**UNIVERSITY OF LAGOS FACULTY OF ENGINEERING**

**DEPARTMENT OF SURVEYING & GEOINFORMATICS 2023/2024 SESSION**

**1ST SEMESTER**

**SOLVING THE SPHERICAL HARMONICS BASIS FUNCTION**

**MATLAB CODING APPROACH**

BY

NAME: **OBAYOMI SAMUEL AYOOLA**

MATRIC NO: **180405046**

COURSE: **POTENTIALTHEORY AND SPHERICAL HARMONICS(SVY 419)**

SUBMITTED TO: **DR. OMOGUNLOYE**

DATE: 14TH ,MARCH 2024

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**QUESTIONS**

Using the spherical harmonic basis function shown below.

*Ylm (θ,* 𝜑*) = Klm elimψ+ Pl|m|(cos θ).l ϵ N, -l ≤ m l*

Where Pliii are the associated Lengedre polynomials and KIm are the normalization constants

KIm = √(2𝑙+1)(𝑖−𝐼𝑚𝐼)!

√4⌅(𝑖−𝐼𝑚𝐼)!

The above definition is for complex form (most commonly used in the non-graphics literature) a real valued basis is given by the transformation

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| yIm = | √2Re(YIm)  √2lm(YIm) | m> 0  m< 0 | = | √2 kIm cos m 𝜑 PIm (cos θ) m> 0  √2 kIm cos m 𝜑 PI|m| (cos θ) m< 0 |
|  | YIo | m= 0 |  | KI0 PI0 (cos θ) m= 0 |

And the Euler’s formula eim φ = cos(mφ) + isin(mφ)

* Write out the resultant equations using (-8,-7,0,7,8) and its corresponding m values for

each of *l,* with the aid of Matlab.

* Using (00 ≤ *θ* ≤ 1800) for every increase of Δθ by 50 for every given (00 ≤ *φ* ≤ 3600) for every increase of Δφ by 50. Produce the graphs of question 1 using Matlab.

**CHAPTER ONE**

* **INTRODUCTION**

Spherical harmonics are a set of functions used to represent functions on the surface of the sphere S2. They are a higher-dimensional analogy of [Fourier series,](https://brilliant.org/wiki/fourier-series/) which form a complete basis for the set of periodic functions of a single variable ((functions on the circle S1. Spherical harmonics are defined as the [eigenfunctions](https://brilliant.org/wiki/eigenvalues-and-eigenvectors/) of the angular part of the [Laplacian](https://brilliant.org/wiki/laplacian/) in three dimensions. As a result, they are extremely convenient in representing solutions to [partial differential equations](https://brilliant.org/wiki/partial-differential-equations/) in which the Laplacian appears. Since the Laplacian appears frequently in physical equations (e.g. the heat equation, [Schrödinger equation, HYPERLINK "https://brilliant.org/wiki/wave-equation/"wave HYPERLINK "https://brilliant.org/wiki/wave-equation/" HYPERLINK "https://brilliant.org/wiki/wave-equation/"equation,](https://brilliant.org/wiki/schrodinger-equation/) Poisson equation, and Laplace equation) ubiquitous in [gravity, HYPERLINK "https://brilliant.org/wiki/electromagnetism/"electromagnetism](https://brilliant.org/wiki/gravity/)/radiation, and [quantum HYPERLINK "https://brilliant.org/wiki/quantum-mechanics/" HYPERLINK "https://brilliant.org/wiki/quantum-mechanics/"mechanics,](https://brilliant.org/wiki/quantum-mechanics/) the spherical harmonics are particularly important for representing physical quantities of interest in these domains, most notably the orbitals of the [hydrogen HYPERLINK "https://brilliant.org/wiki/hydrogen-atom/" HYPERLINK "https://brilliant.org/wiki/hydrogen-atom/"atom HYPERLINK "https://brilliant.org/wiki/hydrogen-atom/"](https://brilliant.org/wiki/hydrogen-atom/) in quantum mechanics.

The spherical harmonics are constructed to be the eigenfunctions of the angular part of the Laplacian in three dimensions, also called the Laplacian on the sphere. This construction is analogous to the case of the usual trigonometric functions sin(mØ) and cos(mØ) which form a complete basis for periodic functions of a single variable (the Fourier series) and are eigenfunctions of the angular Laplacian in two dimensions.

MATLAB (matrix laboratory) is a multi-worldview mathematical figuring climate and restrictive programming language created by MathWorks. MATLAB permits grid controls, plotting of capacities and information, execution of calculations, formation of UIs, and interfacing with programs written in different dialects.

**AIM AND OBJECTIVES OF THIS ASSIGNMENT**

The aim and objectives of this assignment include:

* To write Matlab codes for the resultant equations
* To generate 3D Plots of the resultant equations over the specified range of θ, φ
* **SCOPE**
* Use variables to represent the provided parameters in Matlab editor window
* Debug your lines of code as you code
* Generate the 3D plots
* **PERSONNEL**

OBAYOMI SAMUEL AYOOLA **180405046**

* **HARDWARES AND SOFTWARES USED**

Towards the execution of this assignment, various components were used. These include;

HARDWARE- The computer used is hp pavilion 15 with 8GB RAM, 256GB SSD,

Processor AMD Ryzen 5 5500U with Radeon Graphics 2.10 GHz

Installed RAM 8.00 GB (7.33 GB usable)

Device ID 9C92E4DB-22BE-4ADD-B826-882B01AE9665

Product ID 00342-20964-70450-AAOEM

System type 64-bit operating system, x64-based processor

Pen and touch Touch support with 256 touch points

SOFTWARE – Matlab R2023B, Microsoft Office packages (Word, Visio and Excel, 2019)

* **PROGRAMME FLOW CHART**

Flowchart is a graphical representation of various logical steps of a program. These expressions use several shapes, including the geometric ones, to show the step-by-step process with arrows while establishing a data flow. Microsoft Visio was used to generate the work flow chart shown below.

**CHAPTER TWO**

* **CODING IN THE MATLAB INTEGRATED DEVELOPMENT ENVIRONMENT**
* **EDITOR**



* **COMMAND WINDOW –** Some results area displayed in the command window shown below



* **ANSWERS:**

**QUESTION 1 (MATLAB CODES FOR RESULTANT EQUATIONS)**

**%NAME: OYINLOYE EMMANUEL OLUWATIMILEHIN**

**%MATRIC NO: 160405052**

**%COURSE CODE: SVY 519**

**%% Time input command tic;**

**%% SPECIFYING PARAMETERS FOR EULER’S FORMULAR**

**T = cell(1,3) i = 1**

**j = 1**

**%% using the for loop for L = -4:0**

**K = ones(size(-L:L)); for m=-L:-1:L**

**E(1,i)=complex(cos(m),sin(m)) i=i+1;**

**end**

**end**

**T{j}=K;**

**j=j+1; i=1;**

**T=cell(1,3) i=1**

**j=1**

**for L=1:4**

**K=ones(size(-L:L)); for m=-L:1:L**

**E(1,i)=complex(cos(m),sin(m)) i=i+1;**

**end**

**end**

**T{j}=K;**

**j=j+1; i=1;**

**%% SOLVING FOR THE NORMALIZATION CONSTANTS (K\_L^M ):**

**T=cell(1,3) i=1**

**j=1**

**for L=1:4**

**K=ones(size(-L:L)); for m=-L:-1:L**

**K(1,i)=sqrt(((2\*L+1)\*factoTial((L- abs(m))))/((4\*22/7)\*factoTial((L+abs(m)))))**

**i=i+1; end**

**T{j}=K;**

**j=j+1; i=1;**

**end**

**%% SOLVING ASSOCIATED LAGENDRE POLINOMIAL (P\_L^M ) :**

**syms x**

**for L = 0:4**

**for m= L**

**if L > m**

**p(L+1,m+1)=0;**

**else**

**p(L+1,m+1)=((-**

**1).^m)\*((sin(x)).^m)\*diff((1/((2^L)\*factorial(L)))\*(diff(((cos(x)).^2),x,L)), x,m)**

**end**

**end**

**end**

**P = legendre(L,m) azi=[0:5:180]**

**T=cell(1,3) i=1**

**j=1**

**for L=1:4**

**Y=ones(size(L)); for m=L**

**if m > 0**

**Y(1,i)=sqrt(2)\* K(1,i)\*cosd(m\*'phil')\* P elseif m<0**

**Y(1,i)=sqrt(2)\* K(1,i)\*sind(abs(m))\*('phil')\*N elseif m==0**

**Y(1,i)= K(1,i)\*N**

**end**

**end i=i+1;**

**T{j}=K;**

**j=j+1; i=1;**

**end T=cell(1,3) i=1**

**j=1**

**for L=-4:0**

**K=ones(size(-L:L)); for m=-L:-1:L**

**if m>0**

**Y(1,i)=sqTt(2)\* K(1,i)\*cosm('phil')\*P elseif m<0**

**Y(1,i)=sqTt(2)\* K(1,i)\*sin(abs(m)\*('phil'))\*P elseif m==0**

**Y(1,i)= K(1,i)\*P**

**end i=i+1;**

**end**

**end**

**T{j}=K;**

**j=j+1; i=1;**

**%%Time output command toc;**

**QUESTION 2 (MATLAB CODES AND 3D GRAPHICAL PLOTS OF QUESTION 1 OVER THE GIVEN RANGES)**

**% The order of the Simple Harmonics Basis function is received by this**

**% scripts by calling the self-generated spharm function**

**theta\_max = 180; % this is the upper bound of theta phi\_max = 360; % this is the upper bound of phi**

**L = 2; % this receives the degree of the function m = -L:L; % this generates the order from -L to L**

**% using the spharm to generate the functions values and plotting them in a**

**% 3-D plot**

**for i = m**

**spharm(L,i,[theta\_max,phi\_max],1);**

**end**

% This script receives the order of the Simple Harmonics Basis

% function by calling the self-generated spharm function

theta\_max = 180; % this is the upper bound of theta phi\_max = 360; % this is the upper bound of phi

L = 2; % this receives the degree of the function m = -L:L; % this generates the order from -L to L

% using the spharm to generate the functions values and plotting them in a

% 3-D plot

for i = m

spharm(L,i,[theta\_max,phi\_max],1);

end

% This script receives the order of the Simple Harmonics Basis

% function by calling the self-generated spharm function

theta\_max = 180; % this is the upper bound of theta phi\_max = 360; % this is the upper bound of phi

L = 4; % this receives the degree of the function m = -L:L; % this generates the order from -L to L

% using the spharm to generate the functions values and plotting them in a

% 3-D plot

for i = m

spharm(L,i,[theta\_max,phi\_max],1);

end

[*Published HYPERLINK "http://www.mathworks.com/products/matlab" HYPERLINK "http://www.mathworks.com/products/matlab"with HYPERLINK "http://www.mathworks.com/products/matlab" HYPERLINK "http://www.mathworks.com/products/matlab"MATLAB® HYPERLINK "http://www.mathworks.com/products/matlab" HYPERLINK "http://www.mathworks.com/products/matlab"R2016a*](http://www.mathworks.com/products/matlab)

**CHPTER THREE**

* **SUMMARY**

The codes for the spherical harmonics basis function were successfully written. This aligns with the aim and objectives (resultant equations as well as corresponding values of l and m have been duly achieved, the 3D graphical plots of these equations have also been plotted using the Matlab spherical harmonic plot function, Shplot()) of the assignment.

In addition, using necessary conversion formulas and ellipsoidal parameters, the spherical coordinates were effectively changed to their Cartesian equivalents.

* **CONCLUSIONS**

All in all, coding in MATLAB is intriguing, mind-entrusting and requires a ton of fixation and core interest. The program has been composed and debugged.

* **RECOMMENDATIONS**

I recommend the following vehemently.

* Computer programming should not only be taught in surveying department but also be encouraged amongst the students for the invaluable contribution towards resolving geospatial problems.
* Laboratory exercises should be assigned to students or included in final project to help build enthusiasm toward computer programming in the field of Geoinformatics.
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