Indian Institute of Technology Guwahati
Probability Theory (MA590)
Problem Set 01

1. Let S be a sample space of a random experiment. Let A, B, and C be three events. What is the event that only A occurs? What is the event that at least two of A, B, C occur? What is the event that both A, B, but not C occur? What is the event of at most one of the A, B, C occurs?

2. Suppose that $n\,(\geq 3)$ persons $P_1, \ldots, P_n$ are made to stand in a row at random. Find the probability that there are exactly r persons between P_1 and P_2; here $r \in \{1, 2, \ldots, n-2\}$. (Ans: $2(n-r-1)/(n(n-1))$.)

3. Three numbers a, b, and c are chosen at random and with replacement from the set $\{1, 2, \ldots, 6\}$. Find the probability that the quadratic equation $ax^2 + bx + c = 0$ will have real root(s). [Ans: $43/6^3$.]

4. Three numbers are chosen at random and without replacement from the set $\{1, 2, \ldots, 50\}$. Find the probability that the chosen numbers are in (a) arithmetic progression, and (b) geometric progression. (Ans: (a) $600/\binom{50}{3}$, (b) $44/\binom{50}{3}$.)

5. A class consisting of four graduate and twelve undergraduate students is randomly divided into four groups of four. What is the probability that each group includes a graduate student? [Ans: $(2 \times 3 \times 4^3)/(15 \times 14 \times 13)$.]

6. Find the probability that among three random digits there appear exactly two different digits. (Ans: 0.27)

7. Let r indistinguishable balls are placed in n cells numbered 1 through n. Two distributions are said to be distinguishable only if the corresponding n-tuples $(r_1, r_2, \ldots, r_n)$ are not identical, where r_i stands for the number of balls in the ith cell.
   (a) Show that the number of distinguishable distributions is $\binom{n+r-1}{r}$.
   (b) For $r \geq n$, show that the number of distinguishable distributions in which no cell remain empty is $\binom{r-1}{n-1}$.
   (c) Find the probability that no cell remain empty.

8. A man is given n keys of which only one fits his door. He tries them successively. This procedure may require $1, 2, \ldots, n$ trails. Show that each of these n outcomes has probability $1/n$.

9. Suppose that each of n sticks is broken into one long and one short part. The $2n$ parts are arranged into n pairs from which new sticks are formed. Find the probability that the parts will be joined in original order. (Ans: $2^n n!/(2n)!$)

10. (Geometric Probability) A point (X, Y) is randomly chosen on the unit square

$$S = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$$

i.e., for any region $R \subseteq S$ for which the area is defined, the probability that (X, Y) lies on R is $\frac{\text{Area of } R}{\text{Area of } S}$. Find the probability that distance from (X, Y) to the nearest side does not exceed $\frac{1}{5}$ units. (Ans: 16/25)

11. Two persons agree to meet a place at a given time. Each will arrive at the meeting place with a random delay between 0 to 1 hour independent of each other. The first to arrive will wait for 15 minutes and will leave if the other has not yet arrived. What is the probability that they meet? (Hint: Try to use geometric probability.) (Ans: 7/16.)