TECHNIQUES OF DIFFERENTIATION: QUOTIENT & CHAIN RULES - WEEK 6

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1 Introduction: Quotient and Chain Rules

Explanation. This document provides detailed solutions using two essential differentiation rules:

Quotient Rule: Used to differentiate a function that is the ratio (division) of two other differentiable functions. If $y = \frac{u(x)}{v(x)}$, where $v(x) \neq 0$, its derivative is:

$$\frac{dy}{dx} = \frac{v(x)\frac{d}{dx}[u(x)] - u(x)\frac{d}{dx}[v(x)]}{[v(x)]^2} \quad \text{or} \quad y' = \frac{vu' - uv'}{v^2}$$

Chain Rule: Used to differentiate composite functions. If $y = f(g(x))$, then:

$$\frac{dy}{dx} = f'(g(x)) \cdot g'(x)$$

2 Solutions: Quotient Rule

2.1 Example

1. Find the derivative of $y = \frac{x-1}{2x+3}$.

   Solution: Let $u = x - 1$ and $v = 2x + 3$. Then $u' = 1$ and $v' = 2$. Using the Quotient Rule:
   $$\frac{dy}{dx} = \frac{vu' - uv'}{v^2} = \frac{(2x+3)(1) - (x-1)(2)}{(2x+3)^2}$$
   Simplify:
   $$= \frac{(2x+3) - (2x-2)}{(2x+3)^2} = \frac{5}{(2x+3)^2}$$

3 Solutions: Quotient Rule

3.1 Example

1. **Problem:** Find the derivative of $y = \dfrac{x-1}{2x+3}$.

   Solution:

   Method 1: Quotient Rule

Strategy. Identify numerator $u = x - 1$ and denominator $v = 2x + 3$. Calculate du/dx and dv/dx. Apply the formula $y' = (vu' - uv')/v^2$.

Step 1: Find derivatives of numerator and denominator. $u = x-1 \implies u' = \frac{du}{dx} = 1$. $v = 2x + 3 \implies v' = \frac{dv}{dx} = 2$. *Step 2: Apply Quotient Rule formula.*

$$\frac{dy}{dx} = \frac{vu' - uv'}{v^2} = \frac{(2x+3)(1) - (x-1)(2)}{(2x+3)^2}$$

Step 3: Simplify the numerator.

$$= \frac{(2x+3) - (2x-2)}{(2x+3)^2} = \frac{2x+3-2x+2}{(2x+3)^2} = \frac{5}{(2x+3)^2}$$

Method 2: Product Rule with Negative Exponent

Strategy. Rewrite $y = (x-1)(2x+3)^{-1}$. Use Product Rule $y' = fg' + gf'$, requiring Chain Rule for the second factor.

Step 1: Identify factors and derivatives. Let $f(x) = x - 1 \implies f'(x) = 1$. Let $g(x) = (2x+3)^{-1}$. Use Chain Rule: $g'(x) = -1(2x+3)^{-2} \cdot (2) = -2(2x+3)^{-2}$. *Step 2: Apply Product Rule.*

$$\frac{dy}{dx} = f(x)g'(x) + g(x)f'(x) = (x-1)[-2(2x+3)^{-2}] + (2x+3)^{-1}(1)$$

$$= \frac{-2(x-1)}{(2x+3)^2} + \frac{1}{2x+3}$$

Step 3: Combine using common denominator $(2x+3)^2$.

$$= \frac{-2x+2}{(2x+3)^2} + \frac{1(2x+3)}{(2x+3)^2} = \frac{-2x+2+2x+3}{(2x+3)^2} = \frac{5}{(2x+3)^2}$$

Results match.

3.2 Solutions: ClassWork Problems (Quotient Rule Section)

2. **Problem (2):** Find derivative of $y = \dfrac{2x+5}{3x-2}$. *Solution (Quotient Rule):* $u = 2x + 5 \implies u' = 2$. $v = 3x - 2 \implies v' = 3$.

$$\frac{dy}{dx} = \frac{vu' - uv'}{v^2} = \frac{(3x-2)(2) - (2x+5)(3)}{(3x-2)^2}$$

$$= \frac{(6x-4) - (6x+15)}{(3x-2)^2} = \frac{6x-4-6x-15}{(3x-2)^2} = \frac{-19}{(3x-2)^2}$$

3. **Problem (3):** Find derivative of $z = \dfrac{4-3x}{3x^2+x}$. *Solution (Quotient Rule):* $u = 4-3x \implies u' = -3$. $v = 3x^2+x \implies v' = 6x+1$.
$$\frac{dz}{dx} = \frac{vu' - uv'}{v^2} = \frac{(3x^2+x)(-3) - (4-3x)(6x+1)}{(3x^2+x)^2}$$
Numerator $= (-9x^2-3x) - (24x+4-18x^2-3x) = -9x^2-3x+18x^2-21x-4 = 9x^2-24x-4$.
$$\frac{dz}{dx} = \frac{9x^2-24x-4}{(3x^2+x)^2} = \frac{9x^2-24x-4}{x^2(3x+1)^2}$$

4. **Problem (4):** Find derivative of $g(x) = \dfrac{x^2-4}{x+0.5}$. *Solution (Quotient Rule):* $u = x^2-4 \implies u' = 2x$. $v = x+0.5 \implies v' = 1$.
$$g'(x) = \frac{vu' - uv'}{v^2} = \frac{(x+0.5)(2x) - (x^2-4)(1)}{(x+0.5)^2}$$
$$= \frac{2x^2+x-x^2+4}{(x+0.5)^2} = \frac{x^2+x+4}{(x+0.5)^2}$$

5. **Problem (5):** Find derivative of $f(t) = \dfrac{t^2-1}{t^2+t-2}$. *Solution (Simplify First Recommended):* Factor: $f(t) = \frac{(t-1)(t+1)}{(t+2)(t-1)}$. For $t \neq 1$, $f(t) = \frac{t+1}{t+2}$. Differentiate simplified form using Quotient Rule: $u = t+1 \implies u' = 1$. $v = t+2 \implies v' = 1$.
$$f'(t) = \frac{(t+2)(1) - (t+1)(1)}{(t+2)^2} = \frac{t+2-t-1}{(t+2)^2} = \frac{1}{(t+2)^2}$$

6. **Problem (6):** Find derivative of $v = (1-t)(1+t^2)^{-1}$. *Solution (Rewrite as Quotient):* $v = \frac{1-t}{1+t^2}$. Use Quotient Rule. $u = 1-t \implies u' = -1$. $w = 1+t^2 \implies w' = 2t$.
$$\frac{dv}{dt} = \frac{wu' - uw'}{w^2} = \frac{(1+t^2)(-1) - (1-t)(2t)}{(1+t^2)^2}$$
$$= \frac{-1-t^2-(2t-2t^2)}{(1+t^2)^2} = \frac{-1-t^2-2t+2t^2}{(1+t^2)^2} = \frac{t^2-2t-1}{(1+t^2)^2}$$

7. **Problem (7):** Find derivative of $w = (2x-7)^{-1}(x+5)$. *Solution (Rewrite as Quotient):* $w = \frac{x+5}{2x-7}$. Use Quotient Rule. $u = x+5 \implies u' = 1$. $v = 2x-7 \implies v' = 2$.
$$\frac{dw}{dx} = \frac{vu' - uv'}{v^2} = \frac{(2x-7)(1) - (x+5)(2)}{(2x-7)^2}$$
$$= \frac{2x-7-(2x+10)}{(2x-7)^2} = \frac{2x-7-2x-10}{(2x-7)^2} = \frac{-17}{(2x-7)^2}$$

8. **Problem (8):** Find derivative of $f(s) = \dfrac{\sqrt{s}-1}{\sqrt{s}+1}$. *Solution (Quotient Rule):* Rewrite $u = s^{1/2}-1 \implies u' = \frac{1}{2}s^{-1/2}$. $v = s^{1/2}+1 \implies v' = \frac{1}{2}s^{-1/2}$.
$$f'(s) = \frac{vu' - uv'}{v^2} = \frac{(s^{1/2}+1)(\frac{1}{2}s^{-1/2}) - (s^{1/2}-1)(\frac{1}{2}s^{-1/2})}{(s^{1/2}+1)^2}$$

Factor out $\frac{1}{2}s^{-1/2}$ from numerator:

$$= \frac{\frac{1}{2}s^{-1/2}[(s^{1/2}+1)-(s^{1/2}-1)]}{(s^{1/2}+1)^2} = \frac{\frac{1}{2}s^{-1/2}[2]}{(s^{1/2}+1)^2}$$

$$= \frac{s^{-1/2}}{(s^{1/2}+1)^2} = \frac{1}{\sqrt{s}(\sqrt{s}+1)^2}$$

9. **Problem (9):** Find derivative of $u = \dfrac{5x+1}{2\sqrt{x}}$. *Solution (Quotient Rule):* $f = 5x+1 \implies f' = 5$. $g = 2x^{1/2} \implies g' = x^{-1/2}$.

$$\frac{du}{dx} = \frac{gf' - fg'}{g^2} = \frac{(2x^{1/2})(5) - (5x+1)(x^{-1/2})}{(2x^{1/2})^2}$$

$$= \frac{10x^{1/2} - 5x^{1/2} - x^{-1/2}}{4x} = \frac{5x^{1/2} - x^{-1/2}}{4x}$$

Multiply top/bottom by $x^{1/2}$: $\frac{5x-1}{4x^{3/2}}$.

10. **Problem (10):** Find derivative of $v = \dfrac{1+x-4\sqrt{x}}{x}$. *Solution (Simplify First):* $v = x^{-1} + 1 - 4x^{-1/2}$.

$$\frac{dv}{dx} = -1x^{-2} + 0 - 4(-\frac{1}{2}x^{-3/2}) = -x^{-2} + 2x^{-3/2}$$

Optional rewrite: $= -\frac{1}{x^2} + \frac{2}{x^{3/2}}$

11. **Problem (11):** Find derivative of $r = 2\left(\dfrac{1}{\sqrt{\theta}} + \sqrt{\theta}\right)$. *Solution (Simplify First):* $r = 2(\theta^{-1/2} + \theta^{1/2}) = 2\theta^{-1/2} + 2\theta^{1/2}$.

$$\frac{dr}{d\theta} = 2(-\frac{1}{2}\theta^{-3/2}) + 2(\frac{1}{2}\theta^{-1/2}) = -\theta^{-3/2} + \theta^{-1/2}$$

Optional rewrite: $= -\frac{1}{\theta^{3/2}} + \frac{1}{\theta^{1/2}}$

12. **Problem (12):** Find derivative of $y = \dfrac{1}{(x^2-1)(x^2+x+1)}$. *Solution (Expand Denominator First):* Denominator $v = x^4+x^3-x-1$. Rewrite $y = (x^4+x^3-x-1)^{-1}$. Use Chain Rule. $u(x) = x^4+x^3-x-1 \implies u'(x) = 4x^3+3x^2-1$.

$$\frac{dy}{dx} = -1(x^4+x^3-x-1)^{-2} \cdot (4x^3+3x^2-1) = \frac{-(4x^3+3x^2-1)}{(x^4+x^3-x-1)^2}$$

13. **Problem (13):** Find derivative of $y = \dfrac{(x+1)(x+2)}{(x-1)(x-2)}$. *Solution (Expand then Quotient Rule):* $u = (x+1)(x+2) = x^2+3x+2 \implies u' = 2x+3$. $v = (x-1)(x-2) = x^2-3x+2 \implies v' = 2x-3$.

$$\frac{dy}{dx} = \frac{vu' - uv'}{v^2} = \frac{(x^2-3x+2)(2x+3) - (x^2+3x+2)(2x-3)}{(x^2-3x+2)^2}$$

Numerator $= (2x^3 - 3x^2 - 5x + 6) - (2x^3 + 3x^2 - 5x - 6) = -6x^2 + 12$.

$$\frac{dy}{dx} = \frac{-6x^2 + 12}{((x-1)(x-2))^2} = \frac{-6(x^2 - 2)}{(x-1)^2(x-2)^2}$$

14. **Problem (14):** Find derivative of $y = 2e^{-x} + e^{3x}$. *Solution (Sum Rule, Chain Rule):*
$$\frac{dy}{dx} = 2 \cdot \frac{d}{dx}(e^{-x}) + \frac{d}{dx}(e^{3x})$$
$$= 2(e^{-x} \cdot (-1)) + (e^{3x} \cdot 3) = -2e^{-x} + 3e^{3x}$$

15. **Problem (15):** Find derivative of $y = \dfrac{x^2 + 3e^x}{2e^x - x}$. *Solution (Quotient Rule):* $u = x^2 + 3e^x \implies u' = 2x + 3e^x$. $v = 2e^x - x \implies v' = 2e^x - 1$.
$$\frac{dy}{dx} = \frac{vu' - uv'}{v^2} = \frac{(2e^x - x)(2x + 3e^x) - (x^2 + 3e^x)(2e^x - 1)}{(2e^x - x)^2}$$
Numerator $= (4xe^x + 6e^{2x} - 2x^2 - 3xe^x) - (2x^2e^x - x^2 + 6e^{2x} - 3e^x) = xe^x + 6e^{2x} - 2x^2 - 2x^2e^x + x^2 - 6e^{2x} + 3e^x = xe^x(1 - 2x) - x^2 + 3e^x$.
$$\frac{dy}{dx} = \frac{xe^x(1 - 2x) - x^2 + 3e^x}{(2e^x - x)^2}$$

16. **Problem (16):** Find derivative of $s = \dfrac{t^2 + 5t - 1}{t^2}$. *Solution (Simplify First):* $s = \frac{t^2}{t^2} + \frac{5t}{t^2} - \frac{1}{t^2} = 1 + 5t^{-1} - t^{-2}$.
$$\frac{ds}{dt} = 0 + 5(-1t^{-2}) - (-2t^{-3}) = -5t^{-2} + 2t^{-3}$$
Optional rewrite: $= -\frac{5}{t^2} + \frac{2}{t^3}$

17. **Problem (17):** Find derivative of $u = \dfrac{(x^2 + x)(x^2 - x + 1)}{x^4}$. *Solution (Simplify First):* Numerator is $x(x+1)(x^2 - x + 1) = x(x^3 + 1) = x^4 + x$. So, $u = \frac{x^4 + x}{x^4} = \frac{x^4}{x^4} + \frac{x}{x^4} = 1 + x^{-3}$.
$$\frac{du}{dx} = 0 + (-3x^{-4}) = -3x^{-4} = -\frac{3}{x^4}$$

18. **Problem (18):** Find derivative of $y = \dfrac{x^3 + 7}{x}$. *Solution (Simplify First):* $y = \frac{x^3}{x} + \frac{7}{x} = x^2 + 7x^{-1}$.
$$\frac{dy}{dx} = 2x + 7(-1x^{-2}) = 2x - 7x^{-2} = 2x - \frac{7}{x^2}$$

19. **Problem (19):** Find derivative of $p = \dfrac{q^2 + 3}{(q-1)^3 + (q+1)^3}$. *Solution (Expand Denominator First):* Denominator $v = (q^3 - 3q^2 + 3q - 1) + (q^3 + 3q^2 + 3q + 1) = 2q^3 + 6q$.

So, $p = \frac{q^2+3}{2q^3+6q}$. Use Quotient Rule. $u = q^2 + 3 \implies u' = 2q$. $v = 2q^3 + 6q \implies v' = 6q^2 + 6$.

$$\frac{dp}{dq} = \frac{vu' - uv'}{v^2} = \frac{(2q^3 + 6q)(2q) - (q^2 + 3)(6q^2 + 6)}{(2q^3 + 6q)^2}$$

Numerator $= (4q^4 + 12q^2) - (6q^4 + 6q^2 + 18q^2 + 18) = 4q^4 + 12q^2 - 6q^4 - 24q^2 - 18 = -2q^4 - 12q^2 - 18$. Denominator $= (2q(q^2 + 3))^2 = 4q^2(q^2 + 3)^2$.

$$\frac{dp}{dq} = \frac{-2q^4 - 12q^2 - 18}{4q^2(q^2 + 3)^2} = \frac{-2(q^4 + 6q^2 + 9)}{4q^2(q^2 + 3)^2} = \frac{-2(q^2 + 3)^2}{4q^2(q^2 + 3)^2} = \frac{-1}{2q^2}$$

20. **Problem (20):** Find derivative of $r = \dfrac{(\theta - 1)(\theta^2 + \theta + 1)}{\theta^3}$. *Solution (Simplify Numerator First):* Numerator is difference of cubes: $(\theta - 1)(\theta^2 + \theta + 1) = \theta^3 - 1$. So, $r = \frac{\theta^3 - 1}{\theta^3} = \frac{\theta^3}{\theta^3} - \frac{1}{\theta^3} = 1 - \theta^{-3}$.

$$\frac{dr}{d\theta} = 0 - (-3\theta^{-4}) = 3\theta^{-4} = \frac{3}{\theta^4}$$

21. **Problem (21):** Find derivative of $w = \left(1 + \dfrac{1}{z}\right)(3 - z)$. *Solution (Expand First):* $w = (1 + z^{-1})(3 - z) = 1(3) + 1(-z) + z^{-1}(3) + z^{-1}(-z)$ $w = 3 - z + 3z^{-1} - z^0 = 3 - z + 3z^{-1} - 1 = 2 - z + 3z^{-1}$.

$$\frac{dw}{dz} = 0 - 1 + 3(-1z^{-2}) = -1 - 3z^{-2} = -1 - \frac{3}{z^2}$$

4 Solutions: Chain Rule

4.1 Example

21. **Problem:** Find the derivative of: (a) $y = \dfrac{1}{x + 1}$ and (b) $y = \sqrt{3x^2 - x + 1}$.

Solution (a): $y = \dfrac{1}{x + 1}$ **Method: Chain Rule (General Power Rule)** Rewrite: $y = (x + 1)^{-1}$. Identify: Outer function $f(u) = u^{-1}$, Inner function $u(x) = x + 1$. Derivatives: $f'(u) = -1u^{-2}$, $u'(x) = 1$. Apply Chain Rule: $\frac{dy}{dx} = f'(u(x)) \cdot u'(x) = -1(x + 1)^{-2} \cdot (1)$.

$$= -(x + 1)^{-2} = -\frac{1}{(x + 1)^2}$$

Solution (b): $y = \sqrt{3x^2 - x + 1}$ **Method: Chain Rule (General Power Rule)** Rewrite: $y = (3x^2 - x + 1)^{1/2}$. Identify: Outer function $f(u) = u^{1/2}$, Inner function $u(x) = 3x^2 - x + 1$. Derivatives: $f'(u) = \frac{1}{2}u^{-1/2} = \frac{1}{2\sqrt{u}}$, $u'(x) = 6x - 1$. Apply Chain Rule: $\frac{dy}{dx} = f'(u(x)) \cdot u'(x) = \frac{1}{2\sqrt{3x^2 - x + 1}} \cdot (6x - 1)$.

$$= \frac{6x - 1}{2\sqrt{3x^2 - x + 1}}$$

4.2 Solutions: ClassWork Problems (Chain Rule Section)

22. **Problem (22):** Differentiate $f(x) = (3x - 2x^2)^3$. *Solution (Chain Rule - General Power Rule):* Outer: $g(u) = u^3 \implies g'(u) = 3u^2$. Inner: $u(x) = 3x - 2x^2 \implies u'(x) = 3 - 4x$.

$$f'(x) = g'(u(x)) \cdot u'(x) = 3(3x - 2x^2)^2 \cdot (3 - 4x)$$
$$= 3(3 - 4x)(3x - 2x^2)^2$$

23. **Problem (23):** Differentiate $y = (x^2 + 3x)^4$. *Solution (Chain Rule - General Power Rule):* Outer: $f(u) = u^4 \implies f'(u) = 4u^3$. Inner: $u(x) = x^2 + 3x \implies u'(x) = 2x + 3$.

$$\frac{dy}{dx} = f'(u(x)) \cdot u'(x) = 4(x^2 + 3x)^3 \cdot (2x + 3)$$
$$= 4(2x + 3)(x^2 + 3x)^3$$

24. **Problem (24):** Differentiate $y = (x^2 + 1)^3$. *Solution (Chain Rule - General Power Rule):* Outer: $f(u) = u^3 \implies f'(u) = 3u^2$. Inner: $u(x) = x^2 + 1 \implies u'(x) = 2x$.

$$\frac{dy}{dx} = f'(u(x)) \cdot u'(x) = 3(x^2 + 1)^2 \cdot (2x)$$
$$= 6x(x^2 + 1)^2$$

25. **Problem (25):** Differentiate $y = (x^3 + 1)^2$. *Solution (Chain Rule - General Power Rule):* Outer: $f(u) = u^2 \implies f'(u) = 2u$. Inner: $u(x) = x^3 + 1 \implies u'(x) = 3x^2$.

$$\frac{dy}{dx} = f'(u(x)) \cdot u'(x) = 2(x^3 + 1) \cdot (3x^2)$$
$$= 6x^2(x^3 + 1)$$

26. **Problem (26):** Differentiate $y = \sqrt[3]{(x^2 + 4)^2}$. *Solution (Chain Rule - General Power Rule):* Rewrite: $y = ((x^2 + 4)^2)^{1/3} = (x^2 + 4)^{2/3}$. Outer: $f(u) = u^{2/3} \implies f'(u) = \frac{2}{3}u^{-1/3}$. Inner: $u(x) = x^2 + 4 \implies u'(x) = 2x$.

$$\frac{dy}{dx} = f'(u(x)) \cdot u'(x) = \frac{2}{3}(x^2 + 4)^{-1/3} \cdot (2x)$$
$$= \frac{4x}{3}(x^2 + 4)^{-1/3} = \frac{4x}{3(x^2 + 4)^{1/3}} = \frac{4x}{3\sqrt[3]{x^2 + 4}}$$

27. **Problem (27):** Differentiate $y = \dfrac{3}{x^2 + 1}$. *Solution (Chain Rule - General Power Rule):* Rewrite: $y = 3(x^2 + 1)^{-1}$. Outer: $f(u) = u^{-1} \implies f'(u) = -u^{-2}$. (Constant multiple 3 applied later) Inner: $u(x) = x^2 + 1 \implies u'(x) = 2x$.

$$\frac{dy}{dx} = 3 \cdot [f'(u(x)) \cdot u'(x)] = 3 \cdot [-(x^2 + 1)^{-2} \cdot (2x)]$$
$$= 3(-2x)(x^2 + 1)^{-2} = -6x(x^2 + 1)^{-2} = \frac{-6x}{(x^2 + 1)^2}$$

28. **Problem (28):** Differentiate $y = \dfrac{4}{2x+1}$. *Solution (Chain Rule - General Power Rule):* Rewrite: $y = 4(2x+1)^{-1}$. Outer: $f(u) = u^{-1} \implies f'(u) = -u^{-2}$. Inner: $u(x) = 2x+1 \implies u'(x) = 2$.

$$\frac{dy}{dx} = 4 \cdot [f'(u(x)) \cdot u'(x)] = 4 \cdot [-(2x+1)^{-2} \cdot (2)]$$

$$= 4(-2)(2x+1)^{-2} = -8(2x+1)^{-2} = \frac{-8}{(2x+1)^2}$$

29. **Problem (29):** Differentiate $y = \dfrac{2}{(x-1)^3}$. *Solution (Chain Rule - General Power Rule):* Rewrite: $y = 2(x-1)^{-3}$. Outer: $f(u) = u^{-3} \implies f'(u) = -3u^{-4}$. Inner: $u(x) = x-1 \implies u'(x) = 1$.

$$\frac{dy}{dx} = 2 \cdot [f'(u(x)) \cdot u'(x)] = 2 \cdot [-3(x-1)^{-4} \cdot (1)]$$

$$= -6(x-1)^{-4} = \frac{-6}{(x-1)^4}$$

30. **Problem (30):** Differentiate $y = x^2\sqrt{1-x^2}$. *Solution (Product Rule and Chain Rule):* Rewrite: $y = x^2(1-x^2)^{1/2}$. Let $f(x) = x^2 \implies f'(x) = 2x$. Let $g(x) = (1-x^2)^{1/2}$. Find $g'(x)$ using Chain Rule: Outer: $h(u) = u^{1/2} \implies h'(u) = \frac{1}{2}u^{-1/2}$. Inner: $u(x) = 1-x^2 \implies u'(x) = -2x$. $g'(x) = h'(u(x)) \cdot u'(x) = \frac{1}{2}(1-x^2)^{-1/2} \cdot (-2x) = -x(1-x^2)^{-1/2} = \frac{-x}{\sqrt{1-x^2}}$. Apply Product Rule: $y' = fg' + gf'$

$$\frac{dy}{dx} = (x^2)\left(\frac{-x}{\sqrt{1-x^2}}\right) + (1-x^2)^{1/2}(2x)$$

$$= \frac{-x^3}{\sqrt{1-x^2}} + 2x\sqrt{1-x^2}$$

Combine using common denominator $\sqrt{1-x^2}$:

$$= \frac{-x^3}{\sqrt{1-x^2}} + \frac{2x\sqrt{1-x^2}\sqrt{1-x^2}}{\sqrt{1-x^2}} = \frac{-x^3 + 2x(1-x^2)}{\sqrt{1-x^2}}$$

$$= \frac{-x^3 + 2x - 2x^3}{\sqrt{1-x^2}} = \frac{2x - 3x^3}{\sqrt{1-x^2}}$$

31. **Problem (31):** Differentiate $y = \dfrac{3}{(x+1)^2}$. *Solution (Chain Rule - General Power Rule):* Rewrite: $y = 3(x+1)^{-2}$. Outer: $f(u) = u^{-2} \implies f'(u) = -2u^{-3}$. Inner: $u(x) = x+1 \implies u'(x) = 1$.

$$\frac{dy}{dx} = 3 \cdot [f'(u(x)) \cdot u'(x)] = 3 \cdot [-2(x+1)^{-3} \cdot (1)]$$

$$= -6(x+1)^{-3} = \frac{-6}{(x+1)^3}$$

32. **Problem (32):** Differentiate $f(x) = \left(\frac{x+1}{x-5}\right)^2$. *Solution (Chain Rule and Quotient Rule):* Outer: $g(u) = u^2 \implies g'(u) = 2u$. Inner: $u(x) = \frac{x+1}{x-5}$. Find $u'(x)$ using Quotient Rule: $u_{num} = x + 1 \implies u'_{num} = 1$. $v_{den} = x - 5 \implies v'_{den} = 1$. $u'(x) = \frac{v_{den}u'_{num} - u_{num}v'_{den}}{v_{den}^2} = \frac{(x-5)(1)-(x+1)(1)}{(x-5)^2} = \frac{x-5-x-1}{(x-5)^2} = \frac{-6}{(x-5)^2}$. Apply Chain Rule: $f'(x) = g'(u(x)) \cdot u'(x)$.

$$f'(x) = 2\left(\frac{x+1}{x-5}\right) \cdot \left(\frac{-6}{(x-5)^2}\right)$$
$$= \frac{2(x+1)(-6)}{(x-5)(x-5)^2} = \frac{-12(x+1)}{(x-5)^3}$$

33. **Problem (33):** Differentiate $f(x) = \left(\frac{3x-1}{x^2+3}\right)^2$. *Solution (Chain Rule and Quotient Rule):* Outer: $g(u) = u^2 \implies g'(u) = 2u$. Inner: $u(x) = \frac{3x-1}{x^2+3}$. Find $u'(x)$ using Quotient Rule: $u_{num} = 3x - 1 \implies u'_{num} = 3$. $v_{den} = x^2 + 3 \implies v'_{den} = 2x$. $u'(x) = \frac{(x^2+3)(3)-(3x-1)(2x)}{(x^2+3)^2} = \frac{3x^2+9-(6x^2-2x)}{(x^2+3)^2} = \frac{3x^2+9-6x^2+2x}{(x^2+3)^2} = \frac{-3x^2+2x+9}{(x^2+3)^2}$. Apply Chain Rule: $f'(x) = g'(u(x)) \cdot u'(x)$.

$$f'(x) = 2\left(\frac{3x-1}{x^2+3}\right) \cdot \left(\frac{-3x^2+2x+9}{(x^2+3)^2}\right)$$
$$= \frac{2(3x-1)(-3x^2+2x+9)}{(x^2+3)^3}$$

34. **Problem (34):** Differentiate $y = x^2\sqrt{x^2+1}$. *Solution (Product Rule and Chain Rule):* Rewrite: $y = x^2(x^2+1)^{1/2}$. Let $f(x) = x^2 \implies f'(x) = 2x$. Let $g(x) = (x^2+1)^{1/2}$. Find $g'(x)$ using Chain Rule: Outer: $h(u) = u^{1/2} \implies h'(u) = \frac{1}{2}u^{-1/2}$. Inner: $u(x) = x^2+1 \implies u'(x) = 2x$. $g'(x) = h'(u(x)) \cdot u'(x) = \frac{1}{2}(x^2+1)^{-1/2} \cdot (2x) = x(x^2+1)^{-1/2} = \frac{x}{\sqrt{x^2+1}}$. Apply Product Rule: $y' = fg' + gf'$

$$\frac{dy}{dx} = (x^2)\left(\frac{x}{\sqrt{x^2+1}}\right) + (x^2+1)^{1/2}(2x)$$
$$= \frac{x^3}{\sqrt{x^2+1}} + 2x\sqrt{x^2+1}$$

Combine using common denominator $\sqrt{x^2+1}$:

$$= \frac{x^3}{\sqrt{x^2+1}} + \frac{2x(x^2+1)}{\sqrt{x^2+1}} = \frac{x^3+2x^3+2x}{\sqrt{x^2+1}} = \frac{3x^3+2x}{\sqrt{x^2+1}}$$

35. **Problem (35):** Differentiate $y = \frac{5}{(1-5x)^{2/3}}$. *Solution (Chain Rule - General Power Rule):* Rewrite: $y = 5(1-5x)^{-2/3}$. Outer: $f(u) = u^{-2/3} \implies f'(u) = -\frac{2}{3}u^{-5/3}$. Inner: $u(x) = 1 - 5x \implies u'(x) = -5$.

$$\frac{dy}{dx} = 5 \cdot [f'(u(x)) \cdot u'(x)] = 5 \cdot [-\frac{2}{3}(1-5x)^{-5/3} \cdot (-5)]$$
$$= 5 \cdot [\frac{10}{3}(1-5x)^{-5/3}] = \frac{50}{3}(1-5x)^{-5/3} = \frac{50}{3(1-5x)^{5/3}}$$

36. **Problem (36):** Differentiate $y = (2x-1)^{3/4}$. *Solution (Chain Rule - General Power Rule):* Outer: $f(u) = u^{3/4} \implies f'(u) = \frac{3}{4}u^{-1/4}$. Inner: $u(x) = 2x - 1 \implies u'(x) = 2$.

$$\frac{dy}{dx} = f'(u(x)) \cdot u'(x) = \frac{3}{4}(2x-1)^{-1/4} \cdot (2)$$
$$= \frac{6}{4}(2x-1)^{-1/4} = \frac{3}{2}(2x-1)^{-1/4} = \frac{3}{2(2x-1)^{1/4}} = \frac{3}{2\sqrt[4]{2x-1}}$$

37. **Problem (37):** Differentiate $y = (4x^2+1)^{-1/2}$. *Solution (Chain Rule - General Power Rule):* Outer: $f(u) = u^{-1/2} \implies f'(u) = -\frac{1}{2}u^{-3/2}$. Inner: $u(x) = 4x^2 + 1 \implies u'(x) = 8x$.

$$\frac{dy}{dx} = f'(u(x)) \cdot u'(x) = -\frac{1}{2}(4x^2+1)^{-3/2} \cdot (8x)$$
$$= -4x(4x^2+1)^{-3/2} = \frac{-4x}{(4x^2+1)^{3/2}}$$

38. **Problem (38):** Differentiate $y = (x-6)^{-1/3}$. *Solution (Chain Rule - General Power Rule):* Outer: $f(u) = u^{-1/3} \implies f'(u) = -\frac{1}{3}u^{-4/3}$. Inner: $u(x) = x - 6 \implies u'(x) = 1$.

$$\frac{dy}{dx} = f'(u(x)) \cdot u'(x) = -\frac{1}{3}(x-6)^{-4/3} \cdot (1)$$
$$= -\frac{1}{3}(x-6)^{-4/3} = \frac{-1}{3(x-6)^{4/3}}$$

39. **Problem (39):** Differentiate $y = \dfrac{x^{1/2}}{(1-2x)^{1/3}}$. *Solution (Quotient Rule and Chain Rule):* $u = x^{1/2} \implies u' = \frac{1}{2}x^{-1/2}$. $v = (1-2x)^{1/3}$. Find v' using Chain Rule: Outer $f(w) = w^{1/3} \implies f'(w) = \frac{1}{3}w^{-2/3}$. Inner $w = 1 - 2x \implies w' = -2$. $v' = f'(w(x)) \cdot w'(x) = \frac{1}{3}(1-2x)^{-2/3} \cdot (-2) = -\frac{2}{3}(1-2x)^{-2/3}$. Apply Quotient Rule: $y' = \frac{vu'-uv'}{v^2}$

$$y' = \frac{(1-2x)^{1/3}(\frac{1}{2}x^{-1/2}) - (x^{1/2})(-\frac{2}{3}(1-2x)^{-2/3})}{((1-2x)^{1/3})^2}$$
$$= \frac{\frac{1}{2}x^{-1/2}(1-2x)^{1/3} + \frac{2}{3}x^{1/2}(1-2x)^{-2/3}}{(1-2x)^{2/3}}$$

Factor out common terms with lowest powers from numerator: $x^{-1/2}$ and $(1-2x)^{-2/3}$.

$$= \frac{x^{-1/2}(1-2x)^{-2/3}[\frac{1}{2}(1-2x)^{1/3-(-2/3)} + \frac{2}{3}x^{1/2-(-1/2)}]}{(1-2x)^{2/3}}$$
$$= \frac{x^{-1/2}(1-2x)^{-2/3}[\frac{1}{2}(1-2x)^1 + \frac{2}{3}x^1]}{(1-2x)^{2/3}}$$
$$= \frac{x^{-1/2}[\frac{1}{2} - x + \frac{2}{3}x]}{(1-2x)^{2/3}(1-2x)^{2/3}} = \frac{x^{-1/2}[\frac{1}{2} - \frac{1}{3}x]}{(1-2x)^{4/3}}$$

Multiply numerator/denominator by 6 to clear fractions inside:

$$= \frac{6x^{-1/2}[\frac{1}{2} - \frac{1}{3}x]}{6(1-2x)^{4/3}} = \frac{3x^{-1/2} - 2x^{1/2}}{6(1-2x)^{4/3}} = \frac{3-2x}{6x^{1/2}(1-2x)^{4/3}}$$

40. **Problem (40):** Differentiate $y = \frac{(3-7x)^{3/2}}{2x}$. *Solution (Quotient Rule and Chain Rule):* $u = (3-7x)^{3/2}$. Use Chain Rule for u': Outer $f(w) = w^{3/2} \implies f'(w) = \frac{3}{2}w^{1/2}$. Inner $w = 3-7x \implies w' = -7$. $u' = f'(w(x)) \cdot w'(x) = \frac{3}{2}(3-7x)^{1/2} \cdot (-7) = -\frac{21}{2}(3-7x)^{1/2}$. $v = 2x \implies v' = 2$. Apply Quotient Rule: $y' = \frac{vu'-uv'}{v^2}$

$$y' = \frac{(2x)(-\frac{21}{2}(3-7x)^{1/2}) - (3-7x)^{3/2}(2)}{(2x)^2}$$

$$= \frac{-21x(3-7x)^{1/2} - 2(3-7x)^{3/2}}{4x^2}$$

Factor out common term $-(3-7x)^{1/2}$ from numerator:

$$= \frac{-(3-7x)^{1/2}[21x + 2(3-7x)^1]}{4x^2}$$

$$= \frac{-(3-7x)^{1/2}[21x+6-14x]}{4x^2} = \frac{-\sqrt{3-7x}(7x+6)}{4x^2}$$

5 Solutions: Word Problems / Applications

41. **Problem (41):** *(Example 1 - Word Problem)* You deposit N1000 in an account with an annual interest rate of r (in decimal form) compounded monthly. At the end of 5 years, the balance is $A = 1000\left(1+\frac{r}{12}\right)^{60}$. Find the rates of change of A with respect to r when (a) r = 0.08, (b) r = 0.10, (c) r = 0.12.
Solution:

Strategy. We need to find the derivative $\frac{dA}{dr}$. The function involves a term raised to a power, where the base depends on r. This requires the Constant Multiple Rule and the Chain Rule (specifically, the General Power Rule). Let the inner function be $u(r) = 1 + \frac{r}{12}$ and the outer function be $f(u) = u^{60}$.

Step 1: Find the derivative $\frac{dA}{dr}$.

$$A = 1000\left(1 + \frac{1}{12}r\right)^{60}$$

Let $u(r) = 1 + \frac{1}{12}r$. Then $u'(r) = \frac{d}{dr}(1) + \frac{d}{dr}(\frac{1}{12}r) = 0 + \frac{1}{12}(1) = \frac{1}{12}$. Let the outer function be $g(u) = 1000u^{60}$. Then $g'(u) = 1000 \cdot (60u^{59}) = 60000u^{59}$. Apply Chain Rule: $\frac{dA}{dr} = g'(u(r)) \cdot u'(r)$.

$$\frac{dA}{dr} = 60000\left(1+\frac{r}{12}\right)^{59} \cdot \left(\frac{1}{12}\right)$$

Simplify the constant: $60000 \cdot \frac{1}{12} = 5000$.

$$\frac{dA}{dr} = 5000\left(1+\frac{r}{12}\right)^{59}$$

This derivative represents the rate at which the account balance changes for a small change in the annual interest rate r.

Step 2: Evaluate $\frac{dA}{dr}$ for the given values of r. (a) When $r = 0.08$:

$$\frac{dA}{dr} = 5000\left(1 + \frac{0.08}{12}\right)^{59} \approx 5000(1 + 0.0066667)^{59} \approx 5000(1.0066667)^{59}$$

$$\approx 5000(1.485947) \approx 7429.74$$

The units are N / (decimal rate unit). To interpret this as per 1

(b) When $r = 0.10$:

$$\frac{dA}{dr} = 5000\left(1 + \frac{0.10}{12}\right)^{59} \approx 5000(1 + 0.0083333)^{59} \approx 5000(1.0083333)^{59}$$

$$\approx 5000(1.63339) \approx 8166.95$$

Rate per 1

(c) When $r = 0.12$:

$$\frac{dA}{dr} = 5000\left(1 + \frac{0.12}{12}\right)^{59} = 5000(1 + 0.01)^{59} = 5000(1.01)^{59}$$

$$\approx 5000(1.795856) \approx 8979.28$$

Rate per 1

42. **Problem (42):** *(Example 2 - Word Problem)* Average daily pollutant level P (ppm) is modeled by $P = 0.25\sqrt{0.5n^2 + 5n + 25}$, where n is the number of residents in thousands. Find the rate at which P is increasing when the population is 12,000.

Solution:

Strategy. We need to find $\frac{dP}{dn}$ and evaluate it at $n = 12$ (since n is in thousands and population is 12,000). The function is a constant multiple times a square root, requiring the Chain Rule (General Power Rule).

Step 1: Rewrite P with fractional exponent.

$$P = 0.25(0.5n^2 + 5n + 25)^{1/2}$$

Step 2: Apply Chain Rule. Outer function $f(u) = 0.25u^{1/2} \implies f'(u) = 0.25(\frac{1}{2}u^{-1/2}) = 0.125u^{-1/2} = \frac{0.125}{\sqrt{u}}$. Inner function $u(n) = 0.5n^2 + 5n + 25 \implies u'(n) = 0.5(2n) + 5(1) + 0 = n + 5$. Apply $\frac{dP}{dn} = f'(u(n)) \cdot u'(n)$.

$$\frac{dP}{dn} = \frac{0.125}{\sqrt{0.5n^2 + 5n + 25}} \cdot (n + 5)$$

$$\frac{dP}{dn} = \frac{0.125(n + 5)}{\sqrt{0.5n^2 + 5n + 25}}$$

Step 3: Evaluate $\frac{dP}{dn}$ when $n = 12$.

$$\left.\frac{dP}{dn}\right|_{n=12} = \frac{0.125(12 + 5)}{\sqrt{0.5(12)^2 + 5(12) + 25}}$$

Calculate denominator: $0.5(144) + 60 + 25 = 72 + 60 + 25 = 157$.

$$= \frac{0.125(17)}{\sqrt{157}} = \frac{2.125}{\sqrt{157}}$$

Approximate the value: $\sqrt{157} \approx 12.53$.

$$\approx \frac{2.125}{12.53} \approx 0.1696$$

The units are ppm / (thousand residents).

Interpretation: The pollutant level is increasing at a rate of approximately 0.170 ppm for every thousand additional residents when the population is 12,000. *(Note: The slide solution "12.7 ppm" is likely the value of P itself at n=12, not the rate dP/dn. Let's check P(12):* $P(12) = 0.25\sqrt{157} \approx 0.25(12.53) \approx 3.13$ *ppm. The slide value seems unrelated or misinterpreted.)*

43. **Problem (43):** *(Example 3 - Word Problem)* Number N of bacteria after t days is $N = 400\left[1 - \frac{3}{(t^2+2)^2}\right]$. Find dN/dt for t=0, 1, 2, 3, 4 and conclude.

Solution:

Strategy. Rewrite N using negative exponents. Find the derivative dN/dt using differentiation rules (Constant Multiple, Sum/Difference, Chain Rule). Evaluate dN/dt at the given times.

Step 1: Rewrite N(t).

$$N(t) = 400[1 - 3(t^2+2)^{-2}]$$

Distribute the 400:

$$N(t) = 400 - 1200(t^2+2)^{-2}$$

Step 2: Differentiate N(t) with respect to t.

$$\frac{dN}{dt} = \frac{d}{dt}(400) - \frac{d}{dt}[1200(t^2+2)^{-2}]$$

$$\frac{dN}{dt} = 0 - 1200 \cdot \frac{d}{dt}[(t^2+2)^{-2}]$$

Use Chain Rule for $(t^2+2)^{-2}$: Outer $f(u) = u^{-2} \implies f'(u) = -2u^{-3}$. Inner $u(t) = t^2 + 2 \implies u'(t) = 2t$. $\frac{d}{dt}[(t^2+2)^{-2}] = f'(u(t)) \cdot u'(t) = -2(t^2+2)^{-3} \cdot (2t) = -4t(t^2+2)^{-3}$. Substitute back:

$$\frac{dN}{dt} = -1200[-4t(t^2+2)^{-3}]$$

$$\frac{dN}{dt} = \frac{4800t}{(t^2+2)^3}$$

Step 3: Evaluate dN/dt at t = 0, 1, 2, 3, 4.

- t=0: $\frac{dN}{dt} = \frac{4800(0)}{(0^2+2)^3} = \frac{0}{8} = 0$.
- t=1: $\frac{dN}{dt} = \frac{4800(1)}{(1^2+2)^3} = \frac{4800}{3^3} = \frac{4800}{27} \approx 177.78$.
- t=2: $\frac{dN}{dt} = \frac{4800(2)}{(2^2+2)^3} = \frac{9600}{6^3} = \frac{9600}{216} \approx 44.44$.

- t=3: $\frac{dN}{dt} = \frac{4800(3)}{(3^2+2)^3} = \frac{14400}{11^3} = \frac{14400}{1331} \approx 10.82$.
- t=4: $\frac{dN}{dt} = \frac{4800(4)}{(4^2+2)^3} = \frac{19200}{18^3} = \frac{19200}{5832} \approx 3.29$.

Step 4: Complete Table and Conclude.

t	0	1	2	3	4
dN/dt	0	177.78	44.44	10.82	3.29

Conclusion: The rate of growth of the bacteria population (dN/dt) is initially zero, increases rapidly by day 1, and then decreases significantly over the following days. The growth slows down as time progresses.

44. **Problem (44):** *(ClassWork 1 - Word Problem)* Repeat Example 3 for $N = 400t\left[1 - \frac{3}{(t^2+2)^2}\right]$. Complete table for dN/dt.
Solution:

Strategy. This function is different from Example 3 due to the factor of t outside. Expand $N(t)$ first, then use the Product Rule and Chain Rule to find dN/dt. Evaluate at the required points.

Step 1: Expand N(t).

$$N(t) = 400t[1 - 3(t^2+2)^{-2}] = 400t - 1200t(t^2+2)^{-2}$$

Step 2: Differentiate N(t) term by term.

$$\frac{dN}{dt} = \frac{d}{dt}(400t) - \frac{d}{dt}[1200t(t^2+2)^{-2}]$$

The first term is $\frac{d}{dt}(400t) = 400$. For the second term, $1200t(t^2+2)^{-2}$, use the Product Rule: Let $f(t) = 1200t \implies f'(t) = 1200$. Let $g(t) = (t^2+2)^{-2}$. Use Chain Rule to find $g'(t)$: Outer $h(u) = u^{-2} \implies h'(u) = -2u^{-3}$. Inner $u(t) = t^2+2 \implies u'(t) = 2t$. $g'(t) = h'(u(t)) \cdot u'(t) = -2(t^2+2)^{-3} \cdot (2t) = -4t(t^2+2)^{-3}$. Apply Product Rule $(fg)' = fg' + gf'$:

$$\frac{d}{dt}[1200t(t^2+2)^{-2}] = (1200t)[-4t(t^2+2)^{-3}] + (t^2+2)^{-2}[1200]$$

$$= -4800t^2(t^2+2)^{-3} + 1200(t^2+2)^{-2}$$

Step 3: Combine parts for dN/dt.

$$\frac{dN}{dt} = 400 - [-4800t^2(t^2+2)^{-3} + 1200(t^2+2)^{-2}]$$

$$\frac{dN}{dt} = 400 + \frac{4800t^2}{(t^2+2)^3} - \frac{1200}{(t^2+2)^2}$$

Step 4: Evaluate dN/dt at t = 0, 1, 2, 3, 4.

- t=0: $\frac{dN}{dt} = 400 + \frac{0}{(2)^3} - \frac{1200}{(2)^2} = 400 - \frac{1200}{4} = 400 - 300 = 100$. (Matches slide)
- t=1: $\frac{dN}{dt} = 400 + \frac{4800(1)^2}{(1^2+2)^3} - \frac{1200}{(1^2+2)^2} = 400 + \frac{4800}{27} - \frac{1200}{9} = 400 + \frac{1600}{9} - \frac{1200}{9} = 400 + \frac{400}{9} = \frac{3600+400}{9} = \frac{4000}{9} \approx 444.44$. (Matches slide rounded 444.5)

- t=2: $\frac{dN}{dt} = 400 + \frac{4800(2)^2}{(2^2+2)^3} - \frac{1200}{(2^2+2)^2} = 400 + \frac{19200}{6^3} - \frac{1200}{6^2} = 400 + \frac{19200}{216} - \frac{1200}{36} = 400 + \frac{800}{9} - \frac{100}{3} = 400 + \frac{800-300}{9} = 400 + \frac{500}{9} = \frac{4100}{9} \approx 455.56$. (Slide value 633.4 is likely incorrect).
- t=3: $\frac{dN}{dt} = 400 + \frac{4800(3)^2}{(3^2+2)^3} - \frac{1200}{(3^2+2)^2} = 400 + \frac{43200}{11^3} - \frac{1200}{11^2} = 400 + \frac{43200}{1331} - \frac{1200}{121} \approx 400 + 32.456 - 9.917 \approx 422.54$.
- t=4: $\frac{dN}{dt} = 400 + \frac{4800(4)^2}{(4^2+2)^3} - \frac{1200}{(4^2+2)^2} = 400 + \frac{76800}{18^3} - \frac{1200}{18^2} = 400 + \frac{76800}{5832} - \frac{1200}{324} \approx 400 + 13.168 - 3.704 \approx 409.46$.

Step 5: Complete Table and Conclude.

t	0	1	2	3	4
dN/dt	100	444.44	455.56	422.54	409.46

(Using calculated values) Conclusion: With this model ($N = 400t[...]$), the rate of growth starts at 100, increases to a maximum around t=2, and then starts decreasing.

45. **Problem (45):** *(ClassWork 2 - Word Problem)* Value V of a machine t years after purchase is inversely proportional to $\sqrt{t+1}$. Initial value is N10,000. (a) Write V(t). (b) Find rate of depreciation at t=1. (c) Find rate of depreciation at t=3.

Solution:

Strategy. "Inversely proportional" means $V = \frac{k}{\sqrt{t+1}}$ for some constant k. Use the initial condition (t=0, V=10000) to find k. The rate of depreciation is the negative of the derivative, $-\frac{dV}{dt}$. Find $\frac{dV}{dt}$ using Chain rule.

Step 1: Find the function V(t). We have $V(t) = \frac{k}{(t+1)^{1/2}}$. At $t = 0$, $V(0) = 10000$. $10000 = \frac{k}{(0+1)^{1/2}} = \frac{k}{1^{1/2}} = \frac{k}{1} = k$. So, $k = 10000$. The function is $V(t) = \frac{10000}{(t+1)^{1/2}} = 10000(t+1)^{-1/2}$.

Step 2: Find the derivative $\frac{dV}{dt}$. Use Chain Rule (General Power Rule). Outer: $f(u) = u^{-1/2} \implies f'(u) = -\frac{1}{2}u^{-3/2}$. Inner: $u(t) = t+1 \implies u'(t) = 1$.

$$\frac{dV}{dt} = 10000 \cdot [f'(u(t)) \cdot u'(t)] = 10000 \cdot [-\frac{1}{2}(t+1)^{-3/2} \cdot (1)]$$

$$= -5000(t+1)^{-3/2} = \frac{-5000}{(t+1)^{3/2}}$$

Step 3: Find the rate of depreciation at t=1. Rate of depreciation $= -\frac{dV}{dt}$. At $t = 1$: $\frac{dV}{dt} = \frac{-5000}{(1+1)^{3/2}} = \frac{-5000}{2^{3/2}} = \frac{-5000}{2\sqrt{2}} = \frac{-2500}{\sqrt{2}}$. Rate of depreciation $= -(\frac{-2500}{\sqrt{2}}) = \frac{2500}{\sqrt{2}} = \frac{2500\sqrt{2}}{2} = 1250\sqrt{2}$. Approximation: $1250 \times 1.4142 \approx 1767.75$. The rate is N1767.75 / year.

Step 4: Find the rate of depreciation at t=3. At $t = 3$: $\frac{dV}{dt} = \frac{-5000}{(3+1)^{3/2}} = \frac{-5000}{4^{3/2}} = \frac{-5000}{(\sqrt{4})^3} = \frac{-5000}{2^3} = \frac{-5000}{8} = -625$. Rate of depreciation $= -(-625) = 625$. The rate is N625 / year.

46. **Problem (46):** *(ClassWork 3 - Word Problem)* Repeat ClassWork 2 given V is inversely proportional to the cube root of t+1.

Solution:

Strategy. Now $V = \frac{k}{\sqrt[3]{t+1}} = k(t+1)^{-1/3}$. Use initial condition to find k. Find $-\frac{dV}{dt}$.

Step 1: Find V(t). $V(0) = 10000$. $10000 = \frac{k}{(0+1)^{1/3}} = \frac{k}{1} = k$. So $k = 10000$. $V(t) = 10000(t+1)^{-1/3}$.

Step 2: Find $\frac{dV}{dt}$. Use Chain Rule. Outer: $f(u) = u^{-1/3} \implies f'(u) = -\frac{1}{3}u^{-4/3}$. Inner: $u(t) = t + 1 \implies u'(t) = 1$.

$$\frac{dV}{dt} = 10000 \cdot [f'(u(t)) \cdot u'(t)] = 10000 \cdot [-\frac{1}{3}(t+1)^{-4/3} \cdot (1)]$$

$$= -\frac{10000}{3}(t+1)^{-4/3} = \frac{-10000}{3(t+1)^{4/3}}$$

Step 3: Find rate of depreciation ($-\frac{dV}{dt}$) at t=1. At $t = 1$: $\frac{dV}{dt} = \frac{-10000}{3(1+1)^{4/3}} = \frac{-10000}{3(2^{4/3})} = \frac{-10000}{3\sqrt[3]{16}} = \frac{-10000}{3 \cdot 2\sqrt[3]{2}} = \frac{-5000}{3\sqrt[3]{2}}$. Rate $= -\frac{dV}{dt} = \frac{5000}{3\sqrt[3]{2}}$. Approximation: $\sqrt[3]{2} \approx 1.2599$. Rate $\approx \frac{5000}{3 \times 1.2599} \approx \frac{5000}{3.7797} \approx 1322.83$. Rate is N1322.83 / year. (Matches slide (b))

Step 4: Find rate of depreciation ($-\frac{dV}{dt}$) at t=3. At $t = 3$: $\frac{dV}{dt} = \frac{-10000}{3(3+1)^{4/3}} = \frac{-10000}{3(4^{4/3})} = \frac{-10000}{3(\sqrt[3]{4^4})} = \frac{-10000}{3(\sqrt[3]{256})}$. $4^{4/3} = (4^{1/3})^4 \approx (1.5874)^4 \approx 6.3496$. $\frac{dV}{dt} \approx \frac{-10000}{3(6.3496)} \approx \frac{-10000}{19.0488} \approx -524.97$. Rate $= -\frac{dV}{dt} \approx 524.97$. Rate is N524.97 / year.

47. **Problem (47):** *(ClassWork 4 - Word Problem)* Average annual rate r for credit cards is modeled by $r = \sqrt{-1.7409t^4 + 18.070t^3 - 52.68t^2 + 10.9t + 249}$, where $t = 0$ corresponds to 2000. (a) Find dr/dt. Which rule(s) did you use?
Solution:

Strategy. This requires the Chain Rule (General Power Rule). The outermost function is the square root (power 1/2). The inner function is the polynomial inside the square root.

Step 1: Rewrite r(t).

$$r(t) = (-1.7409t^4 + 18.070t^3 - 52.68t^2 + 10.9t + 249)^{1/2}$$

Step 2: Apply Chain Rule. Let inner function $u(t) = -1.7409t^4 + 18.070t^3 - 52.68t^2 + 10.9t + 249$. Let outer function $f(u) = u^{1/2}$. Find derivatives: $u'(t) = \frac{du}{dt} = -1.7409(4t^3) + 18.070(3t^2) - 52.68(2t) + 10.9(1) + 0$ $u'(t) = -6.9636t^3 + 54.21t^2 - 105.36t + 10.9$. $f'(u) = \frac{1}{2}u^{-1/2} = \frac{1}{2\sqrt{u}}$. Apply $\frac{dr}{dt} = f'(u(t)) \cdot u'(t)$.

$$\frac{dr}{dt} = \frac{1}{2\sqrt{-1.7409t^4 + 18.070t^3 - 52.68t^2 + 10.9t + 249}} \cdot (-6.9636t^3 + 54.21t^2 - 105.36t + 10.9)$$

$$\frac{dr}{dt} = \frac{-6.9636t^3 + 54.21t^2 - 105.36t + 10.9}{2\sqrt{-1.7409t^4 + 18.070t^3 - 52.68t^2 + 10.9t + 249}}$$

Step 3: Rules Used. The primary rule used is the **Chain Rule** (specifically the General Power Rule). To find the derivative of the inner polynomial function, the **Sum/Difference Rule**, **Constant Multiple Rule**, and **Power Rule** were applied.