MA 101: Mathematics I
Solutions of selective problems in Tutorial Sheet-2

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9. Let the sequence (a_n) be defined by

$$a_1 = 1, a_{n+1} = \left(\frac{3+a_n^2}{2}\right)^{1/2}, \quad n \geq 1.$$

Show that (a_n) converges to $\sqrt{3}$.

Solution. Using the principle of mathematical induction, we find that $a_n \leq \sqrt{3}$ for all $n \geq 1$. Also, $a_n \geq 1$ for all n. We now find that $a_{n+1}^2 - a_n^2 = \frac{3}{2} - \frac{a_n^2}{2} \geq 0$, and hence $a_{n+1} \geq a_n$ for all n. This proves that the sequence is convergent. Let $x_n \to \ell$. Then, $\ell^2 = 3$. Since ℓ is positive, so $\ell = \sqrt{3}$. $\square$

11. For $a \in \mathbb{R}$, let $x_1 = a$ and $x_{n+1} = \dfrac{1}{4}(x_n^2 + 3)$ for all $n \geq 2$. Examine the convergence of the sequence $\{x_n\}$ for different values of a. Also, find $\lim\limits_{n\to\infty} x_n$ whenever it exists.

Solution. If $\{x_n\}$ converges, then $\ell = \lim x_n$ satisfies $\ell^2 - 4\ell + 3 = 0$. Hence $\ell = 1$ or $\ell = 3$.
We have $x_{n+1} - x_n = \frac{1}{4}(x_n^2 - x_{n-1}^2)$ for all $n > 1$. Also $x_2 - x_1 = \frac{1}{4}(a-1)(a-3)$.
Case 1: If $a > 3$ then $x_2 > x_1$ and we get $x_{n+1} > x_n$ for all n. If $\{x_n\}$ converges, then $\ell = \lim x_n = \sup\{x_n : n \in \mathbb{N}\} \geq x_1 = a > 3$, which is not possible. Hence, if $a > 3$ then $\{x_n\}$ can't converge.
Case 2: If $a = 3$, then $x_n = 3$ for all $n \in \mathbb{N}$, and hence $\{x_n\}$ converges to 3.
Case 3: If $1 < a < 3$, then $x_2 < x_1$ and we get $x_{n+1} < x_n$ for all $n \in \mathbb{N}$. Also in this case $x_n > 1$ for all $n \in \mathbb{N}$. (Because $x_{n+1} - 1 = \frac{1}{4}(x_n^2 - 1)$ for all $n \in \mathbb{N}$ and $x_1 > 1$.) Hence $\{x_n\}$ converges to 1. Note that $x_n \not\to 3$ as $\lim x_n = \inf\{x_n : n \in \mathbb{N}\} \leq x_1 = a < 3$.
Case 4: If $0 \leq a \leq 1$, then $x_2 \geq x_1$ and we get $x_{n+1} \geq x_n$ for all $n \in \mathbb{N}$. Also in this case $x_n \leq 1$ for all $n \in \mathbb{N}$. Hence $\{x_n\}$ converges to 1.
Case 5: The case for $a < 0$ is treated by considering $-a$ in place of a, because x_2 is same irrespective of whether we choose $x_1 = a$ or $x_1 = -a$. Hence we can say that for $-1 \leq a \leq 0$, $x_n \to 1$, for $-3 < a < -1$, $x_n \to 1$, for $a = -3$, $x_n \to 3$ and for $a < -3$, $\{x_n\}$ does not converge. $\square$

12. Let $x_1 = 6$ and $x_{n+1} = 5 - \frac{6}{x_n}$ for all $n \in \mathbb{N}$. Examine whether the sequence (x_n) is convergent. Also, find $\lim\limits_{n\to\infty} x_n$ if (x_n) is convergent.

Solution. We have $x_1 > 3$ and if we assume that $x_k > 3$ for some $k \in \mathbb{N}$, then $x_{k+1} > 5 - 2 = 3$. Hence by the principle of mathematical induction, $x_n > 3$ for all $n \in \mathbb{N}$. Therefore (x_n) is bounded below. Again, $x_2 = 4 < x_1$ and if we assume that $x_{k+1} < x_k$ for some $k \in \mathbb{N}$, then $x_{k+2} - x_{k+1} = 6(\frac{1}{x_k} - \frac{1}{x_{k+1}}) < 0 \Rightarrow x_{k+2} < x_{k+1}$. Hence by the principle of mathematical induction, $x_{n+1} < x_n$ for all $n \in \mathbb{N}$. Therefore (x_n) is decreasing. Consequently (x_n) is convergent. Let $\ell = \lim\limits_{n\to\infty} x_n$. Then $\lim\limits_{n\to\infty} x_{n+1} = 5 - \frac{6}{\lim\limits_{n\to\infty} x_n} \Rightarrow \ell = 5 - \frac{6}{\ell}$ (since $x_n > 3$ for all $n \in \mathbb{N}$, $\ell \neq 0$) $\Rightarrow (\ell - 2)(\ell - 3) = 0 \Rightarrow \ell = 2$ or $\ell = 3$. But $x_n > 3$ for all $n \in \mathbb{N}$, so $\ell \geq 3$. Therefore $\ell = 3$. $\square$