Indian Institute of Technology Guwahati
Probability Theory (MA590)
Problem Set 05

1. Let X be a random variable with CDF

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{x^2}{2} & \text{if } 0 \le x < 1 \\ \frac{x+2}{6} & \text{if } 1 \le x < 2 \\ 1 & \text{if } x \ge 2. \end{cases}$$

   (a) Using the distribution function, evaluate $P(\{X = 1\})$, $P(\{1 < X < 2\})$, $P(\{1 \le X < 2\})$, $P(\{1 < X \le 2\})$, $P(\{1 \le X \le 2\})$ and $P(\{X \ge 1\})$. (Ans: 0, 1/6, 1/6, 1/2, 1/2, 1/2)

   (b) Is the RV X a DRV?

   (c) Is the RV X a CRV?

2. Let us select five cards at random and without replacement from an ordinary deck of playing cards. Let X be the number of hearts among the five selected cards.

   (a) Find the p.m.f. of X.

   (b) Determine $P(\{X \le 1\})$.

3. Let X be CRV with PDF

$$f_X(x) = \begin{cases} k - |x| & \text{if } |x| < 0.5 \\ 0 & \text{otherwise,} \end{cases}$$

   where $k \in \mathbb{R}$.

   (a) Find the value of constant k. (Ans: 5/4.)

   (b) Using the PDF, evaluate $P(\{X < 0\})$, $P(\{X \le 0\})$, $P(\{0 < X \le \frac{1}{4}\})$, $P(\{0 \le X < \frac{1}{4}\})$, and $P(\{-\frac{1}{8} \le X \le \frac{1}{4}\})$. (Ans: 1/2, 1/2, 9/32, 9/32, 25/32.)

   (c) Find the conditional probabilities $P(\{X > \frac{1}{4}\}|\{|X| > \frac{2}{5}\})$ and $P(\frac{1}{10} < X < 1|\{\frac{1}{10} < X < \frac{1}{5}\})$. (Ans: 1/2, 1.)

   (d) Find the CDF of X.

4. Consider a quiz game where a person is given two questions (*viz.*, Q1 and Q2) and must decide which one to answer first. Suppose that the questions are answered independently. Q1 will be answered correctly with probability 0.8, and then if the person answer it correctly, he/she will receive Rs. 100.00. Q2 will be answered correctly with probability 0.5, and the person will receive as prize Rs. 200.00. If the first question attempted is answered incorrectly, the quiz terminates. If the first question is answered correctly, the person is allowed to attempt other question. Which question should be answered first to maximize the expected value of total prize money received? (Ans: Easier one first)

5. If the weather is good, which happens with probability 0.6, Alice walks the 2 miles to class at a speed of 5 miles per hour, and otherwise rides her motorcycle at a speed of 30 miles per hour. What is the mean of time to get the class? (Ans: 16 minutes.)

6. Consider a target comprising of three concentric circles of radii $1/\sqrt{3}$, 1, and $\sqrt{3}$ feet. Shots within the inner circle earn 4 points, within the next ring 3 points and within the third ring 2 points. Shots outside the target do not earn any point. Let X denote the distance (in feet) of the hit from the center and suppose that X has the PDF

$$f_X(x) = \begin{cases} \frac{2}{\pi(1+x^2)} & \text{if } x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Find the expected score in a single shot. (Ans: $\frac{13}{6}$)

7. Find the expected number of throws of a fair die required to obtain a 6. (Ans: 6.)

8. From a box containing N identical tickets, numbered 1, 2, $\ldots$, N, n ($\leq N$) tickets are drawn with replacement. Let X be the largest number drawn. Find $E(X)$. (Ans: $E(X) = N - \dfrac{1}{N^n} \displaystyle\sum_{i=1}^{N-1} i^n$)

9. Find $E(X)$, $E(X^2)$ and $Var(X)$ for each of the following cases.
   (a) $X \sim Bernoulli(p)$.
   (b) $X \sim Bin(n,\, p)$.
   (c) $X \sim Geo(p)$.
   (d) $X \sim Poi(\lambda)$.
   (e) $X \sim U(a,\, b)$.
   (f) $X \sim Exp(\lambda)$.
   (g) $X \sim N(\mu,\, \sigma^2)$.
   (h) $X \sim Gamma(\alpha,\, \beta)$.
   (i) $X \sim Best(\alpha,\, \beta)$.

10. Let $X \sim \text{Bin}(n,\, p)$, where n is a positive integer and $p \in (0, 1)$. Let $Y_1 = X^2$, and $Y_2 = \sqrt{X}$. Find the PMFs of Y_1 and Y_2.

11. Let X be a random variable with PMF

$$f_X(x) = \begin{cases} \frac{1}{2}\left(\frac{2}{3}\right)^x & \text{if } x = 0,\, 1,\, 2,\, \ldots \\ 0 & \text{otherwise.} \end{cases}$$

Find the CDF of $Y = \frac{X}{X+1}$ and hence determine the PMF of Y.

12. Let the random variable X have the PDF

$$f_X(x) = \begin{cases} \frac{1+x}{2} & \text{if } -1 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Find the distribution functions and hence the PDFs (provided they exist) of $X^+ = \max\{X,\, 0\}$, $X^- = \max\{-X,\, 0\}$, $Y_1 = |X|$, and $Y_2 = X^2$.

13. Let X be random variable whose first two raw moments exist. Then show that

(a) $Var(X) \geq 0$.

(b) $E(X^2) \geq (E(X))^2$.

(c) $V(a + bX) = b^2 V(X)$, for two real constants a and b.

14. Let X be a random variable with PDF

$$f_X(x) = \begin{cases} \frac{x}{2} & \text{if } 0 < x \leq 1 \\ \frac{1}{2} & \text{if } 1 < x < 2 \\ \frac{3-x}{2} & \text{if } 2 < x < 3 \\ 0 & \text{otherwise.} \end{cases}$$

Find the expected value of $Y = X^2 - 5X + 3$. (Ans: -11/6)

15. Let X be a CRV with PDF $f_X(x)$ that is symmetric about $\mu \in \mathbb{R}$, *i.e.*, $f_X(\mu + x) = f_X(\mu - x)$, for all $x \in (-\infty, \infty)$. If $E(X)$ finite, then show that $E(X) = \mu$.

16. Let X be a CRV such that $E(|X|^\beta) < \infty$ for some $\beta > 0$. Then show that $E(|X|^\alpha) < \infty$ for all $\alpha \in (0,\, \beta]$.

17. Find the MGFs of $X \sim Poi(\lambda)$, $X \sim U(a,\, b)$.

18. Find the raw moments of the random variable that has the MGF $M(t) = (1 - t)^{-3}$ for $t < 1$. (Ans: The r-th raw moment is $\frac{(r+2)!}{2}$.)

19. Suppose we know that the number of items produced in a factory during a week is a RV with mean 500. Find an upper bound on the probability that this week's production will be at least 1000? If the variance of a week's production is known to equal to 100, then give a lower bound on the probability that this week's production will be between 400 and 600? (Hint: Moment Inequalities)

20. Let the random variable X have the MGF

$$M_X(t) = \frac{1}{8}e^{-t} + \frac{1}{4}e^{t} + \frac{1}{8}e^{2t} + \frac{1}{2}e^{3t} \text{ for } t \in \mathbb{R}.$$

Find the distribution of X.

21. If the MGF of a random variable X is

$$M_X(t) = \frac{1}{3t}\left(e^{t} - e^{-2t}\right) \text{ for } t \neq 0.$$

Find the PDF of $Y = X^2$. (Hint: Try to identify the distribution of X from its MGF.)

22. Let X be a random variable with MGF $M(t)$, $|t| < h$, for some $h > 0$. Prove that $P(X \geq a) \leq e^{-at}M(t)$ for $0 < t < h$, and $P(X \leq a) \leq e^{-at}M(t)$ for $-h < t < 0$. (Hint: Markov Inequality)

23. Let X be a random variable such that $P(X \leq 0) = 0$ and $\mu = E(X)$ is finite. Show that $P(X \geq 2\mu) \leq 0.5$. (Hint: Use Markov inequality.)

24. If X is a random variable such that $E(X) = 3$ and $E(X^2) = 13$, then determine a lower bound for $P(-2 < X < 8)$. (Hint: Use Chebyshev inequality. Ans: $\frac{21}{25}$.)