MA 101 (Mathematics-I)

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Introduction

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Result: If limit exists, then it is unique.

Example: $\lim_{x\to 1}(\frac{3x}{2} - 1) = \frac{1}{2}$.
Let $\varepsilon > 0$. We have to find $\delta > 0$ such that
$0 < |x-1| < \delta \Rightarrow |f(x) - \ell| < \varepsilon$ holds with $\ell = 1/2$.
Working backwards,

$$\frac{3}{2}|x-1| < \varepsilon \text{ whenever } |x-1| < \delta := \frac{2}{3}\varepsilon.$$

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Theorem (Sequential criterion)

Let $D \subseteq \mathbb{R}$ and let $x_0 \in \mathbb{R}$ such that for some $h > 0$, $(x_0 - h, x_0 + h) \setminus \{x_0\} \subseteq D$. Let $f : D \to \mathbb{R}$. Then the following are equivalent.

(a) $\lim_{x\to x_0} f(x) = \ell$.

(b) *For any sequence (x_n) in D with $x_n \neq x_0$ for all $n \geq 1$ and $x_n \to x_0$, the sequence $(f(x_n))$ converges to ℓ.*

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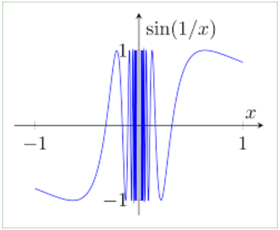

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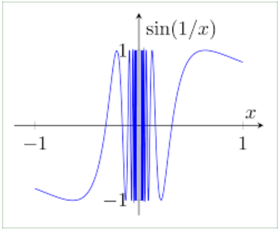

Solution: Let $x_n = \frac{2}{(4n+1)\pi}$ and $y_n = \frac{1}{n\pi}$ for all $n \in \mathbb{N}$. Then $x_n \to 0$ and $y_n \to 0$.

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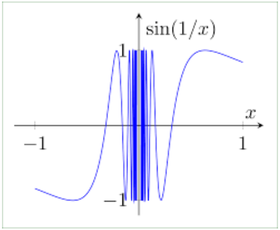

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Since $\sin \frac{1}{x_n} = 1$ and $\sin \frac{1}{y_n} = 0$ for all $n \in \mathbb{N}$, we get $\sin \frac{1}{x_n} \to 1$ and $\sin \frac{1}{y_n} \to 0$.

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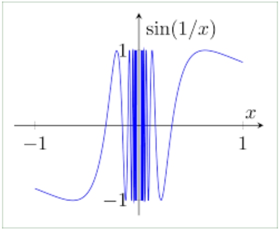

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Therefore by the sequential criterion for limit, $\lim_{x \to 0} \sin \frac{1}{x}$ does not exist.

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Result: Let $f : D \to \mathbb{R}$. Suppose that $\lim_{x \to x_0} f(x) = \ell$. Then there exists some $\delta > 0$ such that f is bounded on $(x_0 - \delta, x_0 + \delta) \setminus \{x_0\}$. That is, there exists $M > 0$ such that $|f(x)| < M$ for all $x \in (x_0 - \delta, x_0 + \delta)$ with $x \neq x_0$.

Theorem (Limit Theorems)

Let $D \subseteq \mathbb{R}$ *and let* $x_0 \in \mathbb{R}$ *such that for some* $h > 0$, $(x_0 - h, x_0 + h) \setminus \{x_0\} \subseteq D$. *Let* $f, g, j : D \to \mathbb{R}$. *Suppose that* $\lim\limits_{x \to x_0} f(x) = \ell$ *and* $\lim\limits_{x \to x_0} g(x) = m$. *Then*

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(4) *If* $f(x) \leq j(x) \leq g(x)$ *for all* $x \in (x_0 - h, x_0 + h) \setminus \{x_0\}$ *and* $\ell = m$, *then* $\lim\limits_{x \to x_0} j(x) = \ell$.

Result: Suppose that $f(x)$ is bounded in $(x_0 - h, x_0 + h) \setminus \{x_0\}$ for some $h > 0$ and $\lim_{x \to x_0} g(x) = 0$. Then $\lim_{x \to x_0} f(x)g(x) = 0$.

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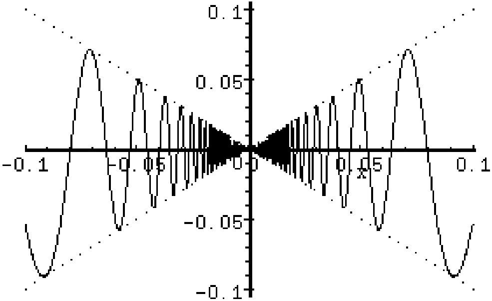

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Result: $\lim\limits_{x \to x_0} f(x) = \ell \Leftrightarrow \lim\limits_{x \to x_0+} f(x) = \lim\limits_{x \to x_0-} f(x) = \ell$.

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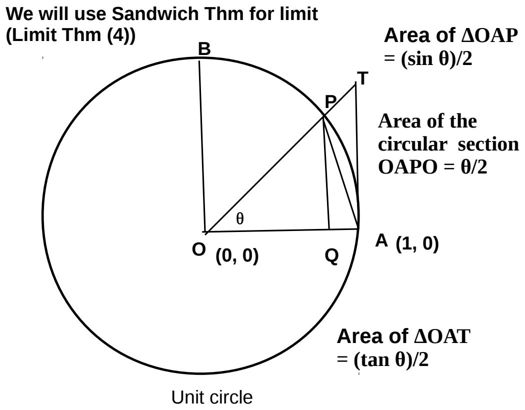

Area of ΔOAP < Area of the circular section OAPO < Area of ΔOAT

Limits at infinity and infinite limits

Definition: $f(x)$ has limit ℓ as x approaches $+\infty$, if for any given $\varepsilon > 0$, there exists $M > 0$ such that

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Definition: A function $f(x)$ approaches ∞ ($f(x) \to \infty$) as $x \to x_0$ if, for every real $M > 0$, there exists $\delta > 0$ such that

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For (ii), let $x_n = \frac{1}{\frac{\pi}{2}+2n\pi}$ and $y_n = \frac{1}{n\pi}$. Then $x_n, y_n \to 0$ as $n \to \infty$.

But $\lim\limits_{n\to\infty} f(x_n) = \dfrac{1}{x_n^2} \to \infty$ and $\lim\limits_{n\to\infty} f(y_n) = 0$.

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Theorem
Suppose that $\lim_{x\to x_0} f(x) = \ell$. *If* $\ell \neq 0$, *then there exists some* δ *such that* $f(x) \neq 0$ *for all* $x \in (x_0 - \delta, x_0 + \delta) \setminus \{x_0\}$.

Continuous functions

Let D be a nonempty subset of $\mathbb{R}$ and $f : D \to \mathbb{R}$. We say that f is continuous at $x_0 \in D$ if for each $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(x_0)| < \varepsilon$ for all $x \in D$ satisfying $|x - x_0| < \delta$.

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Let $f : [a, b] \to \mathbb{R}$. Then f is continuous at $c \in (a, b)$ if $\lim\limits_{x \to c} f(x) = f(c)$. f is continuous at a if $\lim\limits_{x \to a+} f(x) = f(a)$. Similarly, f is continuous at b if $\lim\limits_{x \to b-} f(x) = f(b)$.

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Sequential criterion of continuity: $f : D \to \mathbb{R}$ is continuous at $x_0 \in D$ if and only if for every sequence (x_n) in D such that $x_n \to x_0$, we have $f(x_n) \to f(x_0)$.

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We say that $f : D \to \mathbb{R}$ is continuous if f is continuous at each $x_0 \in D$.

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Result: If $f : D \to \mathbb{R}$ is continuous at $x_0 \in D$ and $f(x_0) \neq 0$, then there exists $\delta > 0$ such that $f(x) \neq 0$ for all $x \in D$ satisfying $|x - x_0| < \delta$.

Examples

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(4) $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$

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Intermediate value theorem

Result (Nested intervals property): If $I_n = [a_n, b_n]$, $n \in \mathbb{N}$, is a nested sequence of intervals, then there exists a number $\xi \in \mathbb{R}$ such that $\xi \in I_n$ for all $n \in \mathbb{N}$.

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Intermediate value theorem: Let I be an interval of $\mathbb{R}$ and let $f : I \to \mathbb{R}$ be continuous. If $a, b \in I$ with $a < b$ and if $f(a) < k < f(b)$, then there exists $c \in (a, b)$ such that $f(c) = k$.

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Examples:

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Examples:

- a The equation $x^2 = x \sin x + \cos x$ has at least two real roots.
- b (Fixed point) If $f : [a, b] \to [a, b]$ is continuous, then there exists $c \in [a, b]$ such that $f(c) = c$.

Recall that a function $f : D \to \mathbb{R}$ is called bounded if there exists $M > 0$ such that $|f(x)| < M$ for all $x \in D$.

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Result: If $f : [a, b] \to \mathbb{R}$ is continuous, then the supremum and the infimum of $f(x)$ are attained in $[a, b]$. That is, there exist $x_0, y_0 \in [a, b]$ such that $f(x_0) \leq f(x) \leq f(y_0)$ for all $x \in [a, b]$.

Limit point of a set: Let $A \subseteq \mathbb{R}$. A real number x is called a limit point of A if there exists a sequence (x_n) in A converging to x.

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Result: Let A be a closed and bounded subset of $\mathbb{R}$. If $f : A \to \mathbb{R}$ is continuous, then f is bounded.

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Result: Let A be a closed and bounded subset of $\mathbb{R}$. If $f : A \to \mathbb{R}$ is continuous, then f is bounded.

Remark: The above result is not true if A is bounded but not closed. For example $f(x) = 1/x$ on $(0, 1)$. Also, the result is not true if A is closed but not bounded. For example, $f(x) = x$ on $\mathbb{R}$.