MA 590 — Mid-Semester Examination

Probability Theory — 9:00–11:00 IST

IIT GUWAHATI — February 26, 2023

Model Answers of Mid-Semester Examination

1. If an aircraft is present in a certain area, a radar detects it and generates an alarm signal with probability 0.99. If an aircraft is not present, the radar generates a (false) alarm, with probability 0.10. We assume that an aircraft is present with probability 0.05.

   (a) (1 point) What is the probability of no aircraft presence and a false alarm?

   (b) (1 point) What is the probability of aircraft presence and no detection?

   (c) (2 points) If the radar generates a alarm, what is the probability of the presence of an aircraft?

   Solution: Let A denote the event that the radar generate an alarm signal and B denote the event that a aircraft is present in the area. Then, the followings are given: $P(A|B) = 0.99$, $P(A|B^c) = 0.10$, and $P(B) = 0.05$.
   (a) The required probability is $P(A \cap B^c) = P(A|B^c)P(B^c) = 0.0950$.
   (b) The required probability is $P(A^c \cap B) = P(A^c|B)P(B) = 0.0005$.
   (c) The required probability is $P(B \cap A) = \dfrac{P(A|B)P(B)}{P(A|B)P(B) + P(A|B^c)P(B^c)} = \dfrac{99}{289}$.

2. A biased coin is tossed repeatedly. Each time there is a probability p of a head turning up independent of other tosses. For $n \geq 1$, let p_n be the probability that an even number of heads has occurred after n tosses and $p_0 = 1$. (zero is an even number.)

   (a) (4 points) Find the recurrence relation between p_n and p_{n-1} for $n \geq 1$.

   (b) (2 points) Find the value of p_{21} in terms of p.

   Solution: (a) Let A_n denote the event that even number of heads turn up after n tosses. Then

$$
\begin{aligned}
p_n &= P(A_n) \\
&= P\left(A_n|A_{n-1}\right) P(A_{n-1}) + P\left(A_n|A^c_{n-1}\right) P(A^c_{n-1}) \\
&= (1-p)p_{n-1} + p(1-p_{n-1}) \\
&= p + (1-2p)p_{n-1}.
\end{aligned}
$$

   (b)

$$
\begin{aligned}
p_{21} &= p + (1-2p)p_{20} \\
&= p + (1-2p)p + (1-2p)^2 p_{19} \\
&\vdots \\
&= p + (1-2p)p + \ldots + (1-2p)^{20}p + (1-2p)^{21} \\
&= \frac{1}{2} + \frac{1}{2}(1-2p)^{21}.
\end{aligned}
$$

3. (4 points) Let $F(\cdot)$ be a cumulative distribution function of a random variable. Let a function $H : \mathbb{R} \to \mathbb{R}$ is defined by

$$H(x) = \alpha F(x) + (1-\alpha)\left(F(x)\right)^2,$$

where $0 < \alpha < 1$. Then prove that $H(\cdot)$ is a cumulative distribution function of some random variable.

Solution:

$H(x_2) - H(x_1) = \alpha\left(F(x_2) - F(x_1)\right) + (1-\alpha)\left(F(x_2) + F(x_1)\right)\left(F(x_2) - F(x_1)\right) \geq 0$ for all $x_1 < x_2$.
$\lim_{x\to\infty} H(x) = \alpha \lim_{x\to\infty} F(x) + (1-\alpha) \lim_{x\to\infty} F^2(x) = 1.$
$\lim_{x\to-\infty} H(x) = \alpha \lim_{x\to-\infty} F(x) + (1-\alpha) \lim_{x\to-\infty} F^2(x) = 0.$
$\lim_{x\to y+} H(x) = \alpha \lim_{x\to y+} F(x) + (1-\alpha) \lim_{x\to y+} F^2(x) = \alpha F(y) + (1-\alpha)\left(F(y)\right)^2 = H(y)$ for all $y \in \mathbb{R}$.

Therefore, $H(\cdot)$ is the CDF of a random variable.

4. (3 points) Consider the probability space $(\mathcal{S}, \mathcal{P}(\mathcal{S}), P)$, where $\mathcal{S} = \{1, 2, 3, 4\}$ and $P(\{1\}) = \frac{1}{4}$. Let X be the random variable defined on the above probability space as $X(1) = 1$, $X(2) = X(3) = 2$ and $X(4) = 3$. If $P(X \leq 2) = \frac{3}{4}$, then find the value of $P(\{1, 4\})$.

Solution:

$$P(X \leq 2) = \frac{3}{4} \implies P(\{1, 2, 3\}) = \frac{3}{4}.$$

Therefore, $P(\{2, 3\}) = P(\{1, 2, 3\}) - P(\{1\}) = \frac{1}{2}$, and hence, $P(\{1, 4\}) = \frac{1}{2}$.

5. Let X be a random variable with probability density function

$$f(x) = \begin{cases} e^{-x} & 0 < x < \infty \\ 0 & \text{otherwise.} \end{cases}$$

Let $Y = 1 - e^{-X}$.

(a) (5 points) Find the cumulative distribution function of Y.

(b) (2 points) Find the expectation of Y.

Solution: (a) For $y < 0$, $P(Y \leq y) = P(X \in \emptyset) = 0$. For $0 \leq y < 1$, $P(Y \leq y) = P\left(1 - e^{-X} \leq y\right) = P(X = -\ln(1-y)) = 1 - e^{\ln(1-y)} = y$. For $y \geq 1$, $P(Y \leq y) = P(0 < X < \infty) = 1$. Therefore, the CDF of Y is

$$F_Y(y) = \begin{cases} 0 & \text{if } y < 0 \\ y & \text{if } 0 \leq y < 1 \\ 1 & \text{if } y \geq 1. \end{cases}$$

(b) Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(y) = \begin{cases} 1 & \text{if } 0 < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $f(y) \geq 0$ and $F_Y(y) = \int_{-\infty}^{y} f(t)dt$ for all $y \in \mathbb{R}$. Therefore, Y is a CRV with PDF $f(\cdot)$. Thus,

$$E(Y) = \int_{-\infty}^{\infty} y f(y) dy = \frac{1}{2}.$$

6. (6 points) Let $\mathcal{S}$ be a sample space and $\mathcal{F}$ be a σ-algebra defined on $\mathcal{S}$. Let $P : \mathcal{F} \to \mathbb{R}$ be a set function satisfying the following properties:

   (a) $P(E) \geq 0$ for all $E \in \mathcal{F}$.

   (b) $P(\mathcal{S}) = 1$.

   (c) For $n \geq 2$ and for disjoint events $E_1, E_2, \ldots, E_n$, $P\left(\cup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i)$.

   (d) If $\{E_n\}_{n\geq 1}$ is a sequence of decreasing events such that $\cap_{i=1}^{\infty} E_i = \emptyset$, then $\lim_{n\to\infty} P(E_n) = 0$.

   Show that P is a probability.

   Solution: Here we need to show the following: For disjoint $A_1, A_2 \ldots \in \mathcal{F}$, $P(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$.

$$\begin{aligned} P(\cup_{i=1}^{\infty} A_i) &= P\left[(\cup_{i=1}^{n} A_i) \cup (\cup_{i=n+1}^{\infty} A_i)\right] \\ &= \sum_{i=1}^{n} P(A_i) + P(\cup_{i=n+1}^{\infty} A_i). \end{aligned} \tag{1}$$

   Now define $B_n = \cup_{i=n}^{\infty} A_i$ for $n \geq 1$. Clearly $\{B_n\}_{n\geq 1}$ is a decreasing sequence of sets. We claim that $\cap_{n=1}^{\infty} B_n = \phi$. If not, then there exist a $\omega \in \cap_{n=1}^{\infty} B_n$ and hence $\omega \in B_n$ for all $n \geq 1$. In particular, $\omega \in B_1 = \cup_{i=1}^{\infty} A_i$. As A_i's are disjoint, $\omega \in A_i$ for exactly one i, say i_0. Then $\omega \notin A_i$ for $i > i_0$ and hence $\omega \notin B_n$ for $n > i_0$. This is a contradiction to the fact that $\omega \in B_n$ for all $n \geq 1$.

   This implies that $\lim_{n\to\infty} P\left(\cup_{i=n+1}^{\infty} A_i\right) = \lim_{n\to\infty} P(B_{n+1}) = 0$. Taking limit $n \to \infty$ on both sides of (1), we get

$$P(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i).$$