INTEGRATION BY PARTS SOLUTIONS

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1 Introduction: Integration by Parts (IBP)

Explanation. Integration by Parts is a technique specifically designed to find the integral of a product of two functions. It's derived directly from the product rule for differentiation.

Recall the Product Rule: $\frac{d}{dx}[u(x)v(x)] = u'(x)v(x) + u(x)v'(x)$.

Writing this using differentials ($\mathrm{d}u = u'(x)\mathrm{d}x, \mathrm{d}v = v'(x)\mathrm{d}x$): $d(uv) = v\,\mathrm{d}u + u\,\mathrm{d}v$.

If we integrate both sides with respect to the underlying variable (like x):

$\int d(uv) = \int v\,\mathrm{d}u + \int u\,\mathrm{d}v$.

Since the integral of a differential is just the function itself, $\int d(uv) = uv$.

So, $uv = \int v\,\mathrm{d}u + \int u\,\mathrm{d}v$.

Rearranging this equation to solve for one of the integrals gives the **Integration by Parts Formula**:

$$\boxed{\int u\ \mathrm{d}v = uv - \int v\ \mathrm{d}u}$$

The Goal: The purpose of this formula is to replace one integral, $\int u\mathrm{d}v$, which might be hard to solve directly, with a different integral, $\int v\mathrm{d}u$, which is hopefully easier to solve.

Choosing u and $\mathrm{d}v$: This selection is the most critical part of the process.

- You must split the entire original integrand (including the $\mathrm{d}x$) into two parts: u and $\mathrm{d}v$.
- The part chosen as $\mathrm{d}v$ **must** be something you know how to integrate to find v.
- The part chosen as u will be differentiated to find $\mathrm{d}u$.
- **Strategic Choice:** The ideal choice makes the **new** integral, $\int v\mathrm{d}u$, simpler than the original one. This often happens when differentiating u makes it significantly simpler (like $x^2 \to 2x$), while integrating $\mathrm{d}v$ doesn't make v overly complicated.
- **LIATE Mnemonic:** A very useful (though not foolproof) guideline for choosing u is the acronym LIATE. Look at the types of functions in your product and choose u based on the type that appears first in this list:
  - **L**: **L**ogarithmic functions (like $\ln x, \log_b x$). These are often good choices for u because their derivatives ($1/x$) are simpler algebraic functions, while their integrals are more complex and often require IBP themselves.

I: **I**nverse trigonometric functions (like $\arcsin x, \arctan x$). Similar to logs, their derivatives are algebraic, while their integrals are harder.

A: **A**lgebraic functions (polynomials like $x^3, 2x - 1$, roots like $\sqrt{x}$). Differentiation reduces their degree, making them simpler. Integration increases their degree.

T: **T**rigonometric functions (like $\sin x, \cos x, \sec^2 x$). Differentiation and integration often cycle between sin and cos, or involve other trig functions.

E: **E**xponential functions (like e^x, a^x). These generally stay similar upon differentiation or integration (up to constants).

Once u is chosen according to LIATE, everything else in the integrand (including $\mathrm{d}x$) automatically becomes $\mathrm{d}v$.

2 Solutions: Examples and Class Work

1. *(Example 1)* Solve $\displaystyle\int xe^x \, \mathrm{d}x$.

Solution:

Strategy. We have a product of x (Algebraic) and e^x (Exponential). Following LIATE, Algebraic (A) comes before Exponential (E). Therefore, we choose $u = x$. This is a good choice because its derivative, $\mathrm{d}u = \mathrm{d}x$, is simpler than $u = x$. The remaining part, $e^x\mathrm{d}x$, becomes $\mathrm{d}v$, which is easy to integrate. We expect one application of IBP to solve this.

Method 1: Standard IBP

Step 1: Define u and $\mathrm{d}v$. Let $u = x$. Let $\mathrm{d}v = e^x\mathrm{d}x$.

Step 2: Compute $\mathrm{d}u$ and v.

Differentiate u: $\mathrm{d}u = \left(\frac{d}{dx}(x)\right)\mathrm{d}x = (1)\mathrm{d}x = \mathrm{d}x$.

Integrate $\mathrm{d}v$: $v = \int \mathrm{d}v = \int e^x\mathrm{d}x = e^x$. *(Constant C omitted until the final answer)*

Step 3: Apply IBP Formula: $\int u\mathrm{d}v = uv - \int v\mathrm{d}u$.

$$\int xe^x \, \mathrm{d}x = (x)(e^x) - \int (e^x)(\mathrm{d}x)$$

$$= xe^x - \int e^x\mathrm{d}x$$

Step 4: Solve the new integral. This integral is simpler than the original.

$$\int e^x\mathrm{d}x = e^x$$

Step 5: Combine results and add the constant of integration C.

$$\int xe^x \, \mathrm{d}x = xe^x - (e^x) + C = xe^x - e^x + C$$

Final Answer (Optional Factoring): $e^x(x-1) + C$.

Method 2: Tabular Method

Explanation. The Tabular Method is also applicable here, though slightly overkill for a single application. It's useful for verifying the standard method. We set up columns for alternating signs, derivatives of u, and integrals of $\mathrm{d}v$.

Let $u = x$, $\mathrm{d}v = e^x \mathrm{d}x$.

Sign	u & derivatives	$\mathrm{d}v = e^x$ & integrals
+	x	e^x
-	1	e^x
+	0	e^x

Multiply diagonally (Sign × u-term × v-integral in **next** row):

$$= (+)(x)(e^x) + (-)(1)(e^x) + C$$

$$= xe^x - e^x + C$$

Result: $e^x(x-1) + C$. Both methods agree.

2. *(Example 2)* Solve $\displaystyle\int xe^{2x}\,\mathrm{d}x$.

   Solution:

   Strategy. Product of Algebraic (x) and Exponential (e^{2x}). LIATE: A before E, so $u = x$. Derivative $\mathrm{d}u = \mathrm{d}x$ is simpler. $\mathrm{d}v = e^{2x}\mathrm{d}x$ is integrable. IBP once.

   Method 1: Standard IBP

   Step 1: Define $u, \mathrm{d}v$. $u = x$, $\mathrm{d}v = e^{2x}\mathrm{d}x$.

   Step 2: Compute $\mathrm{d}u, v$. $\mathrm{d}u = \mathrm{d}x$.

   $v = \int e^{2x}\,\mathrm{d}x$. (Use substitution $w = 2x, \mathrm{d}w = 2\mathrm{d}x \implies \mathrm{d}x = \frac{1}{2}\mathrm{d}w$.

   $\int e^w \left(\frac{1}{2}\mathrm{d}w\right) = \frac{1}{2}e^w = \frac{1}{2}e^{2x}$).

   So, $v = \frac{1}{2}e^{2x}$.

   Step 3: Apply Formula. $\int xe^{2x}\mathrm{d}x = x\left(\frac{1}{2}e^{2x}\right) - \int \left(\frac{1}{2}e^{2x}\right)\mathrm{d}x = \frac{1}{2}xe^{2x} - \frac{1}{2}\int e^{2x}\mathrm{d}x$.

   Step 4: Solve new integral. $\int e^{2x}\mathrm{d}x = \frac{1}{2}e^{2x}$.

   Step 5: Combine + C. $\int xe^{2x}\mathrm{d}x = \frac{1}{2}xe^{2x} - \frac{1}{2}\left(\frac{1}{2}e^{2x}\right) + C = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C$.

   Factored: $\dfrac{e^{2x}}{4}(2x-1) + C$.

Method 2: Tabular Method Let $u = x, \mathrm{d}v = e^{2x}\mathrm{d}x$.

Sign	u & derivatives	$\mathrm{d}v = e^{2x}$ & integrals
+	x	e^{2x}
-	1	$\frac{1}{2}e^{2x}$
+	0	$\frac{1}{4}e^{2x}$

$$= (+)(x)\left(\frac{1}{2}e^{2x}\right) + (-)(1)\left(\frac{1}{4}e^{2x}\right) + C = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C$$

Factored: $\frac{e^{2x}}{4}(2x-1)+C$. Both methods agree.

3. *(ClassWork 1)* Solve $\displaystyle\int x^2 \ln x \, \mathrm{d}x$.

   Solution:

   Strategy. Product of Algebraic (x^2) and Logarithmic ($\ln x$). LIATE: L before A, so $u = \ln x$. This is good because $\mathrm{d}u = (1/x)\,\mathrm{d}x$ is simpler, and $\mathrm{d}v = x^2\mathrm{d}x$ is easy to integrate. Tabular method is not suitable here because the derivatives of $\ln x$ do not terminate at zero.

   Method 1: Standard IBP

   Step 1: Define $u, \mathrm{d}v$. $u = \ln x$, $\mathrm{d}v = x^2\mathrm{d}x$.

   Step 2: Compute $\mathrm{d}u, v$. $\mathrm{d}u = \frac{1}{x}\mathrm{d}x$, $v = \int x^2\mathrm{d}x = \frac{x^3}{3}$.

   Step 3: Apply Formula. $\int x^2 \ln x\mathrm{d}x = (\ln x)(\frac{x^3}{3}) - \int(\frac{x^3}{3})(\frac{1}{x}\mathrm{d}x) = \frac{x^3\ln x}{3} - \int \frac{x^2}{3}\mathrm{d}x$.

   Step 4: Solve new integral. $\int \frac{x^2}{3}\mathrm{d}x = \frac{1}{3}\int x^2\mathrm{d}x = \frac{1}{3}(\frac{x^3}{3}) = \frac{x^3}{9}$.

   Step 5: Combine + C. $\int x^2\ln x\mathrm{d}x = \frac{x^3\ln x}{3} - \frac{x^3}{9} + C$.

   Factored: $\frac{x^3}{9}(3\ln x - 1) + C$.

4. *(ClassWork 2)* Solve $\displaystyle\int x^2 e^x \, \mathrm{d}x$.

   Solution:

   Strategy. Product of Algebraic (x^2) and Exponential (e^x). LIATE: A before E, $u = x^2$. Since u is x^2, its derivative $2x$ is simpler but still contains x. This indicates we'll likely need to apply IBP twice. Both methods are suitable.

   Method 1: Repeated IBP

   First IBP Application:

   Let $u_1 = x^2$, $\mathrm{d}v_1 = e^x\mathrm{d}x$. Then $\mathrm{d}u_1 = 2x\mathrm{d}x$, $v_1 = e^x$.

   $$I = \int x^2e^x\,\mathrm{d}x = (x^2)(e^x) - \int(e^x)(2x\mathrm{d}x) = x^2e^x - 2\int xe^x\mathrm{d}x \quad (*)$$

   The new integral $\int xe^x\mathrm{d}x$ still requires IBP.

   Second IBP Application (for $\int xe^x\mathrm{d}x$):

   Let $u_2 = x$, $\mathrm{d}v_2 = e^x\mathrm{d}x$. Then $\mathrm{d}u_2 = \mathrm{d}x$, $v_2 = e^x$.

   $$\int xe^x\mathrm{d}x = (x)(e^x) - \int(e^x)(\mathrm{d}x) = xe^x - e^x$$

   Combine Results: Substitute the result of the second IBP back into equation $(*)$:

   $$I = x^2e^x - 2(xe^x - e^x) + C$$

Distribute the -2:

$$I = x^2e^x - 2xe^x + 2e^x + C$$

Factored: $e^x(x^2 - 2x + 2) + C$.

Method 2: Tabular Method Let $u = x^2, \mathrm{d}v = e^x\mathrm{d}x$.

Sign	u & derivatives	$\mathrm{d}v = e^x$ & integrals
+	x^2	e^x
-	$2x$	e^x
+	2	e^x
-	0	e^x

Multiply diagonally (Sign × u-term × v-integral in next row) and sum:

$$= (+)(x^2)(e^x) + (-)(2x)(e^x) + (+)(2)(e^x) + C$$

$$= x^2e^x - 2xe^x + 2e^x + C$$

Factored: $e^x(x^2 - 2x + 2) + C$. Both methods agree.

3 Solutions: Practice Problems

5. Solve $\displaystyle\int xe^{3x}\,\mathrm{d}x$.

   Analysis: Algebraic × Exponential. LIATE: $u = x$. IBP once.

   Method 1: Standard IBP

   Let $u = x, \mathrm{d}v = e^{3x}\mathrm{d}x$. $\mathrm{d}u = \mathrm{d}x, v = \int e^{3x}\mathrm{d}x = \frac{1}{3}e^{3x}$.

$$\int xe^{3x}\mathrm{d}x = x\left(\frac{1}{3}e^{3x}\right) - \int\left(\frac{1}{3}e^{3x}\right)\mathrm{d}x = \frac{1}{3}xe^{3x} - \frac{1}{3}\int e^{3x}\mathrm{d}x$$

$$= \frac{1}{3}xe^{3x} - \frac{1}{3}\left(\frac{1}{3}e^{3x}\right) + C = \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + C$$

   Factored: $\frac{e^{3x}}{9}(3x - 1) + C$.

   Method 2: Tabular Method Let $u = x, \mathrm{d}v = e^{3x}\mathrm{d}x$.

Sign	u	$\mathrm{d}v$
+	x	e^{3x}
-	1	$\frac{1}{3}e^{3x}$
+	0	$\frac{1}{9}e^{3x}$

$$= (+)(x)\left(\frac{1}{3}e^{3x}\right) + (-)(1)\left(\frac{1}{9}e^{3x}\right) + C = \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + C$$

6. Solve $\int x^2 e^{3x}\, dx$.

   Analysis: Algebraic (x^2) × Exponential (e^{3x}). LIATE: $u = x^2$. IBP twice.

   Method 1: Repeated IBP

   IBP 1: $u_1 = x^2, dv_1 = e^{3x}dx \implies du_1 = 2x dx, v_1 = \frac{1}{3}e^{3x}$.

   $$I = x^2\left(\frac{1}{3}e^{3x}\right) - \int \frac{1}{3}e^{3x}(2x dx) = \frac{1}{3}x^2e^{3x} - \frac{2}{3}\int xe^{3x}dx \quad (*)$$

   IBP 2 (for $\int xe^{3x}dx$): From Problem 5, result is $\frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x}$.

   Combine: Substitute into (*):

   $I = \frac{1}{3}x^2e^{3x} - \frac{2}{3}\left(\frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x}\right) + C = \frac{1}{3}x^2e^{3x} - \frac{2}{9}xe^{3x} + \frac{2}{27}e^{3x} + C$.

   Factored: $\frac{e^{3x}}{27}(9x^2 - 6x + 2) + C$.

   Method 2: Tabular Method Let $u = x^2, dv = e^{3x}dx$.

Sign	u	dv
+	x^2	e^{3x}
-	$2x$	$\frac{1}{3}e^{3x}$
+	2	$\frac{1}{9}e^{3x}$
-	0	$\frac{1}{27}e^{3x}$

   $$= x^2\left(\frac{1}{3}e^{3x}\right) - 2x\left(\frac{1}{9}e^{3x}\right) + 2\left(\frac{1}{27}e^{3x}\right) + C = \frac{1}{3}x^2e^{3x} - \frac{2}{9}xe^{3x} + \frac{2}{27}e^{3x} + C$$

7. Solve $\int x\ln(2x)\, dx$.

   Analysis: Algebraic × Logarithmic. LIATE: $u = \ln(2x)$. Standard IBP only.

   Method 1: Standard IBP

   Let $u = \ln(2x), dv = x dx \implies du = \left(\frac{1}{x}\right)dx, v = \frac{x^2}{2}$.

   $$\int x\ln(2x)dx = \frac{x^2\ln(2x)}{2} - \int\left(\frac{x^2}{2}\right)\left(\frac{1}{x}\right)dx = \frac{x^2\ln(2x)}{2} - \int\frac{x}{2}dx$$

   $$= \frac{x^2\ln(2x)}{2} - \frac{x^2}{4} + C = \frac{x^2}{4}(2\ln(2x) - 1) + C$$

8. Solve $\int \ln(4x)\, dx$.

   Analysis: Logarithmic only. LIATE: $u = \ln(4x)$, $dv = dx$. Standard IBP only.

   Method 1: Standard IBP

   Let $u = \ln(4x), dv = dx \implies du = \frac{1}{x}dx, v = x$.

$$\int \ln(4x)\mathrm{d}x = x\ln(4x) - \int x\left(\frac{1}{x}\right)\mathrm{d}x = x\ln(4x) - \int 1\mathrm{d}x$$

$$= x\ln(4x) - x + C = x(\ln(4x) - 1) + C$$

9. Solve $\int xe^{-3x}\ \mathrm{d}x$.

Analysis: Algebraic $\times$ Exponential. LIATE: $u = x$. IBP once.

Method 1: Standard IBP

Let $u = x, \mathrm{d}v = e^{-3x}\mathrm{d}x \implies \mathrm{d}u = \mathrm{d}x, v = -\frac{1}{3}e^{-3x}$.

$$\int xe^{-3x}\mathrm{d}x = x\left(-\frac{1}{3}e^{-3x}\right) - \int\left(-\frac{1}{3}e^{-3x}\right)\mathrm{d}x = -\frac{1}{3}xe^{-3x} + \frac{1}{3}\int e^{-3x}\mathrm{d}x$$
$$= -\frac{1}{3}xe^{-3x} + \frac{1}{3}\left(-\frac{1}{3}e^{-3x}\right) + C = -\frac{1}{3}xe^{-3x} - \frac{1}{9}e^{-3x} + C$$

Factored: $-\dfrac{e^{-3x}}{9}(3x+1) + C$.

Method 2: Tabular Method

Let $u = x, \mathrm{d}v = e^{-3x}\mathrm{d}x$.

Sign	u	$\mathrm{d}v$
$+$	x	e^{-3x}
-	1	$-\frac{1}{3}e^{-3x}$
$+$	0	$\frac{1}{9}e^{-3x}$

$$= (+)(x)(-\frac{1}{3}e^{-3x}) + (-)(1)(\frac{1}{9}e^{-3x}) + C = -\frac{1}{3}xe^{-3x} - \frac{1}{9}e^{-3x} + C$$

10. Solve $\int x^2e^{-x}\ \mathrm{d}x$.

Analysis: Algebraic (x^2) $\times$ Exponential (e^{-x}). LIATE: $u = x^2$. IBP twice.

Method 1: Repeated IBP

IBP 1: $u_1 = x^2, \mathrm{d}v_1 = e^{-x}\mathrm{d}x \implies \mathrm{d}u_1 = 2x\mathrm{d}x, v_1 = -e^{-x}$.

$$I = x^2(-e^{-x}) - \int(-e^{-x})(2x\mathrm{d}x) = -x^2e^{-x} + 2\int xe^{-x}\mathrm{d}x$$

IBP 2 (for $\int xe^{-x}\mathrm{d}x$): $u_2 = x, \mathrm{d}v_2 = e^{-x}\mathrm{d}x \implies \mathrm{d}u_2 = \mathrm{d}x, v_2 = -e^{-x}$.

$$\int xe^{-x}\mathrm{d}x = x(-e^{-x}) - \int(-e^{-x})\mathrm{d}x = -xe^{-x} - e^{-x}$$

Combine: $I = -x^2e^{-x} + 2(-xe^{-x} - e^{-x}) + C = -x^2e^{-x} - 2xe^{-x} - 2e^{-x} + C$.

Factored: $-e^{-x}(x^2 + 2x + 2) + C$.

Method 2: Tabular Method Let $u = x^2, \mathrm{d}v = e^{-x}\mathrm{d}x$.

Sign	u	$\mathrm{d}v$
$+$	x^2	e^{-x}
-	$2x$	$-e^{-x}$
$+$	2	e^{-x}
-	0	$-e^{-x}$

$$= (+)(x^2)(-e^{-x}) + (-)(2x)(e^{-x}) + (+)(2)(-e^{-x}) + C = -x^2e^{-x} - 2xe^{-x} - 2e^{-x} + C$$

11. Solve $\int x^2 e^{-2x}\ \mathrm{d}x$.

Analysis: Algebraic (x^2) $\times$ Exponential (e^{-2x}). LIATE: $u = x^2$. IBP twice.

Method 1: Repeated IBP

IBP 1: $u_1 = x^2, \mathrm{d}v_1 = e^{-2x}\mathrm{d}x \implies \mathrm{d}u_1 = 2x\mathrm{d}x, v_1 = -\frac{1}{2}e^{-2x}$.

$$I = x^2(-\frac{1}{2}e^{-2x}) - \int(-\frac{1}{2}e^{-2x})(2x\mathrm{d}x) = -\frac{1}{2}x^2e^{-2x} + \int xe^{-2x}\mathrm{d}x$$

IBP 2 (for $\int xe^{-2x}\mathrm{d}x$): $u_2 = x, \mathrm{d}v_2 = e^{-2x}\mathrm{d}x \implies \mathrm{d}u_2 = \mathrm{d}x, v_2 = -\frac{1}{2}e^{-2x}$.

$$\int xe^{-2x}\mathrm{d}x = x(-\frac{1}{2}e^{-2x}) - \int(-\frac{1}{2}e^{-2x})\mathrm{d}x = -\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x}$$

Combine: $I = -\frac{1}{2}x^2e^{-2x} + (-\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x}) + C$.

Factored: $-\frac{e^{-2x}}{4}(2x^2 + 2x + 1) + C$.

Method 2: Tabular Method Let $u = x^2, \mathrm{d}v = e^{-2x}\mathrm{d}x$.

Sign	u	$\mathrm{d}v$
$+$	x^2	e^{-2x}
-	$2x$	$-\frac{1}{2}e^{-2x}$
$+$	2	$\frac{1}{4}e^{-2x}$
-	0	$-\frac{1}{8}e^{-2x}$

$$\begin{aligned} &= (+)(x^2)(-\frac{1}{2}e^{-2x}) + (-)(2x)(\frac{1}{4}e^{-2x}) + (+)(2)(-\frac{1}{8}e^{-2x}) + C \\ &= -\frac{1}{2}x^2e^{-2x} - \frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + C \end{aligned}$$

12. Solve $\int x^2 e^{2x}\ \mathrm{d}x$.

Analysis: Algebraic (x^2) $\times$ Exponential (e^{2x}). LIATE: $u = x^2$. IBP twice.

Method 1: Repeated IBP

IBP 1: $u_1 = x^2, \mathrm{d}v_1 = e^{2x}\mathrm{d}x \implies \mathrm{d}u_1 = 2x\mathrm{d}x, v_1 = \frac{1}{2}e^{2x}$.

$$I = x^2(\frac{1}{2}e^{2x}) - \int(\frac{1}{2}e^{2x})(2x\mathrm{d}x) = \frac{1}{2}x^2e^{2x} - \int xe^{2x}\mathrm{d}x$$

IBP 2 (for $\int xe^{2x}\mathrm{d}x$): From Problem 2, result is $\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x}$.

Combine: $I = \frac{1}{2}x^2e^{2x} - (\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x}) + C = \frac{1}{2}x^2e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} + C$.

Factored: $\frac{e^{2x}}{4}(2x^2 - 2x + 1) + C$.

Method 2: Tabular Method Let $u = x^2, \mathrm{d}v = e^{2x}\mathrm{d}x$.

Sign	u	$\mathrm{d}v$
+	x^2	e^{2x}
-	$2x$	$\frac{1}{2}e^{2x}$
+	2	$\frac{1}{4}e^{2x}$
-	0	$\frac{1}{8}e^{2x}$

$$= (+)(x^2)(\frac{1}{2}e^{2x}) + (-)(2x)(\frac{1}{4}e^{2x}) + (+)(2)(\frac{1}{8}e^{2x}) + C$$
$$= \frac{1}{2}x^2e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} + C$$

13. Solve $\displaystyle\int \ln(2x)\ \mathrm{d}x$.

Analysis: Logarithmic only. Standard IBP only ($u = \ln(2x)$).

Method 1: Standard IBP

Let $u = \ln(2x), \mathrm{d}v = \mathrm{d}x$. $\mathrm{d}u = \frac{1}{x}\mathrm{d}x, v = x$.

$$\int \ln(2x)\mathrm{d}x = x\ln(2x) - \int x(\frac{1}{x}\mathrm{d}x) = x\ln(2x) - \int 1\mathrm{d}x = x\ln(2x) - x + C$$

Final Answer: $x(\ln(2x) - 1) + C$.

14. Solve $\displaystyle\int \ln(x^2)\ \mathrm{d}x$.

Analysis: Logarithmic. Standard IBP only. Can simplify first.

Method 1: Simplify First

Assume $x > 0$. $\int \ln(x^2)\mathrm{d}x = \int 2\ln x\mathrm{d}x$.

Use IBP: $u = \ln x, \mathrm{d}v = 2\mathrm{d}x$. $\mathrm{d}u = \frac{1}{x}\mathrm{d}x, v = 2x$.

$$\int 2\ln x\mathrm{d}x = (\ln x)(2x) - \int 2x\left(\frac{1}{x}\mathrm{d}x\right) = 2x\ln x - \int 2\mathrm{d}x = 2x\ln x - 2x + C$$

Final Answer: $2x(\ln x - 1) + C$.

Method 2: Direct IBP Let $u = \ln(x^2), \mathrm{d}v = \mathrm{d}x$.

$\mathrm{d}u = \frac{2x}{x^2}\mathrm{d}x = \frac{2}{x}\mathrm{d}x, v = x$.

$$\int \ln(x^2)\mathrm{d}x = x\ln(x^2) - \int x\left(\frac{2}{x}\mathrm{d}x\right) = x\ln(x^2) - \int 2\mathrm{d}x$$
$$= x\ln(x^2) - 2x + C$$

(Equivalent result)

15. **Problem (15):** Solve $\displaystyle\int x \ln x \, dx$. *Analysis:* Algebraic × Logarithmic. LIATE: $u = \ln x$. Standard IBP only. **Method 1: Standard IBP** Let $u = \ln x, dv = x dx$. $du = \frac{1}{x} dx, v = \frac{x^2}{2}$.

$$\int x \ln x dx = (\ln x)(\frac{x^2}{2}) - \int \frac{x^2}{2}\frac{1}{x} dx = \frac{x^2 \ln x}{2} - \int \frac{x}{2} dx$$
$$= \frac{x^2 \ln x}{2} - \frac{1}{2}(\frac{x^2}{2}) + C = \frac{x^2 \ln x}{2} - \frac{x^2}{4} + C$$

Final Answer: $\frac{x^2}{4}(2\ln x - 1) + C$.

16. **Problem (16):** Solve $\displaystyle\int x^3 \ln x \, dx$. *Analysis:* Algebraic × Logarithmic. LIATE: $u = \ln x$. Standard IBP only. **Method 1: Standard IBP** Let $u = \ln x, dv = x^3 dx$. $du = \frac{1}{x} dx, v = \frac{x^4}{4}$.

$$\int x^3 \ln x dx = (\ln x)(\frac{x^4}{4}) - \int \frac{x^4}{4}\frac{1}{x} dx = \frac{x^4 \ln x}{4} - \int \frac{x^3}{4} dx$$
$$= \frac{x^4 \ln x}{4} - \frac{1}{4}(\frac{x^4}{4}) + C = \frac{x^4 \ln x}{4} - \frac{x^4}{16} + C$$

Final Answer: $\frac{x^4}{16}(4\ln x - 1) + C$.

17. **Problem (17):** Solve $\displaystyle\int xe^x \, dx$. *Analysis:* Algebraic × Exponential. LIATE: $u = x$. IBP once. (Same as Problem 1) **Method 1: Standard IBP** Let $u = x, dv = e^x dx$. $du = dx, v = e^x$.

$$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + C$$

Factored: $e^x(x - 1) + C$.

Method 2: Tabular Method Let $u = x, dv = e^x dx$.

Sign	u	dv
+	x	e^x
-	1	e^x
+	0	e^x

$$= (+)(x)(e^x) + (-)(1)(e^x) + C = xe^x - e^x + C$$

18. **Problem (18):** Solve $\displaystyle\int xe^{3x} \, dx$. *Analysis:* Algebraic × Exponential. LIATE: $u = x$. IBP once. (Same as Problem 5) **Method 1: Standard IBP** Let $u = x, dv = e^{3x} dx$. $du = dx, v = \frac{1}{3}e^{3x}$.

$$\int xe^{3x} dx = x(\frac{1}{3}e^{3x}) - \int \frac{1}{3}e^{3x} dx = \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + C$$

Factored: $\frac{e^{3x}}{9}(3x - 1) + C$.

Method 2: Tabular Method Let $u = x, dv = e^{3x} dx$.

Sign	u	$\mathrm{d}v$
+	x	e^{3x}
-	1	$\frac{1}{3}e^{3x}$
+	0	$\frac{1}{9}e^{3x}$

$$= (+)(x)(\frac{1}{3}e^{3x}) + (-)(1)(\frac{1}{9}e^{3x}) + C = \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + C$$

19. **Problem (19):** Solve $\displaystyle\int x^2e^{-x}\ \mathrm{d}x$. *Analysis:* Algebraic (x^2) $\times$ Exponential (e^{-x}). LIATE: $u = x^2$. IBP twice. (Same as Problem 10) **Method 1: Repeated IBP** *IBP 1:* $u_1 = x^2, \mathrm{d}v_1 = e^{-x}\mathrm{d}x \implies \mathrm{d}u_1 = 2x\mathrm{d}x, v_1 = -e^{-x}$.

$$I = x^2(-e^{-x}) - \int(-e^{-x})(2x\mathrm{d}x) = -x^2e^{-x} + 2\int xe^{-x}\mathrm{d}x$$

IBP 2 (for $\int xe^{-x}\mathrm{d}x$): $u_2 = x, \mathrm{d}v_2 = e^{-x}\mathrm{d}x \implies \mathrm{d}u_2 = \mathrm{d}x, v_2 = -e^{-x}$.

$$\int xe^{-x}\mathrm{d}x = x(-e^{-x}) - \int(-e^{-x})\mathrm{d}x = -xe^{-x} - e^{-x}$$

Combine: $I = -x^2e^{-x} + 2(-xe^{-x} - e^{-x}) + C = -x^2e^{-x} - 2xe^{-x} - 2e^{-x} + C$.
Factored: $-e^{-x}(x^2 + 2x + 2) + C$.

Method 2: Tabular Method Let $u = x^2, \mathrm{d}v = e^{-x}\mathrm{d}x$.

Sign	u	$\mathrm{d}v$
+	x^2	e^{-x}
-	$2x$	$-e^{-x}$
+	2	e^{-x}
-	0	$-e^{-x}$

$$= (+)(x^2)(-e^{-x}) + (-)(2x)(e^{-x}) + (+)(2)(-e^{-x}) + C = -x^2e^{-x} - 2xe^{-x} - 2e^{-x} + C$$

20. **Problem (20):** Solve $\displaystyle\int(x^2 - 2x + 1)e^{2x}\ \mathrm{d}x$. *Analysis:* Note $x^2 - 2x + 1 = (x-1)^2$. Algebraic $\times$ Exponential. Needs IBP twice. **Method 1: Repeated IBP**

Let $u_1 = (x-1)^2, \mathrm{d}v_1 = e^{2x}\mathrm{d}x \implies \mathrm{d}u_1 = 2(x-1)\mathrm{d}x, v_1 = \frac{1}{2}e^{2x}$.

$$I = \frac{1}{2}(x-1)^2e^{2x} - \int\frac{1}{2}e^{2x}\cdot 2(x-1)\mathrm{d}x = \frac{1}{2}(x-1)^2e^{2x} - \int(x-1)e^{2x}\mathrm{d}x$$

Let $u_2 = x - 1, \mathrm{d}v_2 = e^{2x}\mathrm{d}x \implies \mathrm{d}u_2 = \mathrm{d}x, v_2 = \frac{1}{2}e^{2x}$.

$$\int(x-1)e^{2x}\mathrm{d}x = (x-1)(\frac{1}{2}e^{2x}) - \int\frac{1}{2}e^{2x}\mathrm{d}x = \frac{1}{2}(x-1)e^{2x} - \frac{1}{4}e^{2x}$$

Combine: $I = \frac{1}{2}(x-1)^2e^{2x} - [\frac{1}{2}(x-1)e^{2x} - \frac{1}{4}e^{2x}] + C$

$$= \frac{e^{2x}}{4}[2(x-1)^2 - 2(x-1) + 1] + C = \frac{e^{2x}}{4}(2x^2 - 6x + 5) + C$$

Method 2: Tabular Method Let $u = (x-1)^2, \mathrm{d}v = e^{2x}\mathrm{d}x$.

Sign	u	$\mathrm{d}v$
+	$(x-1)^2$	e^{2x}
-	$2(x-1)$	$\frac{1}{2}e^{2x}$
+	2	$\frac{1}{4}e^{2x}$
-	0	$\frac{1}{8}e^{2x}$

$$= (x-1)^2(\frac{1}{2}e^{2x}) - 2(x-1)(\frac{1}{4}e^{2x}) + 2(\frac{1}{8}e^{2x}) + C$$
$$= \frac{1}{2}(x-1)^2e^{2x} - \frac{1}{2}(x-1)e^{2x} + \frac{1}{4}e^{2x} + C = \frac{e^{2x}}{4}(2x^2 - 6x + 5) + C$$

21. **Problem (21):** Solve $\int x^3e^x\ \mathrm{d}x$. *Analysis:* Algebraic (x^3) $\times$ Exponential (e^x). Needs IBP 3 times. **Method 1: Repeated IBP (Full Detail)** *IBP 1:* Let $u_1 = x^3, \mathrm{d}v_1 = e^x\mathrm{d}x$. Then $\mathrm{d}u_1 = 3x^2\mathrm{d}x, v_1 = e^x$.

$$I = \int x^3e^x\mathrm{d}x = x^3e^x - \int e^x(3x^2\mathrm{d}x) = x^3e^x - 3\int x^2e^x\mathrm{d}x \quad (1)$$

IBP 2 (for $\int x^2e^x\mathrm{d}x$): Let $u_2 = x^2, \mathrm{d}v_2 = e^x\mathrm{d}x$. Then $\mathrm{d}u_2 = 2x\mathrm{d}x, v_2 = e^x$.

$$\int x^2e^x\mathrm{d}x = x^2e^x - \int e^x(2x\mathrm{d}x) = x^2e^x - 2\int xe^x\mathrm{d}x \quad (2)$$

IBP 3 (for $\int xe^x\mathrm{d}x$): Let $u_3 = x, \mathrm{d}v_3 = e^x\mathrm{d}x$. Then $\mathrm{d}u_3 = \mathrm{d}x, v_3 = e^x$.

$$\int xe^x\mathrm{d}x = xe^x - \int e^x\mathrm{d}x = xe^x - e^x \quad (3)$$

Combine Backwards: Substitute (3) into (2):

$$\int x^2e^x\mathrm{d}x = x^2e^x - 2(xe^x - e^x) = x^2e^x - 2xe^x + 2e^x$$

Substitute this result into (1):

$$I = x^3e^x - 3(x^2e^x - 2xe^x + 2e^x) + C$$
$$I = x^3e^x - 3x^2e^x + 6xe^x - 6e^x + C$$

Factored: $e^x(x^3 - 3x^2 + 6x - 6) + C$.

Method 2: Tabular Method Let $u = x^3, \mathrm{d}v = e^x\mathrm{d}x$.

Sign	u	$\mathrm{d}v$
+	x^3	e^x
-	$3x^2$	e^x
+	$6x$	e^x
-	6	e^x
+	0	e^x

$$= x^3e^x - 3x^2e^x + 6xe^x - 6e^x + C = e^x(x^3 - 3x^2 + 6x - 6) + C$$

22. **Problem (22):** Solve $\int p^4e^{-p}\,dp$. *Analysis:* Algebraic (p^4) × Exponential (e^{-p}). Needs IBP 4 times. **Method 1: Repeated IBP (Full Detail)** *IBP 1:* Let $u_1 = p^4, \mathrm{d}v_1 = e^{-p}\mathrm{d}p$. Then $\mathrm{d}u_1 = 4p^3\mathrm{d}p, v_1 = -e^{-p}$.

$$I = \int p^4e^{-p}\mathrm{d}p = p^4(-e^{-p}) - \int(-e^{-p})(4p^3\mathrm{d}p) = -p^4e^{-p} + 4\int p^3e^{-p}\mathrm{d}p \quad (1)$$

IBP 2 (for $\int p^3e^{-p}\mathrm{d}p$): Let $u_2 = p^3, \mathrm{d}v_2 = e^{-p}\mathrm{d}p$. Then $\mathrm{d}u_2 = 3p^2\mathrm{d}p, v_2 = -e^{-p}$.

$$I_2 = \int p^3e^{-p}\mathrm{d}p = p^3(-e^{-p}) - \int(-e^{-p})(3p^2\mathrm{d}p) = -p^3e^{-p} + 3\int p^2e^{-p}\mathrm{d}p \quad (2)$$

IBP 3 (for $\int p^2e^{-p}\mathrm{d}p$): Let $u_3 = p^2, \mathrm{d}v_3 = e^{-p}\mathrm{d}p$. Then $\mathrm{d}u_3 = 2p\mathrm{d}p, v_3 = -e^{-p}$.

$$I_3 = \int p^2e^{-p}\mathrm{d}p = p^2(-e^{-p}) - \int(-e^{-p})(2p\mathrm{d}p) = -p^2e^{-p} + 2\int pe^{-p}\mathrm{d}p \quad (3)$$

IBP 4 (for $\int pe^{-p}\mathrm{d}p$): Let $u_4 = p, \mathrm{d}v_4 = e^{-p}\mathrm{d}p$. Then $\mathrm{d}u_4 = \mathrm{d}p, v_4 = -e^{-p}$.

$$I_4 = \int pe^{-p}\mathrm{d}p = p(-e^{-p}) - \int(-e^{-p})\mathrm{d}p = -pe^{-p} + \int e^{-p}\mathrm{d}p = -pe^{-p} - e^{-p} \quad (4)$$

Combine Backwards: Substitute (4) into (3): $I_3 = -p^2e^{-p} + 2(-pe^{-p} - e^{-p}) = -p^2e^{-p} - 2pe^{-p} - 2e^{-p} = -e^{-p}(p^2+2p+2)$. Substitute I_3 into (2): $I_2 = -p^3e^{-p} + 3[-e^{-p}(p^2+2p+2)] = -p^3e^{-p} - 3p^2e^{-p} - 6pe^{-p} - 6e^{-p} = -e^{-p}(p^3+3p^2+6p+6)$. Substitute I_2 into (1): $I = -p^4e^{-p} + 4[-e^{-p}(p^3+3p^2+6p+6)] + C$ $I = -p^4e^{-p} - 4p^3e^{-p} - 12p^2e^{-p} - 24pe^{-p} - 24e^{-p} + C$. *Factored:* $-e^{-p}(p^4 + 4p^3 + 12p^2 + 24p + 24) + C$.

Method 2: Tabular Method Let $u = p^4, \mathrm{d}v = e^{-p}\mathrm{d}p$.

Sign	$u = p^4$	$\mathrm{d}v = e^{-p}\mathrm{d}p$
+	p^4	e^{-p}
-	$4p^3$	$-e^{-p}$
+	$12p^2$	e^{-p}
-	$24p$	$-e^{-p}$
+	24	e^{-p}
-	0	$-e^{-p}$

$$= p^4(-e^{-p}) - (4p^3)(e^{-p}) + (12p^2)(-e^{-p}) - (24p)(e^{-p}) + (24)(-e^{-p}) + C$$
$$= -p^4e^{-p} - 4p^3e^{-p} - 12p^2e^{-p} - 24pe^{-p} - 24e^{-p} + C$$

Factored: $-e^{-p}(p^4 + 4p^3 + 12p^2 + 24p + 24) + C$.

23. **Problem (23):** Solve $\int(x^2-5x)e^x\,dx$. *Analysis:* Algebraic (x^2-5x) × Exponential (e^x). Needs IBP twice. **Method 1: Repeated IBP** *IBP 1:* $u_1 = x^2 - 5x, \mathrm{d}v_1 = e^x\mathrm{d}x \implies \mathrm{d}u_1 = (2x-5)\mathrm{d}x, v_1 = e^x$.

$$I = (x^2 - 5x)e^x - \int e^x(2x-5)\mathrm{d}x$$

IBP 2 (for $\int(2x-5)e^x\mathrm{d}x$): $u_2 = 2x-5, \mathrm{d}v_2 = e^x\mathrm{d}x \implies \mathrm{d}u_2 = 2\mathrm{d}x, v_2 = e^x$.

$$\int(2x-5)e^x\mathrm{d}x = (2x-5)e^x - \int e^x(2\mathrm{d}x) = (2x-5)e^x - 2e^x$$

Combine: $I = (x^2-5x)e^x - [(2x-5)e^x - 2e^x] + C = e^x(x^2-5x-2x+5+2) + C$

$$= e^x(x^2-7x+7) + C$$

Method 2: Tabular Method Let $u = x^2-5x, \mathrm{d}v = e^x\mathrm{d}x$.

Sign	u	$\mathrm{d}v$
+	x^2-5x	e^x
-	$2x-5$	e^x
+	2	e^x
-	0	e^x

$= (x^2-5x)e^x - (2x-5)e^x + 2e^x + C = e^x(x^2-5x-2x+5+2) + C = e^x(x^2-7x+7) + C$

24. **Problem (24):** Solve $\displaystyle\int(r^2+r+1)e^r\,\mathrm{d}r$. *Analysis:* Algebraic (r^2+r+1) × Exponential (e^r). Needs IBP twice.

Method 1: Repeated IBP

IBP 1: $u_1 = r^2+r+1, \mathrm{d}v_1 = e^r\mathrm{d}r \implies \mathrm{d}u_1 = (2r+1)\mathrm{d}r, v_1 = e^r$.

$$I = (r^2+r+1)e^r - \int e^r(2r+1)\mathrm{d}r$$

IBP 2 (for $\int(2r+1)e^r\mathrm{d}r$): $u_2 = 2r+1, \mathrm{d}v_2 = e^r\mathrm{d}r \implies \mathrm{d}u_2 = 2\mathrm{d}r, v_2 = e^r$.

$$\int(2r+1)e^r\mathrm{d}r = (2r+1)e^r - \int e^r(2\mathrm{d}r) = (2r+1)e^r - 2e^r$$

Combine: $I = (r^2+r+1)e^r - [(2r+1)e^r - 2e^r] + C = e^r(r^2+r+1-2r-1+2) + C$

$$= e^r(r^2-r+2) + C$$

Method 2: Tabular Method Let $u = r^2+r+1, \mathrm{d}v = e^r\mathrm{d}r$.

Sign	u	$\mathrm{d}v$
+	r^2+r+1	e^r
-	$2r+1$	e^r
+	2	e^r
-	0	e^r

$= (r^2+r+1)e^r - (2r+1)e^r + 2e^r + C = e^r(r^2+r+1-2r-1+2) + C = e^r(r^2-r+2) + C$

25. **Problem (25):** Solve $\int x^5 e^x \, dx$. *Analysis:* Algebraic (x^5) × Exponential (e^x). Needs IBP 5 times. **Method 1: Repeated IBP (Full Detail)** *IBP 1:* $u_1 = x^5, dv_1 = e^x dx \implies du_1 = 5x^4 dx, v_1 = e^x$.

$$I = x^5 e^x - 5 \int x^4 e^x dx$$

IBP 2: $u_2 = x^4, dv_2 = e^x dx \implies du_2 = 4x^3 dx, v_2 = e^x$.

$$\int x^4 e^x dx = x^4 e^x - 4 \int x^3 e^x dx$$

$I = x^5 e^x - 5[x^4 e^x - 4 \int x^3 e^x dx] = x^5 e^x - 5x^4 e^x + 20 \int x^3 e^x dx$. *IBP 3:* $u_3 = x^3, dv_3 = e^x dx \implies du_3 = 3x^2 dx, v_3 = e^x$.

$$\int x^3 e^x dx = x^3 e^x - 3 \int x^2 e^x dx$$

$I = x^5 e^x - 5x^4 e^x + 20[x^3 e^x - 3 \int x^2 e^x dx] = x^5 e^x - 5x^4 e^x + 20x^3 e^x - 60 \int x^2 e^x dx$. *IBP 4:* $u_4 = x^2, dv_4 = e^x dx \implies du_4 = 2x dx, v_4 = e^x$.

$$\int x^2 e^x dx = x^2 e^x - 2 \int x e^x dx$$

$I = x^5 e^x - 5x^4 e^x + 20x^3 e^x - 60[x^2 e^x - 2 \int x e^x dx]$
$I = x^5 e^x - 5x^4 e^x + 20x^3 e^x - 60x^2 e^x + 120 \int x e^x dx$.
IBP 5: $u_5 = x, dv_5 = e^x dx \implies du_5 = dx, v_5 = e^x$.

$$\int x e^x dx = x e^x - \int e^x dx = x e^x - e^x$$

Combine All:
$I = x^5 e^x - 5x^4 e^x + 20x^3 e^x - 60x^2 e^x + 120[x e^x - e^x] + C$.
$I = x^5 e^x - 5x^4 e^x + 20x^3 e^x - 60x^2 e^x + 120x e^x - 120 e^x + C$.
Factored: $e^x(x^5 - 5x^4 + 20x^3 - 60x^2 + 120x - 120) + C$.

Method 2: Tabular Method Let $u = x^5, dv = e^x dx$.

Sign	$u = x^5$	$dv = e^x dx$
+	x^5	e^x
-	$5x^4$	e^x
+	$20x^3$	e^x
-	$60x^2$	e^x
+	$120x$	e^x
-	120	e^x
+	0	e^x

$$= x^5 e^x - 5x^4 e^x + 20x^3 e^x - 60x^2 e^x + 120x e^x - 120 e^x + C$$
$$= e^x(x^5 - 5x^4 + 20x^3 - 60x^2 + 120x - 120) + C$$

26. **Problem (26):** Solve $\displaystyle\int t^2 e^{4t}\, \mathrm{d}t$. *Analysis:* Algebraic (t^2) $\times$ Exponential (e^{4t}). Needs IBP twice. **Method 1: Repeated IBP** *IBP 1:* $u_1 = t^2, \mathrm{d}v_1 = e^{4t}\mathrm{d}t \implies \mathrm{d}u_1 = 2t\mathrm{d}t, v_1 = \frac{1}{4}e^{4t}$.

$$I = t^2(\frac{1}{4}e^{4t}) - \int \frac{1}{4}e^{4t}(2t\mathrm{d}t) = \frac{1}{4}t^2e^{4t} - \frac{1}{2}\int te^{4t}\mathrm{d}t$$

IBP 2 (for $\int te^{4t}\mathrm{d}t$): $u_2 = t, \mathrm{d}v_2 = e^{4t}\mathrm{d}t \implies \mathrm{d}u_2 = \mathrm{d}t, v_2 = \frac{1}{4}e^{4t}$.

$$\int te^{4t}\mathrm{d}t = t(\frac{1}{4}e^{4t}) - \int \frac{1}{4}e^{4t}\mathrm{d}t = \frac{1}{4}te^{4t} - \frac{1}{16}e^{4t}$$

Combine: $I = \frac{1}{4}t^2e^{4t} - \frac{1}{2}[\frac{1}{4}te^{4t} - \frac{1}{16}e^{4t}] + C = \frac{1}{4}t^2e^{4t} - \frac{1}{8}te^{4t} + \frac{1}{32}e^{4t} + C$. *Factored:* $\frac{e^{4t}}{32}(8t^2 - 4t + 1) + C$.

Method 2: Tabular Method Let $u = t^2, \mathrm{d}v = e^{4t}\mathrm{d}t$.

Sign	$u = t^2$	$\mathrm{d}v = e^{4t}\mathrm{d}t$
+	t^2	e^{4t}
-	$2t$	$\frac{1}{4}e^{4t}$
+	2	$\frac{1}{16}e^{4t}$
-	0	$\frac{1}{64}e^{4t}$

$$= t^2(\frac{1}{4}e^{4t}) - (2t)(\frac{1}{16}e^{4t}) + 2(\frac{1}{64}e^{4t}) + C$$
$$= \frac{1}{4}t^2e^{4t} - \frac{1}{8}te^{4t} + \frac{1}{32}e^{4t} + C = \frac{e^{4t}}{32}(8t^2 - 4t + 1) + C$$