Indian Institute of Technology Guwahati
Probability Theory (MA590)
Problem Set 04

1. Consider an experiment involving two successive rolls of a 4-sided die in which all 16 possible outcomes are equally likely and have probability 1/16.

   (a) Are the events $A = \{\text{1st roll results in 1}\}$ and $B = \{\text{2nd roll results in 2}\}$ independent?

   (b) Are the events $A = \{\text{1st roll results in 1}\}$ and $B = \{\text{sum of the two rolls is a 5}\}$ independent?

   (c) Are the events $A = \{\text{maximum of the two rolls is 2}\}$ and $B = \{\text{minimum of the two rolls is 2}\}$ independent?

2. Let $S = (0, 1)$ and $P(I) =$ length of I, where I is an interval in S. Let $A = (0, 1/2)$, $B = (1/4, 1)$ and $C = (1/4, 11/12)$. Show that $P(A \cap B \cap C) = P(A)P(B)P(C)$, but $P(A \cap B) \neq P(A)P(B)$. (Note: It implies that $P(A \cap B \cap C) = P(A)P(B)P(C)$ is not sufficient for mutual independence of A, B, and C.)

3. Consider two independent fair coin tosses, in which all four possible outcomes are equally likely. Let

$$\begin{aligned} H_1 &= \{\text{1st toss is a head}\}, \\ H_2 &= \{\text{2nd toss is a head}\}, \\ D &= \{\text{the two tosses have different results}\}. \end{aligned}$$

   Find $P(H_1)$, $P(H_2)$, $P(H_1 \cap H_2)$, $P(H_1|D)$, $P(H_2|D)$, and $P(H_1 \cap H_2|D)$. (Ans: $P(H_1) = 0.5$, $P(H_2)0.5$, $P(H_1 \cap H_2) = 0.25$, $P(H_1|D) = 0.5$, $P(H_2|D) = 0.5$, and $P(H_1 \cap H_2|D) = 0$.) (Note: Independent does not imply conditionally independent.)

4. There are two coins, a blue and a red one. We choose one of the two at random, each being chosen with probability 1/2, and proceed with two independent tosses. The coins are biased. With the blue coin, the probability of heads in any given toss is 0.99, whereas for the red coin it is 0.01. Let D be the event that the blue coin was selected. Let H_i, $i = 1, 2$, be the event that the ith toss resulted in head. Find $P(H_1)$, $P(H_2)$, $P(H_1 \cap H_2)$, $P(H_1|D)$, $P(H_2|D)$, and $P(H_1 \cap H_2|D)$. (Ans: $P(H_1) = 0.5$, $P(H_2) = 0.5$, $P(H_1 \cap H_2) = 0.4901$, $P(H_1|D) = 0.99$, $P(H_2|D) = 0.99$, and $P(H_1 \cap H_2|D) = 0.9801$.) (Note: Conditional independence does not imply independence.)

5. Let A, B, and C be three events such that $P(B \cap C) > 0$. Prove or disprove each of the following: (a) $P(A \cap B|C) = P(A|B \cap C)P(B|C)$; (b) $P(A \cap B|C) = P(A|C)P(B|C)$ if A and B are independent events.

6. For independent events $A_1, \ldots, A_n$, show that:

$$P\left(\bigcap_{i=1}^{n} A_i^c\right) \leq e^{-\sum_{i=1}^{n} P(A_i)}.$$

   (Hint: Use the inequality $1 - x \leq e^{-x}$ for $x \in [0, 1]$.)

7. Let A, B, and C be three events such that A and B are negatively (positively) associated and B and C are negatively (positively) associated. Can we conclude that, in general, A and C are negatively (positively) associated?

8. Let A and B be two events. Show that if A and B are positively (negatively) associated then A and B^c are negatively (positively) associated.

9. An individual uses the following gambling system. He bets Re. 1. If he wins, he quits. If he loses, he makes the same bet a second time only this time he bets Rs. 2, and then regardless the result of the second match he quits the game. Assuming that he has a probability 0.5 to win each bet, find the probability that he goes home a winner. (Ans: 3/4.)

10. Suppose that each of the three persons tosses a coin. If the outcome of one of the tosses differ from the other outcomes, then the game ends. If not, then the persons start over and re-toss their coins. Assuming that the coins are fair, what is the probability that the game will end with first round of tosses? (Ans: 3/4.)

11. Consider an empty box in which four balls are to be placed (one-by-one) according to the following scheme. A fair die is cast each time and the number of spots on the upper face is noted. If the upper face shows up 2 or 5 spots then a white ball is placed in the box. Otherwise a black ball is placed in the box. Given that the first ball placed in the box was white find the probability that the box will contain exactly two black balls.

12. A student appears in the examinations of four subjects Biology, Chemistry, Physics and Mathematics. Suppose that probabilities of the student clearing examinations in these subjects are $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{5}$ respectively. Assuming that the performances of the student in four subjects are independent, find the probability that the student will clear examination(s) of (a) all the subjects; (b) no subject; (c) exactly one subject; (d) exactly two subjects; (e) at least one subject.

13. A k-out-of-n system is a system comprising of n components that functions if and only if at least $k \in \{1, 2, \ldots, n\}$ of the components function. A 1-out-of-n system is called a parallel system and an n-out-of-n system is called a series system. Consider n components $C_1, \ldots, C_n$ that function independently. At any given time t the probability that the component C_i will be functioning is $p_i(t) \in (0, 1)$ and the probability that it will not be functioning at time t is $1 - p_i(t)$, $i = 1, \ldots, n$.

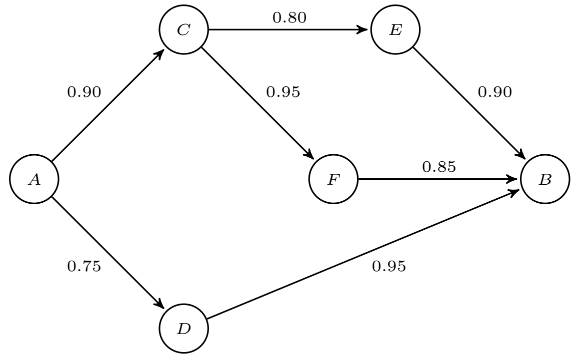
    (a) Find the probability that a parallel system comprising of components $C_1, \ldots, C_n$ will function at time t.

    (b) Find the probability that a series system comprising of components $C_1, \ldots, C_n$ will function at time t.

    (c) If $p_i(t) = p(t)$, $i = 1, \ldots, n$ find the probability that a k-out-of-n system comprising of components $C_1, \ldots, C_n$ will function at time t.

    (Ans: (a) $1 - \prod_{i=1}^{n}(1 - p_i(t))$, (b) $\prod_{i=1}^{n} p_i(t)$, and (c) $\sum_{i=k}^{n} \binom{n}{i} (p(t))^i (1 - p(t))^{n-i}$.)

14. A computer networks connects two nodes A and B through intermediate nodes C, D, E, and F as shown in the figure. For every pair of directly connected nodes, say i and j, there is a given probability p_{ij} that the link form i to j is up. Assume that the link failure are independent each other. What is the probability that there is a path from A to B?



(Ans: 0.957.)