

Indian Institute of Technology Guwahati
Probability Theory (MA 683)
Problem Set 04

1. Prove that

(a) $(X + Y)^+ \leq X^+ + Y^+$

(b) $(X + Y)^- \leq X^- + Y^-$

(c) $X^+ \leq (X + Y)^+ + Y^-$

2. Show that a monotone function $f : \mathbb{R} \rightarrow \mathbb{R}$ is Borel measurable.

3. Let X be a discrete random variable such that $P(X = x_i) = p_i$, $i = 1, 2, \dots$, and $\sum_{i=1}^{\infty} p_i = 1$. Then show that $EX = \sum_{i=1}^{\infty} x_i p_i$ provided X is integrable.

4. Show that $|E(X)| \leq E(|X|)$ for any simple random variable X .

5. Let X be simple random variable with n different values x_1, x_2, \dots, x_n . Show that for any function $g : \mathbb{R} \rightarrow \mathbb{R}$,

$$E(g(X)) = \sum_{i=1}^n g(x_i) f(x_i),$$

where $f(x) = P(\{X = x\})$ is the mass function.

6. Show that $E(\mathbb{I}_A) = P(A)$ for any event A . Verify the identity $1 - \mathbb{I}_{A \cup B} = (1 - \mathbb{I}_A)(1 - \mathbb{I}_B)$ and use it to prove that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

7. Show that $E|X|^r = r \int_0^{\infty} t^{r-1} P(|X| > t) dt$.

8. If $E|X| < \infty$ and $\lim_{n \rightarrow \infty} P(A_n) = 0$, then $\lim_{n \rightarrow \infty} \int_{A_n} X dP = 0$.

9. Every integrable and symmetric random variable has mean 0.

10. Give an example to show that $E(\sum_{n=1}^{\infty} X_n) = \sum_{n=1}^{\infty} EX_n$ is not true in general.

11. If X is a non-negative random variable in \mathcal{L}_p for all $p > 0$, and

$$g(p) = \log EX^p, \text{ for all } p \geq 0,$$

then g is convex on $[0, \infty)$.

12. Give an example of three events A_1, A_2 , and A_3 on some probability space such that they are pairwise independent but not independent.

13. If the events $\{E_\alpha : \alpha \in A\}$ are independent, then so are the events $\{F_\alpha : \alpha \in A\}$, where each F_α may be E_α or E_α^c .

14. For any random variable X and positive numbers a and t , show that

$$P(X \geq a) \leq e^{-at} Ee^{tX}.$$

15. For any $X_i \in \mathcal{L}_2$ for $i = 1, 2, \dots, n$, show that

$$\text{Var} \left(\sum_{i=1}^n X_i \right) = \sum_{i=1}^n \text{Var}(X_i) + 2 \sum_{1 \leq i < j \leq n} \text{Cov}(X_i, X_j).$$

16. If $X_i \in \mathcal{L}_2$ for $i = 1, 2, \dots, n$ such that $\text{Cov}(X_i, X_j) = 0$ for all $i \neq j$, then show that

$$\text{Var} \left(\sum_{i=1}^n X_i \right) = \sum_{i=1}^n \text{Var}(X_i).$$

17. Prove that if f and g are non-decreasing functions and X is a random variable with $Ef(X)$, $Eg(X)$, and $Ef(X)g(X)$ finite, then $\text{Cov}(f(X), g(X)) \geq 0$.

18. If $X' = aX + b$ and $Y' = cX + d$, verify that $\rho(X', Y') = \pm \rho(X, Y)$ according as $ac > 0$ or $ac < 0$.

19. If X, Y are random variables such that $0 < \text{Var}(X), \text{Var}(Y) < \infty$, then $\rho(X, Y) = 1$ iff

$$\frac{X - EX}{\sigma_X} = \frac{Y - EY}{\sigma_Y} \quad \text{a.e. } (P).$$

20. If $X = \sin Z$ and $Y = \cos Z$, where $P(Z = \pm 1) = P(Z = \pm 2) = \frac{1}{4}$, then $\rho(X, Y) = 0$ despite X, Y being functionally related.

21. For arbitrary real numbers a_i, b_i ($1 \leq i \leq n$), prove that

$$\sum_{i=1}^n |a_i b_i| \leq \left(\sum_{i=1}^n |a_i|^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^n |b_i|^q \right)^{\frac{1}{q}},$$

provided $\frac{1}{p} + \frac{1}{q} = 1$, $p > 0$, $q > 0$.

22. Show that for $0 < a < b < d$ and any non-negative random variable Y ,

$$EY^b \leq (EY^a)^{\frac{d-b}{d-a}} (EY^d)^{\frac{b-a}{d-a}}.$$

23. Let $\{X_n\}_{n \geq 1}$ be a sequence of positive random variables and $\{c_n\}_{n \geq 1}$ be a sequence of positive real constants. If $\sum_{n=1}^{\infty} c_n^\alpha EY_n^\alpha < \infty$ for $\alpha = \alpha_1$ and $\alpha = \alpha_2$ ($0 < \alpha_1 < \alpha_2$), then show that $\sum_{n=1}^{\infty} c_n^\alpha EY_n^\alpha < \infty$ for all $\alpha \in [\alpha_1, \alpha_2]$.

24. (Minkowski's inequality) If $X_1 \in \mathcal{L}_p$ and $X_2 \in \mathcal{L}_p$ for some $p \geq 1$, then

$$(E|X_1 + X_2|^p)^{\frac{1}{p}} \leq (E|X_1|^p)^{\frac{1}{p}} + (E|X_2|^p)^{\frac{1}{p}}.$$

[Hint: Apply Holder's inequality to $E|X_i||X_1 + X_2|^{p-1}$.]

25. Let X be a random variable defined on a probability space $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+), P)$ such that $\lim_{\omega \rightarrow \infty} X(\omega) = \theta \in \mathbb{R}$. Show that

$$\lim_{n \rightarrow \infty} \int_0^a X dP = \theta P(0 \leq X \leq a) \quad \text{for all } a > 0.$$

26. Show that

- (a) $\{X_n\}_{n \geq 1}$ is u.i. iff $\{|X_n|\}_{n \geq 1}$ is u.i.
 - (b) If $\{X_n\}_{n \geq 1}$ and $\{Y_n\}_{n \geq 1}$ are u.i., so is $\{X_n + Y_n\}_{n \geq 1}$.
 - (c) If $\{X_n\}_{n \geq 1}$ is u.i., so is any sub sequence of $\{X_n\}_{n \geq 1}$.
 - (d) $\{X_n\}_{n \geq 1}$ is u.i. iff it is u.i. from above and from below.
 - (e) If $|X_n| \leq Y$ for all $n \geq 1$, where $EY < \infty$, then $\{X_n\}_{n \geq 1}$ is u.i.
27. If $\{|X_n|^\beta\}_{n \geq 1}$ is u.i. for some $\beta \geq 1$ and $S_n = \sum_{i=1}^n X_i$, then $\left\{ \left| \frac{S_n}{n} \right|^\beta \right\}_{n \geq 1}$ is uniformly integrable. In particular, if $\{X_n\}_{n \geq 1}$ are identically distributed random variables in \mathcal{L}_p for some $p \geq 1$, then $\left\{ \left| \frac{S_n}{n} \right|^p \right\}_{n \geq 1}$ is uniformly integrable.
28. Let $\{X_n\}_{n \geq 1}$ be a sequence of independent random variables with $EX_n = 0$ and $EX_n^2 = 1$. If $\{X_n^2\}_{n \geq 1}$ is uniformly integrable, then $\left\{ \frac{S_n^2}{n} \right\}_{n \geq 1}$ is uniformly integrable, where $S_n = \sum_{i=1}^n X_i$.