Innovative Assignment
2CSDE56 - Graph Theory

Nihar Markana
18BCE116
Tirth Hihoriya
18BCE244

Department of Computer Science
Institute of Technology, Nirma University

18bce116@nirmauni.ac.in 18BCE244@nirmauni.ac.in

Contents

Threshold Graphs

1 Introduction

Threshold graphs form a class of networks with a very elegant structure and can be depicted in layers of groups of vertices which are similar to each other. These graphs were first described by Chvatal and Hammer in 1977. Threshold graphs can be defined in a number of equivalent ways. Every definition will help us understand the structure of such graphs from a particular point of view. One of these definitions is used to construct and enumerate such graphs from a binary sequence and also leads us to test whether or not a given graph is a threshold graph. [1]

Graphs have important applications in modern systems biology and social sciences. Edges are created between interacting genes or people who know each other. However graphs are not objects which are naturally amenable to simple statistical analyses, there is no natural average graph for instance. Being able to predict or replace a graph by hidden (statisticians call them latent) real variables has many advantages.

2 Definitions of Threshold Graphs

2.1 Definition 1

Let G be a graph with vertex set V(G)=$\{1, 2, \ldots, n\}$ and edge set E(G). The graph G is a threshold graph if there exists a real number S (called the "threshold value") and a set of real-valued vertex weights $\{w_1, w_2, \cdots, w_n\}$, such that $\{i, j\} \in E(G)$ if and only if $w_i + w_j > S$. [2]

The following graph is displayed on the left is an example of a threshold graph on 5 vertices. In order to show that it is a threshold graph, we must find S and a set of vertex weights that satisfy the condition given in Definition 1. Let the weights of the vertices be 3, 3, 4, 4, 2 respectively (as show in the graph displayed on the right) and let S = 5.5. We see that the sum of the weights of the two end-vertices of each edge is greater than 5.5. These are displayed as purple edge-weights on the graph on the right. There is no edge between vertex between 2 and 5, and between 1 and 5 because the sum of the weights of the end-vertices is 5 (< 5.5 = S) in both cases.

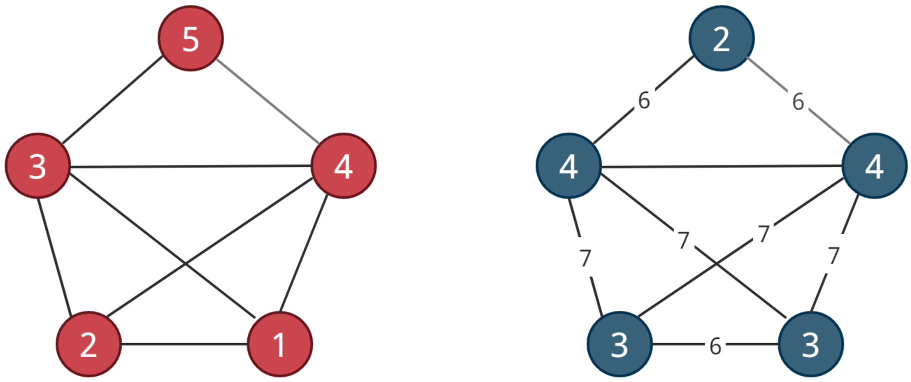

Figure 1

Now consider the following graph.

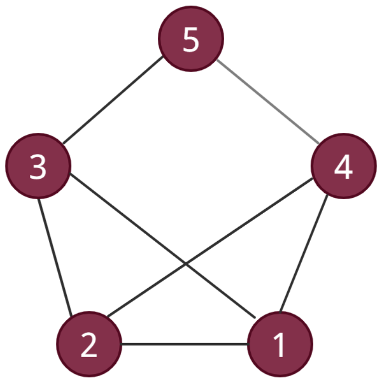

Figure 2

When one tries to find a set of vertex weights and a threshold value S that separates the edges from the non-edges, it turns out to be a difficult task. This is not possible because the graph is not a threshold graph. However, in order to prove this fact, we must prove that no combination of weights and threshold value will satisfy Definition 1. Let us prove this by contradiction. Suppose there are weights $\{w_1, w_2, w_3, w_4, w_5\}$ and S that satisfy Definition 1.

$$\begin{aligned}
&\{1,5\} \notin E(G) \text{ and } \{4,5\} \in E(G) \\
\Rightarrow &w_1 + w_5 \leq S \text{ and } w_4 + w_5 > S \\
\Rightarrow &w_4 > w_1.
\end{aligned}$$

On the other hand,

$$
\begin{aligned}
&\{1,3\} \in E(G) \\
\Rightarrow & w_1 + w_3 > S \\
\Rightarrow & w_4 + w_3 > S \text{ (since } w_4 > w_1)
\end{aligned}
$$

But $\{3, 4\}$ is not an edge of the graph. Hence this is a contradiction and thus we have shown that this is not a threshold graph.

The following equivalent definition describes how any threshold graph can be constructed from a single vertex through two graph operations.

2.2 Definition 2

A threshold graph is a graph that is obtained from the single-vertex graph by repeated addition of an isolated vertex or a dominating vertex.

By adding a dominating vertex, we mean that we add a vertex to the graph and add edges from this vertex to all other vertex already present in the graph. As an example, we are constructing a threshold graph on 5 vertices by starting from the single-vertex graph and adding a dominating vertex, an isolated vertex, a dominating vertex and another dominating vertex. [3]

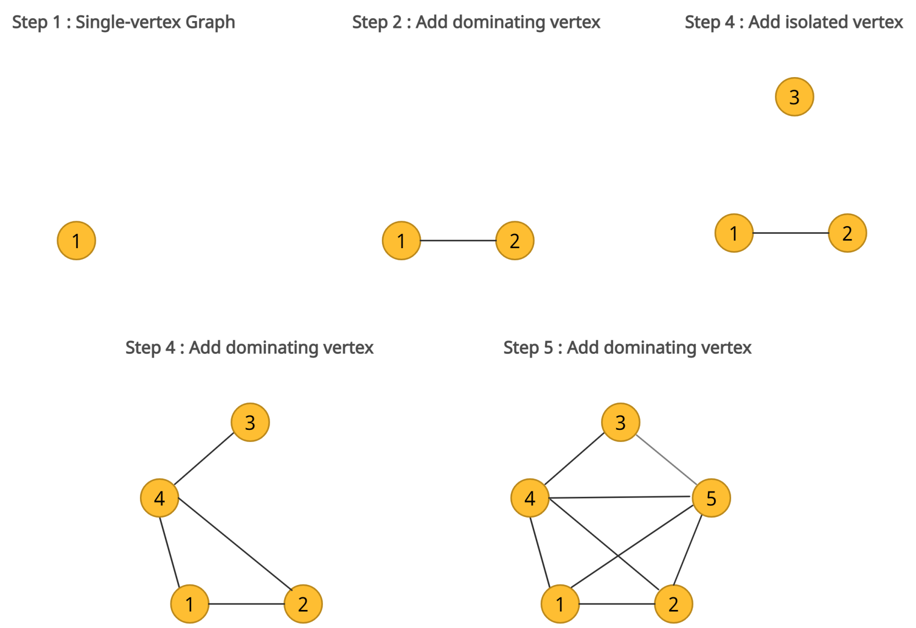

Figure 3

3 The Creation Sequence

Every threshold graph has a unique binary sequence that describes its construction.

3.1 Definition

The creation sequence of a threshold graph G on n vertices is a binary sequence $(c_2, c_3, \ldots, c_n)$ of length n-1 where $c_i = 0$ shows that the i^{th} vertex is added to the graph as an isolated vertex and $c_i = 1$ shows that the i^{th} vertex is added to the graph as a dominating vertex.

Various characteristics of threshold graphs can be deduced from this. First of all, a threshold graph is connected if $c_n = 1$ and in this case, the threshold graph has at least one dominating vertex (i.e. a vertex connected to every other vertex in the graph). Thus a connected threshold graph has a creation sequence of the form $(c_2, c_3, \ldots, c_{n-1}, 1)$ and it can be shown that that there are 2^{n-1} non-isomorphic (distinct/different, ignoring vertex labelling) connected threshold graphs on n vertices. If the restriction of connectedness is relaxed, then there are 2^n non-isomorphic threshold graphs on n vertices.

4 Algorithm

4.1 Generating the threshold graph from creation sequence

Given: Creation sequence $(c_2, c_3, \ldots, c_n)$ of length n, where c_i equals to either 0 or 1

- Step 1: Initialize one isolated vertex.
- Step 2: Scan the sequence from left to right.
- Step 3: Check if c_i=0 then add that vertex in graph without any edge.
- Step 4: Check if c_i=1 then add that vertex in graph in such a way that vertex has an edge with every other vertices.
- Step 5: Repeat step no 3 and 4 till c_n.

For example : Refer Figure 3 for Threshold graph from creation sequence = (1 0 1 1)

4.2 Testing whether or not a graph is a threshold graph

Suppose that we have a threshold graph. By Definition 2, we know that this graph is constructed from a single vertex by a sequence of additions of either isolated or dominating vertices. Now let us deconstruct the threshold graph back to the single-vertex graph. Let the vertices be labelled sequentially according the construction described in Definition 2. If the n^{th} vertex is a dominating vertex (or equivalently the threshold graph is connected), remove it and all its incident edges. Otherwise, if the n^{th} vertex is an isolated vertex, remove it. Either way, you end up with a threshold graph with n-1 vertices. The process of deleting the last vertex is repeated until the graph ends up as a single vertex. [4]

Now suppose that this process is carried on a graph which is not a threshold graph. There is a stage in the process in which the vertex with the highest degree is not a dominating vertex of the graph and the vertex with lowest degree is not an isolated vertex of the graph. The process to decompose the graph to a single-vertex is not possible. This is provides us with a test that determines whether or not a graph is threshold.

Given : Sequence of 0 and 1 and also given number of rows and columns.

- Step 1: Create a graph from adjacency matrix.
- Step 2: Calculate the degree of each vertex.
- Step 3: First find highest degree vertex and check if degree of that vertex is equal to number of total vertices – 1 or (n - 1) then remove that vertex and incident edges of that vertex.
- Step 4: If highest degree vertex is not equal to (n - 1) means it is not dominating vertex then check for lowest degree vertex if lowest degree vertex is equivalent to isolated vertex then remove it.
- Step 5: If highest degree vertex not equal to (n - 1) and also lowest degree vertex is not equivalent to isolated vertex then that graph is not threshold graph and break this loop.

- Step 6: Repeat step till single vertex remain in the graph.
- Step 7: If the loop work properly till single vertex then the graph is threshold graph.

Example :

Given : c(0,1,1,1,1,0,0,0,1,0,0,1,1,0,1,0)
rows = 4
cols = 4

Adjacency matrix :

	1	2	3	4
1	0	1	1	1
2	1	0	0	0
3	1	0	0	1
4	1	0	1	0

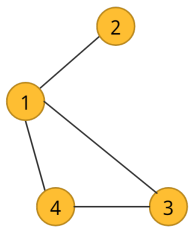

Figure 4

Degree Sequence	3	2	2	1
Vertex no.	1	4	3	2

RHS = Max degree = 3
LHS = Vertices − 1 = 4 − 1 = 3
Here RHS = LHS , Hence remove vertex no 1.

Now, graph becomes

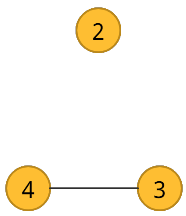

Figure 5

Degree Sequence	1	1	0
Vertex no.	4	3	2

RHS = Max degree = 1
LHS = Vertices − 1 = 3 − 1 = 2
Here RHS != LHS

Now, check for low degree vertex
RHS = Low degree = 0
LHS = Isolated vertex degree = 0
RHS = LHS, Hence remove vertex no 2.

Now, graph becomes

Figure 6

Degree Sequence	1	1
Vertex no.	4	3

RHS = Max degree = 1
LHS = Vertices − 1 = 2 − 1 = 1
Here RHS = LHS, Hence remove vertex no 4.

Now, graph becomes

Figure 7

Hence here only one vertex left which is isolated.
Hence this proves that the graph is threshold graph.

5 Applications

Some of the applications of Threshold Graphs are :

1. Set-packing
2. Parallel processing
3. Resource allocation
4. Scheduling
5. Psychology

6 Conclusion

We have obtained some insight on the structure of threshold graphs from two equivalent definitions. One such definition provides us with a way of constructing threshold graphs and represent them by a unique binary sequence. From such a sequence, we can easily enumerate the number of non-isomorphic threshold graphs on a given number of vertices.

References

[1] D. Boeckner, "Oriented threshold graphs," 2015.

[2] *Data Science genie*, "Threshold graph." `https://datasciencegenie.com/what-is-a-threshold-graph/`, April. 26, 2021. [Online].

[3] M. Andelić and S. Simic, "Some notes on the threshold graphs," *Discrete Mathematics*, vol. 310, pp. 2241–2248, 09 2010.

[4] P. Diaconis, S. Holmes, and S. Janson, "Threshold graph limits and random threshold graphs," 2009.