MA 101 (Mathematics-I)

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Riemann Integration

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This can be done by first breaking up the interval $[a, b]$ into finitely many subintervals, and then underestimating and overestimating the area over each subinterval by computing rectangular areas. The sum of these two areas over all subintervals then produces lower and upper estimates of the required area, and we hope that as we pass these two sums over the limit as the number of subintervals tends to infinity, we arrive at the area we seek.

The theory of Riemann integration is based on bounded real-valued functions defined on a closed and bounded interval. Thus consider $f : [a, b] \to \mathbb{R}$ such that $m \leq f(x) \leq M$ for some real numbers m, M, and for all $x \in [a, b]$.

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Partition: A partition P of an interval $[a, b]$ is a finite set $\{x_0, x_1, x_2, \ldots, x_n\}$ of points satisfying

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$

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The above partition P breaks up an interval into n subintervals $[x_{i-1}, x_i]$, with $1 \leq i \leq n$. We denote by Δx_i the length of the ith subinterval, and by $\|P\|$ the largest of these subinterval lengths. We call $\|P\|$ the norm of P.

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Thus, $\Delta x_i = x_i - x_{i-1}$ and $\|P\| = \max_i \Delta x_i$.

For a fixed partition P of $[a, b]$, set

$$M_i = \sup_{x \in [x_{i-1}, x_i]} f(x); \qquad m_i = \inf_{x \in [x_{i-1}, x_i]} f(x);$$

$$U(f, P) = \sum_{i=1}^{n} M_i \Delta x_i \quad \text{and} \quad L(f, P) = \sum_{i=1}^{n} m_i \Delta x_i.$$

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Since $m \le m_i \le M_i \le M$ for each i, we have

$$m \sum_{i=1}^{n} \Delta x_i \le \sum_{i=1}^{n} m_i \Delta x_i \le \sum_{i=1}^{n} M_i \Delta x_i \le M \sum_{i=1}^{n} \Delta x_i.$$

Hence, $m(b - a) \le L(f, P) \le U(f, P) \le M(b - a)$

for every partition P of $[a, b]$.

Example: Let $f(x) = x^4 - 4x^3 + 10$ for all $x \in [1, 4]$. Then for the partition $P = \{1, 2, 3, 4\}$ of $[1, 4]$, $U(f, P) = 11$ and $L(f, P) = -40$.

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Solution: Since $f'(x) = 4x^2(x - 3)$ for all $x \in [1, 4]$, we have $f'(x) < 0$ for all $x \in (1, 3)$ and $f'(x) > 0$ for all $x \in (3, 4)$. Hence f is strictly decreasing on $[1, 3]$ and strictly increasing on $[3, 4]$.

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Consequently $\sup\{f(x) : x \in [1,2]\} = f(1) = 7$,
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Therefore $U(f, P) = 7(2 - 1) + (-6)(3 - 2) + 10(4 - 3) = 11$,
$L(f, P) = (-6)(2 - 1) + (-17)(3 - 2) + (-17)(4 - 3) = -40$.

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Remark: If $f \in \mathcal{R}[a, b]$ and $m \leq f(x) \leq M$ for $x \in [a, b]$, then

$$m(b - a) \leq \int_a^b f(x)dx \leq M(b - a).$$

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Solution of (b): Let $P = \{x_0, x_1, ..., x_n\}$ be any partition of $[a, b]$. Then $m_i = 0$ and $M_1 = 1, M_i = 0$ for $i = 2, ..., n$ and so $L(f, P) = 0$ and $U(f, P) = x_1 - x_0 = x_1 - a$.

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Therefore f is Riemann integrable on $[a, b]$ and $\int_a^b f(x)\, dx = 0$.

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Hence $\underline{\int_a^b} f(x)\, dx = 0$ and $\overline{\int_a^b} f(x)\, dx = b - a$.

Since $\overline{\int_a^b} f(x)\, dx \neq \underline{\int_a^b} f(x)\, dx$, f is not Riemann integrable.

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$L(f, P_n) = \frac{1}{n}(0 + \frac{1}{n^2} + \cdots + \frac{(n-1)^2}{n^2}) = (1 - \frac{1}{n})(\frac{1}{3} - \frac{1}{6n}) \to \frac{1}{3}$.

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$$U(f, P_n) = \frac{1}{n}(\frac{1}{n^2} + \cdots + \frac{n^2}{n^2}) = (1 + \frac{1}{n})(\frac{1}{3} + \frac{1}{6n}) \to \frac{1}{3}.$$

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Hence f is Riemann integrable on $[0, 1]$ and $\int_0^1 f(x)\, dx = \frac{1}{3}$.

Riemann's criterion for integrability: A bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$ if and only if for each $\varepsilon > 0$, there exists a partition P_ε of $[a, b]$ such that $U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon$.

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Corollary: A bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$ if and only if there exists a sequence (P_n) of partitions of $[a, b]$ such that $\lim_{n\to\infty} [U(f, P_n) - L(f, P_n)] = 0$, in which case

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} U(f, P_n) = \lim_{n\to\infty} L(f, P_n).$$

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(b) Let $f : [a, b] \to \mathbb{R}$ be bounded. If f is continuous except at finitely many points in $[a, b]$, then f is Riemann integrable.

(c) If $f : [a, b] \to \mathbb{R}$ is monotonic, then f is Riemann integrable.

Riemann sum: Let $f : [a, b] \to \mathbb{R}$ be bounded. Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$, and $c_i \in [x_{i-1}, x_i]$ for $i = 1, 2, \ldots, n$. Then

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Result: A bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$ if and only if $\lim_{\|P\| \to 0} S(f, P)$ exists in $\mathbb{R}$.

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Result: A bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$ if and only if $\lim\limits_{\|P\| \to 0} S(f, P)$ exists in $\mathbb{R}$.

Also, in this case, $\int\limits_a^b f = \lim\limits_{\|P\| \to 0} S(f, P)$.

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Solution: Let $f(x) = \frac{1}{1+x}$ for all $x \in [0,1]$. Considering the partition $P_n = \{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n} = 1\}$ of $[0,1]$ for each $n \in \mathbb{N}$ (and taking $c_i = \frac{i}{n}$ for $i = 1, \ldots, n$), we find that

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Since $f : [0,1] \to \mathbb{R}$ is continuous, f is Riemann integrable on $[0,1]$ and hence $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} \frac{1}{n+i} = \lim\limits_{n\to\infty} S(f, P_n) = \int\limits_0^1 f(x)\,dx = \log(1+x)\big|_{x=0}^{1} = \log 2.$

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Suppose that $f, g \in \mathcal{R}[a, b]$ and $\alpha \in \mathbb{R}$.

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(1) Then $\alpha f, f + g \in \mathcal{R}[a, b]$ and

$$\int_a^b (\alpha f)(x)dx = \alpha \int_a^b f(x)dx;$$

$$\int_a^b (f+g)(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx.$$

(2) If $f(x) \leq g(x)$ on $[a, b]$. Then

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(4) If $f \in \mathcal{R}[a, b]$ then $|f| \in \mathcal{R}[a, b]$ and

$$|\int_a^b f(x)dx| \leq \int_a^b |f|(x)dx.$$

Theorem (Mean value theorem)

If $f : [a, b] \to \mathbb{R}$ *is continuous, then there exists a point* $c \in (a, b)$ *such that*

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Theorem (First fundamental theorem of calculus)

Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable and let $F(x) = \int\limits_a^x f(t)\, dt$ for all $x \in [a, b]$. Then $F : [a, b] \to \mathbb{R}$ is continuous. Also, if f is continuous at $x_0 \in [a, b]$, then F is differentiable at x_0 and $F'(x_0) = f(x_0)$.

Corollary: If $f : [a, b] \to \mathbb{R}$ is continuous and $F(x) = \int\limits_a^x f(t)\, dt$ for all $x \in [a, b]$, then F is differentiable on $[a, b]$ and $F' = f$.

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Theorem (Second fundamental theorem of calculus)

Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable on $[a, b]$. If there exists a differentiable function $F : [a, b] \to \mathbb{R}$ such that $F'(x) = f(x)$ for all $x \in [a, b]$, then $\int\limits_a^b f(x)\,dx = F(b) - F(a)$.

Corollary: If $f : [a, b] \to \mathbb{R}$ is continuous and $F(x) = \int\limits_a^x f(t)\, dt$ for all $x \in [a, b]$, then F is differentiable on $[a, b]$ and $F' = f$.

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Remark: It is not true that derivatives are automatically integrable. For example, let $f : [0, 1] \to \mathbb{R}$ be defined by $f(x) = x^2 \sin \frac{1}{x^2}$ for $x \neq 0$ and $f(0) = 0$. Then f is differentiable on $[0, 1]$. It is easy to see that f' is not bounded and hence it is not Riemann integrable.