

# TECHNIQUES OF DIFFERENTIATION: PRODUCT RULE

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## 1 Introduction: The Product Rule

**Explanation.** When we need to differentiate a function that is formed by multiplying two other functions together, we cannot simply differentiate each function separately and multiply the results. We must use the **Product Rule**.

If  $y = f(x) \cdot g(x)$ , where both  $f(x)$  and  $g(x)$  are differentiable functions of  $x$ , then the derivative of  $y$  is given by:

$$\frac{dy}{dx} = f(x) \cdot \frac{d}{dx}[g(x)] + g(x) \cdot \frac{d}{dx}[f(x)]$$

Or using prime notation for derivatives:

$$y' = f(x)g'(x) + g(x)f'(x)$$

In simpler terms: "The derivative of a product is the first function times the derivative of the second, plus the second function times the derivative of the first."

We often use  $u$  and  $v$  to represent the two functions, so if  $y = uv$ , then:

$$\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$$

**Extension to Three Functions:** If  $y = f(x)g(x)h(x)$ , the rule extends:

$$\frac{dy}{dx} = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$$

(Differentiate one function at a time, keeping the others unchanged, and sum the results).

**Alternative Strategy:** Sometimes, it might be easier to first algebraically expand the product into a sum of simpler terms and then differentiate term-by-term using the Power, Sum/Difference, and Constant Multiple rules. We will show this where practical.

## 2 Solutions: Example

1. Find the derivative of  $y = (3x - 2x^2)(5 + 4x)$ .

*Solution:*

**Method 1: Product Rule**

**Strategy.** Identify the two functions being multiplied. Let  $u = (3x - 2x^2)$  and  $v = (5 + 4x)$ . Find the derivatives  $du/dx$  and  $dv/dx$ , then apply the formula  $\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$ .

*Step 1: Identify  $u, v$  and find their derivatives.*

- $u = 3x - 2x^2$
- $v = 5 + 4x$
- $\frac{du}{dx} = \frac{d}{dx}(3x) - \frac{d}{dx}(2x^2) = 3(1) - 2(2x) = 3 - 4x$
- $\frac{dv}{dx} = \frac{d}{dx}(5) + \frac{d}{dx}(4x) = 0 + 4(1) = 4$

*Step 2: Apply the Product Rule formula.*

$$\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$$

$$\frac{dy}{dx} = (3x - 2x^2)(4) + (5 + 4x)(3 - 4x)$$

*Step 3: Expand and simplify the result.*

$$\begin{aligned} \frac{dy}{dx} &= (4 \cdot 3x - 4 \cdot 2x^2) + (5 \cdot 3 + 5(-4x) + 4x \cdot 3 + 4x(-4x)) \\ &= (12x - 8x^2) + (15 - 20x + 12x - 16x^2) \\ &= 12x - 8x^2 + 15 - 8x - 16x^2 \end{aligned}$$

Combine like terms (constants,  $x$  terms,  $x^2$  terms):

$$\begin{aligned} &= 15 + (12x - 8x) + (-8x^2 - 16x^2) \\ &= 15 + 4x - 24x^2 \end{aligned}$$

## Method 2: Expand First

**Strategy.** Multiply the two binomials first to get a single polynomial, then differentiate term by term.

*Step 1: Expand the product.*

$$\begin{aligned} y &= (3x - 2x^2)(5 + 4x) \\ y &= 3x(5) + 3x(4x) - 2x^2(5) - 2x^2(4x) \quad (\text{Using FOIL/Distributive Property}) \\ y &= 15x + 12x^2 - 10x^2 - 8x^3 \end{aligned}$$

Combine like terms:

$$\begin{aligned} y &= -8x^3 + (12x^2 - 10x^2) + 15x \\ y &= -8x^3 + 2x^2 + 15x \end{aligned}$$

*Step 2: Differentiate the expanded polynomial term by term.*

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}(-8x^3) + \frac{d}{dx}(2x^2) + \frac{d}{dx}(15x) \\ &= -8(3x^{3-1}) + 2(2x^{2-1}) + 15(1x^{1-1}) \\ &= -24x^2 + 4x + 15x^0 \\ &= -24x^2 + 4x + 15 \end{aligned}$$

*Result Comparison:* Both methods yield  $15 + 4x - 24x^2$ .

### 3 Solutions: ClassWork Problems

2. Find derivative of  $f(x) = x(x^2 + 3)$ .

*Solution:*

**Method 1: Product Rule**

**Strategy.** Let  $u = x$  and  $v = x^2 + 3$ .

$u = x \implies \frac{du}{dx} = 1$ .  $v = x^2 + 3 \implies \frac{dv}{dx} = 2x + 0 = 2x$ . Apply formula:  $\frac{df}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$

$$\begin{aligned}\frac{df}{dx} &= (x)(2x) + (x^2 + 3)(1) \\ &= 2x^2 + x^2 + 3 = 3x^2 + 3\end{aligned}$$

**Method 2: Expand First**  $f(x) = x(x^2 + 3) = x^3 + 3x$ . Differentiate term-by-term:

$$\frac{df}{dx} = \frac{d}{dx}(x^3) + \frac{d}{dx}(3x) = 3x^2 + 3(1) = 3x^2 + 3$$

*Results match.*

3. Find derivative of  $g(x) = (x - 4)(x + 2)$ .

*Solution:*

**Method 1: Product Rule**

**Strategy.** Let  $u = x - 4$  and  $v = x + 2$ .

$u = x - 4 \implies \frac{du}{dx} = 1 - 0 = 1$ .  $v = x + 2 \implies \frac{dv}{dx} = 1 + 0 = 1$ . Apply formula:

$$\frac{dg}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$$

$$\begin{aligned}\frac{dg}{dx} &= (x - 4)(1) + (x + 2)(1) \\ &= x - 4 + x + 2 = 2x - 2\end{aligned}$$

**Method 2: Expand First**  $g(x) = (x - 4)(x + 2) = x(x) + x(2) - 4(x) - 4(2) = x^2 + 2x - 4x - 8 = x^2 - 2x - 8$ . Differentiate term-by-term:

$$\frac{dg}{dx} = \frac{d}{dx}(x^2) - \frac{d}{dx}(2x) - \frac{d}{dx}(8) = 2x - 2(1) - 0 = 2x - 2$$

*Results match.*

4. Find derivative of  $f(x) = x^2(3x^3 - 1)$ .

*Solution:*

**Method 1: Product Rule**

**Strategy.** Let  $u = x^2$  and  $v = 3x^3 - 1$ .

$u = x^2 \implies \frac{du}{dx} = 2x$ .  $v = 3x^3 - 1 \implies \frac{dv}{dx} = 3(3x^2) - 0 = 9x^2$ . Apply formula:

$$\frac{df}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$$

$$\begin{aligned} \frac{df}{dx} &= (x^2)(9x^2) + (3x^3 - 1)(2x) \\ &= 9x^4 + (3x^3(2x) - 1(2x)) = 9x^4 + (6x^4 - 2x) \\ &= 9x^4 + 6x^4 - 2x = 15x^4 - 2x \end{aligned}$$

**Method 2: Expand First**  $f(x) = x^2(3x^3 - 1) = 3x^5 - x^2$ . Differentiate term-by-term:

$$\frac{df}{dx} = \frac{d}{dx}(3x^5) - \frac{d}{dx}(x^2) = 3(5x^4) - 2x = 15x^4 - 2x$$

*Results match.*

5. Find derivative of  $f(x) = (x^2 + 1)(2x + 5)$ .

*Solution:*

**Method 1: Product Rule**

**Strategy.** Let  $u = x^2 + 1$  and  $v = 2x + 5$ .

$u = x^2 + 1 \implies \frac{du}{dx} = 2x + 0 = 2x$ .  $v = 2x + 5 \implies \frac{dv}{dx} = 2(1) + 0 = 2$ . Apply formula:

$$\frac{df}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$$

$$\begin{aligned} \frac{df}{dx} &= (x^2 + 1)(2) + (2x + 5)(2x) \\ &= (2x^2 + 2) + (4x^2 + 10x) \\ &= 2x^2 + 2 + 4x^2 + 10x = 6x^2 + 10x + 2 \end{aligned}$$

**Method 2: Expand First**  $f(x) = (x^2 + 1)(2x + 5) = x^2(2x) + x^2(5) + 1(2x) + 1(5) = 2x^3 + 5x^2 + 2x + 5$ . Differentiate term-by-term:

$$\begin{aligned} \frac{df}{dx} &= \frac{d}{dx}(2x^3) + \frac{d}{dx}(5x^2) + \frac{d}{dx}(2x) + \frac{d}{dx}(5) \\ &= 2(3x^2) + 5(2x) + 2(1) + 0 = 6x^2 + 10x + 2 \end{aligned}$$

*Results match.*

6. Find derivative of  $y = \frac{1}{x}(x^2 + e^x)$ .

*Solution:*

**Method 1: Product Rule**

**Strategy.** Rewrite as  $y = x^{-1}(x^2 + e^x)$ . Let  $u = x^{-1}$  and  $v = x^2 + e^x$ .

$u = x^{-1} \implies \frac{du}{dx} = -1x^{-2} = -x^{-2} = -\frac{1}{x^2}$ .  $v = x^2 + e^x \implies \frac{dv}{dx} = 2x + e^x$ . (Derivative of  $e^x$  is  $e^x$ ) Apply formula:  $\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$

$$\begin{aligned}\frac{dy}{dx} &= (x^{-1})(2x + e^x) + (x^2 + e^x)(-x^{-2}) \\ &= \frac{1}{x}(2x + e^x) + (x^2 + e^x)\left(-\frac{1}{x^2}\right) \\ &= \left(\frac{2x}{x} + \frac{e^x}{x}\right) - \left(\frac{x^2}{x^2} + \frac{e^x}{x^2}\right) \\ &= \left(2 + \frac{e^x}{x}\right) - \left(1 + \frac{e^x}{x^2}\right) \\ &= 2 + \frac{e^x}{x} - 1 - \frac{e^x}{x^2} = 1 + \frac{e^x}{x} - \frac{e^x}{x^2}\end{aligned}$$

Optional common denominator:  $1 + \frac{xe^x - e^x}{x^2} = \frac{x^2 + xe^x - e^x}{x^2}$ .

**Method 2: Expand First**  $y = \frac{1}{x}(x^2 + e^x) = \frac{x^2}{x} + \frac{e^x}{x} = x + x^{-1}e^x$ . Note: Expanding doesn't eliminate the product  $x^{-1}e^x$ . The first term is easy, but the second term still requires the Product Rule (or Quotient Rule). Let's differentiate  $y = x + x^{-1}e^x$ :

$$\frac{dy}{dx} = \frac{d}{dx}(x) + \frac{d}{dx}(x^{-1}e^x)$$

The first term is 1. For the second term, use Product Rule with  $u = x^{-1}$ ,  $v = e^x$ .  $\frac{du}{dx} = -x^{-2}$ ,  $\frac{dv}{dx} = e^x$ .

$$\frac{d}{dx}(x^{-1}e^x) = u \frac{dv}{dx} + v \frac{du}{dx} = (x^{-1})(e^x) + (e^x)(-x^{-2}) = \frac{e^x}{x} - \frac{e^x}{x^2}$$

Combine all parts:

$$\frac{dy}{dx} = 1 + \left(\frac{e^x}{x} - \frac{e^x}{x^2}\right) = 1 + \frac{e^x}{x} - \frac{e^x}{x^2}$$

Results match.

7. Find derivative of  $y = e^{2x}$ .

*Solution:*

**Strategy.** This function is  $e^u$  where  $u = 2x$ . This is a standard application of the Chain Rule (derivative of outside function evaluated at inside function, times derivative of inside function). The Product Rule doesn't apply here as it's not a product of two distinct functions of  $x$  in the base form.

**Method 1: Chain Rule** Let  $y = e^u$  where  $u = 2x$ .  $\frac{dy}{du} = e^u$ .  $\frac{du}{dx} = \frac{d}{dx}(2x) = 2$ . By Chain Rule,  $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = e^u \cdot 2$ . Substitute back  $u = 2x$ :

$$\frac{dy}{dx} = e^{2x} \cdot 2 = 2e^{2x}$$

8. Find derivative of  $g(x) = (x^2 - 4x + 3)(x - 2)$ .

*Solution:*

**Method 1: Product Rule**

**Strategy.** Let  $u = x^2 - 4x + 3$  and  $v = x - 2$ .

$u = x^2 - 4x + 3 \implies \frac{du}{dx} = 2x - 4$ .  $v = x - 2 \implies \frac{dv}{dx} = 1$ . Apply formula:  $\frac{dg}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$

$$\begin{aligned}\frac{dg}{dx} &= (x^2 - 4x + 3)(1) + (x - 2)(2x - 4) \\ &= x^2 - 4x + 3 + (x(2x) + x(-4) - 2(2x) - 2(-4)) \\ &= x^2 - 4x + 3 + (2x^2 - 4x - 4x + 8) \\ &= x^2 - 4x + 3 + 2x^2 - 8x + 8\end{aligned}$$

Combine like terms:

$$= (x^2 + 2x^2) + (-4x - 8x) + (3 + 8) = 3x^2 - 12x + 11$$

**Method 2: Expand First**  $g(x) = (x^2 - 4x + 3)(x - 2)$   $g(x) = x^2(x) + x^2(-2) - 4x(x) - 4x(-2) + 3(x) + 3(-2)$   $g(x) = x^3 - 2x^2 - 4x^2 + 8x + 3x - 6$   $g(x) = x^3 - 6x^2 + 11x - 6$   
Differentiate term-by-term:

$$\begin{aligned}\frac{dg}{dx} &= \frac{d}{dx}(x^3) - \frac{d}{dx}(6x^2) + \frac{d}{dx}(11x) - \frac{d}{dx}(6) \\ &= 3x^2 - 6(2x) + 11(1) - 0 = 3x^2 - 12x + 11\end{aligned}$$

*Results match.*

9. Find derivative of  $g(x) = (x^2 - 2x + 1)(x^3 - 1)$ .

*Solution: Note:  $x^2 - 2x + 1 = (x - 1)^2$  and  $x^3 - 1 = (x - 1)(x^2 + x + 1)$ . So  $g(x) = (x - 1)^3(x^2 + x + 1)$ . Expanding might be complex.*

**Method 1: Product Rule**

**Strategy.** Let  $u = x^2 - 2x + 1$  and  $v = x^3 - 1$ .

$u = x^2 - 2x + 1 \implies \frac{du}{dx} = 2x - 2$ .  $v = x^3 - 1 \implies \frac{dv}{dx} = 3x^2$ . Apply formula:

$$\begin{aligned}\frac{dg}{dx} &= u \frac{dv}{dx} + v \frac{du}{dx} \\ \frac{dg}{dx} &= (x^2 - 2x + 1)(3x^2) + (x^3 - 1)(2x - 2)\end{aligned}$$

Expand and simplify:

$$\begin{aligned}&= (3x^4 - 6x^3 + 3x^2) + (x^3(2x) + x^3(-2) - 1(2x) - 1(-2)) \\ &= 3x^4 - 6x^3 + 3x^2 + (2x^4 - 2x^3 - 2x + 2) \\ &= 3x^4 - 6x^3 + 3x^2 + 2x^4 - 2x^3 - 2x + 2\end{aligned}$$

Combine like terms:

$$\begin{aligned}&= (3x^4 + 2x^4) + (-6x^3 - 2x^3) + 3x^2 - 2x + 2 \\ &= 5x^4 - 8x^3 + 3x^2 - 2x + 2\end{aligned}$$

**Method 2: Expand First**  $g(x) = (x^2 - 2x + 1)(x^3 - 1)$   $g(x) = x^2(x^3) + x^2(-1) - 2x(x^3) - 2x(-1) + 1(x^3) + 1(-1)$   $g(x) = x^5 - x^2 - 2x^4 + 2x + x^3 - 1$  Rearrange by power:  $g(x) = x^5 - 2x^4 + x^3 - x^2 + 2x - 1$  Differentiate term-by-term:

$$\begin{aligned}\frac{dg}{dx} &= 5x^4 - 2(4x^3) + 3x^2 - 2x + 2(1) - 0 \\ &= 5x^4 - 8x^3 + 3x^2 - 2x + 2\end{aligned}$$

*Results match.*

10. Find derivative of  $f(x) = (x^3 - 3x)(2x^2 + 3x + 5)$ .

*Solution:*

**Method 1: Product Rule**

**Strategy.** Let  $u = x^3 - 3x$  and  $v = 2x^2 + 3x + 5$ .

$u = x^3 - 3x \implies \frac{du}{dx} = 3x^2 - 3$ .  $v = 2x^2 + 3x + 5 \implies \frac{dv}{dx} = 4x + 3$ . Apply formula:

$$\frac{df}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$$

$$\frac{df}{dx} = (x^3 - 3x)(4x + 3) + (2x^2 + 3x + 5)(3x^2 - 3)$$

Expand carefully:

$$\begin{aligned}&= [x^3(4x) + x^3(3) - 3x(4x) - 3x(3)] + [2x^2(3x^2) + 2x^2(-3) + 3x(3x^2) + 3x(-3) + 5(3x^2) + 5(-3)] \\ &= [4x^4 + 3x^3 - 12x^2 - 9x] + [6x^4 - 6x^2 + 9x^3 - 9x + 15x^2 - 15]\end{aligned}$$

Combine like terms:

$$\begin{aligned}&= (4x^4 + 6x^4) + (3x^3 + 9x^3) + (-12x^2 - 6x^2 + 15x^2) + (-9x - 9x) - 15 \\ &= 10x^4 + 12x^3 + (-3x^2) - 18x - 15 \\ &= 10x^4 + 12x^3 - 3x^2 - 18x - 15\end{aligned}$$

**Method 2: Expand First**  $f(x) = (x^3 - 3x)(2x^2 + 3x + 5)$   $f(x) = x^3(2x^2 + 3x + 5) - 3x(2x^2 + 3x + 5)$   $f(x) = (2x^5 + 3x^4 + 5x^3) - (6x^3 + 9x^2 + 15x)$   $f(x) = 2x^5 + 3x^4 + 5x^3 - 6x^3 - 9x^2 - 15x$   $f(x) = 2x^5 + 3x^4 - x^3 - 9x^2 - 15x$  Differentiate term-by-term:

$$\begin{aligned}\frac{df}{dx} &= 2(5x^4) + 3(4x^3) - 3x^2 - 9(2x) - 15(1) \\ &= 10x^4 + 12x^3 - 3x^2 - 18x - 15\end{aligned}$$

*Results match.*

11. Find derivative of  $h(t) = (t^5 - 1)(4t^2 - 7t - 3)$ .

**Method 1: Product Rule**  $u = t^5 - 1 \implies du/dt = 5t^4$ .  $v = 4t^2 - 7t - 3 \implies dv/dt = 8t - 7$ .

$$h'(t) = (t^5 - 1)(8t - 7) + (4t^2 - 7t - 3)(5t^4)$$

$$\begin{aligned}
&= (8t^6 - 7t^5 - 8t + 7) + (20t^6 - 35t^5 - 15t^4) \\
&= 28t^6 - 42t^5 - 15t^4 - 8t + 7
\end{aligned}$$

**Method 2: Expand First**  $h(t) = t^5(4t^2 - 7t - 3) - 1(4t^2 - 7t - 3)$   $h(t) = 4t^7 - 7t^6 - 3t^5 - 4t^2 + 7t + 3$ .

$$\begin{aligned}
h'(t) &= 4(7t^6) - 7(6t^5) - 3(5t^4) - 4(2t) + 7(1) + 0 \\
&= 28t^6 - 42t^5 - 15t^4 - 8t + 7
\end{aligned}$$

*Results match.*

12. Find derivative of  $g(t) = (2t^3 - 1)^2$ . **Method 1: Product Rule** Rewrite  $g(t) = (2t^3 - 1)(2t^3 - 1)$ . Let  $u = 2t^3 - 1, v = 2t^3 - 1$ .  $du/dt = 6t^2, dv/dt = 6t^2$ .

$$\begin{aligned}
g'(t) &= (2t^3 - 1)(6t^2) + (2t^3 - 1)(6t^2) \\
&= 2(2t^3 - 1)(6t^2) = 12t^2(2t^3 - 1) = 24t^5 - 12t^2
\end{aligned}$$

**Method 2: Expand First**  $g(t) = (2t^3 - 1)^2 = (2t^3)^2 - 2(2t^3)(1) + (1)^2 = 4t^6 - 4t^3 + 1$ .

$$g'(t) = 4(6t^5) - 4(3t^2) + 0 = 24t^5 - 12t^2$$

**Method 3: Chain Rule (Preview)** Let outer function be  $f(u) = u^2$ , inner function be  $u(t) = 2t^3 - 1$ .  $f'(u) = 2u$ ,  $u'(t) = 6t^2$ .  $g'(t) = f'(u(t)) \cdot u'(t) = (2u) \cdot (6t^2) = 2(2t^3 - 1)(6t^2) = 12t^2(2t^3 - 1) = 24t^5 - 12t^2$ . *Results match.*

13. Find derivative of  $f(x) = \sqrt[3]{x}(\sqrt{x} + 3)$ .

**Method 1: Product Rule** Rewrite  $f(x) = x^{1/3}(x^{1/2} + 3)$ . Let  $u = x^{1/3}, v = x^{1/2} + 3$ .  $du/dx = \frac{1}{3}x^{-2/3}, dv/dx = \frac{1}{2}x^{-1/2}$ .

$$\begin{aligned}
f'(x) &= (x^{1/3})\left(\frac{1}{2}x^{-1/2}\right) + (x^{1/2} + 3)\left(\frac{1}{3}x^{-2/3}\right) \\
&= \frac{1}{2}x^{(1/3-1/2)} + \frac{1}{3}x^{-2/3}(x^{1/2}) + \frac{1}{3}x^{-2/3}(3) \\
&= \frac{1}{2}x^{-1/6} + \frac{1}{3}x^{(-2/3+1/2)} + x^{-2/3} \\
&= \frac{1}{2}x^{-1/6} + \frac{1}{3}x^{-1/6} + x^{-2/3} \quad (\text{Exponents: } 1/3 - 1/2 = -1/6; -2/3 + 1/2 = -1/6) \\
&= \left(\frac{1}{2} + \frac{1}{3}\right)x^{-1/6} + x^{-2/3} = \frac{5}{6}x^{-1/6} + x^{-2/3}
\end{aligned}$$

*Optional rewrite:*  $\frac{5}{6x^{1/6}} + \frac{1}{x^{2/3}}$ .

**Method 2: Expand First**  $f(x) = x^{1/3}(x^{1/2} + 3) = x^{1/3}x^{1/2} + 3x^{1/3} = x^{(1/3+1/2)} + 3x^{1/3} = x^{5/6} + 3x^{1/3}$ .

$$f'(x) = \frac{5}{6}x^{(5/6-1)} + 3\left(\frac{1}{3}x^{(1/3-1)}\right) = \frac{5}{6}x^{-1/6} + x^{-2/3}$$

*Results match.*



14. Find derivative of  $f(x) = \sqrt[3]{x}(x+1)$ .

**Method 1: Product Rule** Rewrite  $f(x) = x^{1/3}(x+1)$ . Let  $u = x^{1/3}, v = x+1$ .  
 $du/dx = \frac{1}{3}x^{-2/3}$ .  $dv/dx = 1$ .

$$\begin{aligned} f'(x) &= (x^{1/3})(1) + (x+1)\left(\frac{1}{3}x^{-2/3}\right) \\ &= x^{1/3} + \frac{1}{3}x \cdot x^{-2/3} + \frac{1}{3}(1)x^{-2/3} \\ &= x^{1/3} + \frac{1}{3}x^{(1-2/3)} + \frac{1}{3}x^{-2/3} \\ &= x^{1/3} + \frac{1}{3}x^{1/3} + \frac{1}{3}x^{-2/3} = \frac{4}{3}x^{1/3} + \frac{1}{3}x^{-2/3} \end{aligned}$$

*Optional rewrite:*  $\frac{4\sqrt[3]{x}}{3} + \frac{1}{3\sqrt[3]{x^2}}$ .

**Method 2: Expand First**  $f(x) = x^{1/3}(x+1) = x^{1/3}x^1 + x^{1/3} = x^{4/3} + x^{1/3}$ .

$$f'(x) = \frac{4}{3}x^{(4/3-1)} + \frac{1}{3}x^{(1/3-1)} = \frac{4}{3}x^{1/3} + \frac{1}{3}x^{-2/3}$$

*Results match.*

15. Find derivative of  $f(x) = (x^5 - 3x)\left(\frac{1}{x^2}\right)$ .

**Method 1: Product Rule** Rewrite  $f(x) = (x^5 - 3x)(x^{-2})$ . Let  $u = x^5 - 3x, v = x^{-2}$ .  
 $du/dx = 5x^4 - 3$ .  $dv/dx = -2x^{-3}$ .

$$\begin{aligned} f'(x) &= (x^5 - 3x)(-2x^{-3}) + (x^{-2})(5x^4 - 3) \\ &= -2x^5x^{-3} + 6xx^{-3} + 5x^4x^{-2} - 3x^{-2} \\ &= -2x^2 + 6x^{-2} + 5x^2 - 3x^{-2} \\ &= (-2x^2 + 5x^2) + (6x^{-2} - 3x^{-2}) = 3x^2 + 3x^{-2} \end{aligned}$$

*Optional rewrite:*  $3x^2 + \frac{3}{x^2}$ .

**Method 2: Expand First**  $f(x) = (x^5 - 3x)\left(\frac{1}{x^2}\right) = \frac{x^5}{x^2} - \frac{3x}{x^2} = x^3 - 3x^{-1}$ .

$$f'(x) = 3x^2 - 3(-1x^{-2}) = 3x^2 + 3x^{-2}$$

*Results match.*

16. Find derivative of  $f(x) = (3x^3 + 4x)(x - 5)(x + 1)$ .

**Method 1: Extended Product Rule (3 factors)** Let  $F = 3x^3 + 4x$ ,  $G = x - 5$ ,  
 $H = x + 1$ .  $F' = 9x^2 + 4$ .  $G' = 1$ .  $H' = 1$ . Rule:  $(FGH)' = F'GH + FG'H + FGH'$ .

$$f'(x) = (9x^2 + 4)(x - 5)(x + 1) + (3x^3 + 4x)(1)(x + 1) + (3x^3 + 4x)(x - 5)(1)$$

Expand (carefully):  $(9x^2 + 4)(x^2 - 4x - 5) = 9x^4 - 36x^3 - 45x^2 + 4x^2 - 16x - 20 = 9x^4 - 36x^3 - 41x^2 - 16x - 20$ .  $(3x^3 + 4x)(x + 1) = 3x^4 + 3x^3 + 4x^2 + 4x$ .  $(3x^3 + 4x)(x - 5) = 3x^4 - 15x^3 + 4x^2 - 20x$ . Summing these three expansions:

$$f'(x) = (9x^4 - 36x^3 - 41x^2 - 16x - 20) + (3x^4 + 3x^3 + 4x^2 + 4x) + (3x^4 - 15x^3 + 4x^2 - 20x)$$

Combine like terms:  $x^4$ :  $9 + 3 + 3 = 15$ .  $x^3$ :  $-36 + 3 - 15 = -48$ .  $x^2$ :  $-41 + 4 + 4 = -33$ .  $x$ :  $-16 + 4 - 20 = -32$ . Constant:  $-20$ .

$$f'(x) = 15x^4 - 48x^3 - 33x^2 - 32x - 20$$

**Method 2: Expand First** First multiply two factors:  $(x - 5)(x + 1) = x^2 - 4x - 5$ . Now multiply by the first factor:  $f(x) = (3x^3 + 4x)(x^2 - 4x - 5)$   $f(x) = 3x^3(x^2 - 4x - 5) + 4x(x^2 - 4x - 5)$   $f(x) = (3x^5 - 12x^4 - 15x^3) + (4x^3 - 16x^2 - 20x)$   $f(x) = 3x^5 - 12x^4 - 11x^3 - 16x^2 - 20x$ . Differentiate term-by-term:

$$\begin{aligned} f'(x) &= 3(5x^4) - 12(4x^3) - 11(3x^2) - 16(2x) - 20(1) \\ &= 15x^4 - 48x^3 - 33x^2 - 32x - 20 \end{aligned}$$

*Results match.*

17. Find derivative of  $\frac{dR}{dM}$  where  $R = M^2 \left( \frac{C}{2} - \frac{M}{3} \right)$ .

**Method 1: Product Rule** Let  $u = M^2, v = \frac{C}{2} - \frac{M}{3}$ .  $C$  is constant.  $du/dM = 2M$ .  $dv/dM = 0 - \frac{1}{3}(1) = -\frac{1}{3}$ .

$$\begin{aligned} \frac{dR}{dM} &= u \frac{dv}{dM} + v \frac{du}{dM} \\ &= (M^2)\left(-\frac{1}{3}\right) + \left(\frac{C}{2} - \frac{M}{3}\right)(2M) \\ &= -\frac{1}{3}M^2 + \left(\frac{C}{2}(2M) - \frac{M}{3}(2M)\right) \\ &= -\frac{1}{3}M^2 + CM - \frac{2}{3}M^2 \\ &= CM + \left(-\frac{1}{3} - \frac{2}{3}\right)M^2 = CM - \frac{3}{3}M^2 = CM - M^2 \end{aligned}$$

*Factored:  $M(C - M)$ .*

**Method 2: Expand First**  $R = M^2\left(\frac{C}{2}\right) - M^2\left(\frac{M}{3}\right) = \frac{C}{2}M^2 - \frac{1}{3}M^3$ . Differentiate term-by-term w.r.t  $M$ :

$$\frac{dR}{dM} = \frac{C}{2}(2M) - \frac{1}{3}(3M^2) = CM - M^2$$

*Factored:  $M(C - M)$ . Results match.*