

# MA 101 (Mathematics-I)

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# Introduction

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$f : D \rightarrow \mathbb{R}$  is said to be differentiable if  $f$  is differentiable at each  $x_0 \in D$ .

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(c)  $f(x) = \begin{cases} x^2 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$  Then  $f : \mathbb{R} \rightarrow \mathbb{R}$  is differentiable only at 0 and  $f'(0) = 0$ .

## Theorem (Rules for finding derivatives)

*Let  $D \subseteq \mathbb{R}$  and let  $x_0 \in D$  be an interior point. Suppose  $f, g : D \rightarrow \mathbb{R}$  are differentiable at  $x_0$ . Then*

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- (d) (Quotient rule) If  $g(x_0) \neq 0$ , then the function  $f/g$  is differentiable at  $x_0$  and

$$(f/g)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{(g(x_0))^2}.$$

## Theorem (Chain rule for derivative)

*Let  $f : D \rightarrow \mathbb{R}$  and  $g : E \rightarrow \mathbb{R}$ . Let  $f(D) \subseteq E$ . Suppose that  $f$  is differentiable at  $x_0$  and  $g$  is differentiable at  $f(x_0)$ . Then  $g \circ f$  is differentiable at  $x_0$  and  $(g \circ f)'(x_0) = g'(f(x_0))f'(x_0)$ .*



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We have  $\lim_{y \rightarrow f(x_0)} h(y) = g'(f(x_0)) = h(f(x_0))$ , and  $g(y) - g(f(x_0)) = h(y) \times (y - f(x_0))$  for all  $y \in E$ .

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Hence, for  $x \neq x_0$ ,

$$\frac{g(f(x)) - g(f(x_0))}{x - x_0} = h(f(x)) \frac{f(x) - f(x_0)}{x - x_0}$$

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Clearly  $f$  is differentiable at all  $x (\neq 0) \in \mathbb{R}$  and  $f'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}$  for all  $x (\neq 0) \in \mathbb{R}$ .

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For each  $\varepsilon > 0$ , choosing  $\delta = \varepsilon > 0$ , we find that

$$\left| \frac{f(x) - f(0)}{x - 0} \right| = \left| x \sin \frac{1}{x} \right| \leq |x| < \varepsilon \text{ for all } x \in \mathbb{R} \text{ satisfying } 0 < |x| < \delta. \text{ Hence, } f \text{ is differentiable at } 0 \text{ and } f'(0) = 0.$$

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Since  $\frac{1}{2n\pi} \rightarrow 0$  but  $f'(\frac{1}{2n\pi}) \rightarrow -1 \neq f'(0)$ ,  $f' : \mathbb{R} \rightarrow \mathbb{R}$  is not continuous at 0.

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- b The equation  $x^4 + 2x^2 - 6x + 2 = 0$  has exactly two real roots.

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- f  $f'(x) \neq 0$  for all  $x \in I \Rightarrow f$  is one-one on  $I$ .

**Remark:** Note that  $f$  is strictly increasing on  $I$  need not imply that  $f'(x) > 0$  for all  $x \in I$ . For example,  $f : \mathbb{R} \rightarrow \mathbb{R}$  defined by  $f(x) = x^3$  is strictly increasing but  $f'(0) = 0$ .

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- c If  $f(x) = x^3 + x^2 - 5x + 3$  for all  $x \in \mathbb{R}$ , then  $f$  is one-one on  $[1, 5]$  but not one-one on  $\mathbb{R}$ .



## L'Hôpital's rules:

- (1) Let  $f : (a, b) \rightarrow \mathbb{R}$  and  $g : (a, b) \rightarrow \mathbb{R}$  be differentiable at  $x_0 \in (a, b)$ . Also, let  $f(x_0) = g(x_0) = 0$  and  $g'(x_0) \neq 0$ .  
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- (2) Let  $f : (a, b) \rightarrow \mathbb{R}$  and  $g : (a, b) \rightarrow \mathbb{R}$  be differentiable such that  $\lim_{x \rightarrow a+} f(x) = \lim_{x \rightarrow a+} g(x) = 0$  and  $g'(x) \neq 0$  for all  $x \in (a, b)$ . If  $\lim_{x \rightarrow a+} \frac{f'(x)}{g'(x)} = \ell$ , then  $\lim_{x \rightarrow a+} \frac{f(x)}{g(x)} = \ell$ .

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Examples: (a)  $\lim_{x \rightarrow 0} \frac{\sqrt{1+x}-1}{x}$

(b)  $\lim_{x \rightarrow \frac{\pi}{2}} \frac{1-\sin x}{1+\cos 2x}$

**Theorem (Darboux's Theorem)** Let  $I$  be an interval and let  $f : I \rightarrow \mathbb{R}$  be differentiable. Let  $a < b$  be two points in  $I$  and  $k$  is a number between  $f'(a)$  and  $f'(b)$ , then there is atleast one point  $c \in (a, b)$  such that  $f'(c) = k$ .

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**Example:**  $f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$

The derivative of  $f$  is an example of a function, which is not continuous but satisfies intermediate value property.

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### Theorem (Taylor's theorem)

Let  $f : [a, b] \rightarrow \mathbb{R}$  be such that  $f$  and its derivatives  $f', f'', \dots, f^{(n)}$  are continuous on  $[a, b]$  and that  $f^{(n+1)}$  exists on  $(a, b)$ . If  $x_0 \in [a, b]$ , then for any  $x \in [a, b]$  there exists a point  $c$  between  $x$  and  $x_0$  such that

$$\begin{aligned} f(x) &= f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots \\ &\quad \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1} \\ &= P_n(x) + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1} \end{aligned}$$



**Proof:** Let  $x$  and  $x_0$  be given and let  $J$  denote the closed interval with endpoints  $x_0$  and  $x$ . Define  $F$  on  $J$  by

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We have  $F(x) = 0$  and  $F'(t) = -\frac{(x - t)^n}{n!} f^{(n+1)}(t).$

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Then  $f : [0, x] \rightarrow \mathbb{R}$  is twice differentiable and  $f'(t) = \frac{1}{2\sqrt{1+t}}$ ,  
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Since  $0 < \frac{1}{(1+c)^{3/2}} < 1$ , we get  $1 + \frac{x}{2} - \frac{x^2}{8} \leq \sqrt{1+x} \leq 1 + \frac{x}{2}$ .

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$f'(x) = 5x^4 - 20x^3 + 15x^2$  and zeros of  $f'$  are 0, 1, and 3.

$f''(x) = 20x^3 - 60x^2 + 30x$  and  $f''(0) = 0$ ,  $f''(1) = -10 < 0$ ,  $f''(3) = 90 > 0$ .

$f'''(x) = 60x^2 - 120x + 30$  and  $f'''(0) = 30 > 0$ .

**Power series:** A series of the form  $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ , where  $x_0$ ,  $a_n \in \mathbb{R}$  for  $n = 0, 1, 2, \dots$  and  $x \in \mathbb{R}$ .



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**Radius of convergence:** For every power series  $\sum_{n=0}^{\infty} a_n x^n$ , there exists a unique  $R$  satisfying  $0 \leq R \leq \infty$  such that the series converges absolutely if  $|x| < R$  and diverges if  $|x| > R$ .

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Consider the power series  $\sum_{n=0}^{\infty} a_n x^n$ . Let  $\beta = \limsup \sqrt[n]{|a_n|}$  and  $R = \frac{1}{\beta}$  (we define  $R = 0$  if  $\beta = \infty$  and  $R = \infty$  if  $\beta = 0$ ).

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Thus, the power series converges absolutely for  $|x| < 1$ , that is, for  $x \in (-1, 1)$  and diverges for  $|x| > 1$ , that is for  $x \in (-\infty, -1) \cup (1, \infty)$ .

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Again, if  $|x| = 1$ , then  $\sum_{n=1}^{\infty} \left|\frac{x^n}{n^2}\right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$  is convergent.

Therefore, the interval of convergence of the given power series is  $[-1, 1]$ .

Example:  $\sum_{n=1}^{\infty} \frac{(-1)^n}{n \cdot 4^n} (x-1)^n$

Let  $a_n = \frac{(-1)^n}{n \cdot 4^n}$  for all  $n \in \mathbb{N}$ . Clearly,  $\beta = \lim \left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{4}$ .



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Therefore, the radius of convergence of the given power series is 4. Thus the given series is convergent (absolutely) if  $|x - 1| < 4$ , that is, if  $x \in (-3, 5)$  and is not convergent if  $|x - 1| > 4$ , that is, if  $x \in (-\infty, -3) \cup (5, \infty)$ .

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If  $x = 5$ , then  $\sum_{n=1}^{\infty} a_n \cdot 4^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$  is convergent by Leibniz test. Therefore, the interval of convergence of the given power series is  $(-3, 5]$ .

## Theorem (Term by term differentiation of power series)

*A power series can be differentiated term by term within the interval of convergence. In fact, if  $f(x) = \sum_{n=0}^{\infty} a_n x^n$  has radius of convergence  $R$ , then  $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$  for  $|x| < R$ .*

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To prove that the series  $\sum_{n=1}^{\infty} n a_n x^{n-1}$  converges to  $f'(x)$  requires another concept called **uniform convergence** which is beyond the scope of this course.

## Taylor Series

If a function  $f$  has derivatives of all orders at a point  $x_0 \in \mathbb{R}$ , then we can calculate the Taylor coefficients by

$$a_0 = f(x_0), a_n = \frac{f^{(n)}(x_0)}{n!} \text{ for all } n \in \mathbb{N}.$$



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The power series  $\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$  converges to  $f(x)$  for  $|x - x_0| < R$  if and only if the sequence  $(R_n(x))$  of remainders converges to 0 for each  $x$  in  $|x - x_0| < R$ .

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$$a_0 = f(x_0), a_n = \frac{f^{(n)}(x_0)}{n!} \text{ for all } n \in \mathbb{N}.$$

In this way, we obtain a power series  $\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$ .

The power series  $\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$  converges to  $f(x)$  for  $|x - x_0| < R$  if and only if the sequence  $(R_n(x))$  of remainders converges to 0 for each  $x$  in  $|x - x_0| < R$ .

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The Taylor series of a function  $f$  at  $x_0 = 0$  is known as **Maclaurin's series**.

**Example:** The Maclaurin series  $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n-1}}{(2n-1)!}$  of  $\sin x$  converges to  $\sin x$  for all  $x \in \mathbb{R}$ .

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If  $f(x) = \sin x$  for all  $x \in \mathbb{R}$ , then  $f : \mathbb{R} \rightarrow \mathbb{R}$  is infinitely differentiable and  $f^{(2n-1)}(x) = (-1)^{n+1} \cos x$ ,  
 $f^{(2n)}(x) = (-1)^n \sin x$  for all  $x \in \mathbb{R}$  and for all  $n \in \mathbb{N}$ .

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For  $x = 0$ , the Maclaurin series of  $\sin x$  becomes  $0 - 0 + 0 - \dots$ , which clearly converges to  $\sin 0 = 0$ .

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Let  $x (\neq 0) \in \mathbb{R}$ . The remainder term in the Taylor expansion of  $\sin x$  about the point 0 is given by  $R_n(x) = \frac{x^{n+1}}{(n+1)!} f^{(n+1)}(c_n)$ , where  $c_n$  lies between 0 and  $x$ .



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It follows that  $\lim_{n \rightarrow \infty} R_n(x) = 0$ .

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$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

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The Taylor's series of  $f$  about 0 is  $0 + 0 + 0 + \dots$ , which converges to 0, and thus, not to  $f(x)$  for any  $x \neq 0$ .

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Thus, an infinitely differentiable function may not have Taylor series representation.