ON PARACOMPOSITION AND CHANGE OF VARIABLES IN PARADIFFERENTIAL OPERATORS

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Abstract. In this paper we revisit the hypothesis needed to define the "paracomposition" operator, an analogue to the classic pull-back operation in the low regularity setting, first introduced by S. Alinhac in [3]. More precisely we do so in two directions. First we drop the diffeomorphism hypothesis. Secondly we give estimates in global Sobolev and Zygmund spaces. Thus we fully generalize Bony's classic paralinearasition Theorem giving sharp estimates for composition in Sobolev and Zygmund spaces. In order to prove that the new class of operations benefits of symbolic calculus properties when composed by a paradifferential operator, we discuss the pull-back of pseudodifferential and paradifferential operators which then become Fourier Integral Operators. In this discussion we show that those Fourier Integral Operators obtained by pull-back are pseudodifferential or paradifferential operators if and only if they are pulled-back by a diffeomorphism i.e a change of variable. We give a proof to the change of variables in paradifferential operators.
***Keywords*— Composition, Sobolev spaces, Zygmund spaces, Paracomposition, Paradifferential operators, Pseudodifferential operators, Fourier integral operators, Change of variables.**

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1. Introduction

Given a $\rho > 0$ and $C^{1+\rho}$ diffeomorphism $\chi : \Omega_1 \to \Omega_2$ between two open subsets of $\mathbb{R}^d$, Alinhac constructed an operator $\chi^* : \mathfrak{D}'(\Omega_2) \to \mathfrak{D}'(\Omega_1)$ having analogous properties to the usual composition $u \to u \circ \chi$ but with limited dependency on the regularity of χ as for classical paradifferential operators i.e the paraproduct T_a is well defined from $H^s \to H^s$, for all s for a merely in L^∞.
Alinhac's construction was motivated by questions that arose from the study of non linear PDEs for example: the study of the transport of a distribution's wave front by a diffeomorphism with low regularity as in the works of E. Leichtnam in [13], the study of the singularities of solutions to semi-linear hyperbolic evolution problems and the characteristic surfaces of the associated operators(here having low regularity), the main reference being Bony's work on the subject ([6],[7],[8],[9]). More recently in [1] and [2], the Paracomposition appears naturally as the "good variable"[1] after a low regularity change of variable in treating the Cauchy problem for the Water Waves system with rough data. It also appears in our recent proof of the quasi-linearity of the Water Waves system [16].

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[1]The so called good unknown of Alinhac.

Finally the construction of χ^* gives a complete linearization formula to the composition of two functions(with one being a diffeomorphism) generalizing the classic para-linearization Theorem by Bony [6] in a low regularity case:
Bony showed for $u \in C^\infty$ and $\chi \in H^s_{loc}, s > \frac{d}{2}$ (without the diffeomorphism hypothesis):

$$u \circ \chi = T_{u'(\chi)}\chi + \text{ remainder},$$

and Alinhac showed for $u \in C^\sigma_{loc}, \sigma > 1$ and $\chi \in C^{1+\rho}, \rho > 0$ a diffeomorphism:

$$u \circ \chi = \chi^* u + T_{u'(\chi)}\chi + \text{ remainder}. \tag{1.1}$$

Another fundamental result obtained by Alinhac is that the operator χ^* benefits from symbolic calculus properties, that is, it conjugates paradifferential operators. Given T_h a paradifferential operator, Alinhac proved a result in the form:

$$\chi^* T_h u = T_{h^*}\chi^* u + \text{ remainder},$$

where h^* is the pulled back symbol in the case of diffeomorphisms.

The main result of this work generalizes Bony's and Alinhac's work by:

- dropping the diffeomorphism hypothesis with a new operator $\chi^\star : \mathfrak{D}'(\Omega_2) \to \mathfrak{D}'(\Omega_1)$. $\chi^\star$ will coincide with Alinhac's operator χ^* modulo a regular remainder in the case of diffeomorphisms.
- Giving estimates in "global" spaces which were of interest for us in our study of the flow map regularity associated to the Water Waves system.

We will then show that $\chi^\star$ benefits of symbolic calculus properties, for that we will start by discussing the pull-back of pseudodifferential and paradifferential operators by χ which then become Fourier integral operators. In this discussion we show that those Fourier Integral Operators obtained by pull-back are pseudodifferential or paradifferential operators if and only if they are pulled-back by a diffeomorphism i.e a change of variable. We also give a proof to the change of variables in paradifferential operators as we could not find a reference in the literature.

A main application of the paracomposition operator and the paralinearization formula (1.1) now that we have the global estimates is the study of composition in Sobolev spaces with limited regularity. The literature on this problem is rich and our knowledge of it is certainly incomplete but we mainly looked on two recent articles treating this subject [5] and [12] in which they study composition in Sobolev spaces and the geometry of diffeomorphisms groups on manifolds. We will limit the discussion here to the Euclidean space in which the tools presented here significantly improve upon the results from [5] and [12]. First in [12] the composition estimates are proven on $H^n(\mathbb{R}^d) \times D^s(\mathbb{R}^d)$ with $n \in \mathbb{N}$, $s > 1 + \frac{d}{2}$ an integer and

$$D^s(\mathbb{R}^d) = \left\{ \psi - id \in H^s(\mathbb{R}^d), \psi \text{ is a diffeomorphism} \right\}.$$

Here we generalize this to n, s real number and from the paralinearization formula (1.1) it is justified to work in the class $D^s(\mathbb{R}^d)$ which appears naturally but it admits several generalization the simplest one is for example using Zygmund spaces, we also clarify the need of the diffeomorphism hypothesis. More precisely we have the following,

Corollary 1.1. *Consider two real numbers $s \in \mathbb{R}$, $\rho \in R^*_+ \setminus \mathbb{N}$, and take $\phi \in H^s(\mathbb{R}^d)$ and consider $\chi \in W^{1+\rho,\infty}_{loc}(\mathbb{R}^d)$ a diffeomorphism such that $D\chi \in W^{\rho,\infty}(\mathbb{R}^d)$. Then $\phi \circ \chi \in H^{min(s,\rho)}(\mathbb{R}^d)$.*

The result we have is even stronger indeed it's a Kato-Ponce like decomposition of the different terms that appear in the H^s estimates of composition, for example keeping the notations of the previous Corollary and taking $\psi \in D^s(\mathbb{R}^d)$ we can have estimates of the form:

$$\|\phi \circ \psi\|_{H^s} \leq \|D\psi\|_{L^\infty} \|\psi\|_{H^s} + \|D\phi\|_{L^\infty} \|\psi - Id\|_{H^s}.$$

So if only working with Sobolev spaces more sophisticated versions of the previous inequality give,

Corollary 1.2. *Consider a real number $s > 1 + \frac{d}{2}$, and take $\phi \in H^s(\mathbb{R}^d)$ and consider $\chi \in W^{1+s-\frac{d}{2},\infty}_{loc}(\mathbb{R}^d)$ a diffeomorphism such that $D\chi \in W^{1,\infty}(\mathbb{R}^d)$ and $D^2\chi \in H^{s-2}(\mathbb{R}^d)$. Then $\phi \circ \chi \in H^s(\mathbb{R}^d)$.*

Secondly in [5] to prove the well posedness of EPDIFF equation they treat the case of change of variables in pseudodifferential operator with a diffeomorphism with limited regularity. The results are restricted to skew-symmetric operators with compact support and a diffeomorphism in the class $D^s(\mathbb{R}^d)$. Here with the paradifferential calculus and the paracomposition in hand, the more general case of symbols with limited regularity is treated, the pseudodifferential symbols being the the case where the symbols are regular, the ellipticity and symmetry hypothesis dropped and the need of diffeomorphisms justified. More precisely we have

Corollary 1.3. *Consider a real number r, $A \in S^r(\mathbb{R}^d \times \mathbb{R}^d)$ and $\chi \in W^{1+s-\frac{d}{2},\infty}_{loc}(\mathbb{R}^d)$ a diffeomorphism such that $D\chi \in W^{1,\infty}(\mathbb{R}^d)$ and $D^2\chi \in H^{s-2}(\mathbb{R}^d)$. Then the pull back A^* of A by χ defined as*

$$u \in \mathscr{S}, A^*u = [A(u \circ \chi)] \circ \chi^{-1},$$

is extended to a linear bounded operator from $H^s(\mathbb{R}^d)$ to $H^{s-r}(\mathbb{R}^d)$.

1.1. **Heuristics behind Paradifferential calculus and Paracomposition.** For the sake of this discussion let us pretend that ∂_x is left-invertible with a choice of ∂_x^{-1} that acts continuously from H^s to H^{s+1}. We follow here analogous ideas to the ones presented by Shnirelman in [17].

Paraproduct. One way to define the paraproduct of two functions $f, g \in H^s$ with s sufficiently large is: we differentiate fg k times, using the Leibniz formula, and then restore the function fg by the k-th power of ∂_x^{-1}:

$$\begin{aligned} fg &= \partial_x^{-k}\partial_x^k(fg) \\ &= \partial_x^{-k}\big(g\partial_x^k f + k\partial_x g \partial_x^{k-1} f + \cdots + k\partial_x f \partial_x^{k-1} g + g\partial_x^k f\big) \\ &= T_g f + T_f g + R, \end{aligned}$$

where,

$$T_g f = \partial_x^{-k}\big(g\partial_x^k f\big), \quad T_f g = \partial_x^{-k}\big(f\partial_x^k g\big),$$

and R is the sum of all remaining terms.

The key observation is that if $s > \frac{1}{2} + k$, then $g \mapsto T_f g$ is a continuous operator in H^s for $f \in H^{s-k}$. The remainder R is a continuous bilinear operator from H^s to H^{s+1}.

The operator $T_f g$ is called the paraproduct of g and f and can be interpreted as follows. The term $T_f g$ takes into play high frequencies of g compared to those of f and demands more regularity in $g \in H^s$ than $f \in H^{s-k}$ thus the term $T_f g$ bears the "singularities" brought on by g in the product fg. Symmetrically $T_g f$ bears the "singularities" brought on by f in the product fg and the remainder R is a smoother function (H^{s+1}) and does not contribute to the main singularities of the product.

Notice that this definition uses a "general" heuristic from PDE that is the worst terms are the highest order terms (ones involving the highest order of differentiation).

Paracomposition. We again work with $f \in H^s$ and $g \in C^s$ with s large and consider the composition of two functions $f \circ g$ which bears the singularities of both f and g, and our goal is to separate them. We proceed as before by differentiating $f \circ g$ k times, using the Faá di Bruno's formula, and then restore the function fg by the k-th power of ∂_x^{-1}:

$$\begin{aligned} f \circ g &= \partial_x^{-k} \partial_x^k (f \circ g) \\ &= \partial_x^{-k} \big((\partial_x^k f \circ g) \cdot (\partial_x g)^k + \cdots + (\partial_x f \circ g) \cdot \partial_x^k g \big) \\ &= g^* f + T_{\partial_x f \circ g} g + R, \end{aligned}$$

where,

$$g^* f = \partial_x^{-k} \big((\partial_x^k f \circ g) \cdot (\partial_x g)^k \big) \text{ is the paracomposition of } f \text{ by } g$$

and R is the sum of all remaining terms.

Again the key observation is that if $s > \frac{1}{2} + k$, then $f \mapsto g^* f$ is a continuous operator in H^s for $g \in C^{s-k}$. Thus this term bears essentially the singularities of f in $f \circ g$. As before $T_{\partial_x f \circ g} g$ bears essentially the singularities of g in $f \circ g$. The remainder R is a continuous bilinear operator from H^s to H^{s+1}. Thus we have separated the singularities of the composition $f \circ g$.

Change of variable in Paradifferential operators. From what we have seen previously it seems likely that the adequate change of variable for paradifferential operators is one that comes from commuting with the paracomposition by a diffeomorphism. We carry on the previous computation with the trivial paradifferential operator $\partial_x = T_{i\xi}$ and we suppose moreover that g is a diffeomorphism.

$$\begin{aligned} g^* \partial_x f &= \partial_x^{-k} \big((\partial_x^{k+1} f \circ g) \cdot (\partial_x g)^k \big) \\ &= \partial_x^{-k} \big(\partial_x^k [\partial_x^{-k} (\partial_x^{k+1} f \circ g) \cdot (\partial_x g)^{k+1}] \cdot (\partial_x g)^{-1} \big) \\ &= T_{(\partial_x g)^{-1}} T_{i\xi} g^* f, \end{aligned}$$

and we notice that $(\partial_x g)^{-1} i\xi = (\partial_x)^*$ is the usual pull-back formula for pseudo-differential symbols by a diffeomorphism g, giving us the desired symbolic calculus rules.

1.2. **Structure of the paper.** Given the technical nature of the results in this paper we start the paper by a quick overview in sections 2 and 3 of notions of functional analysis and microlocal analysis[2]. Then in section 4 we present the different results on the change of variables in pseudodifferential and paradifferential operators. And finally with all of the tools needed we redefine the para-composition in section 5 and show that it satisfies all of the desired properties. Thus the reader interested in the change of variable/pull-back can directly go to section 4 and if she/he is interested only in the para-composition she/he can go to section 5.

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2. Notations and functional analysis

We present the definitions of the functional spaces that will be used.
We will use the usual definitions and standard notations for the regular functions C^k, C_0^k for those with compact support, the distribution space $\mathscr{D}'$,$\mathscr{E}'$ for those with compact support, $\mathscr{D}'^k$,$\mathscr{E}'^k$ for distributions of order k, Lebesgue spaces (L^p), Sobolev spaces ($H^s, W^{p,q}$) and the Schwartz class $\mathscr{S}$ and it's dual $\mathscr{S}'$. All of those spaces are equipped with their standard topologies.

[2] Thus the reader familiar with those notions can skip the preliminary sections.

Definition 2.1 (Littlewood-Paley decomposition)**.** *Pick $P_0 \in C_0^\infty(\mathbb{R}^d)$ so that $P_0(\xi) = 1$ for $|\xi| < 1$ and 0 for $|\xi| > 2$. We define a dyadic decomposition of unity by:*

$$\text{for } k \geq 1,\ P_{\leq k}(\xi) = \Phi_0(2^{-k}\xi),\ P_k(\xi) = P_{\leq k}(\xi) - P_{\leq k-1}(\xi).$$

Thus,

$$P_{\leq k}(\xi) = \sum_{0 \leq j \leq k} P_j(\xi) \text{ and } 1 = \sum_{j=0}^{\infty} P_j(\xi).$$

Introduce the operator acting on $\mathscr{S}'(\mathbb{R}^d)$:

$$P_{\leq k}u = \mathscr{F}^{-1}(P_{\leq k}(\xi)u) \text{ and } u_k = \mathscr{F}^{-1}(P_k(\xi)u).$$

Thus,

$$u = \sum_k u_k.$$

Finally put $\{k \geq 1, C_k = \text{supp } P_k\}$ the set of rings associated to this decomposition.

Remark 2.1. *An interesting property of the Littlewood-Paley decomposition is that even if the decomposed function is merely a distribution the terms of the decomposition are regular, indeed they all have compact spectrum and thus are entire functions. On classical functions spaces this regularization effect can be "measured" by the following inequalities due to Bernstein.*

Proposition 2.1 (Bernstein's inequalities)**.** *Suppose that $a \in L^p(\mathbb{R}^d)$ has its spectrum contained in the ball $\{|\xi| \leq \lambda\}$. Then $a \in C^\infty$ and for all $\alpha \in \mathbb{N}^d$ and $q \geq p$, there is $C_{\alpha,p,q}$ (independent if λ) such that*

$$\|\partial_x^\alpha a\|_{L^q} \leq C_{\alpha,p,q}\lambda^{|\alpha|+\frac{d}{p}-\frac{d}{q}} \|a\|_{L^p}.$$

In particular,

$$\|\partial_x^\alpha a\|_{L^q} \leq C_\alpha \lambda^{|\alpha|} \|a\|_{L^p}, \text{ and for } p = 2,\ p = \infty$$

$$\|a\|_{L^\infty} \leq C\lambda^{\frac{d}{2}} \|a\|_{L^2}.$$

Proposition 2.2. *For all $\mu > 0$, there is a constant C such that for all $\lambda > 0$ and for all $\alpha \in W^{\mu,\infty}$ with spectrum contained in $\{|\xi| \geq \lambda\}$. one has the following estimate:*

$$\|a\|_{L^\infty} \leq C\lambda^{-\mu} \|a\|_{W^{\mu,\infty}}.$$

Definition 2.2 (Singular support)**.** *$f \in \mathscr{S}'(\mathbb{R}^d)$ is said to be C^∞ in a neighborhood of x, if there exists a neighborhood ω of x such that for all $\psi \in C_0^\infty(\omega)$ we have $\psi f \in C^\infty(\mathbb{R}^d)$.*
The singular support of a distribution f, sing supp f, *is defined as the complementary of such points and is clearly closed.*

Definition 2.3 (Zygmund spaces on $\mathbb{R}^d$)**.** *For $r \in \mathbb{R}$ we define the space $C_*^r(\mathbb{R}^d) \subset \mathscr{S}'(\mathbb{R}^d)$ by:*

$$C_*^r(\mathbb{R}^d) = \left\{u \in \mathscr{S}'(\mathbb{R}^d), \|u\|_r = \sup_q 2^{qr} \|u_q\|_\infty < \infty\right\}$$

equipped with its canonical topology giving it a Banach space structure.
It's a classical result that for $r \notin \mathbb{N}$, $C_*^r(\mathbb{R}^d) = W^{r,\infty}(\mathbb{R}^d)$ the classic Hölder spaces.
We define the local spaces:

$$C_{*,loc}^r(\mathbb{R}^d) = \left\{u \in \mathscr{S}'(\mathbb{R}^d), \forall \psi \in C_0^\infty(\mathbb{R}^d), \psi u \in C_*^r(\mathbb{R}^d)\right\}.$$

Proposition 2.3. *Let B be a ball with center 0. There exists a constant C such that for all $r > 0$ and for all $(u_q)_{q\in\mathbb{N}} \in \mathscr{S}'(\mathbb{R}^d)$ verifying:*

$$\forall q, \operatorname{supp} \hat{u}_q \subset 2^q\ B \text{ and } (2^{qr}\left\|u_q\right\|_\infty)_{q\in\mathbb{N}} \text{ is bounded}$$

$$\text{then, } u = \sum_q u_q \in C^r_*(\mathbb{R}^d) \text{ and } \|u\|_r \leq \frac{C}{1-2^{-r}} sup_{q\in\mathbb{N}} 2^{qr}\left\|u_q\right\|_\infty .$$

For the definition of spaces in open subsets of $\mathbb{R}^d$ we follow the presentation of [10]. Let Ω be an open subset of $\mathbb{R}^d$.

Definition 2.4 (Zygmund spaces on Ω)**.** *For $r \in \mathbb{R}$ we define the space $C^r_*(\Omega) \subset \mathscr{D}'(\Omega)$ by:*

$$C^r_*(\Omega) = \left\{u \in \mathscr{D}'(\Omega), u = U_{|\Omega} \text{ for some } U \in C^r_*(\mathbb{R}^d)\right\}$$

equipped with its canonical topology i.e

$$\|u\|_{C^r_*(\Omega)} = \inf_{\substack{U\in C^r_*(\mathbb{R}^d)\\ U_{|\Omega}=u}} \|U\|_{C^r_*(\mathbb{R}^d)}$$

giving it a Banach space structure.
We define the local spaces:

$$C^r_{*,loc}(\Omega) = \left\{u \in \mathscr{D}'(\Omega), \forall\psi \in C^\infty_0(\Omega), \psi u \in C^r_*(\Omega)\right\}.$$

Definition 2.5 (Sobolev spaces on $\mathbb{R}^d$)**.** *It is also a classical result that for $s \in \mathbb{R}$:*

$$H^s(\mathbb{R}^d) = \left\{u \in \mathscr{S}'(\mathbb{R}^d), |u|_s = \left(\sum_q 2^{2qs}\|u_q\|_{L^2}{}^2\right)^{\frac{1}{2}} < \infty\right\}$$

with the right hand side equipped with its canonical topology giving it a Hilbert space structure and $|\ |_s$ is equivalent to the usual norm on $\|\ \|_{H^s}$.
We define the local spaces:

$$H^s_{loc}(\mathbb{R}^d) = \left\{u \in \mathscr{S}'(\mathbb{R}^d), \forall\psi \in C^\infty_0(\mathbb{R}^d), \psi u \in H^s(\mathbb{R}^d)\right\}.$$

Proposition 2.4. *Let B be a ball with center 0. There exists a constant C such that for all $s > 0$ and for all $(u_q)_{\in\mathbb{N}} \in \mathscr{S}'(\mathbb{R}^d)$ verifying:*

$$\forall q, \operatorname{supp} \hat{u}_q \subset 2^q\ B \text{ and } (2^{qs}\left\|u_q\right\|_{L^2})_{q\in\mathbb{N}} \text{ is in } L^2(\mathbb{N})$$

$$\text{then, } u = \sum_q u_q \in H^s(\mathbb{R}^d) \text{ and } |u|_s \leq \frac{C}{1-2^{-s}}\left(\sum_q 2^{2qs}\|u_q\|_{L^2}{}^2\right)^{\frac{1}{2}}.$$

Definition 2.6 (Sobolev spaces on Ω)**.** *For $s \in \mathbb{R}$ we define the space $H^s(\Omega) \subset \mathscr{D}'(\Omega)$ by:*

$$H^s(\Omega) = \left\{u \in \mathscr{D}'(\Omega), u = U_{|\Omega} \text{ for some } U \in H^s(\mathbb{R}^d)\right\}$$

equipped with its canonical topology i.e

$$\|u\|_{H^s(\Omega)} = \inf_{\substack{U\in H^s(\mathbb{R}^d)\\ U_{|\Omega}=u}} \|U\|_{H^s(\mathbb{R}^d)}$$

giving it a Hilbert space structure[3]*.*
We define the local spaces:

$$H^s_{loc}(\Omega) = \left\{u \in \mathscr{D}'(\Omega), \forall\psi \in C^\infty_0(\Omega), \psi u \in H^s(\Omega)\right\}.$$

[3] This is not immediate from the definition but is a consequence of the fact that $H^s(\Omega)$ can be seen as a quotient of $H^s(\mathbb{R}^d)$ by a closed subset, for a full presentation see [10].

Remark 2.2. *This definition of the functions in an open subset might not seem as the most natural, in fact there are different ways(intrinsically, extrinsically, by interpolation etc...) to define $H^s(\Omega)$ and when no regularity assumption is put on Ω and they don't necessarily match. In [10] they show that when Ω has Lipschitz regularity all the different definitions of $H^s(\Omega)$ coincide.*

We recall the usual nonlinear estimates in Sobolev spaces:

- If $u_j \in H^{s_j}(\mathbb{R}^d), j = 1, 2$, and $s_1 + s_2 > 0$ then $u_1 u_2 \in H^{s_0}(\mathbb{R}^d)$ and if
$$s_0 \leq s_j, j = 1, 2 \text{ and } s_0 \leq s_1 + s_2 - \frac{d}{2},$$
$$\text{then } \|u_1 u_2\|_{H^{s_0}} \leq K \|u_1\|_{H^{s_1}} \|u_2\|_{H^{s_2}},$$
where the last inequality is strict if s_1 or s_2 or $-s_0$ is equal to $\frac{d}{2}$.
- For all C^∞ function F vanishing at the origin, if $u \in H^s(\mathbb{R}^d)$ with $s > \frac{d}{2}$ then
$$\|F(u)\|_{H^s} \leq C(\|u\|_{H^s}),$$
for some non decreasing function C depending only on F.

Finally we present a classic result for operator estimates by Y.Meyer [15]:

Lemma 2.1 (Meyer multipliers)**.** *Let $\delta \in \mathbb{R}$, and suppose we have a sequence $m_p \in C^\infty$ with, for all $k \in \mathbb{N}$,*
$$\sum_{|\alpha|=k} \|\partial^\alpha m_p\|_\infty \leq C_k 2^{p(k+\delta)}.$$
The mapping $M : u \mapsto \sum m_p u_p = Mu$ maps H^s to $H^{s-\delta}$ and C^r_* to $C^{r-\delta}_*$ for all $s, r > \delta$, with operators norms depending only on the C_k for $k \leq E(s-\delta)+1$ or $k \leq E(r-\delta)+1$.

3. Notions of microlocal analysis

In this paragraph we start by reviewing classic notations and results about pseudodifferential calculus, Fourier integral operators and paradifferential calculus, which can be found in [11] , [18], [4] and [14] as an accessible presentation to the theories and from which we follow the presentation.

3.1. **Pseudodifferential Calculus.** We introduce here the basic definitions and symbolic calculus results. We first introduce the classes of regular symbols.

Definition 3.1. *Given $m \in \mathbb{R}, 0 \leq \rho \leq 1$ and $0 \leq \sigma \leq 1$ we denote the symbol class $S^m_{\rho,\sigma}(\mathbb{R}^d \times \mathbb{R}^d)$ the set of all $a \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ such that for all multi-orders α, β we have the estimate:*
$$\left|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)\right| \leq C_{\alpha,\beta}(1+|\xi|)^{m-\rho\beta+\sigma\alpha}.$$
$S^m_{\rho,\sigma}(\mathbb{R}^d \times \mathbb{R}^d)$ is a Fréchet space with the topology defined by the family of semi-norms:
$$M^m_{\beta,\alpha}(a) = \sup_{\mathbb{R}^d} \left|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)(1+|\xi|)^{\rho\beta-m-\sigma\alpha}\right|.$$
Set
$$S^m(\mathbb{R}^d \times \mathbb{R}^d) = S^m_{1,0}(\mathbb{R}^d \times \mathbb{R}^d),$$
$$S^{-\infty}(\mathbb{R}^d \times \mathbb{R}^d) = \bigcap_{m\in\mathbb{R}} S^m(\mathbb{R}^d \times \mathbb{R}^d) \text{ and } S^{+\infty}(\mathbb{R}^d \times \mathbb{R}^d) = \bigcup_{m\in\mathbb{R}} S^m(\mathbb{R}^d \times \mathbb{R}^d)$$
equipped with their canonically induced topology.

Given a symbol $a \in S^m(\mathbb{R}^d \times \mathbb{R}^d)$, we define the pseudodifferential operator:

$$\operatorname{Op}(a)u(x) = a(x,D)u(x) = (2\pi)^{-n} \int_{\mathbb{R}^d} e^{ix.\xi} a(x,\xi)\hat{u}(\xi) d\xi.$$

For $u \in \mathscr{S}(\mathbb{R}^d)$ we have

$$\begin{aligned}
\operatorname{Op}(a)u(x) &= (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ix.\xi} a(x,\xi)\hat{u}(\xi) d\xi \\
&= (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ix.\xi} a(x,\xi) \int_{\mathbb{R}^d} e^{-iy.\xi} u(y) dy d\xi \\
&= \int_{\mathbb{R}^d} \left((2\pi)^{-n} \int_{\mathbb{R}^d} e^{i(x-y).\xi} a(x,\xi) d\xi \right) u(y) dy.
\end{aligned}$$

Thus giving us the following Proposition.

Proposition 3.1. *For $a \in S^m(\mathbb{R}^d \times \mathbb{R}^d)$, $\operatorname{Op}(a)$ has a kernel K defined by*

$$K(x,y) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i(x-y).\xi} a(x,\xi) d\xi = (2\pi)^{-n} \mathscr{F}_\xi a(x, y-x). \tag{3.1}$$

Which can be inverted to give:

$$\begin{aligned}
a(x,\xi) = \mathscr{F}_{y\to\xi} K(x, x-y) &= \int_{\mathbb{R}^d} e^{-iy.\xi} K(x, x-y) dy \\
&= (-1)^d e^{-ix.\xi} \int_{\mathbb{R}^d} e^{iy.\xi} K(x,y) dy. \qquad (3.2)
\end{aligned}$$

Definition 3.2. *Let $m \in \mathbb{R}$, an operator T is said to be of order m if, and only if, for all $\mu \in \mathbb{R}$, it is bounded from $H^\mu(\mathbb{R}^d)$ to $H^{\mu-m}(\mathbb{R}^d)$.*

Theorem 3.1. *If $a \in S^m(\mathbb{R}^d \times \mathbb{R}^d)$, then $a(x,D)$ is an operator of order m. Moreover we have the norm estimate:*

$$\|a(x,D)\|_{H^\mu \to H^{\mu-m}} \leq C M^m_{\mu, m+d/2+1}(a).$$

We will now present the main results in symbolic calculus associated to pseudodifferential operators.

Theorem 3.2. *Let $m, m' \in \mathbb{R}^d$, $a \in S^m(\mathbb{R}^d \times \mathbb{R}^d)$ and $b \in S^{m'}(\mathbb{R}^d \times \mathbb{R}^d)$.*

- *Composition: Then $\operatorname{Op}(a) \circ \operatorname{Op}(b)$ is a pseudodifferential operator of order $m + m'$ with symbol $a\#b$ defined by:*

  $$a\#b(x,\xi) = (2\pi)^{-d} \int_{\mathbb{R}^d \times \mathbb{R}^d} e^{i(x-y).(\xi-\eta)} a(x,\eta) b(y,\xi) dy d\eta.$$

  Moreover,

  $$\operatorname{Op}(a) \circ \operatorname{Op}(b)(x,\xi) - \operatorname{Op}\Big(\sum_{|\alpha|<k} \frac{1}{i^{|\alpha|}\alpha!} (\partial_\xi^\alpha a(x,\xi))(\partial_x^\alpha b(x,\xi))\Big) \text{ is of order } m + m' - k$$

  for all $k \in \mathbb{N}$.
- *Adjoint: The adjoint operator of $\operatorname{Op}(a)$, that will note $\operatorname{Op}(a)^t$ to avoid confusion with the pullback operator defined in this work, is a pseudodifferential operator of order m with symbol a^t defined by:*

  $$a^t(x,\xi) = (2\pi)^{-d} \int_{\mathbb{R}^d \times \mathbb{R}^d} e^{-iy.\xi} a(x-y, \xi-\eta) dy d\eta$$

  Moreover,

  $$\operatorname{Op}(a^t)(x,\xi) - \operatorname{Op}\Big(\sum_{|\alpha|<k} \frac{1}{i^{|\alpha|}\alpha!} (\partial_\xi^\alpha \partial_x^\alpha \bar{a}(x,\xi))\Big) \text{ is of order } m - k$$

  for all $k \in \mathbb{N}$.

Definition 3.3. *Let $(a_j) \in S^{m_j}(\mathbb{R}^d \times \mathbb{R}^d)$ be a series of symbols with $(m_j) \in \mathbb{R}^d$ decreasing to $-\infty$. We say that $a \in S^{m_0}(\mathbb{R}^d \times \mathbb{R}^d)$ is the asymptotic sum of (a_j) if*

$$\forall k \in \mathbb{N}, a - \sum_{j=0}^{k} a_j \in S^{m_{k+1}}(\mathbb{R}^d).$$

We denote $a \sim \sum a_j$

Remark 3.1. *We can now write simply:*

$$a\#b \sim \sum_{|\alpha|} \frac{1}{i^{|\alpha|}\alpha!}(\partial_\xi^\alpha a(x,\xi))(\partial_x^\alpha b(x,\xi))$$

and

$$a^t \sim \sum_{|\alpha|} \frac{1}{i^{|\alpha|}\alpha!}(\partial_\xi^\alpha \partial_x^\alpha \bar{a}(x,\xi)).$$

Proposition 3.2 (Pseudo-local property)**.** *Let $a \in S^m(\mathbb{R}^d \times \mathbb{R}^d)$ and let K be its kernel. Then K is C^∞ for $x \neq y$. In particular, for all $u \in \mathscr{S}'$:*

$$\text{sing supp}\, a(x,D)u \subset \text{sing supp}\, u$$

Proof. Let $x \neq y$, $\psi, \theta \in C_0^\infty(\mathbb{R}^d)$,$\psi = 1$ near x, $\theta = 1$ near y and supp $\psi \cap$supp $\theta = \emptyset$. Then $\tilde{K}(x,y) = \psi(x)K(x,y)\theta(y)$ is the kernel of the operator $\psi a \theta$. By Theorem 3.2, $\psi a \theta \sim 0$ thus is of order $-\infty$ which finishes the proof.

□

Let Ω be an open subset of $\mathbb{R}^d$. We will now define the notion of local symbols and operators in an open set.

Definition 3.4 (Local operators and symbols)**.** *We define $S^m(\Omega \times \mathbb{R}^d)$ to be the set of $a \in C^\infty(\Omega \times \mathbb{R}^d)$ such that for all multi-orders α, β we have the estimate:*

$$\left|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)\right| \leq C_{\alpha,\beta}(1+|\xi|)^{m-\rho\beta+\sigma\alpha}.$$

$S^m(\Omega^d \times \mathbb{R}^d)$ is a Fréchet space with the topology defined by the family of semi-norms:

$$M_{\beta,\alpha}^m(a) = \sup_{\Omega \times \mathbb{R}^d} \left|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)(1+|\xi|)^{\rho\beta-m-\sigma\alpha}\right|.$$

We define the local spaces:

$$S_{loc}^m(\Omega \times \mathbb{R}^d) = \left\{a \in C^\infty(\Omega \times \mathbb{R}^d), \forall \psi \in C_0^\infty(\Omega), \psi a \in S^m(\Omega \times \mathbb{R}^d)\right\},$$

equipped with its canonical topology giving it a Fréchet space structure.

If $a \in S^m(\Omega \times \mathbb{R}^d)$ or $S_{loc}^m(\Omega \times \mathbb{R}^d)$, the usual formula

$$Au(x) = a(x,D)u(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ix.\xi} a(x,\xi)\hat{u}(\xi)d\xi$$

defines an operator respectively from $\mathscr{S}'(\mathbb{R}^d)$,$\mathscr{E}'(\Omega)$ to $\mathscr{D}'(\Omega)$, which can be restricted to an operator $\mathscr{E}'(\Omega) \to \mathscr{D}'(\Omega)$ and $C_0^\infty(\Omega) \to C^\infty(\Omega)$.
The link between such operators and the operators obtained by cut-off from global operators is given by the following Proposition:

Proposition 3.3. *Let $A : v \to C^\infty(\Omega)$ be a continuous linear operator such that for all $\psi, \theta \in C_0^\infty(\Omega)$, $\psi A \theta \in \mathrm{Op}(S^m)$. Then there exists $a' \in S^m(\Omega \times \mathbb{R}^d)$ with A=a'(x,D)+R, where R is an operator with kernel in $C^\infty(\Omega \times \Omega)$.*

Proof. Let $(\psi_j) \in C_0^\infty(\Omega)$ be a partition of unity locally finite over Ω. Put $\psi_j A \psi_k = A_{jk} \in \mathrm{Op}(S^m)$ then

$$Au = \sum_{j,k} \psi_j A \psi_k = \sum_{\substack{j,k \\ \operatorname{supp} \psi_j \cap \psi_k \neq \emptyset}} A_{jk} + \sum_{\substack{j,k \\ \operatorname{supp} \psi_j \cap \psi_k = \emptyset}} A_{jk}.$$

Then

$$a' = \sum_{\substack{j,k \\ \operatorname{supp} \psi_j \cap \psi_k \neq \emptyset}} A_{jk} \in S^m(\Omega \times \mathbb{R}^d)$$

because for $\forall \psi \in C_0^\infty(\Omega)$, $\psi a'$ is a finite sum by definition of a partition of unity locally finite.
The remainder has a kernel:

$$\sum_{\substack{j,k, \\ \operatorname{supp} \psi_j \cap \psi_k = \emptyset}} \psi_j(x) K(x,y) \psi_k(y) \in C^\infty(\Omega \times \Omega)$$

by the pseudo-local property, Proposition 3.2. □

We see from the previous definition that there is subtlety with the support of the functions if one want for example to define A^t. The following class of local operators clarifies that problem:

Definition 3.5 (Properly supported operators)**.** *A continuous linear operator $A : C_0^\infty(\Omega) \to C^\infty(\Omega)$ is said to be properly supported if, for any compact subset $K \subset \Omega$, there exists a compact subset $K' \subset \Omega$ with:*

$$\operatorname{supp} u \subset K \Longrightarrow \operatorname{supp} Au \subset K' \text{ and } u = 0 \text{ on } K' \Longrightarrow Au = 0 \text{ on } K$$

We see that such an operator maps C_0^∞ to C_0^∞ and for example A^t can be extended in a standard way to an operator from $\mathscr{D}'(\Omega)$ to itself.

Proposition 3.4. *Let $A = a(x,D)$ where $a \in S^m_{loc}(\Omega \times \mathbb{R}^d)$. There exists an operator R with kernel in $C^\infty(\Omega \times \Omega)$ such that $A+R$ is properly supported.*

Proof. This is the same proof as Proposition 3.3 because

$$\sum_{\substack{j,k, \\ \operatorname{supp} \psi_j \cap \psi_k = \emptyset}} A_{jk}$$

is properly supported. □

Remark 3.2. *The previous Proposition tells us that for local regularity considerations we can essentially work with properly supported operators for local symbols (modulo a C^∞ kernel) and by Proposition 3.3 we can do the same for operators obtained by cut-off.*

3.2. **Fourier Integral Operators.** Here we will give basic definitions and results as presented in part 1 of Hörmander's [11].
We wish to define operators of the form :

$$\begin{aligned} A_\omega u(x) &= \int e^{iS(x,\xi)} a(x,\xi) \hat{u}(\xi) d\xi \qquad (3.3) \\ &= \int e^{i(S(x,\xi) - y.\xi)} a(x,\xi) u(y) dy \\ &= \int e^{i\omega(x,y,\xi)} a(x,\xi) u(y) dy d\xi \end{aligned}$$

where u is a regular function, a is a symbol and ω is a given function defining the operator A. We can clearly see that for example $\omega = 0$ the integral in not defined for symbols with $m \geq -d$, we thus start by the following definition of suitable phase functions:

Definition 3.6. *Let $\omega(x, y, \xi)$ be a $C^\infty(\Omega \times \Omega \times \mathbb{R}^d)$ map which is positively homogeneous of degree one with respect to ξ. Put:*

$$R_\omega = \left\{(x, y) \in \Omega \times \Omega, \forall \xi \in \mathbb{R}^d \setminus \{0\}, \omega(x, y, \xi) \text{ has no critical point}\right\}^4,$$

and its compliment C_ω, which is the projection on $\Omega \times \Omega$ of the conic set (with respect to ξ) of:

$$C = \left\{(x, y, \xi) \in \Omega \times \Omega \times \mathbb{R}^d \setminus \{0\}, D\omega_\xi(x, y, \xi) = 0\right\}.$$

- *Then ω is called a phase function on $R_\omega \times \mathbb{R}^d$.*
- *ω is called a non-degenerate phase function if at any point in C, the differentials $D(\frac{\partial \omega}{\partial \xi_j}), j = 1, ..., d$, are linearly independent.*
- *ω is called an operator phase function on $R_\omega \times \mathbb{R}^d$ if for each fixed x (or y) it has no critical point (y, ξ) $(or (x, \xi))$ with $\xi \neq 0$.*
- *For $U \subset \Omega$ define $C_\omega U = \{x, (x, y) \in C_\omega \text{ for some } y \in U\}$.*

The main example here are pseudodifferential operators with $\omega(x, y, \xi) = (x-y).\xi$, in that case C_ω is equal to the diagonal $\{(x, x), x \in \Omega\}$, and we see that all of the previous definitions naturally apply in this case.
The following Proposition will give a definition to the weak form of (3.3):

$$< A_\omega u, v >=< op_\omega(a)u, v >= \int e^{i\omega(x,y,\xi)} a(x, y, \xi)u(y)v(x)dxdyd\xi, \;\; u, v \in C_0^\infty(\Omega). \tag{3.4}$$

Proposition 3.5. *Take a symbol $a \in S^m_{\rho,\sigma}(\Omega \times \Omega \times \mathbb{R}^d), \rho > 0, \sigma < 1$, and a phase function ω on $\Omega \times \Omega \times \mathbb{R}^d$ (i.e $R_\omega = \Omega \times \Omega$). Then:*

(1) The oscillatory integral (3.4) exists and is a continuous bilinear form for the C_0^k topologies on u,v if

$$m - k\rho < -N, \;\; m - k(1 - \sigma) < -N.$$

Thus we obtain a continuous linear map A from $C_0^k(\Omega)$ to $\mathscr{D}'^k(\Omega)$ which has a distribution kernel $K_\omega \in \mathscr{D}'^k(\Omega \times \Omega)$ given by the oscillatory integral

$$K_\omega(u) = \int e^{i\omega(x,y,\xi)} a(x, y, \xi)u(x, y)dxdyd\xi, \;\; u \in C_0^\infty(\Omega \times \Omega).$$

(2) If ω has no critical point (y, ξ) for each fixed x, then (3.3) is defined as an oscillatory integral and we obtain a continuous map $A : C_0^k(\Omega) \to C(\Omega)$. By differentiation under the integral sign it follows that A is also continuous map from $C_0^k(\Omega)$ to $C^j(\Omega)$ if

$$m - k\rho < -N - j, \;\; m - k(1 - \sigma) < -N - j.$$

(3) If ω has no critical point (x, ξ) for each fixed y, then the adjoint of A is defined and has the properties listed in point 2, so A is a continuous map of $\mathscr{E}'^j(\Omega)$ into $\mathscr{D}'^k(\Omega)$. In particular A defines a continuous map from $\mathscr{E}'(\Omega)$ into $\mathscr{D}'(\Omega)$.

[4] R_ω is clearly open.

(4) The oscillatory integral:

$$K_\omega(x,y) = \int e^{i\omega(x,y,\xi)} a(x,\xi) d\xi \text{ defines a } C^\infty(\Omega \times \Omega = R_\omega) \text{ map,}$$

it follows that A is an integral operator with C^∞ kernel, so A is a continuous map of $\mathscr{E}'(\Omega)$ to $C^\infty(\Omega)$.

(5) We have the generalization of the pseudo-local property:

$$\text{sing supp } op_\omega(a)u = C_\omega \, \text{sing supp}\, u.$$

When ω is an operator phase function it verifies all the previous properties.

Proposition 3.6. *Let $\omega(x,y,\xi)$ be a $C^\infty(\Omega\times\Omega\times\mathbb{R}^d)$ map which is positively homogeneous of degree one with respect to ξ and a be a symbol in $S^m_{\rho,\sigma}(\Omega\times\Omega\times\mathbb{R}^d)$,$\rho>\sigma$ and that either ω is linear or that $\rho+\sigma=1$. Suppose that a vanishes of infinite order on C then we have the same results as in the previous Proposition with m replaced by $m-\rho+\sigma$.*
If a just vanishes on C then we can find $b \in S^{m-\delta+\rho}_{\rho,\sigma}(\Omega\times\Omega\times\mathbb{R}^d)$ such that we have the formal equality $op_\omega(a)u = op_\omega(b)u$.

As Hörmander summed up, when ω is non degenerate the singularities of the distribution $u \to op_\omega(a)u$ only depend on the Taylor expansion of a on the set C.

The following Proposition, taken from part 2 of [11], gives the natural link between pseudodifferential operators and Fourier Integral operators defined by the phase function $\omega(x,y,\xi) = (x-y).\xi$.

Proposition 3.7. *Consider a real number m and a symbol $c \in S^m(\Omega\times\Omega\times\mathbb{R}^d)$, then:*

$$a(x,\xi) = \int_{\Omega\times\mathbb{R}^d} c(x,y,\eta) e^{i(x-y).(\eta-\xi)} dy d\eta \in S^m(\Omega\times\mathbb{R}^d)$$

and we have:

$$\forall u \in C_0^\infty(\Omega), op_{(x-y).\xi}(c)u = \mathrm{Op}(a)u = (2\pi)^{-d}\int_{\mathbb{R}^d} e^{ix.\xi} a(x,\xi)\hat{u}(\xi)d\xi.$$

Moreover the asymptotic expansion of a is given by:

$$\forall N \in \mathbb{N}, a(x,\xi) - \sum_{|\alpha|<N} \partial_\xi^\alpha \partial_y^\alpha c(x,y,\xi)_{|y=x} \in S^{m-N}(\Omega\times\mathbb{R}^d).$$

Remark 3.3. *In the previous setting c is often called an amplitude.*

We will not give the proof of these Propositions here but we will present the fundamental Lemma behind those results and the idea behind it. The main problem is to define oscillatory integrals of the form:

$$\int e^{i\omega(x,\xi)} a(x,\xi)u(x) dx d\xi, \;\; u \in C_0^\infty(\Omega),$$

We start by remarking that the integral is absolutely convergent if a is of order $m < -N$.

Lemma 3.1. *If ω has no critical point (x,ξ) with $\xi \neq 0$, then one can find a first order differential operator*

$$L = \sum_j h_j \frac{\partial}{\xi_j} + \tilde{h}_j \frac{\partial}{x_j} + c$$

with $h_j \in S^0(\Omega\times\mathbb{R}^d)$ and $\tilde{h}_j, c \in S^{-1}(\Omega\times\mathbb{R}^d)$ such that $L^t e^{i\omega} = e^{i\omega}$.
L is a continuous map from $S^m_{\rho,\sigma}(\Omega\times\Omega\times\mathbb{R}^d)$ to $S^{m-\epsilon}_{\rho,\sigma}(\Omega\times\Omega\times\mathbb{R}^d)$ where $\epsilon = min(\rho, 1-\sigma)$.

Taking a symbol a of order m we compute:

$$\int e^{i\omega(x,\xi)}a(x,\xi)u(x)dxd\xi = \int e^{i\omega(x,\xi)}La(x,\xi)u(x)dxd\xi$$
$$= \int e^{i\omega(x,\xi)}L^k a(x,\xi)u(x)dxd\xi,$$

under the hypothesis $\rho > 0$ and $\sigma < 1$ we have $\epsilon > 0$ and $L^k a \in S^{m-k\epsilon}_{\rho,\sigma}(\Omega\times\Omega\times\mathbb{R}^d)$, taking $m - k\epsilon < -N$ and applying the previous remark we see that the integral is then well defined.

3.3. **Paradifferential Calculus.** We start by the definition of symbols with limited spatial regularity. Let $\mathscr{W} \subset \mathscr{S}'$ be a Banach space.

Definition 3.7. *Given $\rho \geq 0$ and $m \in \mathbb{R}$, $\Gamma^m_{\mathscr{W}}(\mathbb{R}^d)$ denotes the space of locally bounded functions $a(x,\xi)$ on $\mathbb{R}^d \times (\mathbb{R}^d \setminus 0)$, which are C^∞ with respect to ξ for $\xi \neq 0$ and such that, for all $\alpha \in \mathbb{N}^d$ and for all $\xi \neq 0$, the function $x \mapsto \partial^\alpha_\xi a(x,\xi)$ belongs to $\mathscr{W}$ and there exists a constant C_α such that,*

$$\forall\, |\xi| > \frac{1}{2},\ \left\|\partial^\alpha_\xi a(.,\xi)\right\|_{\mathscr{W}} \leq C_\alpha(1+|\xi|)^{m-|\alpha|}$$

Given a symbol a, define the paradifferential operator T_a by

$$\widehat{T_a u}(\xi) = (2\pi)^{-d}\int_{\mathbb{R}^d} \psi(\xi-\eta,\eta)\hat{a}(\xi-\eta,\eta)\hat{u}(\eta)d\eta,$$

where $\hat{a}(\eta,\xi) = \int e^{-ix.\eta}a(x,\xi)dx$ is the Fourier transform of a with respect to the first variable; ψ is a fixed C^∞ function such that:

(1) there are ϵ_1 and ϵ_2 such that $0 < \epsilon_1 < \epsilon_2 < 1$ and

$$\begin{cases} \psi(\xi,\eta) = 1 \text{ for } |\xi| \leq \epsilon_1(1+|\eta|), \\ \psi(\xi,\eta) = 0 \text{ for } |\xi| \geq \epsilon_2(1+|\eta|). \end{cases}$$

(2) for all $(\alpha,\beta) \in \mathbb{N}^d \times \mathbb{N}^d$, there is C_{α_β} such that

$$\forall (\xi,\eta) : \left|\partial^\alpha_\xi \partial^\beta_\eta \psi(\xi,\eta)\right| \leq C_{\alpha,\beta}(1+|\xi|)^{-|\alpha|-|\beta|}.$$

Such a ψ is called an admissible cut-off function.

For quantitative estimates we introduce as in [14]:

Definition 3.8. *For $m \in \mathbb{R}$, $\rho \geq 0$ and $a \in \Gamma^m_{\mathscr{W}}(\mathbb{R}^d)$, we set*

$$M^m_{\mathscr{W}}(a) = \sup_{|\alpha|\leq \frac{d}{2}+1+c}\ \sup_{|\xi|\geq\frac{1}{2}} \left\|(1+|\xi|)^{m-|\alpha|}\partial^\alpha_\xi a(.,\xi)\right\|_{\mathscr{W}},$$

where $c \geq 0$ is a constant fixed by a choice that generally depends on $\mathscr{W}$. We will essentially work with $\mathscr{W} = W^{\rho,\infty}$ and write $\Gamma^m_{\mathscr{W}} = \Gamma^m_\rho$ and $M^m_{W^{\rho,\infty}}(a) = M^m_\rho(a)$ with $c = \rho$.

The main features of symbolic calculus for paradifferential operators given by the following Theorems.

Theorem 3.3. *Take $m \in \mathbb{R}$. if $a \in \Gamma^m_0(\mathbb{R}^d)$, then T_a is of order m. Moreover, for all $\mu \in \mathbb{R}$ there exists a constant K such that*

$$\|T_a\|_{H^\mu \to H^{\mu-m}} \leq K M^m_0(a).$$

Theorem 3.4. *Take $m, m' \in \mathbb{R}$, and $\rho > 0$, $a \in \Gamma^m_\rho(\mathbb{R}^d)$and $b \in \Gamma^{m'}_\rho(\mathbb{R}^d)$.*

- *Composition: Then T_aT_b is a paradifferential operator of order $m+m'$ and $T_aT_b - T_{a\#b}$ is of order $m+m'-\rho$ where $a\#b$ is defined by:*
$$a\#b = \sum_{|\alpha|<\rho} \frac{1}{i^{|\alpha|}\alpha!}\partial_\xi^\alpha a \partial_x^\alpha b$$
Moreover, for all $\mu \in \mathbb{R}$ there exists a constant K such that
$$\|T_aT_b - T_{a\#b}\|_{H^\mu \to H^{\mu-m-m'+\rho}} \leq K M_\rho^m(a) M_\rho^{m'}(b).$$
- *Adjoint: The adjoint operator of T_a, that we will note T_a^t to again avoid confusion with the pull back operator defined in this work, is a paradifferential operator of order m with symbol a^t defined by:*
$$a^t = \sum_{|\alpha|<\rho} \frac{1}{i^{|\alpha|}\alpha!}\partial_\xi^\alpha \partial_x^\alpha \bar{a}$$
Moreover, for all $\mu \in \mathbb{R}$ there exists a constant K such that
$$\left\|T_a^t - T_{a^t}\right\|_{H^\mu \to H^{\mu-m+\rho}} \leq K M_\rho^m(a).$$

If $a = a(x)$ is a function of x only, the paradifferential operator T_a is called a para-product. With a good choice of (ϵ_1, ϵ_2) in the definition of the cut-off function with respect to our choice of the dyadic decomposition of unity in the Littlewood-Paley decomposition we get that when $a = a(x)$, T_a takes the usual form:
$$T_a u = \sum_{k=1}^{\infty} \Phi_{k-1} a u_k.$$
It follows from Theorem 3.4 and the Sobolev embeddings that:

- If $a \in H^\alpha(\mathbb{R}^d)$ and $b \in H^\beta(\mathbb{R}^d)$ with $\alpha, \beta > \frac{d}{2}$, then
$$T_aT_b - T_{ab} \text{ is of order } -\left(min\left\{\alpha,\beta\right\} - \frac{d}{2}\right).$$
- If $a \in H^\alpha(\mathbb{R}^d)$ with $\alpha > \frac{d}{2}$, then
$$T_a^t - T_{a^t} \text{ is of order } -\left(\alpha - \frac{d}{2}\right).$$

An important feature of para-products is that they are well defined for function $a = a(x)$ which are not L^∞ but merely in some Sobolev spaces H^r with $r < \frac{d}{2}$.

Proposition 3.8. *Take $m > 0$. If $a \in H^{\frac{d}{2}-m}(\mathbb{R}^d)$ and $u \in H^\mu(\mathbb{R}^d)$ then $T_a u \in H^{\mu-m}(\mathbb{R}^d)$. Moreover,*
$$\|T_a u\|_{H^{\mu-m}} \leq K \|a\|_{H^{\frac{d}{2}-m}} \|u\|_{H^\mu}$$

A main feature of para-products is the existence of para-linearization Theorems which allow us to replace nonlinear expressions by paradifferential expressions, at the price of error terms which are smoother than the main terms.

Theorem 3.5. *Let $\alpha, \beta \in \mathbb{R}$ be such that $\alpha, \beta > \frac{d}{2}$, then*

- *Bony's Linearization Theorem: for all C^∞ function F, if $a \in H^\alpha(\mathbb{R}^d)$ then*
$$F(a) - F(0) - T_{F'(a)} a \in H^{2\alpha - \frac{d}{2}}(\mathbb{R}^d).$$
- *If $a \in H^\alpha(\mathbb{R}^d)$ and $b \in H^\beta(\mathbb{R}^d)$, then $ab - T_a b - T_b a \in H^{\alpha+\beta-\frac{d}{2}}(\mathbb{R}^d)$. Moreover there exists a positive constant K independent of a and b such that:*
$$\|ab - T_a b - T_b a\|_{H^{\alpha+\beta-\frac{d}{2}}} \leq K \|a\|_{H^\alpha} \|b\|_{H^\beta}.$$

3.4. **Link between pseudodifferential and paradifferential symbols.** As presented in [14] the link between paradifferential operators and pseudodifferential operators will be as follows, for a paradifferential operator $a \in \Gamma_\rho^{m'}(\mathbb{R}^d)$ we can associate a symbol $\sigma_a \in S^m_{1,1}(\mathbb{R}^d \times \mathbb{R}^d)$. All of the results presented above were for the class $S^m_{1,0} \subset S^m_{1,1}$ and don't all generalize easily. To remedy this, the idea in paradifferential calculus is regularization by cut-off in the frequency domain thus σ_a will have an extra spectral localization property that will give them the desired properties as in $S^m_{1,0}$.

Definition-Proposition 3.1. *Take $m \in \mathbb{R}$, $\Sigma^m_{\mathscr{W}}(\mathbb{R}^d)$ denotes the subclass of symbols $\sigma \in \Gamma^m_{\mathscr{W}}(\mathbb{R}^d)$ which satisfy the following spectral condition*

$$\exists \epsilon < 1, \mathscr{F}_x\sigma(\xi,\eta) = 0 \text{ for } |\xi| > \epsilon(|\eta|+1). \tag{3.5}$$

When $\mathscr{W} = W^{r,\infty}(\Omega)$ we write $\Sigma^m_{\mathscr{W}}(\mathbb{R}^d) = \Sigma^m_r(\mathbb{R}^d)$.
When $\mathscr{W} \subset L^\infty(\mathbb{R}^d), \Gamma^m_{\mathscr{W}}(\mathbb{R}^d) \subset \Gamma^m_0(\mathbb{R}^d)$ and $\Sigma^m_{\mathscr{W}}(\mathbb{R}^d) \subset \Sigma^m_0(\mathbb{R}^d)$. Moreover, by the Bernstein inequalities (2.1), $\Sigma^m_0(\mathbb{R}^d) \subset S^m_{1,1}(\mathbb{R}^d)$. More generally, the spectral condition implies that symbols in $\Sigma^m_{\mathscr{W}}(\mathbb{R}^d)$ are smooth in x too.

Remark 3.4. *The interesting fact now is $\Sigma^m_0(\mathbb{R}^d)$ still enjoys all of the symbolic calculus properties announced above for $S^m_{1,0}(\mathbb{R}^d)$.*

Definition-Proposition 3.2. *(Regularization of a symbol) Take $m \in \mathbb{R}$, $a \in \Gamma^m_{\mathscr{W}}$ and ψ an admissible cut-off function. Define σ^ψ_a:*

$$\mathscr{F}_x\sigma_a(\xi,\eta) = \psi(\xi,\eta)\mathscr{F}_x a(\xi,\eta),$$

thus $\mathrm{Op}(\sigma^\psi_a) = T_a$. Moreover for $r \geq 0$, when $\mathscr{W} = W^{r,\infty}(\Omega)$ we have the following properties:

(1) This association is bounded:

$$M^m_r(\sigma^\psi_a) \leq C M^m_r(a).$$

(2) $a - \sigma^\psi_a \in \Gamma^m_0$ and:

$$M^{m-r}_0(\sigma^\psi_a - a) \leq C M^m_r(a).$$

In particular, if ψ_1 and ψ_2 are two admissible cut-off functions then the difference $\sigma^{\psi_1}_a - \sigma^{\psi_2}_a$ belongs to Σ^{m-r}_0 and:

$$M^{m-r}_0(\sigma^{\psi_1}_a - \sigma^{\psi_2}_a) \leq C M^m_r(a).$$

Now we list a couple of important calculus properties to the association $a \mapsto \sigma^\psi_a$. For the following Proposition we consider ψ fixed and drop it from the notations.

Proposition 3.9.

- *For $m \in \mathbb{R}, r \geq 0, \alpha \in \mathbb{N}^d$ of length $|\alpha| \leq r$ and $a \in \Gamma^m_r$:*

$$\partial^\alpha_x \sigma_a = \sigma_{\partial^\alpha_x a} \in \Sigma^m_0.$$

- *For $m \in \mathbb{R}, r \geq 0$ and $\alpha \in \mathbb{N}^d$ of length $|\alpha| \geq r$ the mapping $a \mapsto \partial^\alpha_x \sigma_a$ is bounded from Γ^m_r to $\Sigma^{m+|\alpha|-r}$, more precisely:*

$$M^{m+|\alpha|-r}(\partial^\alpha_x \sigma_a) \leq M^m_r(a).$$

- *For $m \in \mathbb{R}, r \geq 0, \beta \in \mathbb{N}^d$ and $a \in \Gamma^m_r$*

$$\partial^\beta_\xi \sigma_a - \sigma_{\partial^\beta_\xi a} \in \Sigma^{m-|\beta|-r}.$$

From [14] we give an approximation of symbols in $\Sigma^m_0(\mathbb{R}^d)$ by symbols in the Schwartz class.

Lemma 3.2. *For all $\sigma \in \Sigma^m_0$, there is a sequence of symbols $\sigma_n \in \mathscr{S}(\mathbb{R}^d \times \mathbb{R}^d)$ such that*

(1) the family $\{\sigma_n\}$ is bounded in $S^m_{1,1}$,
(2) the σ_n satisfy the spectral condition (3.5) for some $\epsilon < 1$ independent of k,
(3) $\sigma_n \to \sigma$ on compact subsets of $\mathbb{R}^d \times \mathbb{R}^d$.

In order to give the link between Paradifferential calculus and Fourier Integral Operators we start by defining the space of amplitudes for Paradifferential operators.

Definition-Proposition 3.3. *Take $m \in \mathbb{R}$, $A^m_{\mathscr{W}}(\mathbb{R}^d)$ denotes the subclass of symbols $c \in \Gamma^m(\mathscr{W} \times \mathscr{W} \times \mathbb{R}^d)$ which satisfy the following spectral condition*

$$\exists \epsilon < 1, \mathscr{F}_{x,y} c(\xi, \zeta, \eta) = 0 \text{ for } |\xi - \zeta| > \epsilon(|\eta| + 1) \text{ or } |\zeta| > \epsilon(|\eta| + 1). \tag{3.6}$$

When $\mathscr{W} = W^{r,\infty}(\Omega)$ we write $A^m_{\mathscr{W}}(\mathbb{R}^d) = A^m_r(\mathbb{R}^d)$.
By the Bernstein inequalities (2.1), $A^m_0(\mathbb{R}^d) \subset S^m_{1,1}(\mathbb{R}^d)$. More generally, the spectral condition implies that symbols in $A^m_{\mathscr{W}}(\mathbb{R}^d)$ are smooth in x, y too.

Proposition 3.10. *Consider two real numbers $m \in \mathbb{R}$, $r \in \mathbb{R}_+$ and an amplitude $c \in A^m_r(\mathbb{R}^d)$, then:*

$$\sigma(x, \xi) = \int_{\Omega \times \mathbb{R}^d} c(x, y, \eta) e^{i(x-y).(\eta - \xi)} dy d\eta \in \Sigma^m_r(\mathbb{R}^d)$$

and we have:

$$\forall u \in C^\infty_0(\Omega), op_{(x-y).\xi}(c)u = \mathrm{Op}(\sigma)u = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ix.\xi} \sigma(x, \xi) \hat{u}(\xi) d\xi.$$

Moreover the asymptotic expansion of a is given by:

$$\sigma(x, \xi) - \sum_{|\alpha| < N} \partial^\alpha_\xi \partial^\alpha_y c(x, y, \xi)_{|y=x} \in \Sigma^{m-N}_{r-N}(\mathbb{R}^d).$$

Proof. First by Lemma 3.2 we can work with an amplitude c in $\mathscr{S}$. As $\mathscr{S} \subset S^m_{1,0}$ by Proposition 3.7 we have

$$\sigma(x, \xi) = \int_{\mathbb{R}^d \times \mathbb{R}^d} c(x, y, \eta) e^{i(x-y).(\eta - \xi)} dy d\eta \in \mathscr{S}.$$

Moreover writing

$$\mathscr{F}_x \sigma(\xi, \eta) = \int_{\mathbb{R}^d} c(\xi + \eta - \tilde{\eta}, \tilde{\eta} - \eta, \tilde{\eta}) d\tilde{\eta},$$

we see that if c verifies the spectral condition with parameter $\frac{\epsilon}{2}$ then so does σ with parameter ϵ thus $\sigma \in \Sigma^m_r(\mathbb{R}^d)$. The asymptotic expansion comes from the one given in Proposition 3.7 combined by the symbolic calculus rules in Proposition 3.9. $\square$

4. Pull-back of pseudo and para- differential operators

Let Ω, Ω' be two open subsets of $\mathbb{R}^d$. Henceforth we will note all variables in Ω' with a $'$ for clarity in the computations. Let $\chi : \Omega \to \Omega'$ be a C^∞ map, χ gives rise naturally to the pull back operation for functions and kernels:

$$\begin{aligned} &C^\infty(\Omega') \to C^\infty(\Omega) && C^\infty(\Omega' \times \Omega') \to C^\infty(\Omega \times \Omega) \\ &v \mapsto v \circ \chi = v^* && K(x', y') \mapsto K(\chi(x), \chi(y)) |\det D\chi(y)| = K^*(x, y). \end{aligned}$$

This Pull back has the property:

$$\begin{aligned} K^* v^* &= \int_\Omega K(\chi(x), \chi(y)) v(\chi(y)) |\det D\chi(y)| dy \\ &= \int_{\Omega'} K(\chi(x), y') v(y') \# \chi^{-1}(y') dy' = (K(v \# \chi^{-1}))^*. \end{aligned} \tag{4.1}$$

Where $\#\chi^{-1} : \Omega' \to \bar{\mathbb{N}}$ is the function counting the number if pre-images and $v \in C^\infty_0(\Omega')$. We note that the the change of variables is well defined if and only

if one of the two integrals is defined. If χ is a diffeomorphism we have the usual fonctorial property $K^*v^* = (Kv)^*$ which permits the definition of operators with kernels on manifolds.
The classic result on the change of variables in pseudo-differential operators is that for $A \in S^m_{loc}(\Omega' \times \mathbb{R}^d)$ properly supported with kernel K then the operator defined by K^* is a pseudo-differential operator A^* of order m on Ω which is also properly supported. Thus it can be seen as the stability of this sub-class of operators of kernels under the pull back by diffeomorphisms (modulo a C^∞ kernel as in Remark 3.2) and thus are well defined on manifolds by the same process. Before we start by presenting those classic results we will discuss why they are essentially optimal.

We start by computing for a pseudo-differential operator defined by $a \in S^m(\Omega' \times \mathbb{R}^d)$ with kernel K and $\chi : \Omega \to \Omega'$ a C^∞ map:

$$\begin{aligned} K^*u &= \int_\Omega K(\chi(x), \chi(y))u(y)|\det D\chi(y)|dy \\ &= \int_{\Omega\times\Omega} (2\pi)^{-d} e^{i(\chi(x)-\chi(y)).\xi} a(\chi(x), \xi)u(y)|\det D\chi(y)|dyd\xi \end{aligned}$$

thus

$$K^* = op_{(\chi(x)-\chi(y)).\xi}(a(\chi(x), \xi)|\det D\chi(y)|)$$

with

$$a(\chi(x), \xi)|\det D\chi(y)| \in S^m(\Omega \times \Omega \times \mathbb{R}^d),$$

because and all the derivatives of χ are bounded. Put

$$\omega_\chi(x, y, \xi) = (\chi(x) - \chi(y)).\xi,$$

by the definitions on Fourier integral operators we have:

$$C_{\omega_\chi} = \left\{(x, y) \in \Omega^2, \chi(x) = \chi(y)\right\}.$$

We also see that w_χ is non degenerate on $\Omega \times \Omega$ if and only if χ is a local diffeomorphism. To sum up:

Proposition 4.1. *Take $a \in S^m(\Omega' \times \mathbb{R}^d)$ and $\chi \in C^\infty(\Omega, \Omega')$. Then the pull-back of $\mathrm{Op}(a)$ under χ is a Fourier Integral Operator with phase function w_χ and symbol $a(\chi(x), \xi)|\det D\chi(y)| \in S^m(\Omega \times \Omega \times \mathbb{R}^d)$. We have:*

$$C_{\omega_\chi} = \left\{(x, y) \in \Omega^2, \chi(x) = \chi(y)\right\}.$$

Moreover, w_χ is non-degenerate if and only of χ is a local diffeomorphism.

Now we ask the question if there exists a symbol a^* such that:

$$op_{\omega_\chi}(a(\chi(x), \xi)|\det D\chi(y)|) = \mathrm{Op}(a^*).$$

The classic result is that this true if χ is a diffeomorphism. Now we precise that it's essentially optimal as it could be seen by the following two examples:

- The necessity of the injectivity of ξ: we take $\chi = |\ |$ which is a local diffeomorphism from $\mathbb{R} \setminus 0$ in to $\mathbb{R}^+_*$. We compute for $A = Id$ i.e $a = 1$:

  $$op_{\omega_\chi}(a(\chi(x), \xi)|\det D\chi(y)|)u = u(x) + u(-x),$$

  and the part $u(.) \mapsto u(-.)$ is not a pseudo-differential operator.
- The necessity of the local diffeomorphism hypothesis: we take $\chi = x^3$ which is a local diffeomorphism from $\mathbb{R} \setminus 0$ in to $\mathbb{R}$. We compute for $A = \frac{d}{dx}$ i.e $a = i\xi$:

  $$op_{\omega_\chi}(a(\chi(x), \xi)|\det D\chi(y)|)u = \frac{u'(x)}{3x^2},$$

which is a pseudo-differential operator on $\mathbb{R} \setminus 0$ but cannot be extended to one on $\mathbb{R}$ with a regular symbol in 0. [5]

Now we present the classic results of change of variables in pseudo and para-differential operators under the hypothesis that χ is a diffeomorphism as they can be found in [3],[4] and [11].

Theorem 4.1. *Let $\chi : \Omega \to \Omega'$ be a C^∞ diffeomorphism and $A = a(x, D) \in S^m_{loc}(\Omega' \times \mathbb{R}^d)$ a properly supported pseudo-differential operator with kernel K. Then the operator A^* defined by K^* i.e:*

$$\forall u \in C_0^\infty(\Omega), A^* u = \int_\Omega K(\chi(x), \chi(y)) u(y) |\det D\chi(y)| dy$$

is a properly supported pseudo-differential operator with symbol

$$a^*(x, \xi) = (-1)^d e^{-ix.\xi} \int_{\Omega \times \mathbb{R}^d} a(\chi(x), \eta) e^{i(\chi(x) - \chi(y)).\eta + iy.\xi} |\det D\chi(y)| dy d\eta \in S^m_{loc}(\Omega \times \mathbb{R}^d).$$

An expansion of a^* is given by:

$$a^*(x, \xi) \sim \sum_\alpha \frac{1}{\alpha} \partial^\alpha a(\chi(x), D\chi^{-1}(\chi(x))^t \xi) P_\alpha(\chi(x), \xi), \tag{4.2}$$

where,

$$P_\alpha(x', \xi) = D^\alpha_{y'} (e^{i(\chi^{-1}(y') - \chi^{-1}(x') - D\chi^{-1}(x')(y' - x')).\xi})_{|y' = x'}$$

and P_α is polynomial in ξ of degree $\leq \frac{|\alpha|}{2}$, with $P_0 = 1, P_1 = 0$.

Remark 4.1. *This a classic result found commonly in the literature, And as in the Remark 3.4 an analogous result still holds in the class Σ^m_0 as will be shown in the proof of the next Theorem.*

For para-differential operators we have:

Theorem 4.2. *Let $\chi : \Omega \to \Omega'$ be a $W^{1+\rho, \infty}_{loc}$ diffeomorphism with $D\chi \in W^{\rho, \infty}$ and $\rho \geq 0$. Consider $a \in \Gamma^m_r(\mathbb{R}^d)$ a properly supported paradifferential operator. Then there exists a property supported $a^* \in \Gamma^m_{min(r,\rho)}(\mathbb{R}^d)$ defined by:*

$$(T_a u) \circ \chi = T_{a^*}(u \circ \chi) + (R\chi) u,$$

where $R \in \Gamma^m_r(\mathbb{R}^d)$ and $R\chi$ is a term depending essentially on χ and it's explicit formula is given in (5.4).

Moreover a^* has the local expansion:

$$a^*(x, \xi) \sim \sum_{\substack{\alpha \\ |\alpha| \leq \lfloor min(r, \rho) \rfloor}} \frac{1}{\alpha} \partial^\alpha a(\chi(x), D\chi^{-1}(\chi(x))^t \xi) P_\alpha(\chi(x), \xi), \tag{4.3}$$

where,

$$P_\alpha(x', \xi) = D^\alpha_{y'} (e^{i(\chi^{-1}(y') - \chi^{-1}(x') - D\chi^{-1}(x')(y' - x')).\xi})_{|y' = x'}$$

and P_α is polynomial in ξ of degree $\leq \frac{|\alpha|}{2}$, with $P_0 = 1, P_1 = 0$.

An analogous result still holds for para-differential operators modeled on the spaces $a \in C^r_*, r > 0$ and $\chi \in C^{1+\rho}_*$.

As we couldn't find a clear reference to this result in the literature, it is eluded to in [3][6], we give a simple proof of this Theorem.

[5] In fact it can be treated in the more general frame of operators with singular symbols but this goes beyond the scope of this work.

[6] part 3.3 point h, which can be found in pages 114-115.

Proof. Taking ψ a cut-off function with parameters ϵ_1, ϵ_2, and take $u \in C_0^\infty(\Omega)$ compute

$$\begin{aligned}(T_a(u\circ\chi^{-1}))\circ\chi &= op_{(\chi(x)-\chi(y)).\xi}(\sigma_a^\psi(\chi(x),\xi)\,|\det D\chi(y)|)u\\ &= \int_{\Omega\times\mathbb{R}^d} e^{i(\chi(x)-\chi(y))\cdot\xi}\sigma_a^\psi(\chi(x),\xi)\,|\det D\chi(y)|\,u(y)dyd\xi.\end{aligned}$$

As we remarked above the main contribution in this integral will come from $(x, y, \xi) \in C_{\omega_\chi}$ where we recall $\omega_\chi(x, y, \xi) = (\chi(x) - \chi(y)) \cdot \xi$. To show this insert the smooth cut-off function $\theta(x, y)$ supported in a small neighborhood of the diagonal (x, x).

$$\begin{aligned}(T_a(u\circ\chi^{-1}))\circ\chi &= \int_{\Omega\times\mathbb{R}^d} e^{i(\chi(x)-\chi(y))\cdot\xi}\sigma_a^\psi(\chi(x),\xi)\,|\det D\chi(y)|\,u(y)dyd\xi\\ &= \int_{\Omega\times\mathbb{R}^d} e^{i(\chi(x)-\chi(y))\cdot\xi}\theta(x,y)\sigma_a^\psi(\chi(x),\xi)\,|\det D\chi(y)|\,u(y)dyd\xi\\ &+ \int_{\Omega\times\mathbb{R}^d} e^{i(\chi(x)-\chi(y))\cdot\xi}(1-\theta(x,y))\sigma_a^\psi(\chi(x),\xi)\,|\det D\chi(y)|\,u(y)dyd\xi\end{aligned}$$

Now ω_χ has no critical points on the support of $(1 - \theta(x, y))$ and by integration by parts we have:

$$(T_a(u\circ\chi^{-1}))\circ\chi = \int_{\Omega\times\mathbb{R}^d} e^{i(\chi(x)-\chi(y))\cdot\xi}\theta(x,y)\sigma_a^\psi(\chi(x),\xi)\,|\det D\chi(y)|\,u(y)dyd\xi + Ru.$$

with $R \in \Gamma_0^{m-\min(r,\rho)}$. We now analyze when y is close to x. By the mean value Theorem, for y sufficiently close to x, there exists a invertible linear mapping $L_{x,y} \in W^{\rho,\infty}$ such that

$$\begin{cases}\chi(x)-\chi(y) = L_{x,y}\cdot(x-y)\\ L_{x,x} = D\chi(x).\end{cases}$$

Thus we get,

$$\begin{aligned}&(T_a(u\circ\chi^{-1}))\circ\chi\\ &= \int_{\Omega\times\mathbb{R}^d} e^{i(\chi(x)-\chi(y))\cdot\xi}\theta(x,y)\sigma_a^\psi(\chi(x),\xi)\,|\det D\chi(y)|\,u(y)dyd\xi + Ru\\ &= \int_{\Omega\times\mathbb{R}^d} e^{i(x-y)\cdot\xi}\theta(x,y)\sigma_a^\psi(\chi(x),{L_{x,y}^t}^{-1}\xi)\,|\det D\chi(y)|\,\left|\det L_{x,y}^{-1}\right|u(y)dyd\xi + Ru.\end{aligned}$$

We get an operator with an amplitude

$$c(x,y,\xi) = \theta(x,y)\sigma_a^\psi(\chi(x),{L_{x,y}^t}^{-1}\xi)\,|\det D\chi(y)|\,\left|\det L_{x,y}^{-1}\right| \in \Gamma_\rho^m(\mathbb{R}^d).$$

In the frequency domain this amplitude depends on terms coming from $\sigma_a^\psi(\chi(x),{L_{x,y}^t}^{-1}\xi)$, $|\det D\chi(y)|$ and $\left|\det L_{x,y}^{-1}\right|$. Putting all of the high frequency terms depending on χ and χ^{-1} in a term $R\chi$ by defining $\tilde{\psi}$ as the cut-off function in both of the variables $(x - y, y)$ with parameters:

$$c = \min(1, \sup D\chi^{-1}, \sup D\chi),\ \tilde{\epsilon_1} = \frac{\epsilon_1}{c},\ \tilde{\epsilon_2} = \frac{\epsilon_2}{c}.$$

Thus by Proposition 3.2

$$\tilde{c}(x,y,\xi) = \tilde{\psi}(D,\cdot)c \in \Sigma^m_{min(r,\rho)}(\mathbb{R}^d),$$

with

$$c = \tilde{c} + R\chi + R'$$

and $R' \in \Gamma_0^{m-min(r,\rho)}$.
The result then follows from Proposition 3.10. $\square$

5. Paracomposition

5.1. Main results for paracomposition on $\mathbb{R}^d$. We start by a formal computation, as in [18], using the Littlewood-Paley decomposition and two functions u and χ:

$$\begin{aligned} u \circ \chi &= \sum_{k \geq 0} u(\Phi_{k+1}\chi) - u(\Phi_k\chi) = \sum_{j,k} u_j(\Phi_{k+1}\chi) - u_j(\Phi_k\chi) \\ &= \sum_{j<k} u_j(\Phi_{k+1}\chi) - u_j(\Phi_k\chi) + \sum_{j \geq k} u_j(\Phi_{k+1}\chi) - u_j(\Phi_k\chi) \\ &= \underbrace{\sum_{k \geq 1} \Phi_{k-1}u(\Phi_k\chi) - \Phi_{k-1}u(\Phi_{k-1}\chi)}_{1} + \underbrace{\sum_{k \geq 0} u_k(\Phi_k\chi)}_{2}. \end{aligned} \tag{5.1}$$

Remark 5.1. *Heuristically the term 1 has frequencies of u smaller than that of χ and as in classical paradifferential results will depend mainly on the regularity of χ. This is indeed the main term in Bony's para-linearization Theorem modulo a more regular remainder:*

$$\begin{aligned} (1) &= \sum_{k \geq 1} \left(\int_0^1 \Phi_{k-1}u'(\tau\Phi_k\chi + (1-\tau)\Phi_{k-1})\chi d\tau \right) \phi_k\chi \\ &= \underbrace{\sum_{k \geq 1} \Phi_{k-1}(u' \circ \chi)(\phi_k\chi)}_{T_{u'\circ\chi}\chi} \\ &+ \underbrace{\sum_{k \geq 1} \left(\int_0^1 \Phi_{k-1}u'(\tau\Phi_k\chi + (1-\tau)\Phi_{k-1}\chi) - \Phi_{k-1}(u' \circ \chi) d\tau \right) \phi_k\chi}_{R_0}. \end{aligned} \tag{5.2}$$

Same as term 1, heuristically term 2 will essentially depend on the regularity of u, with a remainder depending on χ and u that is more regular when it's well defined. Thus (2) will naturally give rise to the paracomposition operator. To better understand it, let us suppose just for the next computation that χ is linear and invertible:

$$\begin{aligned} (2) &= \sum_{k \geq 0} \int_{\mathbb{R}^d} \phi_k(\xi)\hat{u}(\xi)e^{i\Phi_k\chi(x).\xi}d\xi \\ &= \sum_{k \geq 0} \int_{\mathbb{R}^d} \phi_k(\Phi_k\chi^{-t}\xi)\hat{u}(\Phi_k\chi^{-t}\xi)e^{ix.\xi}|\Phi_k\chi^{-t}(\xi)|d\xi \end{aligned}$$

Thus we essentially have to look at how $\Phi_k\chi^{-t}$ modifies the frequencies and thus how it modifies the rings in the Littlewood-Paley decomposition.
Put $\left\{k \geq 1, C'_k = \operatorname{supp}\ \phi_k(\Phi_k\chi^{-t}.)\right\}$, we have:

$$C'_k \approx \bigcup_{k-N' \leq l \leq k+N} C_l,$$

where N and N' are such that $2^N > sup_{k,\mathbb{R}^d}|\Phi_k\chi'|$ and $2^{N'} > sup_{k,\mathbb{R}^d}|\Phi_k\chi'|^{-1}$ and the natural para-composition operator in this case is obtained by cutting the frequencies according to C'_k, this is exactly the "lemme de recoupe" in Alinhac's work.

Now we define N as in the previous remark and compute:

$$(2) = \underbrace{\sum_{k\geq 0} \sum_{\substack{l\geq 0 \\ l\leq k+N}} \phi_l(D)(u_k \circ \chi)}_{\chi^\star u} \tag{5.3}$$
$$+ \underbrace{\sum_{k\geq 0} \sum_{\substack{l\geq 0 \\ l\leq k+N}} \phi_l(D)[u_k \circ \Phi_k\chi - u_k \circ \chi]}_{R_1} + \underbrace{\sum_k (Id - \Phi_{k+N})(D) u_k \circ \Phi_k \chi}_{R_2}.$$

Theorem 5.1. *Let $\chi : \mathbb{R}^d \to \mathbb{R}^d$ be a $C^{1+\rho}_{*,loc}$ map with $D\chi \in C^\rho_*$ and $\rho > 0$* [7]*. Then for all $\sigma, s \in \mathbb{R}^*_+$ the following maps are continuous:*

$$C^\sigma_*(\mathbb{R}^d) \to C^\sigma_*(\mathbb{R}^d) \qquad C^\sigma_{*,loc}(\mathbb{R}^d) \to C^\sigma_{*,loc}(\mathbb{R}^d)$$
$$u \mapsto \chi^\star u = \sum_{k\geq 0} \sum_{\substack{l\geq 0 \\ l\leq k+N}} \phi_l(D) u_k \circ \chi \qquad u \mapsto \chi^\star u = \sum_{k\geq 0} \sum_{\substack{l\geq 0 \\ l\leq k+N}} \phi_l(D) u_k \circ \chi.$$

If moreover χ is a diffeomorphism then we have the Sobolev estimates:

$$H^s(\mathbb{R}^d) \to H^s(\mathbb{R}^d) \qquad H^s_{loc}(\mathbb{R}^d) \to H^s_{loc}(\mathbb{R}^d)$$
$$u \mapsto \chi^\star u = \sum_{k\geq 0} \sum_{\substack{l\geq 0 \\ l\leq k+N}} \phi_l(D) u_k \circ \chi \qquad u \mapsto \chi^\star u = \sum_{k\geq 0} \sum_{\substack{l\geq 0 \\ l\leq k+N}} \phi_l(D) u_k \circ \chi.$$

Taking $\tilde{\chi} : \mathbb{R}^d \to \mathbb{R}^d$ a $C^{1+\tilde{\rho}}_{*,loc}$ map with $D\tilde{\chi} \in C^{\tilde{\rho}}_*$ and $\tilde{\rho} > 0$, then the previous operation has the natural fonctorial property:

$$\forall u \in C^\sigma_*(\mathbb{R}^d) \cup C^\sigma_{*,loc}(\mathbb{R}^d), \chi^\star \tilde{\chi}^\star u = (\chi \circ \tilde{\chi})^\star u + Ru.$$

with R, $R : C^\sigma_*(\mathbb{R}^d) \to C^{\sigma+min(\rho,\tilde{\rho})}_*(\mathbb{R}^d)$, $R : C^\sigma_{*,loc}(\mathbb{R}^d) \to C^{\sigma+min(\rho,\tilde{\rho})}_{*,loc}(\mathbb{R}^d)$,

and if χ and $\tilde{\chi}$ are diffeomorphisms:

$$R : H^s(\mathbb{R}^d) \to H^{s+min(\rho,\tilde{\rho})}(\mathbb{R}^d), \qquad R : H^s_{loc}(\mathbb{R}^d) \to H^{s+min(\rho,\tilde{\rho})}_{loc}(\mathbb{R}^d).$$

Remark 5.2. *It's natural that the Sobolev estimates only hold when χ is a diffeomorphism because for example even the usual composition operation $u \mapsto u \circ \chi$ is not necessarily continuous on L^p spaces, $p < \infty$. An extra hypothesis that appears in the literature is χ is a local diffeomorphism with all of it's local inverses uniformly bounded in $\dot{W}^{1,\infty}$.*

Theorem 5.2. *Let u be a $W^{1,\infty}(\mathbb{R}^d)$ map and χ be a $C^{1+\rho}_{*,loc}$ map with $D\chi \in C^\rho_*$ and $\rho > 0$. Then:*

$$u \circ \chi(x) = \chi^\star u(x) + T_{u' \circ \chi} \chi(x) + R_0(x) + R_1(x) + R_2(x)$$

where the paracomposition given in the previous Theorem verifies the estimates:

$$\forall \sigma > 0, \|\chi^\star u(x)\|_\sigma \leq C(\|D\chi\|_\infty) \|u(x)\|_\sigma,$$
$$u' \circ \chi \in \Gamma^0_{W^{0,\infty}(\mathbb{R}^d)}(\mathbb{R}^d) \quad \text{for } u \text{ Lipchitz},$$

and the remainders verify the estimates:

- *In Zygmund Spaces, for $\sigma > 0$:*

$$\|R_0\|_{1+\rho+min(1+\rho,\sigma)} \leq C \|D\chi\|_\rho \|u\|_{1+\sigma}$$
$$\text{for } i \in \{1,2\}, \|R_i\|_{1+\rho+\sigma} \leq C(\|D\chi\|_\infty) \|D\chi\|_\rho \|u\|_{1+\sigma}.$$

[7]Clearly when there is no diffeomorphism hypothesis on χ we can choose $\chi : \mathbb{R}^d \to \mathbb{R}^{d'}$ with $d \neq d'$ and have the same results but for clarity we chose to present the same dimensions in this presentation.

- *In Sobolev Spaces, for $s > \frac{d}{2}$ we get the following estimates*
  - *without the diffeomorphism hypothesis:*

$$\|R_0\|_{H^{1+\rho+min(1+\rho,s-\frac{d}{2})}} \leq C \|D\chi\|_\rho \|u\|_{H^{1+s}}$$

$$\|R_1\|_{H^{1+\rho+s}} \leq C(\|D\chi\|_\infty) \|D\chi\|_\rho \|u\|_{H^{1+s}}.$$

  - *Suppose moreover that χ is a diffeomorphism:*

$$\|R_2\|_{H^{1+\rho+s}} \leq C(\|D\chi\|_\infty, \|D\chi^{-1}\|_\infty) \|D\chi\|_\rho \|u\|_{H^{1+s}}.$$

The same estimates hold in the local spaces.

As Alinhac remarked in [3], a particular case of the previous Theorem is Bony para-linearization Theorem but with the extra hypothesis of diffeomorphism, here it's is a full generalization because we dropped the diffeomorphism hypothesis. We find Bony's para-linearization Theorem when $\sigma = +\infty$, in this case only the term $T_{u'\circ\chi}\chi(x)$ appears and $\chi^\star u(x)$ is a part of the remainder. If on the other hand, $\chi \in C^\infty$, the term $T_{u'\circ\chi}\chi(x)$ becomes a part of the remainder and the paracomposition $\chi^\star u(x)$ coincides with the usual composition modulo a regularizing operator. Thus Theorem 5.2 appears as a linearization Theorem of $u\circ\chi$ as the sum of two terms, one depending mainly on the regularity of u (and "less" of χ) and the other depending mainly on the regularity of χ (and "less" of u).

The simplest example here is when $\chi(x) = Ax$ is a linear operator and in that case we see that:

$$u(Ax) \sim (Ax)^* u, \text{ and } T_{u'(Ax)}Ax \sim 0.$$

Remark 5.3. *The proof of Theorem 5.2 tell us that the if in the sum defining $\chi^\star$ we choose a different $N' \geq N$ then the operator is modified by a ρ regularizing operator.*

Theorem 5.3. *Consider $a \in \Gamma^m_\beta(\mathbb{R}^d)$, with $\beta \geq 0$, $\chi : \mathbb{R}^d \to \mathbb{R}^d$ a $C^{1+\rho}_{*,loc}$ map with $D\chi \in C^\rho_*$, $\rho > 0$ and $1+\rho \notin \mathbb{N}$. Then there exists $q \in \Gamma^{m-\beta}_0(\mathbb{R}^d)$ such that we have the following formal symbolic calculus rule:*

$$\chi^\star T_a u = op_{\omega_\chi}\left(\sigma_a(\chi(x),\xi)\frac{|\det D\chi(y)|}{\#\chi^{-1}(\chi(y))}\right)\chi^\star u + op_{\omega_\chi}\left(\sigma_q(\chi(x),\xi)\frac{|\det D\chi(y)|}{\#\chi^{-1}(\chi(y))}\right)\chi^\star u.$$

To join Alinhac's work, the following Proposition makes the link between his definition of the paracomposition operator in the case of a diffeomorphism and the one given here.

Theorem 5.4. *Let u be $W^{1,\infty}(\mathbb{R}^d)$ map and χ be a $C^{1+\rho}_{*,loc}$ diffeomorphism with $D\chi \in C^\rho_*$ and $\rho > 0$. Consider $\tilde{N}$ such that $2^{\tilde{N}} > sup_{k,\mathbb{R}^d} |\Phi_k \chi'|^{-1}$ and $2^{\tilde{N}} > sup_{k,\mathbb{R}^d} |\Phi_k \chi'|$. Put Alinhac's paracomposition operator:*

$$\chi^* u = \sum_{k\geq 1} \sum_{\substack{l\geq 0 \\ |l-k|\leq \tilde{N}}} \phi_l(D) u_k \circ \chi$$

$$\textit{then: } \chi^* u = \chi^\star u + R_3,$$

Where the remainder verifies:

- *In Zygmund Spaces, for $\sigma > 0$:*

$$\|R_3\|_{1+\rho+\sigma} \leq C(\|D\chi^{-1}\|_\infty, \|D\chi\|_\infty) \|D\chi\|_\rho \|u\|_{1+\sigma}.$$

- *In Sobolev Spaces, for $s > \frac{d}{2}$:*

$$\|R_3\|_{H^{1+\rho+s}} \leq C(\|D\chi^{-1}\|_\infty, \|D\chi\|_\infty) \|D\chi\|_\rho \|u\|_{H^{1+s}}.$$

The same estimates hold in the local spaces.
Take $a \in \Gamma^m_\beta(\mathbb{R}^d)$ and q as in Theorem 5.3 then:

$$\chi^\star T_a u = T_{a^*}\chi^\star u + T_{q^*}\chi^\star u$$
$$\chi^* T_a u = T_{a^*}\chi^* u + T_{q'^*}\chi^* u \text{ with } q' \in \Gamma^{m-\beta}_0(\mathbb{R}^d).$$

Remark 5.4. *As in remark 5.3, the proof of Theorem 5.4 tell us that the if in the sum defining χ^* we choose a different $\tilde{N}' \geq \tilde{N}$ then the operator is modified by a ρ regularizing operator.*

Remark 5.5. *As a corollary of Theorem 5.4 we get that in Theorem 4.2:*

$$R\chi = T_{(T_a u)'\circ\chi}\chi - T_{a^*}T_{u'\circ\chi}\chi. \tag{5.4}$$

5.2. **Proofs.** We will give the proof for the estimates in global spaces, for local spaces it is sufficient to see that the given estimates hold under the hypothesis that all the functions used have a compact support and to pass to local spaces estimates it is sufficient to multiply by functions in C^∞_0 which don't modify the estimates given (we don't make any boundary estimates).

Proof of Theorem 5.1 and 5.2. Take $\chi : \mathbb{R}^d \to \mathbb{R}^d$ be a $C^{1+\rho}_*$ map with $\rho > 0$ put $\mathrm{B} = \mathrm{B}(0, N+1)$.
We start by the Zygmund spaces estimates (thus we don't suppose that χ is a diffeomorphism):

$$\|\Phi_{k+N}u_k \circ \chi\|_\infty \leq C\|u_k\|_\infty \leq 2^{-k\sigma}\|u\|_\sigma$$

and supp $\Phi_{k+N}u_k \circ \chi \subset 2^k$ B.
Thus by Proposition 2.3, for $\sigma > 0$:

$$\chi^\star u \in C^\sigma_*(\mathbb{R}^d) \text{ and } \|\chi^\star u\|_\sigma \leq \frac{C(N)}{1-2^{-\sigma}}\|u\|_\sigma .$$

For Sobolev estimates we suppose that χ is a diffeomorphism and by the change of variables formula we have for $s > 0$:

$$\|\Phi_{k+N}u_k \circ \chi\|_{L^2} \leq C(\|D\chi^{-1}\|_\infty)\|u_k\|_{L^2} \leq C(\|D\chi^{-1}\|_\infty)2^{-ks}\|u\|_{H^s}$$

and supp $\Phi_{k+N}u_k \circ \chi \subset 2^k$ B.
Thus by Proposition 2.3, for $\sigma > 0$:

$$\chi^\star u \in H^s(\mathbb{R}^d) \text{ and } \|\chi^\star u\|_{H^s} \leq \frac{C(N, \|D\chi^{-1}\|_\infty)}{1-2^{-s}}\|u\|_{H^s} .$$

Now we compute the estimates on the remainders in the linearization formula.

$$R_0 = \sum_{k\geq 1}\left(\int_0^1 \Phi_{k-1}u'(\tau\Phi_k\chi + (1-\tau)\Phi_{k-1}\chi) - \Phi_{k-1}(u'\circ\chi)d\tau\right)\phi_k\chi = \sum_k r^0_k\chi_k$$

$$\begin{aligned} r^0_k &= \int_0^1 \Phi_{k-1}u'(\tau\Phi_k\chi + (1-\tau)\Phi_{k-1}\chi) - \Phi_{k-1}(u'\circ\chi)d\tau \\ &= \int_0^1\int_0^1 \Phi_{k-1}u''(t\tau(\Phi_k\chi - \Phi_{k-1}\chi) \\ &+ t(\Phi_{k-1}\chi - \chi) - \chi)dt[\tau(\Phi_k\chi - \chi) + (1-\tau)(\Phi_{k-1}\chi - \chi)]d\tau. \end{aligned}$$

Thus if $\sigma \neq 1$:

$$\|r^0_k\|_\infty \leq C2^{k(-\sigma-\rho)}$$

And if $\sigma = 1$:

$$\|r^0_k\|_\infty \leq Ck2^{k(-1-\rho)} \leq C2^{-k},$$

Which sums up in $\left\|r_k^0\right\|_\infty \leq C2^{-min(1+\rho,\sigma)k}$. By the same computations we have analogous estimates on $\left\|\partial^\alpha r_k^0\right\|$ and clearly $r_k^0 \in C^\infty$ which gives the desired estimates on R_0 by Lemma 2.1 and the fact that $r_0^0 = 0$, both in the Sobolev et Zygmund cases without the diffeomorphism hypothesis.

$$\begin{aligned}
R_1 &= \sum_{k\geq 0} \phi_{k+N}(D)[u_k \circ \Phi_k\chi - u_k \circ \chi] \\
&= \sum_{k\geq 0} \phi_{k+N}(D)[(\int_0^1 u_k'(t\Phi_k + (1-t)\chi)dt)(\Phi_k\chi - \chi)] \\
&= \sum_{k\geq 0} \phi_{k+N}(D)[r_k^1(\Phi_k\chi - \chi)].
\end{aligned}$$

We have:

$$\left\|r_k^1\right\|_\infty \leq C2^{-k\sigma}$$

combining this with Propositions 2.3, 2.4 and the fact that $r_0^1 = 0$ we get the desired estimates again in both in the Sobolev et Zygmund cases without the diffeomorphism hypothesis.
The proof of the estimates on R_2 relies on oscillatory integral techniques that come from Lemma 3.1. For the sake of completion we will give the explicit computations without directly using the Lemma.

$$R_2(x) = \sum_k (Id - \Phi_{k+N})(D)u_k \circ \Phi_k\chi(x).$$

We will prove that for $j \geq k+N+1, \nu \geq \rho > 0$, we have:

$$\left\|\phi_j(D)u_k \circ \Phi_k\chi\right\|_\infty \leq C_\nu(\left\|D\chi\right\|_\rho)2^{-j\nu}2^{k(\nu-\rho)}\left\|u_k\right\|_\infty \tag{5.5}$$

which will be sufficient to give the Zygmund estimates on R_2 because we will have:

$$\begin{aligned}
\left\|\phi_j(D)R_2\right\|_\infty \leq & \sum_{\substack{k\geq 0\\ k\leq N-j+1}} \left\|\phi_j(D)u_k \circ \Phi_k\chi\right\|_\infty \\
\leq & \sum_{\substack{k\geq 0\\ k\leq N-j+1}} C_\nu(\left\|D\chi\right\|_\rho)2^{-j\nu}2^{k(\nu-\rho)}\left\|u_k\right\|_\infty \\
\leq & \sum_{\substack{k\geq 0\\ k\leq N-j+1}} C_\nu(\left\|D\chi\right\|_\rho)2^{-j\nu}2^{k(\nu-\rho)}\left\|u_k\right\|_\infty \\
\leq & \sum_{\substack{k\geq 0\\ k\leq N-j+1}} C_\nu(\left\|D\chi\right\|_\rho)2^{-j\nu}2^{k(\nu-\rho-\sigma-1)}\left\|u\right\|_{1+\sigma},
\end{aligned}$$

Taking $\nu > 1+\rho+\sigma$ we dominate the last expression by:

$$C_\nu(\left\|D\chi\right\|_\rho)2^{-j(\rho+\sigma+1)}\left\|u\right\|_{1+\sigma}$$

which gives the desired Zygmund estimate.

For the Sobolev estimates we will prove that:

$$\left\|\phi_j(D)u_k \circ \Phi_k\chi\right\|_2 \leq C_\nu(\left\|D\chi\right\|_\rho)2^{-j\nu}2^{k(\nu-\rho)}\left\|u_k \circ \Phi_k\chi\right\|_2, \tag{5.6}$$

which then necessitates the diffeomorphism hypothesis on χ to have:

$$\left\|\phi_j(D)u_k \circ \Phi_k\chi\right\|_2 \leq C_\nu(\left\|D\chi\right\|_\rho, \left\|D\chi^{-1}\right\|_\infty)2^{-j\nu}2^{k(\nu-\rho)}\left\|u_k\right\|_2,$$

And the desired estimates follow exactly as in the Zygmund case.
Now we prove (5.5) and (5.6),to make the desired estimates we will put in a test function $f \in C_b^\infty$ as it's usually done with oscillatory integral estimates:

$$\phi_j(D) f u_k \circ \Phi_k \chi(x) = \int e^{i(x-y).\xi} \phi_j(\xi)\phi_k(\eta) f(y) \hat{u_k}(\eta) e^{i\Phi_k\chi(y).\eta} d\eta dy d\xi \tag{5.7}$$

Set

$$\omega_k(y,\eta,\xi) = \Phi_k\chi(y).\eta - y.\xi,$$

$$L_k(y,\eta,\xi,\partial_y) = \frac{\Phi_k\chi'(y)^t.\eta - y.\xi}{i\left|\Phi_k\chi'(y)^t.\eta - y.\xi\right|^2}.\nabla_y.$$

Given the definition of N we have:

$$\left|\Phi_k\chi'(y)^t.\eta - y.\xi\right| \geq C(|\eta| + |\xi|) \text{ on } \operatorname{supp}\ \phi_j(\xi)\phi_k(\eta),$$

Thus L_k is well defined and regular, moreover $L_k e^{i\omega_k} = e^{i\omega_k}$. Integrating by parts in (5.6):

$$\phi_j(D) f u_k \circ \Phi_k \chi(x) = \int e^{ix.\xi} \phi_j(\xi)\phi_k(\eta) \hat{u_k}(\eta) e^{iw_k} (L_k^t)^\nu f(y) d\eta dy d\xi.$$

Note that $(L_k^t)^\nu f$ is homogeneous with degree $-\nu$ in (η,ξ), and smooth on the support of $\phi_j(\xi)\phi_k(\eta)$. Also

$$\left|(L_k^t)^\nu f(y)\right| \leq C(\|f\|_\nu, \|D\chi\|_\rho) 2^{\nu-\sigma} \text{ on } |\xi|^2 + |\eta|^2 = 1. \tag{5.8}$$

Next on a box containing $\operatorname{supp}\ \phi_j(\xi)\phi_k(\eta)$, write

$$(L_k^t)^\nu f(y) = \sum_{(\alpha,\beta)\in\Lambda} a_{k\nu\alpha\beta}(y) e^{i\alpha.\xi + i\beta.\eta} = 2^{-j\nu} \sum_{(\alpha,\beta)\in\Lambda} a_{k\nu\alpha\beta}(y) e^{i2^{-j}\alpha.\xi + i2^{-j}\beta.\eta},$$

where Λ is an appropriate lattice and

$$\sum_{(\alpha,\beta)\in\Lambda} \|a_{k\nu\alpha\beta}\|_\infty \leq C(\|f\|_\nu, \|D\chi\|_\rho) 2^{\nu-\sigma}. \tag{5.9}$$

So (5.7) becomes for $j \geq 1$:

$$\begin{aligned}
&\phi_j(D) f u_k \circ \Phi_k\chi(x) && (5.10)\\
&= 2^{-j\nu} \sum_{(\alpha,\beta)\in\Lambda} \int e^{ix.\xi}\phi_j(\xi)\phi_k(\eta)\hat{u_k}(\eta) e^{iw_k} a_{k\nu\alpha\beta}(y) e^{i2^{-j}\alpha.\xi + i2^{-j}\beta.\eta} d\eta dy d\xi \\
&= 2^{-j\nu} \sum_{(\alpha,\beta)\in\Lambda} \int e^{i(x-y).\xi}\phi_j(\xi) u_k(\Phi_k\chi(y) + 2^{-j}\beta) a_{k\nu\alpha\beta}(y) e^{i2^{-j}\alpha.\xi} dy d\xi \\
&= 2^{-j\nu} \sum_{(\alpha,\beta)\in\Lambda} \int e^{i(x-y).\xi} 2^{jn}\hat{\phi}_1(2^j(x-y)+\alpha) u_k(\Phi_k\chi(y) + 2^{-j}\beta) a_{k\nu\alpha\beta}(y) dy. \\
&= 2^{-j\nu} \sum_{(\alpha,\beta)\in\Lambda} (a_{k\nu\alpha\beta} \cdot u_k(\Phi_k\chi + 2^{-j}\beta)) * g_\alpha(x), && (5.11)
\end{aligned}$$

Where $g_\alpha(x) = 2^{jn}\hat{\phi}_1(2^j x + \alpha)$ thus

$$\|g_\alpha\|_{L^1} = 2^{jn} \int \left|\hat{\phi}_1(2^j x + \alpha)\right| dx = \left\|\hat{\phi}_1\right\|_{L^1}. \tag{5.12}$$

For $j = 0$ we have an analog inequality.
Using the classic Young and Hölder inequalities combined with (5.9), (5.12) and taking $f \to 1$ gives us (5.5) and (5.6). This concludes the proof.

Proof of Theorem 5.3. Take $a \in \Gamma^m_\beta(\mathbb{R}^d)$, with $\beta \geq 0$ and $\chi : \mathbb{R}^d \to \mathbb{R}^d$ a $C^{1+\rho}_*$ map with $\rho > 0$. We compute:

$$\chi^\star T_a u = \sum_{k\geq 0} \Phi_{k+N}[(T_a u)_k \circ \chi], \tag{5.13}$$

Note that $(T_a u)_k$ can been as $T_{\phi_k} T_a u$ and seeing this a modification of the cut-off function by Proposition 3.2 we get:

$$(T_a u)_k = T_{\phi_k} T_a u = T_a T_{\phi_k} u + T_{q^k} u, \text{ with } q^k \in \Gamma_0^{m-\beta}(\mathbb{R}^d).$$

Put $q = \sum q^k$ then (5.13) becomes:

$$\chi^\star T_a u = \sum_{k\geq 0} \Phi_{k+N}[(T_a u_k) \circ \chi] + \sum_{k\geq 0} \Phi_{k+N}[(T_{q^k} u_k) \circ \chi].$$

And the formal discussion and computations in part 4 give the desired result.

Proof of Theorem 5.4. The only thing left to prove is the estimate on R_3.

$$R_3 = \sum_k \underbrace{\Phi_{k-\tilde{N}}(D) u_k \circ \Phi_k \chi(x)}_{1} + \phi_N(D) u_k \circ \chi$$

$\phi_N(D) u_k \circ \chi$ is C^∞ so we only have to the estimate to the first term on the left hand side. Estimating 1 is exactly as (5.7) but with ϕ_j substituted by $\Phi_{k-\tilde{N}}$. The core of the estimation relies on the fact that L_k should be well defined and regular on $\operatorname{supp} \Phi_{k-\tilde{N}}(\xi)\phi_k(\eta)$ which is the case give our choice of $\tilde{N}$ and the fact $k \geq 1$. We also have the estimate:

$$\left|\Phi_k \chi'(y)^t.\eta - y.\xi\right| \geq C(|\eta| + |\xi|) \text{ on } \operatorname{supp} \Phi_{k-\tilde{N}}(\xi)\phi_k(\eta).$$

The proof than exactly follows as for R_2.

5.3. **Main results for paracomposition on open subsets.** The previous definition of the operator $\chi^\star$ on functions defined on $\mathbb{R}^d$ relied heavily on the Littelwood-Paley theory which doesn't make it immediately extendable to the open domain case. In [3], Alinhac was able to define such an operator profiting from the continuity of χ^* on the local function spaces and a partition of unity on the open domains. More precisely consider (V_i, Θ_i) a partition of unity locally finite of Ω' then:

$$u \circ \chi = \sum_i \Theta_i u \circ \chi$$

where $\Theta_i u$ is seen as a function of $\mathbb{R}^n$ with the natural extension by 0. In order to have the same natural extension for χ,

$$\chi^{-1}(\operatorname{supp} \Theta_i)$$

needs to be compact we thus have to suppose that χ is a proper map[8]. Under this hypothesis consider $\zeta_i \in C_0^\infty(\Omega)$ such that $\zeta_i = 1$ on $\chi^{-1}(\operatorname{supp} \Theta_i)$:

$$u \circ \chi = \sum_i \zeta_i \Theta_i u \circ \zeta_i \chi, \tag{5.14}$$

where $\zeta_i \chi$ is seen as a function of $\mathbb{R}^n$ with the natural extension by 0.

[8]Note that this extra hypothesis is needed for the methods used to work and is not intrinsic to the problem. Also this hypothesis is immediately verified in the diffeomorphism case treated by Alinhac.

Theorem 5.5. *Let $\chi : \Omega \to \Omega'$ be a $C^{1+\rho}_{*,loc}$ proper map with $D\chi \in C^{\rho}_*$ and $\rho > 0$. Consider (V_i, Θ_i) a partition of unity locally finite of Ω' and ζ_i the associated functions as previously. Then for all $\sigma, s \in \mathbb{R}^*_+$ the following maps are continuous:*

$$C^\sigma_*(\Omega') \to C^\sigma_*(\Omega) \qquad\qquad C^\sigma_{*,loc}(\Omega') \to C^\sigma_{*,loc}(\Omega)$$

$$u \mapsto \chi^\star u = \sum_i \zeta_i \cdot (\zeta_i \chi)^\star \Theta_i u \qquad\qquad u \mapsto \chi^\star u = \sum_i \zeta_i \cdot (\zeta_i \chi)^\star \Theta_i u,$$

if moreover χ is a diffeomorphism then we have the Sobolev estimates:

$$H^s(\Omega') \to H^s(\Omega) \qquad\qquad H^s_{loc}(\Omega') \to H^s_{loc}(\Omega)$$

$$u \mapsto \chi^\star u = \sum_i \zeta_i \cdot (\zeta_i \chi)^\star \Theta_i u \qquad\qquad u \mapsto \chi^\star u = \sum_i \zeta_i \cdot (\zeta_i \chi)^\star \Theta_i u,$$

where $\Theta_i u$ and $\zeta_i \chi$ are treated as functions on $\mathbb{R}^d$. And In the sum defining each $(\zeta_i \chi)^\star$ a choice

$$N_i, 2^{N_i} \geq \sup_{\operatorname{supp} \Theta_i} \chi'$$

is made by the definition in section 5.1, but by remark 5.3 in order to simplify the computations we can take the same

$$N \geq N_i, 2^N \geq \sup_{\Omega} \chi'$$

uniformly for all the operators and this modifies the definition by a ρ regularizing operator.
Making a different choice $(V'_i, \Theta'_i, \zeta'_i)$, which gives a different operator $\chi^\star_1$ then

$$\forall u \in C^\sigma_*(\mathbb{R}^d) \cup C^\sigma_{*,loc}(\mathbb{R}^d), \chi^\star u = \chi^\star_1 u + R' u.$$

with $R'u \in C^\infty$.

Consider $\tilde{\chi} : \Omega' \to \Omega''$ a a $C^{1+\tilde{\rho}}_{*,loc}$ proper map with $D\tilde{\chi} \in C^{\tilde{\rho}}_*$ with $\tilde{\rho} > 0$, then the previous operation has the natural fonctorial property:

$$\forall u \in C^\sigma_*(\Omega'') \cup C^\sigma_{*,loc}(\Omega''), \chi^\star \tilde{\chi}^\star u = (\chi \circ \tilde{\chi})^\star u + \tilde{R} u.$$

$$\text{with } \tilde{R},\ \tilde{R} : C^\sigma_*(\Omega'') \to C^{\sigma+min(\rho,\tilde{\rho})}_*(\Omega),\ \tilde{R} : C^\sigma_{*,loc}(\Omega'') \to C^{\sigma+min(\rho,\tilde{\rho})}_{*,loc}(\Omega),$$

and if χ and $\tilde{\chi}$ are diffeomorphisms:

$$\tilde{R} : H^s(\Omega'') \to H^{s+min(\rho,\tilde{\rho})}(\Omega),\ \tilde{R} : H^s_{loc}(\Omega'') \to H^{s+min(\rho,\tilde{\rho})}_{loc}(\Omega).$$

Theorem 5.6. *Let u be a $W^{1,\infty}(\Omega)$ map and χ be a be a $C^{1+\rho}_{*,loc}$ proper map with $D\chi \in C^\rho_*$ and $\rho > 0$. Then:*

$$u \circ \chi(x) = \chi^\star u(x) + T_{u' \circ \chi} \chi(x) + R_0(x) + R_1(x) + R_2(x)$$

where the paracomposition given in the previous Theorem verifies the estimates:

$$\forall \sigma > 0, \|\chi^\star u(x)\|_\sigma \leq C(\|D\chi\|_\infty) \|u(x)\|_\sigma,$$

$$u' \circ \chi \in \Gamma^0_{W^{0,\infty}(\Omega)}(\mathbb{R}^d) \text{ for } u \text{ Lipchitz.}$$

The remainders are given by:

$$R_0 = \sum_i \sum_{k \geq 1} \zeta_i \bigg(\int_0^1 \Phi_{k-1} \Theta_i u'(\tau \Phi_k \zeta_i \chi + (1-\tau) \Phi_{k-1} \zeta_i \chi) - \Phi_{k-1}(\Theta_i u' \circ \zeta_i \chi) d\tau \bigg) \phi_k \zeta_i \chi,$$

$$R_1 = \sum_i \sum_{k \geq 0} \sum_{\substack{l \geq 0 \\ l \leq k+N}} \zeta_i (\phi_l(D)[\Theta_i u_k \circ \Phi_k \zeta_i \chi - \Theta_i u_k \circ \zeta_i \chi]),$$

$$R_2 = \sum_i \sum_k \zeta_i ((Id - \Phi_{k+N})(D) \Theta_i u_k \circ \Phi_k \zeta_i \chi),$$

and the remainders verify the estimates:

- *In Zygmund Spaces, for $\sigma > 0$:*

$$\|R_0\|_{1+\rho+min(1+\rho,\sigma)} \leq C \|D\chi\|_\rho \|u\|_{1+\sigma}$$

$$for\ i \in \{1,2\}, \|R_i\|_{1+\rho+\sigma} \leq C(\|D\chi\|_\infty) \|D\chi\|_\rho \|u\|_{1+\sigma}.$$

- *In Sobolev Spaces, for $s > \frac{d}{2}$ we get the following estimates*
  - *without the diffeomorphism hypothesis:*

$$\|R_0\|_{H^{1+\rho+min(1+\rho,s-\frac{d}{2})}} \leq C \|D\chi\|_\rho \|u\|_{H^{1+s}}$$

$$\|R_1\|_{H^{1+\rho+s}} \leq C(\|D\chi\|_\infty) \|D\chi\|_\rho \|u\|_{H^{1+s}}.$$

  - *Suppose moreover that χ is a diffeomorphism:*

$$\|R_2\|_{H^{1+\rho+s}} \leq C(\|D\chi\|_\infty, \|D\chi^{-1}\|_\infty) \|D\chi\|_\rho \|u\|_{H^{1+s}}.$$

The same estimates hold in the local spaces.

Theorem 5.7. *Consider $a \in \Gamma^m_\beta(\mathbb{R}^d)$, with $\beta \geq 0$ and $\chi : \Omega \to \Omega'$ a $C^{1+\rho}_{*,loc}$ proper map with $D\chi \in C^\rho_*$, $\rho > 0$ and $1+\rho \notin \mathbb{N}$. Then there exists $q \in \Gamma^{m-\beta}_0(\mathbb{R}^d)$ such that we have the following formal symbolic calculus rule:*

$$\chi^\star T_a u = op_{\omega_\chi}\left(\sigma_a(\chi(x),\xi)\frac{|\det D\chi(y)|}{\#\chi^{-1}(\chi(y))}\right)\chi^\star u + op_{\omega_\chi}\left(\sigma_q(\chi(x),\xi)\frac{|\det D\chi(y)|}{\#\chi^{-1}(\chi(y))}\right)\chi^\star u.$$

Again to join Alinhac's work:

Theorem 5.8. *Let u be $W^{1,\infty}(\Omega)$ map and χ be a $W^{1,\infty}$ diffeomorphism, a $C^{1+\rho}_{*,loc}$ proper map with $D\chi \in C^\rho_*$ and $\rho > 0$. Again, consider (V_i, Θ_i) a partition of unity locally finite of Ω' and ζ_i the associated functions as previously. Put Alinhac's paracomposition operator:*

$$\chi^* u = \sum_i \zeta_i (\zeta_i \chi)^* \Theta_i u \ then:$$

$$\chi^* u = \chi^\star u + R_3,$$

Where the remainder verifies:

- *In Zygmund Spaces, for $\sigma > 0$:*

$$\|R_3\|_{1+\rho+\sigma} \leq C(\|D\chi^{-1}\|_\infty, \|D\chi\|_\infty) \|D\chi\|_\rho \|u\|_{1+\sigma}.$$

- *In Sobolev Spaces, for $s > \frac{d}{2}$:*

$$\|R_3\|_{H^{1+\rho+s}} \leq C(\|D\chi^{-1}\|_\infty, \|D\chi\|_\infty) \|D\chi\|_\rho \|u\|_{H^{1+s}}.$$

The same estimates hold in the local spaces.
Consider $a \in \Gamma^m_\beta(\mathbb{R}^d)$ and q as in Theorem 5.3 then:

$$\chi^\star T_a u = T_{a^*}\chi^\star u + T_{q^*}\chi^\star u$$
$$\chi^* T_a u = T_{a^*}\chi^* u + T_{q'^*}\chi^* u \ with\ q' \in \Gamma^{m-\beta}_0(\mathbb{R}^d).$$

Again we have the same "independence" of the definition of the operator χ^* (modulo a more regular term) with respect to the arbitrary choices made, more precisely, making a different choice $(V'_i, \Theta'_i, \zeta'_i)$ which gives a different operator χ^*_1 then

$$\forall u \in C^\sigma_*(\mathbb{R}^d) \cup C^\sigma_{*,loc}(\mathbb{R}^d), \chi^* u = \chi^*_1 u + R' u.$$

with $R'u \in C^\infty$.

5.4. **Proof.** All of the estimates given come directly for the Theorems of section 5.1. The linearization formulas come from Equation (5.14) and the linearization Theorems in section 5.1. The only thing left to prove is the independency result with respect to the choice of (V_i, Θ_i, ζ_i). We start by the following Lemma:

Lemma 5.1. *Let* $(\Theta, \zeta, \tilde{\zeta}) \in C_0^\infty(\Omega')$ *be such that* $\zeta = 1$ *on* $\chi^{-1}(\operatorname{supp}\Theta)$ *and* $\tilde{\zeta} = 1$ *on* $\operatorname{supp}\zeta$ *then:*

$$\sum_{k\geq 0} \zeta \Phi_{k+N}(D)[(\Theta u)_k \circ \zeta\chi] = \sum_{k\geq 0} \tilde{\zeta}\Phi_{k+N}(D)[(\Theta u)_k \circ \tilde{\zeta}\chi] + F, \;\; F \in C^\infty$$

Proof. Take $\Theta' \in C_0^\infty(\Omega')$ such that $\Theta'\circ\chi = 0$ on supp ζ and $\Theta'\circ\chi = 1$ on supp $\tilde{\zeta} - \zeta$ and compute:

$$\begin{aligned}
&\sum_{k\geq 0} \zeta \Phi_{k+N}(D)[(\Theta u)_k \circ \zeta\chi] \\
&= \sum_{k\geq 0} \tilde{\zeta}\Phi_{k+N}(D)[(\Theta u)_k \circ \tilde{\zeta}\chi] + \sum_{k\geq 0} (\zeta - \tilde{\zeta})\Phi_{k+N}(D)[(\Theta u)_k \circ \tilde{\zeta}\chi] \\
&= \sum_{k\geq 0} \tilde{\zeta}\Phi_{k+N}(D)[(\Theta u)_k \circ \tilde{\zeta}\chi] + \underbrace{\sum_{k\geq 0} (\zeta - \tilde{\zeta})\Phi_{k+N}(D)[(\Theta'(\Theta u)_k) \circ \tilde{\zeta}\chi]}_{F}.
\end{aligned}$$

And we have by integration by parts, $\forall l \in \mathbb{N}$:

$$\Theta'(\Theta u)_k = 2^{-kl} \int \frac{e^{i(x'-y')\xi}}{i(x'-y')^l} \Theta'(x)\Theta(y)\phi_1(2^{-k}\xi)u(y)dyd\xi,$$

$$\text{thus, } \left\|\Theta'(\Theta u)_k\right\|_\infty \leq C_l 2^{-k(l-n)}, \text{ and } F \in C^\infty.$$

□

Given (i,j) such that $\operatorname{supp}\Theta_i \cap \operatorname{supp}\Theta'_j \neq \emptyset$ we define $\tilde{\zeta}_{i,j} \in C_0^\infty(\Omega)$ such that $\tilde{\zeta}_{i,j} = 1$ on $\operatorname{supp}\zeta_i \cup \operatorname{supp}\zeta'_j$.

$$\begin{aligned}
\chi^\star u = \sum_i \zeta_i \cdot (\zeta_i\chi)^\star \Theta_i u &= \sum_{k\geq 0}\sum_{i,j} \zeta_i \Phi_{k+N}(D)[(\Theta_i\Theta'_j u)_k \circ \zeta_i\chi] \\
&= \sum_{k\geq 0}\sum_{i,j} \tilde{\zeta}_{i,j}\Phi_{k+N}(D)[(\Theta_i\Theta'_j u)_k \circ \tilde{\zeta}_{i,j}\chi] + F, \;\; F \in C^\infty \\
&= \sum_{k\geq 0}\sum_{i,j} \zeta'_j\Phi_{k+N}(D)[(\Theta_i\Theta'_j u)_k \circ \zeta'_j\chi] + F + F', \;\; F' \in C^\infty \\
&= \chi_1^\star u + F + F',
\end{aligned}$$

which gives the desired result and ends the proof.

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