CPSC 413 - Fall 2020

Problem Set 5 — Divide and Conquer Algorithms

Total marks: 70

1. **(15 marks)** Consider the following recurrence relation:

$$T(n) = T(\lfloor n/2 \rfloor) + T(\lfloor n/4 \rfloor) + T(\lfloor n/8 \rfloor) + n \ .$$

   (a) **(5 marks)** Use the iteration method to come up with a good guess for a tight asymptotic bound for the recurrence.

   (b) **(10 marks)** Prove that your guess is correct.

2. **(15 marks)** Use the Master Theorem to obtain tight asymptotic bounds on the following recurrences:

   (a) **(3 marks)** $T(n) = 2T(\frac{n}{2}) + n^3$.

   (b) **(3 marks)** $T(n) = 2T(\frac{n}{2}) + n$.

   (c) **(3 marks)** $T(n) = 2T(\frac{n}{2}) + \sqrt{n}$.

   (d) **(3 marks)** $T(n) = 4T(\frac{n}{8}) + \sqrt{n}\lg^2 n$.

   (e) **(3 marks)** $T(n) = 16T(\frac{n}{7}) + n^2$.

   Justify your answers.

3. **(40 marks)** Let X and Y be two arrays, each containing n distinct integers in ascending order. The problem is to compute the median of all $2n$ elements in time $\Theta(\lg n)$.

   (a) **(5 marks)** Give a formal definition (pre- and post-conditions) of the problem described.

   (b) **(10 marks)** Give a divide-and-conquer algorithm for solving the problem.

   (c) **(10 marks)** Prove that your algorithm is correct.

   (d) **(5 marks)** Express the run-time of your algorithm as a recurrence relation. Explain why your recurrence relation is correct.

   (e) **(10 marks)** Prove a tight asymptotic bound on the run-time of your algorithm (proving upper and lower bounds separately if necessary). You may use the master theorem, or a version of the "guess and prove" method.

   Note: Algorithms that do *not* use divide-and-conquer will receive no credit. Divide-and-conquer algorithms with run-time in $\omega(\lg n)$ will receive partial credit.