Indian Institute of Technology Guwahati
Probability Theory (MA 683)
Problem Set 04

1. Prove that
   (a) $(X+Y)^+ \leq X^+ + Y^+$
   (b) $(X+Y)^- \leq X^- + Y^-$
   (c) $X^+ \leq (X+Y)^+ + Y^-$
2. Show that a monotone function $f : \mathbb{R} \to \mathbb{R}$ is Borel measurable.
3. Let X be a discrete random variable such that $P(X = x_i) = p_i$, $i = 1, 2, \ldots$, and $\sum_{i=1}^{\infty} p_i = 1$. Then show that $EX = \sum_{i=1}^{\infty} x_i p_i$ provided X is integrable.
4. Show that $|E(X)| \leq E(|X|)$ for any random variable X.
5. Let X be simple random variable with n different values $x_1, x_2, \ldots, x_n$. Show that for any Borel measurable function $g : \mathbb{R} \to \mathbb{R}$,
   $$E(g(X)) = \sum_{i=1}^{n} g(x_i) f(x_i),$$
   where $f(x) = P(\{X = x\})$ is the mass function.
6. Show that $E(\mathbb{I}_A) = P(A)$ for any event A. Verify the identity $1 - \mathbb{I}_{A \cup B} = (1 - \mathbb{I}_A)(1 - \mathbb{I}_B)$ and use it to prove that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
7. Show that $E\,|X|^r = r \displaystyle\int_0^{\infty} t^{r-1} P(|X| > t) dt$.
8. If $E|X| < \infty$ and $\lim_{n\to\infty} P(A_n) = 0$, then $\lim_{n\to\infty} \displaystyle\int_{A_n} X dP = 0$.
9. Every integrable and symmetric (with respect to 0) random variable has mean 0.
10. Give an example to show that $E\left(\sum_{n=1}^{\infty} X_n\right) = \sum_{n=1}^{\infty} EX_n$ is not true in general.
11. If X is a non-negative random variable in $\mathcal{L}_p$ for all $p > 0$, and
    $$g(p) = \log EX^p, \text{ for all } p \geq 0,$$
    then g is convex on $[0, \infty)$.
12. Give an example of three events A_1, A_2, and A_3 on some probability space such that they are pairwise independent but not independent.
13. If the events $\{E_\alpha : \alpha \in A\}$ are independent, then so are the events $\{F_\alpha : \alpha \in A\}$, where each F_α may be E_α or E_α^c.

14. For any random variable X and positive numbers a and t, show that

$$P(X \geq a) \leq e^{-at} Ee^{tX}.$$

15. For any $X_i \in \mathcal{L}_2$ for $i = 1, 2, \ldots, n$, show that

$$Var\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} Var(X_i) + 2 \sum_{1 \leq i < j \leq n} Cov(X_i, X_j).$$

16. If $X_i \in \mathcal{L}_2$ for $i = 1, 2, \ldots, n$ such that $Cov(X_i, X_j) = 0$ for all $\neq j$, then show that

$$Var\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} Var(X_i).$$

17. Prove that if f and g are non-decreasing functions and X is a random variable with $Ef(X)$, $Eg(X)$, and $Ef(X)g(X)$ finite, then $Cov(f(X), g(X)) \geq 0$.

18. If $X' = aX + b$ and $Y' = cX + d$, verify that $\rho(X', Y') = \pm\rho(X, Y)$ according as $ac > 0$ or $ac < 0$.

19. If X, Y are random variables such that $0 < Var(X), Var(Y) < \infty$, then $\rho(X, Y) = 1$ iff

$$\frac{X - EX}{\sigma_X} = \frac{Y - EY}{\sigma_Y} \quad a.e.\ (P).$$

20. If $X = \sin Z$ and $Y = \cos Z$, where $P(Z = \pm 1) = P(Z = \pm 2) = \frac{1}{4}$, then $\rho(X, Y) = 0$ despite X, Y being functionally related.

21. For arbitrary real numbers a_i, b_i $(1 \leq i \leq n)$, prove that

$$\sum_{i=1}^{n} |a_i b_i| \leq \left(\sum_{i=1}^{n} |a_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |b_i|^q\right)^{\frac{1}{q}},$$

provided $\frac{1}{p} + \frac{1}{q} = 1$, $p > 0$, $q > 0$.

22. Show that for $0 < a < b < d$ and any non-negative random variable Y,

$$EY^b \leq (EY^a)^{\frac{d-b}{d-a}} \left(EY^d\right)^{\frac{b-a}{d-a}}.$$

23. Let $\{X_n\}_{n\geq 1}$ be a sequence of positive random variables and $\{c_n\}_{n\geq 1}$ be a sequence of positive real constants. If $\sum_{n=1}^{\infty} c_n^{\alpha} EY_n^{\alpha} < \infty$ for $\alpha = \alpha_1$ and $\alpha = \alpha_2$ $(0 < \alpha_1 < \alpha_2)$, then show that $\sum_{n=1}^{\infty} c_n^{\alpha} EY_n^{\alpha} < \infty$ for all $\alpha \in [\alpha_1, \alpha_2]$.

24. (Minkowski's inequality) If $X_1 \in \mathcal{L}_p$ and $X_2 \in \mathcal{L}_p$ for some $p \geq 1$, then

$$(E|X_1 + X_2|^p)^{\frac{1}{p}} \leq (E|X_1|^p)^{\frac{1}{p}} + (E|X_2|^p)^{\frac{1}{p}}.$$

[Hint: Apply Holder's inequality to $E|X_i||X_1 + X_2|^{p-1}$.]

25. Let X be a random variable defined on a probability space $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+), P)$ such that $\lim_{\omega\to\infty} X(\omega) = \theta \in \mathbb{R}$. For $n \geq 1$, define

$$X_n(\omega) = X(n\omega) \text{ for all } \omega \in \mathbb{R}_+.$$

Then show that

$$\lim_{n\to\infty} \int_0^a X_n dP = \theta P([0, a]) \text{ for all } a > 0.$$

26. Show that
    (a) $\{X_n\}_{n\geq 1}$ is u.i. iff $\{|X_n|\}_{n\geq 1}$ is u.i.
    (b) If $\{X_n\}_{n\geq 1}$ and $\{Y_n\}_{n\geq 1}$ are u.i., so is $\{X_n + Y_n\}_{n\geq 1}$.
    (c) If $\{X_n\}_{n\geq 1}$ is u.i., so is any sub sequence of $\{X_n\}_{n\geq 1}$.
    (d) $\{X_n\}_{n\geq 1}$ is u.i. iff it is u.i. from above and from below.
    (e) If $|X_n| \leq Y$ for all $n \geq 1$, where $EY < \infty$, then $\{X_n\}_{n\geq 1}$ is u.i.

27. With the help of an example, show that boundedness in $\mathcal{L}_1$ is not enough for uniform integrability.

28. If $\left\{|X_n|^\beta\right\}_{n\geq 1}$ is u.i. for some $\beta \geq 1$ and $S_n = \sum_{i=1}^n X_i$, then $\left\{\left|\frac{S_n}{n}\right|^\beta\right\}_{n\geq 1}$ is uniformly integrable. In particular, if $\{X_n\}_{n\geq 1}$ are identically distributed random variables in $\mathcal{L}_p$ for some $p \geq 1$, then $\left\{\left|\frac{S_n}{n}\right|^p\right\}_{n\geq 1}$ is uniformly integrable.

29. Let $\{X_n\}_{n\geq 1}$ be a sequence of independent random variables with $EX_n = 0$ and $EX_n^2 = 1$. If $\{X_n^2\}_{n\geq 1}$ is uniformly integrable, then $\left\{\frac{S_n^2}{n}\right\}_{n\geq 1}$ is uniformly integrable, where $S_n = \sum_{i=1}^n X_i$.

30. Show that a family of random variables Υ is uniformly integrable if and only if there exists a Borel function $\varphi : [0, \infty) \to [0, \infty)$ such that

$$\lim_{x\to\infty} \frac{\varphi(x)}{x} = \infty \qquad \text{and} \qquad \sup_{X\in\Upsilon} E(\varphi(|X|)) < \infty.$$

Moreover, if it exists, the function φ can be chosen in the class of non-decreasing convex functions.
Note 1: A Borel function φ is called a test function of uniform integrability if $\lim\limits_{x\to\infty} \dfrac{\varphi(x)}{x} = \infty$.
Note 2: The sequence of random variables $\{X_n\}_{n\geq 1}$ in the definition of u.i. and in u.i. criterion can be replaced by a family of random variables, say Υ. In this case, $\sup_{n\geq 1}$ should be replaced by $\sup_{X\in\Upsilon}$.

31. For $p > 1$, let Υ be a nonempty family of random variables bounded in $\mathcal{L}_p$, *i.e.*, such that $\sup_{X\in\Upsilon} E|X|^p < \infty$. Then Υ is uniformly integrable.

32. Let Υ be a nonempty uniformly integrable family of random variables. Show that $conv\,\Upsilon$ is uniformly integrable, where $conv\,\Upsilon$ is the set of all random variables of the form $X = \sum_{i=1}^n \alpha_i X_i$, for some $n \in \mathbb{N}$, $\alpha_i \geq 0$, $i = 1, 2, \ldots, n$, $\sum_{i=1}^n \alpha_i = 1$ and $X_1, X_2, \ldots, X_n \in \Upsilon$.