

APPLICATIONS OF INTEGRATION SOLUTIONS

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1 Introduction: Definite Integrals and Area

Explanation. This section focuses on evaluating definite integrals and using them to calculate the area of regions bounded by curves.

The Definite Integral: The definite integral of a function $f(x)$ from $x = a$ to $x = b$, denoted $\int_a^b f(x)dx$, represents the *net signed area* between the curve $y = f(x)$ and the x-axis over the interval $[a, b]$. "Net signed area" means area above the x-axis counts positively, and area below the x-axis counts negatively.

The Fundamental Theorem of Calculus (FTC), Part 2 (Evaluation Theorem): This theorem provides the primary method for evaluating definite integrals. If $f(x)$ is continuous on $[a, b]$ and $F(x)$ is any antiderivative of $f(x)$ (meaning $F'(x) = f(x)$), then:

$$\int_a^b f(x)dx = F(b) - F(a)$$

The process is:

1. Find an antiderivative $F(x)$ of the integrand $f(x)$.
2. Evaluate $F(x)$ at the upper limit b and the lower limit a .
3. Subtract the value at the lower limit from the value at the upper limit.

Notation: $[F(x)]_a^b = F(b) - F(a)$.

Area Application: If a function $f(x)$ is *non-negative* ($f(x) \geq 0$) and continuous on $[a, b]$, then the definite integral $\int_a^b f(x)dx$ directly gives the geometric area of the region under the curve $y = f(x)$, above the x-axis, between $x = a$ and $x = b$.

$$\text{Area} = \int_a^b f(x)dx \quad (\text{if } f(x) \geq 0 \text{ on } [a, b])$$

If $f(x)$ is negative on the interval, the integral will be negative, and the area is the absolute value of the integral, $\text{Area} = |\int_a^b f(x)dx| = -\int_a^b f(x)dx$. If the function crosses the x-axis within $[a, b]$, the area must be calculated by splitting the integral at the x-intercepts and taking the absolute value of the parts below the axis.

2 Solutions: Examples

1. Find the area of the region bounded by the x-axis and the graph of $f(x) = x^2 - 1$ for $1 \leq x \leq 2$.

Solution:

Strategy. Check if $f(x) \geq 0$ on $[1, 2]$. If so, $\text{Area} = \int_1^2 f(x)dx$. Evaluate using FTC.

For $x \in [1, 2]$, $x^2 \geq 1$, so $f(x) = x^2 - 1 \geq 0$.

Step 1: Set up integral.

$$\text{Area} = \int_1^2 (x^2 - 1) dx$$

Step 2: Antiderivative. $F(x) = \frac{x^3}{3} - x$.

Step 3: Evaluate. $\text{Area} = [\frac{x^3}{3} - x]_1^2 = (\frac{8}{3} - 2) - (\frac{1}{3} - 1) = \frac{2}{3} - (-\frac{2}{3}) = \frac{4}{3}$.

Final Answer: The area is $4/3$.

2. Evaluate the definite integral $\int_0^1 (4t + 1)^2 dt$.

Solution:

Method 1: Expand First

Step 1: Expand. $(4t + 1)^2 = 16t^2 + 8t + 1$.

Step 2: Antiderivative. $F(t) = \frac{16}{3}t^3 + 4t^2 + t$.

Step 3: Evaluate. $[\frac{16}{3}t^3 + 4t^2 + t]_0^1 = (\frac{16}{3} + 4 + 1) - (0) = \frac{16}{3} + 5 = \frac{31}{3}$.

Method 2: U-Substitution

Let $u = 4t + 1 \implies du = 4dt \implies dt = \frac{1}{4}du$.

Limits: $t = 0 \implies u = 1$; $t = 1 \implies u = 5$.

Integral becomes $\int_1^5 u^2(\frac{1}{4}du) = \frac{1}{4} \int_1^5 u^2 du$.

Evaluate: $\frac{1}{4}[\frac{u^3}{3}]_1^5 = \frac{1}{4}(\frac{5^3}{3} - \frac{1^3}{3}) = \frac{1}{4}(\frac{125-1}{3}) = \frac{1}{4}(\frac{124}{3}) = \frac{31}{3}$.

Results match.

3 Solutions: ClassWork Problems

3.1 Area Problems

3. Find the area bounded by $y = x - x^2$ and x-axis.

Solution:

Find bounds: $x(1 - x) = 0 \implies x = 0, x = 1$. Bounds are $[0, 1]$.

Check sign: $y(0.5) = 0.5 - 0.25 = 0.25 > 0$. $y \geq 0$ on $[0, 1]$.

$\text{Area} = \int_0^1 (x - x^2) dx$.

Antiderivative: $F(x) = \frac{x^2}{2} - \frac{x^3}{3}$.

Evaluate: $[\frac{x^2}{2} - \frac{x^3}{3}]_0^1 = (\frac{1}{2} - \frac{1}{3}) - 0 = \frac{1}{6}$.

Final Answer: Area is $1/6$.

4. Find the area bounded by $y = 1 - x^4$ and x-axis.

Solution:

Find bounds: $1 - x^4 = 0 \implies x^4 = 1 \implies x = \pm 1$. Bounds are $[-1, 1]$.

Check sign: $y(0) = 1 > 0$. $y \geq 0$ on $[-1, 1]$.

Area = $\int_{-1}^1 (1 - x^4) dx$.

Antiderivative: $F(x) = x - \frac{x^5}{5}$.

Evaluate: $[x - \frac{x^5}{5}]_{-1}^1 = (1 - \frac{1}{5}) - (-1 - (-\frac{1}{5})) = \frac{4}{5} - (-\frac{4}{5}) = \frac{8}{5}$.

(Or use symmetry: $2 \int_0^1 (1 - x^4) dx = 2[x - x^5/5]_0^1 = 2(1 - 1/5) = 8/5$.)

Final Answer: Area is $8/5$.

5. Find the area bounded by $y = \frac{1}{x^2}$, x-axis, $x = 1$, $x = 2$.

Solution:

Bounds $[1, 2]$. $y = x^{-2} > 0$ on $[1, 2]$.

Area = $\int_1^2 x^{-2} dx$.

Antiderivative: $F(x) = \frac{x^{-1}}{-1} = -1/x$.

Evaluate: $[-\frac{1}{x}]_1^2 = (-\frac{1}{2}) - (-\frac{1}{1}) = -\frac{1}{2} + 1 = \frac{1}{2}$.

Final Answer: Area is $1/2$.

6. Find the area bounded by $y = \frac{2}{\sqrt{x}}$, x-axis, $x = 1$, $x = 4$.

Solution:

Bounds $[1, 4]$. $y = 2x^{-1/2} > 0$ on $[1, 4]$.

Area = $\int_1^4 2x^{-1/2} dx$.

Antiderivative: $F(x) = 2 \frac{x^{1/2}}{1/2} = 4x^{1/2} = 4\sqrt{x}$.

Evaluate: $[4\sqrt{x}]_1^4 = 4\sqrt{4} - 4\sqrt{1} = 4(2) - 4(1) = 8 - 4 = 4$.

Final Answer: Area is 4.

7. Find the area bounded by $y = 3e^{-x/2}$, x-axis, $x = 1$, $x = 4$.

Solution:

Bounds $[1, 4]$. $y > 0$ on $[1, 4]$.

Area = $\int_1^4 3e^{-x/2} dx$.

Antiderivative: $F(x) = 3 \frac{e^{-x/2}}{-1/2} = -6e^{-x/2}$.

Evaluate: $[-6e^{-x/2}]_1^4 = (-6e^{-4/2}) - (-6e^{-1/2}) = -6e^{-2} + 6e^{-1/2}$.

Final Answer: $6e^{-1/2} - 6e^{-2} \approx 2.827$.

8. Find the area bounded by $y = 2e^{x/2}$, x-axis, $x = 1$, $x = 3$.

Solution:

Bounds $[1, 3]$. $y > 0$ on $[1, 3]$.

$$\text{Area} = \int_1^3 2e^{x/2} dx.$$

$$\text{Antiderivative: } F(x) = 2 \frac{e^{x/2}}{1/2} = 4e^{x/2}.$$

$$\text{Evaluate: } [4e^{x/2}]_1^3 = 4e^{3/2} - 4e^{1/2}.$$

$$\text{Final Answer: } 4e^{3/2} - 4e^{1/2} \approx 11.332.$$

9. Find the area bounded by $y = \frac{x^2 + 4}{x}$, x-axis, $x = 1$, $x = 4$. *Solution:* Bounds $[1, 4]$.
 $y = x + 4/x > 0$ on $[1, 4]$.

$$\text{Area} = \int_1^4 (x + \frac{4}{x}) dx.$$

$$\text{Antiderivative: } F(x) = \frac{x^2}{2} + 4 \ln |x|.$$

$$\text{Evaluate: } [\frac{x^2}{2} + 4 \ln |x|]_1^4 = (\frac{16}{2} + 4 \ln 4) - (\frac{1}{2} + 4 \ln 1) = (8 + 4 \ln 4) - (\frac{1}{2} + 0) = \frac{15}{2} + 4 \ln 4.$$

$$\text{Final Answer: } \frac{15}{2} + 4 \ln 4 \approx 13.045.$$

10. Find the area bounded by $y = \frac{x-2}{x}$, x-axis, $x = 2$, $x = 4$.

Solution:

Bounds $[2, 4]$. $y = 1 - 2/x$. For $x \geq 2$, $0 < 2/x \leq 1$, so $y \geq 0$.

$$\text{Area} = \int_2^4 (1 - \frac{2}{x}) dx.$$

$$\text{Antiderivative: } F(x) = x - 2 \ln |x|.$$

$$\text{Evaluate: } [x - 2 \ln |x|]_2^4 = (4 - 2 \ln 4) - (2 - 2 \ln 2) = 2 - 2 \ln 4 + 2 \ln 2.$$

$$= 2 - 2 \ln(2^2) + 2 \ln 2 = 2 - 4 \ln 2 + 2 \ln 2 = 2 - 2 \ln 2.$$

$$\text{Final Answer: } 2 - 2 \ln 2 \approx 0.614.$$

3.2 Solutions: Definite Integral Evaluation

11. Evaluate $\int_0^1 2x \, dx$.

$$\text{Solution: Antiderivative } F(x) = x^2.$$

$$\text{Result: } [x^2]_0^1 = 1^2 - 0^2 = 1.$$

12. Evaluate $\int_2^7 3v \, dv$.

$$\text{Solution: Antiderivative } F(v) = \frac{3}{2}v^2.$$

$$\text{Result: } [\frac{3}{2}v^2]_2^7 = \frac{3}{2}(49) - \frac{3}{2}(4) = \frac{3}{2}(45) = \frac{135}{2}.$$

13. Evaluate $\int_{-1}^1 (x-2) \, dx$.

$$\text{Solution: Antiderivative } F(x) = \frac{x^2}{2} - 2x.$$

$$\text{Result: } [\frac{x^2}{2} - 2x]_{-1}^1 = (\frac{1}{2} - 2) - (\frac{1}{2} + 2) = -4.$$

14. Evaluate $\int_2^5 (-3x + 4) \, dx$.

Solution: Antiderivative $F(x) = -\frac{3}{2}x^2 + 4x$.

Result: $[-\frac{3}{2}x^2 + 4x]_2^5 = (-\frac{75}{2} + 20) - (-\frac{12}{2} + 8) = -\frac{35}{2} - 2 = -\frac{39}{2}$.

15. Evaluate $\int_{-1}^1 (2t - 1)^2 \, dt$.

Solution: Expand: $4t^2 - 4t + 1$.

Antiderivative $F(t) = \frac{4}{3}t^3 - 2t^2 + t$.

Result: $[\frac{4}{3}t^3 - 2t^2 + t]_{-1}^1 = (\frac{4}{3} - 2 + 1) - (-\frac{4}{3} - 2 - 1) = \frac{1}{3} - (-\frac{13}{3}) = \frac{14}{3}$.

16. Evaluate $\int_0^2 (1 - 2x)^2 \, dx$. *Solution:* Expand: $1 - 4x + 4x^2$.

Antiderivative $F(x) = x - 2x^2 + \frac{4}{3}x^3$.

Result: $[x - 2x^2 + \frac{4}{3}x^3]_0^2 = (2 - 8 + \frac{32}{3}) - 0 = -6 + \frac{32}{3} = \frac{14}{3}$.

17. Evaluate $\int_0^3 (x - 2)^3 \, dx$.

Solution (U-Sub): $u = x - 2, du = dx$.

Limits $x = 0 \rightarrow u = -2, x = 3 \rightarrow u = 1$. $\int_{-2}^1 u^3 du = [\frac{u^4}{4}]_{-2}^1 = \frac{1^4}{4} - \frac{(-2)^4}{4} = \frac{1}{4} - \frac{16}{4} = -\frac{15}{4}$.

18. Evaluate $\int_2^5 (x - 3)^3 \, dx$.

Solution (U-Sub): $u = x - 3, du = dx$.

Limits $x = 2 \rightarrow u = -1, x = 5 \rightarrow u = 2$. $\int_{-1}^2 u^3 du = [\frac{u^4}{4}]_{-1}^2 = \frac{2^4}{4} - \frac{(-1)^4}{4} = \frac{16}{4} - \frac{1}{4} = \frac{15}{4}$.

19. Evaluate $\int_{-1}^1 (\sqrt[3]{t} - 2) \, dt$.

Solution: Rewrite $\int_{-1}^1 (t^{1/3} - 2) dt$.

Antiderivative $F(t) = \frac{t^{4/3}}{4/3} - 2t = \frac{3}{4}t^{4/3} - 2t$.

Result: $[\frac{3}{4}t^{4/3} - 2t]_{-1}^1 = (\frac{3}{4} - 2) - (-\frac{3}{4} + 2) = -\frac{5}{4} - \frac{11}{4} = -4$.

20. Evaluate $\int_1^4 \frac{u - 2}{\sqrt{u}} \, du$.

Solution: Simplify $\int_1^4 (u^{1/2} - 2u^{-1/2}) du$.

Antiderivative $F(u) = \frac{u^{3/2}}{3/2} - 2\frac{u^{1/2}}{1/2} = \frac{2}{3}u^{3/2} - 4u^{1/2}$.

Result: $[\frac{2}{3}u^{3/2} - 4u^{1/2}]_1^4 = (\frac{2}{3}(8) - 4(2)) - (\frac{2}{3} - 4) = (\frac{16}{3} - 8) - (-\frac{10}{3}) = -\frac{8}{3} + \frac{10}{3} = \frac{2}{3}$.

21. Evaluate $\int_{-1}^0 (t^{1/3} - t^{2/3}) dt$.

Solution: Antiderivative $F(t) = \frac{t^{4/3}}{4/3} - \frac{t^{5/3}}{5/3} = \frac{3}{4}t^{4/3} - \frac{3}{5}t^{5/3}$.

Result: $[\frac{3}{4}t^{4/3} - \frac{3}{5}t^{5/3}]_{-1}^0 = (0) - (\frac{3}{4}(1) - \frac{3}{5}(-1)) = -(\frac{3}{4} + \frac{3}{5}) = -\frac{27}{20}$.

22. Evaluate $\int_0^4 (x^{1/2} + x^{1/4}) dx$.

Solution: Antiderivative $F(x) = \frac{x^{3/2}}{3/2} + \frac{x^{5/4}}{5/4} = \frac{2}{3}x^{3/2} + \frac{4}{5}x^{5/4}$.

Result: $[\frac{2}{3}x^{3/2} + \frac{4}{5}x^{5/4}]_0^4 = (\frac{2}{3}(4^{3/2}) + \frac{4}{5}(4^{5/4})) - 0 = \frac{2}{3}(8) + \frac{4}{5}(4 \cdot 4^{1/4}) = \frac{16}{3} + \frac{16 \cdot 2^{1/2}}{5} = \frac{16}{3} + \frac{16\sqrt{2}}{5}$.

23. Evaluate $\int_0^1 e^{-2x} dx$.

Solution: Antiderivative $F(x) = -\frac{1}{2}e^{-2x}$.

Result: $[-\frac{1}{2}e^{-2x}]_0^1 = (-\frac{1}{2}e^{-2}) - (-\frac{1}{2}e^0) = -\frac{1}{2}e^{-2} + \frac{1}{2} = \frac{1}{2}(1 - e^{-2})$.

24. Evaluate $\int_1^2 e^{1-x} dx$.

Solution: Antiderivative $F(x) = -e^{1-x}$.

Result: $[-e^{1-x}]_1^2 = (-e^{-1}) - (-e^0) = -e^{-1} + 1 = 1 - 1/e$.

25. Evaluate $\int_1^e \frac{e^{3/x}}{x^2} dx$.

Solution (U-Sub): $u = 3/x, du = -3/x^2 dx \implies \frac{1}{x^2} dx = -\frac{1}{3} du$.

Limits: $x = 1 \rightarrow u = 3, x = e \rightarrow u = 3/e$. $\int_3^{3/e} e^u (-\frac{1}{3} du) = -\frac{1}{3}[e^u]_3^{3/e} = -\frac{1}{3}(e^{3/e} - e^3) = \frac{1}{3}(e^3 - e^{3/e})$.

26. Evaluate $\int_{-1}^1 (e^x - e^{-x}) dx$.

Solution: Integrand $f(x) = e^x - e^{-x}$ is odd: $f(-x) = e^{-x} - e^x = -f(x)$.

Interval is symmetric. Integral = 0.

27. Evaluate $\int_0^1 \frac{e^{-x}}{\sqrt{e^{-x} + 1}} dx$.

Solution (U-Sub): $u = e^{-x} + 1, du = -e^{-x} dx \implies e^{-x} dx = -du$.

Limits: $x = 0 \rightarrow u = 2, x = 1 \rightarrow u = e^{-1} + 1$.

$\int_2^{1+1/e} \frac{-du}{\sqrt{u}} = -\int_2^{1+1/e} u^{-1/2} du = -[2u^{1/2}]_2^{1+1/e} = -2\sqrt{1+1/e} - (-2\sqrt{2}) = 2\sqrt{2} - 2\sqrt{1+1/e}$.

28. Evaluate $\int_0^2 \frac{e^{2x}}{e^{2x} + 1} dx$.

Solution (U-Sub): $u = e^{2x} + 1, du = 2e^{2x}dx \implies e^{2x}dx = \frac{1}{2}du$.

Limits: $x = 0 \rightarrow u = 2, x = 2 \rightarrow u = e^4 + 1$. $\int_2^{e^4+1} \frac{1}{u}(\frac{1}{2}du) = \frac{1}{2}[\ln |u|]_2^{e^4+1} = \frac{1}{2}(\ln(e^4 + 1) - \ln 2) = \frac{1}{2}\ln(\frac{e^4+1}{2})$.

29. Evaluate $\int_1^2 \frac{x}{1+4x^2} dx$.

Solution (U-Sub): $u = 1 + 4x^2, du = 8xdx \implies xdx = \frac{1}{8}du$.

Limits: $x = 1 \rightarrow u = 5, x = 2 \rightarrow u = 17$. $\int_5^{17} \frac{1}{u}(\frac{1}{8}du) = \frac{1}{8}[\ln |u|]_5^{17} = \frac{1}{8}(\ln 17 - \ln 5) = \frac{1}{8}\ln(\frac{17}{5})$.

30. Evaluate $\int_0^1 \frac{e^{2x}}{e^{2x} + 1} dx$.

Solution (U-Sub): $u = e^{2x} + 1, du = 2e^{2x}dx \implies e^{2x}dx = \frac{1}{2}du$.

Limits: $x = 0 \rightarrow u = 2, x = 1 \rightarrow u = e^2 + 1$. $\int_2^{e^2+1} \frac{1}{u}(\frac{1}{2}du) = \frac{1}{2}[\ln |u|]_2^{e^2+1} = \frac{1}{2}(\ln(e^2 + 1) - \ln 2) = \frac{1}{2}\ln(\frac{e^2+1}{2})$.