Department of Mathematics
Indian Institute of Technology Guwahati
MA 101: Mathematics I
Tutorial Sheet-6
July-November 2023

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1. Prove that at most one of the functions f and g below can be a derivative of a function:
$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 1 & \text{if } x = 0, \end{cases} \qquad g(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

2. Let I be an interval and let $f : I \to \mathbb{R}$ be a differentiable function. Show that if the derivative of f is never 0 on I, then f is strictly monotone on I.

3. Let $f : [-1, 1] \to \mathbb{R}$ be defined by $f(x) = \begin{cases} 1 & \text{if } x = \frac{1}{n} \text{ for some } n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$

   (a) Show that f is Riemann integrable on $[-1, 1]$ and that $\int\limits_{-1}^{1} f(x)\,dx = 0$.

   (b) If $F(x) = \int\limits_{-1}^{x} f(t)\,dt$ for all $x \in [-1, 1]$, then show that $F : [-1, 1] \to \mathbb{R}$ is differentiable, and in particular, $F'(0) = f(0)$, although f is not continuous at 0.

4. Let $f : [a, b] \to \mathbb{R}$ be continuous such that $f(x) \geq 0$ for all $x \in [a, b]$ and $\int\limits_{a}^{b} f(x)\,dx = 0$. Show that $f(x) = 0$ for all $x \in [a, b]$.
Equivalently, if $f : [a, b] \to \mathbb{R}$ is continuous such that $f(x) \geq 0$ for all $x \in [a, b]$ and $f(c) \neq 0$ for some $c \in [a, b]$, then $\int\limits_{a}^{b} f(x)\,dx > 0$.
(The above result need not be true if f is assumed to be only Riemann integrable on $[a, b]$.)

5. Let $f : [0, 1] \to \mathbb{R}$ be defined by $f(x) = \begin{cases} x & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$
Examine whether f is Riemann integrable on $[0, 1]$.

6. If $f : [0, 1] \to \mathbb{R}$ is Riemann integrable, then find $\lim\limits_{n\to\infty} \int\limits_{0}^{1} x^n f(x)\,dx$.

7. If $f : [0, 2\pi] \to \mathbb{R}$ is continuous such that $\int\limits_{0}^{\frac{\pi}{2}} f(x)\,dx = 0$, then show that there exists $c \in (0, \frac{\pi}{2})$ such that $f(c) = 2\cos 2c$.

8. Evaluate the limit: $\lim\limits_{n\to\infty} \left(\frac{1^8+3^8+\cdots+(2n-1)^8}{n^9} \right)$.

9. Let $f : [0, \infty) \to \mathbb{R}$ be continuous. If $x \sin(\pi x) = \displaystyle\int_0^{x^2} f(t)dt$, find the value of $f(4)$.