DIFFERENTIATION RULES

NUTM Nexus Writing Team

February 17, 2025

1 Introduction: Differentiation Rules

Explanation. This document provides solutions using fundamental differentiation rules. These rules allow us to find derivatives without using the limit definition directly, which is often tedious. The key rules are:

1. **Constant Rule:** The derivative of any constant function $f(x) = c$ is zero.

$$\frac{d}{dx}(c) = 0$$

2. **Power Rule:** For any real number n, the derivative of $f(x) = x^n$ is nx^{n-1}.

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

(To apply this: bring the original exponent n down as a multiplier, then subtract 1 from the original exponent to get the new exponent). This works for integers, fractions, negative numbers, and irrational numbers as exponents. Remember to rewrite roots and fractions as powers first (e.g., $\sqrt{x} = x^{1/2}$, $1/x^3 = x^{-3}$).

3. **Constant Multiple Rule:** The derivative of a constant c times a function $u(x)$ is the constant times the derivative of the function.

$$\frac{d}{dx}(c \cdot u(x)) = c \cdot \frac{d}{dx}(u(x)) = c \cdot \frac{du}{dx}$$

(Constants "pass through" the differentiation process).

4. **Sum Rule:** The derivative of a sum of functions is the sum of their derivatives.

$$\frac{d}{dx}(u(x) + v(x)) = \frac{du}{dx} + \frac{dv}{dx}$$

5. **Difference Rule:** The derivative of a difference of functions is the difference of their derivatives.

$$\frac{d}{dx}(u(x) - v(x)) = \frac{du}{dx} - \frac{dv}{dx}$$

(This follows from the Sum and Constant Multiple rules, since $u - v = u + (-1)v$).

6. **Derivative of e^x:** The derivative of the natural exponential function is itself.

$$\frac{d}{dx}(e^x) = e^x$$

We combine these rules to differentiate polynomials and other functions term by term.

2 Solutions: Examples

1. *(Example 1)* Differentiate the following powers of x:
   (a) x^3
   (b) $x^{2/3}$
   (c) $x^{\sqrt{2}}$
   (d) $\frac{1}{x^4}$
   (e) $x^{-4/3}$
   (f) $\sqrt{x^{2+\pi}}$

   Solution: We apply the Power Rule $\frac{d}{dx}(x^n) = nx^{n-1}$ to each part.

   (a) $f(x) = x^3$

   Strategy. Apply Power Rule with $n = 3$.

$$\frac{d}{dx}(x^3) = 3 \cdot x^{3-1} = 3x^2$$

   (b) $f(x) = x^{2/3}$

   Strategy. Apply Power Rule with $n = 2/3$.

$$n - 1 = \frac{2}{3} - 1 = \frac{2}{3} - \frac{3}{3} = -\frac{1}{3}$$
$$\frac{d}{dx}(x^{2/3}) = \frac{2}{3}x^{2/3-1} = \frac{2}{3}x^{-1/3}$$

   (Optional rewrite: $\dfrac{2}{3x^{1/3}}$ *or* $d\frac{2}{3\sqrt[3]{x}}$*)*

   (c) $f(x) = x^{\sqrt{2}}$

   Strategy. Apply Power Rule with $n = \sqrt{2}$.

$$\frac{d}{dx}(x^{\sqrt{2}}) = \sqrt{2} \cdot x^{\sqrt{2}-1}$$

   (This cannot be simplified further algebraically)

   (d) $f(x) = \dfrac{1}{x^4}$

   Strategy. First rewrite using a negative exponent, then apply Power Rule.

   Rewrite: $f(x) = x^{-4}$. Apply Power Rule with $n = -4$.

$$n - 1 = -4 - 1 = -5$$
$$\frac{d}{dx}(x^{-4}) = (-4)x^{-4-1} = -4x^{-5}$$

   (Optional rewrite: $-d\frac{4}{x^5}$*)*

   (e) $f(x) = x^{-4/3}$

Strategy. Apply Power Rule with $n = -4/3$.

$$n - 1 = -\frac{4}{3} - 1 = -\frac{4}{3} - \frac{3}{3} = -\frac{7}{3}$$
$$\frac{d}{dx}(x^{-4/3}) = -\frac{4}{3}x^{-4/3-1} = -\frac{4}{3}x^{-7/3}$$

(Optional rewrite: $-\dfrac{4}{3x^{7/3}}$*)*

(f) $f(x) = \sqrt{x^{2+\pi}}$

Strategy. Rewrite using fractional exponents, then apply Power Rule.

Rewrite: $f(x) = (x^{2+\pi})^{1/2} = x^{(2+\pi)/2}$. *(Using* $(a^m)^n = a^{mn}$*)* Let $n = \frac{2+\pi}{2}$. Apply Power Rule: $\frac{d}{dx}(x^n) = nx^{n-1}$.

$$n - 1 = \frac{2+\pi}{2} - 1 = \frac{2+\pi}{2} - \frac{2}{2} = \frac{\pi}{2}$$
$$\frac{d}{dx}(x^{(2+\pi)/2}) = \left(\frac{2+\pi}{2}\right)x^{\pi/2}$$

(Note: The solution on the slide appears to interpret $\sqrt{x^{2+\pi}}$ *as* $\sqrt{x^2 \cdot x^\pi}$ *or assumes a different original problem like* $x^{\sqrt{2+\pi}}$*. Based on standard notation,* $x^{(2+\pi)/2}$ *is the correct interpretation of the question as written on slide 6. The slide 7 solution corresponds to* $\frac{d}{dx}x^{1+(\pi/2)}$*, which is different.)*

2. *(Example 2)* Compute $\dfrac{d}{dx}(3x^2)$.

   Solution:

   Strategy. This involves a constant (3) multiplying a function (x^2). Use the Constant Multiple Rule followed by the Power Rule.

   Step 1: Apply Constant Multiple Rule: $\frac{d}{dx}(c \cdot u) = c \cdot \frac{du}{dx}$.

   $$\frac{d}{dx}(3x^2) = 3 \cdot \frac{d}{dx}(x^2)$$

   Step 2: Apply Power Rule to $\frac{d}{dx}(x^2)$ with $n = 2$.

   $$\frac{d}{dx}(x^2) = 2x^{2-1} = 2x^1 = 2x$$

   Step 3: Combine results.

   $$\frac{d}{dx}(3x^2) = 3 \cdot (2x) = 6x$$

3. *(Example 3)* Find the derivative of the polynomial $y = x^3 + \dfrac{4}{3}x^2 - 5x + 1$.

   Solution:

Strategy. This is a polynomial, which is a sum and difference of terms involving constants and powers of x. We use the Sum/Difference Rule, Constant Multiple Rule, and Power Rule term by term. Let $\frac{dy}{dx}$ denote the derivative.

Step 1: Apply Sum/Difference Rule. Differentiate each term separately.

$$\frac{dy}{dx} = \frac{d}{dx}(x^3) + \frac{d}{dx}(\frac{4}{3}x^2) - \frac{d}{dx}(5x) + \frac{d}{dx}(1)$$

Step 2: Apply Constant Multiple Rule and Power Rule to each term.

- $\frac{d}{dx}(x^3)$: Power Rule with $n = 3$. Result is $3x^{3-1} = 3x^2$.
- $\frac{d}{dx}(\frac{4}{3}x^2)$: Constant Multiple Rule first: $\frac{4}{3} \cdot \frac{d}{dx}(x^2)$. Then Power Rule with $n = 2$: $\frac{d}{dx}(x^2) = 2x^{2-1} = 2x$. Combine: $\frac{4}{3} \cdot (2x) = \frac{8}{3}x$.
- $\frac{d}{dx}(5x)$: Rewrite as $\frac{d}{dx}(5x^1)$. Constant Multiple: $5 \cdot \frac{d}{dx}(x^1)$. Power Rule with $n = 1$: $\frac{d}{dx}(x^1) = 1x^{1-1} = 1x^0 = 1$. Combine: $5 \cdot 1 = 5$.
- $\frac{d}{dx}(1)$: Constant Rule. Derivative of a constant is 0.

Step 3: Combine the results.

$$\frac{dy}{dx} = 3x^2 + \frac{8}{3}x - 5 + 0$$

$$\frac{dy}{dx} = 3x^2 + \frac{8}{3}x - 5$$

3 Solutions: ClassWork Problems

4. Find derivative of $y = -x^2 + 3$. Assume y depends on x.

   Solution:

   Strategy. Differentiate term by term using Sum/Difference, Constant Multiple, Power, and Constant rules.

$$\frac{dy}{dx} = \frac{d}{dx}(-x^2) + \frac{d}{dx}(3)$$

$$= (-1) \cdot \frac{d}{dx}(x^2) + 0 \quad \textit{(Constant Multiple, Constant Rule)}$$

$$= (-1) \cdot (2x^{2-1}) \quad \textit{(Power Rule, n=2)}$$

$$= -2x$$

5. Find derivative of $y = x^2 + x + 8$. Assume y depends on x.

   Solution:

   Strategy. Differentiate term by term.

$$\frac{dy}{dx} = \frac{d}{dx}(x^2) + \frac{d}{dx}(x^1) + \frac{d}{dx}(8)$$

$$= (2x^{2-1}) + (1x^{1-1}) + 0 \quad \textit{(Power Rule, Constant Rule)}$$

$$= 2x^1 + 1x^0 + 0 = 2x + 1(1) = 2x + 1$$

6. Find derivative of $s = 5t^3 - 3t^5$. Assume s depends on t.

   Solution:

   Strategy. Differentiate with respect to t term by term.

$$\begin{aligned}\frac{ds}{dt} &= \frac{d}{dt}(5t^3) - \frac{d}{dt}(3t^5)\\ &= 5 \cdot \frac{d}{dt}(t^3) - 3 \cdot \frac{d}{dt}(t^5) \quad \textit{(Constant Multiple Rule)}\\ &= 5(3t^{3-1}) - 3(5t^{5-1}) \quad \textit{(Power Rule)}\\ &= 15t^2 - 15t^4\end{aligned}$$

7. Find derivative of $w = 3z^7 - 7z^3 + 21z^2$. Assume w depends on z.

   Solution:

   Strategy. Differentiate polynomial term by term with respect to z.

$$\begin{aligned}\frac{dw}{dz} &= \frac{d}{dz}(3z^7) - \frac{d}{dz}(7z^3) + \frac{d}{dz}(21z^2)\\ &= 3(7z^{7-1}) - 7(3z^{3-1}) + 21(2z^{2-1}) \quad \textit{(Const. Multiple, Power Rule)}\\ &= 21z^6 - 21z^2 + 42z\end{aligned}$$

8. Find derivative of $y = \dfrac{4x^3}{3} - x + 2e^x$. Assume y depends on x.

   Solution:

   Strategy. Differentiate term by term. Note the e^x term.

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(\frac{4}{3}x^3) - \frac{d}{dx}(x^1) + \frac{d}{dx}(2e^x)\\ &= \frac{4}{3} \cdot \frac{d}{dx}(x^3) - 1 \cdot x^{1-1} + 2 \cdot \frac{d}{dx}(e^x) \quad \textit{(Const. Multiple, Power Rule)}\\ &= \frac{4}{3}(3x^{3-1}) - 1x^0 + 2(e^x) \quad \textit{(Power Rule, Deriv of } e^x\textit{)}\\ &= \frac{4}{3}(3x^2) - 1 + 2e^x\\ &= 4x^2 - 1 + 2e^x\end{aligned}$$

9. Find derivative of $y = \dfrac{x^3}{3} + \dfrac{x^2}{2} + \dfrac{x}{4}$. Assume y depends on x.

   Solution:

   Strategy. Rewrite terms as constant multiples, then differentiate term by term.

Rewrite: $y = \frac{1}{3}x^3 + \frac{1}{2}x^2 + \frac{1}{4}x^1$.

$$\frac{dy}{dx} = \frac{d}{dx}(\frac{1}{3}x^3) + \frac{d}{dx}(\frac{1}{2}x^2) + \frac{d}{dx}(\frac{1}{4}x^1)$$

$$= \frac{1}{3}(3x^{3-1}) + \frac{1}{2}(2x^{2-1}) + \frac{1}{4}(1x^{1-1}) \quad \textit{(Const. Multiple, Power Rule)}$$

$$= \frac{1}{3}(3x^2) + \frac{1}{2}(2x) + \frac{1}{4}(1x^0)$$

$$= x^2 + x + \frac{1}{4}$$

10. Find derivative of $w = 3z^{-2} - \frac{1}{z}$. Assume w depends on z.

Solution:

Strategy. Rewrite terms with negative exponents, then differentiate term by term.

Rewrite: $w = 3z^{-2} - 1z^{-1}$.

$$\frac{dw}{dz} = \frac{d}{dz}(3z^{-2}) - \frac{d}{dz}(1z^{-1})$$

$$= 3 \cdot \frac{d}{dz}(z^{-2}) - 1 \cdot \frac{d}{dz}(z^{-1})$$

Apply Power Rule: $n = -2 \implies n - 1 = -3$; $n = -1 \implies n - 1 = -2$.

$$= 3(-2z^{-2-1}) - 1(-1z^{-1-1})$$

$$= 3(-2z^{-3}) - (-1z^{-2})$$

$$= -6z^{-3} + z^{-2}$$

Optional rewrite: $= -\frac{6}{z^3} + \frac{1}{z^2}$

11. Find derivative of $s = -2t^{-1} + \frac{4}{t^2}$. Assume s depends on t.

Solution:

Strategy. Rewrite terms with negative exponents, then differentiate term by term.

Rewrite: $s = -2t^{-1} + 4t^{-2}$.

$$\frac{ds}{dt} = \frac{d}{dt}(-2t^{-1}) + \frac{d}{dt}(4t^{-2})$$

$$= -2 \cdot \frac{d}{dt}(t^{-1}) + 4 \cdot \frac{d}{dt}(t^{-2})$$

Apply Power Rule: $n = -1 \implies n - 1 = -2$; $n = -2 \implies n - 1 = -3$.

$$= -2(-1t^{-1-1}) + 4(-2t^{-2-1})$$

$$= -2(-1t^{-2}) + 4(-2t^{-3})$$

$$= 2t^{-2} - 8t^{-3}$$

Optional rewrite: $= \frac{2}{t^2} - \frac{8}{t^3}$

12. Find derivative of $y = 6x^2 - 10x - 5x^{-2}$. Assume y depends on x.

    Solution:

    Strategy. Differentiate polynomial and power terms individually.

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(6x^2) - \frac{d}{dx}(10x^1) - \frac{d}{dx}(5x^{-2})\\ &= 6(2x^{2-1}) - 10(1x^{1-1}) - 5(-2x^{-2-1}) \quad \textit{(Const. Multiple, Power Rule)}\\ &= 12x^1 - 10(1x^0) - 5(-2x^{-3})\\ &= 12x - 10(1) + 10x^{-3}\\ &= 12x - 10 + 10x^{-3}\end{aligned}$$

    Optional rewrite: $= 12x - 10 + \dfrac{10}{x^3}$

13. Find derivative of $y = 4 - 2x - x^{-3}$. Assume y depends on x.

    Solution:

    Strategy. Differentiate term by term.

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(4) - \frac{d}{dx}(2x^1) - \frac{d}{dx}(x^{-3})\\ &= 0 - 2(1x^{1-1}) - (-3x^{-3-1})\\ &= 0 - 2(1x^0) - (-3x^{-4})\\ &= -2(1) + 3x^{-4}\\ &= -2 + 3x^{-4}\end{aligned}$$

    Optional rewrite: $= -2 + \dfrac{3}{x^4}$

14. Find derivative of $r = \dfrac{1}{3s^2} - \dfrac{5}{2s}$. Assume r depends on s.

    Solution:

    Strategy. Rewrite with negative exponents, then differentiate term by term with respect to s.

    Rewrite: $r = \frac{1}{3}s^{-2} - \frac{5}{2}s^{-1}$.

$$\begin{aligned}\frac{dr}{ds} &= \frac{d}{ds}(\frac{1}{3}s^{-2}) - \frac{d}{ds}(\frac{5}{2}s^{-1})\\ &= \frac{1}{3}\cdot\frac{d}{ds}(s^{-2}) - \frac{5}{2}\cdot\frac{d}{ds}(s^{-1})\end{aligned}$$

    Apply Power Rule: $n = -2 \implies n - 1 = -3$; $n = -1 \implies n - 1 = -2$.

$$\begin{aligned}&= \frac{1}{3}(-2s^{-2-1}) - \frac{5}{2}(-1s^{-1-1})\\ &= \frac{1}{3}(-2s^{-3}) - \frac{5}{2}(-1s^{-2})\\ &= -\frac{2}{3}s^{-3} + \frac{5}{2}s^{-2}\end{aligned}$$

    Optional rewrite: $= -\dfrac{2}{3s^3} + \dfrac{5}{2s^2}$

15. Find derivative of $r = \dfrac{12}{\theta} - \dfrac{4}{\theta^3} + \dfrac{1}{\theta^4}$. Assume r depends on θ.

Solution:

Strategy. Rewrite with negative exponents, then differentiate term by term with respect to θ.

Rewrite: $r = 12\theta^{-1} - 4\theta^{-3} + 1\theta^{-4}$.

$$\frac{dr}{d\theta} = \frac{d}{d\theta}(12\theta^{-1}) - \frac{d}{d\theta}(4\theta^{-3}) + \frac{d}{d\theta}(1\theta^{-4})$$
$$= 12(-1\theta^{-1-1}) - 4(-3\theta^{-3-1}) + 1(-4\theta^{-4-1}) \quad \textit{(Const. Multiple, Power Rule)}$$
$$= -12\theta^{-2} - (-12\theta^{-4}) + (-4\theta^{-5})$$
$$= -12\theta^{-2} + 12\theta^{-4} - 4\theta^{-5}$$

Optional rewrite: $= -\dfrac{12}{\theta^2} + \dfrac{12}{\theta^4} - \dfrac{4}{\theta^5}$

16. Find derivative of $y = \dfrac{x^4}{2} - \dfrac{3}{2}x^2 - x$. Assume y depends on x.

Solution:

Strategy. Rewrite as constant multiples, differentiate term by term.

Rewrite: $y = \frac{1}{2}x^4 - \frac{3}{2}x^2 - 1x^1$.

$$\frac{dy}{dx} = \frac{d}{dx}(\frac{1}{2}x^4) - \frac{d}{dx}(\frac{3}{2}x^2) - \frac{d}{dx}(1x^1)$$
$$= \frac{1}{2}(4x^{4-1}) - \frac{3}{2}(2x^{2-1}) - 1(1x^{1-1})$$
$$= \frac{1}{2}(4x^3) - \frac{3}{2}(2x) - 1(1x^0)$$
$$= 2x^3 - 3x - 1(1)$$
$$= 2x^3 - 3x - 1$$

17. Find derivative of $y = \dfrac{x^5}{120}$. Assume y depends on x.

Solution:

Strategy. Rewrite as constant multiple, use power rule.

Rewrite: $y = \frac{1}{120}x^5$.

$$\frac{dy}{dx} = \frac{d}{dx}(\frac{1}{120}x^5)$$
$$= \frac{1}{120} \cdot \frac{d}{dx}(x^5) \quad \textit{(Constant Multiple Rule)}$$
$$= \frac{1}{120}(5x^{5-1}) \quad \textit{(Power Rule, n=5)}$$
$$= \frac{1}{120}(5x^4) = \frac{5}{120}x^4$$

Simplify fraction: $\frac{5}{120} = \frac{1}{24}$.

$$= \frac{1}{24}x^4$$

4 Solution: Application

17. The reaction R to a dose M is $R = M^2\left(\frac{C}{2} - \frac{M}{3}\right)$, where C is constant. Find the sensitivity $\frac{dR}{dM}$.

Solution:

Strategy. First, expand the expression for R to get a polynomial in M. Then, differentiate the resulting polynomial term by term with respect to M, treating C as a constant.

Step 1: *Expand R(M).*

$$R = M^2 \cdot \frac{C}{2} - M^2 \cdot \frac{M}{3}$$
$$R = \frac{C}{2}M^2 - \frac{1}{3}M^3$$

Step 2: *Differentiate R with respect to M*, term by term.

$$\frac{dR}{dM} = \frac{d}{dM}\left(\frac{C}{2}M^2\right) - \frac{d}{dM}\left(\frac{1}{3}M^3\right)$$

Step 3: *Apply Constant Multiple and Power Rules.* For the first term, $\frac{C}{2}$ is a constant. Apply power rule to M^2 $(n = 2)$:

$$\frac{d}{dM}\left(\frac{C}{2}M^2\right) = \frac{C}{2} \cdot \frac{d}{dM}(M^2) = \frac{C}{2} \cdot (2M^{2-1}) = \frac{C}{2} \cdot (2M) = CM$$

For the second term, $\frac{1}{3}$ is a constant. Apply power rule to M^3 $(n = 3)$:

$$\frac{d}{dM}\left(\frac{1}{3}M^3\right) = \frac{1}{3} \cdot \frac{d}{dM}(M^3) = \frac{1}{3} \cdot (3M^{3-1}) = \frac{1}{3} \cdot (3M^2) = M^2$$

Step 4: *Combine the results.*

$$\frac{dR}{dM} = CM - M^2$$

Step 5: *Optional factoring.*

$$\frac{dR}{dM} = M(C - M)$$