notations for the regular functions $C^k$, $C^k_b$ for bounded ones and $C^k_0$ for those with compact support, the distribution space $\mathscr{D}'$,$\mathscr{E}'$ for those with compact support, $\mathscr{D}'^k$,$\mathscr{E}'^k$ for distributions of order $k$, Lebesgue spaces ($L^p$), Sobolev spaces ($H^s, W^{p,q}$) and the Schwartz class $\mathscr{S}$ and it's dual $\mathscr{S}'$. All of those spaces are equipped with their standard topologies.*

*We also recall the Landau notation the expression $O_{\|\ \|}(X)$ is used to denote any quantity bounded in $\|\ \|$ by $CX$, thus $Y=O_{\|\ \|}(X)$ is equivalent to $\|Y\|\le CX$.*

*In order not to confuse the pullback (change of variables) operation with the adjoint we will denote adjoint operators with $^\top$ instead of $^*$.*

### A.1. Pseudodifferential operators.

We introduce here the basic definitions and symbolic calculus results. We first introduce the classes of regular symbols.

**Definition A.1.** *Given $m\in\mathbb{R}, 0\le\rho\le 1$ and $0\le\sigma\le 1$ we denote the symbol class $S^m_{\rho,\sigma}(\mathbb{D}\times\hat{\mathbb{D}})$ the set of all $a\in C^\infty(\mathbb{D}\times\hat{\mathbb{D}})$ such that for all multi-orders $\alpha,\beta$ we have the estimate:*
$$\left|\partial_x^\alpha\partial_\xi^\beta a(x,\xi)\right| \le C_{\alpha,\beta}(1+|\xi|)^{m-\rho\beta+\sigma\alpha}.$$
*$S^m_{\rho,\sigma}(\mathbb{D}\times\hat{\mathbb{D}})$ is a Fréchet space with the topology defined by the family of semi-norms:*
$$M^m_{\beta,\alpha}(a) = \sup_{\mathbb{D}\times\hat{\mathbb{D}}}\left|\partial_x^\alpha\partial_\xi^\beta a(x,\xi)(1+|\xi|)^{\rho\beta-m-\sigma\alpha}\right|.$$

*Set*

$$S^m(\mathbb{D}\times\hat{\mathbb{D}}) = S^m_{1,0}(\mathbb{D}\times\hat{\mathbb{D}}),$$

$$S^{-\infty}(\mathbb{D}\times\hat{\mathbb{D}}) = \bigcap_{m\in\mathbb{R}} S^m(\mathbb{D}\times\hat{\mathbb{D}}) \quad \textit{and} \quad S^{+\infty}(\mathbb{D}\times\hat{\mathbb{D}}) = \bigcup_{m\in\mathbb{R}} S^m(\mathbb{D}\times\hat{\mathbb{D}})$$

*equipped with their canonically induced topology.*

Given a symbol $a \in S^m(\mathbb{D}\times\hat{\mathbb{D}})$, we define the pseudodifferential operator:

$$\mathrm{Op}(a)u(x) = a(x,D)u(x) = (2\pi)^{-n}\int_{\hat{\mathbb{D}}} e^{ix.\xi}a(x,\xi)\hat{u}(\xi)d\xi.$$

**Definition A.2.** *Let $m \in \mathbb{R}$, an operator $T$ is said to be of order $m$ if, and only if, for all $\mu \in \mathbb{R}$, it is bounded from $H^\mu(\mathbb{D})$ to $H^{\mu-m}(\mathbb{D})$.*

**Theorem A.1.** *If $a \in S^m(\mathbb{D}\times\hat{\mathbb{D}})$, then $a(x,D)$ is an operator of order $m$. Moreover we have the norm estimate:*

$$\|a(x,D)\|_{H^\mu\to H^{\mu-m}} \le CM^m_{\mu,m+d/2+1}(a)$$

We will now present the main results in symbolic calculus associated to pseudodifferential operators.

**Theorem A.2.** *Let $m, m' \in \mathbb{R}^d$, $a \in S^m(\mathbb{D}\times\hat{\mathbb{D}})$and $b \in S^{m'}(\mathbb{D}\times\hat{\mathbb{D}})$.*

- *Composition: Then* $\mathrm{Op}(a)\circ\mathrm{Op}(b)$ *is a pseudodifferential operator of order $m+m'$ with symbol $a\#b$ defined by:*

  $$a\#b(x,\xi) = (2\pi)^{-d}\int_{\mathbb{D}\times\hat{\mathbb{D}}} e^{i(x-y).(\xi-\eta)}a(x,\eta)b(y,\xi)dyd\eta$$

  *Moreover,*

  $$\mathrm{Op}(a)\circ\mathrm{Op}(b)(x,\xi) - \mathrm{Op}(\sum_{|\alpha|<k}\frac{1}{i^{|\alpha|}\alpha!}(\partial^\alpha_\xi a(x,\xi))(\partial^\alpha_x b(x,\xi))) \textit{ is of order } m+m'-k$$

  *for all $k \in \mathbb{N}$.*
- *Adjoint: The adjoint operator of* $\mathrm{Op}(a)$, $\mathrm{Op}(a)^\top$ *is a pseudodifferential operator of order $m$ with symbol $a^\top$ defined by:*

  $$a^\top(x,\xi) = (2\pi)^{-d}\int_{\mathbb{D}\times\hat{\mathbb{D}}} e^{-iy.\xi}a(x-y,\xi-\eta)dyd\eta$$

  *Moreover,*

  $$\mathrm{Op}(a^\top)(x,\xi) - \mathrm{Op}(\sum_{|\alpha|<k}\frac{1}{i^{|\alpha|}\alpha!}(\partial^\alpha_\xi\partial^\alpha_x\bar{a}(x,\xi))) \textit{ is of order } m-k$$

  *for all $k \in \mathbb{N}$.*

**Definition A.3.** *Let $(a_j) \in S^{m_j}(\mathbb{D}\times\hat{\mathbb{D}})$ be a series of symbols with $(m_j) \in \mathbb{R}^d$ decreasing to $-\infty$. We say that $a \in S^{m_0}(\mathbb{D}\times\hat{\mathbb{D}})$ is the asymptotic sum of $(a_j)$ if*

$$\forall k \in \mathbb{N}, a - \sum_{j=0}^k a_j \in S^{m_{k+1}}(\mathbb{D}\times\hat{\mathbb{D}}).$$

*We denote $a \sim \sum a_j$*

**Remark A.1.** *We can now write simply:*

$$a\#b \sim \sum_{|\alpha|}\frac{1}{i^{|\alpha|}\alpha!}(\partial^\alpha_\xi a(x,\xi))(\partial^\alpha_x b(x,\xi))$$

*and*

$$a^\top \sim \sum_{|\alpha|}\frac{1}{i^{|\alpha|}\alpha!}(\partial^\alpha_\xi\partial^\alpha_x\bar{a}(x,\xi)).$$

A.2. **Paradifferential operators.** We start by the definition of symbols with limited spatial regularity. Let $\mathscr{W} \subset \mathscr{S}'$ be a Banach space.

**Definition A.4.** *Given $m \in \mathbb{R}$, $\Gamma^m_{\mathscr{W}}(\mathbb{D})$ denotes the space of locally bounded functions $a(x,\xi)$ on $\mathbb{D} \times (\hat{\mathbb{D}} \setminus 0)$, which are $C^\infty$ with respect to $\xi$ for $\xi \neq 0$ and such that, for all $\alpha \in \mathbb{N}^d$ and for all $\xi \neq 0$, the function $x \mapsto \partial_\xi^\alpha a(x,\xi)$ belongs to $\mathscr{W}$ and there exists a constant $C_\alpha$ such that,*

$$\forall \, |\xi| > \frac{1}{2}, \left\| \partial_\xi^\alpha a(.,\xi) \right\|_{\mathscr{W}} \leq C_\alpha (1+|\xi|)^{m-|\alpha|}$$

*Given a symbol $a$, define the paradifferential operator $T_a$ by*

$$\widehat{T_a u}(\xi) = (2\pi)^{-d} \int_{\hat{\mathbb{D}}} \theta(\xi-\eta,\eta) \hat{a}(\xi-\eta,\eta) \psi(\eta) \hat{u}(\eta) d\eta,$$

*where $\hat{a}(\eta,\xi) = \int e^{-ix.\eta} a(x,\xi) dx$ is the Fourier transform of $a$ with respect to the first variable; $\theta$ and $\psi$ are two fixed $C^\infty$ functions such that:*

$$\psi(\eta) = 0 \text{ for } |\eta| \leq 1, \ \psi(\eta) = 1 \text{ for } |\eta| \geq 2,$$

*and $\theta(\xi,\eta)$ is homogeneous of degree 0 and satisfies for $0 < \epsilon_1 < \epsilon_2$ small enough,*

$$\theta(\xi,\eta) = 1 \text{ if } |\xi| \leq \epsilon_1 |\eta|, \ \theta(\xi,\eta) = 0 \text{ if } |\xi| \geq \epsilon_2 |\eta|.$$

For quantitative estimates we introduce as in [11]:

**Definition A.5.** *For $m \in \mathbb{R}$, $\rho \geq 0$ and $a \in \Gamma^m_{\mathscr{W}}(\mathbb{D})$, we set*

$$M^m_{\mathscr{W}}(a) = \sup_{|\alpha| \leq \frac{d}{2}+1+c} \sup_{|\xi| \geq \frac{1}{2}} \left\| (1+|\xi|)^{m-|\alpha|} \partial_\xi^\alpha a(.,\xi) \right\|_{\mathscr{W}}, \text{ with } c > 0.$$

*We will essentially work with $\mathscr{W} = W^{\rho,\infty}$ and write $M^m_{W^{\rho,\infty}}(a) = M^m_\rho(a)$ with $c = \rho$.*

The main features of symbolic calculus for paradifferential operators are given by the following Theorems.

**Theorem A.3.** *Let $m \in \mathbb{R}$. if $a \in \Gamma^m_0(\mathbb{D})$, then $T_a$ is of order $m$. Moreover, for all $\mu \in \mathbb{R}$ there exists a constant $K$ such that*

$$\|T_a\|_{H^\mu \to H^{\mu-m}} \leq K M^m_0(a).$$

**Theorem A.4.** *Let $m, m' \in \mathbb{R}$, and $\rho > 0$, $a \in \Gamma^m_\rho(\mathbb{D})$and $b \in \Gamma^{m'}_\rho(\mathbb{D})$.*

- *Composition: Then $T_a T_b$ is a paradifferential operator of order $m+m'$ and $T_a T_b - T_{a\#b}$ is of order $m+m'-\rho$ where $a\#b$ is defined by:*

$$a\#b = \sum_{|\alpha|<\rho} \frac{1}{i^{|\alpha|}\alpha!} \partial_\xi^\alpha a \partial_x^\alpha b$$

*Moreover, for all $\mu \in \mathbb{R}$ there exists a constant $K$ such that*

$$\|T_a T_b - T_{a\#b}\|_{H^\mu \to H^{\mu-m-m'+\rho}} \leq K M^m_\rho(a) M^{m'}_\rho(b).$$

- *Adjoint: The adjoint operator of $T_a$, $T_a^\top$ is a paradifferential operator of order $m$ with symbol $a^\top$ defined by:*

$$a^\top = \sum_{|\alpha|<\rho} \frac{1}{i^{|\alpha|}\alpha!} \partial_\xi^\alpha \partial_x^\alpha \bar{a}$$

*Moreover, for all $\mu \in \mathbb{R}$ there exists a constant $K$ such that*

$$\left\| T_a^\top - T_{a^\top} \right\|_{H^\mu \to H^{\mu-m+\rho}} \leq K M^m_\rho(a).$$

If $a = a(x)$ is a function of $x$ only, the paradifferential operator $T_a$ is called a paraproduct. It follows from Theorem A.4 and the Sobolev embedding that:

- If $a \in H^{\alpha}(\mathbb{D})$ and $b \in H^{\beta}(\mathbb{D})$ with $\alpha, \beta > \frac{d}{2}$, then
$$T_a T_b - T_{ab} \text{ is of order } -\left(min\{\alpha,\beta\} - \frac{d}{2}\right).$$
- If $a \in H^{\alpha}(\mathbb{D})$ with $\alpha > \frac{d}{2}$, then
$$T_a^{\top} - T_{a^{\top}} \text{ is of order } -\left(\alpha - \frac{d}{2}\right).$$
- If $a \in W^{r,\infty}(\mathbb{D})$, $r \in \mathbb{N}$ then:
$$\|au - T_a u\|_{H^r} \le C\,\|a\|_{W^{r,\infty}}\,\|u\|_{L^2}\,.$$

An important feature of paraproducts is that they are well defined for function $a = a(x)$ which are not $L^{\infty}$ but merely in some Sobolev spaces $H^r$ with $r < \frac{d}{2}$.

**Proposition A.1.** *Let $m > 0$. If $a \in H^{\frac{d}{2}-m}(\mathbb{D})$ and $u \in H^{\mu}(\mathbb{D})$ then $T_a u \in H^{\mu-m}(\mathbb{D})$. Moreover,*
$$\|T_a u\|_{H^{\mu-m}} \le K\,\|a\|_{H^{\frac{d}{2}-m}}\,\|u\|_{H^{\mu}}$$

A main feature of paraproducts is the existence of paralinearisation Theorems which allow us to replace nonlinear expressions by paradifferential expressions, at the price of error terms which are smoother than the main terms.

**Theorem A.5.** *Let $\alpha, \beta \in \mathbb{R}$ be such that $\alpha, \beta > \frac{d}{2}$, then*

- *Bony's Linearization Theorem for all $C^{\infty}$ function $F$, if $a \in H^{\alpha}(\mathbb{D})$ then*
$$F(a) - F(0) - T_{F'(a)}a \in H^{2\alpha - \frac{d}{2}}(\mathbb{D}).$$
- *If $a \in H^{\alpha}(\mathbb{D})$ and $b \in H^{\beta}(\mathbb{D})$, then $ab - T_a b - T_b a \in H^{\alpha+\beta-\frac{d}{2}}(\mathbb{D})$. Moreover there exists a positive constant $K$ independent of $a$ and $b$ such that:*
$$\|ab - T_a b - T_b a\|_{H^{\alpha+\beta-\frac{d}{2}}} \le K\,\|a\|_{H^{\alpha}}\,\|b\|_{H^{\beta}}\,.$$

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