Indian Institute of Technology Guwahati
Probability Theory (MA 683)
Problem Set 01

1. Let $\Omega = \{1, 2, 3, 4\}$. Is any of the following families of sets a σ-algebra?

$$\begin{aligned}\mathcal{F}_1 &= \{\phi, \{1, 2\}, \{3, 4\}\},\\ \mathcal{F}_2 &= \{\phi, \Omega, \{1\}, \{2, 3, 4\}\{1, 2\}, \{3, 4\}\},\\ \mathcal{F}_3 &= \{\phi, \Omega, \{1\}, \{2\}, \{1, 3, 4\}\{2, 3, 4\}\{1, 2\}, \{3, 4\}\}.\end{aligned}$$

2. Let $\Omega = (0, 1)$. Is any of the following families of sets a σ-algebra?

$$\begin{aligned}\mathcal{F}_1 &= \{\phi, \Omega, (0, 1/2), (1/2, 1)\},\\ \mathcal{F}_2 &= \{\phi, \Omega, (0, 1/2), [1/2, 1), (0, 2/3), [2/3, 1)\},\\ \mathcal{F}_3 &= \{\phi, \Omega, (0, 2/3), [2/3, 1)\}.\end{aligned}$$

3. Let $\Omega = [0, 1]$. Adding as few sets as possible, complete the family of sets $\{\Omega, [0, 1/2), \{1\}\}$ to obtain a sigma algebra.

4. Let $I_0 = \Phi$ and $I_n = \{1, 2, \ldots, n\}$ for $n \geq 1$. Is $\mathcal{F} = \{I_n : n \geq 0\} \cup \{\mathbb{N} \setminus I_n : n \geq 0\}$ an algebra?

5. Let Ω be a finite non-empty set. Let $\mathcal{F} \subset \mathcal{P}(\Omega)$ be an algebra. Show that $\mathcal{F}$ is a σ-algebra.

6. Show that the union of two σ-algebras is never a σ-algebra unless one is included into the other.

7. Let Ω be a non-empty set and let $\mathcal{A} = \{A_i : i \in \mathbb{N}\}$ be a partition of Ω. Let $\mathcal{F} = \{\cup_{i\in J} A_i : J \subset \mathbb{N}\}$, where for $J = \emptyset$, $\cup_{i\in J} A_i = \emptyset$. Show that $\mathcal{F}$ is a σ-algebra. Also, show that $\mathcal{F} = \sigma\langle\mathcal{A}\rangle$.

8. Let Ω be a non-empty set and let $\mathcal{B} = \{B_i : 1 \leq i \leq k < \infty\} \subset \mathcal{P}(\Omega)$, $\mathcal{B}$ not necessarily a partition. Find $\sigma\langle\mathcal{B}\rangle$.

9. Let $\Omega = \mathbb{N}$ and $A_i = \{j : j \in \mathbb{N}, j \geq i\}$, $i \in \mathbb{N}$. Show that $\sigma\langle\mathcal{A}\rangle = \mathcal{P}(\Omega)$, where $\mathcal{A} = \{A_i : i \in \mathbb{N}\}$.

10. Let Ω be a non-empty set and $\mathcal{B}$ be a σ-algebra on Ω. Let $A \subset \Omega$ and $\mathcal{B}_A = \{B \cap A : B \in \mathcal{B}\}$. Show that $\mathcal{B}_A$ is a σ-algebra on A. The σ-algebra $\mathcal{B}_A$ is called the trace σ-algebra of $\mathcal{B}$ on A.

11. Let $\Omega = \mathbb{N}$, $\mathcal{F} = \mathcal{P}(\Omega)$, and $A_n = \{j : j \in \mathbb{N},\ j \geq n\}$, $n \in \mathbb{N}$. Let μ be the counting measure on $(\Omega, \mathcal{F})$. Verify that $\lim_{n\to\infty} \mu(A_n) \neq \mu(\cap_{n\geq 1} A_n)$.

12. Let Ω be a non-empty set and let $\mathcal{C} \subset \mathcal{P}(\Omega)$ be a semialgebra. Let

$$\mathcal{A}(\mathcal{C}) = \left\{A : A = \cup_{i=1}^{k} B_i,\ B_i \in \mathcal{C}, i = 1,\ 2,\ \ldots,\ k,\ k \in \mathbb{N}\right\}.$$

Show that $\mathcal{A}(\mathcal{C})$ is the smallest algebra containing $\mathcal{C}$.

13. Let $\Omega = \mathbb{N}$ and let $\mathcal{F}$ be the family of all subset of Ω. Is

$$\mathbb{P}(A) = \liminf_{n\to\infty} \frac{\#(A \cap \{1, 2, \cdots, n\})}{n}$$

a probability measure on $\mathcal{F}$?

14. Let $(\Omega,\ \mathcal{F})$ be a measurable space and μ be a measure defined on $\mathcal{F}$. Let $A_1,\ A_2,\ \ldots$ be a sequence of sets from $\mathcal{F}$ such that $\mu(A_1) + \mu(A_2) + \ldots < \infty$ and let $B_n = A_n \cup A_{n+1} \cup \ldots$. Then $\mu(B_1 \cap B_2 \cap \ldots) = 0$.