Indian Institute of Technology Guwahati
Probability Theory (MA 683)
Problem Set 02

1. Prove that $\mu^* = \overline{\mu}$ on $\mathcal{A}$ (Notations are as used in lectures).

2. Suppose that Ω is a non-empty set, $\mathcal{A}$ is a algebra of subsets of Ω and $\mu : \mathcal{A} \to [0, \infty]$ is a set function not identically $+\infty$. Show that the following conditions are equivalent:

   (a) μ is finitely additive;

   (b) $\mu(\Phi) = 0$ and for all $A, C \in \mathcal{A}$, we have $\mu(A \cup C) + \mu(A \cap C) = \mu(A) + \mu(C)$.

3. Suppose that Ω is a non empty set and $\tilde{\mu}$ is an outer measure on $\mathcal{P}(\Omega)$.

   (a) Let $A \subset \Omega$ be a set such that $\tilde{\mu}(A) < \infty$ and suppose that C is $\tilde{\mu}$ measurable, satisfies $A \subset C$ and $\tilde{\mu}(A) = \tilde{\mu}(C)$. Show that

   $$\tilde{\mu}(A \cap E) = \tilde{\mu}(C \cap E), \ \forall \ \tilde{\mu}\,\text{measurable set}\, E$$

   (b) Let $A \subset \Omega$ be a set such that $\tilde{\mu}(A) < \infty$ Show that

   $$A\,\text{is}\,\tilde{\mu}\,\text{measurable} \iff \tilde{\mu}(D \cap E) = \tilde{\mu}(D) + \tilde{\mu}(E), \ D \subset A, \ E \subset \Omega \setminus A$$

4. If $\mathbb{P}_k, k = 1, 2, \ldots$ are probability measures on $(\Omega, \mathcal{F})$, $\alpha_k \geq 0$ for all $k \geq 1$ and $\sum_{k=1}^{\infty} \alpha_k = 1$, then $\mathbb{P}(A) := \sum_{k=1}^{\infty} \alpha_k \mathbb{P}_k(A)$ is a probability measure on $(\Omega, \mathcal{F})$.

5. Suppose that $(\Omega, \mathcal{F})$ is a measurable space and $\mu : \mathcal{F} \to \mathbb{R}_+$ is an finitely additive set function. For every $A \in \mathcal{F}$, we set

   $$m(A) := \inf\{\lim_{n\to\infty} \mu(C_n)\},$$

   where the infimum is taken over all increasing sequences $\{C_n\}_{n\geq1} \subset \mathcal{F}$ such that $A = \cup_{n\geq1} C_n$. Show that $m : \mathcal{F} \to \mathbb{R}_+$ is a measure.

6. Suppose that Ω is an uncountable set and

   $$\mathcal{F} := \{A \subset X : \text{if A or A}^{\text{c}}\,\text{is finite or countable}\}.$$

   Show that $\mathcal{F}$ is a σ-algebra. Let $m : \mathcal{F} \to \mathbb{R}$ be defined by

   $$m(A) = \begin{cases} 0 & \text{if A is finite or countable} \\ 1 & \text{if A}^{\text{c}}\,\text{is finite or countable} \end{cases}$$

   Show that $(\Omega, \mathcal{F}, m)$ is a measure space.

7. Give an example of a sequence $\{\mathcal{F}_n\}_{n\geq1}$ of sigma-algebras which is strictly increasing, i.e. $\mathcal{F}_n \neq \mathcal{F}_{n+1}$, for all n and such that $\cup_{n\geq1} \mathcal{F}_n$ is not a sigma-algebra.