

THE DERIVATIVE AS LIMIT OF RATE OF CHANGE - WEEK 3

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1 Introduction: Basic Differentiation Rules

Explanation. This section focuses on finding derivatives, which represent the instantaneous rate of change of a function. While the derivative is formally defined using limits, we typically use differentiation rules for efficiency. The main rules used here are:

1. **Power Rule:** For any real number n , $\frac{d}{dx}(x^n) = nx^{n-1}$. (Bring power down, reduce power by 1).

2. **Constant Multiple Rule:** $\frac{d}{dx}(c \cdot f(x)) = c \cdot \frac{d}{dx}(f(x))$. (Constants factor out).

3. **Sum/Difference Rule:** $\frac{d}{dx}(f(x) \pm g(x)) = \frac{d}{dx}(f(x)) \pm \frac{d}{dx}(g(x))$. (Differentiate term by term).

4. **Constant Rule:** $\frac{d}{dx}(c) = 0$. (Derivative of a constant is zero).

5. **Chain Rule (Preview/Implied):** For functions like $(expression)^n$ or $\sqrt{expression}$, we technically need the Chain Rule: $\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x)$. For the power rule case, $\frac{d}{dx}(u^n) = nu^{n-1} \cdot \frac{du}{dx}$. We will apply this where necessary, especially in ClassWork II and Assignment I.

For polynomials, we apply rules 1-4 term by term. For other functions, we might need to rewrite them using exponents (e.g., $\sqrt{x} = x^{1/2}$, $1/x^n = x^{-n}$) before applying the rules.

2 Solutions: ClassWork I

1. **Problem (1):** Find derivative of $y = x^4 - 3x^2 + 8x + 6$. *Solution:* Apply Sum/Difference, Constant Multiple, and Power Rules term-by-term.

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(x^4) - \frac{d}{dx}(3x^2) + \frac{d}{dx}(8x^1) + \frac{d}{dx}(6) \\ &= (4x^{4-1}) - 3(2x^{2-1}) + 8(1x^{1-1}) + 0 \\ &= 4x^3 - 6x^1 + 8(1x^0) = 4x^3 - 6x + 8(1) \\ &= 4x^3 - 6x + 8\end{aligned}$$

(Note: Slide 5 solution has $-4x$ instead of $-6x$, likely a typo on the slide).

2. **Problem (2):** Find derivative of $y = 4x^2 - 5x$. *Solution:* Differentiate term-by-term.

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(4x^2) - \frac{d}{dx}(5x^1) \\ &= 4(2x^{2-1}) - 5(1x^{1-1}) = 8x^1 - 5(1x^0) \\ &= 8x - 5\end{aligned}$$

3. **Problem (3):** Find derivative of $y = x^2 - 4x + 10$. *Solution:* Differentiate term-by-term.

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(x^2) - \frac{d}{dx}(4x^1) + \frac{d}{dx}(10) \\ &= (2x^{2-1}) - 4(1x^{1-1}) + 0 = 2x^1 - 4(1x^0) \\ &= 2x - 4\end{aligned}$$

4. **Problem (4):** Find derivative of $y = x^2 + 6x + 5$. *Solution:* Differentiate term-by-term.

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(x^2) + \frac{d}{dx}(6x^1) + \frac{d}{dx}(5) \\ &= (2x^{2-1}) + 6(1x^{1-1}) + 0 = 2x^1 + 6(1x^0) \\ &= 2x + 6\end{aligned}$$

5. **Problem (5):** Find derivative of $y = x^2 - 2x - 3$. *Solution:* Differentiate term-by-term.

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(x^2) - \frac{d}{dx}(2x^1) - \frac{d}{dx}(3) \\ &= (2x^{2-1}) - 2(1x^{1-1}) - 0 = 2x^1 - 2(1x^0) \\ &= 2x - 2\end{aligned}$$

6. **Problem (6):** Find derivative of $y = 4x^2 + 3x + 5$. *Solution:* Differentiate term-by-term.

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(4x^2) + \frac{d}{dx}(3x^1) + \frac{d}{dx}(5) \\ &= 4(2x^{2-1}) + 3(1x^{1-1}) + 0 = 8x^1 + 3(1x^0) \\ &= 8x + 3\end{aligned}$$

7. **Problem (7):** Find derivative of $y = x^2 - 4x + 3$. *Solution:* Differentiate term-by-term.

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(x^2) - \frac{d}{dx}(4x^1) + \frac{d}{dx}(3) \\ &= (2x^{2-1}) - 4(1x^{1-1}) + 0 = 2x^1 - 4(1x^0) \\ &= 2x - 4\end{aligned}$$

8. **Problem (8):** Find derivative of $y = 2x^2 - 8x + 4$. *Solution:* Differentiate term-by-term.

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(2x^2) - \frac{d}{dx}(8x^1) + \frac{d}{dx}(4) \\ &= 2(2x^{2-1}) - 8(1x^{1-1}) + 0 = 4x^1 - 8(1x^0) \\ &= 4x - 8\end{aligned}$$

9. **Problem (9):** Find derivative of $y = 3x^2 - 6x + 5$. *Solution:* Differentiate term-by-term.

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(3x^2) - \frac{d}{dx}(6x^1) + \frac{d}{dx}(5) \\ &= 3(2x^{2-1}) - 6(1x^{1-1}) + 0 = 6x^1 - 6(1x^0) \\ &= 6x - 6\end{aligned}$$

10. **Problem (10):** Find derivative of $y = 4x^3 - 30x^2 + 74x - 60$. *Solution:* Differentiate term-by-term.

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(4x^3) - \frac{d}{dx}(30x^2) + \frac{d}{dx}(74x^1) - \frac{d}{dx}(60) \\ &= 4(3x^{3-1}) - 30(2x^{2-1}) + 74(1x^{1-1}) - 0 \\ &= 12x^2 - 60x^1 + 74(1x^0) = 12x^2 - 60x + 74\end{aligned}$$

11. **Problem (11):** Find derivative of $y = 2x^2 + 7x - 5$. *Solution:* Differentiate term-by-term.

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(2x^2) + \frac{d}{dx}(7x^1) - \frac{d}{dx}(5) \\ &= 2(2x^{2-1}) + 7(1x^{1-1}) - 0 = 4x^1 + 7(1x^0) \\ &= 4x + 7\end{aligned}$$

12. **Problem (12):** Find derivative of $y = x^3 - 6x^2$. *Solution:* Differentiate term-by-term.

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(x^3) - \frac{d}{dx}(6x^2) \\ &= (3x^{3-1}) - 6(2x^{2-1}) = 3x^2 - 12x^1 \\ &= 3x^2 - 12x\end{aligned}$$

13. **Problem (13):** Find derivative of $y = 3x^2 - 12$. *Solution:* Differentiate term-by-term.

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(3x^2) - \frac{d}{dx}(12) \\ &= 3(2x^{2-1}) - 0 = 6x^1 \\ &= 6x\end{aligned}$$

14. **Problem (14):** Find derivative of $y = x^2 - 3x + 4$. *Solution:* Differentiate term-by-term.

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(x^2) - \frac{d}{dx}(3x^1) + \frac{d}{dx}(4) \\ &= (2x^{2-1}) - 3(1x^{1-1}) + 0 = 2x^1 - 3(1x^0) \\ &= 2x - 3\end{aligned}$$

15. **Problem (15):** Find derivative of $y = x^2 - 4x + 5$. *Solution:* Differentiate term-by-term.

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(x^2) - \frac{d}{dx}(4x^1) + \frac{d}{dx}(5) \\ &= (2x^{2-1}) - 4(1x^{1-1}) + 0 = 2x^1 - 4(1x^0) \\ &= 2x - 4\end{aligned}$$

3 Solutions: ClassWork II

16. **Problem (16):** Find derivative of $y = x^2 + 4x - 1$. *Solution:* Differentiate term-by-term.

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(x^2) + \frac{d}{dx}(4x^1) - \frac{d}{dx}(1) \\ &= (2x^{2-1}) + 4(1x^{1-1}) - 0 = 2x^1 + 4(1x^0) \\ &= 2x + 4\end{aligned}$$

17. **Problem (17):** Find derivative of $y = x^3 + 3x^2 + 1$. *Solution:* Differentiate term-by-term.

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(x^3) + \frac{d}{dx}(3x^2) + \frac{d}{dx}(1) \\ &= (3x^{3-1}) + 3(2x^{2-1}) + 0 = 3x^2 + 6x^1 \\ &= 3x^2 + 6x\end{aligned}$$

18. **Problem (18):** Find derivative of $y = x^2 + 2$. *Solution:* Differentiate term-by-term.

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(x^2) + \frac{d}{dx}(2) \\ &= (2x^{2-1}) + 0 = 2x^1 \\ &= 2x\end{aligned}$$

(Note: Slide 9 solution $2x+1$ seems to correspond to a different problem, perhaps x^2+x+2)

19. **Problem (19):** Find derivative of $y = 3x^2 + 6x$. *Solution:* Differentiate term-by-term.

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(3x^2) + \frac{d}{dx}(6x^1) \\ &= 3(2x^{2-1}) + 6(1x^{1-1}) = 6x^1 + 6(1x^0) \\ &= 6x + 6\end{aligned}$$

20. **Problem (20):** Find derivative of $y = 5x^4 + 12x^3 + 6x^2 + 14x$. *Solution:* Differentiate term-by-term.

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(5x^4) + \frac{d}{dx}(12x^3) + \frac{d}{dx}(6x^2) + \frac{d}{dx}(14x^1) \\ &= 5(4x^{4-1}) + 12(3x^{3-1}) + 6(2x^{2-1}) + 14(1x^{1-1}) \\ &= 20x^3 + 36x^2 + 12x^1 + 14(1x^0) \\ &= 20x^3 + 36x^2 + 12x + 14\end{aligned}$$

21. **Problem (21):** Find derivative of $y = 1 - 2x - x^2$. *Solution:* Differentiate term-by-term.

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(1) - \frac{d}{dx}(2x^1) - \frac{d}{dx}(x^2) \\ &= 0 - 2(1x^{1-1}) - (2x^{2-1}) = -2(1x^0) - 2x^1 \\ &= -2 - 2x\end{aligned}$$

22. **Problem (22):** Find derivative of $y = (x^2 + 1)^2$. *Solution: Method 1: Expand First*
 $y = (x^2)^2 + 2(x^2)(1) + (1)^2 = x^4 + 2x^2 + 1$. Differentiate term-by-term:

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(x^4) + \frac{d}{dx}(2x^2) + \frac{d}{dx}(1) \\ &= 4x^3 + 2(2x) + 0 = 4x^3 + 4x\end{aligned}$$

Method 2: Chain Rule Outer: $f(u) = u^2 \implies f'(u) = 2u$. Inner: $u(x) = x^2 + 1 \implies u'(x) = 2x$.

$$\begin{aligned}\frac{dy}{dx} &= f'(u(x)) \cdot u'(x) = 2(x^2 + 1) \cdot (2x) = 4x(x^2 + 1) \\ &= 4x(x^2) + 4x(1) = 4x^3 + 4x\end{aligned}$$

Results match.

23. **Problem (23):** Find derivative of $y = x^6 + 4x^3 + 5$. *Solution:* Differentiate term-by-term.

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(x^6) + \frac{d}{dx}(4x^3) + \frac{d}{dx}(5) \\ &= 6x^{6-1} + 4(3x^{3-1}) + 0 = 6x^5 + 12x^2\end{aligned}$$

24. **Problem (24):** Find derivative of $y = (x^2 - 4)^2$. *Solution: Method 1: Expand First*
 $y = (x^2)^2 - 2(x^2)(4) + (4)^2 = x^4 - 8x^2 + 16$. Differentiate term-by-term:

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(x^4) - \frac{d}{dx}(8x^2) + \frac{d}{dx}(16) \\ &= 4x^3 - 8(2x) + 0 = 4x^3 - 16x\end{aligned}$$

Method 2: Chain Rule Outer: $f(u) = u^2 \implies f'(u) = 2u$. Inner: $u(x) = x^2 - 4 \implies u'(x) = 2x$.

$$\begin{aligned}\frac{dy}{dx} &= f'(u(x)) \cdot u'(x) = 2(x^2 - 4) \cdot (2x) = 4x(x^2 - 4) \\ &= 4x(x^2) - 4x(4) = 4x^3 - 16x\end{aligned}$$

Results match.

25. **Problem (25):** Find derivative of $y = 6x^5 + 12x^2$. *Solution:* Differentiate term-by-term.

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(6x^5) + \frac{d}{dx}(12x^2) \\ &= 6(5x^{5-1}) + 12(2x^{2-1}) = 30x^4 + 24x^1 \\ &= 30x^4 + 24x\end{aligned}$$

26. **Problem (26):** Find derivative of $y = x^3 + 3x$. *Solution:* Differentiate term-by-term.

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(x^3) + \frac{d}{dx}(3x^1) \\ &= (3x^{3-1}) + 3(1x^{1-1}) = 3x^2 + 3(1x^0) \\ &= 3x^2 + 3\end{aligned}$$

27. **Problem (27):** Find derivative of $y = -x^3 + 3x^2 + 9x + 5$. *Solution:* Differentiate term-by-term.

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(-x^3) + \frac{d}{dx}(3x^2) + \frac{d}{dx}(9x^1) + \frac{d}{dx}(5) \\ &= -1(3x^{3-1}) + 3(2x^{2-1}) + 9(1x^{1-1}) + 0 \\ &= -3x^2 + 6x^1 + 9(1x^0) = -3x^2 + 6x + 9\end{aligned}$$

28. **Problem (28):** Find derivative of $y = 2x^3 - 24x + 5$. *Solution:* Differentiate term-by-term.

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(2x^3) - \frac{d}{dx}(24x^1) + \frac{d}{dx}(5) \\ &= 2(3x^{3-1}) - 24(1x^{1-1}) + 0 = 6x^2 - 24(1x^0) \\ &= 6x^2 - 24\end{aligned}$$

29. **Problem (29):** Find derivative of $y = (x - 1)^3(x - 2)$. *Solution: Method 1: Product Rule (with Chain Rule)* Let $u = (x - 1)^3, v = x - 2$. Find du/dx using Chain Rule: outer $f(w) = w^3 \implies f'(w) = 3w^2$. Inner $w = x - 1 \implies w' = 1$. $du/dx = 3(x - 1)^2(1) = 3(x - 1)^2$. $dv/dx = 1$. Apply Product Rule: $y' = uv' + vu'$.

$$\frac{dy}{dx} = (x - 1)^3(1) + (x - 2)[3(x - 1)^2]$$

Factor out $(x - 1)^2$:

$$= (x - 1)^2[(x - 1) + 3(x - 2)]$$

$$= (x-1)^2[x-1+3x-6] = (x-1)^2(4x-7)$$

Expanding $(x-1)^2(4x-7) = (x^2-2x+1)(4x-7) = 4x^3-7x^2-8x^2+14x+4x-7 = 4x^3-15x^2+18x-7$.

Method 2: Expand First $y = (x^3-3x^2+3x-1)(x-2)$ $y = x(x^3-3x^2+3x-1) - 2(x^3-3x^2+3x-1)$ $y = (x^4-3x^3+3x^2-x) - (2x^3-6x^2+6x-2)$ $y = x^4-3x^3+3x^2-x-2x^3+6x^2-6x+2$ $y = x^4-5x^3+9x^2-7x+2$. Differentiate:

$$\begin{aligned}\frac{dy}{dx} &= 4x^3 - 5(3x^2) + 9(2x) - 7(1) + 0 \\ &= 4x^3 - 15x^2 + 18x - 7\end{aligned}$$

Results match.

30. **Problem (30):** Find derivative of $y = 6(x+2)(x-2)$. *Solution: Method 1: Expand First (Easiest)* Recognize $(x+2)(x-2)$ is a difference of squares: $x^2-2^2 = x^2-4$. $y = 6(x^2-4) = 6x^2-24$. Differentiate:

$$\frac{dy}{dx} = \frac{d}{dx}(6x^2) - \frac{d}{dx}(24) = 6(2x) - 0 = 12x$$

Method 2: Product Rule Let $u = 6(x+2) = 6x+12 \implies du/dx = 6$. Let $v = x-2 \implies dv/dx = 1$.

$$\begin{aligned}\frac{dy}{dx} &= uv' + vu' = (6x+12)(1) + (x-2)(6) \\ &= 6x+12+6x-12 = 12x\end{aligned}$$

Results match.

4 Solutions: Assignment I

31. **Problem (31):** Find derivative of $t = f(u) = 6u^{3/2}$. Assume t depends on u . *Solution (Constant Multiple, Power Rule):*

$$\frac{dt}{du} = \frac{d}{du}(6u^{3/2}) = 6 \cdot \frac{d}{du}(u^{3/2})$$

Apply Power Rule with $n = 3/2$. $n-1 = 3/2-1 = 1/2$.

$$= 6 \cdot \left(\frac{3}{2}u^{3/2-1}\right) = 6 \cdot \frac{3}{2}u^{1/2} = 9u^{1/2}$$

Optional rewrite: $9\sqrt{u}$.

32. **Problem (32):** Find derivative of $w = f(p) = (p^2+4)^{1/2}$. Assume w depends on p . *Solution (Chain Rule):* Outer: $f(u) = u^{1/2} \implies f'(u) = \frac{1}{2}u^{-1/2}$. Inner: $u(p) = p^2+4 \implies u'(p) = 2p$.

$$\begin{aligned}\frac{dw}{dp} &= f'(u(p)) \cdot u'(p) = \frac{1}{2}(p^2+4)^{-1/2} \cdot (2p) \\ &= p(p^2+4)^{-1/2} = \frac{p}{(p^2+4)^{1/2}} = \frac{p}{\sqrt{p^2+4}}\end{aligned}$$

33. **Problem (33):** Find derivative of $b = f(v) = -5 + 3v - \frac{3}{2}v^2 - 7v^3$. Assume b depends on v . *Solution:* Differentiate term-by-term with respect to v .

$$\begin{aligned}\frac{db}{dv} &= \frac{d}{dv}(-5) + \frac{d}{dv}(3v^1) - \frac{d}{dv}\left(\frac{3}{2}v^2\right) - \frac{d}{dv}(7v^3) \\ &= 0 + 3(1v^0) - \frac{3}{2}(2v^1) - 7(3v^2) \\ &= 3(1) - 3v - 21v^2 = 3 - 3v - 21v^2\end{aligned}$$

34. **Problem (34):** Find derivative of $a = f(c) = \frac{2}{c^2 - 1}$. Assume a depends on c . *Solution:*

Method 1: Quotient Rule Let $u = 2 \implies u' = 0$. Let $v = c^2 - 1 \implies v' = 2c$.

$$\frac{da}{dc} = \frac{vu' - uv'}{v^2} = \frac{(c^2 - 1)(0) - (2)(2c)}{(c^2 - 1)^2} = \frac{0 - 4c}{(c^2 - 1)^2} = \frac{-4c}{(c^2 - 1)^2}$$

Method 2: Chain Rule Rewrite $a = 2(c^2 - 1)^{-1}$. Outer: $f(u) = u^{-1} \implies f'(u) = -u^{-2}$. Inner: $u(c) = c^2 - 1 \implies u'(c) = 2c$.

$$\begin{aligned}\frac{da}{dc} &= 2 \cdot [f'(u(c)) \cdot u'(c)] = 2 \cdot [-(c^2 - 1)^{-2} \cdot (2c)] \\ &= 2(-2c)(c^2 - 1)^{-2} = -4c(c^2 - 1)^{-2} = \frac{-4c}{(c^2 - 1)^2}\end{aligned}$$

Results match.

35. **Problem (35):** Find derivative of $y = f(x) = 17x^2 - 10x + 15$. *Solution:* Differentiate term-by-term.

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(17x^2) - \frac{d}{dx}(10x^1) + \frac{d}{dx}(15) \\ &= 17(2x) - 10(1) + 0 = 34x - 10\end{aligned}$$