Indian Institute of Technology Guwahati
Probability Theory (MA590)
Problem Set 06

1. Check weather the following functions are CDFs of 2-dim random vector or not.

$$\text{(a) } F(x,\, y) = \begin{cases} 1 & \text{if } x + 2y \geq 1 \\ 0 & \text{if } x + 2y < 1. \end{cases}$$

$$\text{(b) } F(x,\, y) = \begin{cases} 0 & \text{if } x < 0 \text{ or } x + y < 1 \text{ or } y < 0 \\ 1 & \text{otherwise.} \end{cases}$$

2. Let $F(\cdot, \cdot)$ be the CDFs of a two-dimensional random vector $(X,\, Y)$, and let $F_1(\cdot)$ and $F_2(\cdot)$, respectively, be the marginal CDFs of X and Y. Define $U(x,\, y) = \min\{F_1(x),\, F_2(y)\}$ and $L(x,\, y) = \max\{F_1(x) + F_2(y) - 1,\, 0\}$. Prove the followings.

   (a) $L(x,\, y) \leq F(x,\, y) \leq U(x,\, y)$.

   (b) $L(x,\, y)$ and $U(x,\, y)$ are CDFs of 2-dimensional random vector.

   (c) The marginal distributions of $L(\cdot, \cdot)$ and $U(\cdot, \cdot)$ are same as that of $F(\cdot, \cdot)$.

3. Let the random variable X have CDF $F_1(\cdot)$ and let $Y = g(X)$ have distribution function $F_2(\cdot)$, where $g(\cdot)$ is some function. Prove that

   (a) If $g(\cdot)$ is increasing, $F_{X,Y}(x,\, y) = \min\{F_1(x),\, F_2(y)\}$.

   (b) If $g(\cdot)$ is decreasing, $F_{X,Y}(x,\, y) = \max\{F_1(x) + F_2(y) - 1,\, 0\}$.

4. Consider the following joint PMF of the random vector $(X,\, Y)$.

$x \backslash y$	1	2	3	4
4	0.08	0.11	0.09	0.03
5	0.04	0.12	0.21	0.05
6	0.09	0.06	0.08	0.04

   (a) Find $P(X + Y < 8)$, $P(X + Y > 7)$, $P(XY \leq 14)$, $P(X + Y < 8 | X = 4)$.

   (b) Find the $Corr\,(X, Y)$

5. Three balls are randomly placed in three empty boxes B_1, B_2, and B_3. Let N denote the total number of boxes which are occupied and let X_i denote the number of balls in the box B_i, $i = 1,\, 2,\, 3$.

   (a) Find the joint PMF of $(N,\, X_1)$.

   (b) Find the joint PMF of $(X_1,\, X_2)$.

   (c) Find the marginal distributions of N and X_2.

(d) Find the marginal PMF of X_1 from the joint PMF of (X_1, X_2).

6. For the bivariate negative binomial distribution, the PMF is given by

$$f_{X,Y}(x, y) = \begin{cases} \frac{(x+y+k-1)!}{x!y!(k-1)!}\theta_1^x\theta_2^y(1-\theta_1-\theta_2)^k & \text{if } x \in \{0, 1, 2, \ldots\},\ y \in \{0, 1, 2, \ldots\} \\ 0 & \text{otherwise,} \end{cases}$$

k is a positive integer, $0 < \theta_1 < 1$, $0 < \theta_2 < 1$, and $0 < \theta_1 + \theta_2 < 1$. Find both the marginal distributions.

7. For the bivariate beta random vector (X, Y) having PDF

$$f_{X,Y}(x, y) = \begin{cases} \frac{\Gamma(\theta_1+\theta_2+\theta_3)}{\Gamma(\theta_1)\Gamma(\theta_2)\Gamma(\theta_3)}x^{\theta_1-1}y^{\theta_2-1}(1-x-y)^{\theta_3-1} & \text{if } x > 0,\ y > 0,\ x+y < 1 \\ 0 & \text{otherwise,} \end{cases}$$

where $\theta_i > 0$, $i = 1, 2, 3$. Find both the marginal PDFs.

8. Show that

$$f(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}e^{-\frac{1}{2(1-\rho^2)}\left\{\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2-2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right)+\left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right\}} \text{if } (x_1, x_2) \in \mathbb{R}^2$$

is a PDF of a two-dimensional random vector for $\mu_i \in \mathbb{R}$, $i = 1, 2$; $\sigma_i > 0$, $i = 1, 2$; and $\rho \in (-1, 1)$. Assuming that the JPDF of (X_1, X_2) is $f(\cdot, \cdot)$, find the marginal PDFs of X_1 and X_2.

9. Let X_1, X_2, X_3 have the JPDF

$$f(x_1, x_2, x_3) = \begin{cases} 48x_1x_2x_3 & \text{if } 0 < x_1 < x_2 < x_3 < 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the marginal PDFs of X_1, X_2 and X_3.

(b) Find the JPDFs of (X_1, X_2), (X_2, X_3), and (X_1, X_3).

10. Let X and Y be discrete random variables with JPMF f. Show that

$$E(\ln f_X(X)) \geq E(\ln f_Y(X)),$$

where f_X and f_Y denote marginal PMFs of X and Y, respectively. Hint: $\ln x \leq x - 1$.