PARTIAL FRACTIONS AND ITS INTEGRATION

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1 Introduction: Partial Fractions

Explanation. Partial Fraction Decomposition is an algebraic technique used to break down a complex rational function (a ratio of polynomials) into a sum of simpler rational functions. This is extremely useful because the simpler fractions are often much easier to integrate.

Pre-requisite: The degree of the numerator polynomial must be strictly less than the degree of the denominator polynomial. If not, perform polynomial long division first to get a polynomial plus a proper rational function (where the remainder term satisfies the degree condition).

The Process:

1. **Factor the Denominator:** Completely factor the denominator into linear factors (like $ax + b$) and irreducible quadratic factors (like $ax^2 + bx + c$ where $b^2 - 4ac < 0$, meaning it cannot be factored further using real numbers).

2. **Set up the Decomposition Form:** Based on the factors in the denominator, write the rational function as a sum of simpler fractions with unknown constants (A, B, C, etc.) in the numerators. The rules for the form depend on the type and repetition of the factors:

   - **Distinct Linear Factor:** For each unique factor $(ax+b)$ in the denominator, include a term $\frac{A}{ax+b}$ in the decomposition, where A is an unknown constant.
   - **Repeated Linear Factor:** If a linear factor $(ax+b)$ appears k times, i.e., $(ax+b)^k$, you must include k terms in the decomposition, one for each power from 1 to k: $\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_k}{(ax+b)^k}$.
   - **Distinct Irreducible Quadratic Factor:** For each unique factor $(ax^2 + bx + c)$ that cannot be factored further, include a term $\frac{Ax+B}{ax^2+bx+c}$ in the decomposition (note the linear numerator).
   - **Repeated Irreducible Quadratic Factor:** If an irreducible quadratic factor $(ax^2 + bx + c)$ appears k times, i.e., $(ax^2 + bx + c)^k$, include k terms: $\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_kx+B_k}{(ax^2+bx+c)^k}$.

3. **Solve for the Unknown Constants (A, B, C...):**

   - **Clear Denominators:** Multiply both sides of the equation (original fraction = sum of partial fractions) by the fully factored original denominator. This results in an equation involving only polynomials.
   - **Find Constants:** There are two main methods, often used in combination:
     - *Method 1: Substituting Convenient Values (Heaviside Method):* Substitute the roots of the linear factors (the values of x that make those factors zero) into the equation after clearing denominators. This often allows you to solve for the constants associated with those linear factors directly.

- *Method 2: Equating Coefficients:* Expand the entire right side of the equation (after clearing denominators) and collect terms by powers of x (e.g., all x^2 terms together, all x terms together, all constant terms together). The coefficients of each power of x on the right side must equal the coefficients of the corresponding power of x in the original numerator. This creates a system of linear equations which you can solve for the unknown constants A, B, C, etc.

For repeated factors or irreducible quadratic factors, you often need to use a combination of substituting convenient values and equating coefficients.

4. **Write the Final Decomposition:** Substitute the numerical values you found for A, B, C... back into the decomposition form you set up in Step 2.

Integration: After finding the partial fraction decomposition, the original integral becomes the integral of a sum of simpler terms. Integrate each term separately. Remember the common integrals:

- $\int \frac{A}{ax+b} \mathrm{d}x = \frac{A}{a} \ln|ax+b| + C$ (using a simple u-substitution $u = ax + b$)
- $\int \frac{A}{(ax+b)^k} \mathrm{d}x = \int A(ax+b)^{-k} \mathrm{d}x = \frac{A}{a} \frac{(ax+b)^{-k+1}}{-k+1} + C$ (for $k \neq 1$, using power rule with u-sub $u = ax + b$)
- Integrals with irreducible quadratics $\int \frac{Ax+B}{ax^2+bx+c} \mathrm{d}x$ often require splitting the numerator, completing the square in the denominator, and using substitutions leading to ln and arctan forms.

2 Solutions: Examples - Partial Fraction Decomposition

1. **Problem:** Write the partial fraction decomposition for $\dfrac{x+7}{x^2-x-6}$.

   Solution:

   Strategy. Check degrees (1 ¡ 2). Factor denominator. Use distinct linear factor form. Solve for constants.

   Step 1: Factor Denom. $x^2 - x - 6 = (x-3)(x+2)$. *Step 2: Set up Form.* $\frac{x+7}{(x-3)(x+2)} = \frac{A}{x-3} + \frac{B}{x+2}$. *Step 3: Solve for Constants.* Multiply by $(x-3)(x+2)$: $x + 7 = A(x+2) + B(x-3)$. Let $x = 3$: $10 = A(5) \implies A = 2$. Let $x = -2$: $5 = B(-5) \implies B = -1$. *Step 4: Write Decomposition.* $\frac{x+7}{x^2-x-6} = \frac{2}{x-3} - \frac{1}{x+2}$.

2. **Problem:** Write the partial fraction decomposition for $\dfrac{x+8}{x^2+7x+12}$.

   Solution:

   Strategy. Check degrees (1 ¡ 2). Factor denominator. Use distinct linear factor form. Solve for constants.

Step 1: Factor Denom. $x^2 + 7x + 12 = (x + 3)(x + 4)$. *Step 2: Set up Form.* $\frac{x+8}{(x+3)(x+4)} = \frac{A}{x+3} + \frac{B}{x+4}$. *Step 3: Solve.* Multiply by $(x + 3)(x + 4)$: $x + 8 = A(x + 4) + B(x + 3)$. Let $x = -3$: $5 = A(1) \implies A = 5$. Let $x = -4$: $4 = B(-1) \implies B = -4$. *Step 4: Write Decomposition.* $\frac{x+8}{x^2+7x+12} = \frac{5}{x+3} - \frac{4}{x+4}$.

3. **Problem:** Write the form of the partial fraction decomposition for $\dfrac{5x^2 + 20x + 6}{x(x + 1)^2}$. *Solution:*

   Strategy. Denom has distinct linear x and repeated linear $(x + 1)^2$. Write form.

   Form: $\frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$.

4. **Problem:** Write the form of the partial fraction decomposition for $\dfrac{3x^2 + 7x + 4}{x^3 + 4x^2 + 4x}$. *Solution:*

   Strategy. Factor denominator. Identify factors. Write form.

   Step 1: Factor Denom. $x(x^2 + 4x + 4) = x(x + 2)^2$. Factors: Distinct linear x, Repeated linear $(x + 2)^2$. *Step 2: Set up Form.* $\frac{A}{x} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$.

3 Solutions: ClassWork - Partial Fraction Decomposition

5. **Problem:** Decompose $\dfrac{2(x + 20)}{x^2 - 25}$. *Solution: Step 1: Expand/Factor.* Num$= 2x + 40$. Denom$=(x-5)(x+5)$. Degree 1 ¡ 2. *Step 2: Set up.* $\frac{2x+40}{(x-5)(x+5)} = \frac{A}{x-5} + \frac{B}{x+5}$. *Step 3: Clear Denom.* $2x + 40 = A(x + 5) + B(x - 5)$. *Step 4: Solve.* Let $x = 5 : 10 + 40 = A(10) \implies 50 = 10A \implies A = 5$. Let $x = -5 : -10 + 40 = B(-10) \implies 30 = -10B \implies B = -3$. *Step 5: Decompose.* $\frac{5}{x-5} - \frac{3}{x+5}$.

6. **Problem:** Decompose $\dfrac{3x + 11}{x^2 - 2x - 3}$. *Solution: Step 1: Factor.* Denom$=(x - 3)(x + 1)$. Degree 1 ¡ 2. *Step 2: Set up.* $\frac{3x+11}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1}$. *Step 3: Clear Denom.* $3x + 11 = A(x + 1) + B(x - 3)$. *Step 4: Solve.* Let $x = 3 : 9 + 11 = A(4) \implies 20 = 4A \implies A = 5$. Let $x = -1 : -3 + 11 = B(-4) \implies 8 = -4B \implies B = -2$. *Step 5: Decompose.* $\frac{5}{x-3} - \frac{2}{x+1}$.

7. **Problem:** Decompose $\dfrac{8x + 3}{x^2 - 3x}$. *Solution: Step 1: Factor.* Denom$=x(x - 3)$. Degree 1 ¡ 2. *Step 2: Set up.* $\frac{8x+3}{x(x-3)} = \frac{A}{x} + \frac{B}{x-3}$. *Step 3: Clear Denom.* $8x + 3 = A(x - 3) + Bx$. *Step 4: Solve.* Let $x = 0 : 3 = A(-3) \implies A = -1$. Let $x = 3 : 27 = B(3) \implies B = 9$. *Step 5: Decompose.* $-\frac{1}{x} + \frac{9}{x-3}$.

8. **Problem:** Decompose $\dfrac{10x+3}{x^2+x}$. *Solution: Step 1: Factor.* Denom=$x(x+1)$. Degree 1 ¡ 2. *Step 2: Set up.* $\frac{10x+3}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$. *Step 3: Clear Denom.* $10x+3 = A(x+1)+Bx$. *Step 4: Solve.* Let $x=0 : 3 = A(1) \implies A = 3$. Let $x=-1 : -7 = B(-1) \implies B = 7$. *Step 5: Decompose.* $\frac{3}{x} + \frac{7}{x+1}$.

9. **Problem:** Decompose $\dfrac{4x-13}{x^2-3x-10}$. *Solution: Step 1: Factor.* Denom=$(x-5)(x+2)$. Degree 1 ¡ 2. *Step 2: Set up.* $\frac{4x-13}{(x-5)(x+2)} = \frac{A}{x-5} + \frac{B}{x+2}$. *Step 3: Clear Denom.* $4x-13 = A(x+2)+B(x-5)$. *Step 4: Solve.* Let $x=5 : 7 = A(7) \implies A = 1$. Let $x=-2 : -21 = B(-7) \implies B = 3$. *Step 5: Decompose.* $\frac{1}{x-5} + \frac{3}{x+2}$.

10. **Problem (10):** Decompose $\dfrac{7x+5}{6(2x^2+3x+1)}$. *Solution: Step 1: Factor.* Denom=$6(2x+1)(x+1)$. Degree 1 ¡ 2. *Step 2: Set up.* $\frac{7x+5}{6(2x+1)(x+1)} = \frac{1}{6}\left[\frac{A}{2x+1} + \frac{B}{x+1}\right]$. Solve for $\frac{7x+5}{(2x+1)(x+1)}$. *Step 3: Clear Denom (inner).* $7x+5 = A(x+1)+B(2x+1)$. *Step 4: Solve.* Let $x=-1 : -2 = B(-1) \implies B = 2$. Let $x=-1/2 : 1.5 = A(0.5) \implies A = 3$. *Step 5: Decompose (Full).* $\frac{1}{6}\left[\frac{3}{2x+1} + \frac{2}{x+1}\right] = \frac{1}{2(2x+1)} + \frac{1}{3(x+1)}$.

11. **Problem (11):** Decompose $\dfrac{3x^2-2x-5}{x^3+x^2}$. *Solution: Step 1: Factor.* Denom=$x^2(x+1)$. Repeated linear x^2, distinct linear $x+1$. Degree 2 ¡ 3. *Step 2: Set up.* $\frac{3x^2-2x-5}{x^2(x+1)} = \frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+1}$. *Step 3: Clear Denom.* $3x^2-2x-5 = Ax(x+1)+B(x+1)+Cx^2$. *Step 4: Solve.* Let $x=0 : -5 = B(1) \implies B = -5$. Let $x=-1 : 3+2-5 = C(1) \implies C = 0$. Expand and equate x^2 coeffs: $3x^2... = Ax^2...+Cx^2 \implies 3 = A+C \implies 3 = A+0 \implies A = 3$. *Step 5: Decompose.* $\frac{3}{x} - \frac{5}{x^2}$.

12. **Problem (12):** Decompose $\dfrac{3x^2-x+1}{x(x+1)^2}$. *Solution: Step 1: Factors.* Distinct linear x, repeated linear $(x+1)^2$. Degree 2 ¡ 3. *Step 2: Set up.* $\frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$. *Step 3: Clear Denom.* $3x^2-x+1 = A(x+1)^2+Bx(x+1)+Cx$. *Step 4: Solve.* Let $x=0 : 1 = A(1) \implies A = 1$. Let $x=-1 : 3+1+1 = C(-1) \implies C = -5$. Expand: $3x^2-x+1 = A(x^2+2x+1)+B(x^2+x)+Cx$. Equate x^2 coeffs: $3 = A+B \implies 3 = 1+B \implies B = 2$. *Step 5: Decompose.* $\frac{1}{x} + \frac{2}{x+1} - \frac{5}{(x+1)^2}$.

13. **Problem (13):** Decompose $\dfrac{x+1}{3(x-2)^2}$. *Solution: Step 1: Factors.* Constant 1/3, repeated linear $(x-2)^2$. Degree 1 ¡ 2. *Step 2: Set up.* $\frac{1}{3}\left[\frac{A}{x-2} + \frac{B}{(x-2)^2}\right]$. Solve for $\frac{x+1}{(x-2)^2}$. *Step 3: Clear Denom (inner).* $x+1 = A(x-2)+B$. *Step 4: Solve.* Let $x=2 : 3 = B$. Equate x coeffs: $1 = A$. *Step 5: Decompose (Full).* $\frac{1}{3}\left[\frac{1}{x-2} + \frac{3}{(x-2)^2}\right] = \frac{1}{3(x-2)} + \frac{1}{(x-2)^2}$.

14. **Problem (14):** Decompose $\dfrac{3x-4}{(x-5)^2}$. *Solution: Step 1: Factors.* Repeated linear $(x-5)^2$. Degree 1 ¡ 2. *Step 2: Set up.* $\frac{A}{x-5} + \frac{B}{(x-5)^2}$. *Step 3: Clear Denom.* $3x-4 = A(x-5)+B$.

Step 4: Solve. Let $x = 5 : 11 = B$. Equate x coeffs: $3 = A$. *Step 5: Decompose.* $\frac{3}{x-5} + \frac{11}{(x-5)^2}$.

15. **Problem (15):** Decompose $\dfrac{8x^2 + 15x + 9}{(x+1)^3}$. *Solution: Step 1: Factors.* Repeated linear $(x+1)^3$. Degree 2 ¡ 3. *Step 2: Set up.* $\frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}$. *Step 3: Clear Denom.* $8x^2 + 15x + 9 = A(x+1)^2 + B(x+1) + C$. *Step 4: Solve.* Let $x = -1 : 8 - 15 + 9 = C \implies C = 2$. Expand: $8x^2 + 15x + 9 = A(x^2 + 2x + 1) + B(x+1) + C = Ax^2 + (2A + B)x + (A + B + C)$. Equate coeffs: $x^2 : A = 8$. $x : 15 = 2A + B = 16 + B \implies B = -1$. (Check const: $A + B + C = 8 - 1 + 2 = 9$. Correct). *Step 5: Decompose.* $\frac{8}{x+1} - \frac{1}{(x+1)^2} + \frac{2}{(x+1)^3}$.

16. **Problem (16):** Decompose $\dfrac{6x^2 - 5x}{(x+2)^3}$. *Solution: Step 1: Factors.* Repeated linear $(x+2)^3$. Degree 2 ¡ 3. *Step 2: Set up.* $\frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{(x+2)^3}$. *Step 3: Clear Denom.* $6x^2 - 5x = A(x+2)^2 + B(x+2) + C$. *Step 4: Solve.* Let $x = -2 : 24 + 10 = C \implies C = 34$. Expand: $6x^2 - 5x = A(x^2 + 4x + 4) + B(x+2) + C = Ax^2 + (4A + B)x + (4A + 2B + C)$. Equate coeffs: $x^2 : A = 6$. $x : -5 = 4A + B = 24 + B \implies B = -29$. (Check const: $4A + 2B + C = 24 - 58 + 34 = 0$. Correct). *Step 5: Decompose.* $\frac{6}{x+2} - \frac{29}{(x+2)^2} + \frac{34}{(x+2)^3}$.

4 Solutions: Integration by Partial Fractions

Explanation. Now we combine the algebraic decomposition with integration. 1. Decompose the rational function integrand. 2. Integrate the sum of the simpler terms, typically using $\int \frac{1}{u} du = \ln|u|$ or $\int u^n du = \frac{u^{n+1}}{n+1}$. Remember: $\int \frac{A}{ax+b} \mathrm{d}x = \frac{A}{a} \ln|ax+b| + C$ and $\int A(ax+b)^n \mathrm{d}x = \frac{A}{a} \frac{(ax+b)^{n+1}}{n+1} + C$ for $n \neq -1$.

17. **Problem (17):** Evaluate $\displaystyle\int \frac{1}{x^2 - 1} \mathrm{d}x$. *Solution: Step 1: Decompose.* $\frac{1}{x^2-1} = \frac{1/2}{x-1} - \frac{1/2}{x+1}$. *Step 2: Integrate.*

$$\int \left(\frac{1/2}{x-1} - \frac{1/2}{x+1} \right) \mathrm{d}x = \frac{1}{2} \ln|x-1| - \frac{1}{2} \ln|x+1| + C$$
$$= \frac{1}{2} \ln\left| \frac{x-1}{x+1} \right| + C$$

18. **Problem (18):** Evaluate $\displaystyle\int \frac{4}{x^2 - 4} \mathrm{d}x$. *Solution: Step 1: Decompose.* $\frac{4}{(x-2)(x+2)} = \frac{1}{x-2} - \frac{1}{x+2}$. *Step 2: Integrate.*

$$\int \left(\frac{1}{x-2} - \frac{1}{x+2} \right) \mathrm{d}x = \ln|x-2| - \ln|x+2| + C$$
$$= \ln\left| \frac{x-2}{x+2} \right| + C$$

19. **Problem (19):** Evaluate $\int \frac{-2}{x^2-16}\,\mathrm{d}x$. *Solution: Step 1: Decompose.* $\frac{-2}{(x-4)(x+4)} = \frac{-1/4}{x-4} + \frac{1/4}{x+4}$. *Step 2: Integrate.*

$$\begin{aligned}\int \left(-\frac{1/4}{x-4} + \frac{1/4}{x+4}\right)\mathrm{d}x &= -\frac{1}{4}\ln|x-4| + \frac{1}{4}\ln|x+4| + C\\ &= \frac{1}{4}\ln\left|\frac{x+4}{x-4}\right| + C\end{aligned}$$

20. **Problem (20):** Evaluate $\int \frac{-4}{x^2-4}\,\mathrm{d}x$. *Solution: Step 1: Decompose.* $\frac{-4}{(x-2)(x+2)} = \frac{-1}{x-2} + \frac{1}{x+2}$. *Step 2: Integrate.*

$$\begin{aligned}\int \left(-\frac{1}{x-2} + \frac{1}{x+2}\right)\mathrm{d}x &= -\ln|x-2| + \ln|x+2| + C\\ &= \ln\left|\frac{x+2}{x-2}\right| + C\end{aligned}$$

21. **Problem (21):** Evaluate $\int \frac{1}{2x^2-x}\,\mathrm{d}x$. *Solution: Step 1: Decompose.* $\frac{1}{x(2x-1)} = -\frac{1}{x} + \frac{2}{2x-1}$. *Step 2: Integrate.*

$$\int (-\frac{1}{x} + \frac{2}{2x-1})\mathrm{d}x = -\ln|x| + 2\int \frac{1}{2x-1}\mathrm{d}x$$
$$= -\ln|x| + 2(\frac{1}{2}\ln|2x-1|) + C = -\ln|x| + \ln|2x-1| + C$$
$$= \ln\left|\frac{2x-1}{x}\right| + C$$

22. **Problem (22):** Evaluate $\int \frac{2}{x^2-2x}\,\mathrm{d}x$. *Solution: Step 1: Decompose.* $\frac{2}{x(x-2)} = -\frac{1}{x} + \frac{1}{x-2}$. *Step 2: Integrate.*

$$\int (-\frac{1}{x} + \frac{1}{x-2})\mathrm{d}x = -\ln|x| + \ln|x-2| + C = \ln\left|\frac{x-2}{x}\right| + C$$

23. **Problem:** Evaluate $\int \frac{10}{x^2-10x}\,\mathrm{d}x$. *Solution: Step 1: Decompose.* $\frac{10}{x(x-10)} = -\frac{1}{x} + \frac{1}{x-10}$. *Step 2: Integrate.*

$$\int (-\frac{1}{x} + \frac{1}{x-10})\mathrm{d}x = -\ln|x| + \ln|x-10| + C = \ln\left|\frac{x-10}{x}\right| + C$$

24. **Problem:** Evaluate $\int \frac{5}{x^2+x-6}\,\mathrm{d}x$. *Solution: Step 1: Decompose.* $\frac{5}{(x+3)(x-2)} = -\frac{1}{x+3} + \frac{1}{x-2}$. *Step 2: Integrate.*

$$\int (-\frac{1}{x+3} + \frac{1}{x-2})\mathrm{d}x = -\ln|x+3| + \ln|x-2| + C = \ln\left|\frac{x-2}{x+3}\right| + C$$

25. **Problem:** Evaluate $\int \frac{3}{x^2+x-2}\,\mathrm{d}x$. *Solution: Step 1: Decompose.* $\frac{3}{(x+2)(x-1)} = -\frac{1}{x+2} + \frac{1}{x-1}$. *Step 2: Integrate.*

$$\int (-\frac{1}{x+2} + \frac{1}{x-1})\mathrm{d}x = -\ln|x+2| + \ln|x-1| + C = \ln\left|\frac{x-1}{x+2}\right| + C$$

26. **Problem (26):** Evaluate $\int \frac{1}{4x^2-9}\,\mathrm{d}x$. *Solution: Step 1: Decompose.* $\frac{1}{(2x-3)(2x+3)} = \frac{1/6}{2x-3} - \frac{1/6}{2x+3}$. *Step 2: Integrate.*

$$\begin{aligned}\int (\frac{1/6}{2x-3} - \frac{1/6}{2x+3})\mathrm{d}x &= \frac{1}{6}\int \frac{1}{2x-3}\mathrm{d}x - \frac{1}{6}\int \frac{1}{2x+3}\mathrm{d}x \\ &= \frac{1}{6}(\frac{1}{2}\ln|2x-3|) - \frac{1}{6}(\frac{1}{2}\ln|2x+3|) + C \\ &= \frac{1}{12}\ln|2x-3| - \frac{1}{12}\ln|2x+3| + C = \frac{1}{12}\ln\left|\frac{2x-3}{2x+3}\right| + C\end{aligned}$$

27. **Problem (27):** Evaluate $\int \frac{5-x}{2x^2+x-1}\,\mathrm{d}x$. *Solution: Step 1: Decompose.* $\frac{5-x}{(2x-1)(x+1)} = \frac{3}{2x-1} - \frac{2}{x+1}$. *Step 2: Integrate.*

$$\int (\frac{3}{2x-1} - \frac{2}{x+1})\mathrm{d}x = 3\int \frac{1}{2x-1}\mathrm{d}x - 2\int \frac{1}{x+1}\mathrm{d}x$$
$$= 3(\frac{1}{2}\ln|2x-1|) - 2(\ln|x+1|) + C = \frac{3}{2}\ln|2x-1| - 2\ln|x+1| + C$$

28. **Problem (28):** Evaluate $\int \frac{x+1}{x^2+4x+3}\,\mathrm{d}x$. *Solution: Step 1: Simplify.* Factor: $\frac{x+1}{(x+1)(x+3)} = \frac{1}{x+3}$ (for $x \neq -1$). *Step 2: Integrate.*

$$\int \frac{1}{x+3}\mathrm{d}x = \ln|x+3| + C$$

29. **Problem (29):** Evaluate $\int \frac{x^2-4x-4}{x^3-4x}\,\mathrm{d}x$. *Solution: Step 1: Decompose.* $\frac{x^2-4x-4}{x(x-2)(x+2)} = \frac{1}{x} - \frac{1}{x-2} + \frac{1}{x+2}$. *Step 2: Integrate.*

$$\begin{aligned}\int (\frac{1}{x} - \frac{1}{x-2} + \frac{1}{x+2})\mathrm{d}x &= \ln|x| - \ln|x-2| + \ln|x+2| + C \\ &= \ln\left|\frac{x(x+2)}{x-2}\right| + C\end{aligned}$$

30. **Problem (30):** Evaluate $\int \frac{x^2+12x+12}{x^3-4x}\,\mathrm{d}x$. *Solution: Step 1: Decompose.* Denominator $x(x-2)(x+2)$. $\frac{x^2+12x+12}{x(x-2)(x+2)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+2}$. Clear den.: $x^2 + 12x + 12 =$

$A(x-2)(x+2)+Bx(x+2)+Cx(x-2)$. $x=0 \implies 12=A(-4) \implies A=-3$. $x=2 \implies 4+24+12=B(2)(4) \implies 40=8B \implies B=5$. $x=-2 \implies 4-24+12=C(-2)(-4) \implies -8=8C \implies C=-1$. Decomposition: $-\frac{3}{x}+\frac{5}{x-2}-\frac{1}{x+2}$. *Step 2: Integrate.*

$$\int(-\frac{3}{x}+\frac{5}{x-2}-\frac{1}{x+2})\mathrm{d}x = -3\ln|x|+5\ln|x-2|-\ln|x+2|+C$$
$$= \ln\left|\frac{(x-2)^5}{x^3(x+2)}\right|+C$$

31. **Problem (31):** Evaluate $\int \frac{x+2}{x^2-4x}\,\mathrm{d}x$. *Solution: Step 1: Decompose.* Factor: $\frac{x+2}{x(x-4)}=\frac{A}{x}+\frac{B}{x-4}$. $x+2=A(x-4)+Bx$. $x=0 \implies 2=A(-4) \implies A=-1/2$. $x=4 \implies 6=B(4) \implies B=3/2$. Decomposition: $-\frac{1/2}{x}+\frac{3/2}{x-4}$. *Step 2: Integrate.*

$$\int(-\frac{1/2}{x}+\frac{3/2}{x-4})\mathrm{d}x = -\frac{1}{2}\ln|x|+\frac{3}{2}\ln|x-4|+C$$

32. **Problem (32):** Evaluate $\int \frac{4x^2+2x-1}{x^3+x^2}\,\mathrm{d}x$. *Solution: Step 1: Decompose.* Factor: $\frac{4x^2+2x-1}{x^2(x+1)}=\frac{3}{x}-\frac{1}{x^2}+\frac{1}{x+1}$. *Step 2: Integrate.*

$$\int(\frac{3}{x}-x^{-2}+\frac{1}{x+1})\mathrm{d}x = 3\ln|x|-\frac{x^{-1}}{-1}+\ln|x+1|+C$$
$$= 3\ln|x|+\frac{1}{x}+\ln|x+1|+C$$

33. **Problem (33):** Evaluate $\int \frac{2x-3}{(x-1)^2}\,\mathrm{d}x$. *Solution: Step 1: Decompose.* $\frac{2x-3}{(x-1)^2}=\frac{2}{x-1}-\frac{1}{(x-1)^2}$. *Step 2: Integrate.*

$$\int\left(\frac{2}{x-1}-(x-1)^{-2}\right)\mathrm{d}x = 2\ln|x-1|-\frac{(x-1)^{-1}}{-1}+C$$
$$= 2\ln|x-1|+\frac{1}{x-1}+C$$

34. **Problem (34):** Evaluate $\int \frac{x^4}{(x-1)^3}\,\mathrm{d}x$. *Solution: Step 1: Long Division + PFD.* $\frac{x^4}{(x-1)^3}=(x+3)+\frac{6x^2-8x+3}{(x-1)^3}$. Remainder decomposition: $\frac{6}{x-1}+\frac{4}{(x-1)^2}+\frac{1}{(x-1)^3}$. Full expression: $x+3+\frac{6}{x-1}+4(x-1)^{-2}+(x-1)^{-3}$. *Step 2: Integrate.*

$$\int\left(x+3+\frac{6}{x-1}+4(x-1)^{-2}+(x-1)^{-3}\right)\mathrm{d}x$$
$$= \frac{x^2}{2}+3x+6\ln|x-1|+4\frac{(x-1)^{-1}}{-1}+\frac{(x-1)^{-2}}{-2}+C$$
$$= \frac{x^2}{2}+3x+6\ln|x-1|-\frac{4}{x-1}-\frac{1}{2(x-1)^2}+C$$

35. **Problem (35):** Evaluate $\int \frac{3x^2+3x+1}{x(x^2+2x+1)}\,\mathrm{d}x$. *Solution: Step 1: Simplify Denom.* $x(x+1)^2$. *Step 2: Decompose.* $\frac{3x^2+3x+1}{x(x+1)^2} = \frac{1}{x} + \frac{2}{x+1} - \frac{1}{(x+1)^2}$. *Step 3: Integrate.*

$$\int \left(\frac{1}{x} + \frac{2}{x+1} - (x+1)^{-2}\right) \mathrm{d}x$$

$$= \ln|x| + 2\ln|x+1| - \frac{(x+1)^{-1}}{-1} + C = \ln|x| + 2\ln|x+1| + \frac{1}{x+1} + C$$

36. **Problem (36):** Evaluate $\int \frac{3x}{x^2-6x+9}\,\mathrm{d}x$. *Solution: Step 1: Factor Denom.* $(x-3)^2$. *Step 2: Decompose.* $\frac{3x}{(x-3)^2} = \frac{3}{x-3} + \frac{9}{(x-3)^2}$. *Step 3: Integrate.*

$$\int \left(\frac{3}{x-3} + 9(x-3)^{-2}\right) \mathrm{d}x = 3\ln|x-3| + 9\frac{(x-3)^{-1}}{-1} + C$$

$$= 3\ln|x-3| - \frac{9}{x-3} + C$$