Indian Institute of Technology Guwahati
Probability Theory (MA 683)
Problem Set 03

1. Let Ω_i, $i = 1, 2$ be two nonempty sets and $T : \Omega_1 \to \Omega_2$ be a map. Then for any collection $\{A_\alpha : \alpha \in I\}$ of subsets of Ω_2, show that

$$T^{-1}\left(\cup_{\alpha\in I} A_i\right) = \cup_{\alpha\in I} T^{-1}(A_\alpha) \qquad \text{and} \qquad T^{-1}\left(\cap_{\alpha\in I} A_i\right) = \cap_{\alpha\in I} T^{-1}(A_\alpha).$$

   Further, $\left(T^{-1}(A)\right)^c = T^{-1}(A^c)$ for all $A \subset \Omega_2$.

2. Let Ω_i, $i = 1, 2$ be two nonempty sets and $T : \Omega_1 \to \Omega_2$ be a map.
   (a) Prove that $A \subset T^{-1}(T(A))$ for all $A \subset \Omega_1$ with set equality holding if T is one-to-one.
   (b) Prove that $T\left(T^{-1}(B)\right) \subset B$ for all $B \subset \Omega_2$ with equality if T is onto.

3. Let $\Omega = \{1, 2, 3, 4\}$ and $\mathcal{F} = \{\Phi, \Omega, \{1\}, \{2, 3, 4\}\}$. Is $X(w) = 1 + w$ a random variable with respect to the σ-algebra $\mathcal{F}$? If not, give an example of a non-constant function which is.

4. For each of the function below, find the smallest sigma algebra on $\Omega = \{-2, -1, 0, 1, 2\}$ with respect to which the function is a random variable:
   (a) $X(w) = w^2$
   (b) $X(w) = w + 1$
   (c) $X(w) = |w|$
   (d) $X(w) = 2w$

5. What is the smallest number of elements of a sigma algebra if a function $X : \Omega \to \mathbb{R}$ taking exactly n different values is to be a random variable with respect to this sigma algebra?

6. Let $\Omega = [0, 1]$ with the sigma algebra $\mathcal{G}$ of Borel sets B contained in $[0, 1]$ such that $B = 1 - B$. (By $1 - B$ we denote the set $\{1 - x : x \in B\}$.)
   (a) Is $X(w) = w$ a random variable on Ω with respect to $\mathcal{G}$?
   (b) Is $Y(w) = |w - 1/2|$ a random variable on Ω with respect to $\mathcal{G}$?
   (c) Is $X(w) = 2w$ a random variable on Ω with respect to $\mathcal{G}$?

7. Prove that if X is real measurable function on a measurable space $(\Omega, \mathcal{F})$, so is $|X|$. Is the converse true?

8. Two dice are rolled. Let X be the larger of two numbers shown. Compute $P_X([2, 4])$.

9. Let $\Omega = [0, 1]$ with Borel sigma algebra and Lebesgue measure. Find $P_X([0, 1/2))$ for $X(w) = w^2$.

10. Let X be the number of tosses of a fair coin up to and including the first toss showing head. Find $P_X(2\mathbb{N})$, where $2\mathbb{N}$ is the set of even non-negative integers.

11. Let $(\Omega, \mathcal{F}, P)$ be a probability space. Find the cumulative distribution function of each of the random variables below.

(a) $X(w) = 1$ for $w \in A$ and $X(w) = 2$ otherwise, where $P(A) = 1/3$

(b) $X(w) = c_k$ with probability α_k, for $k = 1, 2, \ldots, n$, where $c_1 < c_2 < \ldots < c_n$ and $\alpha_1 + \alpha_2 + \ldots + \alpha_n = 1$.

(c) Let $\Omega = [0, 1]$ with Borel sigma algebra and Lebesgue measure, $X(w) = 2w - 1$.

12. Let $\{A_i\}_{i \geq 1}$ be a collection of disjoint sets in a measurable space $(\Omega, \mathcal{F})$.

    (a) Let $\{g_i\}_{i \geq 1}$ be a collection of $\langle \mathcal{F}, \mathcal{B}(\mathbb{R}) \rangle$-measurable functions from Ω to $\mathbb{R}$. Show that $\sum_{i=1}^{\infty} g_i I_{A_i}$ converges on $\mathbb{R}$ and is $\langle \mathcal{F}, \mathcal{B}(\mathbb{R}) \rangle$-measurable.

    (b) Let $\mathcal{G} = \sigma \langle \{A_i : i \geq 1\} \rangle$. Show that $h : \Omega \to \mathbb{R}$ is $\langle \mathcal{G}, \mathcal{B}(\mathbb{R}) \rangle$-measurable iff g is constant on each A_i.

13. Let $g : \Omega \to \bar{\mathbb{R}}$ be such that for every $r \in \mathbb{R}$, $g^{-1}((-\infty, r]) \in \mathcal{F}$. Show that g is $\left\langle \mathcal{F}, \mathcal{B}\left(\bar{\mathbb{R}}\right) \right\rangle$-measurable.

14. Let $(\Omega_i, \mathcal{F}_i)$, $i = 1, 2$ be measurable spaces and let $T : \Omega_1 \to \Omega_2$ be a $\langle \mathcal{F}_1, \mathcal{F}_2 \rangle$-measurable function from Ω_1 to Ω_2. Then, for any measure μ on $(\Omega_1, \mathcal{F}_1)$, the set function μT^{-1}, defined by

$$\mu T^{-1}(A) = \mu \left(T^{-1}(A) \right), \ A \in \mathcal{F}_2$$

    is a measure on $\mathcal{F}_2$.

15. Give an example of a discrete random variable, where the cumulative distribution function of the random variable is not a step function.

16. Show that a given cumulative distribution function can be written as a weighted sum of a discrete and a continuous cumulative distribution functions.