Assignment 6

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November 20, 2019

1) Let $L = \{a^i b^j c^{i+j} : i, j \geq 0\}$. Create a grammar G and prove that $L(G) = L$.

Context Free Grammar G is:

$S \to aSc | T$

$T \to bTc | \epsilon$

Before we prove that $L(G) = L$, it will be useful to prove by induction on i that $S \Rightarrow^\star a^i T c^i$ for all $i \geq 0$

Proof. Base Case (i = 0): Since we have the production $S \to T$, we can derive $S \Rightarrow^\star T$, where $T = a^0 T c^0$.

Inductive Step: Suppose $k \geq 0$ is an integer such that $S \Rightarrow^\star a^k T c^k$ (Inductive Hypothesis).

We need to show that $S \Rightarrow^\star a^{k+1} T c^{k+1}$.

Using the inductive hypothesis and the production $S \to aSc$ in our grammar, we can derive $S \Rightarrow aSc \Rightarrow^\star aa^k T c^k c$

So $S \Rightarrow^\star a^{k+1} T c^{k+1}$, as required.

Conclusion: Thus, $S \Rightarrow^\star a^i T c^i$ for all $i \geq 0$ $\square$

Also, before proving $L(G) = L$, it will be useful to consider the production $T \to bTc | \epsilon$ to be the grammar G_T.

Then $L(G_T) = \{b^i c^i : i \geq 0\}$.

Now we will prove that $L(G) = L$ by first proving that $L(G) \subseteq L$ and then proving that $L \subseteq L(G)$

Proof. Suppose $w \in L(G)$. Then $S \Rightarrow^\star w$

Note that for any derivation of a word in $L(G)$, initially the production $S \to aSc$ is used 0 or more times until the production $S \to T$ is used (and then the

productions $T \to bTc|\epsilon$ are used to finish deriving the word).
This means that in the derivation of any word in $x \in L(G)$, there exists a sentinel form $a^iTc^i, i \geq 0$ such that $S \Rightarrow^\star a^iTc^i \Rightarrow^\star x$
Then since $S \Rightarrow^\star w$, we get $S \Rightarrow^\star a^iTc^i \Rightarrow^\star w$ for some $i \geq 0$.
Now we can rewrite $w = a^iyc^i$, where $T \Rightarrow^\star y$
Since T is the seed variable of G_T (the grammar that we defined before the proof), it means that $y \in L(G_T)$, so $y = b^jc^j$ for some $j \geq 0$.
Then $w = a^ib^jc^jc^i = a^ib^jc^{i+j} \in L$
Thus, $L(G) \subseteq L$ $\square$

Now we will prove that $L \subseteq L(G)$

Proof. Suppose $w \in L$. Then $w = a^ib^jc^{i+j}$ for some $i, j \geq 0$
(We need to show that $S \Rightarrow^\star w$)
Now, we know that $S \Rightarrow^\star a^nTc^n$ for all $n \geq 0$
Then $S \Rightarrow^\star a^iTc^i$
Since $b^jc^j \in L(G_T)$, it means that $T \Rightarrow^\star b^jc^j$
So $S \Rightarrow^\star a^iTc^i \Rightarrow^\star a^ib^jc^jc^i$
Then since $w = a^ib^jc^jc^i = a^ib^jc^{i+j}$, it means that $S \Rightarrow^\star w$
Thus, $L \subseteq L(G)$ $\square$

Since we have proved that $L(G) \subseteq L$ and $L \subseteq L(G)$, it means that $L(G) = L$

2a) Let $L_1 = \{w \in \{a, b\}^\star : w = w^r\}$. This is the language of palindromes over $\{a, b\}$, and from class we know that the the language of palindromes are always non-regular.
Since L_1 can be generated by context-free grammars, L_1 is also context-free.

For example, a context-free grammar that generates L_1 is:

$S \to \epsilon$
$S \to a$
$S \to b$
$S \to aSa$
$S \to bSb$

Note that $prefix(L_1) = \{w \in \Sigma^\star\}$

Proof. Suppose $x \in \Sigma^\star$.
Then x is a prefix of the palindrome xx^r.
Thus, all words over Σ are prefices of palindromes, so $prefix(L_1) = \{w \in \Sigma^\star\}$ $\square$

Also, $prefix(L_1)$ can be generated by the regular expression $(a+b)\star$, so that means $prefix(L_1)$ is regular.
So L_1 is a non-regular, context free language and $prefix(L_1)$ is regular.

2b) Let $L_2 = \{a^i b^i : i \geq 0\}$. In class we have shown that this language is non-regular.
This language is context-free because it can be generated by context free grammars
A grammar that generates L_2 is:

$S \to aSb|\epsilon$

Note that $prefix(L_2) = \{a^i b^j : i \geq j\}$

Proof. Suppose $w \in L_2$. Then $w = a^i b^i$ for some $i \geq 0$
We can rewrite w as $w = xy$ such that x is a prefix of w and $x, y \in \Sigma^\star$
Then we have two cases:
Case 1: x does not contain any $b's$. Then $x = a^m b^0$ and $y = a^{i-m} b^i$ for some $m \geq 0$
Case 2: x contains both $a's$ and $b's$. Then y cannot contain any $a's$, so $x = a^i b^{i-n}$ and $y = b^n$ for some $n \geq 0$
So in both cases, the prefix $x = a^m b^n$ such that $m \geq n$ and $m, n \geq 0$
So $prefix(L_2) = \{a^i b^j : i \geq j\}$ $\square$

Now we will prove that $prefix(L_2)$ is non-regular using the pumping lemma for regular languages and contradiction.

Proof. Suppose $prefix(L_2)$ is regular. Then it has a pumping length $n \geq 0$.
Let $w = a^n b^n$. Then $w \in prefix(L_2)$ because $n \geq n$, and note that $|w| = |a^n b^n| = 2n \geq n$
Now suppose $w = xyz$ such that $|xy| \leq n$ and $|y| \geq 1$ (so $y \neq \epsilon$)
Then since $w = a^n b^n$ and $|xy| \leq n$, we must have that $xy = a^i a^j$ for some $0 < i + j \leq n$ (that is, xy can only consist of $a's$).
Let $x = a^i$ and $y = a^j$ (where $i \geq 0$ and $j \geq 1$ since $y \neq \epsilon$).

Then $z = a^{n-i-j}b^n$.
Now, consider the word $w^{(0)} = xy^0z$
This is equal to $a^i a^{n-i-j}b^n = a^{n-j}b^n$
Since $j \geq 1$, then $n - j < n$, so $w^{(0)}$ is not an element of $prefix(L_2)$.
So the pumping lemma does not hold for this language, which contradicts the assumption that $prefix(L_2)$ is regular.
Thus, $prefix(L_2)$ is not regular. $\square$

So we have proved that L_2 is a non-regular, context free language and $prefix(L_2)$ is non-regular.