

PARTIAL FRACTIONS AND ITS INTEGRATION - WEEK 10

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1 Introduction: Partial Fractions

Explanation. Partial Fraction Decomposition is an algebraic technique used to break down a complex rational function (a ratio of polynomials) into a sum of simpler rational functions. This is extremely useful because the simpler fractions are often much easier to integrate.

Pre-requisite: The degree of the numerator polynomial must be strictly less than the degree of the denominator polynomial. If not, perform polynomial long division first to get a polynomial plus a proper rational function (where the remainder term satisfies the degree condition).

The Process:

1. **Factor the Denominator:** Completely factor the denominator into linear factors (like $ax + b$) and irreducible quadratic factors (like $ax^2 + bx + c$ where $b^2 - 4ac < 0$, meaning it cannot be factored further using real numbers).
2. **Set up the Decomposition Form:** Based on the factors in the denominator, write the rational function as a sum of simpler fractions with unknown constants (A, B, C, etc.) in the numerators. The rules for the form depend on the type and repetition of the factors:
 - **Distinct Linear Factor:** For each unique factor $(ax+b)$ in the denominator, include a term $\frac{A}{ax+b}$ in the decomposition, where A is an unknown constant.
 - **Repeated Linear Factor:** If a linear factor $(ax+b)$ appears k times, i.e., $(ax+b)^k$, you must include k terms in the decomposition, one for each power from 1 to k : $\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_k}{(ax+b)^k}$.
 - **Distinct Irreducible Quadratic Factor:** For each unique factor $(ax^2 + bx + c)$ that cannot be factored further, include a term $\frac{Ax+B}{ax^2+bx+c}$ in the decomposition (note the linear numerator).
 - **Repeated Irreducible Quadratic Factor:** If an irreducible quadratic factor $(ax^2 + bx + c)$ appears k times, i.e., $(ax^2 + bx + c)^k$, include k terms: $\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_kx+B_k}{(ax^2+bx+c)^k}$.
3. **Solve for the Unknown Constants (A, B, C...):**
 - **Clear Denominators:** Multiply both sides of the equation (original fraction = sum of partial fractions) by the fully factored original denominator. This results in an equation involving only polynomials.
 - **Find Constants:** There are two main methods, often used in combination:

- *Method 1: Substituting Convenient Values (Heaviside Method):* Substitute the roots of the linear factors (the values of x that make those factors zero) into the equation after clearing denominators. This often allows you to solve for the constants associated with those linear factors directly.
- *Method 2: Equating Coefficients:* Expand the entire right side of the equation (after clearing denominators) and collect terms by powers of x (e.g., all x^2 terms together, all x terms together, all constant terms together). The coefficients of each power of x on the right side must equal the coefficients of the corresponding power of x in the original numerator. This creates a system of linear equations which you can solve for the unknown constants A, B, C, etc.

For repeated factors or irreducible quadratic factors, you often need to use a combination of substituting convenient values and equating coefficients.

4. **Write the Final Decomposition:** Substitute the numerical values you found for A, B, C... back into the decomposition form you set up in Step 2.

Integration: After finding the partial fraction decomposition, the original integral becomes the integral of a sum of simpler terms. Integrate each term separately. Remember the common integrals:

- $\int \frac{A}{ax+b} dx = \frac{A}{a} \ln|ax+b| + C$ (using a simple u-substitution $u = ax+b$)
- $\int \frac{A}{(ax+b)^k} dx = \int A(ax+b)^{-k} dx = \frac{A}{a} \frac{(ax+b)^{-k+1}}{-k+1} + C$ (for $k \neq 1$, using power rule with u-sub $u = ax+b$)
- Integrals with irreducible quadratics $\int \frac{Ax+B}{ax^2+bx+c} dx$ often require splitting the numerator, completing the square in the denominator, and using substitutions leading to \ln and \arctan forms.

2 Solutions: Examples - Partial Fraction Decomposition

1. **Problem (1):** Write the partial fraction decomposition for $\frac{x+7}{x^2-x-6}$.

Solution:

Strategy. Check degrees ($1 \nless 2$). Factor denominator. Use distinct linear factor form. Solve for constants.

Step 1: Factor Denom. $x^2-x-6 = (x-3)(x+2)$. *Step 2: Set up Form.* $\frac{x+7}{(x-3)(x+2)} = \frac{A}{x-3} + \frac{B}{x+2}$. *Step 3: Solve for Constants.* Multiply by $(x-3)(x+2)$: $x+7 = A(x+2) + B(x-3)$. Let $x = 3$: $10 = A(5) \implies A = 2$. Let $x = -2$: $5 = B(-5) \implies B = -1$. *Step 4: Write Decomposition.* $\frac{x+7}{x^2-x-6} = \frac{2}{x-3} - \frac{1}{x+2}$.

2. **Problem (2):** Write the partial fraction decomposition for $\frac{x+8}{x^2+7x+12}$.

Solution:

Strategy. Check degrees ($1 \nless 2$). Factor denominator. Use distinct linear factor form. Solve for constants.

Step 1: Factor Denom. $x^2 + 7x + 12 = (x + 3)(x + 4)$. *Step 2: Set up Form.* $\frac{x+8}{(x+3)(x+4)} = \frac{A}{x+3} + \frac{B}{x+4}$. *Step 3: Solve.* Multiply by $(x + 3)(x + 4)$: $x + 8 = A(x + 4) + B(x + 3)$. Let $x = -3$: $5 = A(1) \implies A = 5$. Let $x = -4$: $4 = B(-1) \implies B = -4$. *Step 4: Write Decomposition.* $\frac{x+8}{x^2+7x+12} = \frac{5}{x+3} - \frac{4}{x+4}$.

3. **Problem (3):** Write the form of the partial fraction decomposition for $\frac{5x^2 + 20x + 6}{x(x + 1)^2}$.

Solution:

Strategy. Denom has distinct linear x and repeated linear $(x + 1)^2$. Write form.

Form: $\frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$.

4. **Problem (4):** Write the form of the partial fraction decomposition for $\frac{3x^2 + 7x + 4}{x^3 + 4x^2 + 4x}$.

Solution:

Strategy. Factor denominator. Identify factors. Write form.

Step 1: Factor Denom. $x(x^2 + 4x + 4) = x(x + 2)^2$. Factors: Distinct linear x , Repeated linear $(x + 2)^2$. *Step 2: Set up Form.* $\frac{A}{x} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$.

3 Solutions: ClassWork - Partial Fraction Decomposition

5. **Problem (5):** Decompose $\frac{2(x + 20)}{x^2 - 25}$. *Solution: Step 1: Expand/Factor.* Num = $2x + 40$. Denom = $(x - 5)(x + 5)$. Degree 1 \nmid 2. *Step 2: Set up.* $\frac{2x+40}{(x-5)(x+5)} = \frac{A}{x-5} + \frac{B}{x+5}$. *Step 3: Clear Denom.* $2x + 40 = A(x + 5) + B(x - 5)$. *Step 4: Solve.* Let $x = 5$: $10 + 40 = A(10) \implies 50 = 10A \implies A = 5$. Let $x = -5$: $-10 + 40 = B(-10) \implies 30 = -10B \implies B = -3$. *Step 5: Decompose.* $\frac{5}{x-5} - \frac{3}{x+5}$.

6. **Problem (6):** Decompose $\frac{3x + 11}{x^2 - 2x - 3}$. *Solution: Step 1: Factor.* Denom = $(x - 3)(x + 1)$. Degree 1 \nmid 2. *Step 2: Set up.* $\frac{3x+11}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1}$. *Step 3: Clear Denom.* $3x + 11 = A(x + 1) + B(x - 3)$. *Step 4: Solve.* Let $x = 3$: $9 + 11 = A(4) \implies 20 = 4A \implies A = 5$. Let $x = -1$: $-3 + 11 = B(-4) \implies 8 = -4B \implies B = -2$. *Step 5: Decompose.* $\frac{5}{x-3} - \frac{2}{x+1}$.

7. **Problem (7):** Decompose $\frac{8x + 3}{x^2 - 3x}$. *Solution: Step 1: Factor.* Denom = $x(x - 3)$. Degree 1 \nmid 2. *Step 2: Set up.* $\frac{8x+3}{x(x-3)} = \frac{A}{x} + \frac{B}{x-3}$. *Step 3: Clear Denom.* $8x + 3 = A(x - 3) + Bx$. *Step 4: Solve.* Let $x = 0$: $3 = A(-3) \implies A = -1$. Let $x = 3$: $27 = B(3) \implies B = 9$. *Step 5: Decompose.* $-\frac{1}{x} + \frac{9}{x-3}$.

8. **Problem (8):** Decompose $\frac{10x+3}{x^2+x}$. *Solution: Step 1: Factor.* Denom= $x(x+1)$. Degree 1 \nmid 2. *Step 2: Set up.* $\frac{10x+3}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$. *Step 3: Clear Denom.* $10x+3 = A(x+1) + Bx$. *Step 4: Solve.* Let $x = 0 : 3 = A(1) \implies A = 3$. Let $x = -1 : -7 = B(-1) \implies B = 7$. *Step 5: Decompose.* $\frac{3}{x} + \frac{7}{x+1}$.
9. **Problem (9):** Decompose $\frac{4x-13}{x^2-3x-10}$. *Solution: Step 1: Factor.* Denom= $(x-5)(x+2)$. Degree 1 \nmid 2. *Step 2: Set up.* $\frac{4x-13}{(x-5)(x+2)} = \frac{A}{x-5} + \frac{B}{x+2}$. *Step 3: Clear Denom.* $4x-13 = A(x+2) + B(x-5)$. *Step 4: Solve.* Let $x = 5 : 7 = A(7) \implies A = 1$. Let $x = -2 : -21 = B(-7) \implies B = 3$. *Step 5: Decompose.* $\frac{1}{x-5} + \frac{3}{x+2}$.
10. **Problem (10):** Decompose $\frac{7x+5}{6(2x^2+3x+1)}$. *Solution: Step 1: Factor.* Denom= $6(2x+1)(x+1)$. Degree 1 \nmid 2. *Step 2: Set up.* $\frac{7x+5}{6(2x+1)(x+1)} = \frac{1}{6} \left[\frac{A}{2x+1} + \frac{B}{x+1} \right]$. Solve for $\frac{7x+5}{(2x+1)(x+1)}$. *Step 3: Clear Denom (inner).* $7x+5 = A(x+1) + B(2x+1)$. *Step 4: Solve.* Let $x = -1 : -2 = B(-1) \implies B = 2$. Let $x = -1/2 : 1.5 = A(0.5) \implies A = 3$. *Step 5: Decompose (Full).* $\frac{1}{6} \left[\frac{3}{2x+1} + \frac{2}{x+1} \right] = \frac{1}{2(2x+1)} + \frac{1}{3(x+1)}$.
11. **Problem (11):** Decompose $\frac{3x^2-2x-5}{x^3+x^2}$. *Solution: Step 1: Factor.* Denom= $x^2(x+1)$. Repeated linear x^2 , distinct linear $x+1$. Degree 2 \nmid 3. *Step 2: Set up.* $\frac{3x^2-2x-5}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$. *Step 3: Clear Denom.* $3x^2-2x-5 = Ax(x+1) + B(x+1) + Cx^2$. *Step 4: Solve.* Let $x = 0 : -5 = B(1) \implies B = -5$. Let $x = -1 : 3+2-5 = C(1) \implies C = 0$. Expand and equate x^2 coeffs: $3x^2 \dots = Ax^2 \dots + Cx^2 \implies 3 = A + C \implies 3 = A + 0 \implies A = 3$. *Step 5: Decompose.* $\frac{3}{x} - \frac{5}{x^2}$.
12. **Problem (12):** Decompose $\frac{3x^2-x+1}{x(x+1)^2}$. *Solution: Step 1: Factors.* Distinct linear x , repeated linear $(x+1)^2$. Degree 2 \nmid 3. *Step 2: Set up.* $\frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$. *Step 3: Clear Denom.* $3x^2-x+1 = A(x+1)^2 + Bx(x+1) + Cx$. *Step 4: Solve.* Let $x = 0 : 1 = A(1) \implies A = 1$. Let $x = -1 : 3+1+1 = C(-1) \implies C = -5$. Expand: $3x^2-x+1 = A(x^2+2x+1) + B(x^2+x) + Cx$. Equate x^2 coeffs: $3 = A + B \implies 3 = 1 + B \implies B = 2$. *Step 5: Decompose.* $\frac{1}{x} + \frac{2}{x+1} - \frac{5}{(x+1)^2}$.
13. **Problem (13):** Decompose $\frac{x+1}{3(x-2)^2}$. *Solution: Step 1: Factors.* Constant $1/3$, repeated linear $(x-2)^2$. Degree 1 \nmid 2. *Step 2: Set up.* $\frac{1}{3} \left[\frac{A}{x-2} + \frac{B}{(x-2)^2} \right]$. Solve for $\frac{x+1}{(x-2)^2}$. *Step 3: Clear Denom (inner).* $x+1 = A(x-2) + B$. *Step 4: Solve.* Let $x = 2 : 3 = B$. Equate x coeffs: $1 = A$. *Step 5: Decompose (Full).* $\frac{1}{3} \left[\frac{1}{x-2} + \frac{3}{(x-2)^2} \right] = \frac{1}{3(x-2)} + \frac{1}{(x-2)^2}$.
14. **Problem (14):** Decompose $\frac{3x-4}{(x-5)^2}$. *Solution: Step 1: Factors.* Repeated linear $(x-5)^2$. Degree 1 \nmid 2. *Step 2: Set up.* $\frac{A}{x-5} + \frac{B}{(x-5)^2}$. *Step 3: Clear Denom.* $3x-4 = A(x-5) + B$.

Step 4: Solve. Let $x = 5 : 11 = B$. Equate x coeffs: $3 = A$. *Step 5: Decompose.*
 $\frac{3}{x-5} + \frac{11}{(x-5)^2}$.

15. **Problem (15):** Decompose $\frac{8x^2 + 15x + 9}{(x+1)^3}$. *Solution: Step 1: Factors.* Repeated linear $(x+1)^3$. Degree 2 \div 3. *Step 2: Set up.* $\frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}$. *Step 3: Clear Denom.* $8x^2 + 15x + 9 = A(x+1)^2 + B(x+1) + C$. *Step 4: Solve.* Let $x = -1 : 8 - 15 + 9 = C \implies C = 2$. Expand: $8x^2 + 15x + 9 = A(x^2 + 2x + 1) + B(x+1) + C = Ax^2 + (2A+B)x + (A+B+C)$. Equate coeffs: $x^2 : A = 8$. $x : 15 = 2A + B = 16 + B \implies B = -1$. (Check const: $A + B + C = 8 - 1 + 2 = 9$. Correct). *Step 5: Decompose.* $\frac{8}{x+1} - \frac{1}{(x+1)^2} + \frac{2}{(x+1)^3}$.

16. **Problem (16):** Decompose $\frac{6x^2 - 5x}{(x+2)^3}$. *Solution: Step 1: Factors.* Repeated linear $(x+2)^3$. Degree 2 \div 3. *Step 2: Set up.* $\frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{(x+2)^3}$. *Step 3: Clear Denom.* $6x^2 - 5x = A(x+2)^2 + B(x+2) + C$. *Step 4: Solve.* Let $x = -2 : 24 + 10 = C \implies C = 34$. Expand: $6x^2 - 5x = A(x^2 + 4x + 4) + B(x+2) + C = Ax^2 + (4A+B)x + (4A+2B+C)$. Equate coeffs: $x^2 : A = 6$. $x : -5 = 4A + B = 24 + B \implies B = -29$. (Check const: $4A + 2B + C = 24 - 58 + 34 = 0$. Correct). *Step 5: Decompose.* $\frac{6}{x+2} - \frac{29}{(x+2)^2} + \frac{34}{(x+2)^3}$.

4 Solutions: Integration by Partial Fractions

Explanation. Now we combine the algebraic decomposition with integration. 1. Decompose the rational function integrand. 2. Integrate the sum of the simpler terms, typically using $\int \frac{1}{u} du = \ln |u|$ or $\int u^n du = \frac{u^{n+1}}{n+1}$. Remember: $\int \frac{A}{ax+b} dx = \frac{A}{a} \ln |ax+b| + C$ and $\int A(ax+b)^n dx = \frac{A}{a} \frac{(ax+b)^{n+1}}{n+1} + C$ for $n \neq -1$.

17. **Problem (17):** Evaluate $\int \frac{1}{x^2 - 1} dx$. *Solution: Step 1: Decompose.* $\frac{1}{x^2 - 1} = \frac{1/2}{x-1} - \frac{1/2}{x+1}$. *Step 2: Integrate.*

$$\begin{aligned} \int \left(\frac{1/2}{x-1} - \frac{1/2}{x+1} \right) dx &= \frac{1}{2} \ln |x-1| - \frac{1}{2} \ln |x+1| + C \\ &= \frac{1}{2} \ln \left| \frac{x-1}{x+1} \right| + C \end{aligned}$$

18. **Problem (18):** Evaluate $\int \frac{4}{x^2 - 4} dx$. *Solution: Step 1: Decompose.* $\frac{4}{(x-2)(x+2)} = \frac{1}{x-2} - \frac{1}{x+2}$. *Step 2: Integrate.*

$$\begin{aligned} \int \left(\frac{1}{x-2} - \frac{1}{x+2} \right) dx &= \ln |x-2| - \ln |x+2| + C \\ &= \ln \left| \frac{x-2}{x+2} \right| + C \end{aligned}$$

19. **Problem (19):** Evaluate $\int \frac{-2}{x^2 - 16} dx$. *Solution: Step 1: Decompose.* $\frac{-2}{(x-4)(x+4)} = \frac{-1/4}{x-4} + \frac{1/4}{x+4}$. *Step 2: Integrate.*

$$\begin{aligned} \int \left(-\frac{1/4}{x-4} + \frac{1/4}{x+4} \right) dx &= -\frac{1}{4} \ln |x-4| + \frac{1}{4} \ln |x+4| + C \\ &= \frac{1}{4} \ln \left| \frac{x+4}{x-4} \right| + C \end{aligned}$$

20. **Problem (20):** Evaluate $\int \frac{-4}{x^2 - 4} dx$. *Solution: Step 1: Decompose.* $\frac{-4}{(x-2)(x+2)} = \frac{-1}{x-2} + \frac{1}{x+2}$. *Step 2: Integrate.*

$$\begin{aligned} \int \left(-\frac{1}{x-2} + \frac{1}{x+2} \right) dx &= -\ln |x-2| + \ln |x+2| + C \\ &= \ln \left| \frac{x+2}{x-2} \right| + C \end{aligned}$$

21. **Problem (21):** Evaluate $\int \frac{1}{2x^2 - x} dx$. *Solution: Step 1: Decompose.* $\frac{1}{x(2x-1)} = -\frac{1}{x} + \frac{2}{2x-1}$. *Step 2: Integrate.*

$$\begin{aligned} \int \left(-\frac{1}{x} + \frac{2}{2x-1} \right) dx &= -\ln |x| + 2 \int \frac{1}{2x-1} dx \\ &= -\ln |x| + 2 \left(\frac{1}{2} \ln |2x-1| \right) + C = -\ln |x| + \ln |2x-1| + C \\ &= \ln \left| \frac{2x-1}{x} \right| + C \end{aligned}$$

22. **Problem (22):** Evaluate $\int \frac{2}{x^2 - 2x} dx$. *Solution: Step 1: Decompose.* $\frac{2}{x(x-2)} = -\frac{1}{x} + \frac{1}{x-2}$. *Step 2: Integrate.*

$$\int \left(-\frac{1}{x} + \frac{1}{x-2} \right) dx = -\ln |x| + \ln |x-2| + C = \ln \left| \frac{x-2}{x} \right| + C$$

23. **Problem (23):** Evaluate $\int \frac{10}{x^2 - 10x} dx$. *Solution: Step 1: Decompose.* $\frac{10}{x(x-10)} = -\frac{1}{x} + \frac{1}{x-10}$. *Step 2: Integrate.*

$$\int \left(-\frac{1}{x} + \frac{1}{x-10} \right) dx = -\ln |x| + \ln |x-10| + C = \ln \left| \frac{x-10}{x} \right| + C$$

24. **Problem (24):** Evaluate $\int \frac{5}{x^2 + x - 6} dx$. *Solution: Step 1: Decompose.* $\frac{5}{(x+3)(x-2)} = -\frac{1}{x+3} + \frac{1}{x-2}$. *Step 2: Integrate.*

$$\int \left(-\frac{1}{x+3} + \frac{1}{x-2}\right) dx = -\ln|x+3| + \ln|x-2| + C = \ln\left|\frac{x-2}{x+3}\right| + C$$

25. **Problem (25):** Evaluate $\int \frac{3}{x^2 + x - 2} dx$. *Solution: Step 1: Decompose.* $\frac{3}{(x+2)(x-1)} = -\frac{1}{x+2} + \frac{1}{x-1}$. *Step 2: Integrate.*

$$\int \left(-\frac{1}{x+2} + \frac{1}{x-1}\right) dx = -\ln|x+2| + \ln|x-1| + C = \ln\left|\frac{x-1}{x+2}\right| + C$$

26. **Problem (26):** Evaluate $\int \frac{1}{4x^2 - 9} dx$. *Solution: Step 1: Decompose.* $\frac{1}{(2x-3)(2x+3)} = \frac{1/6}{2x-3} - \frac{1/6}{2x+3}$. *Step 2: Integrate.*

$$\begin{aligned} \int \left(\frac{1/6}{2x-3} - \frac{1/6}{2x+3}\right) dx &= \frac{1}{6} \int \frac{1}{2x-3} dx - \frac{1}{6} \int \frac{1}{2x+3} dx \\ &= \frac{1}{6} \left(\frac{1}{2} \ln|2x-3|\right) - \frac{1}{6} \left(\frac{1}{2} \ln|2x+3|\right) + C \\ &= \frac{1}{12} \ln|2x-3| - \frac{1}{12} \ln|2x+3| + C = \frac{1}{12} \ln\left|\frac{2x-3}{2x+3}\right| + C \end{aligned}$$

27. **Problem (27):** Evaluate $\int \frac{5-x}{2x^2 + x - 1} dx$. *Solution: Step 1: Decompose.* $\frac{5-x}{(2x-1)(x+1)} = \frac{3}{2x-1} - \frac{2}{x+1}$. *Step 2: Integrate.*

$$\begin{aligned} \int \left(\frac{3}{2x-1} - \frac{2}{x+1}\right) dx &= 3 \int \frac{1}{2x-1} dx - 2 \int \frac{1}{x+1} dx \\ &= 3 \left(\frac{1}{2} \ln|2x-1|\right) - 2(\ln|x+1|) + C = \frac{3}{2} \ln|2x-1| - 2 \ln|x+1| + C \end{aligned}$$

28. **Problem (28):** Evaluate $\int \frac{x+1}{x^2 + 4x + 3} dx$. *Solution: Step 1: Simplify. Factor:* $\frac{x+1}{(x+1)(x+3)} = \frac{1}{x+3}$ (for $x \neq -1$). *Step 2: Integrate.*

$$\int \frac{1}{x+3} dx = \ln|x+3| + C$$

29. **Problem (29):** Evaluate $\int \frac{x^2 - 4x - 4}{x^3 - 4x} dx$. *Solution: Step 1: Decompose.* $\frac{x^2 - 4x - 4}{x(x-2)(x+2)} = \frac{1}{x} - \frac{1}{x-2} + \frac{1}{x+2}$. *Step 2: Integrate.*

$$\begin{aligned} \int \left(\frac{1}{x} - \frac{1}{x-2} + \frac{1}{x+2}\right) dx &= \ln|x| - \ln|x-2| + \ln|x+2| + C \\ &= \ln\left|\frac{x(x+2)}{x-2}\right| + C \end{aligned}$$

30. **Problem (30):** Evaluate $\int \frac{x^2 + 12x + 12}{x^3 - 4x} dx$. *Solution: Step 1: Decompose.* Denominator $x(x - 2)(x + 2)$. $\frac{x^2+12x+12}{x(x-2)(x+2)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+2}$. Clear den.: $x^2 + 12x + 12 = A(x - 2)(x + 2) + Bx(x + 2) + Cx(x - 2)$. $x = 0 \implies 12 = A(-4) \implies A = -3$. $x = 2 \implies 4 + 24 + 12 = B(2)(4) \implies 40 = 8B \implies B = 5$. $x = -2 \implies 4 - 24 + 12 = C(-2)(-4) \implies -8 = 8C \implies C = -1$. Decomposition: $-\frac{3}{x} + \frac{5}{x-2} - \frac{1}{x+2}$. *Step 2: Integrate.*

$$\begin{aligned} \int \left(-\frac{3}{x} + \frac{5}{x-2} - \frac{1}{x+2} \right) dx &= -3 \ln |x| + 5 \ln |x-2| - \ln |x+2| + C \\ &= \ln \left| \frac{(x-2)^5}{x^3(x+2)} \right| + C \end{aligned}$$

31. **Problem (31):** Evaluate $\int \frac{x+2}{x^2-4x} dx$. *Solution: Step 1: Decompose.* Factor: $\frac{x+2}{x(x-4)} = \frac{A}{x} + \frac{B}{x-4}$. $x+2 = A(x-4) + Bx$. $x = 0 \implies 2 = A(-4) \implies A = -1/2$. $x = 4 \implies 6 = B(4) \implies B = 3/2$. Decomposition: $-\frac{1/2}{x} + \frac{3/2}{x-4}$. *Step 2: Integrate.*

$$\int \left(-\frac{1/2}{x} + \frac{3/2}{x-4} \right) dx = -\frac{1}{2} \ln |x| + \frac{3}{2} \ln |x-4| + C$$

32. **Problem (32):** Evaluate $\int \frac{4x^2 + 2x - 1}{x^3 + x^2} dx$. *Solution: Step 1: Decompose.* Factor: $\frac{4x^2+2x-1}{x^2(x+1)} = \frac{3}{x} - \frac{1}{x^2} + \frac{1}{x+1}$. *Step 2: Integrate.*

$$\begin{aligned} \int \left(\frac{3}{x} - x^{-2} + \frac{1}{x+1} \right) dx &= 3 \ln |x| - \frac{x^{-1}}{-1} + \ln |x+1| + C \\ &= 3 \ln |x| + \frac{1}{x} + \ln |x+1| + C \end{aligned}$$

33. **Problem (33):** Evaluate $\int \frac{2x-3}{(x-1)^2} dx$. *Solution: Step 1: Decompose.* $\frac{2x-3}{(x-1)^2} = \frac{2}{x-1} - \frac{1}{(x-1)^2}$. *Step 2: Integrate.*

$$\begin{aligned} \int \left(\frac{2}{x-1} - (x-1)^{-2} \right) dx &= 2 \ln |x-1| - \frac{(x-1)^{-1}}{-1} + C \\ &= 2 \ln |x-1| + \frac{1}{x-1} + C \end{aligned}$$

34. **Problem (34):** Evaluate $\int \frac{x^4}{(x-1)^3} dx$. *Solution: Step 1: Long Division + PFD.* $\frac{x^4}{(x-1)^3} = (x+3) + \frac{6x^2-8x+3}{(x-1)^3}$. Remainder decomposition: $\frac{6}{x-1} + \frac{4}{(x-1)^2} + \frac{1}{(x-1)^3}$. Full expression: $x+3 + \frac{6}{x-1} + 4(x-1)^{-2} + (x-1)^{-3}$. *Step 2: Integrate.*

$$\int \left(x+3 + \frac{6}{x-1} + 4(x-1)^{-2} + (x-1)^{-3} \right) dx$$

$$\begin{aligned}
&= \frac{x^2}{2} + 3x + 6 \ln |x-1| + 4 \frac{(x-1)^{-1}}{-1} + \frac{(x-1)^{-2}}{-2} + C \\
&= \frac{x^2}{2} + 3x + 6 \ln |x-1| - \frac{4}{x-1} - \frac{1}{2(x-1)^2} + C
\end{aligned}$$

35. **Problem (35):** Evaluate $\int \frac{3x^2 + 3x + 1}{x(x^2 + 2x + 1)} dx$. *Solution: Step 1: Simplify Denom. $x(x+1)^2$. Step 2: Decompose. $\frac{3x^2+3x+1}{x(x+1)^2} = \frac{1}{x} + \frac{2}{x+1} - \frac{1}{(x+1)^2}$. Step 3: Integrate.*

$$\begin{aligned}
&\int \left(\frac{1}{x} + \frac{2}{x+1} - (x+1)^{-2} \right) dx \\
&= \ln |x| + 2 \ln |x+1| - \frac{(x+1)^{-1}}{-1} + C = \ln |x| + 2 \ln |x+1| + \frac{1}{x+1} + C
\end{aligned}$$

36. **Problem (36):** Evaluate $\int \frac{3x}{x^2 - 6x + 9} dx$. *Solution: Step 1: Factor Denom. $(x-3)^2$. Step 2: Decompose. $\frac{3x}{(x-3)^2} = \frac{3}{x-3} + \frac{9}{(x-3)^2}$. Step 3: Integrate.*

$$\begin{aligned}
&\int \left(\frac{3}{x-3} + 9(x-3)^{-2} \right) dx = 3 \ln |x-3| + 9 \frac{(x-3)^{-1}}{-1} + C \\
&= 3 \ln |x-3| - \frac{9}{x-3} + C
\end{aligned}$$