Chapter 4: Determinants and Inverse of a Matrix

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1 Determinant of a Matrix

The determinant is a special number that can be calculated from a matrix. The matrix has to be square (same number of rows and columns) like this one:

$$\begin{bmatrix} 3 & 4 \\ 8 & 6 \end{bmatrix}$$

A Matrix
(this one has 2 rows and 2 columns).
Let us calculate the determinant of that matrix:

$$3 \times 6 - 8 \times 4 = 18 - 32 = -14$$

Or if you prefer with line-by-line alignment:

$$3 \times 6 - 8 \times 4 = 18 - 32$$
$$= -14$$

What is it for?

The determinant helps us find the inverse of a matrix, tells us things about the matrix that are useful in systems of linear equations, calculus and more.

Calculating the Determinant

First of all the matrix must be square (i.e. have the same number of rows as columns). Then it is just arithmetic.

For a 2×2 Matrix

For a 2×2 matrix (2 rows and 2 columns):

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The determinant is:

$$|A| = ad - bc$$

For a 3×3 Matrix

For a 3×3 matrix (3 rows and 3 columns):

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

The determinant is:

$$|A| = a(ei - fh) - b(di - fg) + c(dh - eg)$$

Example:

$$D = \begin{bmatrix} 6 & 1 & 1 \\ 4 & -2 & 5 \\ 2 & 8 & 7 \end{bmatrix}$$

$$
\begin{aligned}
|D| &= 6 \times ((-2) \times 7 - 5 \times 8) - 1 \times (4 \times 7 - 5 \times 2) + 1 \times (4 \times 8 - (-2) \times 2) \\
&= 6 \times (-14 - 40) - 1 \times (28 - 10) + 1 \times (32 + 4) \\
&= 6 \times (-54) - 1 \times 18 + 1 \times 36 \\
&= -324 - 18 + 36 \\
&= -306
\end{aligned}
$$

For 4×4 Matrices and Higher

The pattern continues for 4×4 matrices:

- plus a times the determinant of the matrix that is **not** in a's row or column,

- minus b times the determinant of the matrix that is **not** in b's row or column,

- plus c times the determinant of the matrix that is **not** in c's row or column,

- minus d times the determinant of the matrix that is **not** in d's row or column,

Example:

$$A = \begin{pmatrix} 2 & 1 & 0 & 3 \\ -1 & 2 & 1 & 0 \\ 3 & 0 & 2 & -2 \\ 1 & 1 & -1 & 4 \end{pmatrix}$$

The determinant of a 4×4 matrix A can be computed using cofactor expansion along the first row:

$$|A| = 2 \cdot \begin{vmatrix} 2 & 1 & 0 \\ 0 & 2 & -2 \\ 1 & -1 & 4 \end{vmatrix} - 1 \cdot \begin{vmatrix} -1 & 1 & 0 \\ 3 & 2 & -2 \\ 1 & -1 & 4 \end{vmatrix} + 0 \cdot \begin{vmatrix} -1 & 2 & 0 \\ 3 & 0 & -2 \\ 1 & 1 & 4 \end{vmatrix} - 3 \cdot \begin{vmatrix} -1 & 2 & 1 \\ 3 & 0 & 2 \\ 1 & 1 & -1 \end{vmatrix}$$

Step-by-step calculation of $|A|$

Given:

$$A = \begin{pmatrix} 2 & 1 & 0 & 3 \\ -1 & 2 & 1 & 0 \\ 3 & 0 & 2 & -2 \\ 1 & 1 & -1 & 4 \end{pmatrix}$$

Expand along first row:

$$|A| = 2 \cdot M_{11} - 1 \cdot M_{12} + 0 \cdot M_{13} - 3 \cdot M_{14}$$

where M_{1j} = determinant of 3×3 minor after deleting row 1, column j.

1. Minor M_{11}:

$$M_{11} = \det \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & -2 \\ 1 & -1 & 4 \end{pmatrix}$$

$$M_{11} = 2 \cdot (2 \cdot 4 - (-2) \cdot (-1)) - 1 \cdot (0 \cdot 4 - (-2) \cdot 1) + 0$$
$$= 2 \cdot (8 - 2) - 1 \cdot (0 + 2)$$
$$= 2 \cdot 6 - 2 = 12 - 2 = 10$$

2. Minor M_{12}:

$$M_{12} = \det \begin{pmatrix} -1 & 1 & 0 \\ 3 & 2 & -2 \\ 1 & -1 & 4 \end{pmatrix}$$

$$M_{12} = (-1) \cdot (2 \cdot 4 - (-2) \cdot (-1)) - 1 \cdot (3 \cdot 4 - (-2) \cdot 1) + 0$$
$$= (-1) \cdot (8 - 2) - 1 \cdot (12 + 2)$$
$$= (-1) \cdot 6 - 1 \cdot 14 = -6 - 14 = -20$$

3. Minor M_{14}:

$$M_{14} = \det \begin{pmatrix} -1 & 2 & 1 \\ 3 & 0 & 2 \\ 1 & 1 & -1 \end{pmatrix}$$

$$M_{14} = (-1) \cdot (0 \cdot (-1) - 2 \cdot 1) - 2 \cdot (3 \cdot (-1) - 2 \cdot 1) + 1 \cdot (3 \cdot 1 - 0 \cdot 1)$$
$$= (-1) \cdot (0 - 2) - 2 \cdot (-3 - 2) + 1 \cdot (3 - 0)$$

$$= (-1) \cdot (-2) - 2 \cdot (-5) + 3$$
$$= 2 + 10 + 3 = 15$$

Combine:
$$|A| = 2 \cdot 10 - 1 \cdot (-20) - 3 \cdot 15$$
$$= 20 + 20 - 45 = 40 - 45$$

$$\boxed{|A| = -5}$$

2 Inverse of a Matrix

What is the Inverse of a Matrix?

Just like a number has a reciprocal:

$$\text{Reciprocal of a Number} \quad \text{(note: } \frac{1}{8} \text{ can also be written } 8^{-1})$$

When we multiply a number by its reciprocal we get 1:

$$8 \times \frac{1}{8} = 1$$

When we multiply a matrix by its inverse we get the **Identity Matrix** (which is like "1" for matrices):

$$A \times A^{-1} = I$$

Identity Matrix

We just mentioned the "Identity Matrix". It is the matrix equivalent of the number "1":

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(A 3×3 Identity Matrix)

- It is "square" (has same number of rows as columns),
- It has 1s on the diagonal and 0s everywhere else,
- Its symbol is the capital letter I.

The Identity Matrix can be any square size: 2×2, 3×3, 4×4, and so on.

Multiplying a matrix by I (assuming the sizes match) does not change the matrix, just like multiplying a number by 1 does not change the number.

Remember it must be true that:

$$AA^{-1} = I$$

Why Do We Need an Inverse?

Because with matrices we **don't divide**! Seriously, there is no concept of dividing by a matrix.

But we can multiply by an inverse, which achieves the same thing.

Example: Inverse of a 2×2 Matrix

For a general 2×2 matrix:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

The inverse is given by:

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \quad \text{provided } ad - bc \neq 0$$

Numerical example:

$$B = \begin{pmatrix} 4 & 7 \\ 2 & 6 \end{pmatrix}$$

First, find the determinant:

$$\det(B) = (4)(6) - (7)(2) = 24 - 14 = 10$$

Then:

$$B^{-1} = \frac{1}{10} \begin{pmatrix} 6 & -7 \\ -2 & 4 \end{pmatrix} = \begin{pmatrix} 0.6 & -0.7 \\ -0.2 & 0.4 \end{pmatrix}$$

Let's verify $BB^{-1} = I$:

$$\begin{pmatrix} 4 & 7 \\ 2 & 6 \end{pmatrix} \begin{pmatrix} 0.6 & -0.7 \\ -0.2 & 0.4 \end{pmatrix} = \begin{pmatrix} 4(0.6) + 7(-0.2) & 4(-0.7) + 7(0.4) \\ 2(0.6) + 6(-0.2) & 2(-0.7) + 6(0.4) \end{pmatrix}$$

$$= \begin{pmatrix} 2.4 - 1.4 & -2.8 + 2.8 \\ 1.2 - 1.2 & -1.4 + 2.4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

Bigger Matrices

The inverse of a 2×2 matrix is easy ... compared to larger matrices (such as a 3×3, 4×4, and so on).

For those larger matrices there are three main methods to work out the inverse:

1. **Inverse of a Matrix using Elementary Row Operations (Gauss–Jordan)**

2. **Inverse of a Matrix using Minors, Cofactors and Adjugate**

3. **Use a computer** (such as the Matrix Calculator)

Conclusion

- The inverse of A is A^{-1} only when

$$AA^{-1} = A^{-1}A = I$$

- To find the inverse of a 2×2 matrix:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \Rightarrow \quad A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

(swap the positions of a and d, put negatives in front of b and c, and divide everything by the determinant $ad - bc$)

- Sometimes there is no inverse at all (when the determinant is zero).

Sources

- https://www.mathsisfun.com/algebra/matrix-inverse.html

- https://www.mathsisfun.com/algebra/matrix-determinant.html