Chapter 5: Eigenvalues and Diagonalization

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1. Simple Definitions

Eigenvalue

For a square matrix A of size nn, an **eigenvalue** is a scalar λ such that:

$$A\mathbf{v} = \lambda\mathbf{v}$$

for some non-zero vector $\mathbf{v}$. This means multiplying A by $\mathbf{v}$ only scales $\mathbf{v}$ without changing its direction.

Eigenvector

An **eigenvector** is a non-zero vector $\mathbf{v}$ that satisfies:

$$A\mathbf{v} = \lambda\mathbf{v}$$

for some eigenvalue λ. It represents a "special direction" where the matrix acts like simple multiplication.

Diagonalization

A matrix A is **diagonalizable** if it can be written as:

$$A = PDP^{-1}$$

where:

- D is a diagonal matrix (all off-diagonal entries are zero)
- P is an invertible matrix whose columns are eigenvectors of A
- The diagonal entries of D are the eigenvalues of A

2. Applications in Real Life and AI

Applications in Real Life

- **Vibrations Analysis**: Eigenvalues represent natural frequencies of mechanical systems
- **Quantum Mechanics**: Eigenvalues represent possible measurements of observable quantities
- **Google PageRank**: Eigenvectors determine the importance of web pages
- **Image Compression**: Principal Component Analysis (PCA) uses eigenvectors to reduce data size

Applications in Artificial Intelligence

- **PCA (Dimensionality Reduction)**: Finds directions (eigenvectors) with maximum variance
- **Face Recognition**: Eigenfaces method uses eigenvectors of face image covariance matrix
- **Recommendation Systems**: Matrix factorization techniques use eigenvalue decomposition
- **Neural Networks**: Eigenvalues help analyze training stability and convergence

3. Complete Example with Calculations

Given Matrix:

$$A = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$$

Step 1: Find Eigenvalues

Solve $\det(A - \lambda I) = 0$:

$$\det \begin{pmatrix} 4-\lambda & 1 \\ 2 & 3-\lambda \end{pmatrix} = (4-\lambda)(3-\lambda) - 2 = 0$$

$$\lambda^2 - 7\lambda + 12 - 2 = \lambda^2 - 7\lambda + 10 = 0$$

$$\lambda = \frac{7 \pm \sqrt{49 - 40}}{2} = \frac{7 \pm 3}{2}$$

$$\lambda_1 = 5, \quad \lambda_2 = 2$$

Step 2: Find Eigenvectors

For $\lambda_1 = 5$:

$$(A - 5I)\mathbf{v}_1 = \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

From $-x + y = 0$: $x = y$ Choose $x = 1$, then $y = 1$:

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 2$:

$$(A - 2I)\mathbf{v}_2 = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

From $2x + y = 0$: $y = -2x$ Choose $x = 1$, then $y = -2$:

$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Step 3: Construct P and D

$$P = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}, \quad D = \begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix}$$

Step 4: Find P^{-1}

$$\det(P) = (1)(-2) - (1)(1) = -3$$

$$P^{-1} = \frac{1}{-3} \begin{pmatrix} -2 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{pmatrix}$$

Step 5: Verify $A = PDP^{-1}$

$$PDP^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} \frac{10}{3} & \frac{5}{3} \\ \frac{2}{3} & -\frac{2}{3} \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix} = A$$

Practical Use: Compute A^{10} Easily

Instead of 10 matrix multiplications:

$$A^{10} = PD^{10}P^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 5^{10} & 0 \\ 0 & 2^{10} \end{pmatrix} \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{pmatrix}$$

$$5^{10} = 9,765,625, \quad 2^{10} = 1,024$$

Only 3 matrix multiplications needed!

Why is Diagonalization Useful?

Computing A^{10}

To compute A^{10}:

$$A^{10} = PD^{10}P^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 5^{10} & 0 \\ 0 & 2^{10} \end{pmatrix} \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{pmatrix}$$

Instead of 10 matrix multiplications, we just do:

- $5^{10} = 9,765,625$
- $2^{10} = 1,024$
- Then only 3 matrix multiplications

That's the power of diagonalization!

Complexity Comparison

- **Direct computation**: 10 matrix multiplications $\approx 10O(n^3)$ operations
- **With diagonalization**:
  - Diagonal matrix power: n scalar exponentiations
  - Only 3 matrix multiplications
  - Total: $O(n^3) + n$ exponentiations

General Formula

For any positive integer k:

$$A^k = PD^kP^{-1} = P \begin{pmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{pmatrix} P^{-1}$$

This works for any k, large or small!