Department of Mathematics
Indian Institute of Technology Guwahati

MA 101: Mathematics I

Tutorial Sheet-5

July-November 2023

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1. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q}, \\ [x] & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$
   Determine all the points of $\mathbb{R}$ where f is continuous.

2. Let $f : [0,1] \to \mathbb{R}$ be continuous such that $f(0) = f(1)$. Show that
   (a) there exist $x_1, x_2 \in [0,1]$ such that $f(x_1) = f(x_2)$ and $x_1 - x_2 = \frac{1}{2}$.
   (b) there exist $x_1, x_2 \in [0,1]$ such that $f(x_1) = f(x_2)$ and $x_1 - x_2 = \frac{1}{3}$.

3. Let p be an odd degree polynomial with real coefficients in one real variable. If $g : \mathbb{R} \to \mathbb{R}$ is a bounded continuous function, then show that there exists $x_0 \in \mathbb{R}$ such that $p(x_0) = g(x_0)$.

   In particular, this shows that

   (a) every odd degree polynomial with real coefficients in one real variable has at least one real zero.
   (b) the equation $x^9 - 4x^6 + x^5 + \frac{1}{1+x^2} = \sin 3x + 17$ has at least one real root.
   (c) the range of every odd degree polynomial with real coefficients in one real variable is $\mathbb{R}$.

4. Does there exist a continuous function from $(0,1]$ onto $\mathbb{R}$? Justify.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable on $(-\delta, \delta)$ for some $\delta > 0$ and let $f''(0)$ exist. If $f(\frac{1}{n}) = 0$ for all $n \in \mathbb{N}$, then find $f'(0)$ and $f''(0)$.

6. For $n \in \mathbb{N}$, show that the equation $1 - x + \frac{x^2}{2} - \frac{x^3}{3} + \cdots + (-1)^n \frac{x^n}{n} = 0$ has exactly one real root if n is odd and has no real root if n is even.

7. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable such that $f(0) = f(1) = 0$ and $f'(0) > 0$, $f'(1) > 0$. Show that there exist $c_1, c_2 \in (0,1)$ with $c_1 \neq c_2$ such that $f'(c_1) = f'(c_2) = 0$.

8. Let $f : \mathbb{R} \to \mathbb{R}$ be such that $f''(c)$ exists, where $c \in \mathbb{R}$. Show that $\lim\limits_{h \to 0} \frac{f(c+h) - 2f(c) + f(c-h)}{h^2} = f''(c)$. Give an example of an $f : \mathbb{R} \to \mathbb{R}$ and a point $c \in \mathbb{R}$ for which $f''(c)$ does not exist but the above limit exists.

9. Prove that, for $x > 0$,
$$|\log(1+x) - \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1}\frac{x^n}{n}\right)| < \frac{x^{n+1}}{n+1}.$$

10. Test the convergence of the power series: $\sum\limits_{n=1}^{\infty} a_n x^n$, where $a_n = \begin{cases} 2^{-n} & \text{if } n \text{ is even}, \\ 3^{-n} & \text{if } n \text{ is odd}. \end{cases}$

11. Prove that the Maclaurin series for $\cos x$ converges to $\cos x$ for all $x \in \mathbb{R}$.