1 Generation from Univariate Normal Distribution

Recall that the PDF and CDF of univariate normal distribution with mean $\mu\,(\mu \in \mathbb{R})$ and variance $\sigma^2\,(\sigma > 0)$ are

$$\phi_{\mu,\sigma}(x) = \frac{1}{\sigma}\phi(\frac{x-\mu}{\sigma}) = \frac{1}{\sqrt{2\pi}\sigma}e^{-(x-\mu)^2/2\sigma^2} \quad \text{for} \quad x \in \mathbb{R}$$

and

$$\Phi_{\mu,\sigma}(x) = \Phi\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{x} e^{-(t-\mu)^2/2\sigma^2}dt \quad \text{for} \quad x \in \mathbb{R},$$

respectively, where

$$\phi(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}} \quad \text{for} \quad x \in \mathbb{R}$$

and

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{t^2}{2}}dt \quad \text{for} \quad x \in \mathbb{R}.$$

Note that $\phi(\cdot)$ and $\Phi(\cdot)$ are PDF and CDF of standard normal distribution. The word "standard" indicates that the mean is 0 and the variance is 1. If $Z \sim N(0,1)$, then $\mu + \sigma Z \sim N(\mu, \sigma^2)$. Thus, given a method for generating samples $Z_1, Z_2, \ldots$ from the standard normal distribution, we can generate samples $X_1, X_2, \ldots$ from $N(\mu, \sigma^2)$ by setting $X_i = \mu + \sigma Z_i$. It, therefore, suffices to consider methods for sampling from $N(0,1)$.

We now discuss algorithms for generating univariate normal distribution. We assume the availability of a sequence $U_1, U_2, \ldots$ of independent random variables uniformly distributed on the unit interval $(0,1)$ and consider methods for transforming these uniform random variables to normally distributed random variables.

1.1 Using Acceptance Rejection Method

To obtain an upper bound of $\phi(x)$, we need to find an lower bound of $\frac{x^2}{2}$. Notice that

$$\frac{1}{2}\left(|x|-1\right)^2 = \frac{x^2}{2} - |x| + \frac{1}{2} \geq 0.$$

Thus, $\dfrac{x^2}{2} \geq |x| - \dfrac{1}{2}$ for all $x \in \mathbb{R}$. This shows that, for all $x \in \mathbb{R}$,

$$\phi(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}} \leq \frac{1}{\sqrt{2\pi}}e^{-|x|+\frac{1}{2}} = cg(x),$$

where $c = \sqrt{\frac{2e}{\pi}}$ and $g(x) = \frac{1}{2}e^{-|x|}$ for all $x \in \mathbb{R}$. Therefore, we can use acceptance rejection method if we can generate from g easily. To generate from $g(\cdot)$ we can use the following result. The PDF of $X = ZY$ is $g(\cdot)$ if Y and Z are independent random variables such that $Y \sim Exp(1)$ and $P(Z = 1) = P(Z = -1) = \frac{1}{2}$. To see it, notice that the CDF of X is

$$\begin{aligned}
P\left(X \leq x\right) &= P\left(X \leq x | Z = 1\right) P\left(Z = 1\right) + P\left(X \leq x | Z = -1\right) P\left(Z = -1\right)\\
&= \frac{1}{2} P\left(ZY \leq x | Z = 1\right) + \frac{1}{2} P\left(ZY \leq x | Z = -1\right)\\
&= \frac{1}{2} P\left(Y \leq x\right) + \frac{1}{2} P\left(Y \geq -x\right)\\
&= \begin{cases} \frac{1}{2}e^{x} & \text{if } x < 0\\ 1 - \frac{1}{2}e^{-x} & \text{if } x \geq 0. \end{cases}
\end{aligned}$$

This shows that the PDF of X is same as $g(\cdot)$. This result tells us the method of generation from g. First, generate Y from exponential distribution with mean 1 and then assign a random sign to it. Therefore, we have the Algorithm 1.

Algorithm 1 Generation of $N(0, 1)$ by rejecting from Laplace PDF

1: **repeat**
2: generate U from $U(0, 1)$ ▷ Will be used to generate from $Exp(1)$
3: $X \leftarrow -\ln(U)$ ▷ $X \sim Exp(1)$
4: generate V from $U(0, 1)$ ▷ Will be used to generate random sign
5: **if** $V < \frac{1}{2}$ **then** ▷ Assigning signs with equal probability
6: $X \leftarrow -X$
7: **end if**
8: generate W from $U(0, 1)$ ▷ Will be used to implement acceptance rejection step
9: **until** $We^{\frac{1}{2}-|X|} \leq e^{-\frac{X^2}{2}}$
10: **return** X

Note that the condition at Line 9 of the Algorithm 1 does not depend on the sign of X. Thus, we may assign the sign if the candidate is accepted. Moreover, $We^{\frac{1}{2}-|X|} \leq e^{-\frac{X^2}{2}}$ is equivalent to $2\ln W \geq (|X| - 1)^2$. Thus, Algorithm 1 can be modified to obtain a better algorithm given below.

Algorithm 2 Generation of $N(0, 1)$ by rejecting from Laplace PDF

repeat
generate U from $U(0, 1)$ ▷ Will be used to generate from $Exp(1)$
$X \leftarrow -\ln(U)$ ▷ $X \sim Exp(1)$
generate W from $U(0, 1)$ ▷ Needed to implement acceptance rejection step and assign sign
until $2\ln\left(W\right) \geq \left(|X| - 1\right)^2$
if $W < \frac{1}{2}$ **then**
$X \leftarrow -X$
end if
return X

2 Box-Muller Method

Perhaps the simplest method to implement (though not the fastest or necessarily the most convenient) is the one by Box-Muller. This algorithm generates a sample from a bivariate standard normal distribution, each component of which is thus a univariate standard normal. The algorithm is based on the following result.

Theorem 1. *Let $U_1,\, U_2 \overset{i.i.d.}{\sim} U(0,\, 1)$. Define*

$$Z_1 = \sqrt{-2\ln U_1}\cos\left(2\pi U_2\right) \quad \textit{and} \quad Z_2 = \sqrt{-2\ln U_1}\sin\left(2\pi U_2\right).$$

Then $Z_1,\, Z_2 \overset{i.i.d.}{\sim} N(0,\, 1)$.

Proof. Note that

$$\begin{aligned}
\frac{\partial z_1}{\partial u_1} &= -\frac{\cos\left(2\pi u_2\right)}{u_1\sqrt{-2\ln u_1}},\\
\frac{\partial z_2}{\partial u_1} &= -\frac{\sin\left(2\pi u_2\right)}{u_1\sqrt{-2\ln u_1}},\\
\frac{\partial z_1}{\partial u_2} &= 2\pi\sin\left(2\pi u_2\right)\sqrt{-2\ln u_1},\\
\frac{\partial z_2}{\partial u_2} &= 2\pi\cos\left(2\pi u_2\right)\sqrt{-2\ln u_1}.
\end{aligned}$$

Thus, the Jacobian of the transformation is

$$J = \det\begin{pmatrix} \frac{\partial z_1}{\partial u_1} & \frac{\partial z_2}{\partial u_1} \\ \frac{\partial z_1}{\partial u_2} & \frac{\partial z_2}{\partial u_2} \end{pmatrix} = -\frac{2\pi}{u_1}.$$

Therefore, the absolute value of the Jacobian of inverse transformation is

$$\frac{1}{|J|} = \frac{u_1}{2\pi} = \frac{1}{2\pi}e^{-\frac{1}{2}\left(z_1^2+z_2^2\right)}.$$

and hence the JPDF of $(Z_1,\, Z_2)$ is

$$f_{Z_1,\, Z_2}(z_1,\, z_2) = \frac{1}{2\pi}e^{-\frac{1}{2}\left(z_1^2+z_2^2\right)} = \phi(z_1)\phi(z_2) \quad \text{for} \quad z_1 \in \mathbb{R},\, z_2 \in \mathbb{R}.$$

This proves the result. □

Now, based on the theorem, we have the Algorithm 3 to generate a pair of independent standard normal random numbers from a pair of independent $U(0,\, 1)$ random numbers.

Algorithm 3 Box-Muller Method to generate $N(0,\, 1)$ random numbers

generate U_1 and U_2 from $U(0,\, 1)$
$R \leftarrow \sqrt{-2\ln U_1}$
$\theta \leftarrow 2\pi U_2$
$Z_1 \leftarrow R\cos(\theta)$
$Z_2 \leftarrow R\sin(\theta)$
return $(Z_1,\, Z_2)$.

3 Marsaglia and Bray Method

Marsaglia and Bray developed a modification of the Box-Muller method that reduces computing time by avoiding evaluation of the “cos” and “sin” functions. The Marsaglia and Bray method instead uses acceptance rejection method to sample points uniformly in the unit disc and then transforms these points to normal variables. The algorithm is as follows:

Algorithm 4 Marsaglia and Bray Method to generate $N(0,\,1)$ random numbers

repeat
    generate U_1 and U_2 from $U(0,\,1)$
    $U_1 \leftarrow 2U_1 - 1$ and $U_2 \leftarrow 2U_2 - 1$ $\quad\triangleright\ U_i \sim U(-1,\,1)$
until $U_1^2 + U_2^2 \leq 1$ $\quad\triangleright\ (U_1,\,U_2)$ is uniformly distributed on the disc of radius 1 centered at the origin
$Z_1 \leftarrow U_1 \left[\frac{-2\ln(U_1^2+U_2^2)}{U_1^2+U_2^2}\right]^{\frac{1}{2}}$ and $Z_2 \leftarrow U_2 \left[\frac{-2\ln(U_1^2+U_2^2)}{U_1^2+U_2^2}\right]^{\frac{1}{2}}$ $\quad\triangleright\ Z_i \sim N(0,\,1)$ and they are independent
return $(Z_1,\,Z_2)$.