Indian Institute of Technology Guwahati
Probability Theory (MA590)
Problem Set 07

1. Suppose that $X_1, \ldots, X_n$ are independent and identically distributed random variables such that $P(X_i = 0) = 1 - p = 1 - P(X_i = 1)$, $i = 1, \ldots, n$, for some $p \in (0, 1)$. Let X be the number of $X_1, \ldots, X_n$ that are as large as X_1. Find the PMF of X.

2. The joint PDF of (X, Y) is given by
$$f_{X,Y}(x, y) = \begin{cases} 4xy & \text{if } 0 < x < 1,\, 0 < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$
   (a) Verify whether X and Y are independent.
   (b) Find $P(\{0 < X < 0.5,\, 0.25 < Y < 1\})$ and $P(\{X + Y < 1\})$.

3. Let $X = (X_1, X_2, X_3)$ be a random vector with joint PDF
$$f_{X_1,X_2,X_3}(x_1, x_2, x_3) = \frac{1}{(2\pi)^{\frac{3}{2}}} e^{-\frac{1}{2}\left(x_1^2+x_2^2+x_3^2\right)} \left(1 + x_1x_2x_3 e^{-\frac{1}{2}\left(x_1^2+x_2^2+x_3^2\right)}\right) \quad \text{if } (x_1, x_2, x_3) \in \mathbb{R}^3$$
   (a) Are X_1, X_2, and X_3 independent?
   (b) Are X_1, X_2, and X_3 pairwise independent?

4. Let $X_1, \ldots, X_n$ be *i.i.d.* random variables with mean μ and variance σ^2. Then $E\left(\overline{X}\right) = \mu$, $Var\left(\overline{X}\right) = \frac{\sigma^2}{n}$, and $Cov\left(\overline{X}, X_i - \overline{X}\right) = 0$ for all $i = 1, 2, \ldots, n$, where $\overline{X} = \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} X_i$.

5. Let X and Y be jointly distributed random variables with $E(X) = E(Y) = 0$, $E(X^2) = E(Y^2) = 2$, and $Corr(X, Y) = \frac{1}{3}$. Find $Corr\left(\frac{X}{3} + \frac{2Y}{3}, \frac{2X}{3} - \frac{Y}{3}\right)$.

6. Suppose that the random vector (X, Y) is uniformly distributed over the region $A = \{(x, y) : 0 < x < y < 1\}$. Find $Cov(X, Y)$.

7. Let (X, Y) be a continuous random vector with JPDF $f(\cdot, \cdot)$. Show that X and Y are independent if and only if $f(x, y) = g(x)h(y)$ for all $(x, y) \in \mathbb{R}^2$.

8. Let (X, Y) be uniform over the interior of the triangle with vertices $(0, 0), (2, 0)$ and $(1, 2)$. Find $P(X \leq 1, Y \leq 1)$.

9. Two numbers are independently chosen at random between 0 and 1. What is the probability that their product is less than a constant $k (0 < k < 1)$?

10. A vertical board is ruled with horizontal parallel lines at constant distance b apart. A needle of length $a (< b)$ is thrown at random on the board. Find the probability that it will intersect one of the lines.

11. Let (X, Y) be a continuous random vector with JPDF $f_{X,Y}$. Let the corresponding marginal PDFs be f_X and f_Y, respectively. Show that mutual information

$$I = E\left(\ln\left\{\frac{f_{X,Y}(X, Y)}{f_X(X)f_Y(Y)}\right\}\right)$$

satisfies $I \geq 0$, with equality if and only if X and Y are independent.

12. Let X and Y are RVs such that $|\rho(X, Y)| = 1$. Show that $Y = a + bX$ for some real constants a and b.