

MA 101 (Mathematics-I)

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Introduction

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- If the sequence of partial sums (s_n) converges to a limit ℓ , we say that the series converges and its sum is ℓ .
- If (s_n) diverges, we say that the series diverges.

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- ① The geometric series $\sum_{n=1}^{\infty} ar^{n-1}$ (where $a \neq 0$) converges if and only if $|r| < 1$.

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- ② The series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent with sum 1.
- ③ The series $1 - 1 + 1 - 1 + \cdots$ is not convergent.

Cauchy criterion: A series $\sum_{n=1}^{\infty} x_n$ is convergent if and only if for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

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Theorem (Necessary condition for convergence)

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Examples: The following series are not convergent.

$$(a) \sum_{n=1}^{\infty} \frac{n^2+1}{(n+3)(n+4)} \qquad (b) \sum_{n=1}^{\infty} (-1)^n \frac{n}{n+2}$$

Algebraic operations on series: Let $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ be convergent with sums x and y respectively. Then

- a $\sum_{n=1}^{\infty} (x_n + y_n)$ is convergent with sum $x + y$
- b $\sum_{n=1}^{\infty} \alpha x_n$ is convergent with sum αx , where $\alpha \in \mathbb{R}$

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Example:

(1) $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

Theorem (Comparison test)

Let (x_n) and (y_n) be sequences in \mathbb{R} such that for some $n_0 \in \mathbb{N}$, $0 \leq x_n \leq y_n$ for all $n \geq n_0$. Then

- a $\sum_{n=1}^{\infty} y_n$ is convergent $\Rightarrow \sum_{n=1}^{\infty} x_n$ is convergent,
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Example:

- 1 $\sum_{n=1}^{\infty} \frac{1+\sin n}{1+n^2}$ is convergent.

Solution: $0 \leq \frac{1+\sin n}{1+n^2} \leq \frac{2}{n^2}$ for all $n \in \mathbb{N}$.

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Solution: $0 \leq \frac{1+\sin n}{1+n^2} \leq \frac{2}{n^2}$ for all $n \in \mathbb{N}$.

- 2 $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n(n-1)}}$ is not convergent.

Solution: $\frac{1}{\sqrt{n(n-1)}} > \frac{1}{n} > 0$ for all $n \geq 2$.

Theorem (Limit comparison test)

Let (x_n) and (y_n) be sequences of positive real numbers such that $\frac{x_n}{y_n} \rightarrow \ell \in \mathbb{R}$.

- a If $\ell \neq 0$, then $\sum_{n=1}^{\infty} x_n$ is convergent $\Leftrightarrow \sum_{n=1}^{\infty} y_n$ is convergent.
- b If $\ell = 0$, then $\sum_{n=1}^{\infty} y_n$ is convergent $\Rightarrow \sum_{n=1}^{\infty} x_n$ is convergent.

If $\frac{x_n}{y_n}$ diverges to ∞ and $\sum_{n=1}^{\infty} y_n$ is divergent, then $\sum_{n=1}^{\infty} x_n$ is also divergent.

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Example: $\sum_{n=1}^{\infty} \frac{n}{4n^3-2}$ is convergent.

Solution: Let $x_n = \frac{n}{4n^3-2}$ and $y_n = \frac{1}{n^2}$ for all $n \in \mathbb{N}$. Then

$$\lim_{n \rightarrow \infty} \frac{x_n}{y_n} = \lim_{n \rightarrow \infty} \frac{n^3}{4n^3-2} = \lim_{n \rightarrow \infty} \frac{1}{4-\frac{2}{n^3}} = \frac{1}{4} \neq 0.$$

Theorem (Cauchy's condensation test)

Let (x_n) be a decreasing sequence of nonnegative real numbers. Then $\sum_{n=1}^{\infty} x_n$ is convergent if and only if $\sum_{n=1}^{\infty} 2^n x_{2^n}$ is convergent.

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Examples:

- a p -series: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if and only if $p > 1$.
- b $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ is convergent if and only if $p > 1$.

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Theorem (Leibniz's test)

Let (x_n) be a decreasing sequence of positive real numbers such that $x_n \rightarrow 0$. Then the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} x_n$ is convergent.

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By Leibniz's test, the alternating harmonic series

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} \text{ converges.}$$

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Theorem

Every absolutely convergent series is convergent.

Theorem (Comparison test-II)

Let (x_n) be a sequence of real numbers. Then $\sum_{n=1}^{\infty} x_n$ converges absolutely if there is an absolutely convergent series $\sum_{n=1}^{\infty} y_n$ and some $n_0 \in \mathbb{N}$ satisfying $|x_n| \leq |y_n|$ for all $n \geq n_0$.

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Theorem (Limit comparison test-II)

Let (x_n) and (y_n) be sequences of nonzero real numbers such that $\left| \frac{x_n}{y_n} \right| \rightarrow \ell \in \mathbb{R}$.

- a If $\ell \neq 0$, then $\sum_{n=1}^{\infty} x_n$ is absolutely convergent iff $\sum_{n=1}^{\infty} y_n$ is absolutely convergent.
- b If $\ell = 0$, then $\sum_{n=1}^{\infty} y_n$ is absolutely convergent $\Rightarrow \sum_{n=1}^{\infty} x_n$ is absolutely convergent.

Theorem (Ratio Test)

Let $\sum_{n=1}^{\infty} x_n$ be a series of nonzero real numbers. Let

$$a = \liminf \left| \frac{x_{n+1}}{x_n} \right| \quad \text{and} \quad A = \limsup \left| \frac{x_{n+1}}{x_n} \right|.$$

Then

- ① If $A < 1$, then $\sum_{n=1}^{\infty} x_n$ is absolutely convergent.
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- ① If $A < 1$, then $\sum_{n=1}^{\infty} x_n$ is absolutely convergent.
- ② If $a > 1$, then $\sum_{n=1}^{\infty} x_n$ is divergent.

Remark: If $\left| \frac{x_{n+1}}{x_n} \right| \rightarrow \ell$, then $a = A = \ell$.

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Remark: If $\ell = \lim_{n \rightarrow \infty} \left| \frac{x_{n+1}}{x_n} \right| = 1$, then the Ratio test is inconclusive. For example, for both the series $\sum_{n=1}^{\infty} \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$, $\ell = 1$. However, $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

Theorem (Root Test)

Let $\sum_{n=1}^{\infty} x_n$ be a series of real numbers. Let $A = \limsup \sqrt[n]{|x_n|}$. Then

- 1 If $A < 1$, then $\sum_{n=1}^{\infty} x_n$ is absolutely convergent.
- 2 If $A > 1$, then $\sum_{n=1}^{\infty} x_n$ is divergent.
- 3 The test is inconclusive if $A = 1$.

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Remark: If $\sqrt[n]{|x_n|} \rightarrow \ell$, then $A = \limsup \sqrt[n]{|x_n|} = \ell$.

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Example

- ① The series $\sum_{n=1}^{\infty} \frac{(n!)^n}{n^{n^2}}$ is convergent.
- ② The series $\sum_{n=1}^{\infty} \frac{5^n}{3^n + 4^n}$ is not convergent.

Rearrangement of series

A series $\sum_{n=1}^{\infty} y_n$ is called a **rearrangement** of a series $\sum_{n=1}^{\infty} x_n$ if there is a bijection f of \mathbb{N} onto \mathbb{N} such that $y_n = x_{f(n)}$ for all $n \in \mathbb{N}$.

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Example (Tutorial problem): By Leibniz's test, let

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However,

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \cdots = \frac{3}{2}s$$

Theorem (Riemann's rearrangement theorem)

Let $\sum_{n=1}^{\infty} x_n$ be a conditionally convergent series.

- 1 If $s \in \mathbb{R}$, then there exists a rearrangement of terms of $\sum_{n=1}^{\infty} x_n$ such that the rearranged series has the sum s .
- 2 There exists a rearrangement of terms of $\sum_{n=1}^{\infty} x_n$ such that the rearranged series diverges.