ON THE CAUCHY PROBLEM OF DISPERSIVE BURGERS TYPE EQUATIONS

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Abstract. We study the paralinearised weakly dispersive Burgers type equation:

$$\partial_t u + \partial_x [T_u u] - T_{\frac{\partial_x u}{2}} u + \partial_x |D|^{\alpha-1} u = 0, \ \alpha \in]1, 2[,$$

which contains the main non linear "worst interaction" terms, i.e low-high interaction terms, of the usual weakly dispersive Burgers type equation:

$$\partial_t u + u\partial_x u + \partial_x |D|^{\alpha-1} u = 0, \ \alpha \in]1, 2[,$$

with $u_0 \in H^s(\mathbb{D})$, where $\mathbb{D} = \mathbb{T}$ or $\mathbb{R}$.

Through a paradifferential complex Cole-Hopf type gauge transform we introduced in [38], we prove a new a priori estimate in $H^s(\mathbb{D})$ under the control of $\left\| (1 + \|u\|_{L^\infty_x}) \|u\|_{W^{2-\alpha,\infty}_x} \right\|_{L^1_t}$, improving upon the usual hyperbolic control $\|\partial_x u\|_{L^1_t L^\infty_x}$. Thus we eliminate the "standard" wave breaking scenario in case of blow up as conjectured in [28].

For $\alpha \in]2, 3[$ we show that we can completely conjugate the paralinearised dispersive Burgers equation to a semi-linear equation of the form:

$$\partial_t u + \partial_x |D|^{\alpha-1} u = R_\infty(u), \ \alpha \in]2, 3[,$$

where R_∞ is a regularizing operator under the control of $\|u\|_{L^\infty_t C^{2-\alpha}_*}$.

Contents

1. Introduction

This paper is concerned with the well-posedness of the paralinearised "weak" dispersive perturbations of the Burgers equation:

$$\partial_t u + \partial_x [T_u u] - T_{\frac{\partial_x u}{2}} u + \partial_x |D|^{\alpha-1} u = 0, \ u_0 \in H^s, \tag{1.1}$$

which is derived from the "weak" dispersive perturbations of the Burgers equation:

$$\partial_t u + u\partial_x u + \partial_x |D|^{\alpha-1} u = 0, \ \text{where } \alpha \in]1, 2] \text{ and } |D| = \mathrm{Op}(|\xi|). \tag{1.2}$$

For $\alpha = 2$ in (1.2), we have the usual Benjamin-Ono equation, for $\alpha = 3$ we have the KdV equation, for $\alpha = \frac{1}{2}$ and $\alpha = \frac{3}{2}$ respectively this equation is a "toy" model for the system obtained by paralinearization and symetrization of water waves system with and without surface tension [1, 2, 3, 4, 5].

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The Cauchy problem associated to (1.2) have been extensively studied in the literature for a comprehensive and complete overview of those equations and their link to other problems coming from mechanical fluids and dispersive non linear equations in physics we refer to J-C. Saut's [39, 40].

For $\alpha \geq 2$ the Cauchy problem is now very well understood where essentially three main techniques come into play to understand it. A first approach is to use smoothing effects and refined Strichartz estimates see [35, 29]. A second approach is to use time dependent frequency localized spaces as in [24]. The last approach, first introduced by Tao in [45] is to use a gauge transform to eliminate the worst interaction terms, see also [10, 33] for the Benjamin-Ono equation and [17] for $\alpha \in]2,3[$, which to the author's knowledge is the only time the gauge transform was used to improve upon the local well-posedness of dispersive Burgers equation in the fractional dispersion case.

Except for the first approach, in the words of [43], those techniques face major technical difficulties for $\alpha < 2$ and seem to completely fail. The goal of this paper is to show that using the gauge transform introduced in [38], the last approach can be still carried for $1 \leq \alpha < 2$ and it gives $H^s(\mathbb{D})$ under the control of $\left\| (1 + \|u\|_{L^\infty_x}) \|u\|_{W^{2-\alpha,\infty}_x} \right\|_{L^1_t}$, improving upon the known hyperbolic control $\|\partial_x u\|_{L^1_t L^\infty_x}$. To the author's best knowledge this is the first time a gauge transform technique was carried out to improve upon the local well-posedness of the weakly dispersive Burgers equation.

We will also show that for $2 \leq \alpha \leq 3$, this gauge transform can be efficiently used to completely conjugate the paralinearized dispersive Burgers equation to the linear dispersive equation modulo a regular, i.e C^∞, remainder under control of $\|u\|_{L^\infty_t C^{2-\alpha}_*}$. Again to the author's best knowledge this is the first time such a transformation is carried out outside the integrable cases, i.e $\alpha = 2$ and $\alpha = 3$. For those cases, i.e the Benjamin Ono and the KdV equations, suitable Birkhoff coordinates were constructed to "diagonalize" the infinite dimension Hamiltonian, for this we refer to the pioneering works of Gérard, Kappeler and Topalov [13, 14, 25].

First we start with the simplest Cauchy theory for this equation that only uses hyperbolic estimates reads:

Theorem 1.1. *Consider three real numbers $\alpha \in [0, +\infty[$, $s \in]1 + \frac{1}{2}, +\infty[$, $r > 0$ and $u_0 \in H^s(\mathbb{D})$. Then there exists $T > 0$ such that for all v_0 in the ball $B(u_0, r) \subset H^s(\mathbb{D})$ there exists a unique $v \in C([0,T], H^s(\mathbb{D}))$ solving the Cauchy problem:*

$$\begin{cases} \partial_t v + v\partial_x v + |D|^{\alpha-1} \partial_x v = 0 \\ v(0,\cdot) = v_0(\cdot) \end{cases}. \tag{1.3}$$

Moreover we have the estimates for $t \in [0,T]$:

$$\forall 0 \leq \mu \leq s, \|v(t)\|_{H^\mu(\mathbb{D})} \leq \exp(C \|\partial_x v\|_{L^1([0,T], L^\infty(\mathbb{D}))}) \|v_0\|_{H^\mu(\mathbb{D})}. \tag{1.4}$$

Taking two different solutions u, v, and assuming moreover that $u_0 \in H^{s+1}(\mathbb{D})$ then we have:

$$\|(u-v)(t)\|_{H^s(\mathbb{D})} \leq \exp(C \|(u,v)\|_{L^1([0,T], L^\infty(\mathbb{D}))})(1 + Ct \|u\|_{H^{s+1}(\mathbb{D})}) \|u_0 - v_0\|_{H^s(\mathbb{D})}. \tag{1.5}$$

Equation (1.3) is known (see [39]) to be quasi-linear for $\alpha < 3$ i.e the flow map is not regular (C^1) and as such it cannot be solved through a fixed point scheme. This lack of regularity comes from the "bad" low-high frequency interaction $u_{\text{low}} \partial_x u_{\text{high}}$.

Despite of this lack of regularity of the flow map, Tao used in [45] a generalized Cole-Hopf complex transformation to prove global well posedness of the Benjamin-Ono equation in $H^1(\mathbb{R})$. This technique was extensively used to push down the well posedness threshold to $L^2(\mathbb{R})$ for the Benjamin-Ono equation in [24]. This contrasts the fact that for the Burgers equation, i.e $\alpha = 1$, the Cauchy problem is ill-posed in $H^s, s \leq \frac{3}{2}$ as shown in [30].

Thus the general idea is to understand the "interaction" between the nonlinearity and dispersion. The following quantities are conserved by the flow associated to (1.2):

$$\|u(t)\|_{L^2} = \|u_0\|_{L^2}\,, \text{ and,} \tag{1.6}$$

$$H(u) = \int_{\mathbb{R}} \left|D^{\frac{\alpha-1}{2}} u\right|^2 (t,x)dx + \frac{1}{3}\int_{\mathbb{R}} u^3(t,x)dx = H(u_0). \tag{1.7}$$

By the the Sobolev embedding $H^{\frac{1}{6}} \hookrightarrow L^3$, $H(u)$ is well defined for $\alpha \geq 1 + \frac{1}{3}$. Moreover (1.2) is invariant under the scaling transformation:

$$u_\lambda = \lambda^{\alpha-1} u(\lambda^\alpha t, x),$$

for any positive λ. We have $\|u_\lambda(t,\cdot)\|_{\dot{H}^s} = \lambda^{\alpha+s-\frac{3}{2}} \|u(\lambda^\alpha t,\cdot)\|_{\dot{H}^s}$, thus the critical index corresponding to (1.2) is $s_c = \frac{3}{2} - \alpha$. In particular, (1.2) is L^2 critical for $\alpha = \frac{3}{2}$.

In the "low" dispersion case, i.e $\alpha \leq 2$ a complete numerical study was carried out by Klein and Saut in [28] and conjectured among other things the following.

Conjecture 1.1. *(1) For $\alpha \leq 1$ solutions blow up in finite time and do so through a wave breaking scenario, i.e $\lim_{t\to T^*} \|u(t)\|_{L^\infty}$ stays bounded while $\|\partial_x u(t)\|_{L^\infty} \to +\infty$.*

(2) For $\alpha > 1$ we have global in time existence for small initial data.

(3) For $1 < \alpha \leq \frac{3}{2}$ large solutions blow up in finite time and do so through a "dispersive" blow up scenario, i.e $\|u(t)\|_{L^\infty_x} \to +\infty$.

(4) For $\alpha > \frac{3}{2}$ solutions exist globally in time.

In [11] and [21] blow up is proven for $\alpha < 1$ and in [22, 23] it is shown that for $\alpha < \frac{2}{3}$ the only possible blow up scenario is a wave breaking one, a simpler proof can be found in [42].

In [30] using Strichartz estimate well posedness is proved for $s > \frac{3}{2} - \frac{3(\alpha-1)}{8}$ and $\alpha > 1$, proving that even for very low dispersion the threshold of well posedness can be improved which again contrasts with the Burgers equation ($\alpha = 1$) where the equation is shown is shown to be ill-posed for $s = \frac{3}{2}$ in [30].

This was improved upon in [34] using an adapted version of the "I-method" and refined Strichartz estimates in co-normal Bourgain type spaces. They proved well posedness for $s > \frac{3}{2} - \frac{5(\alpha-1)}{4}$ and $\alpha > 1$, thus proving by the conservation of $H(u)$ and scaling, global well posedness for $\alpha > 1 + \frac{6}{7}$. Which was the first and to the authors knowledge only result proving global existing results for $\alpha < 2$.

Finally for $\alpha \geq 2$ the Cauchy problem is much better understood. For the Benjamin Ono equation on $\mathbb{R}$, to the authors knowledge the best known result is L^2 global well-posedness derived in [24]. Recently Patrick Gerard, Thomas Kappeler and Peter Topalov proved in [13] global well-posedness for the periodic Benjamin-Ono equation all the way down to $s > -\frac{1}{2}$ and ill-posedness for $s < -\frac{1}{2} = s_c$, the critical Sobolev exponent. For $\alpha \in]2,3[$ on the real line, the best known local well posedness result is for $\alpha \geq \frac{3}{4}(2-\alpha)$ under a low frequency condition given in [16] and in L^2 without the low frequency condition in [17]. For the KdV equation, for both the periodic and real line cases the Cauchy problem is globally well posed on $H^{-1}(\mathbb{R})$ as shown in [25, 27], which is the best possible well posedness result, i.e the KdV equation is ill-posed for $s < -1$ as shown in [32].

The remarkable well-posedness results for $\alpha = \{2, 3\}$ uses the integrability of the Benjamin-Ono equation and the KdV equation and the construction of Birkhoff coordinates and thus cannot be extended to the case $\alpha \neq \{2, 3\}$.

Remark 1.1. *It's interesting to compare (1.2) to the "fractal" Burgers equation, i.e the Burgers equation with a dissipative term:*

$$\partial_t u + u\partial_x u + (-\Delta)^{\frac{\alpha}{2}} u = 0, \ \alpha \geq 0. \tag{1.8}$$

For $\alpha = 2$, (1.8) is the usual Hopf equation. The local and global Cauchy problem associated to (1.8) is very well understood, we refer to [26] for a complete solution to the problem. For $\alpha < 1$, large solution of (1.8) blow up in finite time and that through a wave breaking mechanism. For $\alpha \geq 1$ solutions exist globally in time.

This contrasts with (1.2) in several directions:

- *the change of local/global well posedness is conjectured to happen at $\alpha = \frac{3}{2}$ and not 1,*
- *the conjectured existence of a new nonlinear blow up regime for $\alpha \in]1, \frac{3}{2}]$,*
- *the drastic difference of the global Cauchy problem between (1.2) and (1.8) for $\alpha = 1$.*

In this paper we will look more closely to the equation:

$$\partial_t u + \partial_x[T_u u] - T_{\frac{\partial_x u}{2}} u + \partial_x |D|^{\alpha-1} u = 0, \ u_0 \in H^s. \tag{1.9}$$

Remark 1.2. *The modifications made to pass from (1.2) to (1.9) are the following:*

- *We replaced the product $\frac{u^2}{2}$ by the paraproduct $T_u u$ and added the term $-T_{\frac{\partial_x u}{2}} u$ to ensure that $\partial_x[T_u u] - T_{\frac{\partial_x u}{2}} u$ is skew symmetric for the L^2 scalar product.*
- *We dropped the remainder terms:*

$$\frac{\partial_x R(u, u)}{2} \text{ and } T_{\frac{\partial_x u}{2}} u.$$

The motivations to study the paralinearised version of the equations are the following:

1. *Equation (1.9) still contains the main "bad" term $u_{low}\partial_x u_{high}$. As remarked in [45] and [10] this is the main term obstructing straightforward estimates in $X^{0^+, \frac{1}{2}^+}$ for $\alpha = 2$.*
2. *Indeed looking through the literature [10] and [24], the neglected terms can be treated in localized Besov-Bourgain type spaces in our threshold of regularity. By contrast the results on (1.9) are simpler to write because they can be completely described in the usual Sobolev spaces.*

Thus to keep our presentation clear and put the key ideas forward in treating the "worst" terms, we opted to present in this paper the results on the paralinearised version of the equation (1.9) and give the results on the "full" equation (1.2) in Bourgain type spaces in a forthcoming work where the action of the gauge transform used here will be studied in such spaces.

Now we give the first theorem of this paper.

Theorem 1.2. *Consider two real numbers $\alpha \in]1, 2[$, $s \in]1 + \frac{1}{2}, +\infty[$. Then for all $v_0 \in H^s$ and $r > 0$ there exists $T > 0$ such that for all u_0 in the ball $B(v_0, r) \subset H^s(\mathbb{D})$ there exists a unique $u \in C([0, T], H^s(\mathbb{D}))$ solving the Cauchy problem:*

$$\partial_t u + iT^{B',b}_{\sigma^{B,b}_{u\xi} + (\sigma^{B,b}_{u\xi})^*} u + \partial_x |D|^{\alpha-1} u = 0, \ u_0 \in H^s, \ b > 0, \tag{1.10}$$

where $B > 2$ is given by Theorem 3.1, $\sigma^{B,b}$ is a cutoff defining paradifferential operators (cf Definition A.1), a^* is the symbolic adjoint given by:

$$a^*(x,\xi) = \frac{1}{2\pi}\int_{\mathbb{D}\times\hat{\mathbb{D}}} e^{-iy.\eta}\bar{a}(x-y,\xi-\eta)dyd\eta,$$

and $B' > B$ is the cutoff corresponding to the left hand side that includes $T^{B,b}$ and it's a adjoint.

The flow map $u_0 \mapsto u$ is continuous from $B(v_0,r)$ to $C([0,T],H^s(\mathbb{D}))$.

Moreover we have the estimate:

$$\|u(t)\|_{H^s} \le e^{C\left\|(1+\|u\|_{L^\infty_x})\|u\|_{W^{2-\alpha,\infty}_x}\right\|_{L^1([0,T])}}\|u_0\|_{H^s}, \text{ for } 1<\alpha<2. \tag{1.11}$$

Remark 1.3.
- *The apriori estimate (1.11) is not enough to improve upon the local well-posedness theory, indeed we need an extra estimate on the difference of two solutions. A straightforward computation shows that taking the difference of two solutions $u-v$ we get:*

$$\partial_t(u-v)+T_{i\xi|\xi|^{\alpha-1}}(u-v)+\underbrace{\partial_x[T_u(u-v)]-T_{\frac{\partial_x u}{2}}(u-v)}_{(1)}+\underbrace{\partial_x[T_{u-v}v]-T_{\frac{\partial_x u-v}{2}}v}_{(2)}=0.$$

  Term (1) can be treated using the gauge transform but term (2) is not a paradifferential operator in the variable $u-v$ and can not be treated in our current restricted paradifferential-Sobolev space setting. Indeed term (2) has the same structure as the residual terms we dropped to get equation (1.9) and has to be treated in the Besov-Bourgain space type spaces which is not done here.
- *The analogue of estimate (1.11) is still valid for $\alpha \ge 2$ but some care is needed as the "negative" Hölder spaces should be replaced by Zygmund spaces. In the real line case the standard Strichartz estimates give for $\alpha = 2$, i.e the Benjamin-Ono equation, the analogue of the well-posedness result of Burq and Planchon [10] and for $\alpha \in [2,3]$ the analogue of [16].*

We turn to the conjugation Theorem for $\alpha \in [2,3]$.

Theorem 1.3. *Consider two real numbers $\alpha \in]2,3[$, $s \in]\frac{1}{2}+2-\alpha,+\infty[$. Then there exists $T>0$ and $r>0$ such that for all u_0 in the ball $B(0,r) \subset H^s(\mathbb{D})$ there exists a unique $u \in C([0,T],H^s(\mathbb{D}))$ solving the Cauchy problem:*

$$\partial_t u + iT^{B,b}_{u\xi}u + \partial_x |D|^{\alpha-1} u = 0,\ u_0 \in H^s,\ b>0, \tag{1.12}$$

where $B \ge 1+\frac{1}{3}$ is given by Theorem 3.1 and the flow map $v_0 \mapsto v$ is continuous from $B(0,r)$ to $C([0,T],H^s(\mathbb{D}))$.

Moreover there exists a symbol $p \in W^{\infty,\infty}_\tau([0,\tau],C^1\Gamma^0_1(\mathbb{D}))$, in the symbol classes with limited regularity in the frequency variable defined in by 3.1, such that for it's hyperbolic flow map $(A^{p(\tau,\cdot)}_\tau)_{\tau=1}$ defined by 2.1 we have:

$$\partial_t(A^{p(\tau,\cdot)}_\tau)_{\tau=1}u + \partial_x |D|^{\alpha-1}(A^{p(\tau,\cdot)}_\tau)_{\tau=1}u = R_\infty(u), \tag{1.13}$$

and there exists a non decreasing functions C_s, such that $R_\infty(u)$ verifies for all $\mu \in \mathbb{R}$:

$$\|R_\infty(u)\|_{H^\mu} \le C_\mu(\|u\|_{L^\infty([0,T],C^{2-\alpha}_*(\mathbb{D}))}).$$

Remark 1.4.
- *The method used here can be pushed to prove that p is in $C^k\Gamma^0_1(\mathbb{D})_{k\in\mathbb{N}}$ for all $k \in \mathbb{N}$, with a smaller radius $r_k > 0$ in Theorem 1.3 but without a lower bound on r_k. We chose to present the computation showing $p \in C^1\Gamma^0_1(\mathbb{D})$, which is the minimal regularity required for the definition of A^p_1.*

- *For the KdV equation this gives local well-posedness in $H^{-\frac{1}{2}}(\mathbb{D})$ which is the analogue of the result proven in [12].*

1.1. **Strategy of the proof.** The starting point is to prove the key apriori estimate (1.11). Let us explain how to do so on the L^2 level first, passing from this to the H^s estimates adds a (significantly) more technical step to the proof.

To get the L^2 estimate we gauge transform the term $T_{i\xi}T_u u - T_{\frac{\partial_x u}{2}} u$ out of the equation. More precisely we are looking for an operator A such that:

$$\partial_t u + T_{i\xi}T_u u - T_{\frac{\partial_x u}{2}} u + T_{i\xi|\xi|^{\alpha-1}} u = 0 \Rightarrow \partial_t u + A^{-1}T_{i\xi|\xi|^{\alpha-1}} A u = R_{2-\alpha} + R_\infty,$$

where A is a unitary operator modulo, at least, an α-regularizing operator, $R_{2-\alpha}$ is of order $2-\alpha$ with $\mathrm{Re}(R_{2-\alpha})$ of order 0.

To find A we follow our construction in [38] and define $A = A_1^p$, where A_τ^p is the flow of a hyperbolic paradifferential equation of the form:

$$\partial_\tau A_\tau^p h_0 - iT_p A_\tau^p h_0 = 0, \ A_0^p h_0 = h_0.$$

Using our result on the Baker-Campbell-Hausdorff formula for this type flow proved in [38], we are looking for p such that:

$$[A_1^p, T_{i\xi|\xi|^{\alpha-1}}] = A_1^p\big(T_{i\xi}T_u - T_{\frac{\partial_x u}{2}}\big) + R.$$

In [38], choosing $p = \frac{\xi|\xi|^{1-\alpha}}{\alpha} U$, where U is a primitive of u, was enough to have R as an $\alpha-1$ regularizing operator when $s > \frac{3}{2}$ which gave us the desired result on the flow map regularity at this threshold. At our threshold of regularity where we only control the $L^2([0,T], W_x^{2-\alpha})$ this would give R as an operator of order 1, that is there is no apparent gain that comes from this transformation.
To remedy this the idea is to construct p implicitly. Indeed using the stability of paradifferential operators by commutation with A_1^p proved in [38], we write:

$$[A_1^p, T_{i\xi|\xi|^{\alpha-1}}] = A_1^p T_{i^c[\xi|\xi|^{\alpha-1}]_1^p}.$$

It's here where the two case $\alpha < 2$ and $\alpha \geq 2$ have to be treated differently. Indeed, on one hand by Proposition 2.2, $i^c[\xi\,|\xi|^{\alpha-1}]_1^p$ belongs to a symbol class of the form $L^\infty_* S_{\min(\alpha-1,1),2-\alpha}$. And on the other hand $u\partial_x + \frac{\partial_x u}{2} \in \Gamma_0^1 = L^\infty_* S_{1,0}$, thus for $\alpha < 2$ there is no hope to solve:

$$i^c[\xi\,|\xi|^{\alpha-1}]_1^p = u\partial_x + \frac{\partial_x u}{2},$$

and for $\alpha \geq 2$ it seems possible through a local implicit function theorem.
The case $\alpha < 2$: We solve the problem approximately, using the ellipticity of $\xi\,|\xi|^{\alpha-1}$, we show that we can fully solve the first term in the Baker-Campbell-Hausdorff expansion of $i^c[\xi\,|\xi|^{\alpha-1}]_1^p$, i.e we solve:

$$[ip, i\xi\,|\xi|^{\alpha-1}] = u\partial_x + \frac{\partial_x u}{2}.$$

This amounts to right inverting a linear operator in the Fréchet space of paradifferential operators. The problem is first reduced to a standard linear inversion in the scale of Banach spaces defining the Fréchet space of paradifferential operators. Then using an explicit approximate parametrix, given by the usual Cole-Hopf choice of gauge transformation, a careful choice of cut-off functions studied in [36] and symbolic calculus we show that a Neumann series can be carried out to correct the right parametrix into a right inverse in one Banach space in the scale. We then use a bootstrap argument to propagate the regularity to the hole scale of Banach spaces and thus the Fréchet space of paradifferential operators.

Getting back to the equation,

- moreover we carefully choose the cut-off used in our paradifferential operators in order not to produce too many remainder terms.
- We use the fact that $u\partial_x + \frac{\partial_x u}{2}$ and $i\xi\,|\xi|^{\alpha-1}$ are L^2 skew-adjoint to ensure that p can be chosen L^2 self-adjoint.

In doing so we construct an L^2 unitary operator $A = A_1^p$ such that:

$$[A_1^p, T_{i\xi|\xi|^{\alpha-1}}] = A_1^p\big(T_{i\xi}T_u - T_{\frac{\partial_x u}{2}}\big) + \underbrace{\int_0^1 A_{1-r}^p[T_{ip}, T_{u\partial_x+\frac{\partial_x u}{2}}]A_{r-1}^p dr}_{(1)}.$$

Crudely at our threshold of regularity using symbolic calculus (1) seems to be of order $1+2-\alpha$ which seems worse that what we had before. The key cancellation is that using the identity:

$$[B,C]^* = [C^*, B^*],$$

we see that (1) is actually L^2 skew-adjoint, which gives the conservation of the L^2 norm.

At our level of regularity $s > 2-\alpha+\frac{1}{2}$ this key cancellation does not hold for higher Sobolev estimates, except in the case of the Benjamin-Ono equation, i.e $\alpha = 2$, indeed in that case:

$$p(x,\xi) = \frac{\mathrm{Op}(\frac{1}{D})P_{\geq b}(D)u}{2} \text{ for } \xi \geq 0,$$

and,

$$\left[T_{ip}^{B',b}, iT^{B',b}_{\sigma_{u\xi}^{B,b}+(\sigma_{u\xi}^{B,b})^*}\right] = T_{iu^2} \text{ for } \xi \geq 0,$$

this "exceptional" algebraic cancellation in the commutator is due to the $\partial_\xi p = 0$ which does not occur for fractional $\alpha \neq 2$. This difficulty was noted in [17] and the proposed solution was to use a gauge transform with indeed $\partial_\xi p = 0$ to eliminate only the lowest frequency terms in $u\partial_x u$, i.e $P_0(D)u\partial_x u$, and treat the remainder terms in carefully chosen function spaces with frequency dependent time localization. Inspired by this idea we show that with the paradifferential setting developed here, the problem can be indeed reduced to a choice of p such that $\partial_\xi p = 0$, which will amount to a simple approximation of the symbol p by step functions in the frequency variable ξ.

The case $\alpha \geq 2$: Using the ellipticity of $\xi\,|\xi|^{\alpha-1}$ and the paradifferential setting constructed we show that:

$$p \mapsto {}^c[\xi\,|\xi|^{\alpha-1}]_1^p$$

is indeed locally surjective around 0, which the key technical result we prove in Theorem 3.2. This is a non trivial problem, equivalent to solving a nonlinear ODE in the Fréchet space of paradifferential symbols. Such an ODE is not generally well posed and to solve such a problem one usually has to look at a Nash-Moser type scheme[1]. In our case the choice of paradifferential setting, inspired by by Hörmander's [20], is shown to be stable by the gauge transformation in Proposition 2.4. Thus We show that the problem can be reduced to a standard implicit function theorem combined with bootstrap argument in order to insure propagation of regularity. The bootstrap trick is the analogue of the one used in the Picard fixed point theorem depending on a parameter.

Thus we get,

$$A_1^p\partial_t u + T_{i\xi|\xi|^{\alpha-1}}A_1^p u = R_\infty.$$

[1] Such a scheme can indeed be carried out here thanks to the tame estimates in Remark 2.3 but we show that this can be avoided here.

We would like to fully conjugate the quasi-linear equation (1.9) to a semi-linear equation, i.e:

$$\partial_t u + T_{i\xi}T_u u - T_{\frac{\partial_x u}{2}} u + T_{i\xi|\xi|^{\alpha-1}} u = 0 \Rightarrow \partial_t \tilde{A} u + T_{i\xi|\xi|^{\alpha-1}} \tilde{A} u = R,$$

Thus compared to first gauge transform we have to treat the term $\partial_t A$. At our threshold of regularity $\partial_t A$ is still of order 1 thus we don't have a gain on the order of the operator. The key idea is to iterate the gauge transform to eliminate those time derivatives, i.e construct $(A_1^{p_n})$ that eliminate the terms $(\partial_t A_1^{p_{n-1}})$ at each step. We then use Theorem 2.1 and the geometric decrease in the norms at each step to prove the convergence:

$$\lim_{n\to+\infty} \prod_{k=1}^{n} A_1^{p_n} = \tilde{A}_1^{\tilde{p}} = \tilde{A}.$$

Remark 1.5. *The continuity of operators of type $T_{e^{i\tau p}}$ on Zygmund space with loss of derivatives were studied by E. Stein [44] and by G. Bourdaud in [9]. In this paper we need explicit estimates taking into play the exact symbol semi-norms. For this we give a complete study the continuity of paradifferential operators defined by symbols in these type of "exotic" symbol classes in Appendix B. Our proofs follow the same lines and methods presented in [44, 47, 31].*

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2. Baker-Campbell-Hausdorff formula: composition and commutator estimates

We will start by giving the propositions defining the operators used in the gauge transforms and the symbolic calculus associated to them. All the main theorems are proved in [38] and we will follow the same presentation, though we make a couple of more precise estimates on the semi-norms used, when we do so a proof is written.

Notation 2.1. *We will essentially compute the conjugation and commutation of operators with a flow map which naturally bring into play Lie derivatives i.e commutators, thus we introduce the following notation for commutation between operators:*

$$\mathfrak{L}_a^0 b = b,\ \mathfrak{L}_a b = [a,b] = a\circ b - b\circ a,\ \mathfrak{L}_a^2 b = [a,[a,b]],\ \mathfrak{L}_a^k b = \underbrace{[a,[\cdots,[a,}_{k \text{ times}} b]]\cdots].$$

In the following propositions the variable $t \in [0,T]$ is the generic time variable that appear in all through the paper and a new variable $\tau \in \mathbb{R}$ will be used and they should not be confused.

We also need to recall the definition of the adjoint of a paradifferential symbol p that we write p^* and is given in (A.8).

We start with the proposition defining the Flow map and its standard properties.

Proposition 2.1. *Consider two real numbers $\delta \leq 1$, $s \in \mathbb{R}$ and a symbol $p \in \Gamma_0^\delta(\mathbb{D})$ such that:*

$$\mathrm{Im}(p) = \frac{p-p^*}{2i} \in \Gamma_0^{\tilde{\delta}}(\mathbb{D}),\ \ \textit{with } \tilde{\delta} \leq 0.$$

The following linear hyperbolic equation is well posed on $\mathbb{R}$:

$$\begin{cases} \partial_\tau h - iT_p h = 0, \\ h(0,\cdot) = h_0(\cdot) \in H^s. \end{cases} \tag{2.1}$$

For $\tau \in \mathbb{R}$, define A^p_τ as the flow map associated to 2.1 i.e,

$$\begin{aligned} A^p_\tau : & H^s(\mathbb{D}) \to H^s(\mathbb{D}) \\ & h_0 \mapsto h(\tau, \cdot). \end{aligned} \tag{2.2}$$

Then for $\tau \in \mathbb{R}$ we have,

(1) $A^p_\tau \in \mathscr{L}(H^s(\mathbb{D}))$ and,

$$\|A^p_\tau\|_{H^s \to H^s} \leq e^{C|\tau| M^0_0(\mathrm{Im}(p))}.$$

(2)

$$iT_p \circ A^p_\tau = A^p_\tau \circ iT_p, \;\; A^p_{\tau+\tau'} = A^p_\tau A^p_{\tau'}.$$

(3) A^p_τ is invertible and,

$$(A^p_\tau)^{-1} = A^p_{-\tau}.$$

Moreover the L^2 adjoint of A^p_τ verifies:

$$(A^p_\tau)^* = A^{(T_p)^*}_{-\tau} = A^p_{-\tau} + R,$$

where R is a $\tilde{\delta}$ regularizing operator and $A^{(T_p)^*}_\tau$ is the flow generated by the Cauchy problem:

$$\begin{cases} \partial_\tau h - i(T_p)^* h = 0, \\ h(0, \cdot) = h_0(\cdot) \in H^s(\mathbb{D}). \end{cases} \tag{2.3}$$

(4) Taking a different symbol $\tilde{p}$ verifying the same hypothesis as p we have:

$$\left\| [A^p_\tau - A^{\tilde{p}}_\tau] h_0 \right\|_{H^s} \leq C\, |\tau|\, e^{C|\tau| M^0_0(\mathrm{Im}(p), \mathrm{Im}(\tilde{p}))} M^\delta_0(p - \tilde{p})\, \|h_0\|_{H^{s+\delta}}. \tag{2.4}$$

Proof. In [38] we worked with $p \in \Gamma^\delta_1$ and the continuity of A^p_τ, i.e point (1), was proved through an energy estimate. Looking closely to the energy estimate we see that that we only need to control the semi norm $M^0_0(\mathrm{Re}(ip))$ and with the hypothesis on p we get the more precise result. □

Remark 2.1. *Under the hypothesis $p \in \Gamma^\delta_1(\mathbb{D})$ and p real valued we have $\tilde{\delta} \leq \delta - 1$ by symbolic calculus and the inequality:*

$$M^{\delta-1}_0(\mathrm{Im}(p)) \leq C M^{\delta-1}_0(\partial_\xi \partial_x p), \;\; \tilde{\delta} \leq 0.$$

Later on we will need to study the continuity of A^p_τ on Hölder/Zygmund spaces. This a non trivial result, indeed hyperbolic flows are not in general continuous on L^p spaces for $p \neq 2$, as it is for example the case for the Shrödinger equation, or equations of the form $\partial_t h + i\,|D|^\alpha h = 0, \alpha \neq 1$ that are not continuous on Zygmund spaces as shown in the Appendix of [3]. To study the continuity of A^p_τ on Hölder/Zygmund we start by studying it's symbol. First we recall the following Lemma we proved in [38] adapting the classic result by Beals on pseudodifferential operators in[7] to the limited regularity setting.

Lemma 2.1. *Consider an operator A continuous from $\mathscr{S}(\mathbb{D})$ to $\mathscr{S}'(\mathbb{D})$ and let $a \in \mathscr{S}'(\mathbb{D} \times \hat{\mathbb{D}})$ the unique symbol associated to A (cf [8] for the uniqueness), i.e, let K be the kernel associated to A then:*

$$u, v \in \mathscr{S}(\mathbb{D}), (Au, v) = K(u \otimes v), \quad a(x, \xi) = \mathscr{F}_{y \to \xi} K(x, x - y).$$

- *If A is continuous from H^m to L^2, with $m \in \mathbb{R}$, and $[\frac{1}{i}\frac{d}{dx}, A]$ is continuous from $H^{m+\delta}$ to L^2 with $\delta < 1$, then $(1 + |\xi|)^{-m} a(x, \xi) \in L^\infty_{x,\xi}(\mathbb{D} \times \hat{\mathbb{D}})$ and we have the estimate:*

$$\left\| (1 + |\xi|)^{-m} a \right\|_{L^\infty_{x,\xi}} \leq C_m \left[\|A\|_{H^m \to L^2} + \left\| \left[\frac{1}{i} \frac{d}{dx}, A \right] \right\|_{H^{m+\delta} \to L^2} \right]. \tag{2.5}$$

- *If A is continuous from H^m to L^2, with $m \in \mathbb{R}$, and $[ix, A]$ is continuous from from $H^{m-\rho}$ to L^2 with $\rho \geq 0$, then $(1+|\xi|)^{-m}a(x,\xi) \in L^\infty_{x,\xi}(\mathbb{D} \times \hat{\mathbb{D}})$ and we have the estimate:*
$$\left\|(1+|\xi|)^{-m}a\right\|_{L^\infty_{x,\xi}} \leq C_m[\|A\|_{H^m \to L^2} + \|[ix, A]\|_{H^{m-\rho}\to L^2}]. \tag{2.6}$$

We can now define the limited-regularity symbol classes to which A^p_τ belongs.

Definition 2.1. *Consider $s \in \mathbb{R}_+$, for $0 \leq \delta, \rho < 1$, we say:*
$$p \in W^{s,\infty}S^m_{\rho,\delta}(\mathbb{D}) \iff \begin{cases} \left|D^k_\xi p(x,\xi)\right| \leq C_k\langle\xi\rangle^{m-\rho k} \\ \left\|D^k_\xi p(\cdot,\xi)\right\|_{W^{s,\infty}} \leq C_k\langle\xi\rangle^{m-\rho k+s\delta} \end{cases}, (x,\xi) \in \mathbb{D}\times\tilde{\mathbb{D}},\ k \geq 0. \tag{2.7}$$
The best constant in (2.7) defines a seminorm denoted by ${}^{\rho,\delta}M^m_s(\cdot;k), k \in \mathbb{N}$ where k is the number of derivatives we make on the frequency variable ξ, we also define the seminorm ${}^{\rho,\delta}M^m_s = {}^{\rho,\delta}M^m_s(\cdot;1)$. We define analogously $W^{s,\infty}S^m_{\rho,\delta}(\mathbb{D}^* \times \tilde{\mathbb{D}})$.

Motivated by Lemma 2.1 we introduce the following family of seminorms:
$${}^{\rho,\delta}H^m_0(p;k) = \sum_{j=0}^k \left\|\mathfrak{L}^j_{ix}p\right\|_{H^m\to H^{j\rho}},$$
$${}^{\rho,\delta}H^m_n(p;k) = \sum_{l=0}^n\sum_{j=0}^k \left\|\mathfrak{L}^j_{ix}\mathfrak{L}^l_{\frac{1}{i}\frac{d}{dx}}p\right\|_{H^m\to H^{j\rho-l\delta}},\ n \in \mathbb{N}, n \leq \rho,$$
and if $s \notin \mathbb{N}$:
$${}^{\rho,\delta}H^m_s(p;k) = H^m_{\lfloor s\rfloor}(p;k) + \sup_{n\in\mathbb{N}} 2^{n(s-\lfloor s\rfloor)}\sum_{j=0}^k \left\|\mathfrak{L}^j_{ix}\mathfrak{L}^{\lfloor s\rfloor}_{\frac{1}{i}\frac{d}{dx}}[P_n(D)p]\right\|_{H^m\to H^{j\rho-l\delta}},$$
where $P_n(D)$ is applied to p in the x variable.

Then ${}^{\rho,\delta}H^m_s(p;k)_{k\in\mathbb{N}}$ induces an equivalent Fréchet topology to ${}^{\rho,\delta}M^m_s(\cdot;k)_{k\in\mathbb{N}}$ on $W^{s,\infty}S^m_{\rho,\delta}$.

In Stein's [44], such symbols are called "exotic", there continuity on different L^p spaces is completely studied but only in the regular case and without explicit estimates depending on the semi-norms. To make such estimates explicit we have given a full proof of such continuity Theorems in Appendix B.

Proposition 2.2. *Consider two real numbers $\delta < 1$, $\rho \geq 0$ and a symbol $p \in \Gamma^\delta_\rho(\mathbb{D})$ such that:*
$$\mathrm{Im}(p) = \frac{p-p^*}{2i} \in \Gamma^{\tilde{\delta}}_0(\mathbb{D}),\ \text{with } \tilde{\delta} \leq 0.$$
Let $A^p_\tau, \tau \in \mathbb{R}$ be the flow map defined by Proposition 2.1, then there exists a symbol $e^{i\tau p}_\otimes \in W^{\rho,\infty}S^0_{1-\delta,\delta}(\mathbb{D}^* \times \tilde{\mathbb{D}})$ such that:
$$A^p_\tau = T^{lim}_{e^{i\tau p}_\otimes} + A^p_\tau(Id - T_1). \tag{2.8}$$
Moreover we have the identities:
$$\begin{cases} \partial_\tau[T^{lim}_{e^{i\tau p}_\otimes}h_0] = iT_pT^{lim}_{e^{i\tau p}_\otimes}h_0, \\ T^{lim}_{e^{i\tau p}_\otimes}h_0\ _{|\tau=0} = T_1h_0. \end{cases} \text{for } h_0 \in H^s(\mathbb{D}), s \in \mathbb{R}. \tag{2.9}$$
$$T^{lim}_{e^{i\tau p}_\otimes} = T_{e^{i\tau p}} + \int_0^\tau A^p_{\tau-s}\big(T_{ip}T_{e^{isp}} - T_{ipe^{isp}}\big)ds. \tag{2.10}$$

Combining the previous Proposition with Theorem B.2 we get the following:

Corollary 2.1. *Consider two real numbers $\delta < 1$, $\rho \geq 0$ and a symbol $p \in \Gamma^\delta_\rho(\mathbb{D})$ such that:*

$$\mathrm{Im}(p) = \frac{p - p^*}{2i} \in \Gamma^{\tilde{\delta}}_0(\mathbb{D}), \text{ with } \tilde{\delta} \leq 0.$$

Then $T^{lim}_{e^{i\tau p}_{\otimes}}$ is continuous from C^s_* to $C^{s-\frac{\delta}{2}}_*$ and from $W^{s+(\frac{1}{2}-\frac{1}{p})\delta,p}$ to $W^{s,p}$ for $s > 0$. Moreover we have the estimate:

$$\left\| T^{lim}_{e^{i\tau p}_{\otimes}} \right\|_{W^{s+(\frac{1}{2}-\frac{1}{p})\delta,p} \to W^{s,p}} \leq K^{\ 1-\delta,\delta} M^0_0(e^{i\tau p}_{\otimes}; 1), \text{ and,}$$

$$\left\| T^{lim}_{e^{i\tau p}_{\otimes}} \right\|_{C^{s+\frac{1}{2}\delta}_* \to C^s_*} \leq K^{\ 1-\delta,\delta} M^0_0(e^{i\tau p}_{\otimes}; 1).$$

Finally another improvement one could make on Proposition 2.2 is on the frequency localization and limit cutoff.

Proposition 2.3. *Consider two real numbers $\delta < 1$, $\rho \geq 0$ and a symbol $p \in \Gamma^\delta_\rho(\mathbb{D})$ such that:*

$$\mathrm{Im}(p) = \frac{p - p^*}{2i} \in \Gamma^{\tilde{\delta}}_0(\mathbb{D}), \text{ with } \tilde{\delta} \leq 0.$$

Let $A^p_\tau, \tau \in \mathbb{R}$ be the flow map defined by Proposition 2.1 with the choice of cutoff $T^{B,n}_{ip}$ with $B \geq 2, b \geq 1$ and $e^{i\tau p}_{\otimes} \in \Gamma^1 S^0_{1-\delta,\delta}(\mathbb{R}^* \times \mathbb{R})$ the symbol given by Proposition 2.2.

$$A^p_\tau = T^{2,b}_{e^{i\tau p}_{\otimes}} + R_\infty, \text{ and,} \tag{2.11}$$

there exists a constant C such that R_∞, verifies,

$$\forall u \in \mathscr{S}', \ \mathrm{supp}\, \mathscr{F}(R_\infty u) \subset B(0, C).$$

Proof. As $B \geq 2$, $i(T^{B,b}_p)^*$ is still a paradifferential operator, thus there exists $(e^{i\tau p}_{\otimes})^*$ such that:

$$(A^p_\tau)^* = A^{(T^{B,b}_p)^*}_\tau = T^{lim}_{(e^{i\tau p}_{\otimes})^*} + A^{(T^{B,b}_p)^*}_\tau (Id - (T^{B,b}_1)^*),$$

On the other hand conjugating (2.8) we have:

$$(A^p_\tau)^* = \left(T^{lim}_{e^{i\tau p}_{\otimes}}\right)^* + (A^p_\tau)^*(Id - (T^{B,b}_1)^*).$$

Now if one calls a^* the symbol of the adjoint of an operator $\mathrm{Op}(a)$, then we have the following identify:

$$\mathscr{F}_x(a^*)(\eta - \xi, \xi) = \overline{\mathscr{F}_x(a)(\xi - \eta, \xi)}. \tag{2.12}$$

Thus writing:

$$\mathscr{F}_x\left(T^{lim}_{(e^{i\tau p}_{\otimes})^*} + A^{(T^{B,b}_p)^*}_\tau (Id - (T^{B,b}_1)^*)\right)(\eta, \xi) = \mathscr{F}_x\left(T^{lim}_{(e^{i\tau p}_{\otimes})^*} + A^{(T^{B,b}_p)^*}_\tau (Id - (T^{B,b}_1)^*)\right)(\eta, \xi),$$

and applying identity (2.12) we get the desired result in the zones $\xi \geq 0, \eta \leq 0$ and $\xi \leq 0, \eta \geq 0$. For the other two zones we see that:

$$\mathscr{F}_x(a)(\eta, \xi) = \overline{\mathscr{F}_x(a)(-\eta, \xi)},$$

and applying the previous result to $\bar{p}$ we obtain the desired result by symmetry. □

The key commutation and conjugation result is given by the following proposition.

Proposition 2.4. *Consider two real numbers $\delta < 1$, $\rho \geq 0$ and a symbol $p \in \Gamma^\delta_\rho(\mathbb{D})$ such that:*

$$\mathrm{Im}(p) = \frac{p - p^*}{2i} \in \Gamma^{\tilde{\delta}}_0(\mathbb{D}), \text{ with } \tilde{\delta} \leq 0.$$

Let $A^p_\tau, \tau \in \mathbb{R}$ be the flow map defined by Proposition 2.1 and take a symbol $b \in \Gamma^\beta_\rho(\mathbb{D}), \beta \in \mathbb{R}$ then we have:

(5) For $\rho \geq 1$, there exists $b^p_\tau \in W^{\rho,\infty}S^{\beta}_{1-\delta,\delta}(\mathbb{D})$ such that:

$$A^p_\tau \circ T_b \circ A^p_{-\tau} = T^{lim}_{b^p_\tau}. \tag{2.13}$$

Moreover we have the estimates:

$$\left\| T^{lim}_{b^p_\tau} - \sum_{k=0}^{\lceil \rho-1 \rceil} \frac{\tau^k}{k!} \mathfrak{L}^k_{iT_p} T_b \right\|_{H^s \to H^{s-\beta-\lceil \rho \rceil \delta+\rho}} \leq C_\rho M^\beta_\rho(b) M^\delta_\rho(p)^{\lceil \rho \rceil}, \tag{2.14}$$

$${}_{1-\delta,\delta}H^\beta_\rho(b^p_\tau;k) \leq C_{\rho,k} e^{\tau C M^0_0(\mathrm{Im}(p))}[H^\beta_\rho(b;k) + H^\beta_\rho(b;k)H^\delta_\rho(p;k)], \ k \in \mathbb{N}, \tag{2.15}$$

where $C_{\rho,k}$ is a constant depending only on ρ and k.

(6) There exists ${}^c b^p_\tau \in W^{\rho-1,\infty}S^{\beta+\delta-1}_{1-\delta,\delta}(\mathbb{D})$ such that:

$$[A^p_\tau, T_b] = A^p_\tau T^{lim}_{{}^c b^p_\tau} \iff T^{lim}_{{}^c b^p_\tau} = T_b - T^{lim}_{b^p_{-\tau}}. \tag{2.16}$$

Moreover we have the estimates:

$$\left\| T^{lim}_{{}^c b^p_\tau} - \sum_{k=1}^{\lceil \rho-1 \rceil} (-1)^{k-1} \frac{\tau^k}{k!} \mathfrak{L}^k_{iT_p} T_b \right\|_{H^s \to H^{s-\beta-\lceil \rho \rceil \delta+\rho}} \leq C_\rho M^\beta_\rho(b) M^\delta_\rho(p)^{\lceil \rho \rceil}, \tag{2.17}$$

$${}_{1-\delta,\delta}H^{\beta+\delta-1}_{\rho-1}({}^c b^p_\tau;k) \leq C_{\rho,k} e^{\tau C M^0_0(\mathrm{Im}(p))}[H^\beta_\rho(b;k) + H^\beta_\rho(b;k)H^\delta_\rho(p;k)], k \in \mathbb{N}, \tag{2.18}$$

where $C_{\rho,k}$ is a constant depending only on ρ and k.

The link between b^p_τ and ${}^c b^p_\tau$ is given by the following:

$$T^{lim}_{{}^c b^p_\tau} = \int_0^\tau A^p_r T_{\mathfrak{L}_{ip} b} A^p_{-r} dr = \int_0^\tau T^{lim}_{(\mathfrak{L}_{ip} b)^p_r} dr,$$

where $\mathfrak{L}_{ip} b$ is the paradifferential symbol associated to $\mathfrak{L}_{iT_p} T_b$ by Theorem A.3. Moreover by Proposition 2.2 we have the following more precise frequency cut-off:

$$A^p_\tau \circ T^{\psi^{B,R}}_b \circ A^p_{-\tau} = T^{\psi^{2\star B \star 2,R}}_{b^p_\tau} + R_\infty, \tag{2.19}$$

where A^p_τ is defined by the choice of cut-off $T^{\psi^{B',R}}_p$, $B' < 1$, $B \star B' = \frac{BB'}{B+B'-1}$ and,

$$\begin{aligned} R^\infty = &- T^{\psi^{2,R}}_{e^{ip}} T^{\psi^{B,R}}_b A^p_{-1}(Id - T^{\psi^{B',R}}_1) - A^p_1(Id - T^{\psi^{B',R}}_1) T^{\psi^{B,R}}_p T^{\psi^{2,R}}_{e^{-ip}} \\ &- A^p_1(Id - T^{\psi^{B',R}}_1) T^{\psi^{B,R}}_b A^p_{-1}(Id - T^{\psi^{B',R}}_1). \end{aligned}$$

Remark 2.2.
- *It is important to notice that the main result of this proposition is the factorization of the A^p_τ terms in (2.13) and (2.16) where the right hand sides contain symbols in the usual classes. This was not apriori the case of the left hand sides containing A^p_τ. In other words we study the stability of Γ^m_ρ under the conjugation by A^p_τ.*
- *In the language of pseudodifferential operators, $T_{b^p_\tau}$ is the asymptotic sum of the series $(\frac{\tau^k}{k!}\mathfrak{L}^k_{iT_p} T_b)$ i.e the Baker-Campbell-Hausdorff formal series. Though $T_{b^p_\tau}$ is not necessarily equal to this sum, for this sum need not converge.*

Remark 2.3. *We would like to note that in the special case $\delta = 0$ we have the refined tame estimates for $k \in \mathbb{N}$:*

$$M^\beta_0(\partial^k_\xi b^p_\tau;0) \leq \sum_{j=0}^k \sum_{l=0}^j \binom{k}{j}\binom{j}{l} M^0_0(\partial^{k-j}_\xi \otimes e^{i\tau p};0) M^\beta_0(\partial^{j-l}_\xi b^p_\tau;0) M^0_0(\partial^l_\xi \otimes e^{-i\tau p};0). \tag{2.20}$$

We won't explicitly use the tameness in our proof as we avoid using a Nash-Moser type scheme but it's worth noting that implicitly it is this condition that ensures *that the constructions in Section 3 converge, for more details on the necessity of this condition we refer to the following complete and instructive article by Hamilton [15].*

Proof. This is the consequences of the Leibniz formula combined with the computation of $[ix, b^p_\tau]$:

$$[ix, A^p_\tau T_b A^p_{-\tau}] = [ix, A^p_\tau]T_b A^p_{-\tau} + A^p_\tau [ix, T_b] A^p_{-\tau} + A^p_\tau T_b [ix, A^p_{-\tau}].$$

□

The different Gateaux derivatives of the operators defined above are given by the following propositions.

Proposition 2.5. *Consider two real numbers $\delta < 1$, $\rho \geq 0$, two symbols $p, p' \in \Gamma^\delta_\rho(\mathbb{D})$ such that,*

$$\mathrm{Im}(p) = \frac{p - p^*}{2i} \in \Gamma^{\tilde{\delta}}_0(\mathbb{R}), \mathrm{Im}(p') = \frac{p' - p'^*}{2i} \in \Gamma^{\tilde{\delta}}_0(\mathbb{R}),\ \tilde{\delta} \leq 0.$$

Let $A^p_\tau, A^{p'}_\tau, \tau \in \mathbb{R}$ be the flow maps defined by Proposition 2.1, then for $\tau \in \mathbb{R}$ we have:

$$A^p_\tau - A^{p'}_\tau = \int_0^\tau A^p_{\tau - r} T_{ip' - p} A^{p'}_r dr. \tag{2.21}$$

Another way to express this is with the Gateaux derivative of $p \mapsto A^p_\tau$ on the Fréchet space $\Gamma^\delta_\rho(\mathbb{D})$ is given by:

$$D_p A^p_\tau(h) = \int_0^\tau A^p_{\tau - r} T_{ih} A^p_r dr. \tag{2.22}$$

Moreover consider an open interval $I \subset \mathbb{R}$, and a symbols $p \in C^1(I, \Gamma^\delta_\rho(\mathbb{D}))$ such that for all $z \in I$:

$$\mathrm{Im}(p(z)) = \frac{p - p^*}{2i} \in \Gamma^{\tilde{\delta}}_0(\mathbb{R}),$$

Let $A^p_\tau, \tau \in \mathbb{R}$ be the flow map defined by Proposition 2.1 then for $\tau \in \mathbb{R}, z \in I$ we have:

$$\partial_z A^p_\tau = \int_0^\tau A^p_{\tau - r} T_{i\partial_z p} A^p_r dr. \tag{2.23}$$

Proposition 2.6. *Consider two real numbers $\delta < 1$, $\rho > 1$, $\rho \notin \mathbb{N}$, and two symbols $p, p' \in \Gamma^\delta_\rho(\mathbb{D})$ verifying:*

$$\mathrm{Im}(p) = \frac{p - p^*}{2i} \in \Gamma^{\tilde{\delta}}_0(\mathbb{R}), \mathrm{Im}(p') = \frac{p' - p'^*}{2i} \in \Gamma^{\tilde{\delta}}_0(\mathbb{R}),\ \tilde{\delta} \leq 0.$$

Let $A^p_\tau, A^{p'}_\tau, \tau \in \mathbb{R}$ be the flow maps defined by Proposition 2.1 and take a symbol $b \in \Gamma^\beta_\rho(\mathbb{R})$ then for $\tau \in \mathbb{R}$ we have:

$$T^{lim}_{b^p_\tau} - T^{lim}_{b^{p'}_\tau} = \int_0^\tau A^p_{\tau - r} \mathscr{L}_{iT_{p-p'}} T^{lim}_{b^{p'}_r} A^p_{r - \tau} dr \tag{2.24}$$

$$= \int_0^\tau \mathscr{L}_{iT^{lim}_{p-(p')^p_{\tau-r}}} T^{lim}_{(b^{p'}_r)^p_{\tau - r}} dr. \tag{2.25}$$

Another way to express this is with the Gateaux derivative of $p \mapsto T^{lim}_{b^p_\tau}$ on the Fréchet space $\Gamma^\delta_\rho(\mathbb{R})$ is given by:

$$D_p T^{lim}_{b^p_\tau}(h) = \int_0^\tau \mathscr{L}_{iT^{lim}_{h^p_{\tau-r}}} T^{lim}_{b^p_\tau} dr = \mathscr{L}_{i\int_0^\tau T^{lim}_{h^p_{\tau-r}} dr} T^{lim}_{b^p_\tau}. \tag{2.26}$$

Writing, $T^{lim}_{cb^p_\tau} = T_b - T^{lim}_{b^p_{-\tau}}$, *and,* $T^{lim}_{cb^{p'}_\tau} = T_b - T^{lim}_{b^{p'}_{-\tau}}$ *we get:*

$$T^{lim}_{cb^p_\tau} - T^{lim}_{cb^{p'}_\tau} = -\int_0^{-\tau} A^p_{-\tau-r}\mathscr{L}_{iT_{p-p'}} T^{lim}_{b^{p'}_r} A^p_{r+\tau} dr \tag{2.27}$$

$$= -\int_0^{-\tau} \mathscr{L}_{iT^{lim}_{p-(p')^p_{-\tau-r}}} T^{lim}_{(b^{p'}_r)^p_{-\tau-r}} dr. \tag{2.28}$$

$$D_p T^{lim}_{cb^p_\tau}(h) = -\int_0^{-\tau} \mathscr{L}_{iT^{lim}_{h^p_{-\tau-r}}} T^{lim}_{b^p_{-\tau}} dr = -\mathscr{L}_{i\int_0^{-\tau} T^{lim}_{h^p_{-\tau-r}} dr} T^{lim}_{b^p_{-\tau}}. \tag{2.29}$$

We now study the composition of two different flows.

Theorem 2.1. *Consider two real numbers* $\delta < 1$, $\rho \geq 0$, *two symbols* $p, p' \in \Gamma^\delta_\rho(\mathbb{D})$ *such that,*

$$\operatorname{Im}(p) = \frac{p - p^*}{2i} \in \Gamma^{\tilde\delta}_0(\mathbb{R}), \operatorname{Im}(p') = \frac{p' - p'^*}{2i} \in \Gamma^{\tilde\delta}_0(\mathbb{R}),\ \ \tilde\delta \leq 0.$$

Then for $\tau \in \mathbb{R}$ *we have:*

$$A^p_\tau A^{p'}_\tau = \tilde A^{p+(p')^p_\tau}_\tau,$$

where $\tilde A^{p+(p')^p_\tau}_\tau$ *is understood as the flow generated by* $iT_p + iT^{lim}_{(p')^p_\tau}$.

Remark 2.4. *Strictly speaking we only presented flows that were generated by operators independent of the* τ *variable which is not the case of* $\tilde A^{p+(p')^p_\tau}_\tau$. *We did so to avoid burdening the presentation, one can see all the results of this section can in verbatim be generalized to operators with Lipschitz dependence on* τ *by the usual Cauchy-Lipschitz theorem.*

Moreover by Appendix A.2 $T^{lim}_{(p')^p_\tau}$ *enjoys all of the same properties as a paradifferential operator with the usual cut-off except the continuity for* $s \leq 0$. *We still recover the continuity for* $s \leq 0$ *by the continuity of* A^p_τ *for* $s \leq 0$. *We can also recover the continuity for* $s \leq 0$ *by Proposition 2.3.*

Proof. Fix $h_0 \in H^s, s \in \mathbb{R}$ and compute:

$$\partial_\tau[A^p_\tau A^{p'}_\tau h_0] = -iT_p[A^p_\tau A^{p'}_\tau h_0] - i[A^p_\tau T_{p'} A^{p'}_\tau h_0],$$

thus by Proposition 2.4,

$$\partial_\tau[A^p_\tau A^{p'}_\tau h_0] = -i(T_p + T^{lim}_{(p')^p_\tau})[A^p_\tau A^{p'}_\tau h_0],$$

and $A^p_\tau A^{p'}_\tau h_0(0,\cdot) = h_0(\cdot)$ which gives the desired result. ∎

3. Implicit construction of symbols

In this section with the different theorems permitting the construction of the gauge transforms used in the subsequent sections.

Theorem 3.1. *Consider two real numbers* $\alpha \geq 1$, $\beta \in \mathbb{R}$ *and a symbol* $a \in \Gamma^\beta_0(\mathbb{D})$. *Then there exists* $B > 1$ *and a symbol* $p \in \Gamma^{\beta+1-\alpha}_1(\mathbb{D})$ *such that,*

$$\sigma^{B,b}_p \otimes \sigma^{B,b}_{\xi|\xi|^{\alpha-1}} - \sigma^{B,b}_{\xi|\xi|^{\alpha-1}} \otimes \sigma^{B,b}_p = \sigma^{B,b}_a, \tag{3.1}$$

where $\otimes$ *is the symbol product defined formally by:*

$$\operatorname{Op}(p)\circ\operatorname{Op}(q) = \operatorname{Op}(p\otimes q),\ \ \textit{where}$$

$$p\otimes q(x,\xi) = \frac{1}{2\pi}\int_{\mathbb{D}\times\hat{\mathbb{D}}} e^{i(x-y).(\xi-\eta)} p(x,\eta) q(y,\xi) dy d\eta.$$

Moreover we have the estimates:

$$M_0^{\beta+1-\alpha}(\partial_x \sigma_p^{B,b};0) \leq \frac{M_0^{\beta}(\sigma_a^{B,b};0)}{B[1-(1-\frac{1}{B})^{\alpha}]}, \tag{3.2}$$

$$\begin{aligned} M_0^{\beta-\alpha}(\partial_\xi \partial_x \sigma_p^{B,b};0) \leq & \frac{M_0^{\beta-1}(\partial_\xi \sigma_a^{B,b};0)}{B[1-(1-\frac{1}{B})^{\alpha}]} \\ & + \alpha \frac{(1+\frac{1}{B})^{\alpha-1}-1}{1-(1-\frac{1}{B})^{\alpha}} M_0^{\beta+1-\alpha}(\partial_x \sigma_p^{B,b};0). \end{aligned} \tag{3.3}$$

Remark 3.1. *The choice of the same cut-off parameters in the right hand side and left hand side of* (3.1) *is not immediate, indeed by the general rule of composition of paradifferential operators given in Proposition A.3, the cut-off on the left hand side is given by* $B \star B = \frac{B^2}{2B-1} > B$*. But in the specific where one of the operators is a Fourier multiplier we have this refined property where the cut-off on the left hand side is indeed given by* B*.*

Proof. First the case $\alpha = 1$ has the immediate solution with the choice of p as the primitive of $\sigma_a^{B,b}$ in the x variable. Henceforth we suppose $\alpha > 1$.

We start by defining the scale of Banach spaces that define the Fréchet space of Paradifferential operators.

Definition 3.1. *Given* $m \in \mathbb{R}$*,* $k \in \mathbb{N}$ *and* $\mathscr{W} \subset \mathscr{S}'$ *a Banach space. Define* $C^k\Gamma^m_{\mathscr{W}}(\mathbb{D} \times \hat{\mathbb{D}} \setminus B(0,R))$ *as the space of locally bounded functions* $a(x,\xi)$ *defined on* $\mathbb{D} \times \hat{\mathbb{D}} \setminus B(0,R)$*, which are* C^k *with respect to* ξ *and such that, for all* $j \leq k$ *and for all* $\xi \geq R$*, the function* $x \mapsto \partial_\xi^\alpha a(x,\xi)$ *belongs to* $\mathscr{W}$ *and there exists a constant* C_k *such that:*

$$\forall\, |\xi| \geq R, j \leq k, \left\| \partial_\xi^j a(.,\xi) \right\|_{\mathscr{W}} \leq C_k(1+|\xi|)^{m-|\alpha|}. \tag{3.4}$$

The space $C^k\Gamma^m_{\mathscr{W}}(\mathbb{D} \times \hat{\mathbb{D}} \setminus B(0,R))$ *is equipped with it's natural Banach space topology by the best constant* C_k*. When* $\mathscr{W} = W^{\rho,\infty}$*, the best constant is the seminorm* $M_\rho^\beta(\cdot;k)$*.*

We define:

$$\psi^{B,b}\Big(\Gamma^m_{\mathscr{W}}(\mathbb{D})\Big) = \Big\{\sigma_p^{B,b}, p \in \Gamma^m_{\mathscr{W}}(\mathbb{D})\Big\},$$

$$\psi^{B,b}\Big(C^k\Gamma^m_{\mathscr{W}}(\mathbb{D} \times \hat{\mathbb{D}} \setminus \mathrm{B}(0,R))\Big) = \Big\{\sigma_p^{B,b}, p \in C^k\Gamma^m_{\mathscr{W}}(\mathbb{D} \times \hat{\mathbb{D}} \setminus \mathrm{B}(0,R))\Big\},$$

equipped with their natural Fréchet and Banach topologies induced by the continuity of the map $p \mapsto \sigma_p^{B,b}$. Now we reinterpret Theorem 3.1 by introducing the linear operator:

$$\begin{aligned} L : & C^k\Gamma_1^{\beta+1-\alpha}(\mathbb{D} \times \hat{\mathbb{D}} \setminus \mathrm{B}(0,b)) \to \psi^{B,b}\Big(C^k\Gamma_0^{\beta}(\mathbb{D} \times \hat{\mathbb{D}} \setminus \mathrm{B}(0,b))\Big) \\ & p \mapsto \sigma_p^{B,b} \otimes \sigma_{\xi|\xi|^{\alpha-1}}^{B,b} - \sigma_{\xi|\xi|^{\alpha-1}}^{B,b} \otimes \sigma_p^{B,b}. \end{aligned}$$

The proof will then proceed in two steps, we first prove that L has a right inverse on the Banach space $C^0\Gamma_1^{\beta+1-\alpha}(\mathbb{R} \setminus \mathrm{B}(0,b))$ and then we prove propagation of regularity in the frequency variable ξ for solutions of the equation (3.1).

To construct a right inverse for L the key idea here is simply that a right hand parametrix is given by the standard Cole-Hopf gauge transform:

$$\begin{aligned} E_{approx} :&C^k\Gamma^\beta_0(\mathbb{D}\times\hat{\mathbb{D}}\setminus \mathrm{B}(0,b)) \rightarrow \psi^{B,b}\Big(C^k\Gamma^{\beta+1-\alpha}_0(\mathbb{D}\times\hat{\mathbb{D}}\setminus \mathrm{B}(0,b))\Big)\\ &a(x,\xi)\rightarrow \frac{|\xi|^{1-\alpha}}{i\alpha}\operatorname{Op}(\frac{1}{D})[\sigma^{B,b}_a(\cdot,\xi)](x,\xi), \end{aligned}$$

where,

$$\mathscr{F}_x(\operatorname{Op}(\frac{1}{D}[\sigma^{B,b}_a(\cdot,\xi)])(\eta)=\frac{1}{i\eta}\mathscr{F}_x(\sigma^{B,b}_a(x,\xi))(\eta),$$

which is well defined as $P_0(D)\operatorname{Op}(\frac{1}{D})[\sigma^{B,b}_a(\cdot,\xi)]=0$ where $P_0(D)$ is the Littelwood-Paley projector defined in Section A.1. We then compute:

$$L\circ E_{approx}=\sigma^{B,b}_{\cdot}(Id-r),\ \text{where:}$$

$$\begin{aligned} r :&C^{\beta+1-\alpha}\Gamma^m_0(\mathbb{D}\times\hat{\mathbb{D}}\setminus \mathrm{B}(0,b)) \rightarrow \psi^{B,b}\Big(C^k\Gamma^{\beta+1-\alpha}_0(\mathbb{D}\times\hat{\mathbb{D}}\setminus \mathrm{B}(0,b))\Big)\\ a&\mapsto \frac{\alpha(\alpha-1)}{4\pi}\int_0^1\int_{\mathbb{D}\times\hat{\mathbb{D}}}e^{i(x-y)\eta}\sigma^{B,b}_{\xi|\xi|^{\alpha-2}}(\xi+t\eta)\psi^{B,b}(\eta,\xi)\sigma^{B,b}_{\frac{1}{i\alpha}\partial_x a|\xi|^{1-\alpha}}(y,\xi)dyd\eta dt. \end{aligned}$$

Let us remark that the remainder can also be written as:

$$r(a)(x,\xi)=\frac{\alpha-1}{2}\underbrace{\operatorname{Op}_x\Big(\int_0^1\sigma^{B,b}_{\xi|\xi|^{\alpha-2}}(\xi+t\eta)\psi^{B,b}(\eta,\xi)dt\Big)}_{(*)}[\sigma^{B,b}_{\frac{1}{i}\partial_x a|\xi|^{1-\alpha}}](x,\xi),$$

where $(*)$ is seen a Fourier multiplier in the x variable for ξ fixed. Thus estimating the semi-norms of $r(a)$ using the continuity of Fourier multipliers combined with the Bernstein inequalities we get we get by the frequency localization of Paradifferential operators:

$$M^{\beta+1-\alpha}_0(r(a);0)\leq\frac{C}{B}M^{\beta+1-\alpha}_0(\sigma^{B,b}_a;0).$$

Thus for B sufficiently large L has a right inverse on $\psi^{B,b}\Big(C^0\Gamma^{\beta+1-\alpha}_1(\mathbb{R}\setminus \mathrm{B}(0,b))\Big)$ given by the Neumann series:

$$E=\sum_{k=0}^{+\infty}E_{approx}r^k.$$

Thus we have constructed a $p\in C^0\Gamma^{\beta+1-\alpha}_1(\mathbb{R}\setminus \mathrm{B}(0,b))$ such that:

$$\sigma^{B,b}_p\otimes\sigma^{B,b}_{\xi|\xi|^{\alpha-1}}-\sigma^{B,b}_{\xi|\xi|^{\alpha-1}}\otimes\sigma^{B,b}_p=\sigma^{B,b}_a.$$

Now we want to prove that $p\in C^k\Gamma^{\beta+1-\alpha}_1(\mathbb{R}\setminus \mathrm{B}(0,b))$ for all k. We start by the following computation that comes from commuting with ix:

$$\sigma^{B,b}_{\partial_\xi p}\otimes\sigma^{B,b}_{\xi|\xi|^{\alpha-1}}-\sigma^{B,b}_{\xi|\xi|^{\alpha-1}}\otimes\sigma^{B,b}_{\partial_\xi p}=\sigma^{B,b}_{\partial_\xi a}+\alpha[\sigma^{B,b}_{|\xi|^{\alpha-1}}\otimes\sigma^{B,b}_p-\sigma^{B,b}_p\otimes\sigma^{B,b}_{|\xi|^{\alpha-1}}].\tag{3.5}$$

Now to get the desired bound we use the following Lemma.

Lemma 3.1. *let $p\in\mathscr{S}'(\mathbb{D}\times\hat{\mathbb{D}})$ be a symbol such that for some cut-off parameters B,b we have:*

$$\sigma^{B,b}_p\otimes\sigma^{B,b}_{\xi|\xi|^{\alpha-1}}-\sigma^{B,b}_{\xi|\xi|^{\alpha-1}}\otimes\sigma^{B,b}_p=\sigma^{B,b}_a,$$

for some $a\in C^0\Gamma^\beta_0(\mathbb{R}\setminus B(0,b))$ wit $\alpha>0$ and $\beta\in\mathbb{R}$.

Then $\partial_x \sigma_p^{B,b} \in C^0\Gamma_0^{\beta+1-\alpha}(\mathbb{R}\setminus B(0,b))$ and moreover we have the estimate:

$$M_0^{\beta+1-\alpha}(\partial_x\sigma_p^{B,b};0) \leq \frac{M_0^\beta(\sigma_a^{B,b};0)}{B[1-(1-\frac{1}{B})^\alpha]}.$$

Proof of Lemma 3.1. Without loss of generality, we suppose $p \in \mathscr{S}$ as the result can be deduced by a standard density argument. We rewrite the identity verified by p as follows:

$$\frac{1}{2\pi}\int_0^1\int_{\mathbb{D}\times\hat{\mathbb{D}}} e^{i(x-y)\eta}\sigma^{B,b}_{|\xi|^{\alpha-1}}(\xi+t\eta)\psi^{B,b}(\eta,\xi)\sigma^{B,b}_{\partial_x p}(y,\xi)dyd\eta dt = \frac{i}{\alpha}\sigma_a^{B,b},$$

thus,

$$\underbrace{\mathrm{Op}_x\left(\int_0^1 \sigma^{B,b}_{|\xi|^{\alpha-1}}(\xi+t\eta)\psi^{B,b}(\eta,\xi)dt\right)}_{(*)}[\sigma^{B,b}_{\partial_x p}](x,\xi) = \frac{i}{\alpha}\sigma_a^{B,b},$$

where $(*)$ is an elliptic Fourier multiplier in the x variable for ξ fixed with the bound:

$$\frac{B}{\alpha}[1-(1-\frac{1}{B})^\alpha]\,|\xi|^{\alpha-1} = \int_0^t \left(1-\frac{t}{B}\right)^{\alpha-1}dt\,|\xi|^{\alpha-1} \leq \int_0^1 \sigma^{B,b}_{|\xi|^{\alpha-1}}(\xi+t\eta)\psi^{B,b}(\eta,\xi)dt$$

which gives the desired result. □

Getting back to the proof of Theorem 3.1 and applying Lemma 3.1 to (3.5) we get $p \in C^1\Gamma_1^{\beta+1-\alpha}(\mathbb{R}\setminus \mathrm{B}(0,b))$ and iterating (3.5) we get $p \in C^k\Gamma_1^{\beta+1-\alpha}(\mathbb{R}\setminus \mathrm{B}(0,b))$ for all k. □

A corollary of Lemma 3.1 is the following unicity result.

Corollary 3.1. *let $p \in \mathscr{S}'(\mathbb{D}\times\hat{\mathbb{D}})$ be a symbol such that for some cut-off parameters B, b we have:*

$$\sigma_p^{B,b}\otimes\sigma^{B,b}_{\xi|\xi|^{\alpha-1}} - \sigma^{B,b}_{\xi|\xi|^{\alpha-1}}\otimes\sigma_p^{B,b} = 0.$$

Then $\sigma_p^{B,b}$ is a Fourier multiplier, i.e there exits a Fourier multiplier m such that:

$$\sigma_p^{B,b}(x,\xi) = \sigma_m^{B,b}(x,\xi) = \psi^{B,b}(0,\xi)m(\xi).$$

We now give the main Theorem that permits the construction of the gauge transform.

Theorem 3.2. *Consider two real numbers $\alpha \geq 1$, $\beta \leq \alpha-1$ and a symbol $a \in \Gamma_0^\beta(\mathbb{R})$. Then there exists $\epsilon > 0, B \geq \frac{4}{3}$ such that for,*

$$M_0^\beta(a;0) \leq \epsilon,$$

there exits a symbol p such that $p \in \Gamma_1^{\beta+1-\alpha}(\mathbb{R})$ and:

$$\sigma^{B,b}_{\xi|\xi|^{\alpha-1}} - \sigma^{2,b}_{\otimes e^{ip}}\otimes\sigma^{B,b}_{\xi|\xi|^{\alpha-1}}\otimes\sigma^{2,b}_{\otimes e^{-ip}} = \sigma_a^{B,b}. \tag{3.6}$$

Moreover we have the estimates:

$$M_0^{\beta+1-\alpha}(\partial_x\sigma_p^{B,b};0) \leq \frac{1+C\epsilon}{\alpha}M_0^\beta(\sigma_a^{B,b};0), \tag{3.7}$$

$$M_0^{\beta-\alpha}(\partial_\xi\partial_x\sigma_p^{B,b};0) \leq (1+CM_0^\beta(\partial_\xi\sigma_a^{B,b};0))\times\left[\frac{M_0^{\beta-1}(\partial_\xi\sigma_a^{B,b};0)}{B[1-(1-\frac{1}{B})^\alpha]}\right. \tag{3.8}$$
$$\left.+\,\alpha\frac{(1+\frac{1}{B})^{\alpha-1}-1}{1-(1-\frac{1}{B})^\alpha}M_0^{\beta+1-\alpha}(\partial_x\sigma_p^{B,b};0)\right].$$

Remark 3.2. *We note that $\beta+1-\alpha \leq 0$ thus the hypothesis on $\mathrm{Im}(p)$ is automatically verified.*

Proof. The proof should in spirit amount to a Nash-Moser scheme as we are looking to prove an implicit function type of result in Fréchet. In our case the problem is simpler due to the following key observation, for $k \geq 0$ the underlining map is well defined on the Banach spaces in the scale defining the Fréchet space of paradifferential operators:

$$\begin{aligned}F :& C^k\Gamma_1^{\beta+1-\alpha}(\mathbb{D}\times\hat{\mathbb{D}}\setminus \mathrm{B}(0,b)) \to \psi^{\frac{4}{3},b}\Big(C^k\Gamma_0^{\beta}(\mathbb{D}\times\hat{\mathbb{D}}\setminus \mathrm{B}(0,b))\Big)\\ &p \mapsto \sigma^{B,b}_{\xi|\xi|^{\alpha-1}} - \sigma^{2,b}_{\otimes e^{ip}}\otimes\sigma^{B,b}_{\xi|\xi|^{\alpha-1}}\otimes\sigma^{2,b}_{\otimes e^{-ip}}\\ F(p) =& \sigma^{\frac{4}{3},b}_{\int_0^1 \sigma^{2,b}_{e^{irp}}\otimes[\sigma^{B,b}_p\otimes\sigma^{B,b}_{\xi|\xi|^{\alpha-1}}-\sigma^{B,b}_{\xi|\xi|^{\alpha-1}}\otimes\sigma^{B,b}_p]\otimes\sigma^{2,b}_{\otimes e^{-irp}}dr}.\end{aligned}$$

The proof will again proceed in two steps, we first prove that F has a right inverse on the Banach space $C^0\Gamma_1^{\beta+1-\alpha}(\mathbb{R}\setminus \mathrm{B}(0,b))$ and then we prove propagation of regularity in the frequency variable ξ for solutions of the equation (3.1).

Noticing that $F(0) = 0$, the goal is thus to prove the local surjectivity of F around the origin. Now that we reduced the problem to Banach spaces, by the inverse function theorem it suffice to find a right inverse to the differential of F at 0. Computing the differential at 0 we get:

$$D_0F(h) = \sigma^{B,b}_h\otimes\sigma^{B,b}_{\xi|\xi|^{\alpha-1}} - \sigma^{B,b}_{\xi|\xi|^{\alpha-1}}\otimes\sigma^{B,b}_h = L(h),$$

Thus by Theorem 3.1 and the local inversion Theorem in Banach spaces we get the desired local surjectivity.

Now we turn to propagation of regularity in the ξ variable, we fix:

$$p\in C^0\Gamma_1^{\beta+1-\alpha}(\mathbb{R}\setminus \mathrm{B}(0,b)) \text{ and } a\in\Gamma_0^{\beta}(\mathbb{R}\setminus \mathrm{B}(0,b)).$$

To make all of the computations rigorous we suppose $p\in\mathscr{S}$ and the desired result is obtained by a density argument. The computation behind the propagation of regularity is the following analogue of (3.5). We start from:

$$\sigma^{B,b}_{\xi|\xi|^{\alpha-1}} - \sigma^{B,b}_{\otimes e^{ip}}\otimes\sigma^{B,b}_{\xi|\xi|^{\alpha-1}}\otimes\sigma^{B,b}_{\otimes e^{-ip}} = \sigma^{B,b}_a,$$

commuting with ix we get:

$$\sigma^{B,b}_{\alpha|\xi|^{\alpha-1}} - \sigma^{B,b}_{\partial_\xi a} = \sigma^{B,b}_{\otimes e^{ip}}\otimes\left[\sigma^{B,b}_{\otimes e^{-ip}}\otimes ix\otimes\sigma^{B,b}_{\otimes e^{ip}}, \sigma^{B,b}_{\xi|\xi|^{\alpha-1}}\right]_{\otimes}\otimes\sigma^{B,b}_{\otimes e^{-ip}},$$

where $[a,b]_{\otimes} = a\otimes b - b\otimes a$. Using the Duhamel formula combined with Proposition 2.2:

$$\sigma^{B,b}_{\otimes e^{-ip}}\otimes ix\otimes\sigma^{B,b}_{\otimes e^{ip}} = ix + \int_0^1 \sigma^{B,b}_{\otimes e^{-irp}}\otimes\sigma^{B,b}_{\partial_\xi p}\otimes\sigma^{B,b}_{\otimes e^{irp}}dr.$$

Thus,

$$\begin{aligned}\sigma^{B,b}_{\otimes e^{-ip}}\otimes\sigma^{B,b}_{\alpha|\xi|^{\alpha-1}}\otimes\sigma^{B,b}_{\otimes e^{ip}} - \sigma^{B,b}_{\alpha|\xi|^{\alpha-1}} - \sigma^{B,b}_{\otimes e^{-ip}}\otimes\sigma^{B,b}_{\partial_\xi a}\otimes\sigma^{B,b}_{\otimes e^{ip}}&\\ = \Big[\int_0^1 \sigma^{B,b}_{\otimes e^{-irp}}\otimes\sigma^{B,b}_{\partial_\xi p}\otimes\sigma^{B,b}_{\otimes e^{irp}}dr, \sigma^{B,b}_{\xi|\xi|^{\alpha-1}}\Big]_{\otimes}.&\end{aligned} \tag{3.9}$$

Applying Lemma 3.1 we get that:

$$\partial_x\left[\int_0^1 \sigma^{B,b}_{\otimes e^{-irp}}\otimes\sigma^{B,b}_{\partial_\xi p}\otimes\sigma^{B,b}_{\otimes e^{irp}}dr\right]\in C^0\Gamma_0^{\beta+1-\alpha}(\mathbb{R}\setminus \mathrm{B}(0,b)),$$

and by the frequency localization of paradifferential operators and the Bernstein inequalities:

$$\int_0^1 \sigma^{B,b}_{\otimes e^{-irp}}\otimes\sigma^{B,b}_{\partial_\xi p}\otimes\sigma^{B,b}_{\otimes e^{irp}}dr\in C^0\Gamma_1^{\beta+1-\alpha}(\mathbb{R}\setminus \mathrm{B}(0,b)).$$

Getting back to the definition of $\otimes e^{-irp}$ as $\beta+1-\alpha \leq 0$:

$$\sigma^{lim,b}_{\otimes e^{irp}} = \sum_{k=0}^{\infty} \frac{i^k r^k}{k!} \otimes^k \sigma^{B,b}_p,$$

thus,

$$\sigma^{lim,b}_{\otimes e^{irp}} = 1 + O_{M^0_0(\cdot;0)}(\epsilon),$$

which gives:

$$\sigma^{B,b}_{\partial_\xi p} \in C^0\Gamma^{\beta+1-\alpha}_1(\mathbb{R} \setminus \mathrm{B}(0,b)).$$

We get the desired result by iteration. □

4. Sobolev estimate on the weakly dispersive Burgers equation

The goal of this section is to prove Theorem 1.2. First it suffice to make the estimate (1.11) for $u_0 \in C^\infty_0$ and deduce the general result by density. We take u a solution to the Cauchy problem (1.10).

By Theorem 3.1 there exists $p \in \Gamma^{2-\alpha}_1(\mathbb{R})$ such that:

$$[T^{B',b}_{ip}, T^{B',b}_{i\xi|\xi|^{\alpha-1}}] = -iT^{B',b}_{\sigma^{B,b}_{u\xi}+(\sigma^{B,b}_{u\xi})^*}, \tag{4.1}$$

where $B' > B$ is the cut-off corresponding to the left hand side that includes $T^{B,b.}$ and it's a adjoint.

Now by Corollary 3.1 as $(T^{B',b}_{\xi|\xi|^{\alpha-1}})^* = T^{B',b}_{\xi|\xi|^{\alpha-1}}$ and the left hand side being L^2 skew-adjoint we get:

$$\left(T^{B',b}_p\right)^* = T^{B',b}_p \text{ in } L^2, \tag{4.2}$$

Henceforth $R_\infty(u)$ will designate a generic infinitely regularizing operator on on Sobolev spaces, i.e $R_\infty(u) : H^\mu \to H^{\mu'}$ for all $\mu, \mu' \in \mathbb{R}$ with the estimate:

$$\|R_\infty(u)\|_{H^\mu \to H^{\mu'}} \leq C_{\mu,\mu',\mu''}(\|u\|_{H^{\mu''}}), \mu'' \in \mathbb{R}.$$

Consider $(A^p_r)_{r\in\mathbb{R}}$ defined by Proposition 2.1 with the choice of cutoff in the paradifferential operator given by B', b. Now by construction we have:

$$A^p_1\partial_t u + T^{B',b}_{i\xi|\xi|^{\alpha-1}} A^p_1 u + \frac{1}{2}\left(\int_0^1 A^p_{1-r}\left[T^{B',b}_{ip}, iT^{B',b}_{\sigma^{B,b}_{u\xi}+(\sigma^{B,b}_{u\xi})^*}\right] A^p_{r-1} dr\right) A^p_1 u = R^\infty(u). \tag{4.3}$$

The key cancellation here is that even though $[T^{B',b}_{ip}, iT^{B',b}_{\sigma^{B,b}_{u\xi}+(\sigma^{B,b}_{u\xi})^*}] \in \Gamma^{3-\alpha}_0$ we have:

$$\left(\int_0^1 A^p_{1-r}\left[T^{B',b}_{ip}, iT^{B',b}_{\sigma^{B,b}_{u\xi}+(\sigma^{B,b}_{u\xi})^*}\right] A^p_{r-1} dr\right)^* = -\int_0^1 A^p_{1-r}\left[T^{B',b}_{ip}, iT^{B',b}_{\sigma^{B,b}_{u\xi}+(\sigma^{B,b}_{u\xi})^*}\right] A^p_{r-1} dr.$$

This gives conservation of the L^2 norm but it presents no concrete gain as the Cauchy problem (1.10) conserves the L^2 norm by a straightforward computation but we find it to be a good self check when making the computations.

At our level of regularity $s > 2-\alpha+\frac{1}{2}$ this key cancellation does not hold for higher Sobolev estimates (except in the "exceptional" case of the Benjamin-Ono equation as noted in the introduction). To remedy this, inspired by [17], we preform an approximation of the symbol p given in (4.1) by step functions in the frequency variable ξ and a Littlewood-Paley decomposition in x.

To get higher Sobolev estimates the goal is estimate the L^2 norm of the k-th dyadic frequency shell of u, for this we start by rewriting (1.10) using p defined by (4.1):

$$\partial_t u - [T^{B',b}_{ip}, T^{B',b}_{i\xi|\xi|^{\alpha-1}}]u + T^{B',b}_{i\xi|\xi|^{\alpha-1}} u = R_\infty, \; u_0 \in H^s, \; b > 0. \tag{4.4}$$

We consider $C_k = [w_1 2^k, w_2 2^k]$ with $w_1 < 1 < w_2$ the k-th dyadic shell, i.e the support of $P_k(D)$. We fix the increasing sequence:

$$a_j = w_1 2^k + j^{\frac{1}{2-\alpha}} 2^{\frac{(1-\alpha)k}{2-\alpha}}, a_{-j} = a_j \text{ for } 0 \leq j \leq \lfloor (w_2 - w_1)^{\frac{1}{2-\alpha}} 2^k \rfloor = j_k,$$

and define the intervals:

$$\begin{cases} I_j = [\frac{2a_{j-1}+a_j}{3}, \frac{2a_j+a_{j+1}}{3}], \text{ for } |j| \leq j_k, \\ I_{j_k} = [j_k, w_2 2^k], I_{-j_k} = [-w_2 2^k, -j_k]. \end{cases}$$

For $|j| \leq j_k$ consider smooth functions $\phi_j : \mathbb{R} \to [0,1]$ supported in I_j and $\phi_j = 0$ for $j \notin [-j_k - 1, j_k + 1]$ such that:

$$\sum_{j \in \mathbb{Z}} \phi_j = 1, \ \left|\partial_\xi^n \phi_j(\xi)\right| \leq C_n |\xi|^{-n}, n \in \mathbb{N}.$$

We now set,

$$p_j^l(x) = P_l(D)\left(\frac{1}{|I_j|}\int_{I_j} p(x,\xi)d\xi\right), |j| \leq j_k, l \leq \lfloor w_2 2^k \rfloor = l_k.$$

We compose (4.4) by the following:

$$\sum_{j=0}^{j_k} \phi_j \prod_{l=1}^{l_k} A_1^{p_j^l} \partial_t u - \sum_{j=0}^{j_k} \phi_j \prod_{l=1}^{l_k} A_1^{p_j^l} [T_{ip}^{B',b}, T_{i\xi|\xi|^{\alpha-1}}^{B',b}] + \underbrace{\sum_{j=0}^{j_k} \phi_j \prod_{l=1}^{l_k} A_1^{p_j^l} T_{i\xi|\xi|^{\alpha-1}}}_{(*)} u = R_\infty, \tag{4.5}$$

and we compute the commutator in $(*)$:

$$\begin{aligned} &\left[\prod_{l=1}^{l_k} A_1^{p_j^l}, T_{i\xi|\xi|^{\alpha-1}}^{B',b}\right] \\ &= \prod_{l=1}^{l_k} A_1^{p_j^l} \sum_{l=1}^{l_k} [T_{ip_j^l}^{B',b}, T_{i\xi|\xi|^{\alpha-1}}^{B',b}] + \prod_{l=1}^{l_k} A_1^{p_j^l} \sum_{l=1}^{l_k} \int_0^1 A_r^{p_j^l} \mathfrak{L}_{ip_j^l}^2 T_{i\xi|\xi|^{\alpha-1}}^{B',b} A_{-r}^{p_j^l} dr \\ &+ \sum_{l=1}^{l_k} \sum_{m=1}^{l-1} \prod_{n=l}^{l_k} A_1^{p_j^n} \mathfrak{L}_{A_1^{p_j^l}} \mathfrak{L}_{A_1^{p_j^m}} T_{i\xi|\xi|^{\alpha-1}}^{B',b} \prod_{n=1}^{l-1} A_1^{p_j^n}. \end{aligned} \tag{4.6}$$

Getting back to (4.5) we get:

$$\begin{aligned} &\underbrace{\sum_{j=0}^{j_k} \phi_j \prod_{l=1}^{l_k} A_1^{p_j^l} \partial_t u}_{(1)} + \underbrace{T_{i\xi|\xi|^{\alpha-1}} \sum_{j=0}^{j_k} \phi_j \prod_{l=1}^{l_k} A_1^{p_j^l} u}_{(2)} \\ &= \underbrace{\sum_{j=0}^{j_k} \phi_j \Big[\prod_{l=1}^{l_k} A_1^{p_j^l} [T_{i(\tilde{\phi}_j p - \sum_{l=1}^{l_k} p_j^l)}^{B',b}, T_{i\xi|\xi|^{\alpha-1}}^{B',b}] u}_{(3)} - \underbrace{\prod_{l=1}^{l_k} A_1^{p_j^l} \sum_{l=1}^{l_k} \int_0^1 A_r^{p_j^l} \mathfrak{L}_{ip_j^l}^2 T_{i\xi|\xi|^{\alpha-1}}^{B',b} A_{-r}^{p_j^l} dru \Big]}_{(4)} \\ &\underbrace{- \sum_{j=0}^{j_k} \phi_j \sum_{l=1}^{l_k} \sum_{m=1}^{l-1} \prod_{n=l}^{l_k} A_1^{p_j^n} \mathfrak{L}_{A_1^{p_j^l}} \mathfrak{L}_{A_1^{p_j^m}} T_{i\xi|\xi|^{\alpha-1}}^{B',b} \prod_{n=1}^{l-1} A_1^{p_j^n} u + R_\infty}_{(5)}, \end{aligned} \tag{4.7}$$

where $\tilde{\phi_j}$ is a bump function with a slightly larger support than ϕ_j.

We now compute the L^2 scalar product of (4.7) with $\sum_{j=0}^{j_k} \phi_j \prod_{l=1}^{l_k} A_1^{p_j^l} u$. Given the complexity of equation 4.7 we will treat each term from (1) to (5) separately.

For term (1):

$$\operatorname{Re}\big((1), \sum_{j=0}^{j_k} \phi_j \prod_{l=1}^{l_k} A_1^{p_j^l} u)_{L^2}\big)$$
$$= \sum_{j=0}^{j_k} \operatorname{Re}\big((\phi_j \prod_{l=1}^{l_k} A_1^{p_j^l} \partial_t u, \phi_j \prod_{l=1}^{l_k} A_1^{p_j^l} u)_{L^2}\big) + 2\operatorname{Re}\big((\phi_{j-1} \prod_{l=1}^{l_k} A_1^{p_{j-1}^l} \partial_t u, \phi_j \prod_{l=1}^{l_k} A_1^{p_j^l} u)_{L^2}\big)$$
$$+ 2\sum_{j=0}^{j_k} \operatorname{Re}\big((\phi_{j+1} \prod_{l=1}^{l_k} A_1^{p_{j+1}^l} \partial_t u, \phi_j \prod_{l=1}^{l_k} A_1^{p_j^l} u)_{L^2}\big),$$

by the L^2 skew symmetry of p (4.2):

$$\operatorname{Re}\big((1), \sum_{j=0}^{j_k} \phi_j \prod_{l=1}^{l_k} A_1^{p_j^l} u)_{L^2}\big) = 5\sum_{j=0}^{j_k} \operatorname{Re}\big((\tilde{\phi_j}\partial_t u, \tilde{\phi_j} u)_{L^2}\big) + R_\infty(u)$$
$$+ \underbrace{2\sum_{j=0}^{j_k} \operatorname{Re}\big(([\phi_{j+1} \prod_{l=1}^{l_k} A_1^{p_{j+1}^l} + \prod_{l=1}^{l_k} A_1^{p_{j-1}^l} - 2\prod_{l=1}^{l_k} A_1^{p_j^l}]\partial_t u, \phi_j \prod_{l=1}^{l_k} A_1^{p_j^l} u)_{L^2}\big)}_{R_1(u)}. \tag{4.8}$$

For $R_1(u)$ we see that the terms,

$$\begin{cases} \phi_{j+1}(D)\phi_j(D)[\phi_{j+1}(D)A^{p_{j+1}^l} - \phi_j(D)A_1^{p_j^l}] \\ \phi_{j-1}(D)\phi_j(D)[\phi_{j-1}(D)A^{p_{j-1}^l} - \phi_j(D)A_1^{p_j^l}] \end{cases}$$

can be seen as residual terms coming from a change of cut-off in the definition of Paradifferential operators. Thus by (A.3) and the choice of a_j we get:

$$\begin{cases} \phi_{j+1}(D)\phi_j(D)[\phi_{j+1}(D)A^{p_{j+1}^l} - \phi_j(D)A_1^{p_j^l}] \\ \phi_{j-1}(D)\phi_j(D)[\phi_{j-1}(D)A^{p_{j-1}^l} - \phi_j(D)A_1^{p_j^l}] \end{cases} \quad \text{are of order } 2-2\alpha,$$

thus, crudely estimating the time derivative by $2^{k\alpha}$ we get the estimate on $R_1(u)$:

$$|R_1(u)| \le C\sum_{j=0}^{j_k}\sum_{l=0}^{j} e^{C\sum_l \|p_j^l\|_{L^\infty_x}} 2^{j(2-\alpha)} \left\|p_j^l\right\|_{L^\infty_x} \left\|\tilde{\phi_j} u\right\|_{L^2}^2 \tag{4.9}$$

For term (2), we have immediately by skew symmetry:

$$\operatorname{Re}\big((2), \sum_{j=0}^{j_k} \phi_j \prod_{l=1}^{l_k} A_1^{p_j^l} u)_{L^2}\big) = 0. \tag{4.10}$$

For term (3), (4) and (5). we have the estimates:

$$\left\|\operatorname{Re}([T^{B',b}_{i(\tilde{\phi_j}p - \sum_{l=1}^{l_k} p_j^l)}, T^{B',b}_{i\xi|\xi|^{\alpha-1}}])\right\|_{L^2\to L^2} \le C2^{j(2-\alpha)} \left\|P_{\le j}(D)u\right\|_{L^\infty_x}, \tag{4.11}$$

$$\left\|\operatorname{Re}(\mathfrak{L}^2_{ip_j^l} T^{B',b}_{i\xi|\xi|^{\alpha-1}})\right\|_{L^2\to L^2} \le C2^{j(2-\alpha)} \left\|P_l(D)u\right\|^2_{L^\infty_x} \tag{4.12}$$

$$\left\|\operatorname{Re}(\mathfrak{L}_{A_1^{p_j^l}} \mathfrak{L}_{A_1^{p_j^m}} T^{B',b}_{i\xi|\xi|^{\alpha-1}})\right\|_{L^2\to L^2}$$
$$\le e^{C\|p_j^l\|_{L^\infty_x} + \|p_j^m\|_{L^\infty_x}} 2^{k(2-\alpha)} \left\|P_l(D)u\right\|_{L^\infty_x} \left\|P_m(D)u\right\|_{L^\infty_x}. \tag{4.13}$$

Finally combining estimates (4.9),(4.10),(4.11),(4.12) and (4.13) we get:

$$\frac{d}{dt}\left\|P_k(D)u\right\|_{L^2}^2 \leq C(1+\left\|P_{\leq k}(D)u\right\|_{L^\infty})2^{j(2-\alpha)}\left\|P_{\leq k}(D)u\right\|_{L^\infty}\left\|P_k(D)u\right\|_{L^2}^2,$$

which gives the desired result by the Littlewood-Paley decomposition and the Gronwall lemma.

5. Complete gauge transform for the dispersive Burgers equation

In contrast to the previous section in the first step to conjugating (1.12) we make the choice, by Theorem 3.1, $p_1 \in \Gamma_1^0(\mathbb{R})$ such that:

$$\sigma^{2,b}_{\otimes e^{ip_1}} \otimes \sigma^{B,b}_{\xi|\xi|^{\alpha-1}} - \sigma^{B,b}_{\xi|\xi|^{\alpha-1}} \otimes \sigma^{2,b}_{\otimes e^{-ip_1}} = \sigma^{B,b}_{iu\xi}, \tag{5.1}$$

$$\operatorname{Im}(p^1) = \frac{p^1 - p^{1*}}{2i} \in \Gamma_0^0(\mathbb{R}),\ M_0^0(\operatorname{Im}(p)) \leq CM_0^0(\partial_x p^1),$$

$$M_0^0(\partial_x p^1(t,\cdot);0) \leq \frac{1+C\epsilon}{\alpha}M_0^{\alpha-1}(\sigma^{B,b}_{iu\xi};0), \tag{5.2}$$

$$M_0^{-1}(\partial_\xi\partial_x p^1;0) \leq (1+CM_0^{\alpha-2}(u;0)) \times \left[\frac{M_0^{\alpha-2}(u;0)}{\alpha} + (\alpha-1)M_0^0(\partial_x p^1;0)\right]. \tag{5.3}$$

where we implicitly used the Bernstein inequalities to see the injection:

$$u \in C_*^{2-\alpha} \Rightarrow \sigma^{B,b}_{iu\xi} \in \Gamma_0^{\alpha-1}.$$

Define $u^1 = A_1^{p^1}u$ and commute with (1.12) we get by Proposition 2.5:

$$\partial_t u^1 + T^{\psi^{B,b}}_{i\xi|\xi|^{\alpha-1}}u^1 - T^{2,b}_{i\int_0^1(\partial_t p^1)_r^{p^1}dr}u^1 = R^{1,\infty},\ u^1(0,\cdot) = A_1^{p^1}u_0(\cdot). \tag{5.4}$$

Remark 5.1. *For the proof of well-posedness we also need to see that for 2 different solutions u, v we have by the proof of Theorem 3.2:*

$$M_0^0([p^1(u)-p^1(v)](t,\cdot);1) \leq e^{C\|(u,v)\|_{C_*^{2-\alpha}}}\left\|\operatorname{Op}(\frac{1}{D})P_{\geq b}[u-v](t,\cdot)\right\|_{C_*^{2-\alpha}}.$$

Let us asses what we obtained from this first gauge transform we eliminated the term:

$$\sigma^{B,b}_{iu\xi} \in \Gamma_0^{\alpha-1}(\mathbb{R})$$

which generates an operator of order $\alpha-1$ and obtained $i\int_0^1(\partial_t p^1)_r^{p^1}dr$ which under mere $C_*^{2-\alpha}$ control on u is still in $\Gamma_0^{\alpha-1}(\mathbb{R})$ which is still of order $\alpha-1$. Thus apriori in the low regularity setting in which we are working, we do not have a gain on the order of the remainder. The key idea is then to iterate this gauge transform, prove that the remainder goes to 0 and that the successive application of this gauge transform converges.

Applying successively Theorem 3.2 we construct a series of symbols $p^j \in \Gamma_1^0(\mathbb{R}), j \geq 2$ such that:

$$\sigma^{2,b}_{\otimes e^{ip_j}} \otimes \sigma^{B,b}_{\xi|\xi|^{\alpha-1}} - \sigma^{B,b}_{\xi|\xi|^{\alpha-1}} \otimes \sigma^{2,b}_{\otimes e^{-ip_j}} = \sigma^{2,b}_{\int_0^1(\partial_t p^{j-1})_r^{p^{j-1}}dr}, \tag{5.5}$$

$$\operatorname{Im}(p^j) = \frac{p^j - p^{j*}}{2i} \in \Gamma_0^0(\mathbb{R}),\ M_0^0(\operatorname{Im}(p^j)) \leq CM_0^0(\partial_x p^j),$$

$$\begin{aligned} M_0^0(\partial_x p^j(t,\cdot);0) \leq & \frac{1+Ce^{CM_0^0(p^{j-1}(t,\cdot);0)}M_0^0(\partial_t p^{j-1}|\xi|^{1-\alpha}(t,\cdot);0)}{\alpha} \\ & \times e^{CM_0^0(p^{j-1}(t,\cdot);0)}M_0^0(\partial_t p^{j-1}|\xi|^{1-\alpha}(t,\cdot);0). \end{aligned} \tag{5.6}$$

$$M_0^{-1}(\partial_\xi\partial_x p^j;0) \leq (1+C\times(*))\times\left[\frac{(*)}{\alpha}\right]+C(\alpha-1)e^{CM_0^0(p^{j-1}(t,\cdot);0)}M_0^0(\partial_x p^j;0)$$

where,

$$\begin{aligned}(*) = e^{CM_0^0(p^{j-1}(t,\cdot);0)}[CM_0^0(\partial_t p^{j-1}\,|\xi|^{1-\alpha}\,(t,\cdot);0)M_0^{-1}(\partial_\xi p^{j-1}(t,\cdot);0) \\ + M_0^{-1}(\partial_\xi\partial_t p^{j-1}\,|\xi|^{1-\alpha}\,(t,\cdot);0)]\end{aligned} \quad (5.7)$$

From this we deduce that for $\|u\|_{C_*^{2-\alpha}} \leq r$ with r sufficiently small we have:

$$M_0^0(\partial_x p^j(t,\cdot);0) \leq C\left(\frac{1+Cr}{\alpha}\right)^j \|u(t,\cdot)\|_{C_*^{2-\alpha}}, \quad (5.8)$$

$$M_0^{-1}(\partial_\xi\partial_x p^j(t,\cdot);0) \leq C\left(\frac{1+Cr}{\alpha}\right)^j \|u(t,\cdot)\|_{C_*^{2-\alpha}}. \quad (5.9)$$

Consider $A_1^{p^j}$ defined by Proposition 2.1 with the choice of cutoff in the paradifferential operator given by B, b and define $u^j = A_1^{p^j}u^{j-1}$, by construction we have:

$$\partial_t u^j + T^{\psi^{B,b}}_{i\xi|\xi|^{\alpha-1}}u^j - T^{\psi^{2,b}}_{i\int_0^1(\partial_t p^j)_r^{p^j}\,dr} = R^{j,\infty},\ u^j(0,\cdot) = A_1^{p^j}u^{j-1}(0,\cdot), \quad (5.10)$$

where,

$$R^{j,\infty} = r^j_\infty + A_1^{p^j}R^{j-1,\infty},$$

where r^j_∞ is generated by the different cut-off errors generated by the difference of $A_1^{p^j}$ and $T_{\otimes e^{ip^j}}$.

We now need to study $\prod_{j=1}^{k} A_1^{p^j}$, for this we define for $\tau\in\mathbb{R}$:

$$\tilde{p^1}(\tau,t,x) = p^1(t,x),\ \tilde{p^k}(\tau,t,x) = p^{k+1}(t,x) + (p^k(t,x))_\tau^{p^{k+1}(t,x)}, \quad (5.11)$$

thus by Proposition 2.1:

$$\prod_{j=1}^{k} A_1^{p_j} = [A_\tau^{\tilde{p^k}(\tau,t,x)}]_{\tau=1}. \quad (5.12)$$

We then have the estimates:

$$\sup_{|\tau|\leq 1} M_0^0(\tilde{p^{k+1}}(t,\cdot);0) \leq M_0^0(p^j(t,\cdot);0) + e^{CM_0^0(p^j(t,\cdot);0)}\sup_{|\tau|\leq 1} M_0^0(\tilde{p^k}(t,\cdot);0), \quad (5.13)$$

$$\begin{aligned}&\sup_{|\tau|\leq 1} M_0^{-1}(\partial_\xi \tilde{p^{k+1}}(t,\cdot);0) \leq M_0^{-1}(\partial_\xi p^j(t,\cdot);0) \\ &+ e^{CM_0^0(p^j(t,\cdot);0)}\left[\sup_{|\tau|\leq 1} M_0^{-1}(\partial_\xi\tilde{p^k}(t,\cdot);0) + CM_0^{-1}(\partial_\xi p^j(t,\cdot);0)\sup_{|\tau|\leq 1} M_0^0(\tilde{p^k}(t,\cdot);0)\right],\end{aligned} \quad (5.14)$$

Thus $(\tilde{p^k})_{k\in\mathbb{N}}$ converges for the seminorm $L^\infty_{|\tau|\leq 1}M_0^0(\cdot,1)$, the iteration of this argument shows that $(\tilde{p^k})_{k\in\mathbb{N}}$ converges for all of the seminorm $W^{m,\infty}_{|\tau|\leq 1}M_1^0(\cdot,1), m\in\mathbb{N}^*$, thus there exits $\tilde{p}$ such that:

$$\tilde{p}\in W^{\infty,\infty}([-1,1];C^1\Gamma_1^0(\mathbb{D})),\quad W^{m,\infty}_\tau M_1^0(\tilde{p}(t,\cdot)) \leq C\|u(t,\cdot)\|_{C_*^{2-\alpha}},$$

$$\mathrm{Im}(\tilde{p}) = \frac{\tilde{p}-\tilde{p}^*}{2i} \in \Gamma_0^0(\mathbb{D}) \text{ and } M_0^0(\mathrm{Im}(\tilde{p})) \leq CM_0^0(\partial_x\tilde{p})$$

Thus passing to the limit in (5.10) we get:

$$\partial_t\left[(A_\tau^{\tilde{p}(\tau,\cdot)})_{\tau=1}u\right] + T^{\psi^{B,b}}_{i\xi|\xi|^{\alpha-1}}\left[(A_\tau^{\tilde{p}(\tau,\cdot)})_{\tau=1}u\right] = R^{\infty,\infty}(u), \quad (5.15)$$

where there exists a non decreasing functions C_s, such that $R^{\infty,\infty}(u)$ verifies for all $\mu \in \mathbb{R}$:

$$\|R^{\infty,\infty}(u)\|_{H^\mu} \leq C_\mu(\|u\|_{L^\infty([0,T],C^{2-\alpha}_*(\mathbb{D}))}),$$

which is the desired conjugation result.

Finally the proof of well-posedness follows from a standard energy estimate on (5.15) combined with Proposition 2.1 and the Lipschtiz estimate:

$$M_0^0([\tilde{p}(u) - \tilde{p}(v)](t,\cdot);1) \leq e^{C\|(u,v)\|_{C^{2-\alpha}_*}} \left\| \mathrm{Op}(\frac{1}{D})P_{\geq b}[u-v](t,\cdot)\right\|_{C^{2-\alpha}_*}.$$

Appendix A. Paradifferential Calculus

In this paragraph we review classic notations and results about paradifferential and pseudodifferential calculus that we need in this paper. We follow the presentations in [18], [19], [46], and [31] which give an accessible and complete presentation.

Notation A.1. *In the following presentation we will use the usual definitions and standard notations for the regular functions C^k, C^k_b for bounded ones and C^k_0 for those with compact support, the distribution space $\mathscr{D}'$,$\mathscr{E}'$ for those with compact support, $\mathscr{D}'^k$,$\mathscr{E}'^k$ for distributions of order k, Lebesgue spaces (L^p), Sobolev spaces ($H^s, W^{p,q}$) and the Schwartz class $\mathscr{S}$ and it's dual $\mathscr{S}'$. All of those spaces are equipped with their standard topologies. We also use the Landau notation $O_{\|\ \|}(X)$.*

For the definition of the periodic symbol classes we will need the following definitions and notations.

Notation A.2. *We will use $\mathbb{D}$ to denote $\mathbb{T}$ or $\mathbb{R}$ and $\hat{\mathbb{D}}$ to denote their duals that is $\mathbb{Z}$ in the case of $\mathbb{T}$ and $\mathbb{R}$ in the case of $\mathbb{R}$. For concision an integral on on $\mathbb{Z}$ i.e $\displaystyle\int_{\mathbb{Z}}$ should be understood as $\displaystyle\sum_{\mathbb{Z}}$. A function a is said to be in $C^\infty(\mathbb{T}\times\mathbb{Z})$ if for every $\xi \in \mathbb{Z}$, $a(\cdot,\xi) \in C^\infty(\mathbb{T})$. For $\xi \in \mathbb{Z}$, ∂_ξ should be understood as the forward difference operator, i.e*

$$\partial_\xi a(\xi) = a(\xi+1) - a(\xi),\ \xi \in \mathbb{Z}.$$

We recall the following simple identities for the Fourier transform on the Torus:

$$\begin{cases} \mathscr{F}_{\mathbb{T}}(\partial_x^\alpha f)(\xi) = \xi^\alpha \mathscr{F}_{\mathbb{T}}(f)(\xi), \xi \in \mathbb{Z}, \\ \mathscr{F}_{\mathbb{T}}((e^{-2i\pi x}-1)^\alpha f)(\xi) = \xi^\alpha \mathscr{F}_{\mathbb{T}}(f)(\xi), \xi \in \mathbb{Z}. \end{cases}$$

A.1. **Littlewood-Paley Theory.**

Definition A.1 (Littlewood-Paley decomposition)**.** *Pick $P_0 \in C^\infty_0(\mathbb{R})$ so that:*

$$P_0(\xi) = 1 \textit{ for } |\xi| < 1 \textit{ and } 0 \textit{ for } |\xi| > 2.$$

We define a dyadic decomposition of unity by:

$$\textit{for } k \geq 1,\ P_{\leq k}(\xi) = \Phi_0(2^{-k}\xi),\ P_k(\xi) = P_{\leq k}(\xi) - P_{\leq k-1}(\xi).$$

Thus,

$$P_{\leq k}(\xi) = \sum_{0\leq j\leq k} P_j(\xi) \textit{ and } 1 = \sum_{j=0}^{\infty} P_j(\xi).$$

Introduce the operator acting on $\mathscr{S}'(\mathbb{R})$:

$$P_{\leq k}u = \mathscr{F}^{-1}(P_{\leq k}(\xi)u) \textit{ and } u_k = \mathscr{F}^{-1}(P_k(\xi)u).$$

Thus,

$$u = \sum_k u_k.$$

Finally put $\{k \geq 1, C_k = \mathrm{supp}\ P_k\}$ the set of rings associated to this decomposition.

Remark A.1. *An interesting property of the Littlewood-Paley decomposition is that even if the decomposed function is merely a distribution the terms of the decomposition are regular, indeed they all have compact spectrum and thus are entire functions. On classical functions spaces this regularization effect can be "measured" by the following Bernstein inequalities.*

Proposition A.1 (Bernstein's inequalities). *Suppose that $a \in L^p(\mathbb{D})$ has its spectrum contained in the ball $\{|\xi| \leq \lambda\}$.*

Then $a \in C^\infty$ and for all $\alpha \in \mathbb{N}$ and $1 \leq p \leq q \leq +\infty$, there is $C_{\alpha,p,q}$ (independent of λ) such that,

$$\|\partial_x^\alpha a\|_{L^q} \leq C_{\alpha,p,q}\lambda^{|\alpha|+\frac{1}{p}-\frac{1}{q}} \|a\|_{L^p}.$$

In particular,

$$\|\partial_x^\alpha a\|_{L^q} \leq C_\alpha \lambda^{|\alpha|} \|a\|_{L^p}, \text{ and for } p=2,\ p=\infty$$

$$\|a\|_{L^\infty} \leq C\lambda^{\frac{1}{2}} \|a\|_{L^2}.$$

If moreover a has it's spectrum is in $\{0 < \mu \leq |\xi| \leq \lambda\}$ then:

$$C_{\alpha,q}^{-1}\mu^{|\alpha|} \|a\|_{L^q} \leq \|\partial_x^\alpha a\|_{L^q} \leq C_{\alpha,q}\lambda^{|\alpha|} \|a\|_{L^q}.$$

Proposition A.2. *For all $\mu > 0$, there is a constant C such that for all $\lambda > 0$ and for all $\alpha \in W^{\mu,\infty}$ with spectrum contained in $\{|\xi| \geq \lambda\}$. one has the following estimate:*

$$\|a\|_{L^\infty} \leq C\lambda^{-\mu} \|a\|_{W^{\mu,\infty}}.$$

Definition A.2 (Zygmund spaces on $\mathbb{D}$). *For $r \in \mathbb{R}$ we define the space:*

$$C_*^r(\mathbb{D}) \subset \mathscr{S}'(\mathbb{D}), \text{ by } C_*^r(\mathbb{D}) = \left\{u \in \mathscr{S}'(\mathbb{D}), \|u\|_r = \sup_k 2^{kr} \|u_k\|_\infty < \infty\right\},$$

equipped with its canonical topology giving it a Banach space structure.
It's a classical result that for $r \notin \mathbb{N}$, $C_*^r(\mathbb{D}) = W^{r,\infty}(\mathbb{D})$ the classic Hölder spaces.

Proposition A.3. *Let B be a ball with center 0. There exists a constant C such that for all $r > 0$ and for all $(u_q)_{q\in\mathbb{N}} \in \mathscr{S}'(\mathbb{D})$ verifying:*

$$\forall q, \operatorname{supp} \hat{u_q} \subset 2^q B \text{ and } (2^{qr} \|u_q\|_\infty)_{q\in\mathbb{N}} \text{ is bounded}$$

$$\text{then, } u = \sum_q u_q \in C_*^r(\mathbb{D}) \text{ and } \|u\|_r \leq \frac{C}{1-2^{-r}} sup_{q\in\mathbb{N}} 2^{qr} \|u_q\|_\infty.$$

Definition A.3 (Sobolev spaces on $\mathbb{D}$). *It is also a classical result that for $s \in \mathbb{R}$:*

$$H^s(\mathbb{D}) = \left\{u \in \mathscr{S}'(\mathbb{D}), |u|_s = \left(\sum_k 2^{2ks} \|u_k\|_{L^2}{}^2\right)^{\frac{1}{2}} < \infty\right\},$$

with the right hand side equipped with its canonical topology giving it a Hilbert space structure and $|\ |_s$ is equivalent to the usual norm on $\|\ \|_{H^s}$.

Proposition A.4. *Let B be a ball with center 0. There exists a constant C such that for all $s > 0$ and for all $(u_q)_{\in\mathbb{N}} \in \mathscr{S}'(\mathbb{D})$ verifying:*

$$\forall q, \operatorname{supp} \hat{u_q} \subset 2^q B \text{ and } (2^{qs} \|u_q\|_{L^2})_{q\in\mathbb{N}} \text{ is in } L^2(\mathbb{N}),$$

$$\text{then, } u = \sum_q u_q \in H^s(\mathbb{D}) \text{ and } |u|_s \leq \frac{C}{1-2^{-s}}\left(\sum_q 2^{2qs} \|u_q\|_{L^2}{}^2\right)^{\frac{1}{2}}.$$

We recall the usual nonlinear estimates in Sobolev spaces:

- If $u_j \in H^{s_j}(\mathbb{D}), j = 1, 2$, and $s_1 + s_2 > 0$ then $u_1 u_2 \in H^{s_0}(\mathbb{D})$ and if
$$s_0 \leq s_j, j = 1, 2 \text{ and } s_0 \leq s_1 + s_2 - \frac{1}{2},$$
$$\text{then } \|u_1 u_2\|_{H^{s_0}} \leq K \|u_1\|_{H^{s_1}} \|u_2\|_{H^{s_2}},$$
where the last inequality is strict if s_1 or s_2 or $-s_0$ is equal to $\frac{1}{2}$.
- For all C^∞ function F vanishing at the origin, if $u \in H^s(\mathbb{D})$ with $s > \frac{1}{2}$ then
$$\|F(u)\|_{H^s} \leq C(\|u\|_{H^s}),$$
for some non decreasing function C depending only on F.

A.2. **Paradifferential operators.** We start by the definition of symbols with limited spatial regularity. Let $\mathscr{W} \subset \mathscr{S}'$ be a Banach space.

Definition A.4. *Given $m \in \mathbb{R}$, $\Gamma^m_{\mathscr{W}}(\mathbb{D})$ denotes the space of locally bounded functions $a(x, \xi)$ on $\mathbb{D} \times (\hat{\mathbb{D}} \setminus 0)$, which are C^∞ with respect to ξ for $\xi \neq 0$ and such that, for all $\alpha \in \mathbb{N}$ and for all $\xi \neq 0$, the function $x \mapsto \partial^\alpha_\xi a(x, \xi)$ belongs to $\mathscr{W}$ and there exists a constant C_α such that, for all $\epsilon > 0$:*
$$\forall\, |\xi| > \epsilon, \left\|\partial^\alpha_\xi a(., \xi)\right\|_{\mathscr{W}} \leq C_{\alpha,\epsilon}(1 + |\xi|)^{m-|\alpha|}. \tag{A.1}$$
The spaces $\Gamma^m_{\mathscr{W}}(\mathbb{D})$ are equipped with their natural Fréchet topology induced by the semi-norms defined by the best constants in (A.1). We will essentially work with $\mathscr{W} = W^{\rho,\infty}$ and write $\Gamma^m_{\mathscr{W}} = \Gamma^m_\rho$, for $\rho < 0$ we use $\mathscr{W} = C^\rho_*$.

For quantitative estimates we introduce as in [31]:

Definition A.5. *For $m \in \mathbb{R}$ and $a \in \Gamma^m_{\mathscr{W}}(\mathbb{D})$, we set,*
$$M^m_{\mathscr{W}}(a; n) = \sup_{|\alpha| \leq n} \sup_{|\xi| \geq \frac{1}{2}} \left\|(1 + |\xi|)^{m-|\alpha|} \partial^\alpha_\xi a(., \xi)\right\|_{\mathscr{W}}, \ \textit{for } n \in \mathbb{N}.$$
We will essentially work with $\mathscr{W} = W^{\rho,\infty}, \rho \geq 0$ and write:
$$\Gamma^m_{W^{\rho,\infty}}(\mathbb{D}) = \Gamma^m_\rho(\mathbb{D}) \ \textit{and} \ M^m_\rho(a) = M^m_{W^{\rho,\infty}}(a; 1).$$
Moreover we introduce the following spaces equipped with their natural Fréchet space structure:
$$C^\infty_b(\mathbb{D}) = \cap_{\rho \geq 0} W^{\rho,\infty},\ \Gamma^m_\infty(\mathbb{D}) = \cap_{\rho \geq 0} \Gamma^m_\rho(\mathbb{D}),\ \Gamma^{-\infty}_\rho(\mathbb{D}) = \cap_{m \in \mathbb{R}} \Gamma^m_\rho(\mathbb{D}) \ \textit{and},$$
$$\Gamma^{-\infty}_\infty(\mathbb{D}) = \cap_{\rho \geq 0} \cap_{m \in \mathbb{R}} \Gamma^m_\rho(\mathbb{D}).$$

Remark A.2. *In higher dimension the 1 in the definition of M^m_ρ should be replaced by $1 + \lfloor \frac{d}{2} \rfloor$.*

Definition A.6. *Define an admissible cutoff function as a function $\psi^{B,b} \in C^\infty$, $B > 1, b > 0$ that verifies:*

(1)
$$\psi^{B,b}(\eta, \xi) = 0 \ \textit{when} \ |\xi| < B|\eta| + b, \ \textit{and} \ \psi^{B,b}(\eta, \xi) = 1 \ \textit{when} \ |\xi| > B|\eta| + b + 1.$$

(2) for all $(\alpha, \beta) \in \mathbb{N}^d \times \mathbb{N}^d$, there is C_{α_β}, with $C_{0,0} \leq 1$, such that:
$$\forall (\xi, \eta) : \left|\partial^\alpha_\xi \partial^\beta_\eta \psi(\xi, \eta)\right| \leq C_{\alpha,\beta}(1 + |\xi|)^{-|\alpha|-|\beta|}. \tag{A.2}$$
$(\psi^{1,b})_{B>1,b>0}$ will be called limit cutoff functions.

Definition-Proposition A.1. *Consider a real numbers $m \in \mathbb{R}$, a symbol $a \in \Gamma^m_{\mathscr{W}}$ and an admissible cutoff function $\psi^{B,b}$ define the paradifferential operator T_a by:*

$$\widehat{T_a u}(\xi) = (2\pi) \int_{\hat{\mathbb{D}}} \psi^{B,b}(\xi - \eta, \eta)\hat{a}(\xi - \eta, \eta)\hat{u}(\eta)d\eta,$$

where $\hat{a}(\eta, \xi) = \int e^{-ix.\eta} a(x, \xi)dx$ is the Fourier transform of a with respect to the first variable. In the language of pseudodifferential operators:

$$T_a u = op(\sigma_a)u, \text{ where } \mathscr{F}_x\sigma_a(\xi, \eta) = \psi^{B,b}(\xi, \eta)\mathscr{F}_x a(\xi, \eta).$$

For a limit cutoff $\psi^{1,b}$ we define:

$$\widehat{T^{lim}_a u}(\xi) = (2\pi) \int_{\hat{\mathbb{D}}} \psi^{1,b}(\xi - \eta, \eta)\hat{a}(\xi - \eta, \eta)\hat{u}(\eta)d\eta,$$

and define analogously σ^{lim}_a. The connection between two different choices of cut-offs is the following:

$$\forall a \in \Gamma^m_\rho, (B, B', b, b') \in [1, +\infty[^2 \times]0, +\infty[^2,\ \sigma_a^{\psi^{B,b}} - \sigma_a^{\psi^{B',b'}} \in \Gamma^{m-\rho}_0. \tag{A.3}$$

An important property of paradifferential operators is their action on functions with localized spectrum.

Lemma A.1. *Consider two real numbers $m \in \mathbb{R}$, $\rho \geq 0$, a symbol $a \in \Gamma^m_0(\mathbb{D})$, an admissible cutoff function $\psi^{B,b}$, a limit cutoff function $\psi^{1,b}$ and $u \in \mathscr{S}(\mathbb{D})$.*

- *For $R >> b$, if $\operatorname{supp} \mathscr{F}u \subset \{|\xi| \leq R\}$, then:*

$$\operatorname{supp} \mathscr{F}T_a u \subset \left\{|\xi| \leq (1 + \frac{1}{B})R - \frac{b}{B}\right\}, \tag{A.4}$$

$$\text{and } \operatorname{supp} \mathscr{F}T^{lim}_a u \subset \left\{|\xi| \leq 2R - \frac{b}{B}\right\}. \tag{A.5}$$

- *For $R >> b$, if $\operatorname{supp} \mathscr{F}u \subset \{|\xi| \geq R\}$, then:*

$$\operatorname{supp} \mathscr{F}T_a u \subset \left\{|\xi| \geq (1 - \frac{1}{B})R + \frac{b}{B}\right\}, \tag{A.6}$$

$$\text{and } \operatorname{supp} \mathscr{F}T^{lim}_a u \subset \left\{|\xi| \geq \frac{b}{B}\right\}. \tag{A.7}$$

The main features of symbolic calculus for paradifferential operators are given by the following Theorems taken from [31] and [36].

Theorem A.1. *Let $m \in \mathbb{R}$. if $a \in \Gamma^m_0(\mathbb{D})$, then T_a is of order m. Moreover, for all $\mu \in \mathbb{R}$ there exists a constant K such that:*

$$\|T_a\|_{H^\mu \to H^{\mu-m}} \leq K M^m_0(a), \text{ and,}$$

$$\|T_a\|_{W^{\mu,\infty} \to W^{\mu-m,\infty}} \leq K M^m_0(a), \mu \notin \mathbb{N}.$$

Theorem A.2. *Take $m \in \mathbb{R}$ and $a \in \Gamma^m_0(\mathbb{D})$, then for all $\mu > 0$ there exists a constant K such that:*

$$\left\|T^{lim}_a\right\|_{H^\mu \to H^{\mu-m}} \leq K M^m_0(a).$$

$$\left\|T^{lim}_a\right\|_{W^{\mu,\infty} \to W^{\mu-m,\infty}} \leq K M^m_0(a), \mu \notin \mathbb{N}.$$

Theorem A.3. *Let $m, m' \in \mathbb{R}$, and $\rho > 0$, $a \in \Gamma^m_\rho(\mathbb{D})$ and $b \in \Gamma^{m'}_\rho(\mathbb{D})$.*

- *Composition: Then T_aT_b is a paradifferential operator with symbol:*

$$a \otimes b \in \Gamma_\rho^{m+m'}(\mathbb{D}), \text{ more precisely,}$$

$$T_a^{\psi^{B,b}} T_b^{\psi^{B',b}} = T_{a\otimes b}^{\psi^{\frac{BB'}{B+B'-1},b}}.$$

Moreover $T_aT_b - T_{a\#b}$ is of order $m+m'-\rho$ where $a\#b$ is defined by:

$$a\#b = \sum_{|\alpha|<\rho} \frac{1}{i^{|\alpha|}\alpha!} \partial_\xi^\alpha a \partial_x^\alpha b,$$

and there exists $r \in \Gamma_0^{m+m'-\rho}(\mathbb{D})$ such that:

$$M_0^{m+m'-\rho}(r) \leq K(M_\rho^m(a)M_0^{m'}(b) + M_\rho^m(a)M_0^{m'}(b)),$$

and we have

$$T_a^{\psi^{B,b}} T_b^{\psi^{B',b}} - T_{a\#b}^{\psi^{\frac{BB'}{B+B'-1},b}} = T_r^{\psi^{\frac{BB'}{B+B'-1},b}},$$

$$T_a^{lim} T_b^{lim} - T_{a\#b}^{lim} = T_r^{lim}.$$

- *Adjoint: The adjoint operator of T_a, T_a^* is a paradifferential operator of order m with symbol a^* defined by:*

$$a^* = \sum_{|\alpha|<\rho} \frac{1}{i^{|\alpha|}\alpha!} \partial_\xi^\alpha \partial_x^\alpha \bar{a}. \tag{A.8}$$

Moreover, for all $\mu \in \mathbb{R}$ there exists a constant K such that

$$\|T_a^* - T_{a^*}\|_{H^\mu \to H^{\mu-m+\rho}} \leq K M_\rho^m(a).$$

If $a = a(x)$ is a function of x only, the paradifferential operator T_a is called a paraproduct. It follows from Theorem A.3 and the Sobolev embedding that:

- If $a \in H^\alpha(\mathbb{D})$ and $b \in H^\beta(\mathbb{D})$ with $\alpha, \beta > \frac{1}{2}$, then

$$T_aT_b - T_{ab} \text{ is of order } -\left(min\left\{\alpha, \beta\right\} - \frac{1}{2}\right).$$

- If $a \in H^\alpha(\mathbb{D})$ with $\alpha > \frac{1}{2}$, then

$$T_a^* - T_{a^*} \text{ is of order } -\left(\alpha - \frac{1}{2}\right).$$

- If $a \in W^{r,\infty}(\mathbb{D})$, $r \in \mathbb{N}$ then:

$$\|au - T_a u\|_{H^r} \leq C \|a\|_{W^{r,\infty}} \|u\|_{L^2}.$$

An important feature of paraproducts is that they are well defined for function $a = a(x)$ which are not L^∞ but merely in some Sobolev spaces H^r with $r < \frac{d}{2}$.

Proposition A.5. *Let $m > 0$. If $a \in H^{\frac{1}{2}-m}(\mathbb{D})$ and $u \in H^\mu(\mathbb{D})$ then $T_a u \in H^{\mu-m}(\mathbb{D})$. Moreover,*

$$\|T_a u\|_{H^{\mu-m}} \leq K \|a\|_{H^{\frac{1}{2}-m}} \|u\|_{H^\mu}$$

A main feature of paraproducts is the existence of paralinearisation Theorems which allow us to replace nonlinear expressions by paradifferential expressions, at the price of error terms which are smoother than the main terms.

Theorem A.4. *Let $\alpha, \beta \in \mathbb{R}$ be such that $\alpha, \beta > \frac{1}{2}$, then*

- *Bony's Linearization Theorem: For all C^∞ function F, if $a \in H^\alpha(\mathbb{D})$ then;*

$$F(a) - F(0) - T_{F'(a)} a \in H^{2\alpha - \frac{1}{2}}(\mathbb{D}).$$

- *If $a \in H^{\alpha}(\mathbb{D})$ and $b \in H^{\beta}(\mathbb{D})$, then $ab - T_a b - T_b a \in H^{\alpha+\beta-\frac{1}{2}}(\mathbb{D})$. Moreover there exists a positive constant K independent of a and b such that:*
$$\|ab - T_a b - T_b a\|_{H^{\alpha+\beta-\frac{1}{2}}} \leq K \|a\|_{H^{\alpha}} \|b\|_{H^{\beta}}.$$

Appendix B. Continuity of limited regularity paradifferential exotic symbols on L^p spaces

We start by giving the following analogue of Theorem 2.1.A of [47].

Theorem B.1. *Consider four real numbers $r > 0, m \in \mathbb{R}$ and $0 \leq \delta, \rho \leq 1$, then for all $a(x,\xi) \in C^r_* S^m_{\rho,\delta}$ such that $a^*(x,\xi) \in C^r_* S^m_{\rho,\delta}$ where:*
$$a^*(x,\xi) = \frac{1}{2\pi} \int_{\mathbb{D}\times\hat{\mathbb{D}}} e^{-iy.\eta} \bar{a}(x-y, \xi-\eta) dy d\eta,$$
then,
$$\mathrm{Op}(a) : W^{s+m+(\frac{1}{2}-\frac{1}{p})(1-\rho),p} \to W^{s,p}, \text{ with } p \in [2, +\infty]$$
provided $0 < s < r$. Furthermore, under these hypothesis,
$$\mathrm{Op}(a) : C^{s+m+\frac{1}{2}(1-\rho)}_* \to C^s_*.$$
Moreover there exists a constant K such that:
$$\|\mathrm{Op}(a)\|_{W^{s+m+(\frac{1}{2}-\frac{1}{p})(1-\rho),p} \to W^{s,p}} \leq K \ {}^*M^{m,r}_{\rho,\delta}(a;1), \text{ and,}$$
$$\|\mathrm{Op}(a)\|_{C^{s+m+\frac{1}{2}(1-\rho)}_* \to C^s_*} \leq K \ {}^*M^{m,r}_{\rho,\delta}(a;1).$$

Remark B.1. *In higher dimension the factor $(\frac{1}{2}-\frac{1}{p})(1-\rho)$ should be adapted to $d(\frac{1}{2}-\frac{1}{p})(1-\rho)$ and the semi norm of order 1 in the ξ variable in the estimates should be adapted to $\lfloor \frac{d}{2} \rfloor + 1$.*

If moreover $\delta < 1$ then all of the previous results extend to L^2 continuity (i.e $s = 0$), this results from an almost orthogonal decomposition combined with a TT^* argument as shown in Theorem 2, Section 2.5 of [44].

It's a result by Hörmander [19] that if $\delta < \rho$ or $\delta = \rho < 1$ the hypothesis on a^* is automatically verified. This hypothesis is also shown to be necessary for $\rho = \delta = 1$.

Proof. We first notice that it suffices to make H^s and C^s_* estimates as the L^p are obtained directly by interpolation.

The key estimate follows from the following adaptation of Lemma 4.3.2 of [31]:

Lemma B.1. *There are constants C and C' such that, for all $\lambda > 0$ and $q \in C^{\infty}(\mathbb{R}^d \times \mathbb{R}^d)$ satisfying:*
$$\operatorname{supp} q \subset \mathbb{R}^d \times \{|\xi| \leq \lambda\}, \quad M = \sup_{|\beta| \leq \tilde{d}} \lambda^{\rho|\beta|} \left\| \partial^{\beta}_{\xi} q \right\|_{L^{\infty}} < \infty, \text{ with } \tilde{d} = \left\lfloor \frac{d}{2} \right\rfloor + 1.$$
Suppose moreover that q and it's derivatives in ξ are uniformly continuous on $\mathbb{R}^d \times \mathbb{R}^d$. Then the function,
$$Q(y) = \int e^{-iy\cdot\xi} q(y,\xi) d\xi,$$
satisfies:
$$\int (1+|\lambda y|^2)^{\tilde{d}} |Q(y)|^2 \, dy \leq C \lambda^d M^2 (1+\lambda^{2(1-\rho)})^{\tilde{d}}, \tag{B.1}$$
and,
$$\|Q\|_{L^1(\mathbb{R}^d)} \leq C' M (1+\lambda^{2(1-\rho)})^{\frac{\tilde{d}}{2}}. \tag{B.2}$$

Proof of Lemma B.1. For $|\alpha| \le \tilde{d}$ we have:

$$y^\alpha Q(y) = \int e^{-iy\cdot\xi} D_\xi^\alpha q(y,\xi) d\xi.$$

At this step we would like to apply Plancherel's theorem to deduce:

$$\int \left|y^{2\alpha}\right| |Q(y)|^2 \, dy \le C\lambda^{d-2\rho\alpha} M^2, \tag{B.3}$$

which is the argument given in [31], as the application of the Plancherel's theorem does not seem immediate to us we opted to expand upon it to make it's application more immediate. We first notice that it suffice to prove (B.3) for $\alpha = 0$. To do so we introduce the function:

$$\tilde{Q}(x,y) = \int e^{-iy\cdot\xi} q(x,\xi) d\xi = \mathscr{F}_\xi q(x,y),$$

in this setting we can apply Plancherel's theorem to deduce:

$$\|Q\|_{L_x^\infty L_y^2} \le \|q\|_{L_x^\infty L_\xi^2}. \tag{B.4}$$

getting back to (B.3) we want to estimate $\left\|\tilde{Q}(y,y)\right\|^2_{L_y^2(\mathbb{R}^d)}$, to do so we estimate uniformly on cubes the norms $\left\|\tilde{Q}(y,y)\right\|^2_{L_y^2(C(y_0,R))}$. For this we define the set:

$$K(y_0,R,\epsilon) = \left\{(x,y), y \in C(y_0,R), |x_j - y_j| \le \epsilon, j \in [1,\cdots,d]\right\}.$$

Thus by the fundamental Theorem of calculus:

$$\frac{1}{c_d \epsilon^d} \int_{K(y_0,R,\epsilon)} \left|\tilde{Q}(x,y)\right|^2 dxdy \underset{\epsilon\to 0}{\longrightarrow} \int_{C(y_0,R)} \left|\tilde{Q}(y,y)\right|^2 dy$$

$$\begin{aligned}\left\|\tilde{Q}(y,y)\right\|^2_{L_y^2(C(y_0,R))} &\le C_{K(y_0,R,\epsilon)} \frac{1}{\epsilon^d} \left\|\tilde{Q}(x,y)\right\|^2_{L^2(K(R,\epsilon))} \\ &\le C_{K(R,\epsilon)} \|q\|^2_{L_x^\infty(C(0,\epsilon);L_\xi^2(C(0,R))},\end{aligned}$$

where the constant $C_{K(y_0,R,\epsilon)}$ can be chosen uniformly in y_0 by the uniform continuity of q. Now to explicit the dependence of $C_{K(R,\epsilon)}$ on the different parameters, by Plancherel's theorem we have:

$$\begin{aligned}\frac{R^d}{c_d' \epsilon^d} \int_{\mathbb{R}^d} \left|\prod_{j=1}^d \operatorname{sinc}\Big(2\xi_j R\Big) e^{iy_0^j \xi_j} * q(x,\xi)\right|^2 dxd\xi &= \frac{1}{c_d\epsilon^d} \int_{K(R,\epsilon)} \left|\tilde{Q}(x,y)\right|^2 dxdy \\ &\underset{\epsilon\to 0}{\longrightarrow} \int_{C(0,R)} \left|\tilde{Q}(y,y)\right|^2 dy.\end{aligned}$$

Thus we cover the diagonal (y,y) in $\mathbb{R}^d \times \mathbb{R}^d$ by compact sets $(K(y_0^i, R_i, \epsilon_i))_{i\in\mathbb{N}}$ where y_0^i, R_i and ϵ_i are chosen in a manner to ensure that the sum of the volume of the different intersections between elements of this cover is summable.

Thus we have:

$$\int \left|y^{2\alpha}\right| |Q(y)|^2 \, dy \le C\lambda^{d-2\rho\alpha} M^2.$$

Multiplying by $\lambda^{2|\alpha|}$ and summing in α, implies (B.1). Since $\tilde{d} > \frac{d}{2}$, the second estimate (B.2) follows. □

Getting back to the proof of Theorem B.1, we are working with $d = \tilde{d} = 1$ in Lemma B.1.

We start by making a Littlewood-Paley type decomposition by writing:

$$a(x,\xi)=a(x,\xi)P_0(\xi)+\sum_{k=1}^{\infty}a(x,\xi)P_k(\xi)=a_0(x,\xi)+\sum_{k=1}^{\infty}a_k(x,\xi).$$

The proof will be divided in two paragraphs where we do the Sobolev and Zygmund estimates respectively.

Continuity in H^s.

Lemma B.2. *For $k\geq 0$, $\mathrm{Op}(a_k)$ maps to L^2 to H^∞. Moreover for all $\alpha\in\mathbb{N}$, there is C_α such that for all $a\in C^r_*S^m_{\rho,\delta}$, $k\geq 0$ and all $f\in L^2$:*

$$\|\partial_x^\alpha \mathrm{Op}(a_k)f\|_{L^2}\leq C_\alpha M^{m,\alpha}_{\rho,\delta}(a;1)\,\|f\|_{L^2}\,2^{k(m+\alpha)} \tag{B.5}$$

Proof of Lemma B.2. Since a_k is compactly supported in ξ, one sees that $\mathrm{Op}(a_k)f$ is given by the convergent integral:

$$\mathrm{Op}(a_k)f(x)=\int A_k(x,y)f(y)dy, \tag{B.6}$$

where the kernel $A_k(x,y)$ is given by the convergent integral:

$$\mathrm{Op}(a_k)=\frac{1}{2\pi}\int e^{i(x-y)\xi}a_k(x,\xi)d\xi. \tag{B.7}$$

Moreover on the support of a_k, $1+|\xi|\simeq 2^k$. Therefore Lemma B.1 can be applied with $\lambda=2^{k+1}$, implying that:

$$\int(1+2^{2k}|x-y|^2)\,|A_k(x,y)|^2\,dy\leq C2^{2km+k+2(1-\rho)k}M^{m,0}_{\rho,\delta}(a;1)^2. \tag{B.8}$$

Hence for $f\in\mathscr{S}(\mathbb{D})$, Cauchy-Schwartz inequality implies that:

$$|\mathrm{Op}(a_k)f(x)|^2\leq C2^{2km+k+2(1-\rho)k}M^{m,0}_{\rho,\delta}(a;1)^2\int\frac{2^{2km+k+2(1-\rho)k}\,|f(y)|^2}{(1+2^{2k}|x-y|^2)}dy. \tag{B.9}$$

The integral $2^k\int(1+2^{2k}|x-y|^2)^{-1}dx=C'$ is finite and independent of k. Thus:

$$\|\mathrm{Op}(a_k)f\|_{L^2}\leq C_\alpha M^{m,0}_{\rho,\delta}(a;1)\,\|f\|_{L^2}\,2^{k(m+(1-\rho))}. \tag{B.10}$$

In order to eliminate the extra factor $2^{k(1-\rho))}$ in (B.10), we use the fundamental TT^* trick, indeed writing $(a_k)^*$ as the formal symbol of the operator $(\mathrm{Op}(a_k))^*$, by the frequency localization we see that $(a_k)^*\in C^r_*S^m_{\rho,\delta}$. Thus $a_k(a_k)^*\in C^r_*S^{2m}_{\rho,\delta}$ and applying the previous estimate, as it's uniform in the choice of symbol, to $\mathrm{Op}(a_k)(\mathrm{Op}(a_k))^*$ we get:

$$\|\mathrm{Op}(a_k)(\mathrm{Op}(a_k))^*f\|_{L^2}\leq C_\alpha M^{m,0}_{\rho,\delta}(a;1)^2\,\|f\|_{L^2}\,2^{k(2m+(1-\rho))}, \tag{B.11}$$

thus by the standard TT^* lemma:

$$\|\mathrm{Op}(a_k)f\|_{L^2}\leq C_\alpha M^{m,0}_{\rho,\delta}(a;1)\,\|f\|_{L^2}\,2^{k(m+\frac{1}{2}(1-\rho))}. \tag{B.12}$$

Iterating this estimate we get (B.5) for $\alpha=0$. The symbol ∂_x^α is $(i\xi+\partial_x)^\alpha a_k(x,\xi)$, which gives the desired estimate for larger α. ☐

Now getting back to the continuity in H^s of $\mathrm{Op}(a)$ we write for $f\in\mathscr{S}(\mathbb{D})$ by the support localization of a_k:

$$\mathrm{Op}(a_k)f=\sum_{|j-k|\leq 3}\mathrm{Op}(a_k)P_jf,$$

Thus by Lemma B.2,

$$\|\partial_x^\alpha\mathrm{Op}(a_k)f\|_{L^2}\leq C_\alpha M^{m,\alpha}_{\rho,\delta}(a;1)\sum_{|j-k|\leq 3}\|P_jf\|_{L^2}\,2^{k(m+\alpha)},$$

then by Definition A.3,

$$\|\partial_x^\alpha \operatorname{Op}(a_k)f\|_{L^2} \leq C_\alpha M^{m,\alpha}_{\rho,\delta}(a;1)2^{k(\alpha-s)}\epsilon_k, \tag{B.13}$$

with,

$$\sum_k \epsilon_k^2 \leq \|f\|_{H^{s+m}}. \tag{B.14}$$

Now to conclude we recall the following Proposition from [31]:

Proposition B.1 (Proposition 4.1.13 of [31])**.** *Let $0 < s$ and let n be an integer, $n > s$. There is a constant C such that, for all sequence $(f_k)_{k\geq 0} \in H^n(\mathbb{D}^d)$ satisfying for all $\alpha \in \mathbb{N}^d, |\alpha| \leq n$:*

$$\|\partial_x^\alpha f_k\|_{L^2(\mathbb{D}^d)} \leq 2^{k(|\alpha|-s)}\epsilon_k, \text{ with } (\epsilon_k) \in l^2, \tag{B.15}$$

the sum $f = \sum f_k$ belongs to $H^s(\mathbb{D}^d)$ and,

$$\|f\|^2_{H^s(\mathbb{D}^d)} \leq C\sum_{k=0}^{\infty}\epsilon_k^2. \tag{B.16}$$

Applying Proposition B.1 to $\operatorname{Op}(a_k)f$ we get the desired Sobolev continuity and the desired estimate.

Continuity in C^s_*. The proof follows the same lines as previously, indeed applying (B.2) to (B.6) we get the following Lemma.

Lemma B.3. *For $k \geq 0$, $\operatorname{Op}(a_k)$ maps to L^∞ to $W^{\infty,\infty}$. Moreover for all $\alpha \in \mathbb{N}$, there is C_α such that for all $a \in C^r_*S^m_{\rho,\delta}$, $k \geq 0$ and all $f \in L^\infty$:*

$$\|\partial_x^\alpha \operatorname{Op}(a_k)f\|_{L^\infty} \leq C_\alpha M^{m,\alpha}_{\rho,\delta}(a;1)\|f\|_{L^\infty} 2^{k(m+\alpha+\frac{1}{2})}. \tag{B.17}$$

Again we have:

$$\|\partial_x^\alpha \operatorname{Op}(a_k)f\|_{L^\infty} \leq C_\alpha M^{m,\alpha}_{\rho,\delta}(a;1)2^{k(\alpha-s)}\|f\|_{C_*^{s+m+\frac{1}{2}}}, \tag{B.18}$$

which gives the desired result and estimate. $\square$

Theorem B.2. *Consider four real numbers $r > 0, m \in \mathbb{R}$ and $0 \leq \delta, \rho \leq 1$, then for all $a(x,\xi) \in \Gamma^0 S^m_{\rho,\delta}$,*

$$T_a : W^{s+m+(\frac{1}{2}-\frac{1}{p})(1-\rho),p} \to W^{s,p}, \text{ with } p \in [2,+\infty], \quad s \in \mathbb{R},$$

$$T_a^{lim} : W^{s+m+(\frac{1}{2}-\frac{1}{p})(1-\rho),p} \to W^{s,p}, \text{ with } p \in [2,+\infty], \quad s > 0.$$

Furthermore, under these hypothesis,

$$T_a : C_*^{s+m+\frac{1}{2}(1-\rho)} \to C^s_*, \quad s \in \mathbb{R},$$

$$T_a^{lim} : C_*^{s+m+\frac{1}{2}(1-\rho)} \to C^s_*, \quad s > 0.$$

Moreover there exists a constant K such that:

$$\|T_a\|_{W^{s+m+(\frac{1}{2}-\frac{1}{p})(1-\rho),p}\to W^{s,p}} \leq K\ M^{m,0}_{\rho,\delta}(a;1), \text{ and,}$$

$$\|T_a\|_{C_*^{s+m+\frac{1}{2}(1-\rho)}\to C^s_*} \leq K\ M^{m,0}_{\rho,\delta}(a;1),$$

$$\left\|T_a^{lim}\right\|_{W^{s+m+(\frac{1}{2}-\frac{1}{p})(1-\rho),p}\to W^{s,p}} \leq K\ M^{m,0}_{\rho,\delta}(a;1), \text{ and,}$$

$$\left\|T_a^{lim}\right\|_{C_*^{s+m+\frac{1}{2}(1-\rho)}\to C^s_*} \leq K\ M^{m,0}_{\rho,\delta}(a;1),$$

Proof. This simply follows from the spectral localization property of paradifferential operators, indeed taking $f \in \mathscr{S}$, then $\mathrm{Op}(\sigma_{ak})f$ is supported in a ring C_k where $|\xi| \sim 2^k$, which is not necessarily the case for $\mathrm{Op}(a_k)f$. The spectral localization property also ensures that the adjoint operator verifies the hypothesis of Theorem B.1. Thus rewriting estimates (B.13) and (B.18) with $\alpha = 0$ then by definition of Sobolev spaces and Zygmund spaces using the Littlewood-Paley decomposition we get:

$$\|T_a\|_{W^{s+m+(\frac{1}{2}-\frac{1}{p})(1-\rho),p} \to W^{s,p}} \leq K\ {}^*M^{m,s}_{\rho,\delta}(\sigma_a; 2), \text{ and,}$$

$$\|T_a\|_{C_*^{s+m+\frac{1}{2}(1-\rho)} \to C_*^s} \leq K\ {}^*M^{m,s}_{\rho,\delta}(\sigma_a; 2), \quad s \in \mathbb{R},$$

which gives the desired result by the Bernstein inequalities.

The proof for T_a^{lim} follows exactly the same lines with the sole difference being that $\mathrm{Op}(\sigma^{lim}_{a\ k})f$ is supported in the ball $\mathrm{B}(0, C2^k)$ and not a ring C_k, which explains the restrictions to $s > 0$ by Propositions A.3 and A.4. ☐

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