MA 101 (Mathematics-I)

Introduction

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Therefore, the rationals $1.4, 1.41, 1.414, 1.4142, 1.41421, \ldots$ are getting closer and closer to $\sqrt{2}$.

Definition (Sequence)

A sequence of real numbers or a sequence in $\mathbb{R}$ is a mapping $f : \mathbb{N} \to \mathbb{R}$. We write x_n for $f(n), n \in \mathbb{N}$ and it is customary to denote a sequence as $\langle x_n \rangle$ or (x_n) or $\{x_n\}$.

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Example

There are different ways of expressing a sequence. For example:

1. *Constant sequence:* $(a, a, a, \ldots)$, *where* $a \in \mathbb{R}$
2. *Sequence defined by listing:* $(1, 4, 8, 11, 52, \ldots)$
3. *Sequence defined by rule:* (x_n), *where* $x_n = 3n^2$ *for all* $n \in \mathbb{N}$
4. *Sequence defined recursively:* (x_n), *where* $x_1 = 4$ *and* $x_{n+1} = 2x_n - 5$ *for all* $n \in \mathbb{N}$

Convergence: What does it mean?

Think of the examples:

1. $(2, 2, 2, \ldots)$
2. $(\frac{1}{n})$
3. $((-1)^n \frac{1}{n})$
4. $(1, 2, 1, 2, \ldots)$
5. $(\sqrt{n})$
6. $((-1)^n (1 - \frac{1}{n}))$

Definition (Convergent sequence)

A sequence (x_n) is said to be convergent if there exists $\ell \in \mathbb{R}$ such that for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ satisfying $|x_n - \ell| < \varepsilon$ for all $n \geq n_0$. We say that ℓ is a limit of (x_n).

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Theorem

Limit of a convergent sequence is unique.

Example

Using the definition of convergence of a sequence, show that $\lim_{n\to\infty} \frac{1}{n} = 0$.

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Example

Consider the sequence (x_n) *where* $x_n = (-1)^n$*. The terms of the sequence are* $-1, 1, -1, 1, -1, 1, \ldots$*. It is intuitively clear that this sequence does not approach to any real number. Therefore, the sequence does not converge. We now establish this fact by using the definition.*

Bounded sequence: Given a sequence (x_n), we can ask whether the set $\{x_1, x_2, x_3, \ldots\}$ is bounded or not. If this set is bounded then we call that the sequence (x_n) is bounded. Equivalently, the sequence (x_n) is bounded if there is a positive number M such that $|x_n| \leq M$ for all $n \in \mathbb{N}$. If (x_n) is not bounded then it is said to be unbounded.

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Theorem

Every convergent sequence is bounded.

Remark

- *From the above theorem, it follows that if a sequence is not bounded then it is not convergent. For example, the sequence $(\sqrt{n})$ is unbounded and hence is not convergent.*
- *Every bounded sequence is not convergent. For example, $((-1)^n)$ is a bounded sequence but it does not converge.*

Limit rules for convergent sequences

Theorem

Let $x_n \to x$ *and* $y_n \to y$. *Then*

(a) $x_n + y_n \to x + y$.

(b) $\alpha x_n \to \alpha x$ *for all* $\alpha \in \mathbb{R}$.

(c) $|x_n| \to |x|$.

(d) $x_n y_n \to xy$.

(e) $\frac{x_n}{y_n} \to \frac{x}{y}$ *if* $y_n \neq 0$ *for all* $n \in \mathbb{N}$ *and* $y \neq 0$.

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Example

The sequence $\left(\frac{2n^2-3n}{3n^2+5n+3}\right)$ *is convergent with limit* $\frac{2}{3}$.

Sandwich theorem

Theorem (Sandwich theorem)

Let (x_n), (y_n), (z_n) be sequences such that $x_n \leq y_n \leq z_n$ for all $n \geq n_0$ for some $n_0 \in \mathbb{N}$. If both (x_n) and (z_n) converge to the same limit ℓ, then (y_n) also converges to ℓ.

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$$\lim_{n\to\infty} \frac{\cos n}{n} = 0.$$

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Example

If $\alpha > 0$, then the sequence $(\alpha^{\frac{1}{n}})$ converges to 1.

Example

The sequence $(n^{\frac{1}{n}})$ *converges to* 1.

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If $p > 0$ *and* $\alpha \in \mathbb{R}$, *then* $\dfrac{n^{\alpha}}{(1+p)^n} \to 0$.

Example

The sequence $(n^{\frac{1}{n}})$ *converges to* 1.

Example

If $p > 0$ *and* $\alpha \in \mathbb{R}$, *then* $\dfrac{n^{\alpha}}{(1+p)^n} \to 0$.

Theorem

Let $r \in \mathbb{R}$. *Then there exists a sequence* (x_n) *of rational numbers such that* $\lim\limits_{n\to\infty} x_n = r$.

Divergent sequences

A sequence (x_n) is said to be divergent if it has no limit.

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Example

- *If* (x_n) *is unbounded then it is divergent.*
- $(\sqrt{n})$, $(3n^2)$, $((-1)^n n^3)$ *are all divergent.*

Example

The sequence $((-1)^n)$ *is not convergent, and so it is a divergent sequence although it is bounded.*

Definition

A sequence (x_n) is said to approach infinity or diverges to infinity if for any real number $M > 0$, there is a positive integer n_0 such that $a_n \geq M$ for all $n \geq n_0$. Similarly, (x_n) is said to approach $-\infty$ or diverges to $-\infty$ if for any real number $M > 0$, there is a positive integer n_0 such that $a_n \leq -M$ for all $n \geq n_0$.

Monotone sequence

Definition

A sequence (x_n) is said to be increasing if $x_{n+1} \geq x_n$ for all $n \in \mathbb{N}$. Similarly, (x_n) is said to be decreasing if $x_{n+1} \leq x_n$ for all $n \in \mathbb{N}$. We say that (x_n) is monotonic if it is either increasing or decreasing.

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1. The sequence $(\frac{1}{n})$ is decreasing.
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3. The sequence $(\cos \frac{n\pi}{3})$ is not monotonic.

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3. The sequence $(\cos \frac{n\pi}{3})$ is not monotonic.
4. The sequence $((-1)^n)$ is not monotonic.

Convergence of Monotone sequences

Theorem

If (x_n) is increasing and not bounded above then (x_n) diverges to ∞. If (x_n) is decreasing and not bounded below then (x_n) diverges to $-\infty$.

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Theorem (Monotone convergence theorem)

Let (x_n) be a sequence of real numbers.

(a) *If (x_n) is increasing and bounded above then (x_n) converges to $\sup\{x_n : n \in \mathbb{N}\}$.*

(b) *If (x_n) is decreasing and bounded below then (x_n) converges to $\inf\{x_n : n \in \mathbb{N}\}$.*

(c) *A monotonic sequence converges if and only if it is bounded.*

Example

Let $x_1 = 1$ *and* $x_{n+1} = \frac{1}{3}(x_n + 1)$ *for all* $n \in \mathbb{N}$. *Then the sequence* (x_n) *is convergent and* $\lim_{n\to\infty} x_n = \frac{1}{2}$.

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Example

Let $x_1 = 1$ *and* $x_{n+1} = \frac{3}{x_n}$ *for all* $n \geq 2$. *Then* (x_n) *diverges.*

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Solution: Let $a_n = (1+1/n)^n$. Then

$$x_n = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{1}{n}\right)^k = 1 + \frac{n}{1} \cdot \frac{1}{n} + \frac{n(n-1)}{2!} \cdot \frac{1}{n^2} + \cdots + \frac{n(n-1)\cdots 2 \cdot 1}{n!} \cdot \frac{1}{n^n}$$

$$= 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \cdots + \frac{1}{n!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{n-1}{n}\right).$$

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Therefore, we have

$$2 \le x_1 < x_2 < \cdots < x_n < x_{n+1} < \cdots$$

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Therefore, we have

$$2 \leq x_1 < x_2 < \cdots < x_n < x_{n+1} < \cdots$$

For $n > 1$, we have

$$2 < x_n < 1 + 1 + \left(\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}}\right) < 2 + \left(1 - \frac{1}{2^{n-1}}\right) < 3.$$

Theorem

Let $x_n \neq 0$ *for all* $n \in \mathbb{N}$ *and let* $L = \lim_{n\to\infty} \left|\frac{x_{n+1}}{x_n}\right|$ *exist.*

1. *If* $L < 1$, *then* $x_n \to 0$.
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Remark

If $L = \lim_{n\to\infty} \left|\frac{x_{n+1}}{x_n}\right| = 1$*, then* (x_n) *may converge or diverge. For example, the sequence* $((-1)^n)$ *diverges and* $L = 1$*. For any nonzero constant sequence,* $L = 1$ *and constant sequences are convergent.*

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Example

If $\alpha \in \mathbb{R}$, *then the sequence* $\left(\frac{\alpha^n}{n!}\right)$ *is convergent.*

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If $L = \lim_{n\to\infty} \left|\frac{x_{n+1}}{x_n}\right| = 1$, then (x_n) may converge or diverge. For example, the sequence $((-1)^n)$ diverges and $L = 1$. For any nonzero constant sequence, $L = 1$ and constant sequences are convergent.

Example

If $\alpha \in \mathbb{R}$, then the sequence $\left(\frac{\alpha^n}{n!}\right)$ is convergent.

Example

The sequence $\left(\frac{2^n}{n^4}\right)$ is not convergent.

Subsequence

Definition (Subsequence)

Let (x_n) be a sequence in $\mathbb{R}$. If (n_k) is a sequence of positive integers such that $n_1 < n_2 < n_3 < \cdots$, then (x_{n_k}) is called a subsequence of (x_n).

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Theorem: If a sequence (x_n) converges to ℓ, then every subsequence of (x_n) must converge to ℓ.

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Theorem: If a sequence (x_n) converges to ℓ, then every subsequence of (x_n) must converge to ℓ.

Remark: From the above theorem, we have the following:

- If (x_n) has a subsequence (x_{n_k}) such that $x_{n_k} \not\to \ell$, then $x_n \not\to \ell$.
- If (x_n) has two subsequences converging to two different limits, then (x_n) cannot be convergent.

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1. If $x_n = (-1)^n(1 - \frac{1}{n})$ for all $n \in \mathbb{N}$, then (x_n) is not convergent.
2. Let (x_n) be a sequence in $\mathbb{R}$. Then (x_{2n}) and (x_{2n-1}) are two subsequences of (x_n). Suppose that $x_{2n} \to \ell \in \mathbb{R}$ and $x_{2n-1} \to \ell$. Then $x_n \to \ell$.

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3. The sequence $(1, \frac{1}{2}, 1, \frac{2}{3}, 1, \frac{3}{4}, \ldots)$ converges to 1.

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Theorem

Every sequence of real numbers has a monotone subsequence.

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3. The sequence $(1, \frac{1}{2}, 1, \frac{2}{3}, 1, \frac{3}{4}, \ldots)$ converges to 1.

Theorem

Every sequence of real numbers has a monotone subsequence.

Theorem (Bolzano-Weierstrass Theorem)

Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.

Cauchy sequence

Definition (Cauchy sequence)

A sequence (x_n) is called a Cauchy sequence if for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $|x_m - x_n| < \varepsilon$ for all $m, n \geq n_0$.

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Theorem (Cauchy's criterion for convergence)

A sequence in $\mathbb{R}$ is convergent if and only if it is a Cauchy sequence.

Cauchy sequence

Theorem

Let (x_n) satisfy either of the following conditions:

1. $|x_{n+1} - x_n| \leq \alpha^n$ *for all* $n \in \mathbb{N}$
2. $|x_{n+2} - x_{n+1}| \leq \alpha |x_{n+1} - x_n|$ *for all* $n \in \mathbb{N}$,

where $0 < \alpha < 1$. *Then (x_n) is a Cauchy sequence.*

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Proof of (1).

For all $m, n \in \mathbb{N}$ with $m > n$, we have

$$\begin{aligned} |x_m - x_n| &\leq |x_n - x_{n+1}| + |x_{n+1} - x_{n+2}| + \cdots + |x_{m-1} - x_m| \\ &\leq \alpha^n + \alpha^{n+1} + \cdots + \alpha^{m-1} \\ &= \frac{\alpha^n}{1-\alpha}(1 - \alpha^{m-n}) < \frac{\alpha^n}{1-\alpha} \end{aligned}$$

$\square$

Cauchy sequence

Proof of (2) For all $m, n \in \mathbb{N}$ with $m > n$, we have

$$
\begin{aligned}
|x_m - x_n| &\le |x_n - x_{n+1}| + |x_{n+1} - x_{n+2}| + \cdots + |x_{m-1} - x_m| \\
&\le (\alpha^{n-1} + \alpha^n + \cdots + \alpha^{m-2})|x_2 - x_1| \\
&= \frac{\alpha^{n-1}}{1-\alpha}(1 - \alpha^{m-n})|x_2 - x_1| \le \frac{\alpha^{n-1}}{1-\alpha}|x_2 - x_1|
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Example: Let (x_n) be a sequence defined as $x_1 = 1$ and $x_{n+1} = 1 + \frac{1}{x_n}$ for $n \in \mathbb{N}$. Then $x_{n+1}x_n = 1 + x_n > 2$. Now,

$$|x_{n+2} - x_{n+1}| = \left|\frac{x_{n+1} - x_n}{x_{n+1}x_n}\right| < \frac{1}{2}|x_{n+1} - x_n|.$$

Hence, (x_n) is a Cauchy sequence.

Limit superior

Let (x_n) be a bounded sequence. Let $y_1 = \sup\{x_1, x_2, \ldots\}$, $y_2 = \sup\{x_2, x_3, \ldots\}$, and so on. That is, for $n \in \mathbb{N}$,

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Remark: Suppose that $|x_n| < M$ for $n \in \mathbb{N}$. Then $-M \leq z_n \leq y_n \leq M$ for all n. Hence,

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Example: Consider the sequence (x_n), where $x_n = (-1)^n$. Clearly, for any n, $y_n = \sup\{x_n, x_{n+1}, \ldots\} = 1$ and $z_n = \inf\{x_n, x_{n+1}, \ldots\} = -1$. Hence,

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Example: Consider the sequence (x_n), where $x_n = \frac{1}{n}$. Clearly, for any n, $y_n = \sup\{\frac{1}{k} : k \geq n\} = \frac{1}{n}$ and $z_n = \inf\{\frac{1}{k} : k \geq n\} = 0$.
Hence, $\limsup x_n = \lim\limits_{n\to\infty} y_n = \lim\limits_{n\to\infty} \frac{1}{n} = 0$ and $\liminf x_n = 0$.

Theorem

Let (a_n) and (b_n) be two bounded sequences.

1. $\liminf a_n \leq \limsup a_n$.
2. *If* $a_n \leq b_n$ *for all* $n \in \mathbb{N}$, *then* $\limsup a_n \leq \limsup b_n$ *and* $\liminf a_n \leq \liminf b_n$.
3. $\limsup(a_n + b_n) \leq \limsup a_n + \limsup b_n$ *and* $\liminf(a_n + b_n) \geq \liminf a_n + \liminf b_n$.

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If (a_n) is a convergent sequence, then

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