MA 323 | Quiz I
Monte Carlo Simulation | 14:00–14:55 IST
IIT GUWAHATI | September 13, 2022

Model Answers of Quiz I

1. (3 points) Generate the complete cycle for the linear congruent generators below and state the observed period.

   (a) $x_{i+1} = 5x_i + 3 \bmod 16$, $x_0 = 7$.

   Solution:

$$x_0 = 7, x_1 = 6, x_2 = 1, x_3 = 8, x_4 = 11, x_5 = 10, x_6 = 5, x_7 = 12, x_8 = 15,$$
$$x_9 = 14, x_{10} = 9, x_{11} = 0, x_{12} = 3, x_{13} = 2, x_{14} = 13, x_{15} = 4, x_{16} = 7.$$

   Period= 16.

   (b) $x_{i+1} = 5x_i + 3 \bmod 16$, $x_0 = 5$.

   Solution:

$$x_0 = 5, x_1 = 12, x_2 = 15, x_3 = 14, x_4 = 9, x_5 = 0, x_6 = 3, x_7 = 2, x_8 = 13,$$
$$x_9 = 4, x_{10} = 7, x_{11} = 6, x_{12} = 1, x_{13} = 8, x_{14} = 11, x_{15} = 10, x_{16} = 5.$$

   Period= 16.

   (c) $x_{i+1} = 5x_i \bmod 16$, $x_0 = 5$.

   Solution:

$$x_0 = 5, x_1 = 9, x_2 = 13, x_3 = 1, x_4 = 5.$$

   Period=4.

2. (3 points) Devise an algorithm based on inverse transformation technique to generate a sample from the following probability density function:

$$f(x) = \begin{cases} x & \text{if } 0 < x \leq 1 \\ 2 - x & \text{if } 1 < x \leq 2 \\ 0 & \text{otherwise.} \end{cases}$$

Solution: The corresponding CDF is

$$F(x) = \begin{cases} 0, \text{ if } x < 0 \\ \frac{x^2}{2}, \text{ if } 0 \leq x < 1 \\ 2x - 1 - \frac{x^2}{2}, \text{ if } 1 \leq x < 2 \\ 1, \text{ if } x \geq 2 \end{cases}$$

Algo:

1. Generate U from $U(0, 1)$.
2. If $U \leq \frac{1}{2}$, set $X = \sqrt{2U}$ (by equating $U = \frac{X^2}{2}$).

3. If $U > \frac{1}{2}$, set $X = 2 - \sqrt{2(1-U)}$ (by equating $2X - 1 - \frac{X^2}{2} = U$ and taking the root that lies between 1 and 2).

4. Return X.

3. (4 points) Devise an algorithm to generate a random number from the probability mass function

$$f(x) = cp(1-p)^{x-1} \quad \text{for } x = n,\, n+1,\, n+2,\, \ldots,$$

where $0 < p < 1$, $n > 1$ is an integer, and c is a normalizing constant.

Solution: As

$$\sum_{k=n}^{\infty} cp(1-p)^{k-1} = 1 \implies c = \frac{1}{(1-p)^{n-1}}.$$

Therefore, for $k = n, n+1, \ldots$,

$$f(k) = p(1-p)^{k-n}.$$

For $k = n, n+1, \ldots$, the corresponding CDF is given by

$$F(k) = P(X \leq k) = \sum_{x=n}^{k} p(1-p)^{x-n} = 1 - (1-p)^{k-n+1}.$$

Let, $c_{k-n+1} = F(k)$, for $k = n, n+1, \ldots$ and $c_0 = 0$.

Algo:

1. Generate U from $U(0,1)$.

2. Find the value of k such that $c_k < U \leq c_{k+1}$.

3. Return $X = k + 1 + n - 1 = k + n$.

4. (5 points) Derive an acceptance-rejection procedure for generating random number from the probability density function that is proportional to $h(\cdot)$, where

$$h(x) = \frac{e^{-\frac{x^2}{2}}}{1+|x|} \text{ for } x \in \mathbb{R}.$$

It may be assumed that there is access to a source of independent random numbers from a standard normal distribution. Comment on the efficiency of your method by computing the acceptance probability, given that $\displaystyle\int_{-\infty}^{\infty} h(s)ds = 1.54$.

Solution: Let $f(x) = ah(x)$, where a is the proportional constant.
Now, as for all $x \in \mathbb{R}, 1 + |x| \geq 1$,

$$\frac{e^{-\frac{x^2}{2}}}{1+|x|} \leq e^{-\frac{x^2}{2}} \iff \frac{ae^{-\frac{x^2}{2}}}{1+|x|} \leq a\sqrt{2\pi}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}, \text{ for } x \in \mathbb{R}.$$

Thus taking $c = a\sqrt{2\pi}$ and $g(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$, $f(x) \leq cg(x)$ for all $x \in \mathbb{R}$.

Moreover, $\frac{f(x)}{cg(x)} = \frac{1}{1+|x|}$.
Algo:

1. Repeat
2. Generate X from $N(0,1)$.
3. Generate U from $U(0,1)$.
4. Until $(1+|X|)U \leq 1$.
5. Return X.

We know that the acceptance probability $= \frac{1}{c} = \frac{1}{a\sqrt{2\pi}}$.

Now, $a\int_{\mathbb{R}} h(x)dx = 1 \implies a = \frac{1}{1.54}$.

Thus, acceptance probability $= \frac{1.54}{\sqrt{2\pi}} \approxeq 0.6144$.

Therefore, about 61.44% of times the generated random number from $N(0,1)$ will be accepted.