Assignment 2

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1) Suppose $M = (Q, \Sigma, \delta, q_0, A)$ and let $\delta^\star : Q \times \Sigma \to Q$ be it's extended transition function.
Suppose $q \in Q$ and $x, y \in \Sigma^\star$
We will use induction on $|x| = n$ to prove that $\delta^\star(q, xy) = \delta^\star(\delta^\star(q, x), y)$

But first, it will be useful to prove the following property:
$\delta^\star(q, ay) = \delta^\star(\delta(q, a), y)$ for any $q \in Q$, $a \in \Sigma$ and $y \in \Sigma^\star$
We will prove this by using induction on $|y| = n$

Proof. Basis (n = 0): If $|y| = 0$, then $y = \epsilon$
Then $\delta^\star(q, ay) = \delta^\star(q, a\epsilon) = \delta^\star(\delta(q, a), \epsilon) = \delta^\star(\delta(q, a), y)$, as required for this case

Inductive step: Suppose $k \geq 0$ is an integer such that $\delta^\star(q, ay) = \delta^\star(\delta(q, a), y)$, where $|y| = k$ (Inductive Hypothesis)
We want to prove that $\delta^\star(q, ay) = \delta^\star(\delta(q, a), y)$ for $|y| = k + 1$
Now, suppose $|y| = k + 1$, and $y = mb$ where $b \in \Sigma, m \in \Sigma^\star$
Then $\delta^\star(q, ay) = \delta^\star(q, amb)$
$= \delta(\delta^\star(q, am), b)$ (by definition of δ)
$= \delta(\delta^\star(\delta(q, a), m), b)$ (by using the inductive hypothesis, since $|m| = |y| - 1 = k$)
$= \delta^\star(\delta(q, a), mb)$ (by definition of δ)
$= \delta^\star(\delta(q, a), y)$, as required
Thus, $\delta^\star(q, ay) = \delta^\star(\delta(q, a), y)$ for any $q \in Q$, $a \in \Sigma$ and $y \in \Sigma^\star$

$\square$

Now we will move on to our main proof:

Proof. Basis (n = 0): Then $|x| = 0$, so $x = \epsilon$
Then $\delta^\star(q, x) = \delta^\star(q, \epsilon) = q$
So $\delta^\star(q, xy) = \delta^\star(q, \epsilon y) = \delta^\star(q, y) = \delta^\star(\delta^\star(q, x), y)$, as required for this case

Inductive step: Suppose $k \geq 0$ is an integer such that $\delta^\star(q, xy) = \delta^\star(\delta^\star(q, x), y)$, where $|x| = k$
We want to prove that $\delta^\star(q, xy) = \delta^\star(\delta^\star(q, x), y)$ for $|x| = k + 1$
Let $x = ma$ where $|x| = k + 1$, $m \in \Sigma^\star$, $|m| = k$ and $a \in \Sigma$
Then $\delta^\star(q, xy) = \delta^\star(q, may)$

$= \delta^\star(\delta^\star(q, m), ay)$ (by the inductive hypothesis, since $|m| = |x| - 1 = k$)
$= \delta^\star(\delta(\delta^\star(q, m), a), y)$ (by using the property proved before)
$= \delta^\star(\delta^\star(q, ma), y)$ (by definition of $\delta^\star$)
$= \delta^\star(\delta^\star(q, x), y)$, as required

Thus, $\delta^\star(q, xy) = \delta^\star(\delta^\star(q, x), y)$ for any $q \in Q$ and $x, y \in \Sigma^\star$

$\square$

2) Suppose $M = (Q, \Sigma, \delta, q_0, A)$ is a deterministic finite state machine such that $L(M) = L$. Suppose that $|Q| = 2$. Let $Q = \{a, b\}$. We have the following cases:
case 1: $A = \emptyset$
Then $\forall x \in \Sigma^\star$, $\delta^\star(q_0, x) \notin A$
So $L = \emptyset$

case 2: $q_0 = q_a, A = \{q_a\}$
Then $q_0 \in A$, so $\delta^\star(q_0, \epsilon) \in A$
So $\epsilon \in L(M) = L$

case 3: $q_0 = q_b, A = \{q_b\}$
Then $q_0 \in A$, so $\delta^\star(q_0, \epsilon) \in A$
So $\epsilon \in L(M) = L$

case 4: $q_0 = q_a, A = \{q_a, q_b\}$
Then $q_0 \in A$, so $\delta^\star(q_0, \epsilon) \in A$
So $\epsilon \in L(M) = L$

case 5: $q_0 = q_b, A = \{q_a, q_b\}$
Then $q_0 \in A$, so $\delta^\star(q_0, \epsilon) \in A$
So $\epsilon \in L(M) = L$

case 6: $q_0 = q_a, A = \{q_b\}$
If $\delta(q_a, a) = q_b$ for some $a \in \Sigma$, then
$\delta(q_a, a) = q_b \Leftrightarrow \delta(\delta^\star(q_0, \epsilon), a) = q_b$
$\Leftrightarrow \delta^\star(q_a, a) = q_b$
$\Leftrightarrow \delta^\star(q_a, a) \in A$ (since q_b is the only element of A)
$\Leftrightarrow a \in L(M) = L$

If $\delta(q_a, a) \neq q_b$ for all $a \in \Sigma$,
then $\delta(q_a, a) = q_a$ for all $a \in \Sigma$
So $\forall x \in \Sigma^\star, \delta^\star(q_0, x) \neq q_b$ (where $q_0 = q_a$)
Then $\delta^\star(q_0, x) \notin A$
So $\forall x \in \Sigma^\star, x \notin L$
So $L = \emptyset$

case 7: $q_0 = q_b, A = \{q_a\}$
If $\delta(q_b, a) = q_a$ for some $a \in \Sigma$, then
$\delta(q_b, a) = q_a \Leftrightarrow \delta(\delta^\star(q_0, \epsilon), a) = q_a$
$\Leftrightarrow \delta^\star(q_b, a) = q_a$
$\Leftrightarrow \delta^\star(q_b, a) \in A$ (since q_a is the only element of A)
$\Leftrightarrow a \in L(M) = L$

If $\delta(q_b, a) \neq q_a$ for all $a \in \Sigma$,
then $\delta(q_b, a) = q_a$ for all $a \in \Sigma$
So $\forall x \in \Sigma^\star, \delta^\star(q_0, x) \neq q_a$ (where $q_0 = q_b$)
Then $\delta^\star(q_0, x) \notin A$
So $\forall x \in \Sigma^\star, x \notin L$
So $L = \emptyset$

Thus, for all cases where $|Q| = 2$, $L = \emptyset$ or $\epsilon \in L$ or $\exists a \in \Sigma$ such that $a \in L$.

3) Suppose L is a finite state language over the aplhabet Σ. Then L is regular, so there exists a finite state machine $M = \{Q, \Sigma, \delta, q_0, A\}$ such that $L(M) = L$. Since L is finite, there exists a word whose length is greater than or equal to the length of all the other words in L. Suppose the length of the longest word in L is n.
We will define each component of M:

$Q = \{q_x : x \in \Sigma^\star, |x| \leq n\} \cup \{q_{ds}\}$
Each state $q_x \in Q$ corresponds to a word $x \in \Sigma^\star$ whose length is less than or equal to the longest word in L. Also, there is a dead state q_{ds} that will loop back to itself whatever the input.

$\Sigma =$ the alphabet that L uses

Let $a \in \Sigma$
If $|x| < n$, then $\delta(q_x, a) = q_y$, where $y = xa$ and $|y| = |x| + 1$
If $|x| = n$, then $\delta(q_x, a) = q_{ds}$

$q_0 = q_x$, where $|x| = 0$. Then $x = \epsilon$, So $q_0 = q_\epsilon$

$A = \{q_x \in Q : x \in L\}$

Proof that $L(M) = L$:

Proof. Suppose $x \in L(M)$.
$\Leftrightarrow \delta^\star(q_0, x) \in A$ (by definition of a machine accepting a word)
Now we have two cases:

Case 1: $x \neq \epsilon$
Then $\delta^\star(q_0, x) \in A \Leftrightarrow \delta^\star(q_0, ay) \in A$ (where $x = ay, a \in \Sigma, y \in \Sigma^\star, |y| = |x| - 1$)
$\Leftrightarrow \delta(\delta^\star(q_0, a), y) \in A$ (by definition of $\delta^\star$)
$\Leftrightarrow \delta(\delta^\star(q_0, \epsilon a), y) \in A$
$\Leftrightarrow \delta(\delta(\delta^\star(q_0, \epsilon), a), y) \in A$ (by definition of $\delta^\star$)
$\Leftrightarrow \delta(\delta(q_0, a), y) \in A$ (by definition of $\delta^\star$)
$\Leftrightarrow \delta(\delta(q_\epsilon, a), y) \in A$ (since $q_0 = q_\epsilon$)
$\Leftrightarrow \delta(q_{\epsilon a}, y) \in A$ (by definition of δ, where $|\epsilon| < |x|$ so $|\epsilon| < n$)
$\Leftrightarrow \delta(q_a, y) \in A$
$\Leftrightarrow q_{ay} \in A$ (by definition of δ ,where $|y| = |x| - 1$ so $|y| < n$)
$\Leftrightarrow q_x \in A$
Now, since $A = \{q_w \in Q : w \in L\}$, and $q_x \in A$, we get
$q_x \in A \Leftrightarrow x \in L$, as required for this case

Case 2: $x = \epsilon$
Then $\delta^\star(q_0, x) \in A \Leftrightarrow \delta^\star(q_0, \epsilon) \in A$
$\Leftrightarrow q_0 \in A$ (by definition of $\delta^\star$)
$\Leftrightarrow q_\epsilon \in A$ (since $q_0 = q_\epsilon$)
Now, since $A = \{q_w \in Q : w \in L\}$, and $q_\epsilon \in A$, we get
$q_x \in A \Leftrightarrow x \in L$, as required for this case

Thus, we have proved that $L(M) = L$

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