TECHNIQUES OF DIFFERENTIATION: PRODUCT RULE

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1 Introduction: The Product Rule

Explanation. When we need to differentiate a function that is formed by multiplying two other functions together, we cannot simply differentiate each function separately and multiply the results. We must use the **Product Rule**.

If $y = f(x) \cdot g(x)$, where both $f(x)$ and $g(x)$ are differentiable functions of x, then the derivative of y is given by:

$$\frac{dy}{dx} = f(x) \cdot \frac{d}{dx}[g(x)] + g(x) \cdot \frac{d}{dx}[f(x)]$$

Or using prime notation for derivatives:

$$y' = f(x)g'(x) + g(x)f'(x)$$

In simpler terms: "The derivative of a product is the first function times the derivative of the second, plus the second function times the derivative of the first."

We often use u and v to represent the two functions, so if $y = uv$, then:

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

Extension to Three Functions: If $y = f(x)g(x)h(x)$, the rule extends:

$$\frac{dy}{dx} = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$$

(Differentiate one function at a time, keeping the others unchanged, and sum the results).

Alternative Strategy: Sometimes, it might be easier to first algebraically expand the product into a sum of simpler terms and then differentiate term-by-term using the Power, Sum/Difference, and Constant Multiple rules. We will show this where practical.

2 Solutions: Example

1. Find the derivative of $y = (3x - 2x^2)(5 + 4x)$.

   Solution:

   Method 1: Product Rule

   Strategy. Identify the two functions being multiplied. Let $u = (3x-2x^2)$ and $v = (5+4x)$. Find the derivatives du/dx and dv/dx, then apply the formula $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$.

Step 1: Identify u, v and find their derivatives.

- $u = 3x - 2x^2$
- $v = 5 + 4x$
- $\frac{du}{dx} = \frac{d}{dx}(3x) - \frac{d}{dx}(2x^2) = 3(1) - 2(2x) = 3 - 4x$
- $\frac{dv}{dx} = \frac{d}{dx}(5) + \frac{d}{dx}(4x) = 0 + 4(1) = 4$

Step 2: Apply the Product Rule formula.

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{dy}{dx} = (3x - 2x^2)(4) + (5 + 4x)(3 - 4x)$$

Step 3: Expand and simplify the result.

$$\begin{aligned}\frac{dy}{dx} &= (4 \cdot 3x - 4 \cdot 2x^2) + (5 \cdot 3 + 5(-4x) + 4x \cdot 3 + 4x(-4x)) \\ &= (12x - 8x^2) + (15 - 20x + 12x - 16x^2) \\ &= 12x - 8x^2 + 15 - 8x - 16x^2\end{aligned}$$

Combine like terms (constants, x terms, $\mathbf{x^2}$ terms):

$$= 15 + (12x - 8x) + (-8x^2 - 16x^2)$$

$$= 15 + 4x - 24x^2$$

Method 2: Expand First

Strategy. Multiply the two binomials first to get a single polynomial, then differentiate term by term.

Step 1: Expand the product.

$$y = (3x - 2x^2)(5 + 4x)$$

$$y = 3x(5) + 3x(4x) - 2x^2(5) - 2x^2(4x) \quad \textit{(Using FOIL/Distributive Property)}$$

$$y = 15x + 12x^2 - 10x^2 - 8x^3$$

Combine like terms:

$$y = -8x^3 + (12x^2 - 10x^2) + 15x$$

$$y = -8x^3 + 2x^2 + 15x$$

Step 2: Differentiate the expanded polynomial term by term.

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(-8x^3) + \frac{d}{dx}(2x^2) + \frac{d}{dx}(15x) \\ &= -8(3x^{3-1}) + 2(2x^{2-1}) + 15(1x^{1-1}) \\ &= -24x^2 + 4x + 15x^0 \\ &= -24x^2 + 4x + 15\end{aligned}$$

Result Comparison: Both methods yield $15 + 4x - 24x^2$.

3 Solutions: ClassWork Problems

2. Find derivative of $f(x) = x(x^2 + 3)$.

   Solution:

   Method 1: Product Rule

   Strategy. Let $u = x$ and $v = x^2 + 3$.

   $u = x \implies \frac{du}{dx} = 1$. $v = x^2 + 3 \implies \frac{dv}{dx} = 2x + 0 = 2x$. Apply formula: $\frac{df}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$$\begin{aligned}\frac{df}{dx} &= (x)(2x) + (x^2 + 3)(1) \\ &= 2x^2 + x^2 + 3 = 3x^2 + 3\end{aligned}$$

   Method 2: Expand First $f(x) = x(x^2 + 3) = x^3 + 3x$. Differentiate term-by-term:

$$\frac{df}{dx} = \frac{d}{dx}(x^3) + \frac{d}{dx}(3x) = 3x^2 + 3(1) = 3x^2 + 3$$

   Results match.

3. Find derivative of $g(x) = (x - 4)(x + 2)$.

   Solution:

   Method 1: Product Rule

   Strategy. Let $u = x - 4$ and $v = x + 2$.

   $u = x - 4 \implies \frac{du}{dx} = 1 - 0 = 1$. $v = x + 2 \implies \frac{dv}{dx} = 1 + 0 = 1$. Apply formula: $\frac{dg}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$$\begin{aligned}\frac{dg}{dx} &= (x - 4)(1) + (x + 2)(1) \\ &= x - 4 + x + 2 = 2x - 2\end{aligned}$$

   Method 2: Expand First $g(x) = (x - 4)(x + 2) = x(x) + x(2) - 4(x) - 4(2) = x^2 + 2x - 4x - 8 = x^2 - 2x - 8$. Differentiate term-by-term:

$$\frac{dg}{dx} = \frac{d}{dx}(x^2) - \frac{d}{dx}(2x) - \frac{d}{dx}(8) = 2x - 2(1) - 0 = 2x - 2$$

   Results match.

4. Find derivative of $f(x) = x^2(3x^3 - 1)$.

   Solution:

   Method 1: Product Rule

   Strategy. Let $u = x^2$ and $v = 3x^3 - 1$.

$u = x^2 \implies \frac{du}{dx} = 2x$. $v = 3x^3 - 1 \implies \frac{dv}{dx} = 3(3x^2) - 0 = 9x^2$. Apply formula: $\frac{df}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$$\frac{df}{dx} = (x^2)(9x^2) + (3x^3 - 1)(2x)$$
$$= 9x^4 + (3x^3(2x) - 1(2x)) = 9x^4 + (6x^4 - 2x)$$
$$= 9x^4 + 6x^4 - 2x = 15x^4 - 2x$$

Method 2: Expand First $f(x) = x^2(3x^3 - 1) = 3x^5 - x^2$. Differentiate term-by-term:

$$\frac{df}{dx} = \frac{d}{dx}(3x^5) - \frac{d}{dx}(x^2) = 3(5x^4) - 2x = 15x^4 - 2x$$

Results match.

5. Find derivative of $f(x) = (x^2 + 1)(2x + 5)$.

   Solution:

   Method 1: Product Rule

   Strategy. Let $u = x^2 + 1$ and $v = 2x + 5$.

   $u = x^2 + 1 \implies \frac{du}{dx} = 2x + 0 = 2x$. $v = 2x + 5 \implies \frac{dv}{dx} = 2(1) + 0 = 2$. Apply formula: $\frac{df}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$$\frac{df}{dx} = (x^2 + 1)(2) + (2x + 5)(2x)$$
$$= (2x^2 + 2) + (4x^2 + 10x)$$
$$= 2x^2 + 2 + 4x^2 + 10x = 6x^2 + 10x + 2$$

Method 2: Expand First $f(x) = (x^2 + 1)(2x + 5) = x^2(2x) + x^2(5) + 1(2x) + 1(5) = 2x^3 + 5x^2 + 2x + 5$. Differentiate term-by-term:

$$\frac{df}{dx} = \frac{d}{dx}(2x^3) + \frac{d}{dx}(5x^2) + \frac{d}{dx}(2x) + \frac{d}{dx}(5)$$
$$= 2(3x^2) + 5(2x) + 2(1) + 0 = 6x^2 + 10x + 2$$

Results match.

6. Find derivative of $y = \frac{1}{x}(x^2 + e^x)$.

   Solution:

   Method 1: Product Rule

   Strategy. Rewrite as $y = x^{-1}(x^2 + e^x)$. Let $u = x^{-1}$ and $v = x^2 + e^x$.

$u = x^{-1} \implies \frac{du}{dx} = -1x^{-2} = -x^{-2} = -\frac{1}{x^2}$. $v = x^2 + e^x \implies \frac{dv}{dx} = 2x + e^x$. *(Derivative of e^x is e^x)* Apply formula: $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$$\begin{aligned}\frac{dy}{dx} &= (x^{-1})(2x + e^x) + (x^2 + e^x)(-x^{-2}) \\ &= \frac{1}{x}(2x + e^x) + (x^2 + e^x)(-\frac{1}{x^2}) \\ &= (\frac{2x}{x} + \frac{e^x}{x}) - (\frac{x^2}{x^2} + \frac{e^x}{x^2}) \\ &= (2 + \frac{e^x}{x}) - (1 + \frac{e^x}{x^2}) \\ &= 2 + \frac{e^x}{x} - 1 - \frac{e^x}{x^2} = 1 + \frac{e^x}{x} - \frac{e^x}{x^2}\end{aligned}$$

Optional common denominator: $1 + \frac{xe^x - e^x}{x^2} = \frac{x^2 + xe^x - e^x}{x^2}$.

Method 2: Expand First $y = \frac{1}{x}(x^2 + e^x) = \frac{x^2}{x} + \frac{e^x}{x} = x + x^{-1}e^x$. *Note: Expanding doesn't eliminate the product $x^{-1}e^x$. The first term is easy, but the second term still requires the Product Rule (or Quotient Rule).* Let's differentiate $y = x + x^{-1}e^x$:

$$\frac{dy}{dx} = \frac{d}{dx}(x) + \frac{d}{dx}(x^{-1}e^x)$$

The first term is 1. For the second term, use Product Rule with $u = x^{-1}, v = e^x$. $\frac{du}{dx} = -x^{-2}$, $\frac{dv}{dx} = e^x$.

$$\frac{d}{dx}(x^{-1}e^x) = u\frac{dv}{dx} + v\frac{du}{dx} = (x^{-1})(e^x) + (e^x)(-x^{-2}) = \frac{e^x}{x} - \frac{e^x}{x^2}$$

Combine all parts:

$$\frac{dy}{dx} = 1 + \left(\frac{e^x}{x} - \frac{e^x}{x^2}\right) = 1 + \frac{e^x}{x} - \frac{e^x}{x^2}$$

Results match.

7. Find derivative of $y = e^{2x}$.

   Solution:

   Strategy. This function is e^u where $u = 2x$. This is a standard application of the Chain Rule (derivative of outside function evaluated at inside function, times derivative of inside function). The Product Rule doesn't apply here as it's not a product of two distinct functions of x in the base form.

   Method 1: Chain Rule Let $y = e^u$ where $u = 2x$. $\frac{dy}{du} = e^u$. $\frac{du}{dx} = \frac{d}{dx}(2x) = 2$. By Chain Rule, $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = e^u \cdot 2$. Substitute back $u = 2x$:

$$\frac{dy}{dx} = e^{2x} \cdot 2 = 2e^{2x}$$

8. Find derivative of $g(x) = (x^2 - 4x + 3)(x - 2)$.

   Solution:

   Method 1: Product Rule

Strategy. Let $u = x^2 - 4x + 3$ and $v = x - 2$.

$u = x^2 - 4x + 3 \implies \frac{du}{dx} = 2x - 4$. $v = x - 2 \implies \frac{dv}{dx} = 1$. Apply formula: $\frac{dg}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$$\frac{dg}{dx} = (x^2 - 4x + 3)(1) + (x - 2)(2x - 4)$$
$$= x^2 - 4x + 3 + (x(2x) + x(-4) - 2(2x) - 2(-4))$$
$$= x^2 - 4x + 3 + (2x^2 - 4x - 4x + 8)$$
$$= x^2 - 4x + 3 + 2x^2 - 8x + 8$$

Combine like terms:

$$= (x^2 + 2x^2) + (-4x - 8x) + (3 + 8) = 3x^2 - 12x + 11$$

Method 2: Expand First $g(x) = (x^2 - 4x + 3)(x - 2)$ $g(x) = x^2(x) + x^2(-2) - 4x(x) - 4x(-2) + 3(x) + 3(-2)$ $g(x) = x^3 - 2x^2 - 4x^2 + 8x + 3x - 6$ $g(x) = x^3 - 6x^2 + 11x - 6$ Differentiate term-by-term:

$$\frac{dg}{dx} = \frac{d}{dx}(x^3) - \frac{d}{dx}(6x^2) + \frac{d}{dx}(11x) - \frac{d}{dx}(6)$$
$$= 3x^2 - 6(2x) + 11(1) - 0 = 3x^2 - 12x + 11$$

Results match.

9. Find derivative of $g(x) = (x^2 - 2x + 1)(x^3 - 1)$.

Solution: Note: $x^2 - 2x + 1 = (x - 1)^2$ *and* $x^3 - 1 = (x - 1)(x^2 + x + 1)$. *So* $g(x) = (x - 1)^3(x^2 + x + 1)$. *Expanding might be complex.*

Method 1: Product Rule

Strategy. Let $u = x^2 - 2x + 1$ and $v = x^3 - 1$.

$u = x^2 - 2x + 1 \implies \frac{du}{dx} = 2x - 2$. $v = x^3 - 1 \implies \frac{dv}{dx} = 3x^2$. Apply formula: $\frac{dg}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$$\frac{dg}{dx} = (x^2 - 2x + 1)(3x^2) + (x^3 - 1)(2x - 2)$$

Expand and simplify:

$$= (3x^4 - 6x^3 + 3x^2) + (x^3(2x) + x^3(-2) - 1(2x) - 1(-2))$$
$$= 3x^4 - 6x^3 + 3x^2 + (2x^4 - 2x^3 - 2x + 2)$$
$$= 3x^4 - 6x^3 + 3x^2 + 2x^4 - 2x^3 - 2x + 2$$

Combine like terms:

$$= (3x^4 + 2x^4) + (-6x^3 - 2x^3) + 3x^2 - 2x + 2$$

$$= 5x^4 - 8x^3 + 3x^2 - 2x + 2$$

Method 2: Expand First $g(x) = (x^2 - 2x + 1)(x^3 - 1)$ $g(x) = x^2(x^3) + x^2(-1) - 2x(x^3) - 2x(-1) + 1(x^3) + 1(-1)$ $g(x) = x^5 - x^2 - 2x^4 + 2x + x^3 - 1$ Rearrange by power: $g(x) = x^5 - 2x^4 + x^3 - x^2 + 2x - 1$ Differentiate term-by-term:

$$\frac{dg}{dx} = 5x^4 - 2(4x^3) + 3x^2 - 2x + 2(1) - 0$$
$$= 5x^4 - 8x^3 + 3x^2 - 2x + 2$$

Results match.

10. Find derivative of $f(x) = (x^3 - 3x)(2x^2 + 3x + 5)$.

Solution:

Method 1: Product Rule

Strategy. Let $u = x^3 - 3x$ and $v = 2x^2 + 3x + 5$.

$u = x^3 - 3x \implies \frac{du}{dx} = 3x^2 - 3$. $v = 2x^2 + 3x + 5 \implies \frac{dv}{dx} = 4x + 3$. Apply formula: $\frac{df}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$$\frac{df}{dx} = (x^3 - 3x)(4x + 3) + (2x^2 + 3x + 5)(3x^2 - 3)$$

Expand carefully:

$$= [x^3(4x) + x^3(3) - 3x(4x) - 3x(3)] + [2x^2(3x^2) + 2x^2(-3) + 3x(3x^2) + 3x(-3) + 5(3x^2) + 5(-3)]$$
$$= [4x^4 + 3x^3 - 12x^2 - 9x] + [6x^4 - 6x^2 + 9x^3 - 9x + 15x^2 - 15]$$

Combine like terms:

$$= (4x^4 + 6x^4) + (3x^3 + 9x^3) + (-12x^2 - 6x^2 + 15x^2) + (-9x - 9x) - 15$$
$$= 10x^4 + 12x^3 + (-3x^2) - 18x - 15$$
$$= 10x^4 + 12x^3 - 3x^2 - 18x - 15$$

Method 2: Expand First $f(x) = (x^3 - 3x)(2x^2 + 3x + 5)$ $f(x) = x^3(2x^2 + 3x + 5) - 3x(2x^2 + 3x + 5)$ $f(x) = (2x^5 + 3x^4 + 5x^3) - (6x^3 + 9x^2 + 15x)$ $f(x) = 2x^5 + 3x^4 + 5x^3 - 6x^3 - 9x^2 - 15x$ $f(x) = 2x^5 + 3x^4 - x^3 - 9x^2 - 15x$ Differentiate term-by-term:

$$\frac{df}{dx} = 2(5x^4) + 3(4x^3) - 3x^2 - 9(2x) - 15(1)$$
$$= 10x^4 + 12x^3 - 3x^2 - 18x - 15$$

Results match.

11. Find derivative of $h(t) = (t^5 - 1)(4t^2 - 7t - 3)$.

Method 1: Product Rule $u = t^5 - 1 \implies du/dt = 5t^4$. $v = 4t^2 - 7t - 3 \implies dv/dt = 8t - 7$.

$$h'(t) = (t^5 - 1)(8t - 7) + (4t^2 - 7t - 3)(5t^4)$$

$$= (8t^6 - 7t^5 - 8t + 7) + (20t^6 - 35t^5 - 15t^4)$$
$$= 28t^6 - 42t^5 - 15t^4 - 8t + 7$$

Method 2: Expand First $h(t) = t^5(4t^2 - 7t - 3) - 1(4t^2 - 7t - 3)$ $h(t) = 4t^7 - 7t^6 - 3t^5 - 4t^2 + 7t + 3$.

$$h'(t) = 4(7t^6) - 7(6t^5) - 3(5t^4) - 4(2t) + 7(1) + 0$$
$$= 28t^6 - 42t^5 - 15t^4 - 8t + 7$$

Results match.

12. Find derivative of $g(t) = (2t^3 - 1)^2$. **Method 1: Product Rule** Rewrite $g(t) = (2t^3 - 1)(2t^3 - 1)$. Let $u = 2t^3 - 1, v = 2t^3 - 1$. $du/dt = 6t^2$, $dv/dt = 6t^2$.

$$g'(t) = (2t^3 - 1)(6t^2) + (2t^3 - 1)(6t^2)$$
$$= 2(2t^3 - 1)(6t^2) = 12t^2(2t^3 - 1) = 24t^5 - 12t^2$$

Method 2: Expand First $g(t) = (2t^3 - 1)^2 = (2t^3)^2 - 2(2t^3)(1) + (1)^2 = 4t^6 - 4t^3 + 1$.

$$g'(t) = 4(6t^5) - 4(3t^2) + 0 = 24t^5 - 12t^2$$

Method 3: Chain Rule (Preview) Let outer function be $f(u) = u^2$, inner function be $u(t) = 2t^3 - 1$. $f'(u) = 2u$, $u'(t) = 6t^2$. $g'(t) = f'(u(t)) \cdot u'(t) = (2u) \cdot (6t^2) = 2(2t^3 - 1)(6t^2) = 12t^2(2t^3 - 1) = 24t^5 - 12t^2$. *Results match.*

13. Find derivative of $f(x) = \sqrt[3]{x}(\sqrt{x} + 3)$.

Method 1: Product Rule Rewrite $f(x) = x^{1/3}(x^{1/2} + 3)$. Let $u = x^{1/3}, v = x^{1/2} + 3$. $du/dx = \frac{1}{3}x^{-2/3}$. $dv/dx = \frac{1}{2}x^{-1/2}$.

$$f'(x) = (x^{1/3})(\frac{1}{2}x^{-1/2}) + (x^{1/2} + 3)(\frac{1}{3}x^{-2/3})$$
$$= \frac{1}{2}x^{(1/3-1/2)} + \frac{1}{3}x^{-2/3}(x^{1/2}) + \frac{1}{3}x^{-2/3}(3)$$
$$= \frac{1}{2}x^{-1/6} + \frac{1}{3}x^{(-2/3+1/2)} + x^{-2/3}$$
$$= \frac{1}{2}x^{-1/6} + \frac{1}{3}x^{-1/6} + x^{-2/3} \quad \text{(Exponents: } 1/3 - 1/2 = -1/6; -2/3 + 1/2 = -1/6)$$
$$= (\frac{1}{2} + \frac{1}{3})x^{-1/6} + x^{-2/3} = \frac{5}{6}x^{-1/6} + x^{-2/3}$$

Optional rewrite: $\frac{5}{6x^{1/6}} + \frac{1}{x^{2/3}}$.

Method 2: Expand First $f(x) = x^{1/3}(x^{1/2}+3) = x^{1/3}x^{1/2}+3x^{1/3} = x^{(1/3+1/2)}+3x^{1/3} = x^{5/6} + 3x^{1/3}$.

$$f'(x) = \frac{5}{6}x^{(5/6-1)} + 3(\frac{1}{3}x^{(1/3-1)}) = \frac{5}{6}x^{-1/6} + x^{-2/3}$$

Results match.

14. Find derivative of $f(x) = \sqrt[3]{x}(x+1)$.

    Method 1: Product Rule Rewrite $f(x) = x^{1/3}(x+1)$. Let $u = x^{1/3}, v = x+1$. $du/dx = \frac{1}{3}x^{-2/3}$. $dv/dx = 1$.

$$f'(x) = (x^{1/3})(1) + (x+1)(\frac{1}{3}x^{-2/3})$$
$$= x^{1/3} + \frac{1}{3}x \cdot x^{-2/3} + \frac{1}{3}(1)x^{-2/3}$$
$$= x^{1/3} + \frac{1}{3}x^{(1-2/3)} + \frac{1}{3}x^{-2/3}$$
$$= x^{1/3} + \frac{1}{3}x^{1/3} + \frac{1}{3}x^{-2/3} = \frac{4}{3}x^{1/3} + \frac{1}{3}x^{-2/3}$$

    Optional rewrite: $\frac{4\sqrt[3]{x}}{3} + \frac{1}{3\sqrt[3]{x^2}}$.

    Method 2: Expand First $f(x) = x^{1/3}(x+1) = x^{1/3}x^1 + x^{1/3} = x^{4/3} + x^{1/3}$.

$$f'(x) = \frac{4}{3}x^{(4/3-1)} + \frac{1}{3}x^{(1/3-1)} = \frac{4}{3}x^{1/3} + \frac{1}{3}x^{-2/3}$$

    Results match.

15. Find derivative of $f(x) = (x^5 - 3x)\left(\frac{1}{x^2}\right)$.

    Method 1: Product Rule Rewrite $f(x) = (x^5 - 3x)(x^{-2})$. Let $u = x^5 - 3x, v = x^{-2}$. $du/dx = 5x^4 - 3$. $dv/dx = -2x^{-3}$.

$$f'(x) = (x^5 - 3x)(-2x^{-3}) + (x^{-2})(5x^4 - 3)$$
$$= -2x^5x^{-3} + 6xx^{-3} + 5x^4x^{-2} - 3x^{-2}$$
$$= -2x^2 + 6x^{-2} + 5x^2 - 3x^{-2}$$
$$= (-2x^2 + 5x^2) + (6x^{-2} - 3x^{-2}) = 3x^2 + 3x^{-2}$$

    Optional rewrite: $3x^2 + \frac{3}{x^2}$.

    Method 2: Expand First $f(x) = (x^5 - 3x)(\frac{1}{x^2}) = \frac{x^5}{x^2} - \frac{3x}{x^2} = x^3 - 3x^{-1}$.

$$f'(x) = 3x^2 - 3(-1x^{-2}) = 3x^2 + 3x^{-2}$$

    Results match.

16. Find derivative of $f(x) = (3x^3 + 4x)(x-5)(x+1)$.

    Method 1: Extended Product Rule (3 factors) Let $F = 3x^3 + 4x$, $G = x - 5$, $H = x + 1$. $F' = 9x^2 + 4$. $G' = 1$. $H' = 1$. Rule: $(FGH)' = F'GH + FG'H + FGH'$.

$$f'(x) = (9x^2 + 4)(x-5)(x+1) + (3x^3 + 4x)(1)(x+1) + (3x^3 + 4x)(x-5)(1)$$

Expand (carefully): $(9x^2+4)(x^2-4x-5) = 9x^4-36x^3-45x^2+4x^2-16x-20 = 9x^4-36x^3-41x^2-16x-20$. $(3x^3+4x)(x+1) = 3x^4+3x^3+4x^2+4x$. $(3x^3+4x)(x-5) = 3x^4-15x^3+4x^2-20x$. Summing these three expansions:

$$f'(x) = (9x^4-36x^3-41x^2-16x-20)+(3x^4+3x^3+4x^2+4x)+(3x^4-15x^3+4x^2-20x)$$

Combine like terms: x^4: $9+3+3=15$. x^3: $-36+3-15=-48$. x^2: $-41+4+4=-33$. x: $-16+4-20=-32$. Constant: -20.

$$f'(x) = 15x^4-48x^3-33x^2-32x-20$$

Method 2: Expand First First multiply two factors: $(x-5)(x+1) = x^2-4x-5$. Now multiply by the first factor: $f(x) = (3x^3+4x)(x^2-4x-5)$ $f(x) = 3x^3(x^2-4x-5)+4x(x^2-4x-5)$ $f(x) = (3x^5-12x^4-15x^3)+(4x^3-16x^2-20x)$ $f(x) = 3x^5-12x^4-11x^3-16x^2-20x$. Differentiate term-by-term:

$$\begin{aligned} f'(x) &= 3(5x^4)-12(4x^3)-11(3x^2)-16(2x)-20(1) \\ &= 15x^4-48x^3-33x^2-32x-20 \end{aligned}$$

Results match.

17. Find derivative of $\dfrac{dR}{dM}$ where $R = M^2\left(\frac{C}{2}-\frac{M}{3}\right)$.

Method 1: Product Rule Let $u = M^2, v = \frac{C}{2}-\frac{M}{3}$. C is constant. $du/dM = 2M$. $dv/dM = 0-\frac{1}{3}(1) = -\frac{1}{3}$.

$$\begin{aligned} \frac{dR}{dM} &= u\frac{dv}{dM}+v\frac{du}{dM} \\ &= (M^2)(-\frac{1}{3})+(\frac{C}{2}-\frac{M}{3})(2M) \\ &= -\frac{1}{3}M^2+(\frac{C}{2}(2M)-\frac{M}{3}(2M)) \\ &= -\frac{1}{3}M^2+CM-\frac{2}{3}M^2 \\ &= CM+(-\frac{1}{3}-\frac{2}{3})M^2 = CM-\frac{3}{3}M^2 = CM-M^2 \end{aligned}$$

Factored: $M(C-M)$.

Method 2: Expand First $R = M^2(\frac{C}{2})-M^2(\frac{M}{3}) = \frac{C}{2}M^2-\frac{1}{3}M^3$. Differentiate term-by-term w.r.t M:

$$\frac{dR}{dM} = \frac{C}{2}(2M)-\frac{1}{3}(3M^2) = CM-M^2$$

Factored: $M(C-M)$. Results match.