MA 101 (Mathematics-I)

Subhamay Saha and Ayon Ganguly
Department of Mathematics
IIT Guwahati

Improper Integrals

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(c) Combination of Type I and Type II is possible.

Example: $\int_1^\infty \frac{1}{x^2-4}dx$

Convergence of Type I improper integrals: Let $f : [a, \infty) \to \mathbb{R}$ be such that $f \in \mathcal{R}[a, x]$ for all $x > a$. If $\lim\limits_{x\to\infty} \int\limits_a^x f(t)\, dt$ exists in $\mathbb{R}$, then $\int\limits_a^\infty f(t)\, dt$ is said to be convergent and

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Similarly, we define convergence of $\int\limits_{-\infty}^b f(t)\,dt$.

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Solution: For all $x > 1$, we have $\int\limits_1^x \frac{1}{t^p}\,dt = \frac{1}{1-p}(x^{1-p} - 1)$ if $p \neq 1$ and $\int\limits_1^x \frac{1}{t}\,dt = \log x$.

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Solution: $\int_0^\infty \frac{1}{1+t^2}\,dt = \lim_{x\to\infty} \int_0^x \frac{1}{1+t^2}\,dt = \lim_{x\to\infty} \tan^{-1} x = \frac{\pi}{2}$.

Theorem (Comparison test)

Suppose that $f, g : [a, \infty) \to \mathbb{R}$ *are such that* $f, g \in \mathcal{R}[a, x]$ *for every* $x > a$ *and* $0 \leq f \leq g$. *If* $\int_a^\infty g(t)\, dt$ *converges, then* $\int_a^\infty f(t)\, dt$ *converges.*

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Example: The improper integral $\int\limits_1^\infty \frac{\sin^2 t}{t^2}\, dt$ converges.

Solution: Since $0 \le \frac{\sin^2 t}{t^2} \le \frac{1}{t^2}$ for all $t \ge 1$ and since $\int\limits_1^\infty \frac{1}{t^2}\, dt$ converges, by the comparison test, $\int\limits_1^\infty \frac{\sin^2 t}{t^2}\, dt$ converges.

Theorem (Dirichlet's test)

Let $f, g : [a, \infty) \to \mathbb{R}$ *be such that*

- (a) f *is decreasing and* $\lim\limits_{t \to \infty} f(t) = 0$, *and*
- (b) g *is continuous and there exists* $M > 0$ *such that* $\left| \int_a^x g(t)\, dt \right| \leq M$ *for all* $x \geq a$.

Then $\int_a^\infty f(t) g(t)\, dt$ *converges.*

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Then $\int\limits_a^\infty f(t) g(t)\, dt$ converges.

Example: The improper integral $\int\limits_1^\infty \frac{\sin t}{t}\, dt$ converges.

Solution: Let $f(t) = \frac{1}{t}$ and $g(t) = \sin t$ for all $t \geq 1$. Then $f : [1, \infty) \to \mathbb{R}$ is decreasing and $\lim\limits_{t \to \infty} f(t) = 0$. For $x \geq 1$,
$\left| \int\limits_1^x g(t)\, dt \right| = |\cos 1 - \cos x| \leq |\cos 1| + |\cos x| \leq 2$. Hence by Dirichlet's test, $\int\limits_1^\infty f(t) g(t)\, dt$ converges.

Absolute convergence: If $\int\limits_a^\infty |f(t)|\,dt$ converges, then $\int\limits_a^\infty f(t)\,dt$ is said to converge absolutely.

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Result: If $\int\limits_a^{\infty} |f(t)|\, dt$ converges, then $\int\limits_a^{\infty} f(t)\, dt$ also converges.

Example: The integral $\int_1^{\infty} \frac{\sin x}{x} dx$ does not converge absolutely.

Solution:

$$\int_1^{\infty} \frac{|\sin x|}{x} dx \geq \int_{\pi}^{\infty} \frac{|\sin x|}{x} dx = \sum_{n=1}^{\infty} \int_{n\pi}^{(n+1)\pi} \frac{|\sin x|}{x} dx$$

$$\geq \sum_{n=1}^{\infty} \frac{1}{(n+1)\pi} \int_{n\pi}^{(n+1)\pi} |\sin x| dx = \sum_{n=2}^{\infty} \frac{1}{n\pi} \int_0^{\pi} \sin x \, dx = \frac{2}{\pi} \sum_{n=2}^{\infty} \frac{1}{n}.$$

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Hence $\int_1^\infty \frac{|\sin x|}{x} dx$ does not converge.

Improper integrals of Type-II: Let $f(x)$ be defined on $[a, b)$ and $f \in \mathcal{R}[a, b-\varepsilon]$ for all $\varepsilon > 0$. Then we define

$$\int_a^b f(x)dx = \lim_{\varepsilon \to 0+} \int_a^{b-\varepsilon} f(x)dx.$$

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Let $f(x)$ be defined on $(a, b]$ and $f \in \mathcal{R}[a+\varepsilon, b]$ for all $\varepsilon > 0$. Then we define

$$\int_a^b f(x)dx = \lim_{\varepsilon \to 0+} \int_{a+\varepsilon}^{b} f(x)dx.$$

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Solution: $\int\limits_0^1 \frac{1}{t^p}\,dt$ exists (in $\mathbb{R}$) as a Riemann integral if $p \le 0$.
So let $p>0$. Then for $0<x<1$, we have
$\int\limits_x^1 \frac{1}{t^p}\,dt = \frac{1}{1-p}(1-x^{1-p})$ if $p \ne 1$ and $\int\limits_x^1 \frac{1}{t}\,dt = -\log x$.

Hence $\lim\limits_{x\to 0+} \int\limits_x^1 \frac{1}{t^p}\,dt = \frac{1}{1-p}$ if $p<1$ and $\lim\limits_{x\to 0+} \int\limits_x^1 \frac{1}{t^p}\,dt = \infty$ if $p \ge 1$. Therefore $\int\limits_0^1 \frac{1}{t^p}\,dt$ converges iff $p<1$.

Theorem (Comparison test for Type-II)

Suppose that $f, g : [a, b) \to \mathbb{R}$ are such that $f, g \in \mathcal{R}[a, b - \varepsilon]$ for every $\varepsilon > 0$ and $0 \leq f \leq g$. If $\int_a^b g(t)\,dt$ converges, then $\int_a^b f(t)\,dt$ converges.

Combination of Type I and Type II: We write the integral as a sum of two integrals where one is Type-I and the other one is Type-II. If both these integrals converge, we say that the integral converges.

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Suppose $f : \mathbb{R} \to \mathbb{R}$ is Riemann integrable over any finite interval $[a, b]$. To integrate over the unbounded interval $(-\infty, \infty)$, we pick any real number c and consider the improper integrals $\int_{-\infty}^{c} f(x)dx$ and $\int_{c}^{\infty} f(x)dx$.

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If both exist, we say that the improper integral $\int_{-\infty}^{\infty} f(x)dx$ exists and define its value by

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{c} f(x)dx + \int_c^\infty f(x)dx.$$

Remark: The above definition does not depend on c.