Assignment 1

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September 28, 2019

1) Prove using induction that $|w^n| = n * |w|$ for all $n \geq 0$

Proof.

Base case (n = 0): $|w^0| = |\epsilon| = 0 = 0 * |w|$ (since $w^0 = \epsilon$, and $|\epsilon| = 0$)

Inductive Step: Suppose $|w^k| = k * |w|$ for all $k \geq 0$ (Inductive Hypothesis)
We need to prove that $|w^{k+1}| = (k+1) * |w|$
Now, note that $|w^{k+1}| = |w^k| + |w|$
Then $|w^k| + |w| = (k * |w|) + |w|$ (by using our inductive hypothesis)
Then $(k * |w|) + |w| = (k+1) * |w|$, as required

Thus, $|w^n| = n * |w|$ for all $n \geq 0$

□

2) Prove that $w^i = w$ if and only if $w = \epsilon$

Proof.

Subproof1: Prove that if $w = \epsilon$ then $(w^i) = w$

Suppose $w = \epsilon$

Now, $\epsilon^0 = \epsilon$

$\epsilon^1 = \epsilon = \epsilon$

$\epsilon^2 = \epsilon\epsilon = \epsilon$

$\epsilon^3 = \epsilon\epsilon\epsilon = \epsilon$

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and so on.

Then $\epsilon^i = \epsilon$ for all $i \geq 0$

So $w^i = \epsilon^i = \epsilon = w$ for all $i \geq 0$

Therefore if $w = \epsilon$, then $w^i = w$ for all $i \geq 0$

Subproof 2: Prove that if $w \neq \epsilon$ then $|w^i| \neq w$

Suppose that $w \neq \epsilon$

Then w is not the empty word, so $|w| \geq 1$

Let $|w| = a$, where $a \geq 1$

By proving question 1), we know that $|w^n| = n * |w|$

Also note when $i \geq 2$, $ia \neq a$

So $|w^i| = ia \neq a = |w|$ when $i \geq 2$

Then since $|w^i| \neq |w|$, then $w^i \neq w$

Thus, $w^i \neq w$ for some i if $w \neq \epsilon$

Conclusion: Therefore, $w^i = w$ if and only if $w = \epsilon$

□

3) An example: $L_1, L_2 = \{aa, a\}$
Then $|L_1 * L_2| = |\{aa, a\} * \{aa, a\}| = |\{aaaa, aaa, aaa, aa\}| = 3$
And $|L_1| * |L_2| = |\{aa, a\}| * |\{aa, a\}| = 2$
So $|L_1 * L_2| < |L_1| * |L_2|$ for this example.
The smallest value of $|L_1| + |L_2|$ such that $|L_1 * L_2| < |L_1|$ is $|L_1| + |L_2| = 4$.
To prove this, we will note the example given above and consider all cases where $|L_1| + |L_2| < 4$.

Proof. There are three main cases when $|L_1| + |L_2| < 4$:

Case 1: $|L_1| + |L_2| = 3$

This case has four subcases:

case 1.1: $|L_1| = 2$ and $|L_2| = 1$

Suppose $L_1 = \{w_1, w_2\}$ and $L_2 = \{w_3\}$, where $w_1, w_2, w_3 \in \{a, b\}^*$ and $w_1 \neq w_2$

Then $|L_1 * L_2| = |\{w_1, w_2\} * \{w_3\}| = |\{w_1 w_3, w_2 w_3\}| = 2 \not< 2 = 2 * 1 = |\{w_1, w_2\}| * |\{w_3\}| = |L_1| * |L_2|$

case 1.2: $|L_1| = 1$ and $|L_2| = 2$

Suppose $L_2 = \{w_1, w_2\}$ and $L_1 = \{w_3\}$, where $w_1, w_2, w_3 \in \{a, b\}^*$ and $w_1 \neq w_2$

Then $|L_1 * L_2| = |\{w_3\} * \{w_1, w_2\}| = |\{w_3 w_1, w_3 w_2\}| = 2 \not< 2 = 1 * 2 = |\{w_3\}| * |\{w_1, w_2\}| = |L_1| * |L_2|$

case 1.3 $|L_1| = 3$ and $|L_2| = 0$

Suppose $L_1 = \{w_1, w_2, w_3\}$ and $L_2 = \emptyset$, where $w_1, w_2, w_3 \in \{a, b\}^*$ and $w_1 \neq w_2,\ w_1 \neq w_3,\ w_2 \neq w_3$

Then $|L_1 * L_2| = |\{w_1, w_2.w_3\} * \emptyset| = |\emptyset| = 0 \not< 0 = 3 * 0 = |L_1| * |L_2|$

case 1.4: $|L_1| = 0$ and $|L_2| = 3$

Suppose $L_2 = \{w_1, w_2, w_3\}$ and $L_1 = \emptyset$, where $w_1, w_2, w_3 \in \{a, b\}^*$ and $w_1 \neq w_2,\ w_1 \neq w_3,\ w_2 \neq w_3$

Then $|L_1 * L_2| = |\emptyset * \{w_1, w_2.w_3\}| = |\emptyset| = 0 \not< 0 = 0 * 3 = |L_1| * |L_2|$

Case 2: $|L_1| + |L_2| = 2$

This case has three subcases:

case 2.1: $|L_1| = 1$ and $|L_2| = 1$

Suppose $L_1 = \{w_1\}$ and $L_2 = \{w_2\}$ where $w_1, w_2 \in \{a, b\}^*$

Then $|L_1 * L_2| = |\{w_1\} * \{w_2\}| = |\{w_1 w_2\}| = 1 \not< 1 = |\{w_1\}| * |\{w_2\}| = |L_1| * |L_2|$

case 2.2: $|L_1| = 2$ and $|L_2| = 0$

Suppose $L_1 = \{w_1, w_2\}$ and $L_2 = \emptyset$, where $w_1, w_2 \in \{a, b\}^*$

Then $|L_1 * L_2| = |L_1 * \emptyset| = 0 \not< 0 = |L_1| * |\emptyset| = |L_1| * |L_2|$

case 2.3: $|L_1| = 0$ and $|L_2| = 2$

Suppose $L_2 = \{w_1, w_2\}$ and $L_1 = \emptyset$, where $w_1, w_2 \in \{a, b\}^*$

Then $|L_1 * L_2| = |\emptyset * L_2| = 0 \not< 0 = |\emptyset| * |L_2| = |L_1| * |L_2|$

Case 3: $|L_1| + |L_2| = 1$

In this case, either $L_1 = \emptyset$ or $L_2 = \emptyset$

If $L_1 = \emptyset$, then $|L_1 * L_2| = |\emptyset * L_2| = |\emptyset| = 0 \not< 0 = |\emptyset| * |L_2| = |L_1| * |L_2|$

And if $L_2 = \emptyset$, then $|L_1 * L_2| = |L_1 * \emptyset| = |\emptyset| = 0 \not< 0 = |L_1| * |\emptyset| = |L_1| * |L_2|$

Case 4: $|L_1| + |L_2| = 0$

In this case, $L_1 = \emptyset$ and $L_2 = \emptyset$, so $|L_1 * L_2| = |\emptyset * \emptyset| = 0 \not< 0 = |\emptyset| * |\emptyset| = |L_1| * |L_2|$

Conclusion: Thus, since $|L_1 * L_2| \not< |L_1| * |L_2|$ for all cases where $|L_1| + |L_2| < 4$, it means that the smallest value of $|L_1| + |L_2|$ such that $|L_1 * L_2| < |L_1|$ is $|L_1| + |L_2| = 4$.

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