MA 101 (Mathematics-I)

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$f : D \to \mathbb{R}$ is said to be differentiable if f is differentiable at each $x_0 \in D$.

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(c) $f(x) = \begin{cases} x^2 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$ *Then $f : \mathbb{R} \to \mathbb{R}$ is differentiable only at 0 and* $f'(0) = 0$.

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(d) *(Quotient rule) If $g(x_0) \neq 0$, then the function f/g is differentiable at x_0 and*

$$(f/g)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{(g(x_0))^2}.$$

Theorem (Chain rule for derivative)

Let $f : D \to \mathbb{R}$ and $g : E \to \mathbb{R}$. Let $f(D) \subseteq E$. Suppose that f is differentiable at x_0 and g is differentiable at $f(x_0)$. Then $g \circ f$ is differentiable at x_0 and $(g \circ f)'(x_0) = g'(f(x_0))f'(x_0)$.

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Proof: We define a function $h : E \to \mathbb{R}$ by

$$h(y) = \begin{cases} \frac{g(y)-g(f(x_0))}{y-f(x_0)}, & y \neq f(x_0); \\ g'(f(x_0)), & y = f(x_0). \end{cases}$$

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We have $\lim\limits_{y \to f(x_0)} h(y) = g'(f(x_0)) = h(f(x_0))$, and $g(y) - g(f(x_0)) = h(y) \times (y - f(x_0))$ for all $y \in E$.

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Hence, for $x \neq x_0$,

$$\frac{g(f(x)) - g(f(x_0))}{x - x_0} = h(f(x))\frac{f(x) - f(x_0)}{x - x_0}$$

Example: Let $f : \mathbb{R} \to \mathbb{R}$ defined by
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Clearly f is differentiable at all $x \ (\neq 0) \in \mathbb{R}$ and $f'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}$ for all $x \ (\neq 0) \in \mathbb{R}$.

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For each $\varepsilon > 0$, choosing $\delta = \varepsilon > 0$, we find that $\left|\frac{f(x)-f(0)}{x-0}\right| = |x \sin \frac{1}{x}| \leq |x| < \varepsilon$ for all $x \in \mathbb{R}$ satisfying $0 < |x| < \delta$. Hence, f is differentiable at 0 and $f'(0) = 0$.

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Since $\frac{1}{2n\pi}\to 0$ but $f'(\frac{1}{2n\pi})\to -1\neq f'(0)$, $f':\mathbb{R}\to\mathbb{R}$ is not continuous at 0.

Definition: $f : D \to \mathbb{R}$ has a local maximum (resp. minimum) at $x_0 \in D$ if there exists $\delta > 0$ such that $f(x) \leq f(x_0)$ (resp. $f(x_0) \leq f(x)$) for all $x \in (x_0 - \delta, x_0 + \delta) \cap D$.

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Rolle's Theorem: Let $f : [a, b] \to \mathbb{R}$. Suppose that

(a) f is continuous on $[a, b]$.

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a. The equation $x^2 = x \sin x + \cos x$ has exactly two real roots.
b. The equation $x^4 + 2x^2 - 6x + 2 = 0$ has exactly two real roots.

Theorem (Mean value theorem) If $f : [a, b] \to \mathbb{R}$ is continuous and if f is differentiable on (a, b), then there exists $c \in (a, b)$ such that $f(b) - f(a) = f'(c)(b - a)$.

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e. $f'(x) < 0$ for all $x \in I \Rightarrow f$ is strictly decreasing on I.
f. $f'(x) \neq 0$ for all $x \in I \Rightarrow f$ is one-one on I.

Remark: Note that f is strictly increasing on I need not imply that $f'(x) > 0$ for all $x \in I$. For example, $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$ is strictly increasing but $f'(0) = 0$.

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b. $\sin x \geq x - \frac{x^3}{6}$ for all $x \in [0, \frac{\pi}{2}]$.
c. If $f(x) = x^3 + x^2 - 5x + 3$ for all $x \in \mathbb{R}$, then f is one-one on $[1, 5]$ but not one-one on $\mathbb{R}$.

L'Hôpital's rules:

(1) Let $f : (a, b) \to \mathbb{R}$ and $g : (a, b) \to \mathbb{R}$ be differentiable at $x_0 \in (a, b)$. Also, let $f(x_0) = g(x_0) = 0$ and $g'(x_0) \neq 0$. Then $\lim\limits_{x \to x_0} \frac{f(x)}{g(x)} = \frac{f'(x_0)}{g'(x_0)}$.

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Examples: (a) $\lim\limits_{x \to 0} \frac{\sqrt{1+x}-1}{x}$ (b) $\lim\limits_{x \to \frac{\pi}{2}} \frac{1-\sin x}{1+\cos 2x}$

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Theorem (Taylor's theorem)

Let $f : [a, b] \to \mathbb{R}$ be such that f and its derivatives $f', f'', \ldots, f^{(n)}$ are continuous on $[a, b]$ and that $f^{(n+1)}$ exists on (a, b). If $x_0 \in [a, b]$, then for any $x \in [a, b]$ there exists a point c between x and x_0 such that

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots$$

$$\cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}$$

$$= P_n(x) + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}$$

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We have $F'(t) = -\dfrac{(x-t)^n}{n!} f^{(n+1)}(t)$.

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Since $0 < \frac{1}{(1+c)^{3/2}} < 1$, we get $1 + \frac{x}{2} - \frac{x^2}{8} \leq \sqrt{1+x} \leq 1 + \frac{x}{2}$.

Theorem: Let $x_0 \in (a, b)$ and let $n \geq 2$. Also, let $f, f', ..., f^{(n)}$ be continuous on (a, b) and $f'(x_0) = f''(x_0) = \cdots = f^{(n-1)}(x_0) = 0$ but $f^{(n)}(x_0) \neq 0$.

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Example: Find local maximum and local minimum values of f, where $f(x) = x^5 - 5x^4 + 5x^3 + 12$ for all $x \in \mathbb{R}$.

Power series: A series of the form $\sum_{n=0}^{\infty} a_n(x-x_0)^n$, where x_0, $a_n \in \mathbb{R}$ for $n = 0, 1, 2, \ldots$ and $x \in \mathbb{R}$.

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Theorem:

a. If $\sum_{n=0}^{\infty} a_n x^n$ converges for $x = x_1 \neq 0$, then it converges absolutely for all $x \in \mathbb{R}$ satisfying $|x| < |x_1|$.
b. If $\sum_{n=0}^{\infty} a_n x^n$ diverges for $x = x_2$, then it diverges for all $x \in \mathbb{R}$ satisfying $|x| > |x_2|$.

Radius of convergence: For every power series $\sum_{n=0}^{\infty} a_n x^n$, there exists a unique R satisfying $0 \leq R \leq \infty$ such that the series converges absolutely if $|x| < R$ and diverges if $|x| > R$.

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Proof of (a) (Method-1): If $x = 0$, then the given series becomes $0 + 0 + \cdots$, which is clearly convergent. Let $x\ (\neq 0) \in \mathbb{R}$ and let $a_n = \frac{x^n}{n^2}$ for all $n \in \mathbb{N}$.

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Then $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = |x|$. Hence by ratio test, $\sum_{n=1}^{\infty} a_n$ is convergent (absolutely) if $|x| < 1$ and is not convergent if $|x| > 1$. Therefore the radius of convergence of the given power series is 1.

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Again, if $|x| = 1$, then $\sum\limits_{n=1}^{\infty} |a_n| = \sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ is convergent and hence $\sum\limits_{n=1}^{\infty} a_n$ is also convergent. Therefore the interval of convergence of the given power series is $[-1, 1]$.

Proof of (b) (Method-2): The given power series is $\sum_{n=1}^{\infty} \frac{(-1)^n}{n \cdot 4^n}(x-1)^n$. Let $a_n = \frac{(-1)^n}{n \cdot 4^n}$ for all $n \in \mathbb{N}$. Clearly, $\beta = \lim \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{4}$.

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Therefore the radius of convergence of the given power series is 4. Thus the given series is convergent (absolutely) if $|x-1| < 4$, that is, if $x \in (-3, 5)$ and is not convergent if $|x-1| > 4$, that is, if $x \in (-\infty, -3) \cup (5, \infty)$.

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If $x = 5$, then $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is convergent by Leibniz test.
Therefore the interval of convergence of the given power series is $(-3, 5]$.

Theorem (Term by term differentiation of power series)

A power series can be differentiated term by term within the interval of convergence. In fact, if $f(x) = \sum_{n=0}^{\infty} a_n x^n$ *has radius of convergence* R*, then* $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$ *for* $|x| < R$*.*

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To prove that the series $\sum_{n=1}^{\infty} n a_n x^{n-1}$ converges to $f'(x)$ requires another concept called uniform convergence which is beyond the scope of this course.

If a function f has derivatives of all orders at a point $c \in \mathbb{R}$, then we can calculate the Taylor coefficients by $a_0 = f(c), a_n = \frac{f^{(n)}(c)}{n!}$ for all $n \in \mathbb{N}$.

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The power series $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$ converges to $f(x)$ for $|x-c| < R$ if and only if the sequence $(R_n(x))$ of remainders converges to 0 for each x in $|x-c| < R$.

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The Taylor series of a function f at $c = 0$ is known as Maclaurin's series.

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It follows that $\displaystyle\lim_{n\to\infty} R_n(x) = 0$.

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Thus, an infinitely differentiable function may not have Taylor series representation.