**Indian Institute of Technology Guwahati**
**Probability Theory (MA590)**
**Problem Set 02**

1. Let $S = \{0, 1, 2, \ldots\}$ be a sample space. Let $\mathcal{F} = \mathcal{P}(S)$. In each of the following cases, verify if $P(\cdot)$ is a probability.

   (a) $P(A) = \sum_{x \in A} \frac{e^{-\lambda}\lambda^x}{x!}$, $A \in \mathfrak{F}$, $\lambda > 0$.

   (b) $P(A) = \sum_{x \in A} p(1-p)^x$, $A \in \mathfrak{F}$, $0 < p < 1$.

   (c) $P(A) = 0$, if $A$ has a finite number of elements, and $P(A) = 1$, if $A$ has infinite number of elements, $A \in \mathfrak{F}$.

2. Let $I$ be any index set. Let $\mathcal{F}_\alpha, \alpha \in I$ be a collection of $\sigma-$algebras on $\mathcal{S}$. Prove that $\bigcap_{\alpha \in I} \mathcal{F}_\alpha$ is a $\sigma-$algebra.

3. Give a counter-example to show that union of two $\sigma-$algebras need not be a sigma algebra.

4. Let $\mathcal{S}$ be the sample space of a random experiment. Let $\mathcal{A}$ be a collection of subsets of $\mathcal{S}$. The smallest $\sigma-$algebra containing $\mathcal{A}$ or the $\sigma-$algebra generated by $\mathcal{A}$ is defined as

$$\sigma(\mathcal{A}) \doteq \bigcap_{\mathcal{A} \subset \mathcal{F},\, \mathcal{F} \text{ is } \sigma-\text{algebra on } \mathcal{S}} \mathcal{F}\,.$$

   (a) Let $A$ be a non-empty subset of $\mathcal{S}$. Write down the smallest $\sigma-$algebra containing $A$.
   (b) Let $A$ and $B$ be two non-empty subsets of $\mathcal{S}$ such that $A \cup B \neq \mathcal{S}$ and $A \cap B \neq \phi$ . Write down the smallest $\sigma-$algebra containing $A$ and $B$.

5. Let $A_1, A_2, \ldots, A_n$ be $n > 1$ events. Then prove that

$$P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P\left(A_i\right).$$

6. Let $A_1, A_2, \ldots$ be a sequence of events. Then prove that

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} P\left(A_i\right).$$

7. (Principle of inclusion and exclusion) Let $A_1, A_2, \ldots, A_n$ be $n > 1$ events. Then prove that

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) - \sum_{\substack{i_1=1 \\ i_1<i_2}}^{n} \sum_{i_2=1}^{n} P\left(A_{i_1} \cap A_{i_2}\right) + \sum_{\substack{i_1=1 \\ i_1<i_2<i_3}}^{n} \sum_{i_2=1}^{n} \sum_{i_3=1}^{n} P\left(A_{i_1} \cap A_{i_2} \cap A_{i_3}\right) - \ldots + (-1)^{n+1} P\left(\bigcap_{i=1}^{n} A_i\right)$$

8. (Bonferroni's Inequality) Given $n\,(> 1)$ events $A_1,\ A_2,\ \ldots,\ A_n$, prove that

$$\sum_{i=1}^{n} P(A_i) - \sum_{\substack{i=1 \\ i<j}}^{n} \sum_{j=1}^{n} P(A_i \cap A_j) \leq P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i).$$

[Hint: To prove the LHS, use induction.]

9. For any two events $A$ and $B$, prove that $P(A \cap B) \geq 1 - P(A^c) - P(B^c)$.

10. Let $A$, $B$, $C$, and $D$ be four events such that $P(A) = 0.6$, $P(B) = 0.5$, $P(C) = 0.4$, $P(A \cap B) = 0.3$, $P(A \cap C) = 0.2$, $P(B \cap C) = 0.2$, $P(A \cap B \cap C) = 0.1$, $P(B \cap D) = P(C \cap D) = 0$, $P(A \cap D) = 0.1$, and $P(D) = 0.2$. Find

    (a) $P(A \cup B \cup C)$ and $P(A^c \cap B^c \cap C^c)$. (Ans: 0.9 and 0.1)

    (b) $P\left((A \cup B) \cap C\right)$ and $P\left(A \cup (B \cap C)\right)$. (Ans: 0.3 and 0.7)

    (c) $P\left((A^c \cup B^c) \cap C^c\right)$ and $P\left((A^c \cap B^c) \cup C^c\right)$. (Ans: 0.4 and 0.7)

    (d) $P(D \cap B \cap C)$ and $P(A \cap C \cap D)$. (Ans: 0 and 0)

    (e) $P(A \cup B \cup D)$ and $P(A \cup B \cup C \cup D)$. (Ans: 0.9 and 1.0)

    (f) $P\left((A \cap B) \cup (C \cap D)\right)$. (Ans: 0.3)

11. Let $(\Omega,\ \mathcal{F},\ P)$ be a probability space and let $A,\ B \in \mathcal{F}$. Show that $P(A \cap B) - P(A)P(B) = P(A)P(B^c) - P(A \cap B^c) = P(A^c)P(B) - P(A^c \cap B) = P\left((A \cup B)^c\right) - P(A^c)P(B^c)$.

12. Suppose that we have $n\,(\geq 2)$ letters and corresponding $n$ addressed envelopes. If these letters are inserted at random in $n$ envelopes, find the probability that no letter is inserted into the correct envelop. (Ans: $\dfrac{1}{2!} - \dfrac{1}{3!} + \dfrac{1}{4!} - \ldots + (-1)^n \dfrac{1}{n!}$.)