MA 101 (Mathematics-I)

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IIT Guwahati

Outline of the Course

Two Topics:

- Single variable calculus
  Will be taught as the first part of the course. Total Number of Lectures= 21 and Tutorials = 6.

  - R. G. Bartle and D. R. Sherbert, Introduction to Real Analysis, Wiley India, 4th Edition, 2014.
  - G. B. Thomas, Jr. and R. L. Finney, Calculus and Analytic Geometry, 6th/ 9th Edition, Narosa/ Pearson Education India, 1996.
  - S. R. Ghorpade and B. V. Limaye, A Course in Calculus and Real Analysis, 5th Indian Reprint, Springer, 2010.
  - W. Rudin, Principles of Mathematical Analysis, 3rd Edition, McGraw Hill Education, 2017.

- Multivariable Calculus
  Will be taught as the second part of the course.

Outline of the Course

Course webpage (Single variable calculus):
https://ayonganguly.github.io/ma101.html

- For Lecture Divisions and Tutorial Groups, Lecture Venues, Tutorial Venues and Class & Exam Time Tables, See Intranet Academic Section Website.
- Tutorial problem sheets will be uploaded in the course webpage. You are expected to try all the problems in the problem sheet before coming to the tutorial class.
  Do not expect the tutor to solve completely all the problems given in the tutorial sheet.

Outline of the Course

Attendance Policy

Attendance in all lecture and tutorial classes is compulsory.

As per Institute guidelines, students who do not meet 75% attendance requirement in the course will NOT be allowed to write the end semester examination and will be awarded F (Fail) grade in the course.

In this course we will strictly follow the Institute guidelines on attendance policy. There will be 42 classes of this course. Therefore, students must attend at least 30 classes.

(Refer: B.Tech. Ordinance Clause 4.1)

Outline of the Course

Marks distribution for Single Variable Calculus part:

Exam	Date	Weightage
Quiz I	August 27, 2023	10%
Quiz II	September 10, 2023	10%
Mid-Sem	September 19, 2023	30%

No make up test for Quizzes and Mid Semester Examination.

Do preserve your (evaluated) answer scripts of Quizzes and Mid Semester Examination of MA101 till the completion of the Course Grading.

Introduction

We denote the set of natural numbers by $\mathbb{N}$, the set of integers by $\mathbb{Z}$, and the set of rational numbers by $\mathbb{Q}$, and we assume familiarity with each of these sets:

- $\mathbb{N} = \{1, 2, 3, \ldots\}$
- $\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$
- $\mathbb{Q} = \{m/n : m \in \mathbb{Z}, n \in \mathbb{N}\}$

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The **Well-Ordering Property of** $\mathbb{N}$ states that every nonempty subset of $\mathbb{N}$ has a least element.

That is, given a nonempty subset S of $\mathbb{N}$, there exists $m \in S$ such that $m \leq k$ for all $k \in S$. The element m is the **least element** of S.

The set of real numbers, denoted by $\mathbb{R}$, is best described geometrically by setting up a one-to-one correspondence with points of a line that stretches infinitely in both directions.
Next, we list three sets of axioms that the set of real numbers follows.

Field Axioms:

1. (Associative laws) $x + (y + z) = (x + y) + z$ and $x(yz) = (xy)z$ for all $x, y, z \in \mathbb{R}$
2. (Commutative laws) $x + y = y + x$ and $xy = yx$ for all $x, y \in \mathbb{R}$
3. (Identities) $x + 0 = x = 0 + x$ and $x \cdot 1 = x = 1 \cdot x$ for all $x \in \mathbb{R}$
4. (Inverses) $x + (-x) = 0 = (-x) + x$ for all $x \in \mathbb{R}$ and $x \cdot \frac{1}{x} = 1 = \frac{1}{x} \cdot x$ for all $x \in \mathbb{R} \setminus \{0\}$
5. (Distributive laws) $x(y + z) = xy + xz$ and $(x + y)z = xz + yz$ for all $x, y, z \in \mathbb{R}$

Order Axioms:

1. For each $x, y \in \mathbb{R}$, exactly one of $x > y,\ x = y,\ x < y$ holds
2. If $x \geq y$, then $x + z \geq y + z$ for all $z \in \mathbb{R}$
3. If $x \geq y$ and $z \geq 0$, then $xz \geq yz$

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From the Order Axioms one can derive the usual inequalities satisfied by the set of real numbers. For example,

Property

1. *If $x \geq y$ and $z \leq 0$, then $xz \leq yz$.*
2. *If $x + \varepsilon \geq y$ holds for all $\varepsilon > 0$, then $x \geq y$ also holds.*

Absolute Value: The absolute value $|a|$ of a real number a is defined as $\max\{a, -a\}$. In particular, $|a| = |-a| \geq 0$ for all $a \in \mathbb{R}$, with equality if and only if $a = 0$. It is easy to see that $-|a| \leq a \leq |a|$ for all real a.

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Two of the most significant properties satisfied by the absolute value function $|\cdot|$ are: (i) $|ab| = |a||b|$ for each pair $a, b \in \mathbb{R}$, and (ii) (Triangle Inequality) $|a + b| \leq |a| + |b|$ for each pair $a, b \in \mathbb{R}$.

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The inequality $|x - a| < \varepsilon$ easily translates to $a - \varepsilon < x < a + \varepsilon$ and the inequality $|x - a| > \varepsilon$ translates to $x > a + \varepsilon$ or $x < a - \varepsilon$.

Bounded sets

Let S be a nonempty subset of $\mathbb{R}$.

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Observation: S is bounded precisely when there is a real number M for which $|a| \leq M$ holds for every $a \in S$.

Least upper bound and greatest lower bound

If S is bounded above, then a number u is said to be a **supremum** (or a **least upper bound**) of S if it satisfies the following conditions:

(a) u is an upper bound of S

(b) if v is any upper bound of S, then $u \leq v$.

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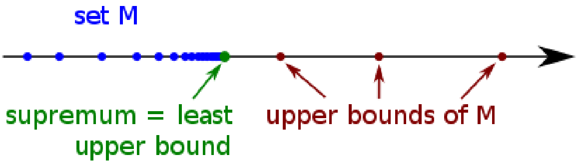
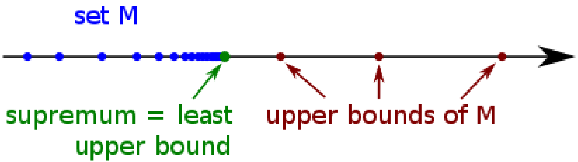
If S is bounded below, then a number w is said to be an **infimum** (or a **greatest lower bound**) of S if it satisfies the following conditions:

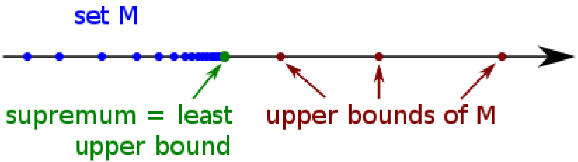
(a) w is a lower bound of S

(b) if t is any lower bound of S, then $t \leq w$.


set M
supremum = least upper bound
upper bounds of M



Facts: A nonempty set S can have atmost one supremum. Similarly, a nonempty set S can have atmost one infimum.



Facts: A nonempty set S can have atmost one supremum. Similarly, a nonempty set S can have atmost one infimum.

If the supremum or the infimum of a set S exists, we will denote them by

$$\sup(S) \quad \text{and} \quad \inf(S).$$

Maximum and minimum

A set S has a maximum when there exists $M \in S$ such that $a \leq M$ for all $a \in S$. Observe that every nonempty set S can have at most one maximum, and that the maximum (if it exists) is also the supremum of S. On the other hand, if the supremum of S exists and $\sup(S) \in S$, then $\sup(S) = \max(S)$.

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Analogous observation to greatest lower bound and to minimum of a set.

Completeness Axiom

It is not possible to prove on the basis of the field and order properties of $\mathbb{R}$ that every nonempty subset of $\mathbb{R}$ that is bounded above has a supremum in $\mathbb{R}$.

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It is a deep and fundamental property of the real number system that this is indeed the case.

The **completeness property of** $\mathbb{R}$ states that every nonempty set of real numbers that has an upper bound also has a supremum in $\mathbb{R}$; and that every nonempty set of real numbers that has a lower bound also has an infimum in $\mathbb{R}$.

Note that instead of $\mathbb{R}$ if we look at $\mathbb{Q}$, then it is an ordered field but not complete.

Proposition

$\sqrt{2}$ *is irrational.*

Example

Let $A = \{p \in \mathbb{Q} : p > 0 \text{ and } p^2 < 2\}$. Then A is bounded above but supremum does not exist in $\mathbb{Q}$.

Example

$\sup\{p \in \mathbb{R} : p > 0 \text{ and } p^2 < 2\} = \sqrt{2}$.

Lemma (Property of supremum)

Let A be a nonempty set of real numbers, and suppose $sup(A)$ exists. Then for every $\varepsilon > 0$, there exists $a \in A$ such that $sup(A) - \varepsilon < a \leq sup(A)$.

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Lemma (Property of infimum)

Let A be a nonempty set of real numbers, and suppose $inf(A)$ exists. Then for every $\varepsilon > 0$, there exists $a \in A$ such that $inf(A) \leq a < inf(A) + \varepsilon$.

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Let $S = \{\frac{1}{n} : n \in \mathbb{N}\}$. *We note that* $0 < \frac{1}{n} \leq 1$ *for all* $n \in \mathbb{N}$. *Thus,* 1 *is an upper bound and since* $1 \in S$ *we have* $sup(S) = max(S) = 1$.

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Clearly, 0 *is a lower bound of* S*. We can use Archimedean property to prove that* 0 *is the greatest lower bound.*

Density of rational and irrational numbers in $\mathbb{R}$

The set of rational numbers is “dense” in $\mathbb{R}$ in the sense that given any two real numbers there is a rational number between them (in fact, there are infinitely many such rational numbers).

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Theorem (The Density Theorem)

If x and y are any real numbers with $x < y$, then there exists a rational number $r \in \mathbb{Q}$ such that $x < r < y$.

Corollary

If x and y are any real numbers with $x < y$, then there exists an irrational number $r \in \mathbb{Q}$ such that $x < r < y$.