INTEGRATION SOLUTIONS

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1 Solutions: Differentiation of Integrals

Explanation: This section applies the first part of the Fundamental Theorem of Calculus (FTC), which relates differentiation and integration. It tells us how to find the derivative of a function that is defined by an integral.

Rule 1 (Basic FTC): If a function y is defined as the integral of another function $f(t)$ from a constant a up to a variable x, like $y = \int_a^x f(t)\,\mathrm{d}t$, then the derivative of y with respect to x is simply the original function f evaluated at x. Mathematically: $\frac{dy}{dx} = f(x)$. You just replace the integration variable t with the upper limit x.

Rule 2 (FTC with Chain Rule): If the upper limit of integration is itself a function of x, say $u(x)$, so $y = \int_a^{u(x)} f(t)\,\mathrm{d}t$, finding the derivative requires the Chain Rule. The derivative is: $\frac{dy}{dx} = f(u(x)) \cdot u'(x)$. You replace t with the upper limit function $u(x)$, and then multiply by the derivative of that upper limit function, $u'(x)$.

Important Property: If the variable limit is the lower one, $\int_{b(x)}^a f(t)\,\mathrm{d}t$, you can swap the limits by introducing a negative sign: $-\int_a^{b(x)} f(t)\,\mathrm{d}t$. Then you can apply the rules above.

1. Find $\frac{dy}{dx}$ if $y = \displaystyle\int_a^x (t^3 + 1)\,\mathrm{d}t$.

   Solution: Identify the situation: The function y is defined by an integral. The lower limit a is a constant. The upper limit is simply x. The function inside the integral (the integrand) is $f(t) = t^3 + 1$.
   Apply the rule: This matches Rule 1 (Basic FTC) exactly. To find $\frac{dy}{dx}$, we replace the integration variable t in the integrand $f(t)$ with the upper limit x.

   $$\frac{dy}{dx} = f(x) = x^3 + 1$$

2. Find $\frac{dy}{dx}$ if $y = \displaystyle\int_{1+3x^2}^4 \frac{1}{2+e^t}\,\mathrm{d}t$.

   Solution: Step 1: *Handle the limits.* The function of x, which is $1 + 3x^2$, is in the lower limit, and the upper limit 4 is a constant. We need the function of x to be the upper limit to apply our rules directly. Use the property $\int_b^a f(t)\,\mathrm{d}t = -\int_a^b f(t)\,\mathrm{d}t$ to swap the limits:

   $$y = -\int_4^{1+3x^2} \frac{1}{2+e^t}\,\mathrm{d}t$$

Step 2: *Identify components for Rule 2.* Now the integral has a constant lower limit (4) and an upper limit that is a function of x. This matches Rule 2 (FTC with Chain Rule), with an extra minus sign out front.

- Integrand: $f(t) = \frac{1}{2+e^t}$
- Upper limit function: $u(x) = 1 + 3x^2$

Step 3: *Find the derivative of the upper limit.* We need $u'(x)$ for the Chain Rule part.

$$u'(x) = \frac{d}{dx}(1 + 3x^2) = \frac{d}{dx}(1) + \frac{d}{dx}(3x^2) = 0 + 3(2x) = 6x$$

Step 4: *Apply Rule 2 (FTC with Chain Rule).* The rule is $\frac{dy}{dx} = f(u(x)) \cdot u'(x)$. Since we have a minus sign from Step 1, our derivative is $\frac{dy}{dx} = -[f(u(x)) \cdot u'(x)]$. Substitute the pieces we found:

$$\begin{aligned}\frac{dy}{dx} &= -\left[f(1+3x^2) \cdot (6x)\right] \\ &= -\left[\frac{1}{2+e^{(1+3x^2)}} \cdot 6x\right] \\ &= -\frac{6x}{2+e^{1+3x^2}}\end{aligned}$$

3. Find the derivative $\frac{d}{dt}\int_0^{t^4} \sqrt{u}\ \mathrm{d}u$.

Solution: Identify the situation: We are differentiating with respect to t. The integral has a constant lower limit (0) and an upper limit that is a function of t, namely $v(t) = t^4$. The integrand is $f(u) = \sqrt{u}$. This requires Rule 2 (FTC with Chain Rule).

Step 1: *Identify components.*

- Integrand: $f(u) = \sqrt{u}$
- Upper limit function: $v(t) = t^4$

Step 2: *Find the derivative of the upper limit function.*

$$v'(t) = \frac{d}{dt}(t^4) = 4t^3$$

Step 3: *Apply Rule 2.* The rule states the derivative is $f(v(t)) \cdot v'(t)$.

$$\begin{aligned}\frac{d}{dt}\int_0^{t^4} \sqrt{u}\ \mathrm{d}u &= f(t^4) \cdot (4t^3) \quad \textit{(Replace } u \textit{ with } v(t) = t^4 \textit{ in } f(u)\textit{)} \\ &= \sqrt{t^4} \cdot 4t^3 \\ &= |t^2| \cdot 4t^3 \quad \textit{(The principal square root is always non-negative)} \\ &= t^2 \cdot 4t^3 \quad \textit{(Since } t^2 \textit{ is never negative, } |t^2| = t^2\textit{)} \\ &= 4t^{2+3} = 4t^5 \quad \textit{(Using exponent rule } x^m \cdot x^n = x^{m+n}\textit{)}\end{aligned}$$

4. Find the derivative $\frac{d}{dx}\int_0^{x^3} e^{-t}\,\mathrm{d}t$.

   Solution: Identify the situation: We are differentiating with respect to x. The integral has a constant lower limit (0) and an upper limit that is a function of x, namely $u(x) = x^3$. The integrand is $f(t) = e^{-t}$. This requires Rule 2 (FTC with Chain Rule).

   Step 1: *Identify components.*

   - Integrand: $f(t) = e^{-t}$
   - Upper limit function: $u(x) = x^3$

   Step 2: *Find the derivative of the upper limit function.*

$$u'(x) = \frac{d}{dx}(x^3) = 3x^2$$

   Step 3: *Apply Rule 2.* The derivative is $f(u(x)) \cdot u'(x)$.

$$\begin{aligned}\frac{d}{dx}\int_0^{x^3} e^{-t}\,\mathrm{d}t &= f(x^3)\cdot(3x^2) \quad \textit{(Replace } t \textit{ with } u(x) = x^3 \textit{ in } f(t)\textit{)}\\ &= e^{-(x^3)}\cdot 3x^2\\ &= 3x^2e^{-x^3}\end{aligned}$$

2 Solutions: Evaluation of Definite Integrals

Explanation: This section uses the second part of the Fundamental Theorem of Calculus (FTC), often called the Evaluation Theorem. It provides a method to calculate the exact numerical value of a definite integral, which geometrically represents the net signed area between the function's graph and the x-axis over the interval $[a, b]$.

The Process: To evaluate $\int_a^b f(x)\,\mathrm{d}x$: 1. **Find an Antiderivative:** Determine a function $F(x)$ such that its derivative $F'(x)$ equals the integrand $f(x)$. This process is called indefinite integration. The most common rule needed here is the **Power Rule for Integration**: $\int x^n \mathrm{d}x = \frac{x^{n+1}}{n+1} + C$, which works for any $n \neq -1$. Remember that $\int k\mathrm{d}x = kx + C$ for any constant k. We use the property of linearity: $\int (c\cdot g(x) + d\cdot h(x))\mathrm{d}x = c\int g(x)\mathrm{d}x + d\int h(x)\mathrm{d}x$.

2. **Evaluate at Limits:** Calculate the value of the antiderivative $F(x)$ at the upper limit of integration ($x = b$) and at the lower limit ($x = a$).

3. **Subtract:** The value of the definite integral is the difference: $F(b) - F(a)$. We often write this using bracket notation: $[F(x)]_a^b = F(b) - F(a)$. (Note: The constant of integration C is not needed for definite integrals because it would cancel: $(F(b) + C) - (F(a) + C) = F(b) - F(a)$.)

1. Evaluate $\int_{-2}^{0}(2x + 5)\,\mathrm{d}x$.

   Solution: Step 1: *Find an antiderivative* $F(x)$ *of the integrand* $f(x) = 2x + 5$. We integrate term by term using the power rule:

- For $2x = 2x^1$: $\int 2x^1 \mathrm{d}x = 2 \cdot \frac{x^{1+1}}{1+1} = 2 \cdot \frac{x^2}{2} = x^2$.
- For $5 = 5x^0$: $\int 5x^0 \mathrm{d}x = 5 \cdot \frac{x^{0+1}}{0+1} = 5 \cdot \frac{x^1}{1} = 5x$.

Combining these, an antiderivative is $F(x) = x^2 + 5x$.

Step 2: *Apply the FTC Evaluation Theorem*: $\int_a^b f(x)\mathrm{d}x = F(b) - F(a)$. Here $a = -2$ (lower limit) and $b = 0$ (upper limit).

$$
\begin{aligned}
\int_{-2}^{0} (2x+5)\ \mathrm{d}x &= [x^2 + 5x]_{a=-2}^{b=0} \quad \textit{(Using bracket notation)} \\
&= F(0) - F(-2) \\
&= ((0)^2 + 5(0)) - ((-2)^2 + 5(-2)) \quad \textit{(Substitute b=0 and a=-2 into F(x))} \\
&= (0+0) - (4-10) \\
&= 0 - (-6) \\
&= 6
\end{aligned}
$$

2. Evaluate $\displaystyle\int_{-3}^{1} \left(5 - \frac{x}{2}\right)\ \mathrm{d}x$.

Solution: Step 1: *Find an antiderivative* $F(x)$ of $f(x) = 5 - \frac{1}{2}x$.

- For $5 = 5x^0$: $\int 5x^0 \mathrm{d}x = 5 \cdot \frac{x^1}{1} = 5x$.
- For $-\frac{1}{2}x = -\frac{1}{2}x^1$: $\int -\frac{1}{2}x^1 \mathrm{d}x = -\frac{1}{2} \cdot \frac{x^{1+1}}{1+1} = -\frac{1}{2} \cdot \frac{x^2}{2} = -\frac{x^2}{4}$.

Combining these, an antiderivative is $F(x) = 5x - \frac{x^2}{4}$.

Step 2: *Apply the FTC Evaluation Theorem.* Here $a = -3$ and $b = 1$.

$$
\begin{aligned}
\int_{-3}^{1} \left(5 - \frac{x}{2}\right)\ \mathrm{d}x &= \left[5x - \frac{x^2}{4}\right]_{-3}^{1} \\
&= F(1) - F(-3) \\
&= \left(5(1) - \frac{(1)^2}{4}\right) - \left(5(-3) - \frac{(-3)^2}{4}\right) \\
&= \left(5 - \frac{1}{4}\right) - \left(-15 - \frac{9}{4}\right) \\
&= \left(\frac{20}{4} - \frac{1}{4}\right) - \left(-\frac{60}{4} - \frac{9}{4}\right) \quad \textit{(Finding common denominators)} \\
&= \left(\frac{19}{4}\right) - \left(-\frac{69}{4}\right) \\
&= \frac{19}{4} + \frac{69}{4} = \frac{19+69}{4} = \frac{88}{4} = 22
\end{aligned}
$$

3. Evaluate $\displaystyle\int_{0}^{2} x(x-3)\ \mathrm{d}x$.

Solution: Step 1: *Expand the integrand* to make it easier to apply the power rule.

$$f(x) = x(x-3) = x^2 - 3x$$

Step 2: *Find an antiderivative* $F(x)$ of $f(x) = x^2 - 3x$.

- $\int x^2 \mathrm{d}x = \frac{x^{2+1}}{2+1} = \frac{x^3}{3}$.
- $\int -3x^1 \mathrm{d}x = -3 \cdot \frac{x^{1+1}}{1+1} = -3 \cdot \frac{x^2}{2} = -\frac{3x^2}{2}$.

So, $F(x) = \frac{x^3}{3} - \frac{3x^2}{2}$.

Step 3: *Apply the FTC Evaluation Theorem.* Here $a = 0$ and $b = 2$.

$$\begin{aligned}
\int_0^2 (x^2 - 3x)\ \mathrm{d}x &= \left[\frac{x^3}{3} - \frac{3x^2}{2}\right]_0^2 \\
&= F(2) - F(0) \\
&= \left(\frac{(2)^3}{3} - \frac{3(2)^2}{2}\right) - \left(\frac{(0)^3}{3} - \frac{3(0)^2}{2}\right) \\
&= \left(\frac{8}{3} - \frac{3(4)}{2}\right) - (0 - 0) \\
&= \left(\frac{8}{3} - \frac{12}{2}\right) = \frac{8}{3} - 6 \\
&= \frac{8}{3} - \frac{18}{3} = \frac{8 - 18}{3} = -\frac{10}{3}
\end{aligned}$$

4. Evaluate $\displaystyle\int_{-1}^{1} (x^2 - 2x + 3)\ \mathrm{d}x$.

   Solution 1: Direct Evaluation

   Step 1: *Find an antiderivative* $F(x)$ of $f(x) = x^2 - 2x + 3$.

$$F(x) = \int (x^2 - 2x + 3)\ \mathrm{d}x = \frac{x^3}{3} - 2\left(\frac{x^2}{2}\right) + 3x = \frac{x^3}{3} - x^2 + 3x$$

   Step 2: *Apply the FTC Evaluation Theorem.* Here $a = -1$ and $b = 1$.

$$\begin{aligned}
\int_{-1}^{1} (x^2 - 2x + 3)\ \mathrm{d}x &= \left[\frac{x^3}{3} - x^2 + 3x\right]_{-1}^{1} \\
&= F(1) - F(-1) \\
&= \left(\frac{1^3}{3} - (1)^2 + 3(1)\right) - \left(\frac{(-1)^3}{3} - (-1)^2 + 3(-1)\right) \\
&= \left(\frac{1}{3} - 1 + 3\right) - \left(-\frac{1}{3} - 1 - 3\right) \\
&= \left(\frac{1}{3} + 2\right) - \left(-\frac{1}{3} - 4\right) \\
&= \left(\frac{1+6}{3}\right) - \left(-\frac{1+12}{3}\right) = \frac{7}{3} - \left(-\frac{13}{3}\right) = \frac{7+13}{3} = \frac{20}{3}
\end{aligned}$$

   Solution 2: Using Symmetry

   Explanation: The interval of integration $[-1, 1]$ is **symmetric** about $x = 0$. We can analyze the integrand $f(x) = x^2 - 2x + 3$. A function $g(x)$ is **even** if $g(-x) = g(x)$ (graph is symmetric about the y-axis). A function $h(x)$ is **odd** if $h(-x) = -h(x)$ (graph is symmetric about the origin). Here, x^2 is even, 3 is even (constants are even), so (x^2+3) is even. The term $-2x$ is odd.

Property 1: For any odd function $h(x)$, $\int_{-a}^{a} h(x)\ dx = 0$.

Property 2: For any even function $g(x)$, $\int_{-a}^{a} g(x)\ dx = 2\int_0^a g(x)\ dx$.

Step 1: *Split the integral and apply symmetry properties.*

$$\int_{-1}^{1} (x^2 - 2x + 3)\ dx = \int_{-1}^{1} \underbrace{(x^2+3)}_{\text{even}}\ dx + \int_{-1}^{1} \underbrace{(-2x)}_{\text{odd}}\ dx$$

$$= 2\int_0^1 (x^2+3)\ dx + 0$$

Step 2: *Evaluate the remaining integral.* Find the antiderivative of x^2+3: $F(x) = \frac{x^3}{3} + 3x$. Apply FTC from 0 to 1:

$$= 2\left[\frac{x^3}{3} + 3x\right]_0^1 = 2(F(1) - F(0))$$

$$= 2\left(\left(\frac{1^3}{3} + 3(1)\right) - \left(\frac{0^3}{3} + 3(0)\right)\right) = 2\left((\frac{1}{3} + 3) - 0\right)$$

$$= 2\left(\frac{1+9}{3}\right) = 2\left(\frac{10}{3}\right) = \frac{20}{3}$$

Both methods confirm the result is $\frac{20}{3}$.

5. Evaluate $\displaystyle\int_0^4 \left(3x - \frac{x^3}{4}\right)\ dx$.

   Solution: Step 1: *Find antiderivative* $F(x)$ of $f(x) = 3x - \frac{1}{4}x^3$.

$$F(x) = \int (3x^1 - \frac{1}{4}x^3)\ dx = 3\left(\frac{x^2}{2}\right) - \frac{1}{4}\left(\frac{x^4}{4}\right) = \frac{3x^2}{2} - \frac{x^4}{16}$$

   Step 2: *Apply FTC Evaluation Theorem* $(a = 0, b = 4)$.

$$\begin{aligned}\int_0^4 \left(3x - \frac{x^3}{4}\right)\ dx &= \left[\frac{3x^2}{2} - \frac{x^4}{16}\right]_0^4 \\ &= F(4) - F(0) \\ &= \left(\frac{3(4^2)}{2} - \frac{4^4}{16}\right) - \left(\frac{3(0)^2}{2} - \frac{0^4}{16}\right) \\ &= \left(\frac{3(16)}{2} - \frac{256}{16}\right) - 0 \\ &= (3\cdot 8) - 16 = 24 - 16 = 8\end{aligned}$$

6. Evaluate $\displaystyle\int_{-2}^{2} (x^3 - 2x + 3)\ dx$.

   Solution: Strategy: Use symmetry since the interval $[-2, 2]$ is symmetric about 0. Step 1: *Analyze the integrand* $f(x) = x^3 - 2x + 3$.

- $x^3 - 2x$: This part is odd because $(-x)^3 - 2(-x) = -x^3 + 2x = -(x^3 - 2x)$.
- 3: This part is even because $f(-x) = 3 = f(x)$.

Step 2: *Split the integral and apply symmetry properties.*

$$\int_{-2}^{2} (x^3 - 2x + 3)\ \mathrm{d}x = \underbrace{\int_{-2}^{2} (x^3 - 2x)\ \mathrm{d}x}_{\text{Integral of odd function} = 0} + \underbrace{\int_{-2}^{2} 3\ \mathrm{d}x}_{\text{Integral of even function} = 2\int_0^2 3\mathrm{d}x}$$

$$= 0 + 2\int_{0}^{2} 3\ \mathrm{d}x$$

Step 3: *Evaluate the remaining integral.* The antiderivative of 3 is $3x$.

$$= 2\left[3x\right]_0^2 = 2(F(2) - F(0)) = 2((3 \cdot 2) - (3 \cdot 0)) = 2(6 - 0) = 12$$

7. Evaluate $\displaystyle\int_0^1 (x^2 + \sqrt{x})\ \mathrm{d}x$.

Solution: Step 1: *Rewrite the integrand* using exponents: $f(x) = x^2 + x^{1/2}$.

Step 2: *Find the antiderivative* $F(x)$ using the power rule.

- $\int x^2 \mathrm{d}x = \frac{x^{2+1}}{2+1} = \frac{x^3}{3}$.
- $\int x^{1/2} \mathrm{d}x = \frac{x^{1/2+1}}{1/2+1} = \frac{x^{3/2}}{3/2} = \frac{2}{3}x^{3/2}$.

So, $F(x) = \frac{x^3}{3} + \frac{2}{3}x^{3/2}$.

Step 3: *Apply the FTC Evaluation Theorem* $(a = 0, b = 1)$.

$$\begin{aligned}\int_0^1 (x^2 + x^{1/2})\ \mathrm{d}x &= \left[\frac{x^3}{3} + \frac{2}{3}x^{3/2}\right]_0^1 \\ &= F(1) - F(0) \\ &= \left(\frac{1^3}{3} + \frac{2}{3}(1)^{3/2}\right) - \left(\frac{0^3}{3} + \frac{2}{3}(0)^{3/2}\right) \\ &= \left(\frac{1}{3} + \frac{2}{3}\right) - (0 + 0) = \frac{3}{3} = 1\end{aligned}$$

8. Evaluate $\displaystyle\int_1^{32} x^{-6/5}\ \mathrm{d}x$.

Solution: Step 1: *Find the antiderivative* $F(x)$ using the Power Rule with $n = -6/5$.

$$n + 1 = -6/5 + 1 = -6/5 + 5/5 = -1/5$$

$$F(x) = \int x^{-6/5}\ \mathrm{d}x = \frac{x^{-1/5}}{-1/5} = -5x^{-1/5}$$

Step 2: *Apply the FTC Evaluation Theorem* $(a = 1, b = 32)$.

$$\begin{aligned}
\int_1^{32} x^{-6/5}\, \mathrm{d}x &= [-5x^{-1/5}]_1^{32} \\
&= F(32) - F(1) \\
&= (-5 \cdot (32)^{-1/5}) - (-5 \cdot (1)^{-1/5}) \\
&= (-5 \cdot (2^5)^{-1/5}) - (-5 \cdot 1) \quad \textit{(Since } 32 = 2^5 \textit{ and } 1^{anypower} = 1) \\
&= (-5 \cdot 2^{(5\times -1/5)}) + 5 \\
&= (-5 \cdot 2^{-1}) + 5 \\
&= (-5 \cdot \frac{1}{2}) + 5 = -\frac{5}{2} + \frac{10}{2} = \frac{5}{2}
\end{aligned}$$

9. Evaluate $\displaystyle\int_1^{-1} (r+1)^2\, \mathrm{d}r$.

   Solution: Step 1: *Notice the limits.* The lower limit (1) is greater than the upper limit (-1). Swap them and negate.

$$\int_1^{-1} (r+1)^2\, \mathrm{d}r = -\int_{-1}^{1} (r+1)^2\, \mathrm{d}r$$

   Step 2: *Expand the integrand* inside the new integral.

$$f(r) = (r+1)^2 = r^2 + 2r + 1$$

   Step 3: *Find the antiderivative* $F(r)$ of $r^2 + 2r + 1$.

$$F(r) = \int (r^2 + 2r + 1)\, \mathrm{d}r = \frac{r^3}{3} + \frac{2r^2}{2} + r = \frac{r^3}{3} + r^2 + r$$

   Step 4: *Apply the FTC* to $\int_{-1}^{1} (r^2 + 2r + 1)\mathrm{d}r$ and include the negative sign from Step 1.

$$\begin{aligned}
-\int_{-1}^{1} (r^2 + 2r + 1)\, \mathrm{d}r &= -[F(r)]_{-1}^{1} \\
&= -(F(1) - F(-1)) \\
&= -\left[\left(\frac{1^3}{3} + 1^2 + 1\right) - \left(\frac{(-1)^3}{3} + (-1)^2 + (-1)\right)\right] \\
&= -\left[(\frac{1}{3} + 1 + 1) - (-\frac{1}{3} + 1 - 1)\right] \\
&= -\left[(\frac{1}{3} + 2) - (-\frac{1}{3} + 0)\right] \\
&= -\left[\frac{7}{3} - (-\frac{1}{3})\right] = -\left[\frac{7}{3} + \frac{1}{3}\right] = -\left[\frac{8}{3}\right] = -\frac{8}{3}
\end{aligned}$$

10. Evaluate $\displaystyle\int_{-\sqrt{3}}^{\sqrt{3}} (t+1)(t^2+4)\, \mathrm{d}t$.

    Solution: Step 1: *Expand the integrand.*

$$f(t) = (t+1)(t^2+4) = t^3 + 4t + t^2 + 4 = t^3 + t^2 + 4t + 4$$

    Step 2: *Check for symmetry.* The interval $[-\sqrt{3}, \sqrt{3}]$ is symmetric.

- Odd part: $t^3 + 4t$
- Even part: $t^2 + 4$

Step 3: *Apply symmetry properties.*

$$\int_{-\sqrt{3}}^{\sqrt{3}} (t^3 + t^2 + 4t + 4)\ \mathrm{d}t = \underbrace{\int_{-\sqrt{3}}^{\sqrt{3}} (t^3 + 4t)\ \mathrm{d}t}_{=0} + \underbrace{\int_{-\sqrt{3}}^{\sqrt{3}} (t^2 + 4)\ \mathrm{d}t}_{=2\int_0^{\sqrt{3}} (t^2+4)\mathrm{d}t}$$

$$= 2\int_0^{\sqrt{3}} (t^2 + 4)\ \mathrm{d}t$$

Step 4: *Find antiderivative* $G(t)$ of the remaining integrand $t^2 + 4$.

$$G(t) = \int (t^2 + 4)\ \mathrm{d}t = \frac{t^3}{3} + 4t$$

Step 5: *Evaluate using FTC.*

$$\begin{aligned}
2\int_0^{\sqrt{3}} (t^2 + 4)\ \mathrm{d}t &= 2[G(t)]_0^{\sqrt{3}} = 2(G(\sqrt{3}) - G(0)) \\
&= 2\left[\left(\frac{(\sqrt{3})^3}{3} + 4\sqrt{3}\right) - (\frac{0^3}{3} + 4(0))\right] \\
&= 2\left[(\frac{3\sqrt{3}}{3} + 4\sqrt{3}) - 0\right] \quad \textit{(Note: } (\sqrt{3})^3 = 3\sqrt{3}) \\
&= 2[\sqrt{3} + 4\sqrt{3}] = 2[5\sqrt{3}] = 10\sqrt{3}
\end{aligned}$$

11. Evaluate $\int_{\sqrt{2}}^{1} \left(\frac{u^7}{2} - \frac{1}{u^5}\right)\ \mathrm{d}u$.

Solution: Step 1: *Rewrite integrand* using negative exponents.

$$f(u) = \frac{1}{2}u^7 - u^{-5}$$

Step 2: *Find the antiderivative* $F(u)$.

$$F(u) = \int (\frac{1}{2}u^7 - u^{-5})\ \mathrm{d}u = \frac{1}{2}\left(\frac{u^8}{8}\right) - \left(\frac{u^{-4}}{-4}\right) = \frac{u^8}{16} + \frac{1}{4u^4}$$

Step 3: *Apply the FTC* $(a = \sqrt{2}, b = 1)$.

$$\begin{aligned}
\int_{\sqrt{2}}^{1} f(u)\ \mathrm{d}u &= [F(u)]_{\sqrt{2}}^{1} = F(1) - F(\sqrt{2}) \\
&= \left(\frac{1^8}{16} + \frac{1}{4(1)^4}\right) - \left(\frac{(\sqrt{2})^8}{16} + \frac{1}{4(\sqrt{2})^4}\right) \\
&= \left(\frac{1}{16} + \frac{1}{4}\right) - \left(\frac{16}{16} + \frac{1}{4(4)}\right) \quad \textit{(Note: } (\sqrt{2})^8 = 16, (\sqrt{2})^4 = 4) \\
&= \left(\frac{1+4}{16}\right) - \left(1 + \frac{1}{16}\right) = \frac{5}{16} - \frac{17}{16} = -\frac{12}{16} = -\frac{3}{4}
\end{aligned}$$

12. Evaluate $\displaystyle\int_{-3}^{-1} \frac{y^5 - 2y}{y^3}\, dy$.

Solution: Step 1: *Simplify the integrand* by dividing term-by-term.

$$f(y) = \frac{y^5}{y^3} - \frac{2y^1}{y^3} = y^{5-3} - 2y^{1-3} = y^2 - 2y^{-2}$$

Step 2: *Find the antiderivative* $F(y)$.

$$F(y) = \int (y^2 - 2y^{-2})\, dy = \frac{y^3}{3} - 2\left(\frac{y^{-1}}{-1}\right) = \frac{y^3}{3} + \frac{2}{y}$$

Step 3: *Apply the FTC* $(a = -3, b = -1)$.

$$\begin{aligned}
\int_{-3}^{-1} f(y)\, dy &= [F(y)]_{-3}^{-1} = F(-1) - F(-3) \\
&= \left(\frac{(-1)^3}{3} + \frac{2}{-1}\right) - \left(\frac{(-3)^3}{3} + \frac{2}{-3}\right) \\
&= \left(-\frac{1}{3} - 2\right) - \left(-\frac{27}{3} - \frac{2}{3}\right) \\
&= \left(-\frac{1+6}{3}\right) - \left(-\frac{27+2}{3}\right) = -\frac{7}{3} - (-\frac{29}{3}) = \frac{-7+29}{3} = \frac{22}{3}
\end{aligned}$$

13. Evaluate $\displaystyle\int_{1}^{\sqrt{2}} \frac{s^2 + \sqrt{s}}{s^2}\, ds$.

Solution: Step 1: *Simplify the integrand.* Rewrite $\sqrt{s} = s^{1/2}$ and divide.

$$f(s) = \frac{s^2}{s^2} + \frac{s^{1/2}}{s^2} = 1 + s^{1/2-2} = 1 + s^{-3/2}$$

Step 2: *Find the antiderivative* $F(s)$ using power rule $(n = -3/2)$.

$$F(s) = \int (1 + s^{-3/2})\, ds = s + \frac{s^{-3/2+1}}{-3/2+1} = s + \frac{s^{-1/2}}{-1/2} = s - 2s^{-1/2} = s - \frac{2}{\sqrt{s}}$$

Step 3: *Apply the FTC* $(a = 1, b = \sqrt{2})$.

$$\begin{aligned}
\int_{1}^{\sqrt{2}} f(s)\, ds &= [F(s)]_1^{\sqrt{2}} = F(\sqrt{2}) - F(1) \\
&= \left(\sqrt{2} - \frac{2}{\sqrt{\sqrt{2}}}\right) - \left(1 - \frac{2}{\sqrt{1}}\right) \\
&= \left(\sqrt{2} - \frac{2}{2^{1/4}}\right) - (1 - 2) \quad \textit{(Note: } \sqrt{\sqrt{s}} = s^{1/4}) \\
&= \sqrt{2} - 2^{1-1/4} - (-1) \quad \textit{(Exponent rule: } a^m/a^n = a^{m-n}) \\
&= \sqrt{2} - 2^{3/4} + 1
\end{aligned}$$

14. Evaluate $\displaystyle\int_1^8 \frac{(x^{1/3}+1)(2-x^{2/3})}{x^{1/3}}\,\mathrm{d}x$.

Solution: Step 1: *Simplify the integrand.* This requires expanding the numerator first, then dividing by the denominator.

Numerator: $(x^{1/3}+1)(2-x^{2/3}) = (x^{1/3})(2) + (x^{1/3})(-x^{2/3}) + (1)(2) + (1)(-x^{2/3}) = 2x^{1/3} - x^{(1/3+2/3)} + 2 - x^{2/3} = 2x^{1/3} - x^1 + 2 - x^{2/3}$. Now divide by $x^{1/3}$:

$$\begin{aligned} f(x) &= \frac{2x^{1/3} - x^1 + 2 - x^{2/3}}{x^{1/3}} \\ &= \frac{2x^{1/3}}{x^{1/3}} - \frac{x^1}{x^{1/3}} + \frac{2}{x^{1/3}} - \frac{x^{2/3}}{x^{1/3}} \\ &= 2 - x^{(1-1/3)} + 2x^{-1/3} - x^{(2/3-1/3)} \quad \textit{(Using exponent rule } x^m/x^n = x^{m-n}\textit{)} \\ &= 2 - x^{2/3} + 2x^{-1/3} - x^{1/3} \end{aligned}$$

Step 2: *Find the antiderivative* $F(x)$ term-by-term using the Power Rule.

$$\begin{aligned} F(x) &= \int (2 - x^{2/3} + 2x^{-1/3} - x^{1/3})\,\mathrm{d}x \\ &= 2x - \frac{x^{2/3+1}}{2/3+1} + 2\frac{x^{-1/3+1}}{-1/3+1} - \frac{x^{1/3+1}}{1/3+1} \\ &= 2x - \frac{x^{5/3}}{5/3} + 2\frac{x^{2/3}}{2/3} - \frac{x^{4/3}}{4/3} \\ &= 2x - \frac{3}{5}x^{5/3} + 2(\frac{3}{2})x^{2/3} - \frac{3}{4}x^{4/3} \\ &= 2x - \frac{3}{5}x^{5/3} + 3x^{2/3} - \frac{3}{4}x^{4/3} \end{aligned}$$

Step 3: *Evaluate* $F(x)$ *at the limits* $a = 1$ and $b = 8$. For $F(8)$: We need powers of $8^{1/3} = \sqrt[3]{8} = 2$.

- $8^{2/3} = (8^{1/3})^2 = 2^2 = 4$
- $8^{4/3} = (8^{1/3})^4 = 2^4 = 16$
- $8^{5/3} = (8^{1/3})^5 = 2^5 = 32$

$$\begin{aligned} F(8) &= 2(8) - \frac{3}{5}(8^{5/3}) + 3(8^{2/3}) - \frac{3}{4}(8^{4/3}) \\ &= 16 - \frac{3}{5}(32) + 3(4) - \frac{3}{4}(16) \\ &= 16 - \frac{96}{5} + 12 - (3\cdot 4) \\ &= 16 - \frac{96}{5} + 12 - 12 = 16 - \frac{96}{5} \\ &= \frac{80}{5} - \frac{96}{5} = -\frac{16}{5} \end{aligned}$$

For $F(1)$: Any power of 1 is 1.

$$
\begin{aligned}
F(1) &= 2(1) - \frac{3}{5}(1^{5/3}) + 3(1^{2/3}) - \frac{3}{4}(1^{4/3}) \\
&= 2 - \frac{3}{5}(1) + 3(1) - \frac{3}{4}(1) \\
&= 2 - \frac{3}{5} + 3 - \frac{3}{4} = (2+3) - \frac{3}{5} - \frac{3}{4} \\
&= 5 - (\frac{3 \cdot 4}{20} + \frac{3 \cdot 5}{20}) \quad \textit{(Common denominator is 20)} \\
&= 5 - (\frac{12+15}{20}) = 5 - \frac{27}{20} = \frac{100}{20} - \frac{27}{20} = \frac{73}{20}
\end{aligned}
$$

Step 4: *Calculate the definite integral $F(8) - F(1)$.*

$$F(8) - F(1) = -\frac{16}{5} - \frac{73}{20}$$

Find a common denominator (20):

$$= -\frac{16 \cdot 4}{5 \cdot 4} - \frac{73}{20} = -\frac{64}{20} - \frac{73}{20} = \frac{-64-73}{20} = -\frac{137}{20}$$