

# INTEGRATION SOLUTIONS - WEEK 7

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## 1 Solutions: Differentiation of Integrals

**Explanation:** This section applies the first part of the Fundamental Theorem of Calculus (FTC), which relates differentiation and integration. It tells us how to find the derivative of a function that is defined by an integral.

**Rule 1 (Basic FTC):** If a function  $y$  is defined as the integral of another function  $f(t)$  from a constant  $a$  up to a variable  $x$ , like  $y = \int_a^x f(t) dt$ , then the derivative of  $y$  with respect to  $x$  is simply the original function  $f$  evaluated at  $x$ . Mathematically:  $\frac{dy}{dx} = f(x)$ . You just replace the integration variable  $t$  with the upper limit  $x$ .

**Rule 2 (FTC with Chain Rule):** If the upper limit of integration is itself a function of  $x$ , say  $u(x)$ , so  $y = \int_a^{u(x)} f(t) dt$ , finding the derivative requires the Chain Rule. The derivative is:  $\frac{dy}{dx} = f(u(x)) \cdot u'(x)$ . You replace  $t$  with the upper limit function  $u(x)$ , and then multiply by the derivative of that upper limit function,  $u'(x)$ .

**Important Property:** If the variable limit is the lower one,  $\int_{b(x)}^a f(t) dt$ , you can swap the limits by introducing a negative sign:  $-\int_a^{b(x)} f(t) dt$ . Then you can apply the rules above.

1. **Problem:** Find  $\frac{dy}{dx}$  if  $y = \int_a^x (t^3 + 1) dt$ .

*Solution: Identify the situation:* The function  $y$  is defined by an integral. The lower limit  $a$  is a constant. The upper limit is simply  $x$ . The function inside the integral (the integrand) is  $f(t) = t^3 + 1$ .

*Apply the rule:* This matches Rule 1 (Basic FTC) exactly. To find  $\frac{dy}{dx}$ , we replace the integration variable  $t$  in the integrand  $f(t)$  with the upper limit  $x$ .

$$\frac{dy}{dx} = f(x) = x^3 + 1$$

2. **Problem:** Find  $\frac{dy}{dx}$  if  $y = \int_{1+3x^2}^4 \frac{1}{2+e^t} dt$ .

*Solution: Step 1: Handle the limits.* The function of  $x$ , which is  $1+3x^2$ , is in the lower limit, and the upper limit 4 is a constant. We need the function of  $x$  to be the upper limit to apply our rules directly. Use the property  $\int_b^a f(t) dt = -\int_a^b f(t) dt$  to swap the limits:

$$y = -\int_4^{1+3x^2} \frac{1}{2+e^t} dt$$

Step 2: *Identify components for Rule 2.* Now the integral has a constant lower limit (4) and an upper limit that is a function of  $x$ . This matches Rule 2 (FTC with Chain Rule), with an extra minus sign out front.

- Integrand:  $f(t) = \frac{1}{2+e^t}$
- Upper limit function:  $u(x) = 1 + 3x^2$

Step 3: *Find the derivative of the upper limit.* We need  $u'(x)$  for the Chain Rule part.

$$u'(x) = \frac{d}{dx}(1 + 3x^2) = \frac{d}{dx}(1) + \frac{d}{dx}(3x^2) = 0 + 3(2x) = 6x$$

Step 4: *Apply Rule 2 (FTC with Chain Rule).* The rule is  $\frac{dy}{dx} = f(u(x)) \cdot u'(x)$ . Since we have a minus sign from Step 1, our derivative is  $\frac{dy}{dx} = -[f(u(x)) \cdot u'(x)]$ . Substitute the pieces we found:

$$\begin{aligned} \frac{dy}{dx} &= -[f(1 + 3x^2) \cdot (6x)] \\ &= -\left[\frac{1}{2 + e^{(1+3x^2)}} \cdot 6x\right] \\ &= -\frac{6x}{2 + e^{1+3x^2}} \end{aligned}$$

3. **Problem:** Find the derivative  $\frac{d}{dt} \int_0^{t^4} \sqrt{u} \, du$ .

*Solution: Identify the situation:* We are differentiating with respect to  $t$ . The integral has a constant lower limit (0) and an upper limit that is a function of  $t$ , namely  $v(t) = t^4$ . The integrand is  $f(u) = \sqrt{u}$ . This requires Rule 2 (FTC with Chain Rule).

Step 1: *Identify components.*

- Integrand:  $f(u) = \sqrt{u}$
- Upper limit function:  $v(t) = t^4$

Step 2: *Find the derivative of the upper limit function.*

$$v'(t) = \frac{d}{dt}(t^4) = 4t^3$$

Step 3: *Apply Rule 2.* The rule states the derivative is  $f(v(t)) \cdot v'(t)$ .

$$\begin{aligned} \frac{d}{dt} \int_0^{t^4} \sqrt{u} \, du &= f(t^4) \cdot (4t^3) \quad (\text{Replace } u \text{ with } v(t) = t^4 \text{ in } f(u)) \\ &= \sqrt{t^4} \cdot 4t^3 \\ &= |t^2| \cdot 4t^3 \quad (\text{The principal square root is always non-negative}) \\ &= t^2 \cdot 4t^3 \quad (\text{Since } t^2 \text{ is never negative, } |t^2| = t^2) \\ &= 4t^{2+3} = 4t^5 \quad (\text{Using exponent rule } x^m \cdot x^n = x^{m+n}) \end{aligned}$$

4. **Problem:** Find the derivative  $\frac{d}{dx} \int_0^{x^3} e^{-t} dt$ .

*Solution: Identify the situation:* We are differentiating with respect to  $x$ . The integral has a constant lower limit (0) and an upper limit that is a function of  $x$ , namely  $u(x) = x^3$ . The integrand is  $f(t) = e^{-t}$ . This requires Rule 2 (FTC with Chain Rule).

Step 1: *Identify components.*

- Integrand:  $f(t) = e^{-t}$
- Upper limit function:  $u(x) = x^3$

Step 2: *Find the derivative of the upper limit function.*

$$u'(x) = \frac{d}{dx}(x^3) = 3x^2$$

Step 3: *Apply Rule 2.* The derivative is  $f(u(x)) \cdot u'(x)$ .

$$\begin{aligned} \frac{d}{dx} \int_0^{x^3} e^{-t} dt &= f(x^3) \cdot (3x^2) \quad (\text{Replace } t \text{ with } u(x) = x^3 \text{ in } f(t)) \\ &= e^{-(x^3)} \cdot 3x^2 \\ &= 3x^2 e^{-x^3} \end{aligned}$$

## 2 Solutions: Evaluation of Definite Integrals

**Explanation:** This section uses the second part of the Fundamental Theorem of Calculus (FTC), often called the Evaluation Theorem. It provides a method to calculate the exact numerical value of a definite integral, which geometrically represents the net signed area between the function's graph and the x-axis over the interval  $[a, b]$ .

**The Process:** To evaluate  $\int_a^b f(x) dx$ : 1. **Find an Antiderivative:** Determine a function  $F(x)$  such that its derivative  $F'(x)$  equals the integrand  $f(x)$ . This process is called indefinite integration. The most common rule needed here is the **Power Rule for Integration:**  $\int x^n dx = \frac{x^{n+1}}{n+1} + C$ , which works for any  $n \neq -1$ . Remember that  $\int k dx = kx + C$  for any constant  $k$ . We use the property of linearity:  $\int (c \cdot g(x) + d \cdot h(x)) dx = c \int g(x) dx + d \int h(x) dx$ .

2. **Evaluate at Limits:** Calculate the value of the antiderivative  $F(x)$  at the upper limit of integration ( $x = b$ ) and at the lower limit ( $x = a$ ).

3. **Subtract:** The value of the definite integral is the difference:  $F(b) - F(a)$ . We often write this using bracket notation:  $[F(x)]_a^b = F(b) - F(a)$ . (Note: The constant of integration  $C$  is not needed for definite integrals because it would cancel:  $(F(b) + C) - (F(a) + C) = F(b) - F(a)$ .)

1. **Problem:** Evaluate  $\int_{-2}^0 (2x + 5) dx$ .

*Solution:* Step 1: *Find an antiderivative*  $F(x)$  of the integrand  $f(x) = 2x + 5$ . We integrate term by term using the power rule:

- For  $2x = 2x^1$ :  $\int 2x^1 dx = 2 \cdot \frac{x^{1+1}}{1+1} = 2 \cdot \frac{x^2}{2} = x^2$ .
- For  $5 = 5x^0$ :  $\int 5x^0 dx = 5 \cdot \frac{x^{0+1}}{0+1} = 5 \cdot \frac{x^1}{1} = 5x$ .

Combining these, an antiderivative is  $F(x) = x^2 + 5x$ .

Step 2: *Apply the FTC Evaluation Theorem:*  $\int_a^b f(x) dx = F(b) - F(a)$ . Here  $a = -2$  (lower limit) and  $b = 0$  (upper limit).

$$\begin{aligned}
 \int_{-2}^0 (2x + 5) dx &= [x^2 + 5x]_{a=-2}^{b=0} \quad (\text{Using bracket notation}) \\
 &= F(0) - F(-2) \\
 &= ((0)^2 + 5(0)) - ((-2)^2 + 5(-2)) \quad (\text{Substitute } b=0 \text{ and } a=-2 \text{ into } F(x)) \\
 &= (0 + 0) - (4 - 10) \\
 &= 0 - (-6) \\
 &= 6
 \end{aligned}$$

2. **Problem:** Evaluate  $\int_{-3}^1 \left(5 - \frac{x}{2}\right) dx$ .

*Solution:* Step 1: Find an antiderivative  $F(x)$  of  $f(x) = 5 - \frac{1}{2}x$ .

- For  $5 = 5x^0$ :  $\int 5x^0 dx = 5 \cdot \frac{x^1}{1} = 5x$ .
- For  $-\frac{1}{2}x = -\frac{1}{2}x^1$ :  $\int -\frac{1}{2}x^1 dx = -\frac{1}{2} \cdot \frac{x^{1+1}}{1+1} = -\frac{1}{2} \cdot \frac{x^2}{2} = -\frac{x^2}{4}$ .

Combining these, an antiderivative is  $F(x) = 5x - \frac{x^2}{4}$ .

Step 2: *Apply the FTC Evaluation Theorem.* Here  $a = -3$  and  $b = 1$ .

$$\begin{aligned}
 \int_{-3}^1 \left(5 - \frac{x}{2}\right) dx &= \left[5x - \frac{x^2}{4}\right]_{-3}^1 \\
 &= F(1) - F(-3) \\
 &= \left(5(1) - \frac{(1)^2}{4}\right) - \left(5(-3) - \frac{(-3)^2}{4}\right) \\
 &= \left(5 - \frac{1}{4}\right) - \left(-15 - \frac{9}{4}\right) \\
 &= \left(\frac{20}{4} - \frac{1}{4}\right) - \left(-\frac{60}{4} - \frac{9}{4}\right) \quad (\text{Finding common denominators}) \\
 &= \left(\frac{19}{4}\right) - \left(-\frac{69}{4}\right) \\
 &= \frac{19}{4} + \frac{69}{4} = \frac{19 + 69}{4} = \frac{88}{4} = 22
 \end{aligned}$$

3. **Problem:** Evaluate  $\int_0^2 x(x - 3) dx$ .

*Solution:* Step 1: *Expand the integrand* to make it easier to apply the power rule.

$$f(x) = x(x - 3) = x^2 - 3x$$

Step 2: Find an antiderivative  $F(x)$  of  $f(x) = x^2 - 3x$ .

- $\int x^2 dx = \frac{x^{2+1}}{2+1} = \frac{x^3}{3}.$
- $\int -3x^1 dx = -3 \cdot \frac{x^{1+1}}{1+1} = -3 \cdot \frac{x^2}{2} = -\frac{3x^2}{2}.$

So,  $F(x) = \frac{x^3}{3} - \frac{3x^2}{2}.$

Step 3: *Apply the FTC Evaluation Theorem.* Here  $a = 0$  and  $b = 2$ .

$$\begin{aligned}
 \int_0^2 (x^2 - 3x) \, dx &= \left[ \frac{x^3}{3} - \frac{3x^2}{2} \right]_0^2 \\
 &= F(2) - F(0) \\
 &= \left( \frac{(2)^3}{3} - \frac{3(2)^2}{2} \right) - \left( \frac{(0)^3}{3} - \frac{3(0)^2}{2} \right) \\
 &= \left( \frac{8}{3} - \frac{3(4)}{2} \right) - (0 - 0) \\
 &= \left( \frac{8}{3} - \frac{12}{2} \right) = \frac{8}{3} - 6 \\
 &= \frac{8}{3} - \frac{18}{3} = \frac{8-18}{3} = -\frac{10}{3}
 \end{aligned}$$

4. **Problem:** Evaluate  $\int_{-1}^1 (x^2 - 2x + 3) \, dx$ .

*Solution 1: Direct Evaluation*

Step 1: *Find an antiderivative  $F(x)$  of  $f(x) = x^2 - 2x + 3$ .*

$$F(x) = \int (x^2 - 2x + 3) \, dx = \frac{x^3}{3} - 2 \left( \frac{x^2}{2} \right) + 3x = \frac{x^3}{3} - x^2 + 3x$$

Step 2: *Apply the FTC Evaluation Theorem.* Here  $a = -1$  and  $b = 1$ .

$$\begin{aligned}
 \int_{-1}^1 (x^2 - 2x + 3) \, dx &= \left[ \frac{x^3}{3} - x^2 + 3x \right]_{-1}^1 \\
 &= F(1) - F(-1) \\
 &= \left( \frac{1^3}{3} - (1)^2 + 3(1) \right) - \left( \frac{(-1)^3}{3} - (-1)^2 + 3(-1) \right) \\
 &= \left( \frac{1}{3} - 1 + 3 \right) - \left( -\frac{1}{3} - 1 - 3 \right) \\
 &= \left( \frac{1}{3} + 2 \right) - \left( -\frac{1}{3} - 4 \right) \\
 &= \left( \frac{1+6}{3} \right) - \left( -\frac{1+12}{3} \right) = \frac{7}{3} - \left( -\frac{13}{3} \right) = \frac{7+13}{3} = \frac{20}{3}
 \end{aligned}$$

*Solution 2: Using Symmetry*

**Explanation:** The interval of integration  $[-1, 1]$  is **symmetric** about  $x = 0$ . We can analyze the integrand  $f(x) = x^2 - 2x + 3$ . A function  $g(x)$  is **even** if  $g(-x) = g(x)$  (graph is symmetric about the y-axis). A function  $h(x)$  is **odd** if  $h(-x) = -h(x)$  (graph

is symmetric about the origin). Here,  $x^2$  is even, 3 is even (constants are even), so  $(x^2 + 3)$  is even. The term  $-2x$  is odd.

Property 1: For any odd function  $h(x)$ ,  $\int_{-a}^a h(x) \, dx = 0$ .

Property 2: For any even function  $g(x)$ ,  $\int_{-a}^a g(x) \, dx = 2 \int_0^a g(x) \, dx$ .

Step 1: *Split the integral and apply symmetry properties.*

$$\begin{aligned} \int_{-1}^1 (x^2 - 2x + 3) \, dx &= \int_{-1}^1 \underbrace{(x^2 + 3)}_{\text{even}} \, dx + \int_{-1}^1 \underbrace{(-2x)}_{\text{odd}} \, dx \\ &= 2 \int_0^1 (x^2 + 3) \, dx + 0 \end{aligned}$$

Step 2: *Evaluate the remaining integral.* Find the antiderivative of  $x^2 + 3$ :  $F(x) = \frac{x^3}{3} + 3x$ . Apply FTC from 0 to 1:

$$\begin{aligned} &= 2 \left[ \frac{x^3}{3} + 3x \right]_0^1 = 2(F(1) - F(0)) \\ &= 2 \left( \left( \frac{1^3}{3} + 3(1) \right) - \left( \frac{0^3}{3} + 3(0) \right) \right) = 2 \left( \left( \frac{1}{3} + 3 \right) - 0 \right) \\ &= 2 \left( \frac{1 + 9}{3} \right) = 2 \left( \frac{10}{3} \right) = \frac{20}{3} \end{aligned}$$

Both methods confirm the result is  $\frac{20}{3}$ .

5. **Problem:** Evaluate  $\int_0^4 \left( 3x - \frac{x^3}{4} \right) \, dx$ .

*Solution:* Step 1: Find antiderivative  $F(x)$  of  $f(x) = 3x - \frac{1}{4}x^3$ .

$$F(x) = \int (3x^1 - \frac{1}{4}x^3) \, dx = 3 \left( \frac{x^2}{2} \right) - \frac{1}{4} \left( \frac{x^4}{4} \right) = \frac{3x^2}{2} - \frac{x^4}{16}$$

Step 2: *Apply FTC Evaluation Theorem* ( $a = 0, b = 4$ ).

$$\begin{aligned} \int_0^4 \left( 3x - \frac{x^3}{4} \right) \, dx &= \left[ \frac{3x^2}{2} - \frac{x^4}{16} \right]_0^4 \\ &= F(4) - F(0) \\ &= \left( \frac{3(4^2)}{2} - \frac{4^4}{16} \right) - \left( \frac{3(0)^2}{2} - \frac{0^4}{16} \right) \\ &= \left( \frac{3(16)}{2} - \frac{256}{16} \right) - 0 \\ &= (3 \cdot 8) - 16 = 24 - 16 = 8 \end{aligned}$$

6. **Problem:** Evaluate  $\int_{-2}^2 (x^3 - 2x + 3) \, dx$ .

*Solution: Strategy:* Use symmetry since the interval  $[-2, 2]$  is symmetric about 0. Step 1: Analyze the integrand  $f(x) = x^3 - 2x + 3$ .

- $x^3 - 2x$ : This part is odd because  $(-x)^3 - 2(-x) = -x^3 + 2x = -(x^3 - 2x)$ .
- 3: This part is even because  $f(-x) = 3 = f(x)$ .

Step 2: *Split the integral and apply symmetry properties.*

$$\begin{aligned} \int_{-2}^2 (x^3 - 2x + 3) \, dx &= \underbrace{\int_{-2}^2 (x^3 - 2x) \, dx}_{\text{Integral of odd function} = 0} + \underbrace{\int_{-2}^2 3 \, dx}_{\text{Integral of even function} = 2 \int_0^2 3 \, dx} \\ &= 0 + 2 \int_0^2 3 \, dx \end{aligned}$$

Step 3: *Evaluate the remaining integral.* The antiderivative of 3 is  $3x$ .

$$= 2[3x]_0^2 = 2(F(2) - F(0)) = 2((3 \cdot 2) - (3 \cdot 0)) = 2(6 - 0) = 12$$

7. **Problem:** Evaluate  $\int_0^1 (x^2 + \sqrt{x}) \, dx$ .

*Solution:* Step 1: *Rewrite the integrand using exponents:*  $f(x) = x^2 + x^{1/2}$ .

Step 2: *Find the antiderivative  $F(x)$  using the power rule.*

- $\int x^2 dx = \frac{x^{2+1}}{2+1} = \frac{x^3}{3}$ .
- $\int x^{1/2} dx = \frac{x^{1/2+1}}{1/2+1} = \frac{x^{3/2}}{3/2} = \frac{2}{3}x^{3/2}$ .

So,  $F(x) = \frac{x^3}{3} + \frac{2}{3}x^{3/2}$ .

Step 3: *Apply the FTC Evaluation Theorem ( $a = 0, b = 1$ ).*

$$\begin{aligned} \int_0^1 (x^2 + x^{1/2}) \, dx &= \left[ \frac{x^3}{3} + \frac{2}{3}x^{3/2} \right]_0^1 \\ &= F(1) - F(0) \\ &= \left( \frac{1^3}{3} + \frac{2}{3}(1)^{3/2} \right) - \left( \frac{0^3}{3} + \frac{2}{3}(0)^{3/2} \right) \\ &= \left( \frac{1}{3} + \frac{2}{3} \right) - (0 + 0) = \frac{3}{3} = 1 \end{aligned}$$

8. **Problem:** Evaluate  $\int_1^{32} x^{-6/5} \, dx$ .

*Solution:* Step 1: *Find the antiderivative  $F(x)$  using the Power Rule with  $n = -6/5$ .*

$$n + 1 = -6/5 + 1 = -6/5 + 5/5 = -1/5$$

$$F(x) = \int x^{-6/5} \, dx = \frac{x^{-1/5}}{-1/5} = -5x^{-1/5}$$

Step 2: *Apply the FTC Evaluation Theorem ( $a = 1, b = 32$ ).*

$$\begin{aligned}
\int_1^{32} x^{-6/5} dx &= [-5x^{-1/5}]_1^{32} \\
&= F(32) - F(1) \\
&= (-5 \cdot (32)^{-1/5}) - (-5 \cdot (1)^{-1/5}) \\
&= (-5 \cdot (2^5)^{-1/5}) - (-5 \cdot 1) \quad (\text{Since } 32 = 2^5 \text{ and } 1^{\text{any power}} = 1) \\
&= (-5 \cdot 2^{(5 \times -1/5)}) + 5 \\
&= (-5 \cdot 2^{-1}) + 5 \\
&= (-5 \cdot \frac{1}{2}) + 5 = -\frac{5}{2} + \frac{10}{2} = \frac{5}{2}
\end{aligned}$$

9. **Problem:** Evaluate  $\int_1^{-1} (r+1)^2 dr$ .

*Solution:* Step 1: *Notice the limits.* The lower limit (1) is greater than the upper limit (-1). Swap them and negate.

$$\int_1^{-1} (r+1)^2 dr = - \int_{-1}^1 (r+1)^2 dr$$

Step 2: *Expand the integrand* inside the new integral.

$$f(r) = (r+1)^2 = r^2 + 2r + 1$$

Step 3: *Find the antiderivative*  $F(r)$  of  $r^2 + 2r + 1$ .

$$F(r) = \int (r^2 + 2r + 1) dr = \frac{r^3}{3} + \frac{2r^2}{2} + r = \frac{r^3}{3} + r^2 + r$$

Step 4: *Apply the FTC* to  $\int_{-1}^1 (r^2 + 2r + 1) dr$  and include the negative sign from Step 1.

$$\begin{aligned}
- \int_{-1}^1 (r^2 + 2r + 1) dr &= -[F(r)]_{-1}^1 \\
&= -(F(1) - F(-1)) \\
&= - \left[ \left( \frac{1^3}{3} + 1^2 + 1 \right) - \left( \frac{(-1)^3}{3} + (-1)^2 + (-1) \right) \right] \\
&= - \left[ \left( \frac{1}{3} + 1 + 1 \right) - \left( -\frac{1}{3} + 1 - 1 \right) \right] \\
&= - \left[ \left( \frac{1}{3} + 2 \right) - \left( -\frac{1}{3} + 0 \right) \right] \\
&= - \left[ \frac{7}{3} - \left( -\frac{1}{3} \right) \right] = - \left[ \frac{7}{3} + \frac{1}{3} \right] = - \left[ \frac{8}{3} \right] = -\frac{8}{3}
\end{aligned}$$

10. **Problem:** Evaluate  $\int_{-\sqrt{3}}^{\sqrt{3}} (t+1)(t^2+4) dt$ .

*Solution:* Step 1: *Expand the integrand.*

$$f(t) = (t+1)(t^2+4) = t^3 + 4t + t^2 + 4 = t^3 + t^2 + 4t + 4$$

Step 2: *Check for symmetry.* The interval  $[-\sqrt{3}, \sqrt{3}]$  is symmetric.



- Odd part:  $t^3 + 4t$
- Even part:  $t^2 + 4$

Step 3: *Apply symmetry properties.*

$$\begin{aligned}\int_{-\sqrt{3}}^{\sqrt{3}} (t^3 + t^2 + 4t + 4) \, dt &= \underbrace{\int_{-\sqrt{3}}^{\sqrt{3}} (t^3 + 4t) \, dt}_{=0} + \underbrace{\int_{-\sqrt{3}}^{\sqrt{3}} (t^2 + 4) \, dt}_{=2 \int_0^{\sqrt{3}} (t^2 + 4) \, dt} \\ &= 2 \int_0^{\sqrt{3}} (t^2 + 4) \, dt\end{aligned}$$

Step 4: *Find antiderivative  $G(t)$  of the remaining integrand  $t^2 + 4$ .*

$$G(t) = \int (t^2 + 4) \, dt = \frac{t^3}{3} + 4t$$

Step 5: *Evaluate using FTC.*

$$\begin{aligned}2 \int_0^{\sqrt{3}} (t^2 + 4) \, dt &= 2[G(t)]_0^{\sqrt{3}} = 2(G(\sqrt{3}) - G(0)) \\ &= 2 \left[ \left( \frac{(\sqrt{3})^3}{3} + 4\sqrt{3} \right) - \left( \frac{0^3}{3} + 4(0) \right) \right] \\ &= 2 \left[ \left( \frac{3\sqrt{3}}{3} + 4\sqrt{3} \right) - 0 \right] \quad (\text{Note: } (\sqrt{3})^3 = 3\sqrt{3}) \\ &= 2[\sqrt{3} + 4\sqrt{3}] = 2[5\sqrt{3}] = 10\sqrt{3}\end{aligned}$$

11. **Problem:** Evaluate  $\int_{\sqrt{2}}^1 \left( \frac{u^7}{2} - \frac{1}{u^5} \right) \, du$ .

*Solution:* Step 1: *Rewrite integrand using negative exponents.*

$$f(u) = \frac{1}{2}u^7 - u^{-5}$$

Step 2: *Find the antiderivative  $F(u)$ .*

$$F(u) = \int \left( \frac{1}{2}u^7 - u^{-5} \right) \, du = \frac{1}{2} \left( \frac{u^8}{8} \right) - \left( \frac{u^{-4}}{-4} \right) = \frac{u^8}{16} + \frac{1}{4u^4}$$

Step 3: *Apply the FTC ( $a = \sqrt{2}, b = 1$ ).*

$$\begin{aligned}\int_{\sqrt{2}}^1 f(u) \, du &= [F(u)]_{\sqrt{2}}^1 = F(1) - F(\sqrt{2}) \\ &= \left( \frac{1^8}{16} + \frac{1}{4(1)^4} \right) - \left( \frac{(\sqrt{2})^8}{16} + \frac{1}{4(\sqrt{2})^4} \right) \\ &= \left( \frac{1}{16} + \frac{1}{4} \right) - \left( \frac{16}{16} + \frac{1}{4(4)} \right) \quad (\text{Note: } (\sqrt{2})^8 = 16, (\sqrt{2})^4 = 4) \\ &= \left( \frac{1+4}{16} \right) - \left( 1 + \frac{1}{16} \right) = \frac{5}{16} - \frac{17}{16} = -\frac{12}{16} = -\frac{3}{4}\end{aligned}$$

12. **Problem:** Evaluate  $\int_{-3}^{-1} \frac{y^5 - 2y}{y^3} dy$ .

*Solution:* Step 1: *Simplify the integrand* by dividing term-by-term.

$$f(y) = \frac{y^5}{y^3} - \frac{2y^1}{y^3} = y^{5-3} - 2y^{1-3} = y^2 - 2y^{-2}$$

Step 2: *Find the antiderivative*  $F(y)$ .

$$F(y) = \int (y^2 - 2y^{-2}) dy = \frac{y^3}{3} - 2 \left( \frac{y^{-1}}{-1} \right) = \frac{y^3}{3} + \frac{2}{y}$$

Step 3: *Apply the FTC* ( $a = -3, b = -1$ ).

$$\begin{aligned} \int_{-3}^{-1} f(y) dy &= [F(y)]_{-3}^{-1} = F(-1) - F(-3) \\ &= \left( \frac{(-1)^3}{3} + \frac{2}{-1} \right) - \left( \frac{(-3)^3}{3} + \frac{2}{-3} \right) \\ &= \left( -\frac{1}{3} - 2 \right) - \left( -\frac{27}{3} - \frac{2}{3} \right) \\ &= \left( -\frac{1+6}{3} \right) - \left( -\frac{27+2}{3} \right) = -\frac{7}{3} - \left( -\frac{29}{3} \right) = \frac{-7+29}{3} = \frac{22}{3} \end{aligned}$$

13. **Problem:** Evaluate  $\int_1^{\sqrt{2}} \frac{s^2 + \sqrt{s}}{s^2} ds$ .

*Solution:* Step 1: *Simplify the integrand.* Rewrite  $\sqrt{s} = s^{1/2}$  and divide.

$$f(s) = \frac{s^2}{s^2} + \frac{s^{1/2}}{s^2} = 1 + s^{1/2-2} = 1 + s^{-3/2}$$

Step 2: *Find the antiderivative*  $F(s)$  using power rule ( $n = -3/2$ ).

$$F(s) = \int (1 + s^{-3/2}) ds = s + \frac{s^{-3/2+1}}{-3/2+1} = s + \frac{s^{-1/2}}{-1/2} = s - 2s^{-1/2} = s - \frac{2}{\sqrt{s}}$$

Step 3: *Apply the FTC* ( $a = 1, b = \sqrt{2}$ ).

$$\begin{aligned} \int_1^{\sqrt{2}} f(s) ds &= [F(s)]_1^{\sqrt{2}} = F(\sqrt{2}) - F(1) \\ &= \left( \sqrt{2} - \frac{2}{\sqrt{\sqrt{2}}} \right) - \left( 1 - \frac{2}{\sqrt{1}} \right) \\ &= \left( \sqrt{2} - \frac{2}{2^{1/4}} \right) - (1 - 2) \quad (\text{Note: } \sqrt{\sqrt{s}} = s^{1/4}) \\ &= \sqrt{2} - 2^{1-1/4} - (-1) \quad (\text{Exponent rule: } a^m/a^n = a^{m-n}) \\ &= \sqrt{2} - 2^{3/4} + 1 \end{aligned}$$

14. **Problem:** Evaluate  $\int_1^8 \frac{(x^{1/3} + 1)(2 - x^{2/3})}{x^{1/3}} dx$ .

*Solution:* Step 1: *Simplify the integrand.* This requires expanding the numerator first, then dividing by the denominator.

Numerator:  $(x^{1/3} + 1)(2 - x^{2/3}) = (x^{1/3})(2) + (x^{1/3})(-x^{2/3}) + (1)(2) + (1)(-x^{2/3}) = 2x^{1/3} - x^{(1/3+2/3)} + 2 - x^{2/3} = 2x^{1/3} - x^1 + 2 - x^{2/3}$ . Now divide by  $x^{1/3}$ :

$$\begin{aligned} f(x) &= \frac{2x^{1/3} - x^1 + 2 - x^{2/3}}{x^{1/3}} \\ &= \frac{2x^{1/3}}{x^{1/3}} - \frac{x^1}{x^{1/3}} + \frac{2}{x^{1/3}} - \frac{x^{2/3}}{x^{1/3}} \\ &= 2 - x^{(1-1/3)} + 2x^{-1/3} - x^{(2/3-1/3)} \quad (\text{Using exponent rule } x^m/x^n = x^{m-n}) \\ &= 2 - x^{2/3} + 2x^{-1/3} - x^{1/3} \end{aligned}$$

Step 2: *Find the antiderivative  $F(x)$  term-by-term using the Power Rule.*

$$\begin{aligned} F(x) &= \int (2 - x^{2/3} + 2x^{-1/3} - x^{1/3}) dx \\ &= 2x - \frac{x^{2/3+1}}{2/3+1} + 2\frac{x^{-1/3+1}}{-1/3+1} - \frac{x^{1/3+1}}{1/3+1} \\ &= 2x - \frac{x^{5/3}}{5/3} + 2\frac{x^{2/3}}{2/3} - \frac{x^{4/3}}{4/3} \\ &= 2x - \frac{3}{5}x^{5/3} + 2\left(\frac{3}{2}\right)x^{2/3} - \frac{3}{4}x^{4/3} \\ &= 2x - \frac{3}{5}x^{5/3} + 3x^{2/3} - \frac{3}{4}x^{4/3} \end{aligned}$$

Step 3: *Evaluate  $F(x)$  at the limits  $a = 1$  and  $b = 8$ .* For  $F(8)$ : We need powers of

$$8^{1/3} = \sqrt[3]{8} = 2.$$

- $8^{2/3} = (8^{1/3})^2 = 2^2 = 4$
- $8^{4/3} = (8^{1/3})^4 = 2^4 = 16$
- $8^{5/3} = (8^{1/3})^5 = 2^5 = 32$

$$\begin{aligned} F(8) &= 2(8) - \frac{3}{5}(8^{5/3}) + 3(8^{2/3}) - \frac{3}{4}(8^{4/3}) \\ &= 16 - \frac{3}{5}(32) + 3(4) - \frac{3}{4}(16) \\ &= 16 - \frac{96}{5} + 12 - (3 \cdot 4) \\ &= 16 - \frac{96}{5} + 12 - 12 = 16 - \frac{96}{5} \\ &= \frac{80}{5} - \frac{96}{5} = -\frac{16}{5} \end{aligned}$$

For  $F(1)$ : Any power of 1 is 1.

$$\begin{aligned}
F(1) &= 2(1) - \frac{3}{5}(1^{5/3}) + 3(1^{2/3}) - \frac{3}{4}(1^{4/3}) \\
&= 2 - \frac{3}{5}(1) + 3(1) - \frac{3}{4}(1) \\
&= 2 - \frac{3}{5} + 3 - \frac{3}{4} = (2 + 3) - \frac{3}{5} - \frac{3}{4} \\
&= 5 - \left(\frac{3 \cdot 4}{20} + \frac{3 \cdot 5}{20}\right) \quad (\text{Common denominator is } 20) \\
&= 5 - \left(\frac{12 + 15}{20}\right) = 5 - \frac{27}{20} = \frac{100}{20} - \frac{27}{20} = \frac{73}{20}
\end{aligned}$$

Step 4: Calculate the definite integral  $F(8) - F(1)$ .

$$F(8) - F(1) = -\frac{16}{5} - \frac{73}{20}$$

Find a common denominator (20):

$$= -\frac{16 \cdot 4}{5 \cdot 4} - \frac{73}{20} = -\frac{64}{20} - \frac{73}{20} = \frac{-64 - 73}{20} = -\frac{137}{20}$$