APPLICATIONS OF INTEGRATION SOLUTIONS

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1 Introduction: Definite Integrals and Area

Explanation. This section focuses on evaluating definite integrals and using them to calculate the area of regions bounded by curves.

The Definite Integral: The definite integral of a function $f(x)$ from $x = a$ to $x = b$, denoted $\int_a^b f(x)\mathrm{d}x$, represents the *net signed area* between the curve $y = f(x)$ and the x-axis over the interval $[a, b]$. "Net signed area" means area above the x-axis counts positively, and area below the x-axis counts negatively.

The Fundamental Theorem of Calculus (FTC), Part 2 (Evaluation Theorem): This theorem provides the primary method for evaluating definite integrals. If $f(x)$ is continuous on $[a, b]$ and $F(x)$ is any antiderivative of $f(x)$ (meaning $F'(x) = f(x)$), then:

$$\int_a^b f(x)\mathrm{d}x = F(b) - F(a)$$

The process is:
1. Find an antiderivative $F(x)$ of the integrand $f(x)$.
2. Evaluate $F(x)$ at the upper limit b and the lower limit a.
3. Subtract the value at the lower limit from the value at the upper limit.

Notation: $[F(x)]_a^b = F(b) - F(a)$.

Area Application: If a function $f(x)$ is *non-negative* ($f(x) \geq 0$) and continuous on $[a, b]$, then the definite integral $\int_a^b f(x)\mathrm{d}x$ directly gives the geometric area of the region under the curve $y = f(x)$, above the x-axis, between $x = a$ and $x = b$.

$$\text{Area} = \int_a^b f(x)\mathrm{d}x \quad (\text{if } f(x) \geq 0 \text{ on } [a, b])$$

If $f(x)$ is negative on the interval, the integral will be negative, and the area is the absolute value of the integral, Area $= |\int_a^b f(x)\mathrm{d}x| = -\int_a^b f(x)\mathrm{d}x$. If the function crosses the x-axis within $[a, b]$, the area must be calculated by splitting the integral at the x-intercepts and taking the absolute value of the parts below the axis.

2 Solutions: Examples

1. Find the area of the region bounded by the x-axis and the graph of $f(x) = x^2 - 1$ for $1 \leq x \leq 2$.

   Solution:

   Strategy. Check if $f(x) \geq 0$ on $[1, 2]$. If so, Area $= \int_1^2 f(x)\mathrm{d}x$. Evaluate using FTC.

For $x \in [1, 2]$, $x^2 \geq 1$, so $f(x) = x^2 - 1 \geq 0$.

Step 1: Set up integral.

$$\text{Area} = \int_1^2 (x^2 - 1)\mathrm{d}x$$

Step 2: Antiderivative. $F(x) = \frac{x^3}{3} - x$.

Step 3: Evaluate. Area $= [\frac{x^3}{3} - x]_1^2 = (\frac{8}{3} - 2) - (\frac{1}{3} - 1) = \frac{2}{3} - (-\frac{2}{3}) = \frac{4}{3}$.

Final Answer: The area is $4/3$.

2. Evaluate the definite integral $\displaystyle\int_0^1 (4t + 1)^2 \, \mathrm{d}t$.

   Solution:

   Method 1: Expand First

   Step 1: Expand. $(4t + 1)^2 = 16t^2 + 8t + 1$.

   Step 2: Antiderivative. $F(t) = \frac{16}{3}t^3 + 4t^2 + t$.

   Step 3: Evaluate. $[\frac{16}{3}t^3 + 4t^2 + t]_0^1 = (\frac{16}{3} + 4 + 1) - (0) = \frac{16}{3} + 5 = \frac{31}{3}$.

   Method 2: U-Substitution

   Let $u = 4t + 1 \implies \mathrm{d}u = 4\mathrm{d}t \implies \mathrm{d}t = \frac{1}{4}\mathrm{d}u$.

   Limits: $t = 0 \implies u = 1$; $t = 1 \implies u = 5$.

   Integral becomes $\int_1^5 u^2(\frac{1}{4}\mathrm{d}u) = \frac{1}{4}\int_1^5 u^2\mathrm{d}u$.

   Evaluate: $\frac{1}{4}[\frac{u^3}{3}]_1^5 = \frac{1}{4}(\frac{5^3}{3} - \frac{1^3}{3}) = \frac{1}{4}(\frac{125-1}{3}) = \frac{1}{4}(\frac{124}{3}) = \frac{31}{3}$.

   Results match.

3 Solutions: ClassWork Problems

3.1 Area Problems

3. Find the area bounded by $y = x - x^2$ and x-axis.

   Solution:

   Find bounds: $x(1 - x) = 0 \implies x = 0, x = 1$. Bounds are $[0, 1]$.

   Check sign: $y(0.5) = 0.5 - 0.25 = 0.25 > 0$. $y \geq 0$ on $[0, 1]$.

   Area $= \int_0^1 (x - x^2)\mathrm{d}x$.

   Antiderivative: $F(x) = \frac{x^2}{2} - \frac{x^3}{3}$.

   Evaluate: $[\frac{x^2}{2} - \frac{x^3}{3}]_0^1 = (\frac{1}{2} - \frac{1}{3}) - 0 = \frac{1}{6}$.

   Final Answer: Area is $1/6$.

4. Find the area bounded by $y = 1 - x^4$ and x-axis.

   Solution:

   Find bounds: $1 - x^4 = 0 \implies x^4 = 1 \implies x = \pm 1$. Bounds are $[-1, 1]$.

   Check sign: $y(0) = 1 > 0$. $y \geq 0$ on $[-1, 1]$.

   Area $= \int_{-1}^{1} (1 - x^4)\mathrm{d}x$.

   Antiderivative: $F(x) = x - \frac{x^5}{5}$.

   Evaluate: $[x - \frac{x^5}{5}]_{-1}^{1} = (1 - \frac{1}{5}) - (-1 - (-\frac{1}{5})) = \frac{4}{5} - (-\frac{4}{5}) = \frac{8}{5}$.

   (Or use symmetry: $2\int_0^1 (1 - x^4)\mathrm{d}x = 2[x - x^5/5]_0^1 = 2(1 - 1/5) = 8/5$.*)*

   Final Answer: Area is 8/5.

5. Find the area bounded by $y = \dfrac{1}{x^2}$, x-axis, $x = 1$, $x = 2$.

   Solution:

   Bounds $[1, 2]$. $y = x^{-2} > 0$ on $[1, 2]$.

   Area $= \int_1^2 x^{-2}\mathrm{d}x$.

   Antiderivative: $F(x) = \frac{x^{-1}}{-1} = -1/x$.

   Evaluate: $[-\frac{1}{x}]_1^2 = (-\frac{1}{2}) - (-\frac{1}{1}) = -\frac{1}{2} + 1 = \frac{1}{2}$.

   Final Answer: Area is 1/2.

6. Find the area bounded by $y = \dfrac{2}{\sqrt{x}}$, x-axis, $x = 1$, $x = 4$.

   Solution:

   Bounds $[1, 4]$. $y = 2x^{-1/2} > 0$ on $[1, 4]$.

   Area $= \int_1^4 2x^{-1/2}\mathrm{d}x$.

   Antiderivative: $F(x) = 2\frac{x^{1/2}}{1/2} = 4x^{1/2} = 4\sqrt{x}$.

   Evaluate: $[4\sqrt{x}]_1^4 = 4\sqrt{4} - 4\sqrt{1} = 4(2) - 4(1) = 8 - 4 = 4$.

   Final Answer: Area is 4.

7. Find the area bounded by $y = 3e^{-x/2}$, x-axis, $x = 1$, $x = 4$.

   Solution:

   Bounds $[1, 4]$. $y > 0$ on $[1, 4]$.

   Area $= \int_1^4 3e^{-x/2}\mathrm{d}x$.

   Antiderivative: $F(x) = 3\frac{e^{-x/2}}{-1/2} = -6e^{-x/2}$.

   Evaluate: $[-6e^{-x/2}]_1^4 = (-6e^{-4/2}) - (-6e^{-1/2}) = -6e^{-2} + 6e^{-1/2}$.

   Final Answer: $6e^{-1/2} - 6e^{-2} \approx 2.827$.

8. Find the area bounded by $y = 2e^{x/2}$, x-axis, $x = 1$, $x = 3$.

   Solution:

   Bounds $[1, 3]$. $y > 0$ on $[1, 3]$.

Area $= \int_1^3 2e^{x/2}\mathrm{d}x$.

Antiderivative: $F(x) = 2\frac{e^{x/2}}{1/2} = 4e^{x/2}$.

Evaluate: $[4e^{x/2}]_1^3 = 4e^{3/2} - 4e^{1/2}$.

Final Answer: $4e^{3/2} - 4e^{1/2} \approx 11.332$.

9. Find the area bounded by $y = \dfrac{x^2+4}{x}$, x-axis, $x = 1$, $x = 4$. *Solution:* Bounds $[1, 4]$. $y = x + 4/x > 0$ on $[1, 4]$.
   Area $= \int_1^4 (x + \frac{4}{x})\mathrm{d}x$.
   Antiderivative: $F(x) = \frac{x^2}{2} + 4\ln|x|$.
   Evaluate: $[\frac{x^2}{2} + 4\ln|x|]_1^4 = (\frac{16}{2} + 4\ln 4) - (\frac{1}{2} + 4\ln 1) = (8 + 4\ln 4) - (\frac{1}{2} + 0) = \frac{15}{2} + 4\ln 4$.
   Final Answer: $\frac{15}{2} + 4\ln 4 \approx 13.045$.

10. Find the area bounded by $y = \dfrac{x-2}{x}$, x-axis, $x = 2$, $x = 4$.
    Solution:
    Bounds $[2, 4]$. $y = 1 - 2/x$. For $x \geq 2$, $0 < 2/x \leq 1$, so $y \geq 0$.
    Area $= \int_2^4 (1 - \frac{2}{x})\mathrm{d}x$.
    Antiderivative: $F(x) = x - 2\ln|x|$.
    Evaluate: $[x - 2\ln|x|]_2^4 = (4 - 2\ln 4) - (2 - 2\ln 2) = 2 - 2\ln 4 + 2\ln 2$.
    $= 2 - 2\ln(2^2) + 2\ln 2 = 2 - 4\ln 2 + 2\ln 2 = 2 - 2\ln 2$.
    Final Answer: $2 - 2\ln 2 \approx 0.614$.

3.2 Solutions: Definite Integral Evaluation

11. Evaluate $\displaystyle\int_0^1 2x \, \mathrm{d}x$.
    Solution: Antiderivative $F(x) = x^2$.
    Result: $[x^2]_0^1 = 1^2 - 0^2 = 1$.

12. Evaluate $\displaystyle\int_2^7 3v \, \mathrm{d}v$.
    Solution: Antiderivative $F(v) = \frac{3}{2}v^2$.
    Result: $[\frac{3}{2}v^2]_2^7 = \frac{3}{2}(49) - \frac{3}{2}(4) = \frac{3}{2}(45) = \frac{135}{2}$.

13. Evaluate $\displaystyle\int_{-1}^1 (x - 2) \, \mathrm{d}x$.
    Solution: Antiderivative $F(x) = \frac{x^2}{2} - 2x$.
    Result: $[\frac{x^2}{2} - 2x]_{-1}^1 = (\frac{1}{2} - 2) - (\frac{1}{2} + 2) = -4$.

14. Evaluate $\displaystyle\int_2^5 (-3x+4)\ \mathrm{d}x$.

    Solution: Antiderivative $F(x) = -\frac{3}{2}x^2 + 4x$.

    Result: $[-\frac{3}{2}x^2 + 4x]_2^5 = (-\frac{75}{2} + 20) - (-\frac{12}{2} + 8) = -\frac{35}{2} - 2 = -\frac{39}{2}$.

15. Evaluate $\displaystyle\int_{-1}^1 (2t-1)^2\ \mathrm{d}t$.

    Solution: Expand: $4t^2 - 4t + 1$.

    Antiderivative $F(t) = \frac{4}{3}t^3 - 2t^2 + t$.

    Result: $[\frac{4}{3}t^3 - 2t^2 + t]_{-1}^1 = (\frac{4}{3} - 2 + 1) - (-\frac{4}{3} - 2 - 1) = \frac{1}{3} - (-\frac{13}{3}) = \frac{14}{3}$.

16. Evaluate $\displaystyle\int_0^2 (1-2x)^2\ \mathrm{d}x$. *Solution:* Expand: $1 - 4x + 4x^2$.

    Antiderivative $F(x) = x - 2x^2 + \frac{4}{3}x^3$.

    Result: $[x - 2x^2 + \frac{4}{3}x^3]_0^2 = (2 - 8 + \frac{32}{3}) - 0 = -6 + \frac{32}{3} = \frac{14}{3}$.

17. Evaluate $\displaystyle\int_0^3 (x-2)^3\ \mathrm{d}x$.

    Solution (U-Sub): $u = x-2, du = dx$.

    Limits $x = 0 \to u = -2$, $x = 3 \to u = 1$. $\int_{-2}^1 u^3 du = [\frac{u^4}{4}]_{-2}^1 = \frac{1^4}{4} - \frac{(-2)^4}{4} = \frac{1}{4} - \frac{16}{4} = -\frac{15}{4}$.

18. Evaluate $\displaystyle\int_2^5 (x-3)^3\ \mathrm{d}x$.

    Solution (U-Sub): $u = x-3, du = dx$.

    Limits $x = 2 \to u = -1$, $x = 5 \to u = 2$. $\int_{-1}^2 u^3 du = [\frac{u^4}{4}]_{-1}^2 = \frac{2^4}{4} - \frac{(-1)^4}{4} = \frac{16}{4} - \frac{1}{4} = \frac{15}{4}$.

19. Evaluate $\displaystyle\int_{-1}^1 (\sqrt[3]{t} - 2)\ \mathrm{d}t$.

    Solution: Rewrite $\int_{-1}^1 (t^{1/3} - 2)dt$.

    Antiderivative $F(t) = \frac{t^{4/3}}{4/3} - 2t = \frac{3}{4}t^{4/3} - 2t$.

    Result: $[\frac{3}{4}t^{4/3} - 2t]_{-1}^1 = (\frac{3}{4} - 2) - (\frac{3}{4} + 2) = -\frac{5}{4} - \frac{11}{4} = -4$.

20. Evaluate $\displaystyle\int_1^4 \frac{u-2}{\sqrt{u}}\ \mathrm{d}u$.

    Solution: Simplify $\int_1^4 (u^{1/2} - 2u^{-1/2})du$.

    Antiderivative $F(u) = \frac{u^{3/2}}{3/2} - 2\frac{u^{1/2}}{1/2} = \frac{2}{3}u^{3/2} - 4u^{1/2}$.

    Result: $[\frac{2}{3}u^{3/2} - 4u^{1/2}]_1^4 = (\frac{2}{3}(8) - 4(2)) - (\frac{2}{3} - 4) = (\frac{16}{3} - 8) - (-\frac{10}{3}) = -\frac{8}{3} + \frac{10}{3} = \frac{2}{3}$.

21. Evaluate $\displaystyle\int_{-1}^{0}(t^{1/3}-t^{2/3})\ \mathrm{d}t$.

Solution: Antiderivative $F(t)=\frac{t^{4/3}}{4/3}-\frac{t^{5/3}}{5/3}=\frac{3}{4}t^{4/3}-\frac{3}{5}t^{5/3}$.

Result: $[\frac{3}{4}t^{4/3}-\frac{3}{5}t^{5/3}]_{-1}^{0}=(0)-(\frac{3}{4}(1)-\frac{3}{5}(-1))=-(\frac{3}{4}+\frac{3}{5})=-\frac{27}{20}$.

22. Evaluate $\displaystyle\int_{0}^{4}(x^{1/2}+x^{1/4})\ \mathrm{d}x$.

Solution: Antiderivative $F(x)=\frac{x^{3/2}}{3/2}+\frac{x^{5/4}}{5/4}=\frac{2}{3}x^{3/2}+\frac{4}{5}x^{5/4}$.

Result: $[\frac{2}{3}x^{3/2}+\frac{4}{5}x^{5/4}]_0^4=(\frac{2}{3}(4^{3/2})+\frac{4}{5}(4^{5/4}))-0=\frac{2}{3}(8)+\frac{4}{5}(4\cdot 4^{1/4})=\frac{16}{3}+\frac{16\cdot 2^{1/2}}{5}=\frac{16}{3}+\frac{16\sqrt{2}}{5}$.

23. Evaluate $\displaystyle\int_{0}^{1}e^{-2x}\ \mathrm{d}x$.

Solution: Antiderivative $F(x)=-\frac{1}{2}e^{-2x}$.

Result: $[-\frac{1}{2}e^{-2x}]_0^1=(-\frac{1}{2}e^{-2})-(-\frac{1}{2}e^{0})=-\frac{1}{2}e^{-2}+\frac{1}{2}=\frac{1}{2}(1-e^{-2})$.

24. Evaluate $\displaystyle\int_{1}^{2}e^{1-x}\ \mathrm{d}x$.

Solution: Antiderivative $F(x)=-e^{1-x}$.

Result: $[-e^{1-x}]_1^2=(-e^{-1})-(-e^{0})=-e^{-1}+1=1-1/e$.

25. Evaluate $\displaystyle\int_{1}^{e}\frac{e^{3/x}}{x^2}\ \mathrm{d}x$.

Solution (U-Sub): $u=3/x, du=-3/x^2dx \implies \frac{1}{x^2}dx=-\frac{1}{3}du$.

Limits: $x=1\to u=3$, $x=e\to u=3/e$. $\int_3^{3/e}e^u(-\frac{1}{3}du)=-\frac{1}{3}[e^u]_3^{3/e}=-\frac{1}{3}(e^{3/e}-e^3)=\frac{1}{3}(e^3-e^{3/e})$.

26. Evaluate $\displaystyle\int_{-1}^{1}(e^{x}-e^{-x})\ \mathrm{d}x$.

Solution: Integrand $f(x)=e^x-e^{-x}$ is odd: $f(-x)=e^{-x}-e^{x}=-f(x)$.

Interval is symmetric. Integral $=0$.

27. Evaluate $\displaystyle\int_{0}^{1}\frac{e^{-x}}{\sqrt{e^{-x}+1}}\ \mathrm{d}x$.

Solution (U-Sub): $u=e^{-x}+1, du=-e^{-x}dx \implies e^{-x}dx=-du$.

Limits: $x=0\to u=2$, $x=1\to u=e^{-1}+1$.

$\int_2^{1+1/e}\frac{-du}{\sqrt{u}}=-\int_2^{1+1/e}u^{-1/2}du=-[2u^{1/2}]_2^{1+1/e}=-2\sqrt{1+1/e}-(-2\sqrt{2})=2\sqrt{2}-2\sqrt{1+1/e}$.

28. Evaluate $\displaystyle\int_0^2 \frac{e^{2x}}{e^{2x}+1}\,\mathrm{d}x$.

    Solution (U-Sub): $u = e^{2x} + 1, du = 2e^{2x}dx \implies e^{2x}dx = \frac{1}{2}du$.

    Limits: $x = 0 \to u = 2$, $x = 2 \to u = e^4 + 1$. $\int_2^{e^4+1} \frac{1}{u}(\frac{1}{2}du) = \frac{1}{2}[\ln|u|]_2^{e^4+1} = \frac{1}{2}(\ln(e^4 + 1) - \ln 2) = \frac{1}{2}\ln(\frac{e^4+1}{2})$.

29. Evaluate $\displaystyle\int_1^2 \frac{x}{1+4x^2}\,\mathrm{d}x$.

    Solution (U-Sub): $u = 1 + 4x^2, du = 8xdx \implies xdx = \frac{1}{8}du$.

    Limits: $x = 1 \to u = 5$, $x = 2 \to u = 17$. $\int_5^{17} \frac{1}{u}(\frac{1}{8}du) = \frac{1}{8}[\ln|u|]_5^{17} = \frac{1}{8}(\ln 17 - \ln 5) = \frac{1}{8}\ln(\frac{17}{5})$.

30. Evaluate $\displaystyle\int_0^1 \frac{e^{2x}}{e^{2x}+1}\,\mathrm{d}x$.

    Solution (U-Sub): $u = e^{2x} + 1, du = 2e^{2x}dx \implies e^{2x}dx = \frac{1}{2}du$.

    Limits: $x = 0 \to u = 2$, $x = 1 \to u = e^2 + 1$. $\int_2^{e^2+1} \frac{1}{u}(\frac{1}{2}du) = \frac{1}{2}[\ln|u|]_2^{e^2+1} = \frac{1}{2}(\ln(e^2 + 1) - \ln 2) = \frac{1}{2}\ln(\frac{e^2+1}{2})$.