**Indian Institute of Technology Guwahati**
**Probability Theory (MA590)**
**Problem Set 10**

1. Let $\{X_n\}$ be a sequence of random variables with $P\left(X_n = n\right) = 1 - \frac{1}{n}$ and $P\left(X_n = 0\right) = \frac{1}{n}$. Does $X_n$ converge to some random variable $X$ in distribution? [Note: This example shows that even if a sequence of distribution functions converges, it may not converge to a distribution function.]

2. Let $X_n \to X$ in $r$th mean, for some $r > 0$. Show that $X_n \to X$ in probability.

3. Let $X_n$ be a sequence of discrete random variables such that $P(X_n = \frac{k}{2^n}) = \frac{1}{2^n}$ for $k = 1,\, 2,\, \ldots,\, 2^n$. Show that $X_n \to X$ in distribution, where $X \sim U(0,\, 1)$.

4. Let $\{X_n\}$ be a sequence of *i.i.d.* random variables with finite variance $\sigma^2$. Let $S_n^2 = \dfrac{1}{n-1}\displaystyle\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2$. Show that $\{S_n^2\}$ converges to $\sigma^2$ almost surely.

5. Let $\{X_n\}$ be a sequence of identically distributed random variables with mean $\mu \in \mathbb{R}$ and variance $\sigma^2 < \infty$, where $\sigma > 0$. Also assume that $\operatorname{Cov}\left(X_i,\, X_j\right) = 0$ for $i \neq j$. Show that $\overline{X}_n \to \mu$ in probability.

6. Let $\{X_n\}$ be a sequence of *i.i.d.* random variables with mean $\mu$ and finite variance $\sigma^2$. Show that $\sqrt{n}\frac{\overline{X}_n - \mu}{S_n} \to Z$ in distribution, where $Z \sim N(0,\, 1)$.

7. Let $X_i$ and $Y_i$, $i = 1,\, 2,\, \ldots$ are independently and identically distributed $U(0,\, 1)$ random variables. Let $N_n = \#\left\{k : 1 \leq k \leq n, X_k^2 + Y_k^2 \leq 1\right\}$. Show that $\frac{4N_n}{n}$ converges to $\pi$ with probability one.

8. Show that

$$\lim_{n\to\infty} e^{-n}\sum_{k=0}^{n}\frac{n^k}{k!} = \frac{1}{2}.$$