

INTEGRATION BY PARTS - WEEK 9

(MAXIMUM DETAIL SOLUTIONS)

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April 27, 2025

1 Introduction: Integration by Parts (IBP)

Explanation. Integration by Parts is a technique specifically designed to find the integral of a product of two functions. It's derived directly from the product rule for differentiation. Recall the Product Rule: $\frac{d}{dx}[u(x)v(x)] = u'(x)v(x) + u(x)v'(x)$. Writing this using differentials ($du = u'(x)dx$, $dv = v'(x)dx$): $d(uv) = v du + u dv$. If we integrate both sides with respect to the underlying variable (like x): $\int d(uv) = \int v du + \int u dv$. Since the integral of a differential is just the function itself, $\int d(uv) = uv$. So, $uv = \int v du + \int u dv$. Rearranging this equation to solve for one of the integrals gives the **Integration by Parts Formula**:

$$\boxed{\int u dv = uv - \int v du}$$

The Goal: The purpose of this formula is to replace one integral, $\int u dv$, which might be hard to solve directly, with a different integral, $\int v du$, which is hopefully easier to solve. **Choosing u and dv :** This selection is the most critical part of the process.

- You must split the entire original integrand (including the dx) into two parts: u and dv .
- The part chosen as dv *must* be something you know how to integrate to find v .
- The part chosen as u will be differentiated to find du .
- **Strategic Choice:** The ideal choice makes the *new* integral, $\int v du$, simpler than the original one. This often happens when differentiating u makes it significantly simpler (like $x^2 \rightarrow 2x$), while integrating dv doesn't make v overly complicated.
- **LIATE Mnemonic:** A very useful (though not foolproof) guideline for choosing u is the acronym LIATE. Look at the types of functions in your product and choose u based on the type that appears first in this list:
 - L:** Logarithmic functions (like $\ln x, \log_b x$). These are often good choices for u because their derivatives ($1/x$) are simpler algebraic functions, while their integrals are more complex and often require IBP themselves.
 - I:** Inverse trigonometric functions (like $\arcsin x, \arctan x$). Similar to logs, their derivatives are algebraic, while their integrals are harder.
 - A:** Algebraic functions (polynomials like $x^3, 2x - 1$, roots like \sqrt{x}). Differentiation reduces their degree, making them simpler. Integration increases their degree.
 - T:** Trigonometric functions (like $\sin x, \cos x, \sec^2 x$). Differentiation and integration often cycle between \sin and \cos , or involve other trig functions.

E: Exponential functions (like e^x, a^x). These generally stay similar upon differentiation or integration (up to constants).

Once u is chosen according to LIATE, everything else in the integrand (including dx) automatically becomes dv .

2 Solutions: Examples and Class Work

1. **Problem:** (*Example 1*) Solve $\int xe^x dx$.

Solution:

Strategy. We have a product of x (Algebraic) and e^x (Exponential). Following LIATE, Algebraic (A) comes before Exponential (E). Therefore, we choose $u = x$. This is a good choice because its derivative, $du = dx$, is simpler than $u = x$. The remaining part, $e^x dx$, becomes dv , which is easy to integrate. We expect one application of IBP to solve this.

Method 1: Standard IBP *Step 1: Define u and dv .* Let $u = x$. Let $dv = e^x dx$.

Step 2: Compute du and v . Differentiate u : $du = (\frac{d}{dx}x)dx = (1)dx = dx$. Integrate dv : $v = \int dv = \int e^x dx = e^x$. (*Constant C omitted until the final answer*)

Step 3: Apply IBP Formula $\int u dv = uv - \int v du$.

$$\begin{aligned}\int xe^x dx &= (x)(e^x) - \int (e^x)(dx) \\ &= xe^x - \int e^x dx\end{aligned}$$

Step 4: Solve the new integral. This integral is simpler than the original.

$$\int e^x dx = e^x$$

Step 5: Combine results and add the constant of integration C .

$$\int xe^x dx = xe^x - (e^x) + C = xe^x - e^x + C$$

Final Answer (Optional Factoring): $e^x(x - 1) + C$.

Method 2: Tabular Method

Explanation. The Tabular Method is also applicable here, though slightly overkill for a single application. It's useful for verifying the standard method. We set up columns for alternating signs, derivatives of u , and integrals of dv .

Let $u = x$, $dv = e^x dx$.

Sign	u & derivatives	$dv = e^x$ & integrals
+	x	e^x
-	1	e^x
+	0	e^x

Multiply diagonally (Sign \times u-term \times v-integral in **next** row):

$$= (+)(x)(e^x) + (-)(1)(e^x) + C$$

$$= xe^x - e^x + C$$

Result: $e^x(x - 1) + C$. Both methods agree.

2. **Problem:** (*Example 2*) Solve $\int xe^{2x} dx$.

Solution:

Strategy. Product of Algebraic (x) and Exponential (e^{2x}). LIATE: A before E, so $u = x$. Derivative $du = dx$ is simpler. $dv = e^{2x}dx$ is integrable. IBP once.

Method 1: Standard IBP *Step 1: Define u, dv . $u = x$, $dv = e^{2x}dx$. Step 2: Compute du, v . $du = dx$. $v = \int e^{2x}dx$. (Use substitution $w = 2x, dw = 2dx \implies dx = \frac{1}{2}dw$. $\int e^w(\frac{1}{2}dw) = \frac{1}{2}e^w = \frac{1}{2}e^{2x}$). So, $v = \frac{1}{2}e^{2x}$. Step 3: Apply Formula. $\int xe^{2x}dx = x(\frac{1}{2}e^{2x}) - \int(\frac{1}{2}e^{2x})dx = \frac{1}{2}xe^{2x} - \frac{1}{2}\int e^{2x}dx$. Step 4: Solve new integral. $\int e^{2x}dx = \frac{1}{2}e^{2x}$. Step 5: Combine + C . $\int xe^{2x}dx = \frac{1}{2}xe^{2x} - \frac{1}{2}(\frac{1}{2}e^{2x}) + C = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C$. Factored: $\frac{e^{2x}}{4}(2x - 1) + C$.*

Method 2: Tabular Method Let $u = x, dv = e^{2x}dx$.

Sign	u & derivatives	$dv = e^{2x}$ & integrals
+	x	e^{2x}
-	1	$\frac{1}{2}e^{2x}$
+	0	$\frac{1}{4}e^{2x}$

$$= (+)(x)(\frac{1}{2}e^{2x}) + (-)(1)(\frac{1}{4}e^{2x}) + C = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C$$

Factored: $\frac{e^{2x}}{4}(2x - 1) + C$. Both methods agree.

3. **Problem:** (*ClassWork 1*) Solve $\int x^2 \ln x dx$.

Solution:

Strategy. Product of Algebraic (x^2) and Logarithmic ($\ln x$). LIATE: L before A, so $u = \ln x$. This is good because $du = 1/x dx$ is simpler, and $dv = x^2 dx$ is easy to integrate. Tabular method is not suitable here because the derivatives of $\ln x$ do not terminate at zero.

Method 1: Standard IBP *Step 1: Define u, dv . $u = \ln x$, $dv = x^2 dx$. Step 2: Compute du, v . $du = \frac{1}{x} dx$, $v = \int x^2 dx = \frac{x^3}{3}$. Step 3: Apply Formula. $\int x^2 \ln x dx = (\ln x)(\frac{x^3}{3}) - \int (\frac{x^3}{3})(\frac{1}{x} dx) = \frac{x^3 \ln x}{3} - \int \frac{x^2}{3} dx$. Step 4: Solve new integral. $\int \frac{x^2}{3} dx = \frac{1}{3} \int x^2 dx = \frac{1}{3}(\frac{x^3}{3}) = \frac{x^3}{9}$. Step 5: Combine + C . $\int x^2 \ln x dx = \frac{x^3 \ln x}{3} - \frac{x^3}{9} + C$. Factored: $\frac{x^3}{9}(3 \ln x - 1) + C$.*

4. **Problem:** (ClassWork 2) Solve $\int x^2 e^x dx$.

Solution:

Strategy. Product of Algebraic (x^2) and Exponential (e^x). LIATE: A before E, $u = x^2$. Since u is x^2 , its derivative $2x$ is simpler but still contains x . This indicates we'll likely need to apply IBP twice. Both methods are suitable.

Method 1: Repeated IBP *First IBP Application:* Let $u_1 = x^2$, $dv_1 = e^x dx$. Then $du_1 = 2x dx$, $v_1 = e^x$.

$$I = \int x^2 e^x dx = (x^2)(e^x) - \int (e^x)(2x dx) = x^2 e^x - 2 \int x e^x dx \quad (*)$$

The new integral $\int x e^x dx$ still requires IBP. *Second IBP Application (for $\int x e^x dx$):* Let $u_2 = x$, $dv_2 = e^x dx$. Then $du_2 = dx$, $v_2 = e^x$.

$$\int x e^x dx = (x)(e^x) - \int (e^x)(dx) = x e^x - e^x$$

Combine Results: Substitute the result of the second IBP back into equation (*):

$$I = x^2 e^x - 2(x e^x - e^x) + C$$

Distribute the -2 :

$$I = x^2 e^x - 2x e^x + 2e^x + C$$

Factored: $e^x(x^2 - 2x + 2) + C$.

Method 2: Tabular Method Let $u = x^2$, $dv = e^x dx$.

Sign	u & derivatives	$dv = e^x$ & integrals
+	x^2	e^x
-	$2x$	e^x
+	2	e^x
-	0	e^x

Multiply diagonally (Sign \times u-term \times v-integral in next row) and sum:

$$\begin{aligned} &= (+)(x^2)(e^x) + (-)(2x)(e^x) + (+)(2)(e^x) + C \\ &= x^2 e^x - 2x e^x + 2e^x + C \end{aligned}$$

Factored: $e^x(x^2 - 2x + 2) + C$. Both methods agree.

3 Solutions: Practice Problems

Note: Numbering corresponds to the renumbered question list (5-26).

5. **Problem (5):** Solve $\int xe^{3x} dx$. *Analysis:* Algebraic \times Exponential. LIATE: $u = x$. IBP once. **Method 1: Standard IBP** Let $u = x, dv = e^{3x} dx$. $du = dx, v = \int e^{3x} dx = \frac{1}{3}e^{3x}$.

$$\begin{aligned}\int xe^{3x} dx &= x\left(\frac{1}{3}e^{3x}\right) - \int \left(\frac{1}{3}e^{3x}\right) dx = \frac{1}{3}xe^{3x} - \frac{1}{3} \int e^{3x} dx \\ &= \frac{1}{3}xe^{3x} - \frac{1}{3}\left(\frac{1}{3}e^{3x}\right) + C = \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + C\end{aligned}$$

Factored: $\frac{e^{3x}}{9}(3x - 1) + C$.

Method 2: Tabular Method Let $u = x, dv = e^{3x} dx$.

Sign	u	dv
+	x	e^{3x}
-	1	$\frac{1}{3}e^{3x}$
+	0	$\frac{1}{9}e^{3x}$

$$= (+)(x)\left(\frac{1}{3}e^{3x}\right) + (-)(1)\left(\frac{1}{9}e^{3x}\right) + C = \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + C$$

6. **Problem (6):** Solve $\int x^2 e^{3x} dx$. *Analysis:* Algebraic (x^2) \times Exponential (e^{3x}). LIATE: $u = x^2$. IBP twice. **Method 1: Repeated IBP** *IBP 1:* $u_1 = x^2, dv_1 = e^{3x} dx \implies du_1 = 2x dx, v_1 = \frac{1}{3}e^{3x}$.

$$I = x^2\left(\frac{1}{3}e^{3x}\right) - \int \frac{1}{3}e^{3x}(2x dx) = \frac{1}{3}x^2e^{3x} - \frac{2}{3} \int xe^{3x} dx \quad (*)$$

IBP 2 (for $\int xe^{3x} dx$): From Problem 5, result is $\frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x}$. *Combine:* Substitute into (*): $I = \frac{1}{3}x^2e^{3x} - \frac{2}{3}\left(\frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x}\right) + C = \frac{1}{3}x^2e^{3x} - \frac{2}{9}xe^{3x} + \frac{2}{27}e^{3x} + C$. *Factored:* $\frac{e^{3x}}{27}(9x^2 - 6x + 2) + C$.

Method 2: Tabular Method Let $u = x^2, dv = e^{3x} dx$.

Sign	u	dv
+	x^2	e^{3x}
-	$2x$	$\frac{1}{3}e^{3x}$
+	2	$\frac{1}{9}e^{3x}$
-	0	$\frac{1}{27}e^{3x}$

$$= x^2\left(\frac{1}{3}e^{3x}\right) - 2x\left(\frac{1}{9}e^{3x}\right) + 2\left(\frac{1}{27}e^{3x}\right) + C = \frac{1}{3}x^2e^{3x} - \frac{2}{9}xe^{3x} + \frac{2}{27}e^{3x} + C$$

7. **Problem (7):** Solve $\int x \ln(2x) \, dx$. *Analysis:* Algebraic \times Logarithmic. LIATE: $u = \ln(2x)$. Standard IBP only. **Method 1: Standard IBP** Let $u = \ln(2x)$, $dv = x \, dx \implies du = \frac{1}{x} \, dx$, $v = \frac{x^2}{2}$.

$$\begin{aligned} \int x \ln(2x) \, dx &= \frac{x^2 \ln(2x)}{2} - \int \frac{x^2}{2} \frac{1}{x} \, dx = \frac{x^2 \ln(2x)}{2} - \int \frac{x}{2} \, dx \\ &= \frac{x^2 \ln(2x)}{2} - \frac{x^2}{4} + C = \frac{x^2}{4} (2 \ln(2x) - 1) + C \end{aligned}$$

8. **Problem (8):** Solve $\int \ln(4x) \, dx$. *Analysis:* Logarithmic only. LIATE: $u = \ln(4x)$, $dv = dx$. Standard IBP only. **Method 1: Standard IBP** Let $u = \ln(4x)$, $dv = dx \implies du = \frac{1}{x} \, dx$, $v = x$.

$$\begin{aligned} \int \ln(4x) \, dx &= x \ln(4x) - \int x \frac{1}{x} \, dx = x \ln(4x) - \int 1 \, dx \\ &= x \ln(4x) - x + C = x(\ln(4x) - 1) + C \end{aligned}$$

9. **Problem (9):** Solve $\int x e^{-3x} \, dx$. *Analysis:* Algebraic \times Exponential. LIATE: $u = x$. IBP once. **Method 1: Standard IBP** Let $u = x$, $dv = e^{-3x} \, dx \implies du = dx$, $v = -\frac{1}{3} e^{-3x}$.

$$\begin{aligned} \int x e^{-3x} \, dx &= x \left(-\frac{1}{3} e^{-3x}\right) - \int \left(-\frac{1}{3} e^{-3x}\right) \, dx = -\frac{1}{3} x e^{-3x} + \frac{1}{3} \int e^{-3x} \, dx \\ &= -\frac{1}{3} x e^{-3x} + \frac{1}{3} \left(-\frac{1}{3} e^{-3x}\right) + C = -\frac{1}{3} x e^{-3x} - \frac{1}{9} e^{-3x} + C \end{aligned}$$

Factored: $-\frac{e^{-3x}}{9} (3x + 1) + C$.

Method 2: Tabular Method Let $u = x$, $dv = e^{-3x} \, dx$.

Sign	u	dv
+	x	e^{-3x}
-	1	$-\frac{1}{3} e^{-3x}$
+	0	$\frac{1}{9} e^{-3x}$

$$= (+)(x)\left(-\frac{1}{3} e^{-3x}\right) + (-)(1)\left(\frac{1}{9} e^{-3x}\right) + C = -\frac{1}{3} x e^{-3x} - \frac{1}{9} e^{-3x} + C$$

10. **Problem (10):** Solve $\int x^2 e^{-x} \, dx$. *Analysis:* Algebraic (x^2) \times Exponential (e^{-x}). LIATE: $u = x^2$. IBP twice. **Method 1: Repeated IBP** *IBP 1:* $u_1 = x^2$, $dv_1 = e^{-x} \, dx \implies du_1 = 2x \, dx$, $v_1 = -e^{-x}$.

$$I = x^2(-e^{-x}) - \int (-e^{-x})(2x \, dx) = -x^2 e^{-x} + 2 \int x e^{-x} \, dx$$

IBP 2 (for $\int xe^{-x}dx$): $u_2 = x, dv_2 = e^{-x}dx \implies du_2 = dx, v_2 = -e^{-x}$.

$$\int xe^{-x}dx = x(-e^{-x}) - \int (-e^{-x})dx = -xe^{-x} - e^{-x}$$

Combine: $I = -x^2e^{-x} + 2(-xe^{-x} - e^{-x}) + C = -x^2e^{-x} - 2xe^{-x} - 2e^{-x} + C$. *Factored:* $-e^{-x}(x^2 + 2x + 2) + C$.

Method 2: Tabular Method Let $u = x^2, dv = e^{-x}dx$.

Sign	u	dv
+	x^2	e^{-x}
-	$2x$	$-e^{-x}$
+	2	e^{-x}
-	0	$-e^{-x}$

$$= (+)(x^2)(-e^{-x}) + (-)(2x)(e^{-x}) + (+)(2)(-e^{-x}) + C = -x^2e^{-x} - 2xe^{-x} - 2e^{-x} + C$$

11. **Problem (11):** Solve $\int x^2e^{-2x} dx$. *Analysis:* Algebraic (x^2) \times Exponential (e^{-2x}). *LIATE:* $u = x^2$. *IBP twice.* **Method 1: Repeated IBP** *IBP 1:* $u_1 = x^2, dv_1 = e^{-2x}dx \implies du_1 = 2xdx, v_1 = -\frac{1}{2}e^{-2x}$.

$$I = x^2(-\frac{1}{2}e^{-2x}) - \int (-\frac{1}{2}e^{-2x})(2xdx) = -\frac{1}{2}x^2e^{-2x} + \int xe^{-2x}dx$$

IBP 2 (for $\int xe^{-2x}dx$): $u_2 = x, dv_2 = e^{-2x}dx \implies du_2 = dx, v_2 = -\frac{1}{2}e^{-2x}$.

$$\int xe^{-2x}dx = x(-\frac{1}{2}e^{-2x}) - \int (-\frac{1}{2}e^{-2x})dx = -\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x}$$

Combine: $I = -\frac{1}{2}x^2e^{-2x} + (-\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x}) + C$. *Factored:* $-\frac{e^{-2x}}{4}(2x^2 + 2x + 1) + C$.

Method 2: Tabular Method Let $u = x^2, dv = e^{-2x}dx$.

Sign	u	dv
+	x^2	e^{-2x}
-	$2x$	$-\frac{1}{2}e^{-2x}$
+	2	$\frac{1}{4}e^{-2x}$
-	0	$-\frac{1}{8}e^{-2x}$

$$\begin{aligned} &= (+)(x^2)(-\frac{1}{2}e^{-2x}) + (-)(2x)(\frac{1}{4}e^{-2x}) + (+)(2)(-\frac{1}{8}e^{-2x}) + C \\ &= -\frac{1}{2}x^2e^{-2x} - \frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + C \end{aligned}$$

12. **Problem (12):** Solve $\int x^2e^{2x} dx$. *Analysis:* Algebraic (x^2) \times Exponential (e^{2x}). *LIATE:* $u = x^2$. *IBP twice.* **Method 1: Repeated IBP** *IBP 1:* $u_1 = x^2, dv_1 = e^{2x}dx \implies$

$$du_1 = 2x dx, v_1 = \frac{1}{2}e^{2x}.$$

$$I = x^2\left(\frac{1}{2}e^{2x}\right) - \int \left(\frac{1}{2}e^{2x}\right)(2x dx) = \frac{1}{2}x^2e^{2x} - \int xe^{2x} dx$$

IBP 2 (for $\int xe^{2x} dx$): From Problem 2, result is $\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x}$. *Combine:* $I = \frac{1}{2}x^2e^{2x} - (\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x}) + C = \frac{1}{2}x^2e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} + C$. *Factored:* $\frac{e^{2x}}{4}(2x^2 - 2x + 1) + C$.

Method 2: Tabular Method Let $u = x^2, dv = e^{2x} dx$.

Sign	u	dv
+	x^2	e^{2x}
-	$2x$	$\frac{1}{2}e^{2x}$
+	2	$\frac{1}{4}e^{2x}$
-	0	$\frac{1}{8}e^{2x}$

$$\begin{aligned} &= (+)(x^2)\left(\frac{1}{2}e^{2x}\right) + (-)(2x)\left(\frac{1}{4}e^{2x}\right) + (+)(2)\left(\frac{1}{8}e^{2x}\right) + C \\ &= \frac{1}{2}x^2e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} + C \end{aligned}$$

13. **Problem (13):** Solve $\int \ln(2x) dx$. *Analysis:* Logarithmic only. Standard IBP only ($u = \ln(2x)$). (Same as Problem 8) **Method 1: Standard IBP** Let $u = \ln(2x), dv = dx$. $du = \frac{1}{x}dx, v = x$.

$$\int \ln(2x) dx = x \ln(2x) - \int x\left(\frac{1}{x}dx\right) = x \ln(2x) - \int 1 dx = x \ln(2x) - x + C$$

Final Answer: $x(\ln(2x) - 1) + C$.

14. **Problem (14):** Solve $\int \ln(x^2) dx$. *Analysis:* Logarithmic. Standard IBP only. Can simplify first. **Method 1: Simplify First** Assume $x > 0$. $\int \ln(x^2) dx = \int 2 \ln x dx$. Use IBP: $u = \ln x, dv = 2dx$. $du = \frac{1}{x}dx, v = 2x$.

$$\int 2 \ln x dx = (\ln x)(2x) - \int 2x\left(\frac{1}{x}dx\right) = 2x \ln x - \int 2 dx = 2x \ln x - 2x + C$$

Final Answer: $2x(\ln x - 1) + C$.

Method 2: Direct IBP Let $u = \ln(x^2), dv = dx$. $du = \frac{2x}{x^2}dx = \frac{2}{x}dx, v = x$.

$$\begin{aligned} \int \ln(x^2) dx &= x \ln(x^2) - \int x\left(\frac{2}{x}dx\right) = x \ln(x^2) - \int 2 dx \\ &= x \ln(x^2) - 2x + C \end{aligned}$$

(Equivalent result)

15. **Problem (15):** Solve $\int x \ln x \, dx$. *Analysis:* Algebraic \times Logarithmic. LIATE: $u = \ln x$. Standard IBP only. **Method 1: Standard IBP** Let $u = \ln x, dv = x dx$. $du = \frac{1}{x} dx, v = \frac{x^2}{2}$.

$$\begin{aligned}\int x \ln x \, dx &= (\ln x)\left(\frac{x^2}{2}\right) - \int \frac{x^2}{2} \frac{1}{x} dx = \frac{x^2 \ln x}{2} - \int \frac{x}{2} dx \\ &= \frac{x^2 \ln x}{2} - \frac{1}{2}\left(\frac{x^2}{2}\right) + C = \frac{x^2 \ln x}{2} - \frac{x^2}{4} + C\end{aligned}$$

Final Answer: $\frac{x^2}{4}(2 \ln x - 1) + C$.

16. **Problem (16):** Solve $\int x^3 \ln x \, dx$. *Analysis:* Algebraic \times Logarithmic. LIATE: $u = \ln x$. Standard IBP only. **Method 1: Standard IBP** Let $u = \ln x, dv = x^3 dx$. $du = \frac{1}{x} dx, v = \frac{x^4}{4}$.

$$\begin{aligned}\int x^3 \ln x \, dx &= (\ln x)\left(\frac{x^4}{4}\right) - \int \frac{x^4}{4} \frac{1}{x} dx = \frac{x^4 \ln x}{4} - \int \frac{x^3}{4} dx \\ &= \frac{x^4 \ln x}{4} - \frac{1}{4}\left(\frac{x^4}{4}\right) + C = \frac{x^4 \ln x}{4} - \frac{x^4}{16} + C\end{aligned}$$

Final Answer: $\frac{x^4}{16}(4 \ln x - 1) + C$.

17. **Problem (17):** Solve $\int x e^x \, dx$. *Analysis:* Algebraic \times Exponential. LIATE: $u = x$. IBP once. (Same as Problem 1) **Method 1: Standard IBP** Let $u = x, dv = e^x dx$. $du = dx, v = e^x$.

$$\int x e^x \, dx = x e^x - \int e^x \, dx = x e^x - e^x + C$$

Factored: $e^x(x - 1) + C$.

Method 2: Tabular Method Let $u = x, dv = e^x dx$.

Sign	u	dv
+	x	e^x
-	1	e^x
+	0	e^x

$$= (+)(x)(e^x) + (-)(1)(e^x) + C = x e^x - e^x + C$$

18. **Problem (18):** Solve $\int x e^{3x} \, dx$. *Analysis:* Algebraic \times Exponential. LIATE: $u = x$. IBP once. (Same as Problem 5) **Method 1: Standard IBP** Let $u = x, dv = e^{3x} dx$. $du = dx, v = \frac{1}{3} e^{3x}$.

$$\int x e^{3x} \, dx = x\left(\frac{1}{3} e^{3x}\right) - \int \frac{1}{3} e^{3x} \, dx = \frac{1}{3} x e^{3x} - \frac{1}{9} e^{3x} + C$$

Factored: $\frac{e^{3x}}{9}(3x - 1) + C$.

Method 2: Tabular Method Let $u = x, dv = e^{3x} dx$.

Sign	u	dv
+	x	e^{3x}
-	1	$\frac{1}{3}e^{3x}$
+	0	$\frac{1}{9}e^{3x}$

$$= (+)(x)\left(\frac{1}{3}e^{3x}\right) + (-)(1)\left(\frac{1}{9}e^{3x}\right) + C = \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + C$$

19. **Problem (19):** Solve $\int x^2 e^{-x} dx$. *Analysis:* Algebraic (x^2) \times Exponential (e^{-x}). LI-ATE: $u = x^2$. IBP twice. (Same as Problem 10) **Method 1: Repeated IBP** *IBP 1:* $u_1 = x^2, dv_1 = e^{-x} dx \implies du_1 = 2x dx, v_1 = -e^{-x}$.

$$I = x^2(-e^{-x}) - \int (-e^{-x})(2x dx) = -x^2 e^{-x} + 2 \int x e^{-x} dx$$

IBP 2 (for $\int x e^{-x} dx$): $u_2 = x, dv_2 = e^{-x} dx \implies du_2 = dx, v_2 = -e^{-x}$.

$$\int x e^{-x} dx = x(-e^{-x}) - \int (-e^{-x}) dx = -x e^{-x} - e^{-x}$$

Combine: $I = -x^2 e^{-x} + 2(-x e^{-x} - e^{-x}) + C = -x^2 e^{-x} - 2x e^{-x} - 2e^{-x} + C$. *Factored:* $-e^{-x}(x^2 + 2x + 2) + C$.

Method 2: Tabular Method Let $u = x^2, dv = e^{-x} dx$.

Sign	u	dv
+	x^2	e^{-x}
-	$2x$	$-e^{-x}$
+	2	e^{-x}
-	0	$-e^{-x}$

$$= (+)(x^2)(-e^{-x}) + (-)(2x)(e^{-x}) + (+)(2)(-e^{-x}) + C = -x^2 e^{-x} - 2x e^{-x} - 2e^{-x} + C$$

20. **Problem (20):** Solve $\int (x^2 - 2x + 1)e^{2x} dx$. *Analysis:* Note $x^2 - 2x + 1 = (x - 1)^2$. Algebraic \times Exponential. Needs IBP twice. **Method 1: Repeated IBP** Let $u_1 = (x - 1)^2, dv_1 = e^{2x} dx \implies du_1 = 2(x - 1) dx, v_1 = \frac{1}{2}e^{2x}$.

$$I = \frac{1}{2}(x - 1)^2 e^{2x} - \int \frac{1}{2}e^{2x} \cdot 2(x - 1) dx = \frac{1}{2}(x - 1)^2 e^{2x} - \int (x - 1)e^{2x} dx$$

Let $u_2 = x - 1, dv_2 = e^{2x} dx \implies du_2 = dx, v_2 = \frac{1}{2}e^{2x}$.

$$\int (x - 1)e^{2x} dx = (x - 1)\left(\frac{1}{2}e^{2x}\right) - \int \frac{1}{2}e^{2x} dx = \frac{1}{2}(x - 1)e^{2x} - \frac{1}{4}e^{2x}$$

Combine: $I = \frac{1}{2}(x - 1)^2 e^{2x} - \left[\frac{1}{2}(x - 1)e^{2x} - \frac{1}{4}e^{2x}\right] + C$

$$= \frac{e^{2x}}{4}[2(x - 1)^2 - 2(x - 1) + 1] + C = \frac{e^{2x}}{4}(2x^2 - 6x + 5) + C$$

Method 2: Tabular Method Let $u = (x - 1)^2, dv = e^{2x} dx$.

Sign	u	dv
+	$(x-1)^2$	e^{2x}
-	$2(x-1)$	$\frac{1}{2}e^{2x}$
+	2	$\frac{1}{4}e^{2x}$
-	0	$\frac{1}{8}e^{2x}$

$$\begin{aligned}
 &= (x-1)^2\left(\frac{1}{2}e^{2x}\right) - 2(x-1)\left(\frac{1}{4}e^{2x}\right) + 2\left(\frac{1}{8}e^{2x}\right) + C \\
 &= \frac{1}{2}(x-1)^2e^{2x} - \frac{1}{2}(x-1)e^{2x} + \frac{1}{4}e^{2x} + C = \frac{e^{2x}}{4}(2x^2 - 6x + 5) + C
 \end{aligned}$$

21. **Problem (21):** Solve $\int x^3 e^x dx$. *Analysis:* Algebraic (x^3) \times Exponential (e^x). Needs IBP 3 times. **Method 1: Repeated IBP (Full Detail)** *IBP 1:* Let $u_1 = x^3$, $dv_1 = e^x dx$. Then $du_1 = 3x^2 dx$, $v_1 = e^x$.

$$I = \int x^3 e^x dx = x^3 e^x - \int e^x (3x^2 dx) = x^3 e^x - 3 \int x^2 e^x dx \quad (1)$$

IBP 2 (for $\int x^2 e^x dx$): Let $u_2 = x^2$, $dv_2 = e^x dx$. Then $du_2 = 2x dx$, $v_2 = e^x$.

$$\int x^2 e^x dx = x^2 e^x - \int e^x (2x dx) = x^2 e^x - 2 \int x e^x dx \quad (2)$$

IBP 3 (for $\int x e^x dx$): Let $u_3 = x$, $dv_3 = e^x dx$. Then $du_3 = dx$, $v_3 = e^x$.

$$\int x e^x dx = x e^x - \int e^x dx = x e^x - e^x \quad (3)$$

Combine Backwards: Substitute (3) into (2):

$$\int x^2 e^x dx = x^2 e^x - 2(x e^x - e^x) = x^2 e^x - 2x e^x + 2e^x$$

Substitute this result into (1):

$$I = x^3 e^x - 3(x^2 e^x - 2x e^x + 2e^x) + C$$

$$I = x^3 e^x - 3x^2 e^x + 6x e^x - 6e^x + C$$

Factored: $e^x(x^3 - 3x^2 + 6x - 6) + C$.

Method 2: Tabular Method Let $u = x^3$, $dv = e^x dx$.

Sign	u	dv
+	x^3	e^x
-	$3x^2$	e^x
+	$6x$	e^x
-	6	e^x
+	0	e^x

$$= x^3 e^x - 3x^2 e^x + 6x e^x - 6e^x + C = e^x(x^3 - 3x^2 + 6x - 6) + C$$

22. **Problem (22):** Solve $\int p^4 e^{-p} dp$. *Analysis:* Algebraic (p^4) \times Exponential (e^{-p}). Needs IBP 4 times. **Method 1: Repeated IBP (Full Detail)** *IBP 1:* Let $u_1 = p^4, dv_1 = e^{-p} dp$. Then $du_1 = 4p^3 dp, v_1 = -e^{-p}$.

$$I = \int p^4 e^{-p} dp = p^4(-e^{-p}) - \int (-e^{-p})(4p^3 dp) = -p^4 e^{-p} + 4 \int p^3 e^{-p} dp \quad (1)$$

IBP 2 (for $\int p^3 e^{-p} dp$): Let $u_2 = p^3, dv_2 = e^{-p} dp$. Then $du_2 = 3p^2 dp, v_2 = -e^{-p}$.

$$I_2 = \int p^3 e^{-p} dp = p^3(-e^{-p}) - \int (-e^{-p})(3p^2 dp) = -p^3 e^{-p} + 3 \int p^2 e^{-p} dp \quad (2)$$

IBP 3 (for $\int p^2 e^{-p} dp$): Let $u_3 = p^2, dv_3 = e^{-p} dp$. Then $du_3 = 2p dp, v_3 = -e^{-p}$.

$$I_3 = \int p^2 e^{-p} dp = p^2(-e^{-p}) - \int (-e^{-p})(2p dp) = -p^2 e^{-p} + 2 \int p e^{-p} dp \quad (3)$$

IBP 4 (for $\int p e^{-p} dp$): Let $u_4 = p, dv_4 = e^{-p} dp$. Then $du_4 = dp, v_4 = -e^{-p}$.

$$I_4 = \int p e^{-p} dp = p(-e^{-p}) - \int (-e^{-p}) dp = -p e^{-p} + \int e^{-p} dp = -p e^{-p} - e^{-p} \quad (4)$$

Combine Backwards: Substitute (4) into (3): $I_3 = -p^2 e^{-p} + 2(-p e^{-p} - e^{-p}) = -p^2 e^{-p} - 2p e^{-p} - 2e^{-p} = -e^{-p}(p^2 + 2p + 2)$. Substitute I_3 into (2): $I_2 = -p^3 e^{-p} + 3[-e^{-p}(p^2 + 2p + 2)] = -p^3 e^{-p} - 3p^2 e^{-p} - 6p e^{-p} - 6e^{-p} = -e^{-p}(p^3 + 3p^2 + 6p + 6)$. Substitute I_2 into (1): $I = -p^4 e^{-p} + 4[-e^{-p}(p^3 + 3p^2 + 6p + 6)] + C = -p^4 e^{-p} - 4p^3 e^{-p} - 12p^2 e^{-p} - 24p e^{-p} - 24e^{-p} + C$. *Factored:* $-e^{-p}(p^4 + 4p^3 + 12p^2 + 24p + 24) + C$.

Method 2: Tabular Method Let $u = p^4, dv = e^{-p} dp$.

Sign	$u = p^4$	$dv = e^{-p} dp$
+	p^4	e^{-p}
-	$4p^3$	$-e^{-p}$
+	$12p^2$	e^{-p}
-	$24p$	$-e^{-p}$
+	24	e^{-p}
-	0	$-e^{-p}$

$$= p^4(-e^{-p}) - (4p^3)(e^{-p}) + (12p^2)(-e^{-p}) - (24p)(e^{-p}) + (24)(-e^{-p}) + C$$

$$= -p^4 e^{-p} - 4p^3 e^{-p} - 12p^2 e^{-p} - 24p e^{-p} - 24e^{-p} + C$$

Factored: $-e^{-p}(p^4 + 4p^3 + 12p^2 + 24p + 24) + C$.

23. **Problem (23):** Solve $\int (x^2 - 5x)e^x dx$. *Analysis:* Algebraic ($x^2 - 5x$) \times Exponential (e^x). Needs IBP twice. **Method 1: Repeated IBP** *IBP 1:* $u_1 = x^2 - 5x, dv_1 = e^x dx \implies du_1 = (2x - 5)dx, v_1 = e^x$.

$$I = (x^2 - 5x)e^x - \int e^x(2x - 5)dx$$

IBP 2 (for $\int (2x - 5)e^x dx$): $u_2 = 2x - 5, dv_2 = e^x dx \implies du_2 = 2dx, v_2 = e^x$.

$$\int (2x - 5)e^x dx = (2x - 5)e^x - \int e^x(2dx) = (2x - 5)e^x - 2e^x$$

$$\begin{aligned} \text{Combine: } I &= (x^2 - 5x)e^x - [(2x - 5)e^x - 2e^x] + C = e^x(x^2 - 5x - 2x + 5 + 2) + C \\ &= e^x(x^2 - 7x + 7) + C \end{aligned}$$

Method 2: Tabular Method Let $u = x^2 - 5x, dv = e^x dx$.

Sign	u	dv
+	$x^2 - 5x$	e^x
-	$2x - 5$	e^x
+	2	e^x
-	0	e^x

$$= (x^2 - 5x)e^x - (2x - 5)e^x + 2e^x + C = e^x(x^2 - 5x - 2x + 5 + 2) + C = e^x(x^2 - 7x + 7) + C$$

24. **Problem (24):** Solve $\int (r^2 + r + 1)e^r dr$. *Analysis:* Algebraic $(r^2 + r + 1) \times$ Exponential (e^r) . Needs IBP twice. **Method 1: Repeated IBP** *IBP 1:* $u_1 = r^2 + r + 1, dv_1 = e^r dr \implies du_1 = (2r + 1)dr, v_1 = e^r$.

$$I = (r^2 + r + 1)e^r - \int e^r(2r + 1)dr$$

IBP 2 (for $\int (2r + 1)e^r dr$): $u_2 = 2r + 1, dv_2 = e^r dr \implies du_2 = 2dr, v_2 = e^r$.

$$\int (2r + 1)e^r dr = (2r + 1)e^r - \int e^r(2dr) = (2r + 1)e^r - 2e^r$$

$$\begin{aligned} \text{Combine: } I &= (r^2 + r + 1)e^r - [(2r + 1)e^r - 2e^r] + C = e^r(r^2 + r + 1 - 2r - 1 + 2) + C \\ &= e^r(r^2 - r + 2) + C \end{aligned}$$

Method 2: Tabular Method Let $u = r^2 + r + 1, dv = e^r dr$.

Sign	u	dv
+	$r^2 + r + 1$	e^r
-	$2r + 1$	e^r
+	2	e^r
-	0	e^r

$$= (r^2 + r + 1)e^r - (2r + 1)e^r + 2e^r + C = e^r(r^2 + r + 1 - 2r - 1 + 2) + C = e^r(r^2 - r + 2) + C$$

25. **Problem (25):** Solve $\int x^5 e^x dx$. *Analysis:* Algebraic (x^5) \times Exponential (e^x). Needs IBP 5 times. **Method 1: Repeated IBP (Full Detail)** *IBP 1:* $u_1 = x^5, dv_1 = e^x dx \implies du_1 = 5x^4 dx, v_1 = e^x$.

$$I = x^5 e^x - 5 \int x^4 e^x dx$$

$$\text{IBP 2: } u_2 = x^4, dv_2 = e^x dx \implies du_2 = 4x^3 dx, v_2 = e^x.$$

$$\int x^4 e^x dx = x^4 e^x - 4 \int x^3 e^x dx$$

$$I = x^5 e^x - 5[x^4 e^x - 4 \int x^3 e^x dx] = x^5 e^x - 5x^4 e^x + 20 \int x^3 e^x dx. \text{ IBP 3: } u_3 = x^3, dv_3 = e^x dx \implies du_3 = 3x^2 dx, v_3 = e^x.$$

$$\int x^3 e^x dx = x^3 e^x - 3 \int x^2 e^x dx$$

$$I = x^5 e^x - 5x^4 e^x + 20[x^3 e^x - 3 \int x^2 e^x dx] = x^5 e^x - 5x^4 e^x + 20x^3 e^x - 60 \int x^2 e^x dx. \text{ IBP 4: } u_4 = x^2, dv_4 = e^x dx \implies du_4 = 2x dx, v_4 = e^x.$$

$$\int x^2 e^x dx = x^2 e^x - 2 \int x e^x dx$$

$$I = x^5 e^x - 5x^4 e^x + 20x^3 e^x - 60[x^2 e^x - 2 \int x e^x dx] = x^5 e^x - 5x^4 e^x + 20x^3 e^x - 60x^2 e^x + 120 \int x e^x dx. \text{ IBP 5: } u_5 = x, dv_5 = e^x dx \implies du_5 = dx, v_5 = e^x.$$

$$\int x e^x dx = x e^x - \int e^x dx = x e^x - e^x$$

$$\text{Combine All: } I = x^5 e^x - 5x^4 e^x + 20x^3 e^x - 60x^2 e^x + 120[x e^x - e^x] + C. I = x^5 e^x - 5x^4 e^x + 20x^3 e^x - 60x^2 e^x + 120x e^x - 120e^x + C. \text{ Factored: } e^x(x^5 - 5x^4 + 20x^3 - 60x^2 + 120x - 120) + C.$$

Method 2: Tabular Method Let $u = x^5, dv = e^x dx$.

Sign	$u = x^5$	$dv = e^x dx$
+	x^5	e^x
-	$5x^4$	e^x
+	$20x^3$	e^x
-	$60x^2$	e^x
+	$120x$	e^x
-	120	e^x
+	0	e^x

$$= x^5 e^x - 5x^4 e^x + 20x^3 e^x - 60x^2 e^x + 120x e^x - 120e^x + C$$

$$= e^x(x^5 - 5x^4 + 20x^3 - 60x^2 + 120x - 120) + C$$

26. **Problem (26):** Solve $\int t^2 e^{4t} dt$. *Analysis:* Algebraic (t^2) \times Exponential (e^{4t}). Needs IBP twice. **Method 1: Repeated IBP** *IBP 1:* $u_1 = t^2, dv_1 = e^{4t} dt \implies du_1 =$

$$2t dt, v_1 = \frac{1}{4}e^{4t}.$$

$$I = t^2\left(\frac{1}{4}e^{4t}\right) - \int \frac{1}{4}e^{4t}(2t dt) = \frac{1}{4}t^2e^{4t} - \frac{1}{2} \int te^{4t} dt$$

$$IBP \ 2 \ (for \ \int te^{4t} dt): \ u_2 = t, dv_2 = e^{4t} dt \implies du_2 = dt, v_2 = \frac{1}{4}e^{4t}.$$

$$\int te^{4t} dt = t\left(\frac{1}{4}e^{4t}\right) - \int \frac{1}{4}e^{4t} dt = \frac{1}{4}te^{4t} - \frac{1}{16}e^{4t}$$

$$Combine: \ I = \frac{1}{4}t^2e^{4t} - \frac{1}{2}\left[\frac{1}{4}te^{4t} - \frac{1}{16}e^{4t}\right] + C = \frac{1}{4}t^2e^{4t} - \frac{1}{8}te^{4t} + \frac{1}{32}e^{4t} + C. \quad Factored: \frac{e^{4t}}{32}(8t^2 - 4t + 1) + C.$$

Method 2: Tabular Method Let $u = t^2, dv = e^{4t} dt$.

Sign	$u = t^2$	$dv = e^{4t} dt$
+	t^2	e^{4t}
-	$2t$	$\frac{1}{4}e^{4t}$
+	2	$\frac{1}{16}e^{4t}$
-	0	$\frac{1}{64}e^{4t}$

$$\begin{aligned} &= t^2\left(\frac{1}{4}e^{4t}\right) - (2t)\left(\frac{1}{16}e^{4t}\right) + 2\left(\frac{1}{64}e^{4t}\right) + C \\ &= \frac{1}{4}t^2e^{4t} - \frac{1}{8}te^{4t} + \frac{1}{32}e^{4t} + C = \frac{e^{4t}}{32}(8t^2 - 4t + 1) + C \end{aligned}$$