# THE DERIVATIVE AS LIMIT OF RATE OF CHANGE

**NUTM Nexus Writing Team**

February 11, 2025

## 1 Introduction: Basic Differentiation Rules

**Explanation.** This section focuses on finding derivatives, which represent the instantaneous rate of change of a function. While the derivative is formally defined using limits, we typically use differentiation rules for efficiency. The main rules used here are:

1. **Power Rule:** For any real number $n$, $\frac{d}{dx}(x^n) = nx^{n-1}$. (Bring power down, reduce power by 1).

2. **Constant Multiple Rule:** $\frac{d}{dx}(c \cdot f(x)) = c \cdot \frac{d}{dx}(f(x))$. (Constants factor out).

3. **Sum/Difference Rule:** $\frac{d}{dx}(f(x) \pm g(x)) = \frac{d}{dx}(f(x)) \pm \frac{d}{dx}(g(x))$. (Differentiate term by term).

4. **Constant Rule:** $\frac{d}{dx}(c) = 0$. (Derivative of a constant is zero).

5. **Chain Rule**: For functions like $(expression)^n$ or $\sqrt{expression}$, we technically need the Chain Rule: $\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x)$.

For the power rule case, $\frac{d}{dx}(u^n) = nu^{n-1} \cdot \frac{du}{dx}$.

We will apply this where necessary, especially in ClassWork II and Assignment I.

For polynomials, we apply rules 1-4 term by term. For other functions, we might need to rewrite them using exponents (e.g., $\sqrt{x} = x^{1/2}, 1/x^n = x^{-n}$) before applying the rules.

## 2 Solutions: ClassWork I

1. Find derivative of $y = x^4 - 3x^2 + 8x + 6$.

   *Solution:* Apply Sum/Difference, Constant Multiple, and Power Rules term-by-term.

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(x^4) - \frac{d}{dx}(3x^2) + \frac{d}{dx}(8x^1) + \frac{d}{dx}(6) \\ &= (4x^{4-1}) - 3(2x^{2-1}) + 8(1x^{1-1}) + 0 \\ &= 4x^3 - 6x^1 + 8(1x^0) = 4x^3 - 6x + 8(1) \\ &= 4x^3 - 6x + 8\end{aligned}$$

2. Find derivative of $y = 4x^2 - 5x$.

   *Solution:* Differentiate term-by-term.

$$\frac{dy}{dx} = \frac{d}{dx}(4x^2) - \frac{d}{dx}(5x^1)$$
$$= 4(2x^{2-1}) - 5(1x^{1-1}) = 8x^1 - 5(1x^0)$$
$$= 8x - 5$$

3. Find derivative of $y = x^2 - 4x + 10$.

   *Solution:* Differentiate term-by-term.

$$\frac{dy}{dx} = \frac{d}{dx}(x^2) - \frac{d}{dx}(4x^1) + \frac{d}{dx}(10)$$
$$= (2x^{2-1}) - 4(1x^{1-1}) + 0 = 2x^1 - 4(1x^0)$$
$$= 2x - 4$$

4. Find derivative of $y = x^2 + 6x + 5$.

   *Solution:* Differentiate term-by-term.

$$\frac{dy}{dx} = \frac{d}{dx}(x^2) + \frac{d}{dx}(6x^1) + \frac{d}{dx}(5)$$
$$= (2x^{2-1}) + 6(1x^{1-1}) + 0 = 2x^1 + 6(1x^0)$$
$$= 2x + 6$$

5. Find derivative of $y = x^2 - 2x - 3$.

   *Solution:* Differentiate term-by-term.

$$\frac{dy}{dx} = \frac{d}{dx}(x^2) - \frac{d}{dx}(2x^1) - \frac{d}{dx}(3)$$
$$= (2x^{2-1}) - 2(1x^{1-1}) - 0 = 2x^1 - 2(1x^0)$$
$$= 2x - 2$$

6. Find derivative of $y = 4x^2 + 3x + 5$.

   *Solution:* Differentiate term-by-term.

$$\frac{dy}{dx} = \frac{d}{dx}(4x^2) + \frac{d}{dx}(3x^1) + \frac{d}{dx}(5)$$
$$= 4(2x^{2-1}) + 3(1x^{1-1}) + 0 = 8x^1 + 3(1x^0)$$
$$= 8x + 3$$

7. Find derivative of $y = x^2 - 4x + 3$.

   *Solution:* Differentiate term-by-term.

$$\frac{dy}{dx} = \frac{d}{dx}(x^2) - \frac{d}{dx}(4x^1) + \frac{d}{dx}(3)$$
$$= (2x^{2-1}) - 4(1x^{1-1}) + 0 = 2x^1 - 4(1x^0)$$
$$= 2x - 4$$

8. Find derivative of $y = 2x^2 - 8x + 4$.

   *Solution:* Differentiate term-by-term.

$$\frac{dy}{dx} = \frac{d}{dx}(2x^2) - \frac{d}{dx}(8x^1) + \frac{d}{dx}(4)$$
$$= 2(2x^{2-1}) - 8(1x^{1-1}) + 0 = 4x^1 - 8(1x^0)$$
$$= 4x - 8$$

9. Find derivative of $y = 3x^2 - 6x + 5$.

   *Solution:* Differentiate term-by-term.

$$\frac{dy}{dx} = \frac{d}{dx}(3x^2) - \frac{d}{dx}(6x^1) + \frac{d}{dx}(5)$$
$$= 3(2x^{2-1}) - 6(1x^{1-1}) + 0 = 6x^1 - 6(1x^0)$$
$$= 6x - 6$$

10. Find derivative of $y = 4x^3 - 30x^2 + 74x - 60$.

    *Solution:* Differentiate term-by-term.

$$\frac{dy}{dx} = \frac{d}{dx}(4x^3) - \frac{d}{dx}(30x^2) + \frac{d}{dx}(74x^1) - \frac{d}{dx}(60)$$
$$= 4(3x^{3-1}) - 30(2x^{2-1}) + 74(1x^{1-1}) - 0$$
$$= 12x^2 - 60x^1 + 74(1x^0) = 12x^2 - 60x + 74$$

11. Find derivative of $y = 2x^2 + 7x - 5$.

    *Solution:* Differentiate term-by-term.

$$\frac{dy}{dx} = \frac{d}{dx}(2x^2) + \frac{d}{dx}(7x^1) - \frac{d}{dx}(5)$$
$$= 2(2x^{2-1}) + 7(1x^{1-1}) - 0 = 4x^1 + 7(1x^0)$$
$$= 4x + 7$$

12. Find derivative of $y = x^3 - 6x^2$.

    *Solution:* Differentiate term-by-term.

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}(x^3) - \frac{d}{dx}(6x^2) \\ &= (3x^{3-1}) - 6(2x^{2-1}) = 3x^2 - 12x^1 \\ &= 3x^2 - 12x \end{aligned}$$

13. Find derivative of $y = 3x^2 - 12$.

    *Solution:* Differentiate term-by-term.

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}(3x^2) - \frac{d}{dx}(12) \\ &= 3(2x^{2-1}) - 0 = 6x^1 \\ &= 6x \end{aligned}$$

14. Find derivative of $y = x^2 - 3x + 4$.

    *Solution:* Differentiate term-by-term.

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}(x^2) - \frac{d}{dx}(3x^1) + \frac{d}{dx}(4) \\ &= (2x^{2-1}) - 3(1x^{1-1}) + 0 = 2x^1 - 3(1x^0) \\ &= 2x - 3 \end{aligned}$$

15. Find derivative of $y = x^2 - 4x + 5$.

    *Solution:* Differentiate term-by-term.

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}(x^2) - \frac{d}{dx}(4x^1) + \frac{d}{dx}(5) \\ &= (2x^{2-1}) - 4(1x^{1-1}) + 0 = 2x^1 - 4(1x^0) \\ &= 2x - 4 \end{aligned}$$

## 3 Solutions: ClassWork II

16. Find derivative of $y = x^2 + 4x - 1$.

    *Solution:* Differentiate term-by-term.

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}(x^2) + \frac{d}{dx}(4x^1) - \frac{d}{dx}(1) \\ &= (2x^{2-1}) + 4(1x^{1-1}) - 0 = 2x^1 + 4(1x^0) \\ &= 2x + 4 \end{aligned}$$

17. Find derivative of $y = x^3 + 3x^2 + 1$.

    *Solution:* Differentiate term-by-term.

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(x^3) + \frac{d}{dx}(3x^2) + \frac{d}{dx}(1)\\ &= (3x^{3-1}) + 3(2x^{2-1}) + 0 = 3x^2 + 6x^1\\ &= 3x^2 + 6x\end{aligned}$$

18. Find derivative of $y = x^2 + 2$.

    *Solution:* Differentiate term-by-term.

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(x^2) + \frac{d}{dx}(2)\\ &= (2x^{2-1}) + 0 = 2x^1\\ &= 2x\end{aligned}$$

19. Find derivative of $y = 3x^2 + 6x$.

    *Solution:* Differentiate term-by-term.

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(3x^2) + \frac{d}{dx}(6x^1)\\ &= 3(2x^{2-1}) + 6(1x^{1-1}) = 6x^1 + 6(1x^0)\\ &= 6x + 6\end{aligned}$$

20. Find derivative of $y = 5x^4 + 12x^3 + 6x^2 + 14x$.

    *Solution:* Differentiate term-by-term.

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(5x^4) + \frac{d}{dx}(12x^3) + \frac{d}{dx}(6x^2) + \frac{d}{dx}(14x^1)\\ &= 5(4x^{4-1}) + 12(3x^{3-1}) + 6(2x^{2-1}) + 14(1x^{1-1})\\ &= 20x^3 + 36x^2 + 12x^1 + 14(1x^0)\\ &= 20x^3 + 36x^2 + 12x + 14\end{aligned}$$

21. Find derivative of $y = 1 - 2x - x^2$.

    *Solution:* Differentiate term-by-term.

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(1) - \frac{d}{dx}(2x^1) - \frac{d}{dx}(x^2)\\ &= 0 - 2(1x^{1-1}) - (2x^{2-1}) = -2(1x^0) - 2x^1\\ &= -2 - 2x\end{aligned}$$

22. Find derivative of $y = (x^2 + 1)^2$.

    *Solution:* **Method 1: Expand First** $y = (x^2)^2 + 2(x^2)(1) + (1)^2 = x^4 + 2x^2 + 1$. Differentiate term-by-term:

$$\frac{dy}{dx} = \frac{d}{dx}(x^4) + \frac{d}{dx}(2x^2) + \frac{d}{dx}(1)$$
$$= 4x^3 + 2(2x) + 0 = 4x^3 + 4x$$

---

**Method 2: Chain Rule** Outer: $f(u) = u^2 \implies f'(u) = 2u$. Inner: $u(x) = x^2 + 1 \implies u'(x) = 2x$.

$$\frac{dy}{dx} = f'(u(x)) \cdot u'(x) = 2(x^2 + 1) \cdot (2x) = 4x(x^2 + 1)$$
$$= 4x(x^2) + 4x(1) = 4x^3 + 4x$$

*Results match.*

23. Find derivative of $y = x^6 + 4x^3 + 5$.

    *Solution:* Differentiate term-by-term.

$$\frac{dy}{dx} = \frac{d}{dx}(x^6) + \frac{d}{dx}(4x^3) + \frac{d}{dx}(5)$$
$$= 6x^{6-1} + 4(3x^{3-1}) + 0 = 6x^5 + 12x^2$$

24. Find derivative of $y = (x^2 - 4)^2$.

    *Solution:* **Method 1: Expand First** $y = (x^2)^2 - 2(x^2)(4) + (4)^2 = x^4 - 8x^2 + 16$. Differentiate term-by-term:

$$\frac{dy}{dx} = \frac{d}{dx}(x^4) - \frac{d}{dx}(8x^2) + \frac{d}{dx}(16)$$
$$= 4x^3 - 8(2x) + 0 = 4x^3 - 16x$$

---

**Method 2: Chain Rule** Outer: $f(u) = u^2 \implies f'(u) = 2u$. Inner: $u(x) = x^2 - 4 \implies u'(x) = 2x$.

$$\frac{dy}{dx} = f'(u(x)) \cdot u'(x) = 2(x^2 - 4) \cdot (2x) = 4x(x^2 - 4)$$
$$= 4x(x^2) - 4x(4) = 4x^3 - 16x$$

*Results match.*

25. Find derivative of $y = 6x^5 + 12x^2$.

    *Solution:* Differentiate term-by-term.

$$\frac{dy}{dx} = \frac{d}{dx}(6x^5) + \frac{d}{dx}(12x^2)$$
$$= 6(5x^{5-1}) + 12(2x^{2-1}) = 30x^4 + 24x^1$$
$$= 30x^4 + 24x$$

26. Find derivative of $y = x^3 + 3x$.

*Solution:* Differentiate term-by-term.

$$\frac{dy}{dx} = \frac{d}{dx}(x^3) + \frac{d}{dx}(3x^1)$$

$$= (3x^{3-1}) + 3(1x^{1-1}) = 3x^2 + 3(1x^0)$$

$$= 3x^2 + 3$$

27. Find derivative of $y = -x^3 + 3x^2 + 9x + 5$.

*Solution:* Differentiate term-by-term.

$$\frac{dy}{dx} = \frac{d}{dx}(-x^3) + \frac{d}{dx}(3x^2) + \frac{d}{dx}(9x^1) + \frac{d}{dx}(5)$$

$$= -1(3x^{3-1}) + 3(2x^{2-1}) + 9(1x^{1-1}) + 0$$

$$= -3x^2 + 6x^1 + 9(1x^0) = -3x^2 + 6x + 9$$

28. Find derivative of $y = 2x^3 - 24x + 5$.

*Solution:* Differentiate term-by-term.

$$\frac{dy}{dx} = \frac{d}{dx}(2x^3) - \frac{d}{dx}(24x^1) + \frac{d}{dx}(5)$$

$$= 2(3x^{3-1}) - 24(1x^{1-1}) + 0 = 6x^2 - 24(1x^0)$$

$$= 6x^2 - 24$$

29. Find derivative of $y = (x-1)^3(x-2)$.

*Solution:* **Method 1: Product Rule (with Chain Rule)** Let $u = (x-1)^3, v = x - 2$. Find $du/dx$ using Chain Rule: outer $f(w) = w^3 \implies f'(w) = 3w^2$. Inner $w = x - 1 \implies w' = 1$. $du/dx = 3(x-1)^2(1) = 3(x-1)^2$. $dv/dx = 1$. Apply Product Rule: $y' = uv' + vu'$.

$$\frac{dy}{dx} = (x-1)^3(1) + (x-2)[3(x-1)^2]$$

Factor out $(x-1)^2$:

$$= (x-1)^2[(x-1) + 3(x-2)]$$

$$= (x-1)^2[x - 1 + 3x - 6] = (x-1)^2(4x-7)$$

*Expanding* $(x-1)^2(4x-7) = (x^2 - 2x + 1)(4x - 7) = 4x^3 - 7x^2 - 8x^2 + 14x + 4x - 7 = 4x^3 - 15x^2 + 18x - 7$.

---

**Method 2: Expand First** $y = (x^3 - 3x^2 + 3x - 1)(x - 2)$ $y = x(x^3 - 3x^2 + 3x - 1) - 2(x^3 - 3x^2 + 3x - 1)$ $y = (x^4 - 3x^3 + 3x^2 - x) - (2x^3 - 6x^2 + 6x - 2)$ $y = x^4 - 3x^3 + 3x^2 - x - 2x^3 + 6x^2 - 6x + 2$ $y = x^4 - 5x^3 + 9x^2 - 7x + 2$. Differentiate:

$$\frac{dy}{dx} = 4x^3 - 5(3x^2) + 9(2x) - 7(1) + 0$$

$$= 4x^3 - 15x^2 + 18x - 7$$

*Results match.*

30. Find derivative of $y = 6(x + 2)(x - 2)$.

*Solution:* **Method 1: Expand First (Easiest)** Recognize $(x + 2)(x - 2)$ is a difference of squares: $x^2 - 2^2 = x^2 - 4$. $y = 6(x^2 - 4) = 6x^2 - 24$. Differentiate:

$$\frac{dy}{dx} = \frac{d}{dx}(6x^2) - \frac{d}{dx}(24) = 6(2x) - 0 = 12x$$

---

**Method 2: Product Rule** Let $u = 6(x + 2) = 6x + 12 \implies du/dx = 6$. Let $v = x - 2 \implies dv/dx = 1$.

$$\frac{dy}{dx} = uv' + vu' = (6x + 12)(1) + (x - 2)(6)$$
$$= 6x + 12 + 6x - 12 = 12x$$

*Results match.*

## 4 Solutions: Assignment I

31. Find derivative of $t = f(u) = 6u^{3/2}$. Assume $t$ depends on $u$.

*Solution (Constant Multiple, Power Rule):*

$$\frac{dt}{du} = \frac{d}{du}(6u^{3/2}) = 6 \cdot \frac{d}{du}(u^{3/2})$$

Apply Power Rule with $n = 3/2$. $n - 1 = 3/2 - 1 = 1/2$.

$$= 6 \cdot (\frac{3}{2}u^{3/2-1}) = 6 \cdot \frac{3}{2}u^{1/2} = 9u^{1/2}$$

*Optional rewrite:* $9\sqrt{u}$.

32. Find derivative of $w = f(p) = (p^2 + 4)^{1/2}$. Assume $w$ depends on $p$.

*Solution (Chain Rule):* Outer: $f(u) = u^{1/2} \implies f'(u) = \frac{1}{2}u^{-1/2}$. Inner: $u(p) = p^2 + 4 \implies u'(p) = 2p$.

$$\frac{dw}{dp} = f'(u(p)) \cdot u'(p) = \frac{1}{2}(p^2 + 4)^{-1/2} \cdot (2p)$$
$$= p(p^2 + 4)^{-1/2} = \frac{p}{(p^2 + 4)^{1/2}} = \frac{p}{\sqrt{p^2 + 4}}$$

33. Find derivative of $b = f(v) = -5 + 3v - \frac{3}{2}v^2 - 7v^3$. Assume $b$ depends on $v$.

*Solution:* Differentiate term-by-term with respect to $v$.

$$\frac{db}{dv} = \frac{d}{dv}(-5) + \frac{d}{dv}(3v^1) - \frac{d}{dv}(\frac{3}{2}v^2) - \frac{d}{dv}(7v^3)$$
$$= 0 + 3(1v^0) - \frac{3}{2}(2v^1) - 7(3v^2)$$
$$= 3(1) - 3v - 21v^2 = 3 - 3v - 21v^2$$

34. Find derivative of $a = f(c) = \frac{2}{c^2 - 1}$. Assume $a$ depends on $c$.

*Solution:* **Method 1: Quotient Rule** Let $u = 2 \implies u' = 0$. Let $v = c^2 - 1 \implies v' = 2c$.

$$\frac{da}{dc} = \frac{vu' - uv'}{v^2} = \frac{(c^2 - 1)(0) - (2)(2c)}{(c^2 - 1)^2} = \frac{0 - 4c}{(c^2 - 1)^2} = \frac{-4c}{(c^2 - 1)^2}$$

---

**Method 2: Chain Rule** Rewrite $a = 2(c^2 - 1)^{-1}$. Outer: $f(u) = u^{-1} \implies f'(u) = -u^{-2}$. Inner: $u(c) = c^2 - 1 \implies u'(c) = 2c$.

$$\frac{da}{dc} = 2 \cdot [f'(u(c)) \cdot u'(c)] = 2 \cdot [-(c^2 - 1)^{-2} \cdot (2c)]$$

$$= 2(-2c)(c^2 - 1)^{-2} = -4c(c^2 - 1)^{-2} = \frac{-4c}{(c^2 - 1)^2}$$

*Results match.*

35. Find derivative of $y = f(x) = 17x^2 - 10x + 15$.

*Solution:* Differentiate term-by-term.

$$\frac{dy}{dx} = \frac{d}{dx}(17x^2) - \frac{d}{dx}(10x^1) + \frac{d}{dx}(15)$$

$$= 17(2x) - 10(1) + 0 = 34x - 10$$