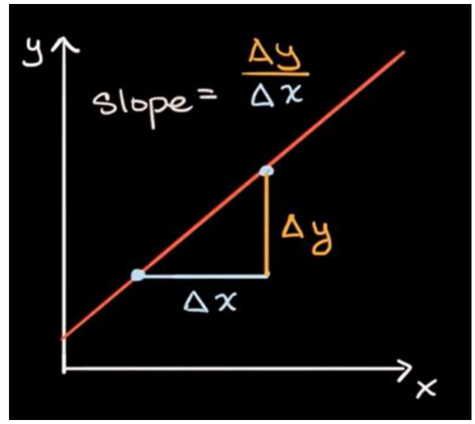
The slop:

For any line which has a constant rate of change

F prime = m = y2-y1/x2-x1



$$= \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

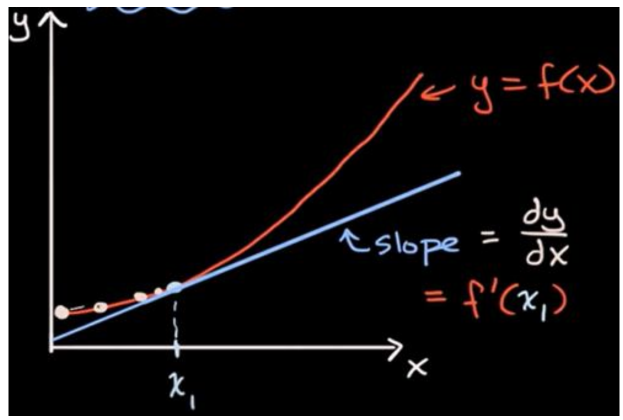
Instantaneous rate of change at specific point for a curve which changing:

dy/dx means changes x and y become close to zero

super small change of y / super small change of x

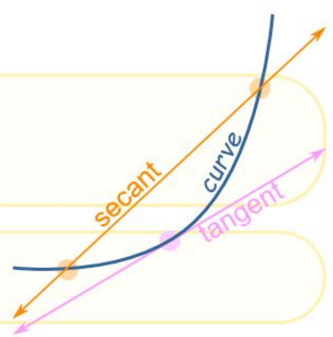
when the changes approach to 0

f prime is the slop of the tangent line at that point.



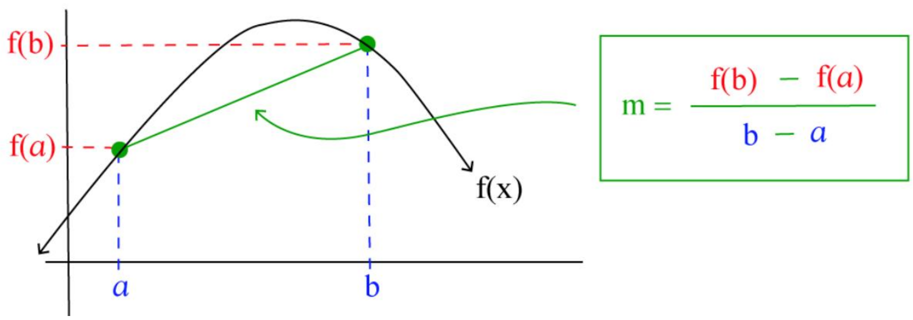
A **tangent line** just touches a curve at a point, matching the curve's slope there. (From the Latin ***tangens*** "touching", like in the word "tangible".)

A **secant line** intersects two or more points on a curve. (From the Latin ***secare*** "cut or sever")



The slope of the secant line passing between two points.

$$\text{Average Rate of Change} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(b) - f(a)}{b - a}$$



Average Vs Instantaneous Rate of Change

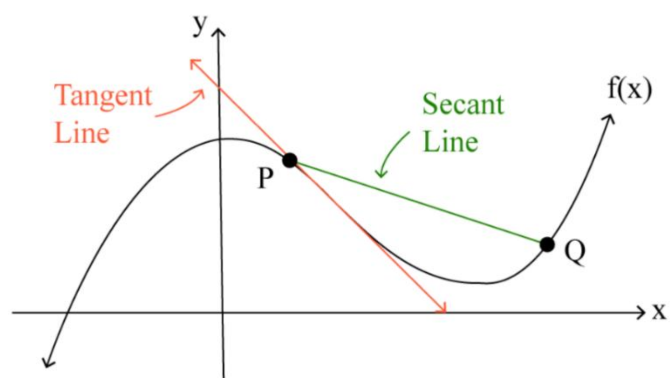
While both used to find the slope,

The average rate of change:

- Calculates the slope of the secant line using the **slope formula** from algebra.
- finds the **slope over an interval**

The instantaneous rate of change:

- Calculates the **slope of the tangent line** using derivatives.
- finds the **slope at a particular point**



Given $f(x) = 1 - 5x - x^3$ find the following

a. **Average rate of change** over $[1,3]$

$$Avg = \frac{f(3) - f(1)}{3-1} = \frac{\left[1 - 5(3) - (3)^3\right] - \left[1 - 5(1) - (1)^3\right]}{3-1} = \frac{(-41) - (-5)}{2} = -18$$

b. **Instantaneous rate of change** at $x = 2$

$$f'(x) = -5 - 3x^2$$

$$f'(2) = -5 - 3(2)^2 = -17$$

Formal definition of the derivative as a limit:

To find the slop of the line at specific point, the **derivative** of function f at x=c is the **limit** of the slope of the secant line from x=c to x=c+h as h approaches 0. Symbolically, this is the **limit** of [f(c)-f(c+h)]/h as h→0.

With a small change of notation, this limit written as:

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

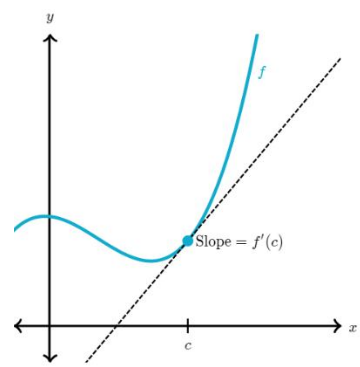
The **derivative of** f(x) **with respect to *x*** is the function f′(x) and is defined as: (the slop of the tangent line)

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

We can calculate the slope of a tangent line using the definition of the derivative of a function f at x = c, (provided that limit exists):

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

Once we've got the slope, we can find the equation of the line.



Example: Finding the equation of the line tangent to the graph of

$f(x) = x^2$ **at** $x = 3$

The expression for the derivative of f(x)

$$f'(3) = \lim_{h \to 0} \frac{(3+h)^2 - 3^2}{h}$$

It gives us the slope of the tangent line.

$f'(3) =$ 6

To find the complete equation, we need a point the line goes through.
We are looking for the equation of the line whose slope is 6, and that goes through the point (3,9). To do that, we can use the definition of slope:

$$6 = \frac{y-9}{x-3}$$

Now we can isolate y:

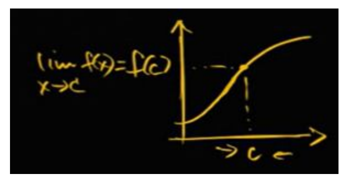
$$6 = \frac{y-9}{x-3}$$

$$6(x-3) = y-9$$

$$6(x-3)+9 = y$$

Therefore, the equation of the line is $y = 6(x-3)+9$.

Continuity



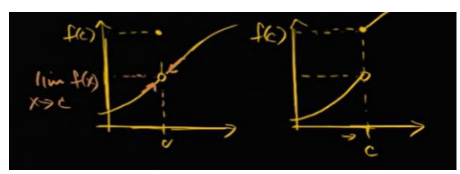
A function is said to be continuous at a point *x* = *a*, if

$\lim_{x\to a}$ f(x) Exists, and

$\lim_{x\to a}$ f(x) = *f*(*a*)

It implies that if the left-hand limit (L.H.L), right hand limit (R.H.L) and the value of the function at *x*=*a* exists and these parameters are equal to each other, then the function *f* is said to be continuous at *x*=*a*.

Discontinuous



If the function is undefined or does not exist, then we say that the function is discontinuous.

Continuity in open interval (a, b)

f(*x*) will be continuous in the open interval (*a*,*b*) if at any point in the given interval the function is continuous.

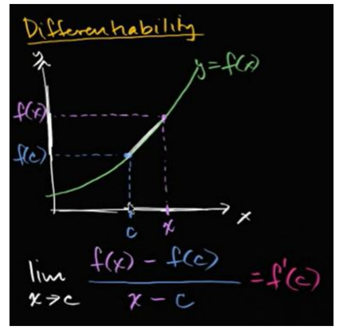
Continuity in closed interval [a, b]

A function *f*(*x*) is said to be continuous in the closed interval [*a*,*b*] if it satisfies the following three conditions.

1) *f*(*x*) is be continuous in the open interval (*a*, *b*)

2) *f*(*x*) is continuous at the point a from right i.e. $\lim_{x\to a}f(x)=f(a)$
3) *f*(*x*) is continuous at the point b from left i.e. $\lim_{x\to b}f(x)=f(b)$

Differentiability



f(*x*) is said to be differentiable at the point *x* = *a* if the derivative f '(a) exists at every point in its domain. It is given by

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

For a function to be differentiable at any point *x*=*a* in its domain, it must be continuous at that particular point but vice-versa is not always true.

Example: Consider the function $f(x) = (2x-3)^{\frac{1}{5}}$.Discuss its continuity and differentiability at $x = \frac{3}{2}$.

Solution: For checking the continuity, we need to check the left hand and right-hand **limits** and the value of the function at a point x=a.

L.H.L. = $\lim\limits_{x \to a^-} f(x) = \lim_{x \to \frac{3}{2}} (2x-3)^{\frac{1}{5}}$

$= \left(2 \times \frac{3}{2} - 3\right)^{\frac{1}{5}}$

= 0

R.H.L. = $\lim\limits_{x \to a^+} f(x) = \lim_{x \to \frac{3}{2}} (2x-3)^{\frac{1}{5}}$

$= \left(2 \times \frac{3}{2} - 3\right)^{\frac{1}{5}}$

= 0

L.H.L = R.H.L = f(a) = 0.

Thus the function is continuous at about the point $x = \frac{3}{2}$.

Now to check differentiability at the given point, we know

$f'(a) = \lim\limits_{h \to 0} \frac{f(a+h)-f(a)}{h}$

= $\lim\limits_{h \to 0} \frac{f(\frac{3}{2}+h)-f(\frac{3}{2})}{h}$

= $\lim\limits_{h \to 0} \frac{\left([2(\frac{3}{2})+h]-3\right)^{\frac{1}{5}} - \left(2(\frac{3}{2})-3\right)^{\frac{1}{5}}}{h}$

= $\lim\limits_{h \to 0} \frac{(3+2h-3)^{\frac{1}{5}} - (3-3)^{\frac{1}{5}}}{h}$

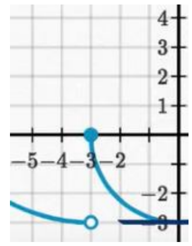
= $\lim\limits_{h \to 0} \frac{(2h)^{\frac{1}{5}} - 0}{h}$

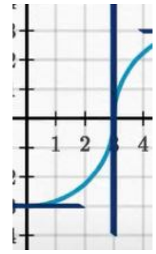
= $\lim\limits_{h \to 0} \frac{2^{\frac{1}{5}}}{h^{\frac{4}{5}}} = \infty$

Thus f is not differentiable at $x = \frac{3}{2}$.

A function f is not differentiable in the below scenario:

1- Vertical tangent.
2- Not continuous function
3- Sharp turn function





Example:

Is the function given below continuous/differentiable at $x = 3$?

$$f(x) = \begin{cases} x^2 & , x < 3 \\ 6x - 9 & , x \geq 3 \end{cases}$$

function f to be continues:
Limit from left side should equal limit from right side by applying the values directly in the function.

Check the function f(3) if equals to the limit for both sides:

$$\lim_{x\to 3^-} x^2 = 9$$

$$\lim_{x\to 3^+} 6x-9 = 9$$

9 Cont.

$$f(3) = \lim_{x\to 3} f(x)$$

function f to be differentiable:
Limit from left side should equal limit from right side.

To define the limit:

differentable

$$\lim_{x\to 3} \frac{f(x) - \overset{9}{f(3)}}{x-3}$$

Test both sides if equals:

$$\lim_{x\to 3^-} \frac{(x+3)\cancel{(x-3)}}{\cancel{x-3}} = 6$$

$$\lim_{x\to 3^+} \frac{\overset{6(x-3)}{\cancel{6x-18}}}{\cancel{x-3}} = 6$$

If a function is differentiable then it's also continuous. This property is very useful when working with functions, because if we know that a function is differentiable, we immediately know that it's also continuous.

The power rule:

to find the derivative of any expression in the form x^n:

$$\frac{d}{dx}[x^n] = n \cdot x^{n-1}$$

*Power Rule

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$f(x) = x^n, \quad n \neq 0$$

$$f'(x) = n x^{n-1}$$

$$\frac{d}{dx}[A f(x)] = A \underbrace{\frac{d}{dx}[f(x)]} = A f'(x)$$

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)] = f'(x) + g'(x)$$

the basic derivative rules

Constant rule	$\frac{d}{dx}k = 0$
Constant multiple rule	$\frac{d}{dx}[k \cdot f(x)] = k \cdot \frac{d}{dx}f(x)$
Sum rule	$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$
Difference rule	$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$

The derivatives of the basic trigonometric functions are;

1. $\frac{d}{dx}(\sin x) = \cos x$
2. $\frac{d}{dx}(\cos x) = -\sin x$
3. $\frac{d}{dx}(\tan x) = \sec^2 x$
4. $\frac{d}{dx}(\cot x) = -\mathrm{cosec}^2 x$
5. $\frac{d}{dx}(\sec x) = \sec x \tan x$
6. $\frac{d}{dx}(\mathrm{cosec}\, x) = -\mathrm{cosec}\, x \cot x$

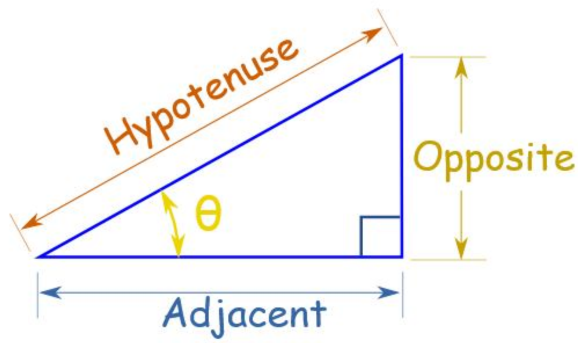
$$\cot x = \frac{\cos x}{\sin x}, \qquad \frac{d}{dx}(\cot x) = -cosec^2 x$$

$$sec(x) = \frac{1}{\cos(x)}, \qquad \frac{d}{dx}(\sec x) = \frac{\sin x}{cos^2 x} = \sec x \tan x$$

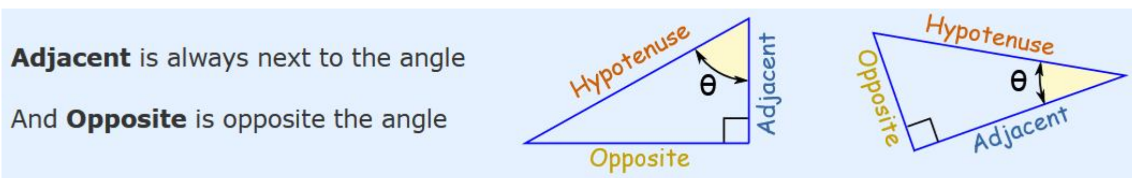
$$cosec(x) = \frac{1}{\sin(x)}, \qquad \frac{d}{dx}(\mathrm{cosec}\, x) = \frac{\cos x}{sin^2 x} = \mathrm{cosec}\, x \cot x$$

Sine, **Cosine** and **Tangent** are the main functions used in Trigonometry and are based on a Right-Angled Triangle.

Before getting stuck into the functions, it helps to give a **name** to each side of a right triangle:

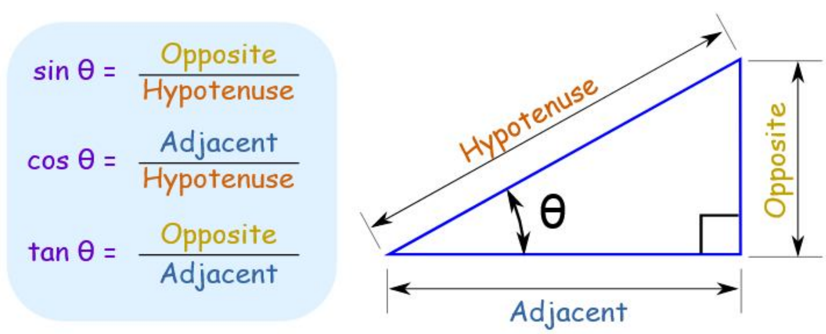


- "Opposite" is opposite to the angle θ
- "Adjacent" is adjacent (next to) to the angle θ
- "Hypotenuse" is the long one



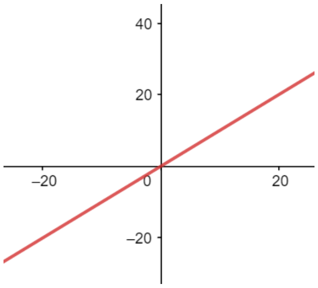
Sine, Cosine and Tangent

Sine, **Cosine** and **Tangent** (often shortened to **sin**, **cos** and **tan**) are each a **ratio of sides** of a right-angled triangle:

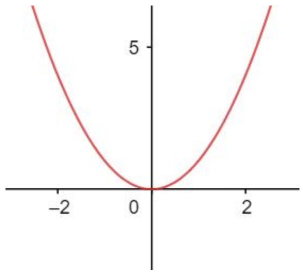


For a given angle ***θ*** each ratio stays the same
no matter how big or small the triangle is

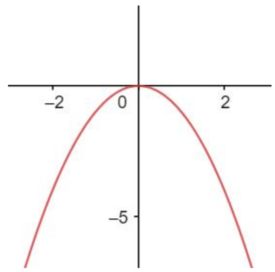
Parent Functions



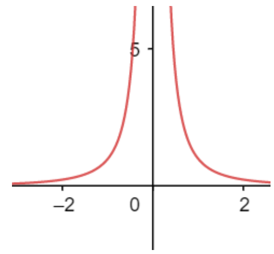
$f(x) = x$
Linear



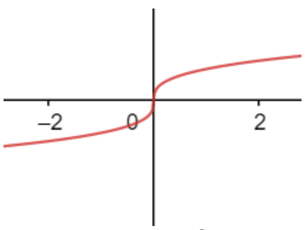
$f(x) = x^2$
quadratic



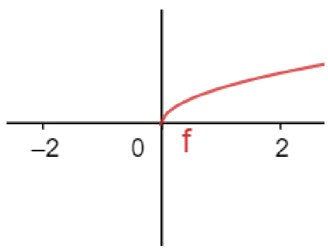
$f(x) = -x^2$



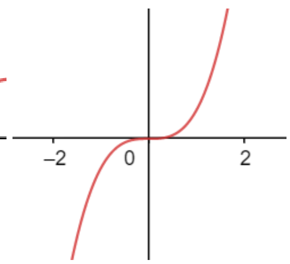
$f(x) = \frac{1}{x^2}$



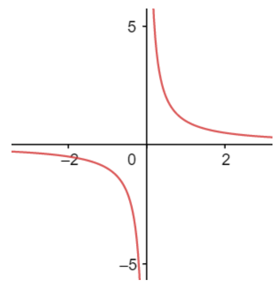
$f(x) = \sqrt[3]{x}$
cube root



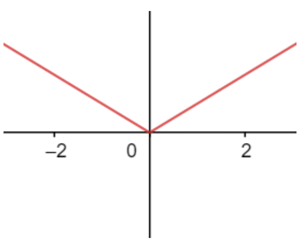
$f(x) = \sqrt{x}$
square root



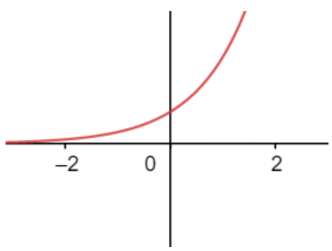
$f(x) = x^3$
Cubic



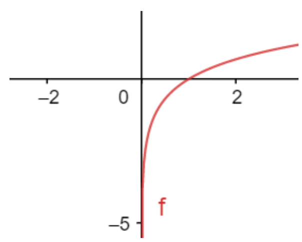
$f(x) = \frac{1}{x}$



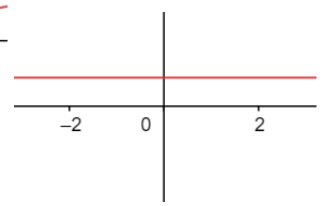
$f(x) = |x|$
Absolute Value



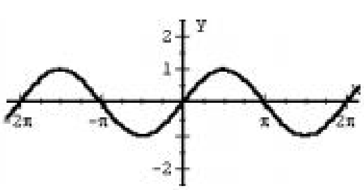
$f(x) = e^x$
exponential



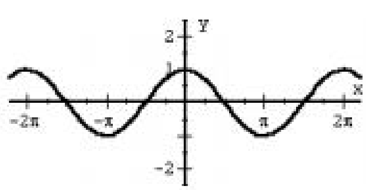
$f(x) = \ln(x)$
natural logarithmic



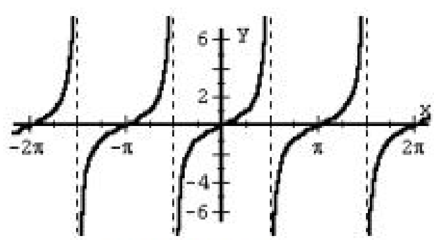
$f(x) = 1$
constant



f(x) = sin x

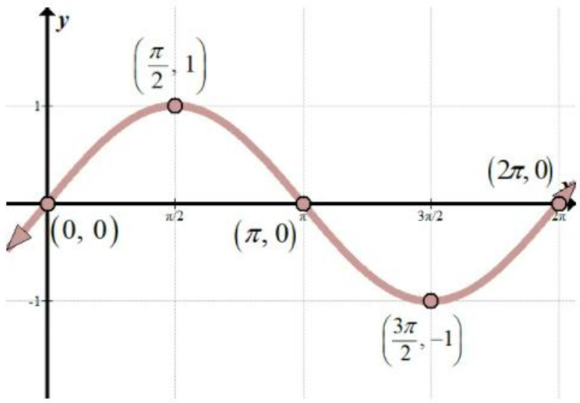


f(x) = cos x

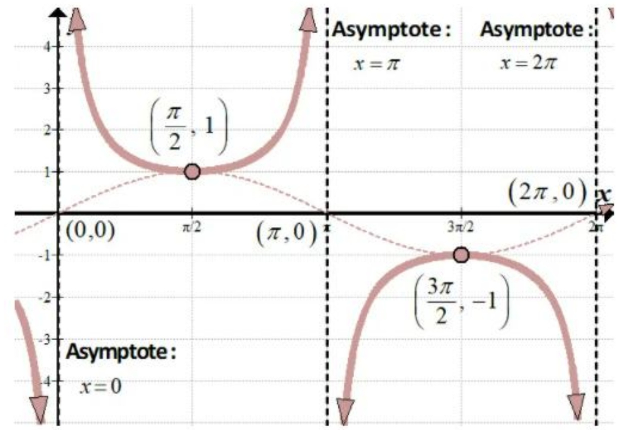


f(x) = tan x

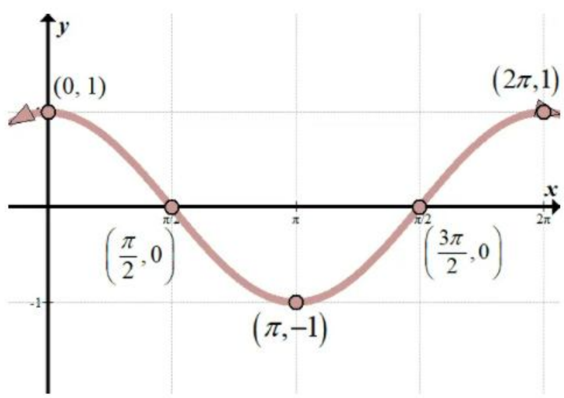
Trigonometric Functions



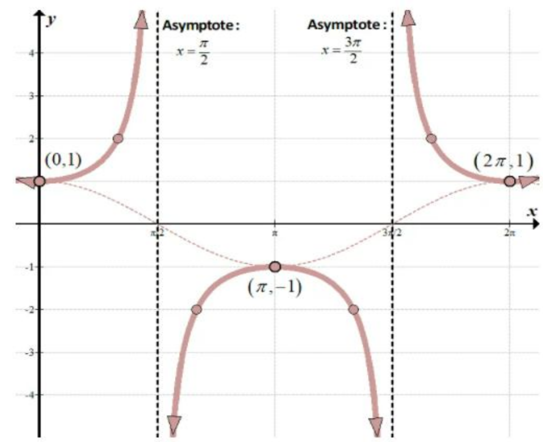
$f(x) = \sin(x)$



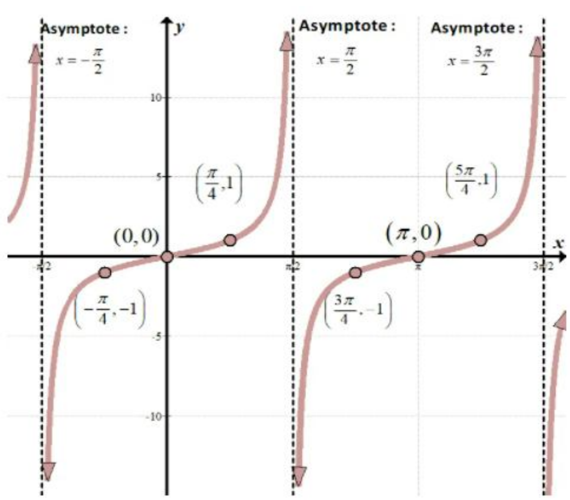
$f(x) = \csc(x) = \dfrac{1}{\sin(x)}$



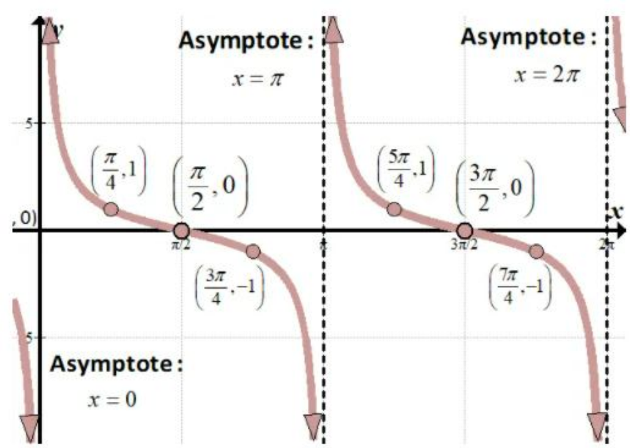
$f(x) = \cos(x)$



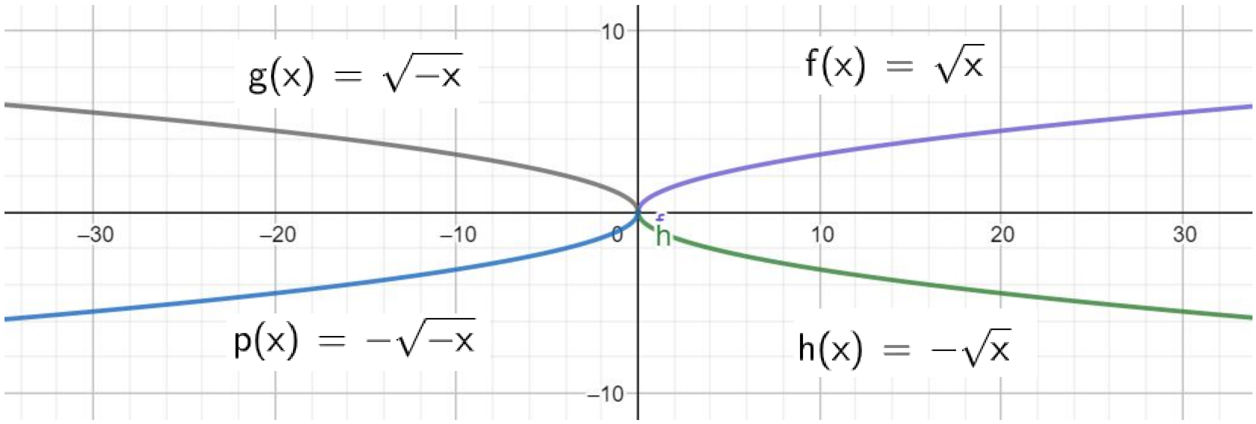
$f(x) = \sec(x) = \dfrac{1}{\cos(x)}$



$f(x) = \tan(x)$



$f(x) = \cot(x)$



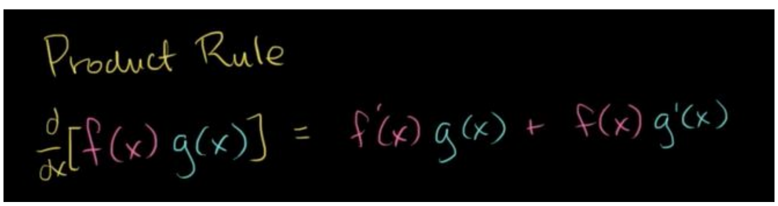
Derivative of e^x: is the only function that is the derivative of itself!

$$f(x) = e^x, \qquad \frac{d}{dx}[e^x] = e^x$$

Derivative of ln(x):

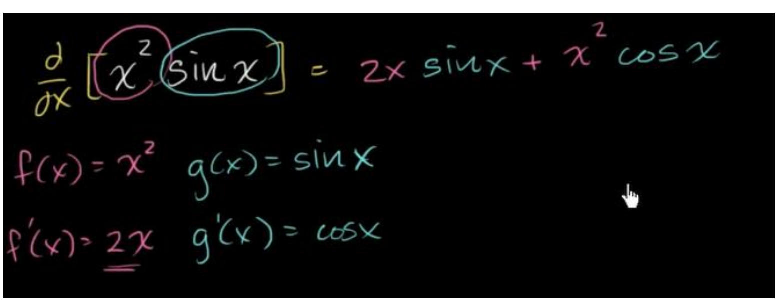
$$f(x) = \ln(x), \qquad \frac{d}{dx}[\ln(x)] = \frac{1}{x}$$

Product rule:



$$\frac{d}{dx}[f(x) \cdot g(x)] = \frac{d}{dx}[f(x)] \cdot g(x) + f(x) \cdot \frac{d}{dx}[g(x)]$$

Example:



Quotient rule:

Quotient Rule

$$f(x)=\frac{u(x)}{v(x)} \qquad f'(x)=\frac{u'(x)\,v(x)-u(x)\,v'(x)}{[v(x)]^2}$$

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{\frac{d}{dx}[f(x)]\cdot g(x)-f(x)\cdot\frac{d}{dx}[g(x)]}{[g(x)]^2}$$

Example:

$$f(x)=\frac{\boxed{x^2}\leftarrow u(x)}{\boxed{\cos x}\leftarrow v(x)} \qquad u'(x)=2x \qquad v'(x)=-\sin x$$

$$f'(x)=\frac{2x\cos x-x^2(-\sin x)}{(\cos x)^2}=\frac{2x\cos x+x^2\sin x}{\cos^2 x}$$

Chain rule

* Chain Rule

$$\frac{d}{dx}[f(g(x))]=f'(g(x))\cdot g'(x)$$

$$\frac{d}{dx}\left[f\Big(g(x)\Big)\right]=f'\Big(g(x)\Big)g'(x)$$

Example

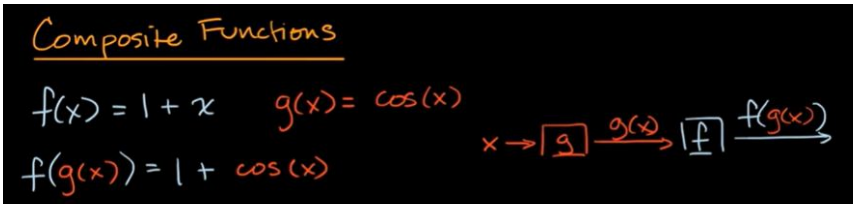
$$\frac{d}{dx}[f(g(x))]=\underbrace{f'(g(x))}\cdot g'(x)$$

$$\frac{d}{dx}\Big[\underbrace{\ln(\overbrace{\sin(x)}^{g(x)})}_{f(g(x))}\Big]=\frac{1}{\sin(x)}\cdot\underbrace{\cos(x)}$$

$$\frac{d}{dx}\Big[\overbrace{\ln(x)}^{f(x)}\,\overbrace{\sin(x)}^{g(x)}\Big]$$

Product Rule

$$f'(g(x))=\frac{1}{\sin(x)} \qquad g'(x)=\cos(x)$$



A function is *composite* if you can write it as $f\big(g(x)\big)$. In other words, it is a function within a function, or a function of a function.

For example, $\cos(x^2)$ is composite, because if we let $f(x) = \cos(x)$ and $g(x) = x^2$, then $\cos(x^2) = f\big(g(x)\big)$.

g is the function within f, so we call g the "inner" function and f the "outer" function.

$$\underbrace{\cos(\overbrace{x^2}^{\text{inner}})}_{\text{outer}}$$

Example:

$$h(x) = \underbrace{(\overbrace{5-6x}^{\text{inner}})^5}_{\text{outer}}$$

$$g(x) = 5 - 6x \qquad \text{inner function}$$

$$f(x) = x^5 \qquad \text{outer function}$$

Because h is composite, we can differentiate it using the chain rule:

$$\frac{d}{dx}\left[f\Big(g(x)\Big)\right] = f'\Big(g(x)\Big) \cdot g'(x)$$

Before applying the rule, let's find the derivatives of the inner and outer functions:

$$g'(x) = -6$$

$$f'(x) = 5x^4$$

Now let's apply the chain rule:

$$\frac{d}{dx}\left[f\Big(g(x)\Big)\right]$$

$$=f'\Big(g(x)\Big)\cdot g'(x)$$

$$=5(5-6x)^4\cdot -6$$

$$=-30(5-6x)^4$$

Derivative of a^x (for any positive base a)

$$\frac{d}{dx}[e^x]=e^x \qquad a=e^{\boxed{\ln a}} \qquad \ln a$$

$$\frac{d}{dx}[a^x]=\frac{d}{dx}\left[\left(e^{\ln a}\right)^x\right]=\frac{d}{dx}\left[e^{\boxed{(\ln a)x}}\right]=e^{(\ln a)x}\cdot \ln a$$

$$=(\ln a)\big(\underbrace{e^{\ln a}}_{a}\big)^x=(\ln a)\,a^x$$

example:

$$\frac{d}{dx}[8\cdot 3^x]=8\cdot(\ln 3)\cdot 3^x=(8\ln 3)\cdot 3^x$$

Derivative of $\log_a x$ (for any positive base a≠1)

$$\frac{d}{dx}[\ln x]=\frac{1}{x} \qquad \boxed{\log_a b=\frac{\log_c b}{\log_c a}}$$

$$\frac{d}{dx}[\log_a x]=\frac{d}{dx}\left[\frac{1}{\ln a}\cdot \ln x\right] \qquad \log_{\square}100=2$$

$$\log 100$$

$$\frac{1}{\ln a}\cdot\frac{d}{dx}[\ln x]=\frac{1}{(\ln a)x} \qquad \ln x=\log_e x$$

$$\log_3 8=\frac{\log_{10}8}{\log_{10}3}=\frac{\ln 8}{\ln 3}$$

Example:

$$f(x)=\log_7 x$$

$$f'(x)=\frac{1}{(\ln 7)x}$$

Derivative rules review

Basic Properties and Formulas

If $f(x)$ and $g(x)$ are differentiable functions (the derivative exists), c and n are any real numbers,

1. $(cf)' = c f'(x)$
2. $(f \pm g)' = f'(x) \pm g'(x)$
3. $(fg)' = f'g + fg'$ – **Product Rule**
4. $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$ – **Quotient Rule**
5. $\frac{d}{dx}(c) = 0$
6. $\frac{d}{dx}(x^n) = n x^{n-1}$ – **Power Rule**
7. $\frac{d}{dx}(f(g(x))) = f'(g(x))g'(x)$

   This is the **Chain Rule**

Common Derivatives

$\frac{d}{dx}(x) = 1$

$\frac{d}{dx}(\sin x) = \cos x$

$\frac{d}{dx}(\cos x) = -\sin x$

$\frac{d}{dx}(\tan x) = \sec^2 x$

$\frac{d}{dx}(\sec x) = \sec x \tan x$

$\frac{d}{dx}(\csc x) = -\csc x \cot x$

$\frac{d}{dx}(\cot x) = -\csc^2 x$

$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

$\frac{d}{dx}(a^x) = a^x \ln(a)$

$\frac{d}{dx}(\mathbf{e}^x) = \mathbf{e}^x$

$\frac{d}{dx}(\ln(x)) = \frac{1}{x}, \quad x > 0$

$\frac{d}{dx}(\ln|x|) = \frac{1}{x}, \quad x \neq 0$

$\frac{d}{dx}(\log_a(x)) = \frac{1}{x \ln a}, \quad x > 0$

Derivative of exponential functions

$$\frac{d}{dx}(e^x) = e^x \ln e = e^x(1) = e^x \qquad \frac{d}{dx}(a^x) = a^x \ln a$$

Derivatives of inverse trig functions

$$\frac{d}{dx}\left[a \sin^{-1}(f(x))\right] = \frac{a \cdot f'(x)}{\sqrt{1 - [f(x)]^2}} \qquad \frac{d}{dx}\left[a \csc^{-1}(f(x))\right] = -\frac{a \cdot f'(x)}{|f(x)|\sqrt{[f(x)]^2 - 1}}$$

$\tan^2 x + 1 = \sec^2 x$, $\cos^2 x + \sin^2 x = 1$, $\sec x = 1/\cos x$

Taylor series

Application: Taylor series expansion represents a function as a series, an infinite sum, of terms that are calculated using the function's derivatives at a point:

$$f(x) = \frac{f(a)}{0!}(x-a)^0 + \frac{f'(a)}{1!}(x-a)^1 + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots$$

Note that the first term is just the function calculated at the point a, it is the 0th order derivative of the function and is called the 0th order term (all terms are referred to by the order of the derivative).

The Taylor series, in the LaGrange remainder formula, for $f(x)$ in a neighborhood of the value a is

$$f(x) = \sum_{k=0}^{n-1}\left[\frac{f^{(k)}(a)(x-a)^k}{k!}\right] + R_n \qquad R_n = \frac{f^{(n)}(c)(x-a)^n}{n!}$$

A good example of the use of the Taylor series expansion is that for $f(x) = e^x$ around the point $a = 0$ (a Taylor series expansion around $a = 0$ is known as a Maclaurin series). Recall $f'(e^x) = e^x$, $e^0 = 1$ then

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots + \frac{x^{n-1}}{(n-1)!} + R_n$$

$R_n = e^c x^n/n!$ and $c \in (0, x)$. The remainder term can be made as small as desired by taking larger n so that the approximation of e^x can be fixed as precise as desired.

Taylor series in two variables

The Taylor series expansion can be extended to 2 or more dimensions, we consider only $\mathbb{R}^2$ here. Consider the case of a function $f(x, y)$. The Taylor series around a point (a, b) gives an approximation of the function in the neighborhood of (a, b) as follows (using Leibniz' notation for clarity)

$$\begin{aligned} f(x,y) = f(a,b) + \frac{\partial f(a,b)}{\partial x}(x-a) + \frac{\partial f(a,b)}{\partial y}(y-b) + \\ + \frac{1}{2}\left[\frac{\partial^2 f(a,b)}{\partial x^2}(x-a)^2 + \frac{\partial^2 f(a,b)}{\partial x \partial y}(x-a)(y-b) + \frac{\partial^2 f(a,b)}{\partial y^2}(y-b)^2\right] + \dots \end{aligned}$$

The general expression for the Taylor series in 2 variables can be written as

$$f(x,y) = \sum_{k=0}^{\infty}\left(\frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}\frac{\partial^n f(a,b)}{\partial x^{n-k}\partial y^k}(x-a)^{n-k}(y-b)^k\right) \qquad \binom{n}{k} = \frac{n!}{(n-k)!k!}$$

An example. Let

$$f(x,y) = x^3 + 3y - y^3 - 3x.$$

We find the second-degree Taylor polynomial as

$$\begin{aligned} f(x,y) \approx\ & f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) + \\ & + \frac{1}{2}\left(f_{xx}(a,b)(x-a)^2 + 2f_{xy}(a,b)(x-a)(y-b) + f_{yy}(a,b)(y-b)^2\right) \end{aligned}$$

Using the point (2,1) for the approximation we have

$$\begin{aligned} f(2,1) &= 8 + 3 - 1 - 6 = 4 \\ f_x &= 3x^2 - 3 = 9 \qquad f_{xx} = 6x = 12 \\ f_y &= 3 - 3y^2 = 0 \qquad f_{yy} = -6y = -6 \\ & \qquad\qquad\qquad\quad\ f_{xy} = 0 \end{aligned}$$

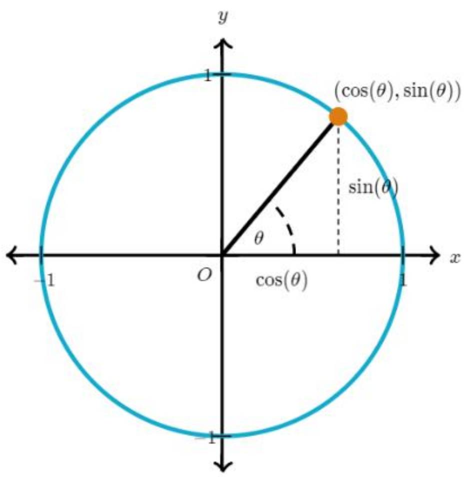
and

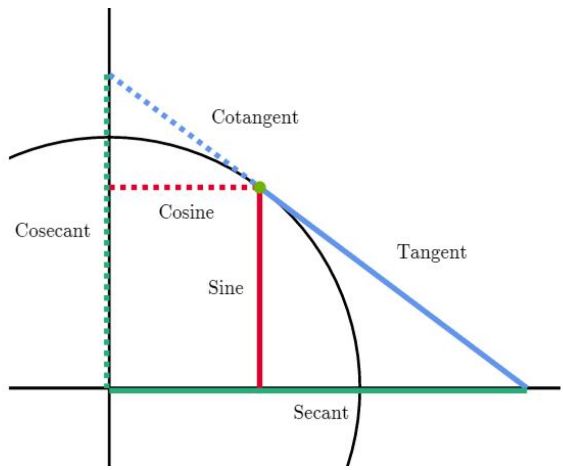
$$\begin{aligned} f(x,y) &\approx 4 + 9(x-2) + 0(y-1) + \frac{1}{2}\left(12(x-2)^2 + 2\cdot 0 - 6(y-1)^2\right) \\ &\approx 4 + 9(x-2) + 6(x-2)^2 - 3(y-1)^2 \end{aligned}$$

unit circle definition of the trigonometric functions

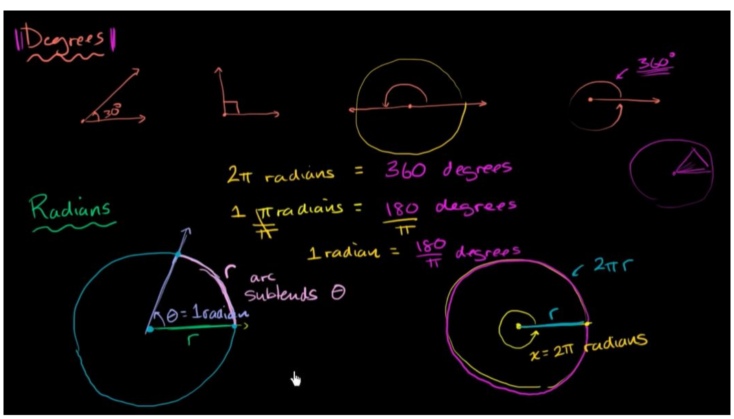
The unit circle definition allows us to extend the domain of sine and cosine to **all real numbers**. The process for determining the sine/cosine of any angle theta θ is as follows:

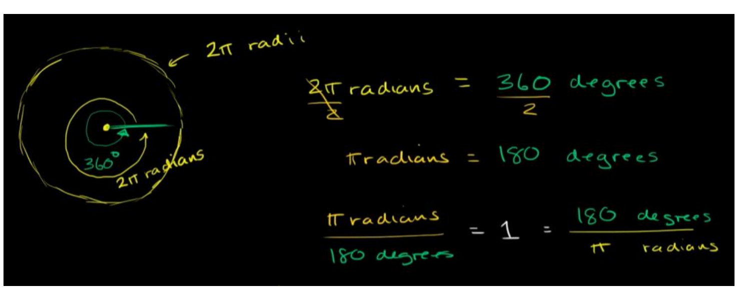
1. Starting from (1,0) move along the unit circle in the counterclockwise direction until the angle that is formed between your position, the origin, and the positive x-axis is equal to θ.
2. sin(θ) is equal to the y-coordinate of your point, and cos(θ) is equal to the x-coordinate.





Radians & Degrees





Trigonometric Values:

$sin(0) = 0$ $\quad sin\left(\frac{\pi}{3}\right) = \sin(60) = \frac{\sqrt{3}}{2}$ $\quad sin\left(\frac{\pi}{4}\right) = \sin(45) = \frac{\sqrt{2}}{2}$ $\quad sin\left(\frac{\pi}{6}\right) = \sin(30) = \frac{1}{2}$

$cos(0) = 1$ $\quad cos\left(\frac{\pi}{3}\right) = \cos(60) = \frac{1}{2}$ $\quad cos\left(\frac{\pi}{4}\right) = \cos(45) = \frac{\sqrt{2}}{2}$ $\quad cos\left(\frac{\pi}{6}\right) = \cos(30) = \frac{\sqrt{3}}{2}$

$$sin\left(\frac{\pi}{2}\right) = \sin(90) = 1$$

$$cos\left(\frac{\pi}{2}\right) = \cos(90) = 0$$

Example:

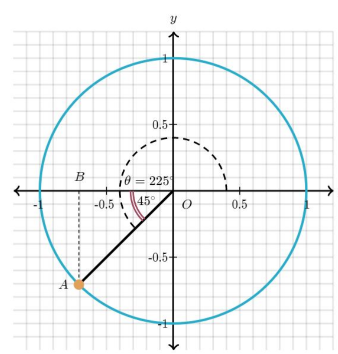
Find the following trigonometric values.

$\cos(225^\circ) =$ ☐ $\quad \sin(225^\circ) =$ ☐

Strategy

- Draw the desired angle on the unit circle and find a corresponding right triangle.
- Solve the right triangle using the known trigonometric values of the special angles 0, degrees, 30, 45 degrees, or their complementary angles (as necessary).
- Determine the sign of the trigonometric values of the angle in question.

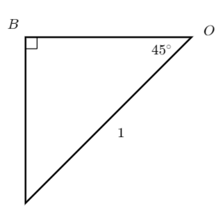
Drawing 225° and finding a corresponding right triangle



In the following drawing, A is the intersection point of the unit circle with the line that forms an angle of 225° with the positive x-axis. This means that the coordinates of A are $\big(\cos(225^\circ), \sin(225^\circ)\big)$.

If we draw a vertical line from A to the x-axis, we obtain the right triangle $\triangle AOB$. We can use this triangle to find the coordinates of A.

Solving $\triangle AOB$



$$BO = \cos(45^\circ) = \frac{\sqrt{2}}{2}$$

$$AB = \sin(45^\circ) = \frac{\sqrt{2}}{2}$$

Relating $\triangle AOB$ back to the unit circle

We now know that A is $\frac{\sqrt{2}}{2}$ units ***below*** the x-axis and $\frac{\sqrt{2}}{2}$ units ***to the left*** of the y-axis.

Therefore, its coordinates are $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$.

Example:

The angle θ_1 is located in Quadrant II, and $\cos(\theta_1) = -\frac{22}{29}$.

What is the value of $\sin(\theta_1)$?
Express your answer exactly.

$\sin(\theta_1) =$

The Strategy

We can use the Pythagorean identity, $\cos^2(\theta) + \sin^2(\theta) = 1$,

In this case, we can find $\sin(\theta_1)$ by doing the following.

- Find $\sin^2(\theta_1)$ using $\cos(\theta_1)$ and the Pythagorean identity.
- Determine $\sin(\theta_1)$ by considering the quadrant of θ_1.

Finding $\sin^2(\theta_1)$

Let's plug in $\cos(\theta_1) = -\frac{22}{29}$ into the equation $\cos^2(\theta) + \sin^2(\theta) = 1$ to solve for $\sin^2(\theta_1)$.

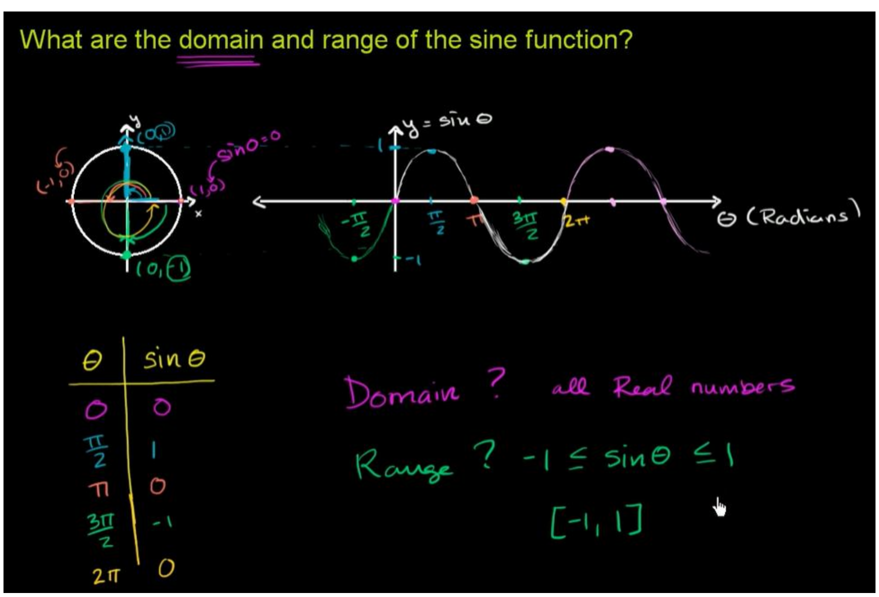
$\cos^2(\theta_1) + \sin^2(\theta_1) = 1$

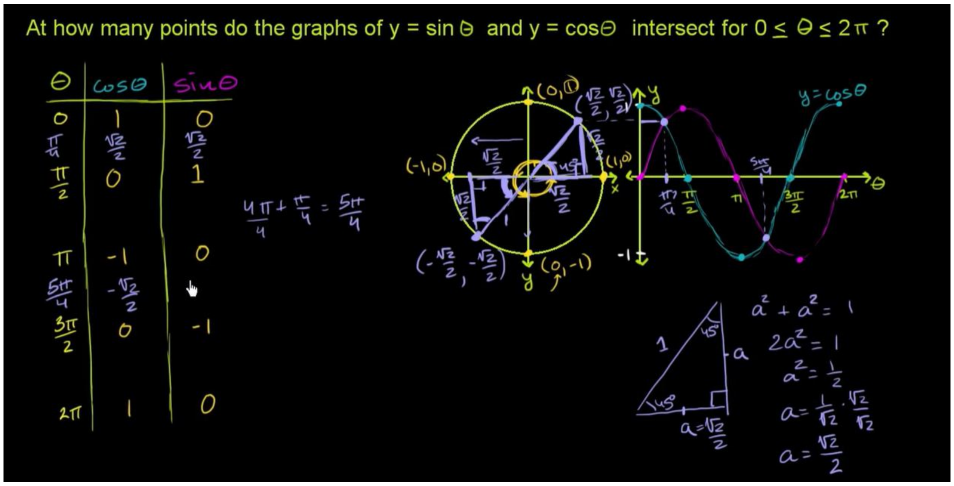
$$\begin{aligned}\sin^2(\theta_1) &= 1 - \cos^2(\theta_1)\\ &= 1 - \left(-\frac{22}{29}\right)^2\\ &= \frac{357}{841}\end{aligned}$$

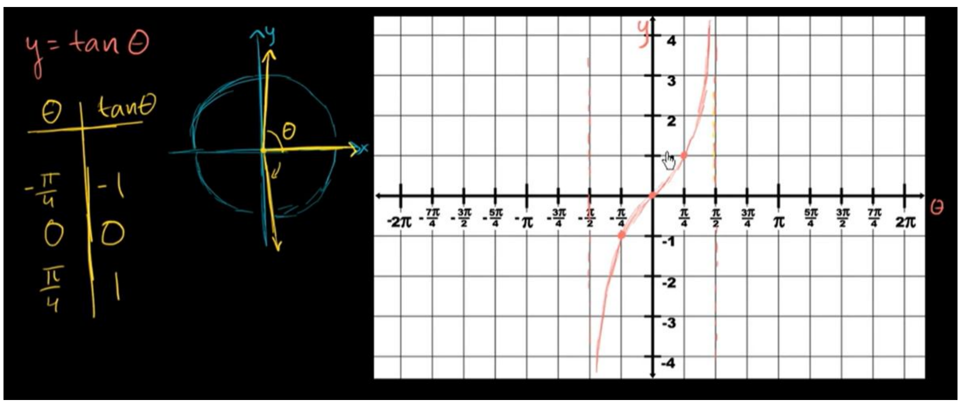
Finding $\sin(\theta_1)$

Since θ_1 is in Quadrant II, $\sin(\theta_1)$ is positive.

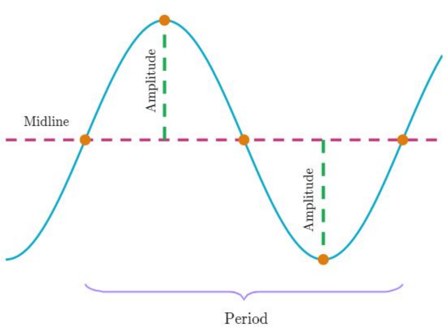
$$\begin{aligned}\sin(\theta_1) &= \sqrt{\sin^2(\theta_1)}\\ &= \sqrt{\frac{357}{841}}\\ &= \frac{\sqrt{357}}{29}\end{aligned}$$

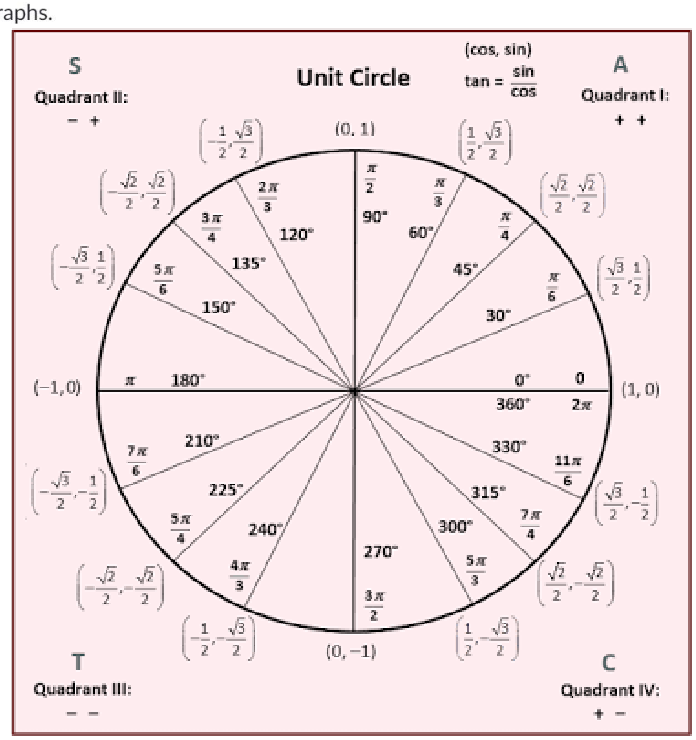




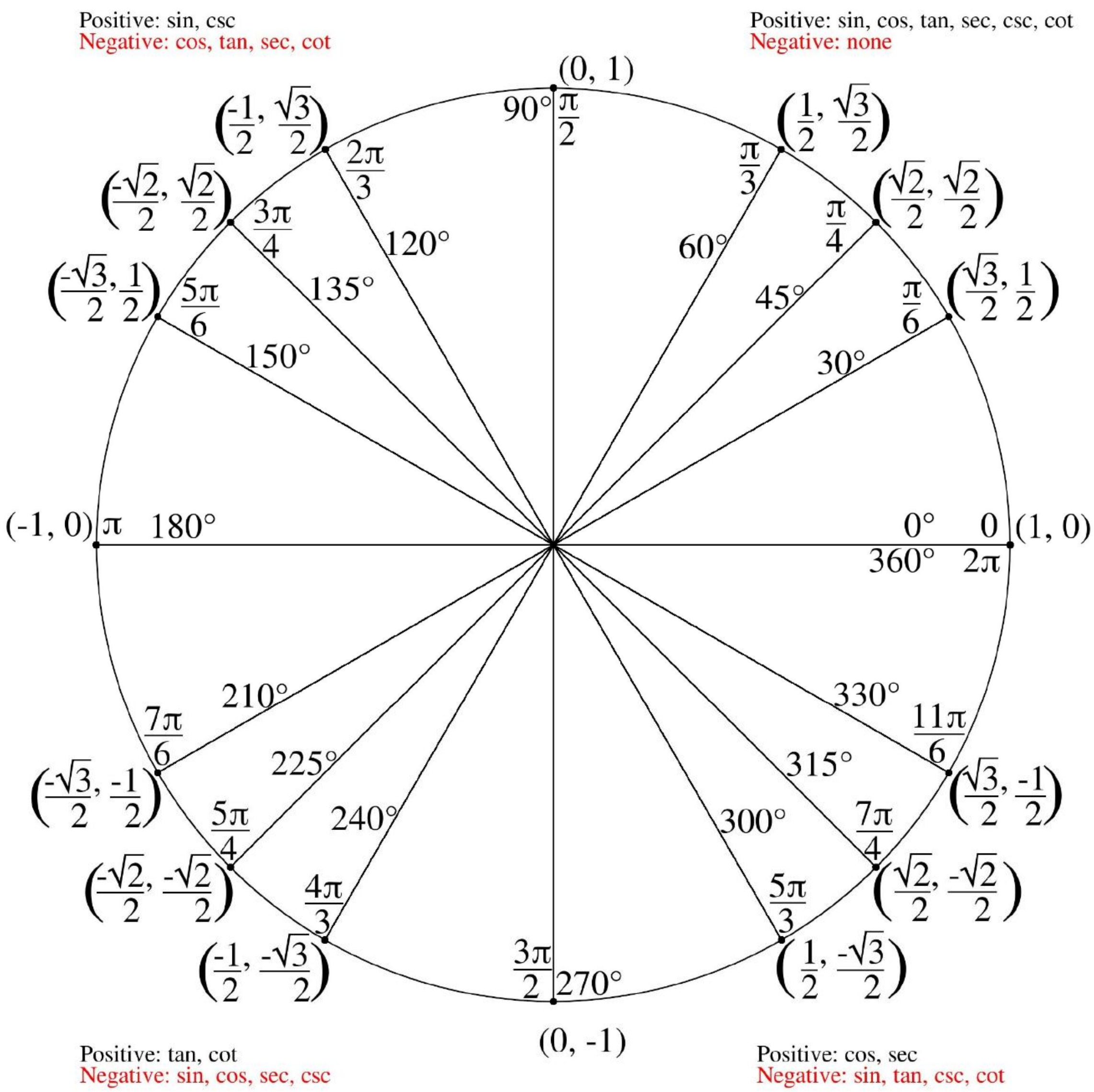


Midline, amplitude, and period are three features of sinusoidal graphs.





The Unit Circle



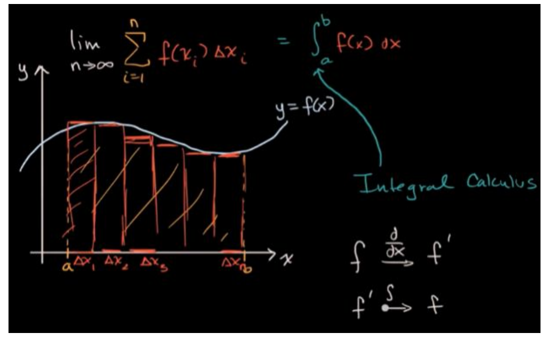
Radians to degrees conversion

$$1° = \frac{\pi}{180} \text{radians}$$

$$\frac{180}{\pi}° = 1 \text{ radians}$$

Introduction to integral calculus

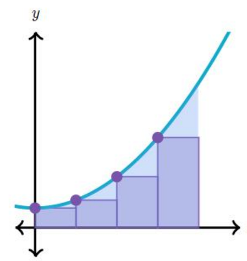
Definite integrals represent the area under the curve of a function and above the *x*-axis



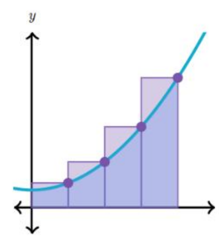
Riemann sums:
A Riemann sum is an approximation of the area under a curve by dividing it into multiple simple shapes (like rectangles or trapezoids).

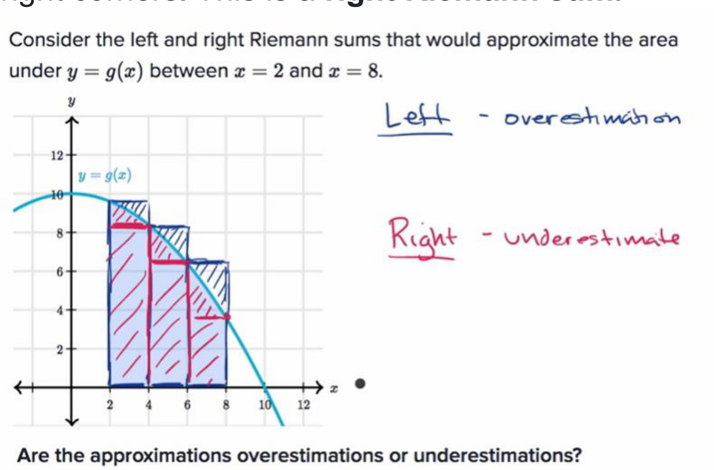
Left & right Riemann sums

To make a Riemann sum, we must choose how we're going to make our rectangles. One possible choice is to make our rectangles touch the curve with their top-left corners. This is called a **left Riemann sum**.



Another choice is to make our rectangles touch the curve with their top-right corners. This is a **right Riemann sum**.

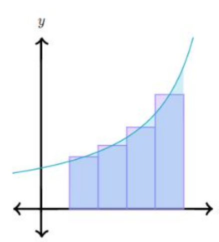



Consider the left and right Riemann sums that would approximate the area under $y = g(x)$ between $x = 2$ and $x = 8$.

Are the approximations overestimations or underestimations?

Riemann sums are approximations of the area under a curve, so they will almost always be slightly more than the actual area (an overestimation) or slightly less than the actual area (an underestimation).

In general, if the function is always increasing or always decreasing on an interval, we can tell whether the Riemann sum approximation will be an overestimation or underestimation based on whether it's a left or a right Riemann sum.



In a **midpoint Riemann sum**, the height of each rectangle is equal to the value of the function at the midpoint of its base.

Example:

Approximate the area between the x-axis and $h(x) = \dfrac{3}{x}$ from $x = 0$ to $x = 1.5$ using a *right Riemann sum* with 3 equal subdivisions.

Solution:

The entire width of the interval $[0, 1.5]$ is 1.5 units, so if we want three equal subdivisions, each rectangle should be 0.5 units wide.

Our first rectangle sits on the interval $[0, 0.5]$. Since we are using a **right** Riemann sum, its top-right vertex should be on the curve. The x-value of that vertex is 0.5, so its y-value is $h(0.5) = \dfrac{3}{0.5} = 6$.

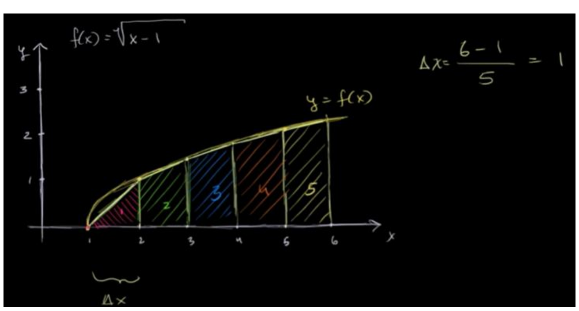
The heights of the other rectangles are, accordingly, $h(1) = \dfrac{3}{1} = 3$ and $h(1.5) = \dfrac{3}{1.5} = 2$.

	First rectangle	Second rectangle	Third rectangle
Width	0.5	0.5	0.5
Height	6	3	2
Area	$0.5 \cdot 6 = 3$	$0.5 \cdot 3 = 1.5$	$0.5 \cdot 2 = 1$

Our approximation is 5.5 units^2.

Trapezoidal sums rule

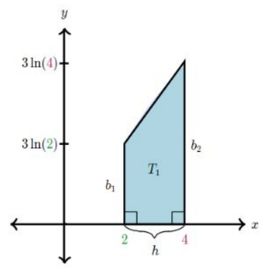
Trapezoidal sums actually give a better approximation, in general, than rectangular sums that use the same number of subdivisions.

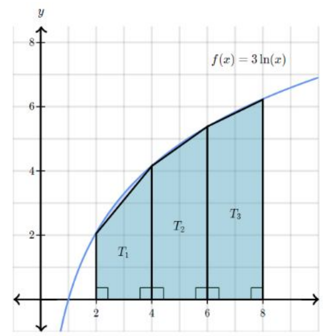


$$\frac{f(1)+f(2)}{2}\Delta x + \frac{f(2)+f(3)}{2}\Delta x + \frac{f(3)+f(4)}{2}\Delta x + \frac{f(4)+f(5)}{2}\Delta x + \frac{f(5)+f(6)}{2}\Delta x$$

An example of the trapezoid rule

Using 3 trapezoids to approximate the area under the function f(x)=3ln(x) on the interval [2,8]





Using the area of a trapezoid is $h(\frac{b_1+b_2}{2})$

Find the Area of T1

x=2 to x=4, so h=2

b_1 is the value of f(x) at x=2, b_1=3 ln (2).

b_2 is the value of f(x) at x=4, b_2=3 ln (4).

$$T_1 = h\left(\frac{b_1+b_2}{2}\right) = 2\left(\frac{3\ln(2)+3\ln(4)}{2}\right) = 3\ln(2) + 3\ln(4)$$

Same for T2, T3

$$T_2 = h\left(\frac{b_1+b_2}{2}\right) = 2\left(\frac{3\ln(4)+3\ln(6)}{2}\right) = 3\ln(4) + 3\ln(6)$$

$$T_3 = h\left(\frac{b_1+b_2}{2}\right) = 2\left(\frac{3\ln(6)+3\ln(8)}{2}\right) = 3\ln(6) + 3\ln(8)$$

Total area approximation = $T_1 + T_2 + T_3$ = 3 (ln 2 + 2 ln 4 + 2 ln 6 + ln 8)

Summation notation (or sigma notation):

Allows us to write a long sum in a single expression.

Rewrite left Riemann using Sigma:

$$\text{Approx. Area} = \underbrace{f(x_0)\Delta x}_{\text{rect } 1} + \underbrace{f(x_1)\Delta x}_{\text{rect } 2} + \underbrace{f(x_2)\Delta x}_{\text{rect } 3} + \cdots + \underbrace{f(x_{n-1})\Delta x}_{\text{rect } n}$$

$$= \boxed{\sum_{i=1}^{n} f(x_{i-1})\Delta x}$$

Stop at $n = 3$

(inclusive)

$$\sum_{n=1}^{3} 2n - 1$$

Start at $n = 1$

Expression for each term in the sum

Summarizing the process of writing a Riemann sum in summation notation

Imagine we want to approximate the area under the graph of f over the interval [a,b] with n equal subdivisions.

Define Δx: Let Δx denote the width of each rectangle, then $\Delta x = \frac{b-a}{n}$.

Define x_i: Let x_i denote the right endpoint of each rectangle, then $x_i = a + \Delta x \cdot i$.

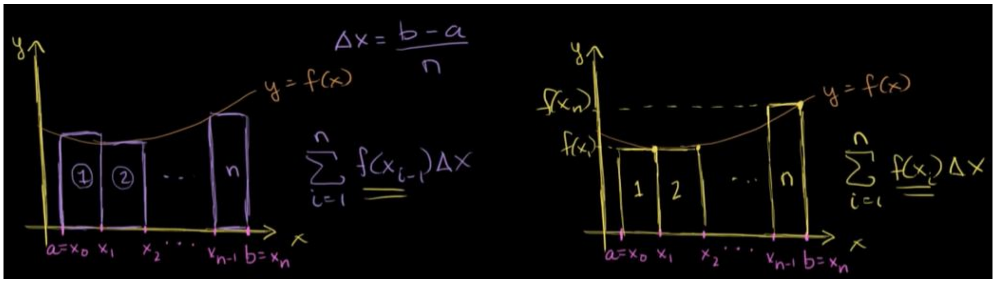
Define area of i^{th} rectangle: The height of each rectangle is then $f(x_i)$, and the area of each rectangle is $\Delta x \cdot f(x_i)$.

Sum the rectangles: Now we use summation notation to add all the areas. The values we use for i are different for left and right Riemann sums:

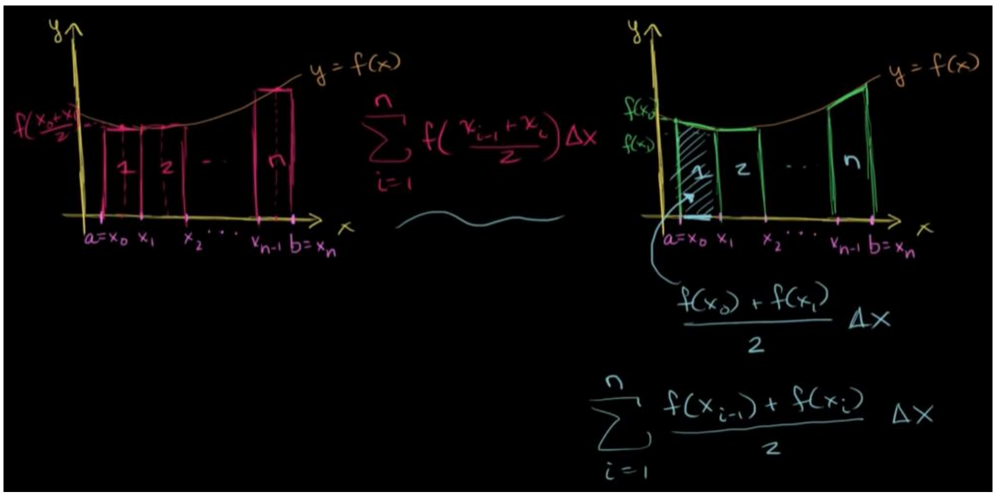
- When we are writing a *right* Riemann sum, we will take values of i from 1 to n.
- However, when we are writing a *left* Riemann sum, we will take values of i from 0 to $n-1$ (these will give us the value of f at the *left* endpoint of each rectangle).

Left Riemann sum	Right Riemann sum
$\sum_{i=0}^{n-1} \Delta x \cdot f(x_i)$	$\sum_{i=1}^{n} \Delta x \cdot f(x_i)$

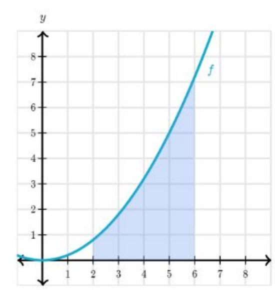
Left and right Reimann sum



Midpoint and trapezoid



Definite integrals represent the area under the curve of a function, and Riemann sums help us approximate such areas.

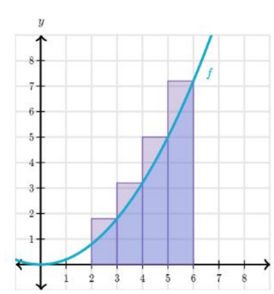


Riemann sums with "infinite" rectangles

Imagine we want to find the area under the graph of $f(x) = \frac{1}{5}x^2$ between $x = 2$ and $x = 6$.

Using definite integral notation, we can represent the exact area:

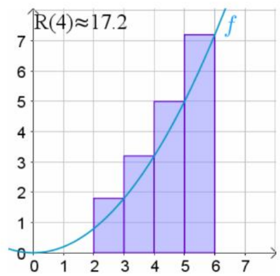
$$\int_2^6 \frac{1}{5}x^2 dx$$



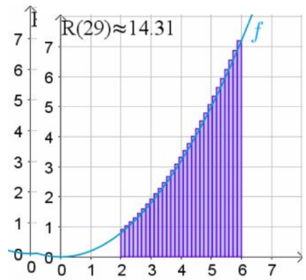
We can approximate this area using Riemann sums. Let *R*(*n*) be the right Riemann sum approximation of our area using *n* equal subdivisions (i.e. *n* rectangles of equal width).

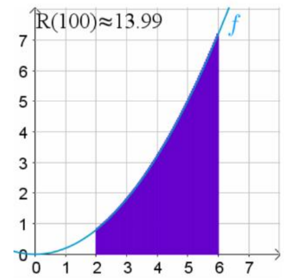
For example, this is *R*(4),you can see it's an overestimation of the actual area.

You can see how the approximation gets closer to the actual area as the number of rectangles goes from 1 to 100:



What if we could take a Riemann sum with *infinite* equal subdivisions? Is that even possible? Well, we can't set *n*=∞ because infinity isn't an actual number, but you might recall we have a way of taking something *to* infinity.





Limits!

$$\lim_{n\to\infty} R(n)$$

Amazing fact #1: This limit really gives us the exact value of $\int_2^6 \frac{1}{5}x^2dx$

Amazing fact #2: It doesn't matter whether we take the limit of a right Riemann sum, a left Riemann sum, or any other common approximation. At infinity, we will always get the exact value of the definite integral.

Quick review: We are looking for Δx, the constant width of any rectangle, and x_i, the x-value of the right edge of the i^{th} rectangle. Then, $f(x_i)$ will give us the height of each rectangle.

$$\Delta x = \frac{6-2}{n} = \frac{4}{n}$$

$$x_i = 2 + \Delta x \cdot i = 2 + \frac{4}{n}i$$

$$f(x_i) = \frac{1}{5}(x_i)^2 = \frac{1}{5}\left(2+\frac{4}{n}i\right)^2$$

So the area of the i^{th} rectangle is $\frac{4}{n}\cdot\frac{1}{5}\left(2+\frac{4}{n}i\right)^2$, and we sum that for values of i from 1 to n:

$$R(n) = \sum_{i=1}^{n}\left(2+\frac{4i}{n}\right)^2\cdot\frac{4}{5n}$$

Now we can represent the actual area as a limit:

$$\int_2^6 \frac{1}{5}x^2\,dx$$

$$= \lim_{n\to\infty} R(n)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(2+\frac{4i}{n}\right)^2 \cdot \frac{4}{5n}$$

By definition, the definite integral is the limit of the Riemann sum

The definite integral of a continuous function f over the interval $[a,b]$, denoted by $\int_a^b f(x)dx$, is the limit of a Riemann sum as the number of subdivisions approaches infinity. That is,

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta x \cdot f(x_i)$$

where $\Delta x = \frac{b-a}{n}$ and $x_i = a + \Delta x \cdot i$.

Example:

write the following definite integral as the limit of a Riemann sum. $\int_{\pi}^{2\pi} \cos(x)\,dx$

Using: $\lim\limits_{n\to\infty} \sum_{i=1}^{n} \Delta x . f(x_i)$, where $\Delta x = \frac{b-a}{n}$, $x_i = a + \Delta x . i$

$$\Delta x = \frac{2\pi - \pi}{n} = \frac{\pi}{n}, \qquad x_i = \pi + \frac{\pi}{n}.i = \pi + \frac{\pi i}{n}$$

Therefore $\int_{\pi}^{2\pi} \cos(x)\,dx = \lim\limits_{n\to\infty} \sum_{i=1}^{n} \frac{\pi}{n}.cos(\pi + \frac{\pi i}{n})$

write a definite integral from the limit of a Riemann sum

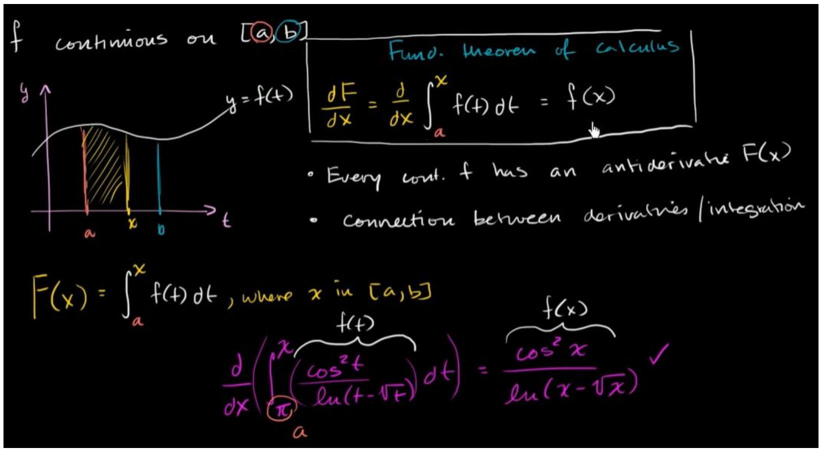
$\lim\limits_{n\to\infty} \sum_{i=1}^{n} \frac{\pi}{n}.cos(\pi + \frac{\pi i}{n})$

$\Delta x = \frac{\pi}{n}$, $x_i = \pi + \frac{\pi i}{n} \gg a = \pi$, f(x)=cos(x)

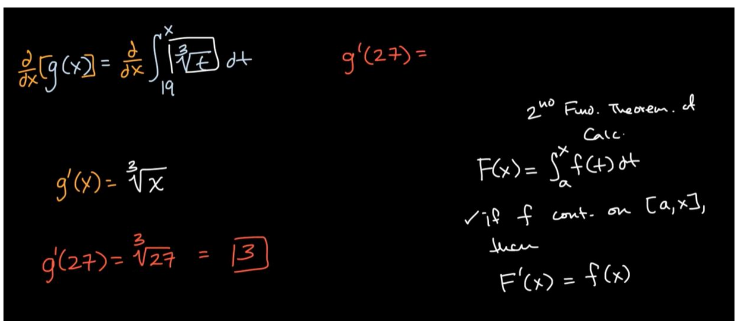
$$\Delta x = \frac{b-a}{n}\ so\ \frac{\pi}{n} = \frac{b-\pi}{n}\ , b = 2\pi$$

Therefor $\int_{\pi}^{2\pi} \cos(x)\,dx$

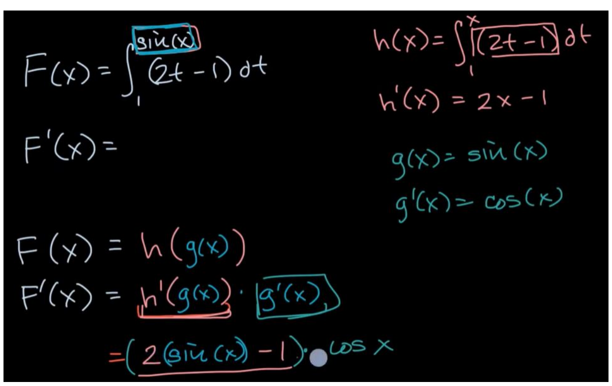
The fundamental theorem of calculus



Finding derivative with fundamental theorem of calculus

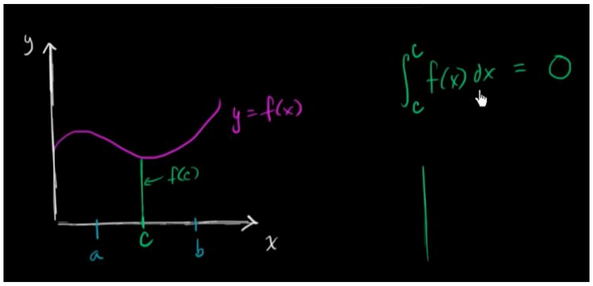


Finding derivative with fundamental theorem of calculus: chain rule

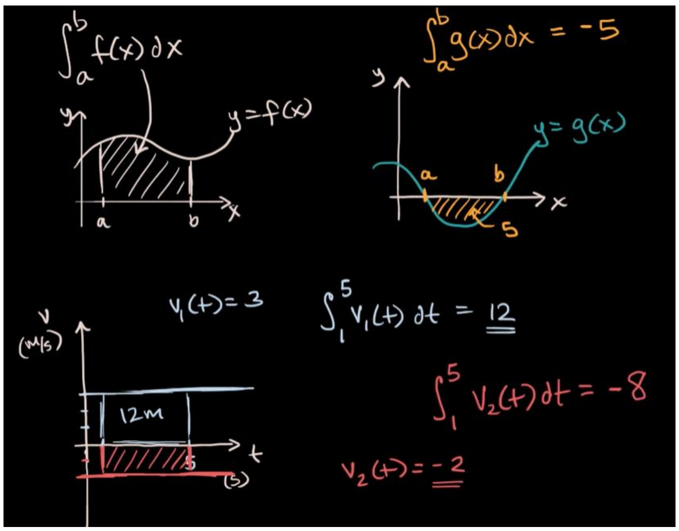


Prosperity of definite integral:

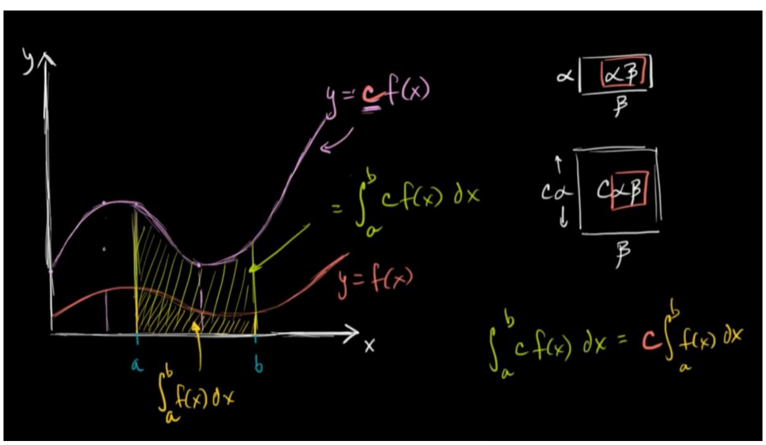
1- Definite integral over a single point



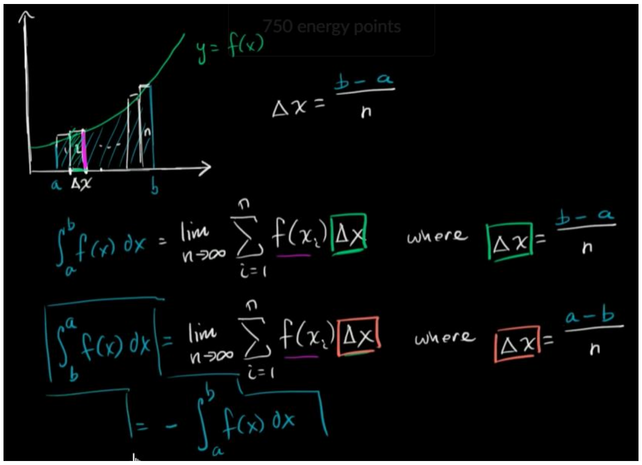
2- Negative definite Integrals



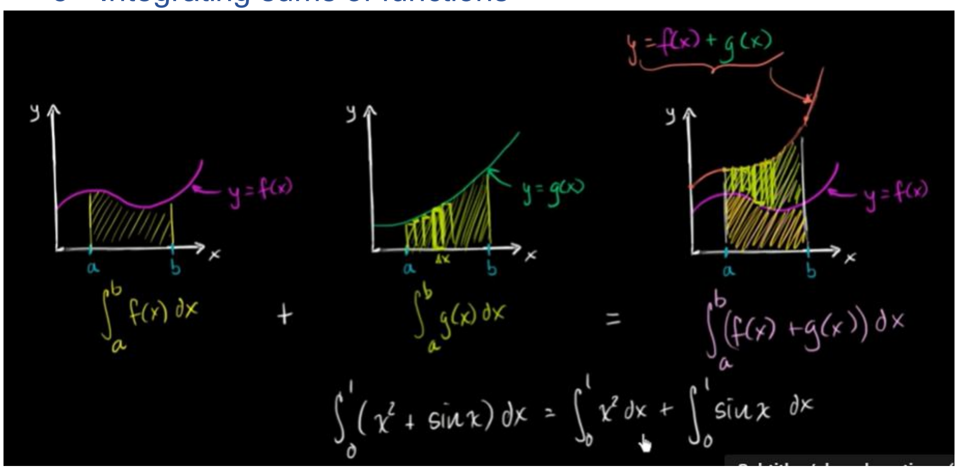
3- Integrating scaled version of function



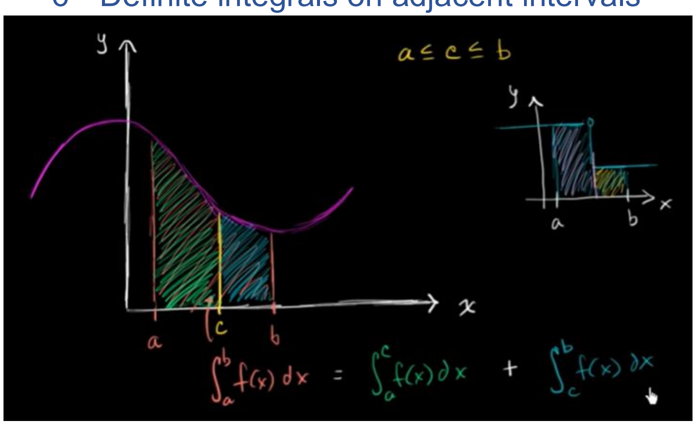
4- Switching bounds of definite integral



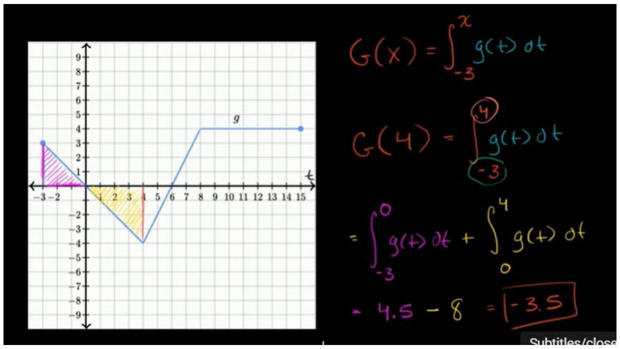
5- Integrating sums of functions



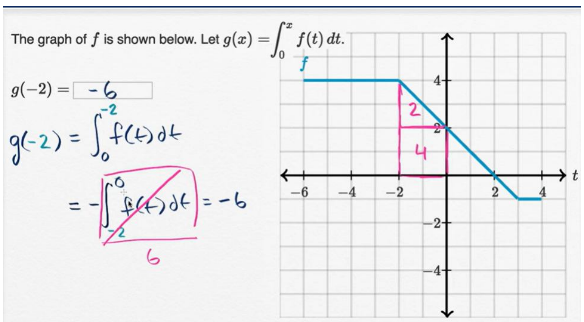
6- Definite integrals on adjacent intervals



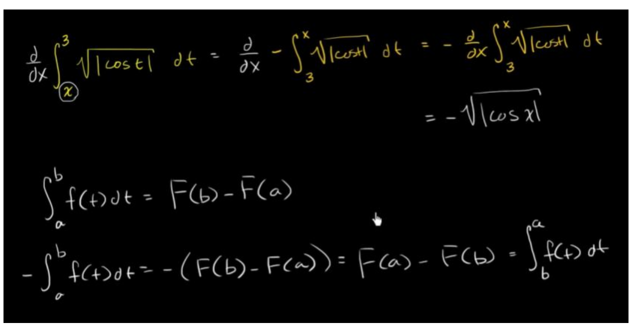
Example



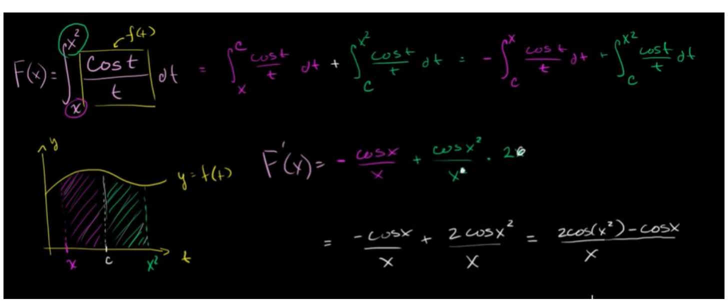
Example



Example



example



Definite integrals properties review

Sum/Difference: $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

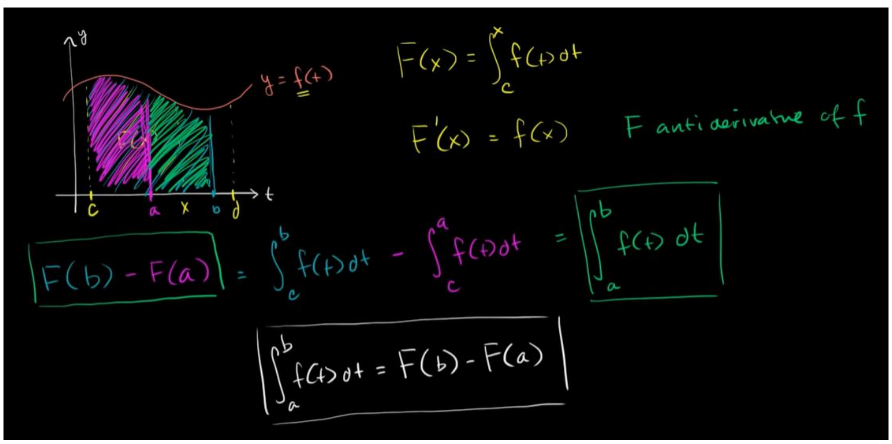
Constant multiple: $\int_a^b k \cdot f(x)dx = k \int_a^b f(x)dx$

Reverse interval: $\int_a^b f(x)dx = -\int_b^a f(x)dx$

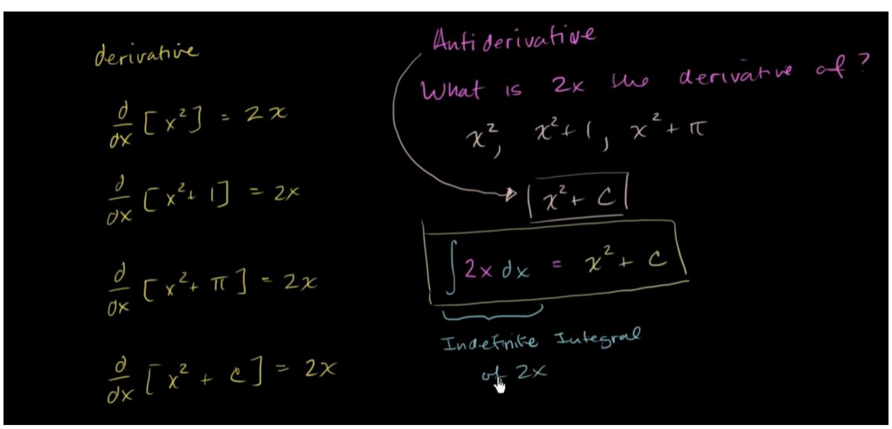
Zero-length interval: $\int_a^a f(x)dx = 0$

Adding intervals: $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$

The fundamental theorem of calculus and definite integrals



Antiderivative



The reverse power rule

where $n \neq -1$:

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$$

Integrals that are logarithmic functions

$$\int \frac{1}{x} \, dx = \ln|x| + C$$

Integrating polynomials

$$\int 3x^7 \, dx = 3\left(\frac{x^{7+1}}{7+1}\right) + C$$

$$= 3\left(\frac{x^8}{8}\right) + C$$

$$= \frac{3}{8}x^8 + C$$

Polynomials

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$$

Radicals

$$\int \sqrt[m]{x^n} \, dx = \int x^{\frac{n}{m}} \, dx$$

$$= \frac{x^{\frac{n}{m}+1}}{\frac{n}{m}+1} + C$$

Exponential functions

$$\int e^x \, dx = e^x + C$$

$$\int a^x \, dx = \frac{a^x}{\ln(a)} + C$$

Integrals that are inverse trigonometric functions

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \arcsin\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$$

Definite integrals: reverse power rule example:

$$\int_a^b f(x)\,dx = \underbrace{F(b)} - \underbrace{F(a)}$$

$$\int_{-3}^{5} 4 \, dx = 4x \Big|_{-3}^{5} = \underbrace{4\cdot 5}_{20} - \underbrace{4(-3)}_{+12} = 32$$

$$\int_{-1}^{3} \underbrace{7x^{2}} \, dx = \frac{7x^3}{3} \Big|_{-1}^{3}$$

$$\underbrace{\frac{7\cdot \cancel{3}^{3}}{\cancel{3}}}_{63} + \underbrace{\frac{7(\cancel{-1})^3}{3}}_{+\frac{7}{3} = 2\frac{1}{3}} = 65\frac{1}{3}$$

Integrals

Definitions

Definite Integral: Suppose $f(x)$ is continuous on $[a,b]$. Divide $[a,b]$ into n subintervals of width Δx and choose x_i^* from each interval.

Then $\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x$.

Anti-Derivative : An anti-derivative of $f(x)$ is a function, $F(x)$, such that $F'(x) = f(x)$.

Indefinite Integral : $\int f(x)dx = F(x)+c$ where $F(x)$ is an anti-derivative of $f(x)$.

Fundamental Theorem of Calculus

Part I : If $f(x)$ is continuous on $[a,b]$ then $g(x) = \int_a^x f(t)dt$ is also continuous on $[a,b]$ and $g'(x) = \frac{d}{dx}\int_a^x f(t)dt = f(x)$.

Part II : $f(x)$ is continuous on $[a,b]$, $F(x)$ is an anti-derivative of $f(x)$ (*i.e.* $F(x) = \int f(x)dx$) then $\int_a^b f(x)dx = F(b)-F(a)$.

Variants of Part I :

$$\frac{d}{dx}\int_a^{u(x)} f(t)dt = u'(x)f[u(x)]$$

$$\frac{d}{dx}\int_{v(x)}^{b} f(t)dt = -v'(x)f[v(x)]$$

$$\frac{d}{dx}\int_{v(x)}^{u(x)} f(t)dt = u'(x)f[u(x)]-v'(x)f[v(x)]$$

Properties

$\int f(x)\pm g(x)dx = \int f(x)dx \pm \int g(x)dx$

$\int_a^b f(x)\pm g(x)dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

$\int_a^a f(x)dx = 0$

$\int_a^b f(x)dx = -\int_b^a f(x)dx$

$\int cf(x)dx = c\int f(x)dx$, c is a constant

$\int_a^b cf(x)dx = c\int_a^b f(x)dx$, c is a constant

$\int_a^b c\,dx = c(b-a)$

$\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|dx$

$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ for any value of c.

If $f(x)\ge g(x)$ on $a\le x\le b$ then $\int_a^b f(x)dx \ge \int_a^b g(x)dx$

If $f(x)\ge 0$ on $a\le x\le b$ then $\int_a^b f(x)dx \ge 0$

If $m\le f(x)\le M$ on $a\le x\le b$ then $m(b-a)\le \int_a^b f(x)dx \le M(b-a)$

Common Integrals

$\int k\,dx = k\,x + c$

$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + c, n \neq -1$

$\int x^{-1}\,dx = \int \frac{1}{x}dx = \ln|x| + c$

$\int \frac{1}{ax+b}dx = \frac{1}{a}\ln|ax+b| + c$

$\int \ln u\,du = u\ln(u) - u + c$

$\int \mathbf{e}^u\,du = \mathbf{e}^u + c$

$\int \cos u\,du = \sin u + c$

$\int \sin u\,du = -\cos u + c$

$\int \sec^2 u\,du = \tan u + c$

$\int \sec u \tan u\,du = \sec u + c$

$\int \csc u \cot u du = -\csc u + c$

$\int \csc^2 u\,du = -\cot u + c$

$\int \tan u\,du = \ln|\sec u| + c$

$\int \sec u\,du = \ln|\sec u + \tan u| + c$

$\int \frac{1}{a^2+u^2}du = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + c$

$\int \frac{1}{\sqrt{a^2-u^2}}du = \sin^{-1}\left(\frac{u}{a}\right) + c$

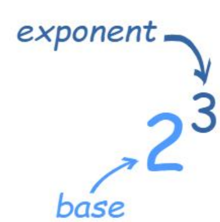
Exponents

Let a and b be real numbers and m and n be integers. Then tl exponents hold, provided that all of the expressions appearing in defined.

1. $a^m a^n = a^{m+n}$	2. $(a^m)^n = a^{mn}$	3. $(ab)^m = a^m b^m$
4. $\frac{a^m}{a^n} = a^{m-n},\ a \neq 0$	5. $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m},\ b \neq 0$	6. $a^{-m} = \frac{1}{a^m},\ a \neq$
7. $a^{\frac{1}{n}} = \sqrt[n]{a}$	8. $a^0 = 1,\ a \neq 0$	9. $a^{\frac{m}{n}} = \sqrt[n]{a^m} =$

where m and n are integers in properties 7 and 9.

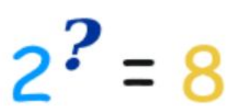
What is an Exponent

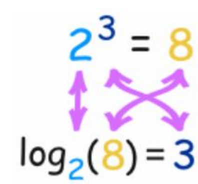


The exponent of a number says **how many times** to use the number in a multiplication.
In this example: $\mathbf{2^3 = 2 \times 2 \times 2 = 8}$

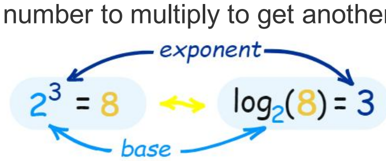
What is a Logarithm

A Logarithm goes the other way.
It asks the question "what exponent produced this?":

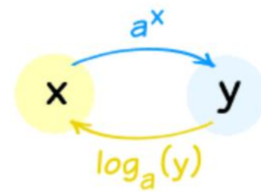




A Logarithm says how many of one number to multiply to get another number



Exponents and Logarithms work well together because they "undo" each other (so long as the base "a" is the same):



Logarithms

Definition: $y = \log_a x$ if and only if $x = a^y$, where $a > 0$.
In other words, logarithms are exponents.

Remarks:

- $\log x$ always refers to log base 10, i.e., $\log x = \log_{10} x$.
- $\ln x$ is called the natural logarithm and is used to represent $\log_e x$, where the irrational number $e \approx 2.71828$. Therefore, $\ln x = y$ if and only if $e^y = x$.
- Most calculators can directly compute logs base 10 and the natural log. For any other base it is necessary to use the change of base formula: $\log_b a = \dfrac{\ln a}{\ln b}$ or $\dfrac{\log_{10} a}{\log_{10} b}$.

Properties of Logarithms (Recall that logs are only defined for positive values of x.)

For the natural logarithm	For logarithms base a
1. $\ln xy = \ln x + \ln y$	1. $\log_a xy = \log_a x + \log_a y$
2. $\ln \frac{x}{y} = \ln x - \ln y$	2. $\log_a \frac{x}{y} = \log_a x - \log_a y$
3. $\ln x^y = y \cdot \ln x$	3. $\log_a x^y = y \cdot \log_a x$
4. $\ln e^x = x$	4. $\log_a a^x = x$
5. $e^{\ln x} = x$	5. $a^{\log_a x} = x$

Useful Identities for Logarithms

For the natural logarithm	For logarithms base a
1. $\ln e = 1$	1. $\log_a a = 1$, for all $a > 0$
2. $\ln 1 = 0$	2. $\log_a 1 = 0$, for all $a > 0$

Natural Log Rules

The ln of a negative number is undefined

ln(0) is undefined

ln(1)=0

ln(∞)= ∞

ln(e)=1

ln(e^x) = x

$e^{\ln(x)}$=x

- ln(x)(y) = ln(x) + ln(y)
- ln(x/y) = ln(x) - ln(y)
- ln(1/x) = −ln(x)
- ln(x^y) = y * ln(x)

Logarithms

The key difference between natural logs and other logarithms is the base being used. **Logarithms typically use a base of 10 (although it can be a different value, which will be specified), while natural logs will always use a base of *e*.**

This means ln(x)=$\log_e(x)$

If you need to convert between logarithms and natural logs, use the following two equations:

- $\log_{10}(x)$ = ln(x) / ln(10)
- ln(x) = $\log_{10}(x)$ / $\log_{10}(e)$

Logarithm Rules:

$\log(xy) = \log(x) + \log\,(y)$

$\log(x/y) = \log(x) - \log\,(y)$

$\log(ax) = \mathrm{a}\log(x)$

$\log(10x) = \mathrm{x}$

$10^{\log\,(x)} = \mathrm{x}$

$e^a e^b = e^{a+b}$

$e^{\ln(xy)} = xy = e^{\ln(x)} e^{\ln(y)} = e^{\ln(x)+\ln(y)}, \ln(xy) = \ln(x) + \ln(y)$

$\frac{e^a}{e^b} = e^{a-b}$

$e^{\ln(x/y)} = \frac{x}{y} = \frac{e^{\ln(x)}}{e^{\ln(y)}} = e^{\ln(x)-\ln(y)}, \ln(x/y) = \ln(x) - \ln(y)$

$(e^a)^b = e^{ab}$

$e^{\ln(x^y)} = x^y = \left(e^{\ln(x)}\right)^y = e^{y\ln(x)}, e^{\ln(x^y)} = y\ln(x)$

Probability

Probability is simply how likely something is to happen.

$$\frac{\text{\# of possibilities that meet my condition}}{\text{\# of equally likely possibilities}}$$

Integration by parts : integrate

$$\int f(x) \times g'(x)\, dx = -\int f'(x) \times g(x)\, dx + f(x) \times g(x)$$

differentiate

$(+C)$

$$\int \ln x \, dx = \int \underbrace{1}_{\text{integrate}} \times \underbrace{\ln x}_{\text{differentiate}}\, dx$$

iBP

$$= -\int \underbrace{x}_{g(x)} \times \underbrace{\frac{1}{x}}_{f'(x)}\, dx + x\ln x$$

$f(x) = \ln x$

$g'(x) = 1$

$$= -\int 1\, dx + x\ln x$$

$$= -x + C + x\ln x$$

11 (d)

$$\int_0^t \overbrace{x^3}^{u}\overbrace{e^x}^{v'}\,dx$$

$$\overset{\text{iBP}}{=} -\underbrace{\int_0^t \underbrace{3x^2}_{u'}\,\underbrace{e^x}_{v}\,dx} + \Big[\underbrace{x^3}_{u}\,\underbrace{e^x}_{v}\Big]_0^t$$

New integral: $\displaystyle\int_0^t \underbrace{3x^2}_{u}\,\underbrace{e^x}_{v'}\,dx$

$$\overset{\text{iBP}}{=} -\int_0^t \underbrace{6x}_{u'}\,\underbrace{e^x}_{v}\,dx + \Big[\underbrace{3x^2}_{u}\,\underbrace{e^x}_{v}\Big]_0^t$$

New integral: $\displaystyle\int_0^t \underbrace{6x}_{u}\,\underbrace{e^x}_{v'}\,dx \overset{\text{iBP}}{=} -\int_0^t 6e^x\,dx + \Big[6xe^x\Big]_0^t$