MA 101 (Mathematics-I)

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Introduction

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- If the sequence of partial sums (s_n) converges to a limit ℓ, we say that the series converges and its sum is ℓ.
- If (s_n) diverges, we say that the series diverges.

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(3) The series $1 - 1 + 1 - 1 + \cdots$ is not convergent.

Cauchy criterion: A series $\sum_{n=1}^{\infty} x_n$ is convergent if and only if for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$|x_{n+1} + \cdots + x_m| < \varepsilon \quad \text{for all} \quad m > n \geq n_0.$$

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Theorem (Necessary condition for convergence)

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Examples: The following series are not convergent.

(a) $\sum_{n=1}^{\infty} \frac{n^2+1}{(n+3)(n+4)}$ (b) $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n+2}$

Algebraic operations on series: Let $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ be convergent with sums x and y respectively. Then

(a) $\sum_{n=1}^{\infty} (x_n + y_n)$ is convergent with sum $x + y$

(b) $\sum_{n=1}^{\infty} \alpha x_n$ is convergent with sum αx, where $\alpha \in \mathbb{R}$

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Theorem (Comparison test)

Let (x_n) and (y_n) be sequences in $\mathbb{R}$ such that for some $n_0 \in \mathbb{N}$, $0 \leq x_n \leq y_n$ for all $n \geq n_0$. Then

(a) $\sum_{n=1}^{\infty} y_n$ *is convergent* $\Rightarrow \sum_{n=1}^{\infty} x_n$ *is convergent,*

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Example:

1. $\sum_{n=1}^{\infty} \frac{1+\sin n}{1+n^2}$ is convergent.
   Solution: $0 \leq \frac{1+\sin n}{1+n^2} \leq \frac{2}{n^2}$ for all $n \in \mathbb{N}$.

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2. $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n(n-1)}}$ is not convergent.
   Solution: $\frac{1}{\sqrt{n(n-1)}} > \frac{1}{n} > 0$ for all $n \geq 2$.

Theorem (Limit comparison test)

Let (x_n) and (y_n) be sequences of positive real numbers such that $\frac{x_n}{y_n} \to \ell \in \mathbb{R}$.

(a) *If $\ell \neq 0$, then $\sum_{n=1}^{\infty} x_n$ is convergent $\Leftrightarrow \sum_{n=1}^{\infty} y_n$ is convergent.*

(b) *If $\ell = 0$, then $\sum_{n=1}^{\infty} y_n$ is convergent $\Rightarrow \sum_{n=1}^{\infty} x_n$ is convergent.*

If $\frac{x_n}{y_n}$ diverges to ∞ and $\sum_{n=1}^{\infty} y_n$ is divergent, then $\sum_{n=1}^{\infty} x_n$ is also divergent.

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Example: $\sum\limits_{n=1}^{\infty} \frac{n}{4n^3-2}$ is convergent.

Solution: Let $x_n = \frac{n}{4n^3-2}$ and $y_n = \frac{1}{n^2}$ for all $n \in \mathbb{N}$. Then $\lim\limits_{n\to\infty} \frac{x_n}{y_n} = \lim\limits_{n\to\infty} \frac{n^3}{4n^3-2} = \lim\limits_{n\to\infty} \frac{1}{4-\frac{2}{n^3}} = \frac{1}{4} \neq 0.$

Theorem (Cauchy's condensation test)

Let (x_n) be a decreasing sequence of nonnegative real numbers. Then $\sum_{n=1}^{\infty} x_n$ is convergent if and only if $\sum_{n=1}^{\infty} 2^n x_{2^n}$ is convergent.

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Examples:

(a) p-series: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if and only if $p > 1$.

(b) $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ is convergent if and only if $p > 1$.

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Theorem (Leibniz's test)

Let (x_n) be a decreasing sequence of positive real numbers such that $x_n \to 0$. Then the alternating series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}x_n$ is convergent.

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By Leibniz's test, the alternating harmonic series $\sum_{n=1}^{\infty}(-1)^{n+1}\ \frac{1}{n}$ converges.

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Theorem

Every absolutely convergent series is convergent.

Theorem (Comparison test-II)

Let (x_n) be a sequence of real numbers. Then $\sum_{n=1}^{\infty} x_n$ converges absolutely if there is an absolutely convergent series $\sum_{n=1}^{\infty} y_n$ and some $n_0 \in \mathbb{N}$ satisfying $|x_n| \leq |y_n|$ for all $n \geq n_0$.

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Theorem (Limit comparison test-II)

Let (x_n) and (y_n) be sequences of nonzero real numbers such that $\left|\dfrac{x_n}{y_n}\right| \to \ell \in \mathbb{R}$.

(a) *If $\ell \neq 0$, then $\sum\limits_{n=1}^{\infty} x_n$ is absolutely convergent iff $\sum\limits_{n=1}^{\infty} y_n$ is absolutely convergent.*

(b) *If $\ell = 0$, then $\sum\limits_{n=1}^{\infty} y_n$ is absolutely convergent $\Rightarrow \sum\limits_{n=1}^{\infty} x_n$ is absolutely convergent.*

Theorem (Ratio Test)

Let $\sum_{n=1}^{\infty} x_n$ *be a series of nonzero real numbers. Let*

$$a = \liminf \left|\frac{x_{n+1}}{x_n}\right| \quad \text{and} \quad A = \limsup \left|\frac{x_{n+1}}{x_n}\right|.$$

Then

1. *If* $A < 1$, *then* $\sum_{n=1}^{\infty} x_n$ *is absolutely convergent.*
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Remark: If $\left|\frac{x_{n+1}}{x_n}\right| \to \ell$, then $a = A = \ell$.

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Remark: If $\ell = \lim_{n\to\infty} \left|\frac{x_{n+1}}{x_n}\right| = 1$, then the Ratio test is inconclusive. For example, for both the series $\sum_{n=1}^{\infty} \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$, $\ell = 1$. However, $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

Theorem (Root Test)

Let $\sum_{n=1}^{\infty} x_n$ *be a series of real numbers. Let* $A = \limsup \sqrt[n]{|x_n|}$. *Then*

1. *If* $A < 1$, *then* $\sum\limits_{n=1}^{\infty} x_n$ *is absolutely convergent.*
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Remark: If $\sqrt[n]{|x_n|} \to \ell$, then $A = \limsup \sqrt[n]{|x_n|} = \ell$.

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Example

1. The series $\sum\limits_{n=1}^{\infty} \frac{(n!)^n}{n^{n^2}}$ is convergent.
2. The series $\sum\limits_{n=1}^{\infty} \frac{5^n}{3^n+4^n}$ is not convergent.

Grouping of series

Given a series $\sum_{n=1}^{\infty} x_n$, we can construct many other series $\sum_{n=1}^{\infty} y_n$ by leaving the order of the terms x_n fixed, but inserting parentheses that group together finite number of terms.

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For example, the following series

$$1-\frac{1}{2}+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{5}-\frac{1}{6}+\frac{1}{7}\right)-\frac{1}{8}+\left(\frac{1}{9}-\cdots+\frac{1}{13}\right)-\cdots$$

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Theorem

Grouping of terms of a convergent series does not change the convergence and the sum. However, a divergent series can become convergent after grouping of terms.

Rearrangement of series

A series $\sum_{n=1}^{\infty} y_n$ is called a **rearrangement** of a series $\sum_{n=1}^{\infty} x_n$ if there is a bijection f of $\mathbb{N}$ onto $\mathbb{N}$ such that $y_n = x_{f(n)}$ for all $n \in \mathbb{N}$.

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Example (Tutorial problem): By Leibniz's test, let

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = s$$

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However,

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \cdots = \frac{3}{2}s$$

Theorem (Riemann's rearrangement theorem)

Let $\sum_{n=1}^{\infty} x_n$ be a conditionally convergent series.

1. *If $s \in \mathbb{R}$, then there exists a rearrangement of terms of $\sum_{n=1}^{\infty} x_n$ such that the rearranged series has the sum s.*
2. *There exists a rearrangement of terms of $\sum_{n=1}^{\infty} x_n$ such that the rearranged series diverges.*