Complex Number:

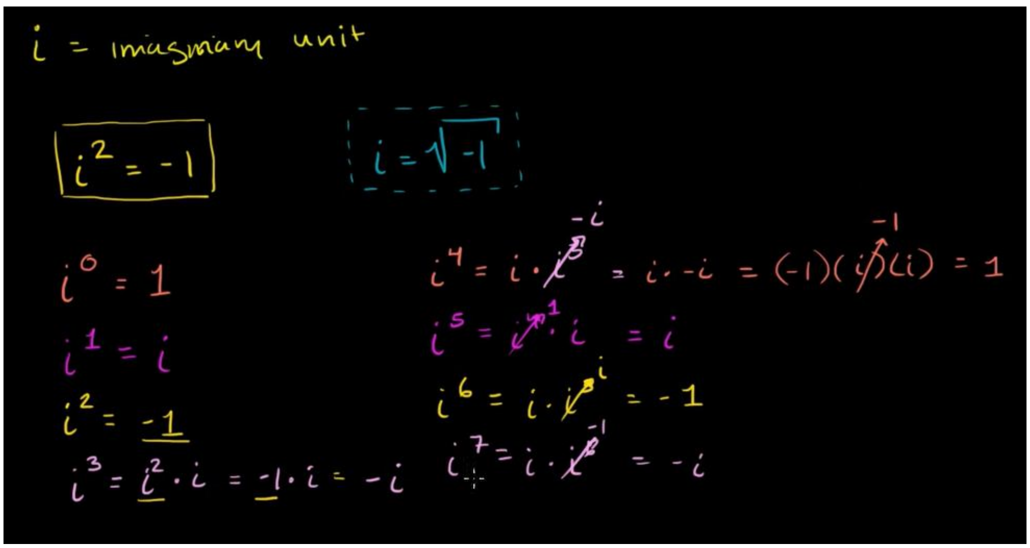
Some quadratic equations do not have any *real number* solutions.

you will never be able to find a *real number* solution to the equation x^2=-1. This is because it is impossible to square a real number and get a negative number!

However, a solution to the equation x^2=-1 does exist in a new number system called **the complex number system**.

The imaginary unit

The backbone of this new number system is the **imaginary unit**, or the number *i*.



i^1	i^2	i^3	i^4	i^5	i^6	i^7	i^8
i	-1	$-i$	1	i	-1	$-i$	1

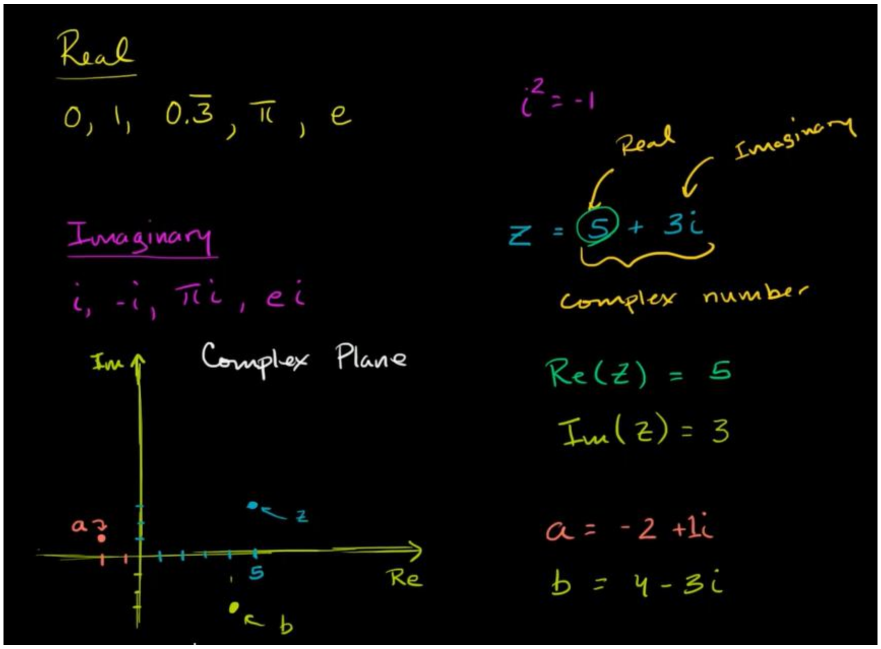
i^{-1}	i^0	i^1	i^2	i^3	i^4	i^5	i^6	i^7	i^8
$-i$	1	i	-1	$-i$	1	i	-1	$-i$	1

The fact that (3i)^2=-9 means that 3i is a square root of -9.

we can see that pure imaginary numbers are the square roots of negative numbers!

The imaginary unit i allows us to find solutions to many equations that do not have real number solutions.

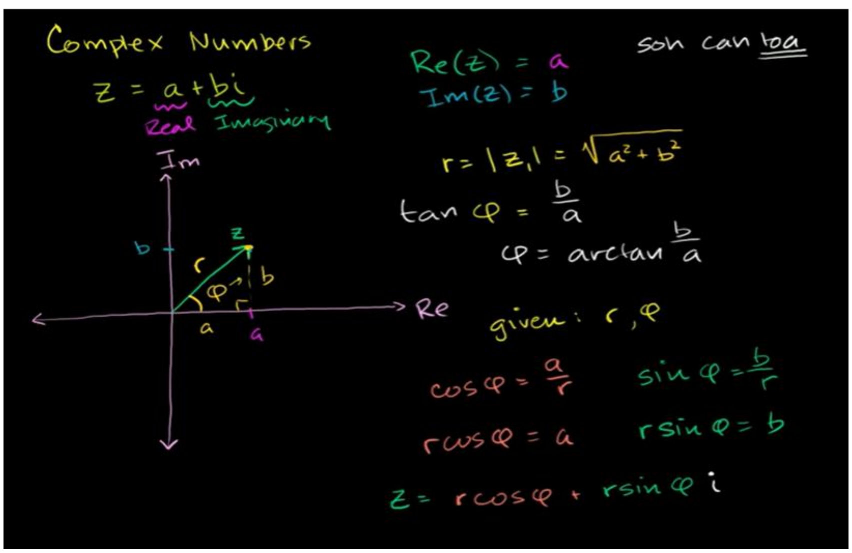
Complex Number:



$$\underset{\substack{\uparrow\\ \text{Real}\\ \text{part}}}{a} + \underset{\substack{\uparrow\\ \text{Imaginary}\\ \text{part}}}{bi}$$

Conjugate of the complex number:

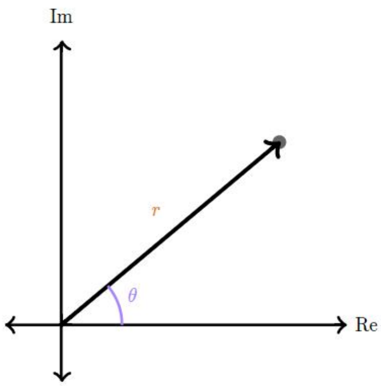
$$z \cdot \bar{z} = (a+bi)(a-bi) = (a)^2 - (bi)^2 = a^2 + b^2 = |z|^2$$



$$z = r\cos\varphi + r\sin\varphi\, i = r e^{i\varphi}$$
$$= r(\underbrace{\cos\varphi + i\sin\varphi}_{e^{i\varphi}})$$

Complex number absolute value & angle

Absolute value of $a + bi$	$\lvert z\rvert = \sqrt{a^2 + b^2}$
Angle of $a + bi$	$\theta = \tan^{-1}\left(\frac{b}{a}\right)$
Rectangular form from absolute value r and angle θ	$r\cos(\theta) + r\sin(\theta)\cdot i$



The absolute value, or modulus, gives the distance of the number from the origin in the complex plane, while its angle, or argument, is the angle the number forms with the positive Real axis.

To find the angle of a complex number, we take the inverse tangent of the ratio of its parts:

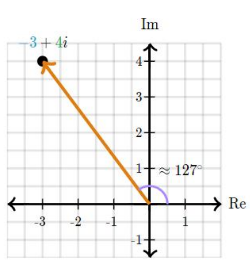
$$\theta = \tan^{-1}\left(\frac{b}{a}\right)$$

Let's find the angle of $-3 + 4i$. First, notice that $-3 + 4i$ is in Quadrant II.

$$\tan^{-1}\left(\frac{4}{-3}\right) \approx -53^\circ$$

-53° is in Quadrant IV, not II. We must add 180° to obtain the opposite angle:

$$-53^\circ + 180^\circ = 127^\circ$$



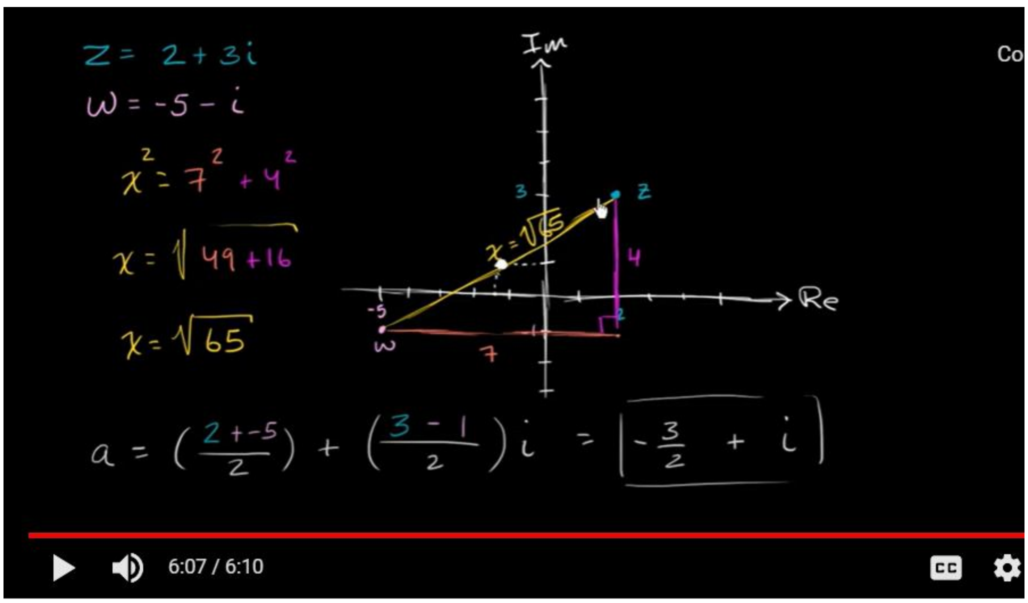
To find the real and imaginary parts of a complex number from its absolute value and angle, we multiply the absolute value by the sine or cosine of the angle:

$$\overbrace{r\cos(\theta)}^{a} + \overbrace{r\sin(\theta)}^{b} \cdot i$$

This results from using trigonometry in the right triangle formed by the number and the Real axis.

For example, this is the rectangular form of the complex number whose absolute value is 2 and angle is 30°:

$$2\cos(30^\circ) + 2\sin(30^\circ)i = \sqrt{3} + 1i$$



Distance & midpoint of complex numbers

A complex number in rectangular form, $z = a + bi$, can be written in polar form as $z = r[\cos\theta + i\sin\theta]$, where r is the absolute value, or *modulus*, and θ is the angle, or *argument*.

Therefore, r and θ can be found using the following formulas:

- $r = \sqrt{a^2 + b^2}$
- $\tan\theta = \dfrac{b}{a}$

[How did we get these equations?]

Similarly, a complex number in polar form, $z = r[\cos\theta + i\sin\theta]$, can be written in rectangular form as $z = a + bi$, using the following formulas:

- $a = r\cos\theta$
- $b = r\sin\theta$

different complex number forms

Rectangular	$a + bi$
Polar	$r\big(\cos(\theta) + i\sin(\theta)\big)$
Exponential	$r \cdot e^{i\theta}$

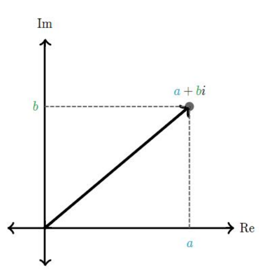
Rectangular form

$a + bi$

The rectangular form of a complex number is a sum of two terms: the number's real part and the number's imaginary part multiplied by i.

As such, it is really useful for adding and subtracting complex numbers.

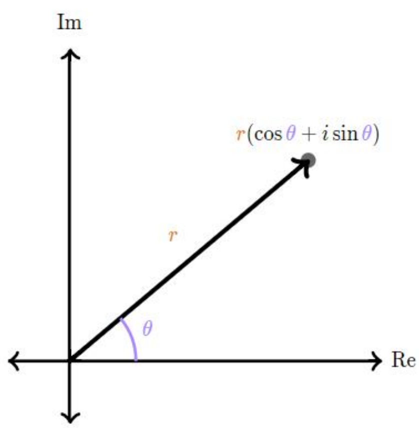
We can also plot a complex number given in rectangular form in the **complex plane**. The real and imaginary parts determine the real and imaginary coordinates of the number.



Polar form

$$r\left(\cos(\theta)+i\cdot\sin(\theta)\right)$$

Polar form emphasizes the graphical attributes of complex numbers: absolute value (the distance of the number from the origin in the complex plane) and angle (the angle that the number forms with the positive Real axis). These are also called modulus and argument.



Note that if we expand the parentheses in the polar representation, we get the number's rectangular form:

$$r\left(\cos(\theta)+i\cdot\sin(\theta)\right)=\overbrace{r\cos(\theta)}^{a}+\overbrace{r\sin(\theta)}^{b}\cdot i$$

This form is really useful for multiplying and dividing complex numbers, because of their special behavior: the product of two numbers with absolute values r_1 and r_2 and angles θ_1 and θ_2 will have an absolute value r_1r_2 and angle $\theta_1+\theta_2$.

Exponential form

$r \cdot e^{i\theta}$

Exponential form uses the same attributes as polar form, absolute value and angle. It only displays them in a different way that is more compact. For example, the multiplicative property can now be written as follows:

$$(r_1 \cdot e^{i\theta_1}) \cdot (r_2 \cdot e^{i\theta_2}) = r_1 r_2 \cdot e^{i(\theta_1+\theta_2)}$$

This form stems from Euler's expansion of the exponential function e^z to any complex number z. The reasoning behind it is quite advanced, but its meaning is simple: for any real number x, we define e^{ix} to be $\cos(x) + i\sin(x)$.

Using this definition, we obtain the equivalence of exponential and polar forms:

$$r \cdot e^{i\theta} = r\big(\cos(\theta) + i\sin(\theta)\big)$$

What complex multiplication looks like

By now we know how to multiply two complex numbers, both in rectangular and polar form. In particular, the polar form tells us that we **multiply magnitudes** and **add angles**:

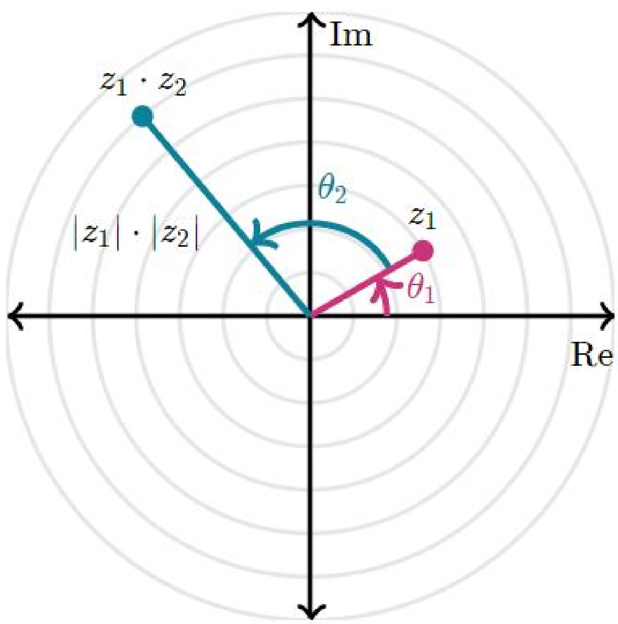
$$r(\cos(\alpha) + i\sin(\alpha)) \cdot s(\cos(\beta) + i\sin(\beta))$$

$$= rs[\cos(\alpha + \beta) + i\sin(\alpha + \beta)]$$

To compute $\frac{z}{w}$, where let's say $z = a + bi$ and $w = c + di$, we learned to multiply both numerator and denominator by the complex conjugate of w, $\overline{w} = c - di$.

$$\frac{z}{w} = \frac{a+bi}{c+di} = \frac{a+bi}{c+di} \cdot \frac{c-di}{c-di} = \frac{(a+bi)(c-di)}{c^2+d^2} = \frac{z \cdot \overline{w}}{|w|^2}$$

In other words, dividing by w is the same as multiplying by $\frac{\overline{w}}{|w|^2}$. Is there a

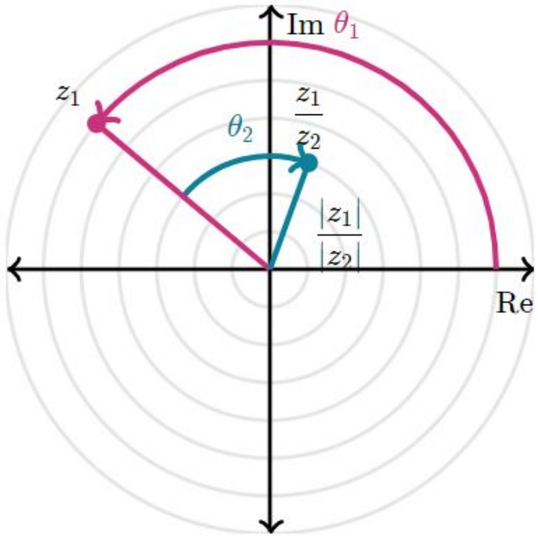


In other words, if the polar forms of z_1 and z_2 are as follows, we can find their product by *multiplying* their moduli and *adding* their arguments:

$$z_1 = r_1(\cos(\theta_1) + i\sin(\theta_1))$$

$$z_2 = r_2(\cos(\theta_2) + i\sin(\theta_2))$$

$$z_1 z_2 = r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$$



In other words, if the polar forms of z_1 and z_2 are as follows, we can find their quotient by *dividing* their moduli and *subtracting* their arguments:

$$z_1 = r_1(\cos(\theta_1) + i\sin(\theta_1))$$

$$z_2 = r_2(\cos(\theta_2) + i\sin(\theta_2))$$

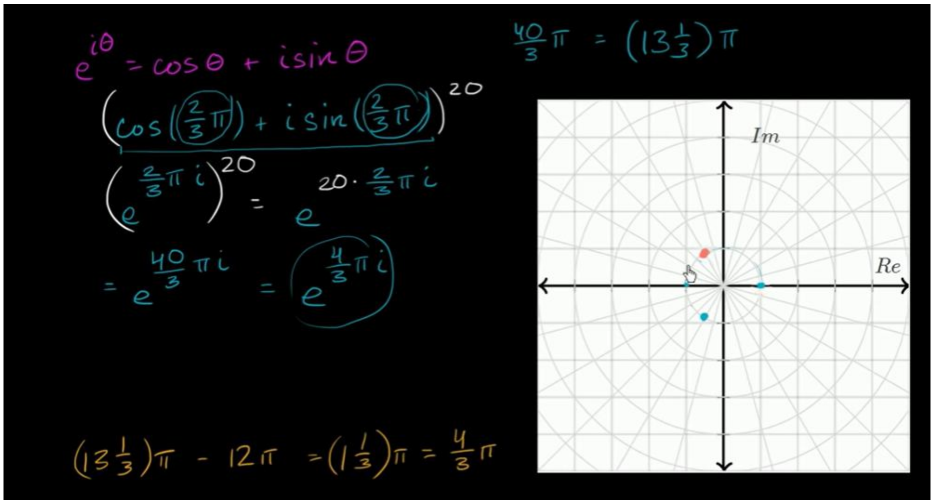
$$\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2))$$

Polar form is really useful for multiplying and dividing complex numbers:

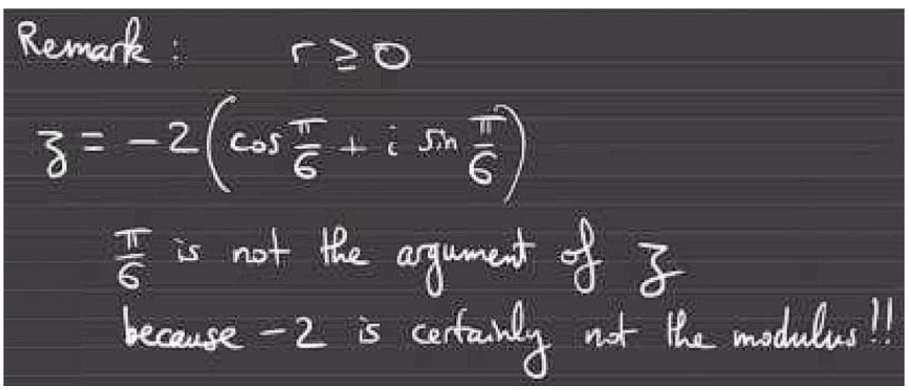
$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$$
$$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$
$$\Downarrow$$
$$z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$$

Powers of complex numbers



$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$$
$$\Downarrow$$
$$(z_1)^n = (r_1)^n[\cos(n\cdot\theta_1) + i\sin(n\cdot\theta_1)]$$



Trigonometric relations

There are many, many fundamental identities for trigonometric functions that can be useful for particular problems. Here are just a few to give a taste of the possibilities.

- the sine function is called an *odd function* - for any angle $\sin(-\alpha) = -\sin(\alpha)$

- cosine function is called an *even function* -for any angle $\cos(-\alpha) = \cos(\alpha)$

Angle-sum and angle-difference relations:

$$\begin{aligned} \sin(\alpha \pm \beta) &= \sin\alpha\cos\beta \pm \cos\alpha\sin\beta \\ \cos(\alpha \pm \beta) &= \cos\alpha\cos\beta \mp \sin\alpha\sin\beta \end{aligned}$$

Double-angle relations:

$$\begin{aligned} \sin 2\alpha &= 2\sin\alpha\cos\alpha \\ \cos 2\alpha &= cos^2\alpha - \sin^2\alpha \end{aligned}$$

Power relations:

$$\begin{aligned} \sin^2\alpha &= \frac{1}{2}(1 - \cos 2\alpha) \\ \cos^2\alpha &= \frac{1}{2}(1 + \cos 2\alpha) \end{aligned}$$