Iterative Algorithms for Large Linear and Multilinear Dynamical Systems

MST MOCASIM

Elhassani Ayoub

Cadi Ayyad university – Faculty of Science and Technology

Supervisors:
Pr. SADEK El Mostafa
Pr. Karim Kreit

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What's a dynamical system

Definition

A dynamical system is a mathematical model describing the time evolution of a system's state.

- Types:
  - Linear vs. Nonlinear
  - Time-invariant vs. Time-varying
  - Continuous vs. Discrete

LTI Dynamical System

A continuous linear time-invariant dynamical system ("Linear Time-Invariant" LTI)

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t), \end{cases} \tag{1}$$

- $A \in \mathbb{R}^{n \times n}$: Large and sparse state matrix
- $B \in \mathbb{R}^{n \times p}$: Input matrix
- $C^T \in \mathbb{R}^{n \times s}$: Output matrix
- $D \in \mathbb{R}^{s \times p}$: Direct transmission matrix
- $x(t) \in \mathbb{R}^{n}$: State vector, belonging to the state space
- $u(t) \in \mathbb{R}^{p}$: Input (or control) vector
- $y(t) \in \mathbb{R}^{s}$: Output (to be measured)
- **System Types:**
  - Single-Input Single-Output (SISO)
  - Multiple-Input Multiple-Output (MIMO)

Transfer function

In order to get this function, we apply the Laplace transform below:

$$\mathcal{L}f(s) = \int_0^\infty f(t)e^{-st}dt. \tag{2}$$

to the state equation (1), we get

$$Y(s) = F(s)U(s), \tag{3}$$

with

$$F(s) = C(sI - A)^{-1}B + D. \tag{4}$$

The function $F(s)$ is called the transfer function of the system (1).

Moment of a Transfer function

For the transfer function expanded around a point $s_0 \in \mathbb{C}$:

$$\eta_j(s_0) = C(s_0 I_n - A)^{-(j+1)} B \quad \forall j \geq 0. \tag{5}$$

In the special case where $s_0 = \infty$:

$$\eta_i(\infty) = CA^i B. \tag{6}$$

In this case, the matrix coefficients $\eta_i(\infty)$ are called the Markov parameters of F.

Stability, controllability, and observability

We set $D = 0$, then the LTI system is written as:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t). \end{cases} \tag{7}$$

Definition

An LTI system is said to be stable if its matrix A is stable.

Proposition

An (LTI) system is controllable if and only if

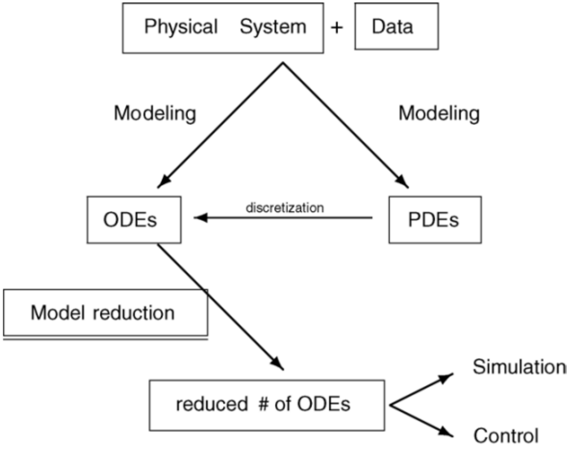
$$rank([B\ AB\ A^2B\ \ldots\ A^{n-1}B]) = n. \tag{8}$$

Proposition

An LTI system is observable if and only if

$$rank([C\ CA\ \ \ldots\ CA^{n-1}]) = n. \tag{9}$$

Model order reduction



Common MOR Methods

1. Truncation-based methods:
   - Balanced Truncation
   - Hankel-norm approximation
2. Projection-based methods:
   - Krylov subspace methods (e.g., Arnoldi, Lanczos)
   - Proper Orthogonal Decomposition (POD)
3. Singular Value Decomposition (SVD) based methods
4. Data-driven methods (e.g., Dynamic Mode Decomposition)

Projection Method for Model Order Reduction

Theoretical Concept

- Project high-dimensional system onto a lower-dimensional subspace
- Preserve essential dynamics while reducing complexity

Projection steps:
Consider a linear time-invariant system:

1. Choose projection matrices $V, W \in \mathbb{R}^{n \times r}$, where $r \ll n$
2. Approximate $x \approx Vz$, where $z \in \mathbb{R}^r$
3. Apply Petrov-Galerkin condition: $W^T(AVz + Bu - V\dot{z}) = 0$

Mathematical Formulation (continued)

Resulting reduced-order model:

$$\begin{cases} \dot{z}(t) = (W^T AV)z(t) + (W^T B)u(t) \\ \hat{y}(t) = (CV)z(t) \end{cases} \tag{10}$$

Block Krylov Subspace

Definition

The block Krylov subspace $\mathbb{K}_m^b(A, V)$ *is a subspace of* $\mathbb{R}^n$ *generated by the columns of the matrices* $V, AV, \ldots, A^{m-1}V$*:*

$$\mathbb{K}_m^b(A, V) = \text{Range}\left(\left[V, AV, \ldots, A^{m-1}V\right]\right). \tag{11}$$

After m steps we get the following results:

- The real matrix of size $n \times mp$ defined by:

$$\mathbb{V}_m = [V_1, \cdots, V_m] \text{ is orthogonal: } \mathbb{V}_m^T \mathbb{V}_m = I_{mp}.$$

- The real matrix $\mathbb{H}_m$ of size $mp \times mp$, whose non-zero blocks H_{ij} of size $p \times p$ are given by the previous algorithm:

$$H_m = (H_{ij})_{1 \leq i,j \leq m}, \quad H_{ij} \equiv 0 \text{ for } i > j+1 \text{ is of Hessenberg type.}$$

Projection Property

- We have the following projection property:

$$A\mathbb{V}_m = \mathbb{V}_m\mathbb{H}_m + V_{m+1}H_{m+1,m}E_m^\top. \tag{12}$$

with

$$E_m^T = [0_p, \cdots, 0_p, I_p].$$

MOR using block Arnoldi method

We consider the linear time invariant (LTI) system:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$, and $C \in \mathbb{R}^{s\times n}$, and assuming that $x(0) = 0$.
The transfer function of this LTI system is given by:

$$F(z) = C\left(zI_n - A\right)^{-1} B. \tag{13}$$

We look for models of low order that approximate the behavior of the original models.

Low-Order Model

Our goal is to find low-order model expressed as follows:

$$\begin{cases} \dot{\hat{x}}(t) = \hat{A}\hat{x}(t) + \widehat{B}u(t) \\ \hat{y}(t) = \widehat{C}\hat{x}(t) \end{cases}$$

where $\hat{A} \in \mathbb{R}^{k\times k}$, $\widehat{B} \in \mathbb{R}^{k\times p}$ and $\widehat{C} \in \mathbb{R}^{s\times k}$ with $k \ll n$.

Approximating the transfer Function

Consider $F(z)$ as $F(z) = CX$ where $X \in \mathbb{R}^{n \times p}$ is the solution of the following linear system:

$$(zI_n - A)\,X = B. \tag{14}$$

Applying the block Arnoldi process to the pairs (A, B) we get the matrix $\mathbb{V}_m$ and an upper block Hessenberg matrix $\mathbb{H}_m$.

Galerkin Condition

Considering the approximation $X \approx \mathbb{V}_m Y_m$, and using the Galerkin condition, we get the projected linear problem:

$$\mathbb{V}_m^T (sI_n - A)\mathbb{V}_m Y_m = \mathbb{V}_m^T B, \tag{15}$$

which is equivalent to:

$$(zI_{ms} - \mathbb{H}_m) Y_m = B_m. \tag{16}$$

Approximate Transfer Function

Then we get the approximate transfer function:

$$F_m(z) = C\mathbb{V}_m = C_m(zI - \mathbb{H}_m)^{-1}B_m, \quad \mathbb{H}_m = \mathbb{V}_m^T A\mathbb{V}_m, \quad C_m = C\mathbb{V}_m. \tag{17}$$

This suggests that the reduced order model can be given by:

$$\begin{cases} \dot{x}_m(t) = A_m x_m(t) + B_m u(t) \\ y_m(t) = C_m x_m(t) \end{cases}$$

where $A_m = \mathbb{H}_m \in \mathbb{R}^{ms\times ms}$, $B_m = \mathbb{V}_m^T B \in \mathbb{R}^{ms\times s}$ and $C_m = C\mathbb{V}_m \in \mathbb{R}^{r\times ms}$.

Global Krylov Subspace

Definition

The global Krylov subspace $\mathcal{K}_m(A, V)$ is the subspace of $\mathbb{R}^{n\times p}$ generated by the matrices $V, AV, \ldots, A^{m-1}V$:

$$\mathcal{K}_m(A, V) = span\{V, AV, \ldots, A^{m-1}V\}.$$

Projection Property

Proposition

Suppose that m steps of the algorithm have been computed. Then we have the following projection property:

$$
\begin{aligned}
A\mathcal{V}_m &= \mathcal{V}_m \diamond \tilde{H}_m & (18)\\
&= \mathcal{V}_m \diamond H_m + h_{m+1,m} V_{m+1} E_m^T & (19)
\end{aligned}
$$

where

$$E_m^T = [0_p, \cdots, 0_p, I_p]$$

and

$$\mathcal{V}_m^T \diamond (A\mathcal{V}_m) = H_m.$$

MOR via Global Arnoldi method

Similarly to block case we look for low-order models that approximate the behavior of the original models.
Using the global Arnoldi process, we approximate the transfer function.
Let $F(z)$ be expressed as $F(z) = CX$, where $X \in \mathbb{R}^{n\times p}$ is the solution of:

$$(zI_n - A)\,X = B.$$

We seek an approximate solution X_m to X via the global Full Orthogonalization Method (Gl-FOM):

$$X_m = \beta \mathcal{V}_m \left[\left(zI_m - H_m \right)^{-1} e_1 \otimes I_s \right],$$

where $\mathcal{V}_m$ and H_m represent the matrix Krylov basis and the Hessenberg matrix obtained by applying the global Arnoldi process to the pair (A, B).

Approximate Transfer Function

Hence, $F(z)$ is approximated by:

$$\begin{aligned} F_m(z) = CX_m &= \beta C\mathcal{V}_m \left[\left(zI_m - H_m \right)^{-1} e_1 \otimes I_s \right] \\ &= C\mathcal{V}_m \left[zI_{ms} - \left(H_m \otimes I_s \right) \right]^{-1} \beta \left(e_1 \otimes I_s \right). \end{aligned}$$

Reduced Order Model

The reduced order model is given by:

$$\begin{cases} \dot{x}_m(t) = A_m x_m(t) + B_m u(t) \\ y_m(t) = C_m x_m(t) \end{cases}$$

where

$$A_m = (H_m \otimes I_s) \in \mathbb{R}^{ms \times ms}, B_m = \beta\,(e_1 \otimes I_s) \in \mathbb{R}^{ms \times s}, C_m = C\mathcal{V}_m \in \mathbb{R}^{r \times ms}. \tag{20}$$

Theoretical Results

Next, we show that the reduced order model approximates the behavior of the original model. We have the following results:

Theoreme

The matrices A_m, B_m and C_m generated by applying the global/block Arnoldi process are such that the first m Markov parameters of the original and the reduced models are the same:

$$CA^jB = C_mA_m^jB_m, \quad \text{for } j = 0, 1, \ldots, m-1.$$

Extended Block Krylov Subspace

Definition

For a matrix $V \in \mathbb{R}^{n\times p}$, the extended block Krylov subspace $K_m(A, V)$ is defined by the span of the columns of matrices $A^k V$, for k ranging from $-m$ to $m-1$:

$$\mathcal{K}_m(A, V) = Range\{A^{-m}V, \ldots, A^{-2}V, A^{-1}V, V, AV, A^2V, \ldots, A^{m-1}V\}. \quad (21)$$

• Let $\mathbb{T}_m = [T_{ij}] \in \mathbb{R}^{2mr \times 2mr}$ be the restriction of the matrix A to the extended Krylov subspace $\mathcal{K}_m(A, V)$, i.e.,

$$\mathbb{T}_m = \mathbb{V}_m^T A \mathbb{V}_m.$$

$\mathbb{T}_m$ is an upper block Hessenberg with $2r \times 2r$ blocks.
• Consider the restriction of the matrix A^{-1} to the extended Krylov subspace $\mathcal{K}_m(A, V)$, i.e.,

$$\mathbb{L}_m = \mathbb{V}_m^T A^{-1} \mathbb{V}_m.$$

The matrix $\mathbb{L}_m = [L_{i,j}]$ is also an upper block Hessenberg matrix.

Proposition

Let $\tilde{\mathbb{T}}_m = \mathbb{V}_{m+1}^T A \mathbb{V}_m$, *and* $\overline{\mathbb{L}}_m = \mathbb{V}_{m+1}^T A^{-1} \mathbb{V}_m$, *suppose that m steps of Algorithm EBA have been carried out. Then we have*

$$\begin{aligned} A\mathbb{V}_m &= \mathbb{V}_{m+1}\tilde{\mathbb{T}}_m \\ &= \mathbb{V}_m \mathbb{T}_m + V_{m+1} T_{m+1,m} E_m^T. \end{aligned} \tag{22}$$

and

$$\begin{aligned} A^{-1}\mathbb{V}_m &= \mathbb{V}_{m+1}\overline{\mathbb{L}}_m \\ &= \mathbb{V}_m \mathbb{L}_m + V_{m+1} L_{m+1,m} \mathbb{E}_m^T. \end{aligned} \tag{23}$$

Extended Global Arnoldi Algorithm

Definition

As in the block case, the classical matrix (global) extended Krylov subspace is enriched by A^{-1} to get:

$$\mathcal{K}_m^e(A, V) := \text{span}\left(\left[V, A^{-1}V, AV, A^{-2}V, \ldots, A^{m-1}V, A^{-m}V\right]\right) \tag{24}$$

So the extended matrix Krylov subspace $\mathcal{K}_m^e(A, V)$ can be seen as a combination of two classical matrix Krylov subspaces:

$$\mathcal{K}_m^e(A, V) = \mathcal{K}_m(A, V) + \mathcal{K}_m\left(A^{-1}, A^{-1}V\right) = \mathcal{K}_m(A, V) + \mathcal{K}_{m+1}\left(A^{-1}, V\right) \tag{25}$$

The Extended Global Arnoldi (EGA) process computes:

- An $n\times\ 2mr$ F-orthonormal block matrix $\mathcal{V}_m = [V_1, \ldots, V_m]$ with $V_i \in \mathbb{R}^{n\times 2r} (i = 1, \ldots, m)$.
- A $2(m+1) \times 2m$ upper block Hessenberg matrix $\overline{\mathcal{H}}_m = [H_{i,j}]$ with $H_{i,j} \in \mathbb{R}^{2\times 2}$.

Proposition

Assume that m steps of the algorithm EGA have been run and let $\overline{\mathcal{T}}_m = \mathcal{V}_{m+1}^T \diamond (A\mathcal{V}_m)$ *and* $\overline{\mathcal{L}}_m = \mathcal{V}_{m+1}^T \diamond \left(A^{-1}\mathcal{V}_m\right)$*, then we have the following relations*

$$\begin{aligned} A\mathcal{V}_m &= \mathcal{V}_{m+1}\left(\overline{\mathcal{T}}_m \otimes I_r\right) \\ &= \mathcal{V}_m\left(\mathcal{T}_m \otimes I_r\right) + V_{m+1}\left(T_{m+1,m}E_m^T \otimes I_r\right) \end{aligned} \tag{26}$$

and

$$\begin{aligned} A^{-1}\mathcal{V}_m &= \mathcal{V}_{m+1}\left(\overline{\mathcal{L}}_m \otimes I_r\right) \\ &= \mathcal{V}_m\left(\mathcal{L}_m \otimes I_r\right) + V_{m+1}^g\left(L_{m+1,m}E_m^T \otimes I_r\right) \end{aligned} \tag{27}$$

MOR via Extended Block Arnoldi Process

Applying the extended block Arnoldi process to the pair (A, B), we can verify that the original transfer function F can be approximated by:

$$\mathbb{F}_m(s) = \mathbb{C}_m \left(sI_{2mr} - \mathbb{T}_m\right)^{-1} \mathbb{B}_m, \tag{28}$$

where $\mathbb{T}_m = \mathbb{V}_m^T A \mathbb{V}_m, \mathbb{C}_m = C\mathbb{V}_m$ and $\mathbb{B}_m = \mathbb{V}_m^T B$.

MOR via Extended Global Arnoldi Process

Similarly, if m iterations of the extended global Arnoldi algorithm are applied to the pair (A, B), then we can approximate F by:

$$\mathcal{F}_m(s) = \mathcal{C}_m \left(sI_{2mr} - (\mathcal{T}_m \otimes I_r)\right)^{-1} \mathcal{B}_m, \tag{29}$$

where $\mathcal{T}_m = \mathcal{V}_m^T \diamond (A\mathcal{V}_m)$, $\mathcal{C}_m = C\mathcal{V}_m$ and $\mathcal{B}_m = \mathcal{V}_m^T \diamond B$.

Developments Around $s = \infty$

The developments of $\mathbb{F}_m$ and $\mathcal{F}_m$ around $s = \infty$ give the following expressions:

$$\mathbb{F}_m(s) = \frac{1}{s}\sum_{i=0}^{\infty} m_i^b s^{-i}, \text{ with } m_i^b = \mathbb{C}_m \mathbb{T}_m^i \mathbb{B}_m$$

and

$$\mathcal{F}_m(s) = \frac{1}{s}\sum_{i=0}^{\infty} m_i^g s^{-i}, \text{ with } m_i^g = \mathcal{C}_m \left(\mathcal{T}_m \otimes I_r\right)^i \mathcal{B}_m.$$

Markov Parameters Matching

Proposition

In this case, one can show that the first m Markov parameters are matched. In the block case:

$$M_i = m_i^b, \quad i = 0, \ldots, m-1$$

and in the global case:

$$M_i = m_i^g, \quad i = 0, \ldots, m-1.$$

Neumann Series Around $s = 0$

The development of the Neumann series of F around $s = 0$ gives the following expression:

$$F(s) = \sum_{i=0}^{\infty} \widetilde{M}_{i+1} s^i$$

The matrix coefficients $\widetilde{M}_i$ are called the moments of F and they are given by:

$$\widetilde{M}_j = -CA^{-j}B, \quad j = 1, 2, \ldots$$

Taylor Series of $\mathbb{F}_m$ and $\mathcal{F}_m$

By considering the Taylor series of $\mathbb{F}_m$ and $\mathcal{F}_m$, we get the following expansion of $\mathbb{F}_m$ around $s = 0$:

$$\mathbb{F}_m(s) = \sum_{i=0}^{\infty} \widetilde{m}_{i+1}^{b} s^i, \text{ with } \widetilde{m}_i^b = -\mathbb{C}_m \mathbb{T}_m^{-i} \mathbb{B}_m$$

while for $\mathcal{F}_m$, we get:

$$\mathcal{F}_m(s) = \sum_{i=0}^{\infty} \widetilde{m}_{i+1}^{g} s^i, \text{ with } \widetilde{m}_i^g = -\mathcal{C}_m \left(\mathcal{T}_m \otimes I_r\right)^{-i} \mathcal{B}_m$$

Moment Matching

Proposition

Let $\widetilde{M}_j$ and $\widetilde{m}_j^b$ be the matrix moments given by the Neumann expansions of F and $\mathbb{F}_m$, respectively around $s = 0$. Then we have:

$$\widetilde{M}_j = \widetilde{m}_j^b, \quad \textit{for } j = 0, \ldots, m-1.$$

Proposition

Let $\widetilde{M}_j$ and $\widetilde{m}_j^g$ be the matrix moments given by the Neumann expansions of F and $\mathcal{F}_m$, respectively around $s = 0$. Then we have:

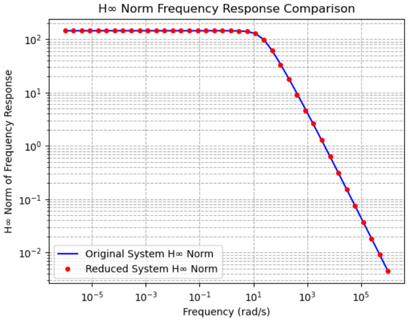
$$\widetilde{M}_j = \widetilde{m}_j^g, \quad \textit{for } j = 0, \ldots, m-1.$$

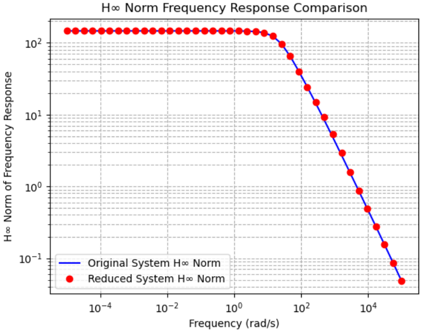
Numerical Test: Example 1

FDM Model:

- Operator: $L_A(u) = \Delta u - f(x,y)\frac{\partial u}{\partial x} - g(x,y)\frac{\partial u}{\partial y} - h(x,y)u$
- Domain: $[0,1] \times [0,1]$ with Dirichlet boundary conditions
- Functions:
  - $f(x,y) = \sin(x+2y)$
  - $g(x,y) = e^{x+y}$
  - $h(x,y) = x + y$
- Grid: $n_0 = 100$ points in each direction, $n = n_0^2$
- B, C: Random matrices $n \times p$ and $p \times n$

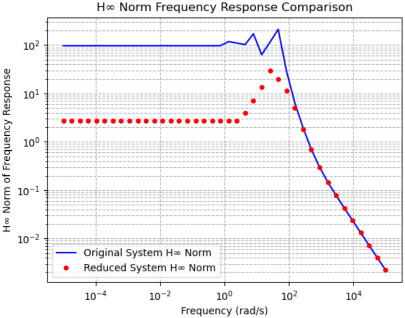
Results: Extended Krylov subspace based Method

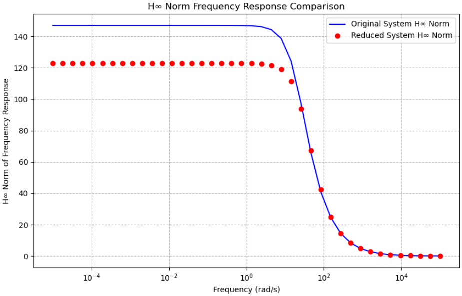




- Parameters: $m = 15$, $p = 8$
- Frequency range: $\omega \in [10^{-5}, 10^{5}]$
- Right: Extended Block / Left: Extended Global

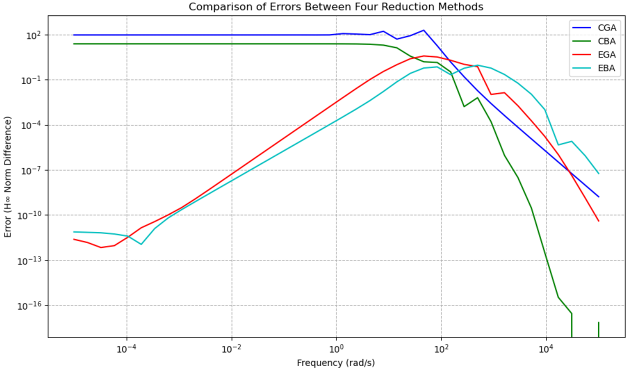
Results: Classical Krylov subspace based Method





- Parameters: $m = 20,\ p = 6$
- Frequency range: $\omega \in [10^{-5}, 10^{5}]$
- Right: Block case / Left: Global case

Comparison of Methods



- EBA: Extended block Arnoldi
- EGA: Extended global Arnoldi
- CGA: Classical global Arnoldi
- CBA: Classical block Arnoldi

Numerical Test: Example 2

Benchmark Models:

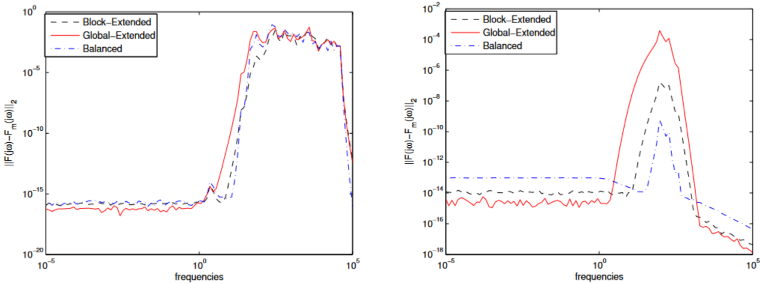
- CDplayer and FOM from NICONET

Matrix A	**size n**	$\|A\|_F$	cond(A)
FOM	1006	$1.82e+04$	1000
CDplayer	120	$2.31e+05$	$1.81e+04$

Table: Test matrices

Comparison of Methods

- Compared methods:
  - Extended block Arnoldi (dashed)
  - Extended global Arnoldi (solid)
  - Balanced truncation (dashed-dotted)
- Error measured: $\|F(i\omega) - F_m(i\omega)\|_2$
- Frequency range: $\omega \in [10^{-5}, 10^5]$
- Parameters:
  - CDplayer: $m = 10,\ r = 2$
  - FOM: $m = 15,\ r = 3$



Multi-linear dynamical systems via Einstein Product

In the discrete case, a MLTI system could be described as follows:

$$\begin{cases} \mathcal{X}_{k+1} = \mathcal{A} *_N \mathcal{X}_k + \mathcal{B} *_M \mathcal{U}_k, \quad \mathcal{X}_0 = 0 \\ \mathcal{Y}_k = \mathcal{C} *_N \mathcal{X}_k \end{cases} \tag{30}$$

Where:

- $*_N$ denotes Einstein product
- $\mathcal{X}_k \in \mathbb{R}^{J_1 \times \ldots \times J_N}$
- $\mathcal{A} \in \mathbb{R}^{J_1 \times \ldots \times J_N \times J_1 \times \ldots \times J_N}$ is a square tensor
- $\mathcal{B} \in \mathbb{R}^{I_1 \times \ldots \times J_N \times K_1 \times \ldots \times K_M}$
- $\mathcal{C} \in \mathbb{R}^{I_1 \times \ldots \times I_M \times J_1 \times \ldots \times J_N}$
- $\mathcal{U}_k \in \mathbb{R}^{K_1 \times \ldots \times K_M}$
- $\mathcal{Y}_k \in \mathbb{R}^{I_1 \times \ldots \times I_M}$

Proposition

The transfer function associated with MLTI dynamical system is given by:

$$\mathcal{F}(s) = \mathcal{C} *_N (s\mathcal{I} - \mathcal{A})^{-1} *_N \mathcal{B} \tag{31}$$

The first step in the approximation of the associated transfer function to the MLTI system is rewriting it as:

$$\mathcal{F}(s) = \mathcal{C} *_N \mathcal{X} \tag{32}$$

where $\mathcal{X}$ verifies the following multi-linear system:

$$(s\mathcal{I} - \mathcal{A}) *_N \mathcal{X} = \mathcal{B} \tag{33}$$

Tensor Krylov Subspace

- To find an approximation to $\mathcal{F}(s)$, we approximate the above multi-linear system using a projection Krylov subspace technique.
- Define the m-th tensor Krylov subspace:

$$\mathcal{K}_m(\mathcal{A}, \mathcal{B}) = \text{span}\left\{\mathcal{B}, \mathcal{A} * \mathcal{B}, \ldots, \mathcal{A}^{\star m-1} * \mathcal{B}\right\} \tag{34}$$

- We focus on the case where $N = M = 2$ for simplicity, considering 4th-order tensors.

Product $\circledast$

Definition

Given a matrix $P \in \mathbb{R}^{m \times n}$ and a tensor $\mathcal{J} \in \mathbb{R}^{K_1 \times K_2 \times K_1 \times K_2}$, the resulting tensor $\mathcal{R} = P \circledast \mathcal{J}$ is defined as follows:

1. $\mathcal{R} \in \mathbb{R}^{K_1 \times mK_2 \times K_1 \times nK_2}$
2. $\mathcal{R}(i, :, i, :) = P \otimes \mathcal{J}(i, :, i, :)$, *for* $i = 1, \ldots, K_1$, *where* $\otimes$ *is the Kronecker product.*

After m steps, we get the following decomposition:

$$\mathcal{A} * \widetilde{\mathcal{V}}_m = \widetilde{\mathcal{V}}_{m+1} * (\bar{H}_m \circledast \mathcal{I}_K) \tag{35}$$

where $\widetilde{\mathcal{V}}_{m+1} \in \mathbb{R}^{J_1 \times J_2 \times K_1 \times (m+1)K_2}$ and $\bar{H}_m \in \mathbb{R}^{m+1 \times m}$ is a Hessenberg matrix.

Extended Tensor Global Krylov Subspace

- The extended tensor global Krylov subspace of dimension $2m$, denoted by $\mathcal{K}_m^e(\mathcal{A}, \mathcal{V})$, is defined as follows:

$$\begin{aligned} \mathcal{K}_m^e(\mathcal{A}, \mathcal{V}) &= \mathcal{K}_m(\mathcal{A}, \mathcal{V}) + \mathcal{K}_m\left(\mathcal{A}^{-1}, \mathcal{A}^{-1} * \mathcal{V}\right) && (36) \\ &= \text{span}\left\{\mathcal{A}^{-\star m} * \mathcal{V}, \ldots, \mathcal{A}^{-1} * \mathcal{V}, \mathcal{V}, \ldots, \mathcal{A}^{\star m-1} * \mathcal{V}\right\} && (37) \end{aligned}$$

with

- $\mathcal{A}^{\star k} = \underbrace{\mathcal{A} * \ldots * \mathcal{A}}_{k}$
- $\mathcal{A}^{\star -k} = \underbrace{\mathcal{A}^{-1} * \ldots * \mathcal{A}^{-1}}_{k}$
- $\mathcal{K}_m(\mathcal{A}, \mathcal{V})$: tensor classic global Krylov subspace associated with the pair $(\mathcal{A}, \mathcal{V})$

Algorithm Explanation

- Algorithm creates an orthonormal tensor basis called $\widetilde{\mathcal{V}}_{2(m+1)}$.
- It ensures that for $i, j = 1, \ldots, m$, the following conditions are met:

$$\mathcal{V}_i^T \diamond \mathcal{V}_j = 0_{2\times 2} \quad (i \neq j) \quad \text{and} \quad \mathcal{V}_i^T \diamond \mathcal{V}_i = I_{2\times 2}$$

- An upper Hessenberg matrix $\bar{H}_m \in \mathbb{R}^{2(m+1)\times 2m}$ is generated with upper block matrices $H_{i,j} \in \mathbb{R}^{2\times 2}$.

After m steps of Algorithm TEGA, we can show that:

$$\begin{aligned} \mathcal{A} * \widetilde{\mathcal{V}}_{2m} &= \widetilde{\mathcal{V}}_{2(m+1)} * \left(\bar{T}_m E_m^T \circledast \mathcal{I}_K \right) \\ &= \widetilde{\mathcal{V}}_{2m} * \left(T_m \circledast \mathcal{I}_k \right) + \mathcal{V}_{m+1} * \left(T_{m+1,m} E_m^T \circledast \mathcal{I}_K \right) \end{aligned} \tag{38}$$

where E_m is the last $2m \times 2$ block column of the identity matrix $I_m \in \mathbb{R}^{2m \times 2m}$, $\bar{T}_m = \tilde{\mathcal{V}}_{2(m+1)}^T \diamond \left(\mathcal{A} * \widetilde{\mathcal{V}}_{2m} \right) \in \mathbb{R}^{2(m+1) \times 2m}$ and T_m is a $2m \times 2m$ block Hessenberg matrix defined by $T_m = \widetilde{\mathcal{V}}_{2m}^T \diamond \left(\mathcal{A} * \widetilde{\mathcal{V}}_{2m} \right) \in \mathbb{R}^{2m \times 2m}$

Original MLTI System

The original MLTI system:

$$\begin{aligned} \mathcal{X}_{k+1} &= \mathcal{A} * \mathcal{X}_k + \mathcal{B} * \mathcal{U}_k, \quad \mathcal{X}_0 = 0 \\ \mathcal{Y}_k &= \mathcal{C} * \mathcal{X}_k \end{aligned}$$

where $\mathcal{A} \in \mathbb{R}^{J_1 \times J_2 \times J_1 \times J_2}$ and $\mathcal{B}, \mathcal{C}^T \in \mathbb{R}^{J_1 \times J_2 \times K_1 \times K_2}$.

Classical Global Krylov Subspace

The reduced MLTI system using the classical global Krylov subspace:

$$\begin{aligned} \hat{\mathcal{X}}_{k+1} &= \hat{\mathcal{A}} * \hat{\mathcal{X}}_k + \hat{\mathcal{B}} * \hat{\mathcal{U}}_k, \quad \hat{\mathcal{X}}_0 = 0 \\ \hat{\mathcal{Y}}_k &= \hat{\mathcal{C}} * \hat{\mathcal{X}}_k \end{aligned}$$

with the associated tensorial structures:

$$\hat{\mathcal{A}} = H_m \circledast \mathcal{I}_K, \quad \hat{\mathcal{B}} = \|\mathcal{B}\| e_1^m \circledast \mathcal{I}_K, \quad \hat{\mathcal{C}} = \mathcal{C} * \widetilde{\mathcal{V}}_m \tag{39}$$

Extended Global Krylov Subspace

The reduced MLTI system using the extended global Krylov subspace:

$$\begin{aligned} \hat{\mathcal{X}}_{k+1} &= \hat{\mathcal{A}} * \hat{\mathcal{X}}_k + \hat{\mathcal{B}} * \hat{\mathcal{U}}_k, \quad \hat{\mathcal{X}}_0 = 0 \\ \hat{\mathcal{Y}}_k &= \hat{\mathcal{C}} * \hat{\mathcal{X}}_k \end{aligned}$$

with the associated tensorial structures:

$$\hat{\mathcal{A}} = T_m \circledast \mathcal{I}_K, \quad \hat{\mathcal{B}} = \left(\widetilde{\mathcal{V}}_{2m}^T \diamond \mathcal{B}\right) \circledast \mathcal{I}_K, \quad \hat{\mathcal{C}} = \mathcal{C} * \widetilde{\mathcal{V}}_{2m} \tag{40}$$

Example 1

- The partial differential equation describing the evolution of heat distribution in a solid medium over time:

$$\begin{cases} \frac{\partial}{\partial t}\phi(t,x) = c^2 \frac{\partial^2}{\partial x^2}\phi(t,x) + \delta(x)u_t, & x \in D \text{ (square } D = [-\pi^2, \pi]), \\ \phi(t,x) = 0, & x \in \partial D, \end{cases} \tag{41}$$

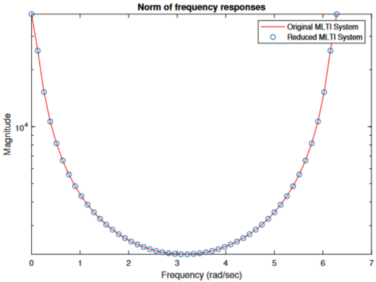
where $c > 0$, u_t is a one-dimensional control input and $\delta(x)$ is the Dirac delta function centered at zero.

Example 1: MLTI System

We consider the following system tensors:

- The tensor $\mathcal{A} \in \mathbb{R}^{N \times N \times N \times N}$ with $N = 128$ is the tensorization of $\frac{c^2 \Delta t}{h^2} \Delta_{dd} \in \mathbb{R}^{N^2 \times N^2}$, where Δ_{dd} is the discrete Laplacian on a rectangular grid with a Dirichlet boundary condition.
- The constructed tensor $\mathcal{B}$ is of dimension $N \times N \times 1 \times 1$. Here, we choose it as a sparse tensor in $\mathbb{R}^{N \times N \times K_1 \times K_2}$ with $K_1 = 3$ and $K_2 = 5$.
- The output tensor $\mathcal{C}$ is chosen as a random tensor in $\mathbb{R}^{K_1 \times K_2 \times N \times N}$ with $K_1 = 3$ and $K_2 = 5$.
- We set $m = 2$, the associated dimension of the extended tensor Krylov subspace.

Example 1 Results



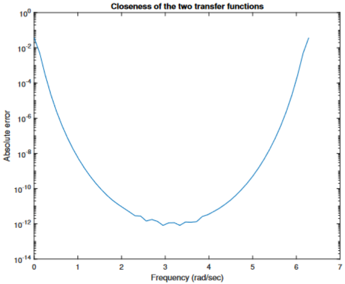


Figure: The frequency responses of the original and reduced MLTI systems (left) and the error norms $\|\mathcal{F} - \mathcal{F}_m\|_\infty$ (right).

Perspectives

1. Trying to use the T-product in place of the Einstein product for multilinear dynamical systems. This could potentially lead to similar or even better results, and it would be interesting to compare the performance and computational efficiency of these two approaches.
2. Studying the preservation of specific system properties (like stability or passivity) in the reduced models.
3. Exploring the use of machine learning techniques in combination with these model reduction methods.
4. Applying these techniques to real-world problems in areas such as control systems, signal processing, or computational fluid dynamics.

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Thank you for your attention