Indian Institute of Technology Guwahati
Probability Theory (MA 683)
Problem Set 05

1. The sequence $\{X_n\}_{n\geq 1}$ converges a.e. if and only if for every $\varepsilon > 0$,
$$\lim_{m\to\infty} P\left(|X_n - X_{n'}| > \varepsilon \text{ for some } n' > n \geq m\right) = 0.$$

2. The sequence $\{X_n\}_{n\geq 1}$ converges in probability if and only if for every $\varepsilon > 0$,
$$\lim_{n\to\infty,\, n'\to\infty} P\left(|X_n - X_{n'}| > \varepsilon\right) = 0.$$

3. The sequence $\{X_n\}_{n\geq 1}$ converges in probability to zero if and only if
$$E\left(\frac{|X_n|}{1+|X_n|}\right) \to 0.$$

4. The function $\rho(\cdot,\, \cdot)$ given by
$$\rho(X, Y) = E\left(\frac{|X-Y|}{1+|X-Y|}\right)$$
is a metric in the space of random variables, provided that we identify random variables that are equal a.e.
Note: A notion of convergence is called metrizable if there exists a metric d such that $X_n \to X$ in the notion of convergence if and only if $d(X_n,\, X) \to 0$. For example, Problems 3 and 4 imply that the probability convergence is metrizable.

5. Let Υ denote the collection of all random variables defined on a probability space $(\Omega,\, \mathcal{F},\, P)$. Let d be a metric defined on Υ such that $\{X_n\}_{n\geq 1} \subset \Upsilon$ converges almost surely to $X \in \Upsilon$ if and only if $d(X_n,\, X) \to 0$ as $n \to \infty$. Then show that $X_n \to 0$ a.e. (P) if $X_n \to 0$ in probability.
Hint: Let $(M,\, d)$ be a metric space. A sequence $\{x_n\}_{n\geq 1}$ in M converges to $x \in M$ if and only if given any subsequence $\{x_{n_k}\}_{k\geq 1}$ of $\{x_n\}_{n\geq 1}$, there exists a further subsequence $\{x_{m_k}\}_{k\geq 1}$ of $\{x_{n_k}\}_{k\geq 1}$ that converges to x.
Note: This result shows that convergence in a.e. is not metrizable in general.

6. If $\{Y_n\}_{n\geq 1}$ are non-negative random variables satisfying $\lim\limits_{n\to\infty} EY_n = 1 = \lim\limits_{n\to\infty} EY_n^p$ for some $p > 1$, then show that $Y_n \to 1$ in p-th mean.

7. If $\{X_n\}_{n\geq 1}$ is a sequence of random variables on some probability space such that
$$\sum_{n=1}^{\infty} E\left|X_n - X\right|^p < \infty,$$
for some $p > 0$, then show that $X_n \to X$ a.e.

8. Let $\{X_n\}_{n\geq 1}$ be a sequence of random variables such that $\sum_{i=1}^{\infty} EX_i < \infty$. Then show that $\{S_n\}_{n\geq 1}$ converges a.e., where $S_n = X_1 + X_2 + \ldots + X_n$.

9. Let $(\Omega, \mathcal{F}, \mu)$ be a probability space and let $\{A_n\}_{n\geq 1} \subseteq \mathcal{F}$ be a sequence of events. Show that

   (a) $\mu\left(\liminf_{n\to\infty} A_n\right) \leq \liminf_{n\to\infty} \mu\left(A_n\right).$

   (b) $\limsup_{n\to\infty} \mu\left(A_n\right) \leq \mu\left(\limsup_{n\to\infty} A_n\right).$

10. Let $X_1, X_2, \ldots$ be a sequence of random variables, with $E(X_n) = 8$ and $Var(X_n) = \frac{1}{\sqrt{n}}$ for all $n \geq 1$. Prove or disprove that $\{X_n\}_{n\geq 1}$ must converge to 8 in probability.

11. If $X_n \to X$ and $Y_n \to Y$ both in probability, then $X_n \pm Y_n \to X \pm Y$ and $X_n Y_n \to XY$ all in probability.

12. Let f be a bounded continuous function on $\mathbb{R}$. Then $X_n \to 0$ in probability implies $Ef(X_n) \to f(0)$.

13. Convergence in $\mathcal{L}_p$ implies convergence in $\mathcal{L}_r$ for $r < p$.

14. (a) If $X_n \to X$ and $Y_n \to Y$, both in $\mathcal{L}_p$, then $X_n \pm Y_n \to X \pm Y$ in $\mathcal{L}_p$.

    (b) If $X_n \to X$ in $\mathcal{L}_p$ and $Y_n \to Y$ in $\mathcal{L}_q$, where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then $X_n Y_n \to XY$ in $\mathcal{L}_1$.

15. If $X_n \to X$ in probability and $X_n \to Y$ in probability, then $X = Y$ a.e.

16. Let $X_n \to X$ a.e. and P_n and P be the induced probability by X_n and X, respectively. With an example show that $\{P_n(I)\}_{n\geq 1}$ may not converge to $P(I)$ for an interval $I \subset \mathbb{R}$.

17. Give an example in which $E(X_n) \to 0$ but there does not exist any subsequence $\{n_k\}_{k\geq 1}$ such that $n_k \to \infty$ and $X_{n_k} \to 0$ in probability.

18. Let $\{X_n\}_{n\geq 1}$ be a sequence of independent random variables with $E(X_n) = 0$ and $Var(X_n) = 1$ for all $n \geq 1$. Prove that for any bounded random variable Y, $\lim_{n\to\infty} EX_nY = 0$. Hint: Consider $E\left(Y - \sum_{i=1}^{n} X_i EX_iY\right)^2$ to obtain $E(Y^2) \geq \sum_{i=1}^{n} \left(EX_iY\right)^2$.