PARTIAL FRACTIONS AND ITS INTEGRATION

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1 Introduction: Partial Fractions

Explanation. Partial Fraction Decomposition is an algebraic technique used to break down a complex rational function (a ratio of polynomials) into a sum of simpler rational functions. This is extremely useful because the simpler fractions are often much easier to integrate. **Prerequisite:** The degree of the numerator polynomial must be strictly less than the degree of the denominator polynomial. If not, perform polynomial long division first to get a polynomial plus a proper rational function (where the remainder term satisfies the degree condition).

The Process:

1. **Factor the Denominator:** Completely factor the denominator into linear factors (like $ax + b$) and irreducible quadratic factors (like $ax^2 + bx + c$ where $b^2 - 4ac < 0$, meaning it cannot be factored further using real numbers).

2. **Set up the Decomposition Form:** Based on the factors in the denominator, write the rational function as a sum of simpler fractions with unknown constants (A, B, C, etc.) in the numerators. The rules for the form depend on the type and repetition of the factors:

   - **Distinct Linear Factor:** For each unique factor $(ax+b)$ in the denominator, include a term $\frac{A}{ax+b}$ in the decomposition, where A is an unknown constant.
   - **Repeated Linear Factor:** If a linear factor $(ax+b)$ appears k times, i.e., $(ax+b)^k$, you must include k terms in the decomposition, one for each power from 1 to k:

   $$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_k}{(ax+b)^k}$$

   .

   - **Distinct Irreducible Quadratic Factor:** For each unique factor $(ax^2 + bx + c)$ that cannot be factored further, include a term $\dfrac{Ax + B}{ax^2 + bx + c}$ in the decomposition (note the linear numerator).
   - **Repeated Irreducible Quadratic Factor:** If an irreducible quadratic factor $(ax^2 + bx + c)$ appears k times, i.e., $(ax^2 + bx + c)^k$, include k terms:

   $$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_kx + B_k}{(ax^2 + bx + c)^k}$$

   .

3. **Solve for the Unknown Constants (A, B, C...):**

   - **Clear Denominators:** Multiply both sides of the equation (original fraction = sum of partial fractions) by the fully factored original denominator. This results in an equation involving only polynomials.

- **Find Constants:** There are two main methods, often used in combination:
  - *Method 1: Substituting Convenient Values (Heaviside Method):* Substitute the roots of the linear factors (the values of x that make those factors zero) into the equation after clearing denominators. This often allows you to solve for the constants associated with those linear factors directly.
  - *Method 2: Equating Coefficients:* Expand the entire right side of the equation (after clearing denominators) and collect terms by powers of x (e.g., all x^2 terms together, all x terms together, all constant terms together). The coefficients of each power of x on the right side must equal the coefficients of the corresponding power of x in the original numerator. This creates a system of linear equations which you can solve for the unknown constants A, B, C, etc.

  For repeated factors or irreducible quadratic factors, you often need to use a combination of substituting convenient values and equating coefficients.

4. **Write the Final Decomposition:** Substitute the numerical values you found for A, B, C... back into the decomposition form you set up in Step 2.

Integration: After finding the partial fraction decomposition, the original integral becomes the integral of a sum of simpler terms. Integrate each term separately. Remember the common integrals:

- $\int \frac{A}{ax+b} \mathrm{d}x = \frac{A}{a} \ln|ax+b| + C$ (using a simple u-substitution $u = ax+b$)
- $\int \frac{A}{(ax+b)^k} \mathrm{d}x = \int A(ax+b)^{-k} \mathrm{d}x = \frac{A}{a} \frac{(ax+b)^{-k+1}}{-k+1} + C$ (for $k \neq 1$, using power rule with u-sub $u = ax+b$)
- Integrals with irreducible quadratics $\int \frac{Ax+B}{ax^2+bx+c} \mathrm{d}x$ often require splitting the numerator, completing the square in the denominator, and using substitutions leading to ln and arctan forms.

2 Solutions: Examples - Partial Fraction Decomposition

1. Write the partial fraction decomposition for $\frac{x+7}{x^2-x-6}$.

   Solution:

   Strategy. Check degrees $(1 < 2)$. Factor denominator. Use distinct linear factor form. Solve for constants.

   Step 1: Factor Denominator $x^2 - x - 6 = (x-3)(x+2)$.

   Step 2: Set up Form. $\frac{x+7}{(x-3)(x+2)} = \frac{A}{x-3} + \frac{B}{x+2}$.

   Step 3: Solve for Constants. Multiply by $(x-3)(x+2)$: $x + 7 = A(x+2) + B(x-3)$.

   Let $x = 3$: $10 = A(5) \implies A = 2$.

   Let $x = -2$: $5 = B(-5) \implies B = -1$.

   Step 4: Write Decomposition.

   $$\frac{x+7}{x^2-x-6} = \frac{2}{x-3} - \frac{1}{x+2}$$

   .

2. Write the partial fraction decomposition for $\dfrac{x+8}{x^2+7x+12}$.

   Solution:

   Strategy. Check degrees $(1 < 2)$. Factor denominator. Use distinct linear factor form. Solve for constants.

   Step 1: Factor Denominator $x^2+7x+12 = (x+3)(x+4)$.

   Step 2: Set up Form. $\frac{x+8}{(x+3)(x+4)} = \frac{A}{x+3} + \frac{B}{x+4}$.

   Step 3: Solve. Multiply by $(x+3)(x+4)$: $x+8 = A(x+4) + B(x+3)$.

   Let $x = -3$: $5 = A(1) \implies A = 5$.

   Let $x = -4$: $4 = B(-1) \implies B = -4$.

   Step 4: Write Decomposition.

   $$\frac{x+8}{x^2+7x+12} = \frac{5}{x+3} - \frac{4}{x+4}$$

   .

3. Write the form of the partial fraction decomposition for $\dfrac{5x^2+20x+6}{x(x+1)^2}$.

   Solution:

   Strategy. Denominator has distinct linear x and repeated linear $(x+1)^2$. Write form.

   Form: $\dfrac{A}{x} + \dfrac{B}{x+1} + \dfrac{C}{(x+1)^2}$.

4. Write the form of the partial fraction decomposition for $\dfrac{3x^2+7x+4}{x^3+4x^2+4x}$.

   Solution:

   Strategy. Factor denominator. Identify factors. Write form.

   Step 1: Factor Denominator $x(x^2+4x+4) = x(x+2)^2$.

   Factors: Distinct linear x, Repeated linear $(x+2)^2$.

   Step 2: Set up Form. $\dfrac{A}{x} + \dfrac{B}{x+2} + \dfrac{C}{(x+2)^2}$.

3 Solutions: ClassWork - Partial Fraction Decomposition

5. Decompose $\dfrac{2(x+20)}{x^2-25}$.

   Solution:

   Step 1: Expand/Factor. Numerator $= 2x+40$. Denominator $= (x-5)(x+5)$. Degree $1 < 2$.

   Step 2: Set up. $\frac{2x+40}{(x-5)(x+5)} = \frac{A}{x-5} + \frac{B}{x+5}$.

   Step 3: Clear Denominator $2x+40 = A(x+5) + B(x-5)$.

   Step 4: Solve. Let $x = 5 : 10 + 40 = A(10) \implies 50 = 10A \implies A = 5$.

   Let $x = -5 : -10 + 40 = B(-10) \implies 30 = -10B \implies B = -3$.

   Step 5: Decompose. $\frac{5}{x-5} - \frac{3}{x+5}$.

6. Decompose $\dfrac{3x+11}{x^2-2x-3}$.

   Solution:

   Step 1: Factor. Denominator $= (x-3)(x+1)$. Degree $1 < 2$.

   Step 2: Set up. $\frac{3x+11}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1}$.

   Step 3: Clear Denominator $3x + 11 = A(x+1) + B(x-3)$.

   Step 4: Solve. Let $x = 3 : 9 + 11 = A(4) \implies 20 = 4A \implies A = 5$.

   Let $x = -1 : -3 + 11 = B(-4) \implies 8 = -4B \implies B = -2$.

   Step 5: Decompose. $\frac{5}{x-3} - \frac{2}{x+1}$.

7. Decompose $\dfrac{8x+3}{x^2-3x}$.

   Solution:

   Step 1: Factor. Denominator $= x(x-3)$. Degree $1 < 2$.

   Step 2: Set up. $\frac{8x+3}{x(x-3)} = \frac{A}{x} + \frac{B}{x-3}$.

   Step 3: Clear Denominator $8x + 3 = A(x-3) + Bx$.

   Step 4: Solve. Let $x = 0 : 3 = A(-3) \implies A = -1$.

   Let $x = 3 : 27 = B(3) \implies B = 9$.

   Step 5: Decompose. $-\frac{1}{x} + \frac{9}{x-3}$.

8. Decompose $\dfrac{10x+3}{x^2+x}$.

   Solution:

   Step 1: Factor. Denominator $= x(x+1)$. Degree $1 < 2$.

   Step 2: Set up. $\frac{10x+3}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$.

   Step 3: Clear Denominator $10x + 3 = A(x+1) + Bx$.

   Step 4: Solve. Let $x = 0 : 3 = A(1) \implies A = 3$. Let $x = -1 : -7 = B(-1) \implies B = 7$.

   Step 5: Decompose. $\frac{3}{x} + \frac{7}{x+1}$.

9. Decompose $\dfrac{4x-13}{x^2-3x-10}$.

   Solution:

   Step 1: Factor. Denominator $= (x-5)(x+2)$. Degree $1 < 2$.

   Step 2: Set up. $\frac{4x-13}{(x-5)(x+2)} = \frac{A}{x-5} + \frac{B}{x+2}$.

   Step 3: Clear Denominator $4x - 13 = A(x+2) + B(x-5)$.

   Step 4: Solve. Let $x = 5 : 7 = A(7) \implies A = 1$.

   Let $x = -2 : -21 = B(-7) \implies B = 3$.

   Step 5: Decompose. $\frac{1}{x-5} + \frac{3}{x+2}$.

10. Decompose $\dfrac{7x+5}{6(2x^2+3x+1)}$.

    Solution:

    Step 1: Factor. Denominator $= 6(2x+1)(x+1)$. Degree $1 < 2$.

    Step 2: Set up. $\frac{7x+5}{6(2x+1)(x+1)} = \frac{1}{6}\left[\frac{A}{2x+1} + \frac{B}{x+1}\right]$. Solve for $\frac{7x+5}{(2x+1)(x+1)}$.

    Step 3: Clear Denominator (inner). $7x+5 = A(x+1) + B(2x+1)$.

    Step 4: Solve. Let $x = -1 : -2 = B(-1) \implies B = 2$.

    Let $x = -1/2 : 1.5 = A(0.5) \implies A = 3$.

    Step 5: Decompose (Full). $\dfrac{1}{6}\left[\dfrac{3}{2x+1} + \dfrac{2}{x+1}\right] = \dfrac{1}{2(2x+1)} + \dfrac{1}{3(x+1)}$.

11. Decompose $\dfrac{3x^2-2x-5}{x^3+x^2}$.

    Solution:

    Step 1: Factor. Denominator $= x^2(x+1)$. Repeated linear x^2, distinct linear $x+1$. Degree $2 < 3$.

    Step 2: Set up. $\frac{3x^2-2x-5}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$.

    Step 3: Clear Denominator $3x^2 - 2x - 5 = Ax(x+1) + B(x+1) + Cx^2$.

    Step 4: Solve. Let $x = 0 : -5 = B(1) \implies B = -5$.

    Let $x = -1 : 3 + 2 - 5 = C(1) \implies C = 0$.

    Expand and equate x^2 coeffs: $3x^2 ... = Ax^2 ... + Cx^2 \implies 3 = A + C \implies 3 = A + 0 \implies A = 3$.

    Step 5: Decompose. $\dfrac{3}{x} - \dfrac{5}{x^2}$.

12. Decompose $\dfrac{3x^2-x+1}{x(x+1)^2}$.

    Solution:

    Step 1: Factors. Distinct linear x, repeated linear $(x+1)^2$. Degree $2 < 3$.

    Step 2: Set up. $\frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$.

    Step 3: Clear Denominator $3x^2 - x + 1 = A(x+1)^2 + Bx(x+1) + Cx$.

    Step 4: Solve. Let $x = 0 : 1 = A(1) \implies A = 1$.

    Let $x = -1 : 3 + 1 + 1 = C(-1) \implies C = -5$.

    Expand: $3x^2 - x + 1 = A(x^2 + 2x + 1) + B(x^2 + x) + Cx$.

    Equate x^2 coeffs: $3 = A + B \implies 3 = 1 + B \implies B = 2$.

    Step 5: Decompose. $\dfrac{1}{x} + \dfrac{2}{x+1} - \dfrac{5}{(x+1)^2}$.

13. Decompose $\dfrac{x+1}{3(x-2)^2}$.

    Solution:

    Step 1: Factors. Constant $1/3$, repeated linear $(x-2)^2$. Degree $1 < 2$.

Step 2: Set up. $\frac{1}{3}\left[\frac{A}{x-2}+\frac{B}{(x-2)^2}\right]$. Solve for $\frac{x+1}{(x-2)^2}$. *Step 3: Clear Denominator (inner).* $x+1=A(x-2)+B$. *Step 4: Solve.* Let $x=2:3=B$. Equate x coeffs: $1=A$. *Step 5: Decompose (Full).* $\frac{1}{3}\left[\frac{1}{x-2}+\frac{3}{(x-2)^2}\right]=\frac{1}{3(x-2)}+\frac{1}{(x-2)^2}$.

14. Decompose $\dfrac{3x-4}{(x-5)^2}$.

    Solution:

    Step 1: Factors. Repeated linear $(x-5)^2$. Degree $1<2$.

    Step 2: Set up. $\frac{A}{x-5}+\frac{B}{(x-5)^2}$. *Step 3: Clear Denominator* $3x-4=A(x-5)+B$. *Step 4: Solve.* Let $x=5:11=B$. Equate x coeffs: $3=A$. *Step 5: Decompose.* $\frac{3}{x-5}+\frac{11}{(x-5)^2}$.

15. Decompose $\dfrac{8x^2+15x+9}{(x+1)^3}$.

    Solution:

    Step 1: Factors. Repeated linear $(x+1)^3$. Degree $2<3$.

    Step 2: Set up. $\frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{(x+1)^3}$. *Step 3: Clear Denominator* $8x^2+15x+9=A(x+1)^2+B(x+1)+C$. *Step 4: Solve.* Let $x=-1:8-15+9=C \implies C=2$. Expand: $8x^2+15x+9=A(x^2+2x+1)+B(x+1)+C=Ax^2+(2A+B)x+(A+B+C)$. Equate coeffs: $x^2:A=8$. $x:15=2A+B=16+B \implies B=-1$. (Check const: $A+B+C=8-1+2=9$. Correct). *Step 5: Decompose.* $\frac{8}{x+1}-\frac{1}{(x+1)^2}+\frac{2}{(x+1)^3}$.

16. Decompose $\dfrac{6x^2-5x}{(x+2)^3}$.

    Solution:

    Step 1: Factors. Repeated linear $(x+2)^3$. Degree $2<3$.

    Step 2: Set up. $\frac{A}{x+2}+\frac{B}{(x+2)^2}+\frac{C}{(x+2)^3}$. *Step 3: Clear Denominator* $6x^2-5x=A(x+2)^2+B(x+2)+C$. *Step 4: Solve.* Let $x=-2:24+10=C \implies C=34$. Expand: $6x^2-5x=A(x^2+4x+4)+B(x+2)+C=Ax^2+(4A+B)x+(4A+2B+C)$. Equate coeffs: $x^2:A=6$. $x:-5=4A+B=24+B \implies B=-29$. (Check const: $4A+2B+C=24-58+34=0$. Correct). *Step 5: Decompose.* $\frac{6}{x+2}-\frac{29}{(x+2)^2}+\frac{34}{(x+2)^3}$.

4 Solutions: Integration by Partial Fractions

Explanation. Now we combine the algebraic decomposition with integration. 1. Decompose the rational function integrand. 2. Integrate the sum of the simpler terms, typically using $\int \frac{1}{u}du=\ln|u|$ or $\int u^n du=\frac{u^{n+1}}{n+1}$. Remember: $\int \frac{A}{ax+b}\mathrm{d}x=\frac{A}{a}\ln|ax+b|+C$ and $\int A(ax+b)^n\mathrm{d}x=\frac{A}{a}\frac{(ax+b)^{n+1}}{n+1}+C$ for $n\neq -1$.

17. **Problem (17):** Evaluate $\displaystyle\int \frac{1}{x^2-1}\,\mathrm{d}x$.

    Solution:

    Step 1: Decompose. $\frac{1}{x^2-1} = \frac{1/2}{x-1} - \frac{1/2}{x+1}$. *Step 2: Integrate.*

$$\begin{aligned}\int \left(\frac{1/2}{x-1} - \frac{1/2}{x+1}\right) \mathrm{d}x &= \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C \\ &= \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C\end{aligned}$$

18. **Problem (18):** Evaluate $\displaystyle\int \frac{4}{x^2-4}\,\mathrm{d}x$.

    Solution:

    Step 1: Decompose. $\frac{4}{(x-2)(x+2)} = \frac{1}{x-2} - \frac{1}{x+2}$. *Step 2: Integrate.*

$$\begin{aligned}\int \left(\frac{1}{x-2} - \frac{1}{x+2}\right) \mathrm{d}x &= \ln|x-2| - \ln|x+2| + C \\ &= \ln\left|\frac{x-2}{x+2}\right| + C\end{aligned}$$

19. **Problem (19):** Evaluate $\displaystyle\int \frac{-2}{x^2-16}\,\mathrm{d}x$.

    Solution:

    Step 1: Decompose. $\frac{-2}{(x-4)(x+4)} = \frac{-1/4}{x-4} + \frac{1/4}{x+4}$. *Step 2: Integrate.*

$$\begin{aligned}\int \left(-\frac{1/4}{x-4} + \frac{1/4}{x+4}\right) \mathrm{d}x &= -\frac{1}{4}\ln|x-4| + \frac{1}{4}\ln|x+4| + C \\ &= \frac{1}{4}\ln\left|\frac{x+4}{x-4}\right| + C\end{aligned}$$

20. **Problem (20):** Evaluate $\displaystyle\int \frac{-4}{x^2-4}\,\mathrm{d}x$.

    Solution:

    Step 1: Decompose. $\frac{-4}{(x-2)(x+2)} = \frac{-1}{x-2} + \frac{1}{x+2}$. *Step 2: Integrate.*

$$\begin{aligned}\int \left(-\frac{1}{x-2} + \frac{1}{x+2}\right) \mathrm{d}x &= -\ln|x-2| + \ln|x+2| + C \\ &= \ln\left|\frac{x+2}{x-2}\right| + C\end{aligned}$$

21. **Problem (21):** Evaluate $\displaystyle\int \frac{1}{2x^2-x}\,\mathrm{d}x$.

    Solution:

Step 1: Decompose. $\frac{1}{x(2x-1)} = -\frac{1}{x} + \frac{2}{2x-1}$. *Step 2: Integrate.*

$$\int (-\frac{1}{x} + \frac{2}{2x-1})\mathrm{d}x = -\ln|x| + 2\int \frac{1}{2x-1}\mathrm{d}x$$
$$= -\ln|x| + 2(\frac{1}{2}\ln|2x-1|) + C = -\ln|x| + \ln|2x-1| + C$$
$$= \ln\left|\frac{2x-1}{x}\right| + C$$

22. **Problem (22):** Evaluate $\int \frac{2}{x^2-2x}\,\mathrm{d}x$.

Solution:

Step 1: Decompose. $\frac{2}{x(x-2)} = -\frac{1}{x} + \frac{1}{x-2}$. *Step 2: Integrate.*

$$\int (-\frac{1}{x} + \frac{1}{x-2})\mathrm{d}x = -\ln|x| + \ln|x-2| + C = \ln\left|\frac{x-2}{x}\right| + C$$

23. Evaluate $\int \frac{10}{x^2-10x}\,\mathrm{d}x$.

Solution:

Step 1: Decompose. $\frac{10}{x(x-10)} = -\frac{1}{x} + \frac{1}{x-10}$. *Step 2: Integrate.*

$$\int (-\frac{1}{x} + \frac{1}{x-10})\mathrm{d}x = -\ln|x| + \ln|x-10| + C = \ln\left|\frac{x-10}{x}\right| + C$$

24. Evaluate $\int \frac{5}{x^2+x-6}\,\mathrm{d}x$.

Solution:

Step 1: Decompose. $\frac{5}{(x+3)(x-2)} = -\frac{1}{x+3} + \frac{1}{x-2}$. *Step 2: Integrate.*

$$\int (-\frac{1}{x+3} + \frac{1}{x-2})\mathrm{d}x = -\ln|x+3| + \ln|x-2| + C = \ln\left|\frac{x-2}{x+3}\right| + C$$

25. Evaluate $\int \frac{3}{x^2+x-2}\,\mathrm{d}x$.

Solution:

Step 1: Decompose. $\frac{3}{(x+2)(x-1)} = -\frac{1}{x+2} + \frac{1}{x-1}$. *Step 2: Integrate.*

$$\int (-\frac{1}{x+2} + \frac{1}{x-1})\mathrm{d}x = -\ln|x+2| + \ln|x-1| + C = \ln\left|\frac{x-1}{x+2}\right| + C$$

26. **Problem (26):** Evaluate $\int \frac{1}{4x^2-9}\,\mathrm{d}x$.

Solution:

Step 1: Decompose. $\frac{1}{(2x-3)(2x+3)} = \frac{1/6}{2x-3} - \frac{1/6}{2x+3}$. *Step 2: Integrate.*

$$\int (\frac{1/6}{2x-3} - \frac{1/6}{2x+3})\mathrm{d}x = \frac{1}{6}\int \frac{1}{2x-3}\mathrm{d}x - \frac{1}{6}\int \frac{1}{2x+3}\mathrm{d}x$$

$$= \frac{1}{6}(\frac{1}{2}\ln|2x-3|) - \frac{1}{6}(\frac{1}{2}\ln|2x+3|) + C$$

$$= \frac{1}{12}\ln|2x-3| - \frac{1}{12}\ln|2x+3| + C = \frac{1}{12}\ln\left|\frac{2x-3}{2x+3}\right| + C$$

27. **Problem (27):** Evaluate $\int \frac{5-x}{2x^2+x-1}\,\mathrm{d}x$.

    Solution:

    Step 1: Decompose. $\frac{5-x}{(2x-1)(x+1)} = \frac{3}{2x-1} - \frac{2}{x+1}$. *Step 2: Integrate.*

$$\int (\frac{3}{2x-1} - \frac{2}{x+1})\mathrm{d}x = 3\int \frac{1}{2x-1}\mathrm{d}x - 2\int \frac{1}{x+1}\mathrm{d}x$$

$$= 3(\frac{1}{2}\ln|2x-1|) - 2(\ln|x+1|) + C = \frac{3}{2}\ln|2x-1| - 2\ln|x+1| + C$$

28. **Problem (28):** Evaluate $\int \frac{x+1}{x^2+4x+3}\,\mathrm{d}x$.

    Solution:

    Step 1: Simplify. Factor: $\frac{x+1}{(x+1)(x+3)} = \frac{1}{x+3}$ (for $x \neq -1$). *Step 2: Integrate.*

$$\int \frac{1}{x+3}\mathrm{d}x = \ln|x+3| + C$$

29. **Problem (29):** Evaluate $\int \frac{x^2-4x-4}{x^3-4x}\,\mathrm{d}x$.

    Solution:

    Step 1: Decompose. $\frac{x^2-4x-4}{x(x-2)(x+2)} = \frac{1}{x} - \frac{1}{x-2} + \frac{1}{x+2}$. *Step 2: Integrate.*

$$\int (\frac{1}{x} - \frac{1}{x-2} + \frac{1}{x+2})\mathrm{d}x = \ln|x| - \ln|x-2| + \ln|x+2| + C$$

$$= \ln\left|\frac{x(x+2)}{x-2}\right| + C$$

30. **Problem (30):** Evaluate $\int \frac{x^2+12x+12}{x^3-4x}\,\mathrm{d}x$.

    Solution:

    Step 1: Decompose. Denominator $x(x-2)(x+2)$. $\frac{x^2+12x+12}{x(x-2)(x+2)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+2}$. Clear den.: $x^2+12x+12 = A(x-2)(x+2) + Bx(x+2) + Cx(x-2)$. $x = 0 \implies 12 = A(-4) \implies A = -3$. $x = 2 \implies 4+24+12 = B(2)(4) \implies 40 = 8B \implies B = 5$.

$x = -2 \implies 4 - 24 + 12 = C(-2)(-4) \implies -8 = 8C \implies C = -1$. Decomposition: $-\frac{3}{x} + \frac{5}{x-2} - \frac{1}{x+2}$. *Step 2: Integrate.*

$$\int (-\frac{3}{x} + \frac{5}{x-2} - \frac{1}{x+2})\mathrm{d}x = -3\ln|x| + 5\ln|x-2| - \ln|x+2| + C$$
$$= \ln\left|\frac{(x-2)^5}{x^3(x+2)}\right| + C$$

31. **Problem (31):** Evaluate $\int \frac{x+2}{x^2-4x}\,\mathrm{d}x$.

    Solution:

    Step 1: Decompose. Factor: $\frac{x+2}{x(x-4)} = \frac{A}{x} + \frac{B}{x-4}$. $x + 2 = A(x-4) + Bx$. $x = 0 \implies 2 = A(-4) \implies A = -1/2$. $x = 4 \implies 6 = B(4) \implies B = 3/2$. Decomposition: $-\frac{1/2}{x} + \frac{3/2}{x-4}$. *Step 2: Integrate.*

    $$\int (-\frac{1/2}{x} + \frac{3/2}{x-4})\mathrm{d}x = -\frac{1}{2}\ln|x| + \frac{3}{2}\ln|x-4| + C$$

32. **Problem (32):** Evaluate $\int \frac{4x^2+2x-1}{x^3+x^2}\,\mathrm{d}x$.

    Solution:

    Step 1: Decompose. Factor: $\frac{4x^2+2x-1}{x^2(x+1)} = \frac{3}{x} - \frac{1}{x^2} + \frac{1}{x+1}$. *Step 2: Integrate.*

    $$\int (\frac{3}{x} - x^{-2} + \frac{1}{x+1})\mathrm{d}x = 3\ln|x| - \frac{x^{-1}}{-1} + \ln|x+1| + C$$
    $$= 3\ln|x| + \frac{1}{x} + \ln|x+1| + C$$

33. **Problem (33):** Evaluate $\int \frac{2x-3}{(x-1)^2}\,\mathrm{d}x$.

    Solution:

    Step 1: Decompose. $\frac{2x-3}{(x-1)^2} = \frac{2}{x-1} - \frac{1}{(x-1)^2}$. *Step 2: Integrate.*

    $$\int \left(\frac{2}{x-1} - (x-1)^{-2}\right)\mathrm{d}x = 2\ln|x-1| - \frac{(x-1)^{-1}}{-1} + C$$
    $$= 2\ln|x-1| + \frac{1}{x-1} + C$$

34. **Problem (34):** Evaluate $\int \frac{x^4}{(x-1)^3}\,\mathrm{d}x$.

    Solution:

    Step 1: Long Division + PFD. $\frac{x^4}{(x-1)^3} = (x+3) + \frac{6x^2-8x+3}{(x-1)^3}$. Remainder decomposition: $\frac{6}{x-1} + \frac{4}{(x-1)^2} + \frac{1}{(x-1)^3}$. Full expression: $x + 3 + \frac{6}{x-1} + 4(x-1)^{-2} + (x-1)^{-3}$. *Step 2:*

Integrate.

$$\begin{aligned} &\int \left(x + 3 + \frac{6}{x-1} + 4(x-1)^{-2} + (x-1)^{-3} \right) \mathrm{d}x \\ &= \frac{x^2}{2} + 3x + 6\ln|x-1| + 4\frac{(x-1)^{-1}}{-1} + \frac{(x-1)^{-2}}{-2} + C \\ &= \frac{x^2}{2} + 3x + 6\ln|x-1| - \frac{4}{x-1} - \frac{1}{2(x-1)^2} + C \end{aligned}$$

35. **Problem (35):** Evaluate $\displaystyle\int \frac{3x^2+3x+1}{x(x^2+2x+1)}\,\mathrm{d}x$.

Solution:

Step 1: Simplify Denominator $x(x+1)^2$. *Step 2: Decompose.* $\frac{3x^2+3x+1}{x(x+1)^2} = \frac{1}{x} + \frac{2}{x+1} - \frac{1}{(x+1)^2}$. *Step 3: Integrate.*

$$\begin{aligned} &\int \left(\frac{1}{x} + \frac{2}{x+1} - (x+1)^{-2} \right) \mathrm{d}x \\ &= \ln|x| + 2\ln|x+1| - \frac{(x+1)^{-1}}{-1} + C = \ln|x| + 2\ln|x+1| + \frac{1}{x+1} + C \end{aligned}$$

36. **Problem (36):** Evaluate $\displaystyle\int \frac{3x}{x^2-6x+9}\,\mathrm{d}x$.

Solution:

Step 1: Factor Denominator $(x-3)^2$. *Step 2: Decompose.* $\frac{3x}{(x-3)^2} = \frac{3}{x-3} + \frac{9}{(x-3)^2}$. *Step 3: Integrate.*

$$\int \left(\frac{3}{x-3} + 9(x-3)^{-2} \right) \mathrm{d}x = 3\ln|x-3| + 9\frac{(x-3)^{-1}}{-1} + C$$

$$= 3\ln|x-3| - \frac{9}{x-3} + C$$