Comparison of Conventional and Subspace based Algorithms to Estimate Direction of Arrival (DOA)

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***Abstract*—This paper deals with two categories of direction of arrival (DOA) estimation techniques: spectral based techniques and subspace based techniques. The space based approaches are limited with resolution, so we can achieve high resolution using the subspace based approaches. Mutiple Signal Classification (MUSIC) and Estimation of Signal Parameters via Rotational Invariance Technique (ESPRIT) are two widely used subspace based methods. In this paper, we discuss the concept of DOA estimation and compare the performance of each algorithm in terms of estimation accuracy.**

***Index Terms*—Direction-of-Arrival, DOA, Delay-and-Sum, MVDR, MUSIC, ESPRIT**

I. Introduction

Direction of arrival (DOA) estimation from data collected by sensor arrays is vital to many applications such as radar, sonar, wireless communications, biomedicine, astronomy communications. The primary goal for formation of an array is to combine the outputs from the sensors so that various parameters of the incident signals can be estimated by enhancing the signal-to-noise ratio (SNR). The direction of arrival angle is one such important parameter, and the objective of DOA esimtaion methods is to determine the direction of arrival without errors.

DOA estimation algorithms can be divided into three categories: spectral-based algorithms, subspace-based algorithms and parametric algorithms. Subspectral-based algorithms are using the concept of null-steering and beamforming and subspace-based algorithms are based on the concept of decomposition of eigenvalues. In the paper titled "Comparison of Conventional and subspace based algorithms to estimate direction of arrival (DOA)"[1], spectral-based algorithms (conventional delay and sum beamformer (CBF) and Minimum Variance Distortionless Response Beamformer (MVDR)) and one of the subspace-based algorithms, Multiple Signal Classification (MUSIC) are discussed and the performance of each algorithm is compared.

In this paper, we will discuss the three approaches and Estimation of Signal Parameters via Rotational Invariance Technique (ESPRIT) [2] we saw in the midterm exam. I implemented the algorithms and applied them to a new scenario where there are two sources and four sensors. The number of time snapshots is 200 and SNR is 10dB. I also calculate the root mean square error (RMSE) to compare their performance in terms of esimation accuracy.

II. System Model for DOA Estimation

DOA estimation methods discussed in this paper use a uniform linear array (ULA). Consider an aray that consist of M sensors that are uniformly spaced on a line as shown in Fig. 1.

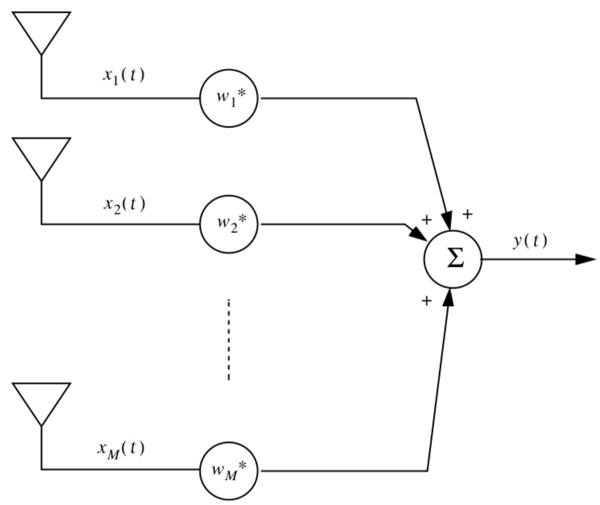

Fig. 1. Classical Beamforming Structure

The array output vector is obtained as

$$\mathbf{x}(t) = \mathbf{a}(\theta)s(t), \tag{1}$$

where $\mathbf{a}(\theta) = [1, \exp^{-j\frac{2\pi}{\lambda}kd\sin\theta}, \ldots, \exp^{-j\frac{2\pi}{\lambda}k(M-1)d\sin\theta}]$ is steering vector and $s(t)$ is the signal transmitted from the source. If N signals impinge on an M-dimensional array from distinct DOAs, $\theta_1, \theta_2, \ldots, \theta_M$, the output vector takes the form of

$$\mathbf{x}(t) = \sum_{i=1}^{N} \mathbf{a}(\theta_i)s_i(t), \tag{2}$$

where $s_i(t)$ denotes the signal from ith source. The output equation can be in a more compact form by defining a steering matrix and a vector of signal waveforms as

$$\mathbf{A}(\theta) = [\mathbf{a}(\theta_1), \ldots, \mathbf{a}(\theta_N)] \tag{3}$$

$$\mathbf{s}(t) = [s_1(t), \ldots, s_N(t)]^T \tag{4}$$

In the presence of an additive noise $\mathbf{n}(t)$, we now get the model commonly used in array processing

$$\mathbf{x}(t) = \mathbf{A}(\theta)s(t) + \mathbf{n}(t). \tag{5}$$

The idea of finding DOAs is to steer the array in one direction at a time and measure the output power. The output power should peak at the DOAs of the sources. The array response is steered by forming a linear combination of the sensor outputs

$$y(t) = \sum_{i=1}^{M} w_i^* x_i(t) = \mathbf{w}^H \mathbf{x}(t), \tag{6}$$

where $\mathbf{w}$ is the weighting vector used for cancelling the phase delay between the sensors and $\mathbf{w}^H$ is the Hermitian of $\mathbf{w}$. K samples of $y(t)$ are taken with time interval T between samples and $t = nT$, where $n = 1, 2, \ldots, K$. The output power is measured by

$$\begin{aligned} P(\mathbf{w}) = \frac{1}{K} \sum_{t=1}^{K} |y(t)|^2 &= \frac{1}{K} \sum_{t=1}^{K} \mathbf{w}^H \mathbf{x}(t) \mathbf{x}(t)^H \mathbf{w} \\ &= \mathbf{w}^H \mathbf{R} \mathbf{w}, \end{aligned} \tag{7}$$

where $\mathbf{R} = \frac{1}{K} \sum_{t=1}^{K} \mathbf{x}(t)\mathbf{x}(t)^H$

III. Spectral based DOA Estimation Algorithms

In spectral based estimation, a spectrum-like function of the parameter of interest is formed. The locations of the highest peaks of the function are the DOA estimates.

A. Conventional Delay and Sum Beamformer (CBF)

This algorithm maximizes the power of the beamforming output for a given signal. Given a signal emanating from direction θ, the problem of maximizing the output power is formulated as

$$\begin{aligned} \max E[\mathbf{w}^H \mathbf{x}(t)\mathbf{x}(t)^H \mathbf{w}] &= \max \left\{ \mathbf{w}^H E[\mathbf{x}(t)\mathbf{x}(t)^H] \mathbf{w} \right\} \\ &= \max \left\{ E\left|s(t)\right|^2 \left|\mathbf{w}^H \mathbf{a}(\theta)\right|^2 + \sigma^2 |\mathbf{w}|^2 \right\} \end{aligned} \tag{8}$$

where σ^2 is the noise covariance and the assumption of spatial white noise is used. For CBF, the resulting solution for $\mathbf{w}$ is the steering vector. The output power for different angles is

$$P(\theta) = \mathbf{a}^H(\theta) \mathbf{R} \mathbf{a}(\theta) \tag{9}$$

The pseudocode of the CBF is depicted in Algorithm 1.

Algorithm 1 Conventional Delay and Sum Beamformer

$\mathbf{R} = \frac{1}{K} \sum_{t=1}^{K} \mathbf{x}(t)\mathbf{x}(t)^H$
$P(\theta) = \mathbf{a}^H(\theta) \mathbf{R} \mathbf{a}(\theta)$
Find peaks

Fig. 2 shows the CBF output power when two sources are located at $\theta_1 = -20°$ and $\theta_2 = 30°$. The power peaks are observed at approximately $-20°$ and $30°$. However, the DOA resolution of beamforming for signals coming from brodeside is [3]

$$\theta_{BW} = \arcsin \frac{\lambda}{Md} \tag{10}$$

When we have a ULA of M = 4 sensors of half-wavelenghting inter-element spacing, sources need to be at least 30° apart in order to be separated by the beamformer.

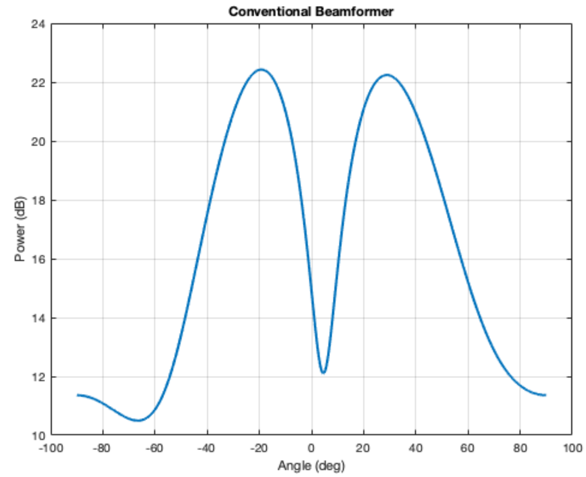

Fig. 2. DOA of two sources at $-20°$and $30°$, M=4, K=200

When two sources are located at $\theta_1 = -5°$ and $\theta_2 = 20°$, the conventional beamformer cannot resolve out the sources within the beamwidth as shown in Fig. 3.

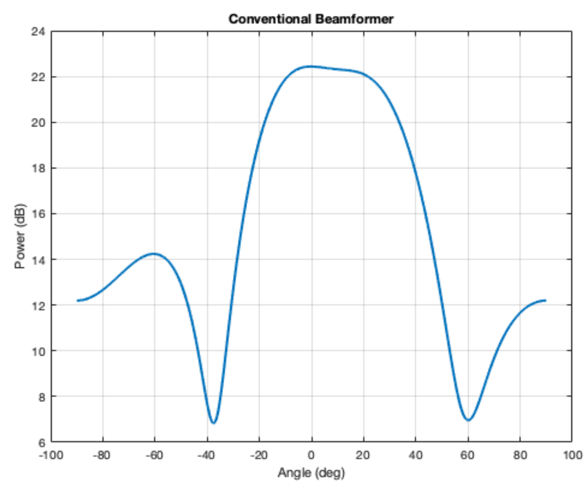

Fig. 3. DOA of two sources at $-5°$and $20°$, M=4, K=200

B. Minimum Variance Distortionless Response Beamformer (MVDR)

To alleviate the limitation of the conventional beamformer, the Minimum Variance Distortionless Response (MVDR) beamformer is proposed by Capon. The idea is that for each possible angle θ, the power in the cost function must be minimized w.r.t $\mathbf{w}$ subject to a single constraint:

$$\begin{aligned} &\min_{\mathbf{w}} P(\mathbf{w}), \\ &\text{subject to } \mathbf{w}^H \mathbf{a}(\theta) = 1 \end{aligned} \tag{11}$$

where $P(\mathbf{w})$ is as defined in Eq. 7. The optimal $\mathbf{w}$ can be obtained using the Lagrange multiplier, resulting in

$$\mathbf{w} = \frac{\mathbf{R}^{-1}\mathbf{a}(\theta)}{\mathbf{a}(\theta)^H\mathbf{R}^{-1}\mathbf{a}(\theta)} \tag{12}$$

By inserting the above equation into Eq. 7, we can obtain the MVDR received power,

$$P(\theta) = \frac{1}{\mathbf{a}^H(\theta)\mathbf{R}^{-1}\mathbf{a}(\theta)} \tag{13}$$

This beamformer attempts to minimize the power contributed by noise and any signals coming from other directions than θ, while maintaining a fixed gain in the "look direction θ" like as a sharp spatial bandpass filter. The pseudocode of the MVDR is described in Algorithm 2.

Algorithm 2 MVDR Beamformer

1: $\mathbf{R} = \frac{1}{K}\sum_{t=1}^{K}\mathbf{x}(t)\mathbf{x}(t)^H$
2: $\mathbf{R}^{-1}$
3: $P(\theta) = \frac{1}{\mathbf{a}^H(\theta)\mathbf{R}^{-1}\mathbf{a}(\theta)}$
4: Find peaks

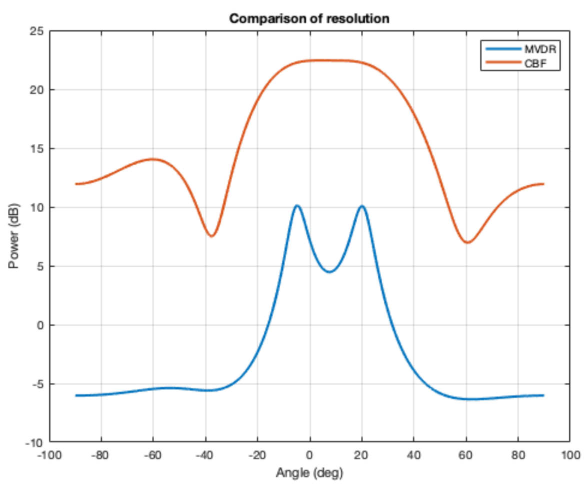

Fig. 4. DOA of two sources at $-5°$and $20°$, M=4, K=200

Here we consider two sources at $-5°$and $20°$ again. From Fig. 4 we observe that MVDR can resolve two sources within the beamwidth, meaning that the resolution of MVDR is higher than that of CBF.

IV. Subspace based DOA Estimation Algorithms

A. Multiple Signal Classification (MUSIC)

This algorithm uses the eigenvalue decomposition of the covariance matrix and the fact that the noise subspace is orthogonal to the signal subspace. The covariance matrix can be expressed as

$$\mathbf{R} = \mathbf{APA}^H + \sigma^2\mathbf{I} = \mathbf{U}_s\mathbf{\Lambda}_s\mathbf{U}_s^H + \sigma^2\mathbf{U}_n\mathbf{U}_n^H, \tag{14}$$

where $\mathbf{U}_s$ and $\mathbf{U}_n$ represents the signal subspace and noise subspace respectively. We assume that $\mathbf{APA}^H$ has full rank. Since the eigenvectors in the noise subspace are orthogonal to $\mathbf{A}$, we have

$$\mathbf{U}_n^H\mathbf{a}(\theta) = 0, \text{ for } \theta \in \{\theta_1, \theta_2, \ldots, \theta_M\} \tag{15}$$

Thus, the output spectrum for MUSIC is defined as

$$P(\theta) = \frac{1}{|\mathbf{U}_n^H\mathbf{a}(\theta)|^2} \tag{16}$$

The pseudocode of the MUSIC algorithm is provided in Algorithm 3.

Algorithm 3 MUSIC Algorithm

1: $\mathbf{R} = \frac{1}{K}\sum_{t=1}^{K}\mathbf{x}(t)\mathbf{x}(t)^H$
2: $\mathbf{R} = \mathbf{U\Lambda U}^H$ {Eigendecomposition}
3: $\mathbf{UU}^H$ {Eigenvectors multiplication}
4: $P(\theta) = \frac{1}{|\mathbf{U}_n^H\mathbf{a}(\theta)|^2}$
5: Find peaks

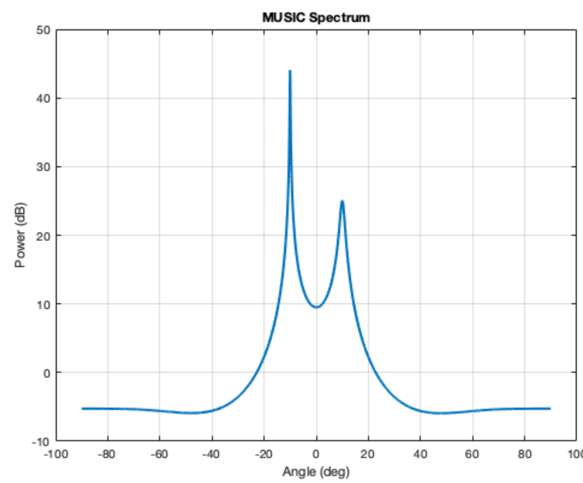

Fig. 5. DOA of two sources at $-10°$and $10°$, M=4, K=200

Fig. 5 shows that two sources resolved by MUSIC beamformer. It can be seen that MUSIC beamformer also can detect two distinct sources within the beamwidth.

B. Estimation of Signal Parameters via Rotational Invariance Technique (ESPRIT)

This algorithm is based on rotational invariance structure of sensors. ESPRIT imposes a particular geometric constraint on the array, but it has a tremendous computational advantage over MUSIC since it does not require an exhaustive search through all possible steering vectors for DOA estimation. Fig. 6 shows an example of subarrays used in ESPRIT. The subarrays are represented by $\mathbf{x}_1$ and $\mathbf{x}_2$. The output of the subarrays $\mathbf{x}_1$ and $\mathbf{x}_2$ can be represented as

$$\mathbf{x}_1 = \mathbf{As} + \mathbf{n}_{x_1} \tag{17}$$

$$\mathbf{x}_2 = \mathbf{A\Phi s} + \mathbf{n}_{x_2}, \tag{18}$$

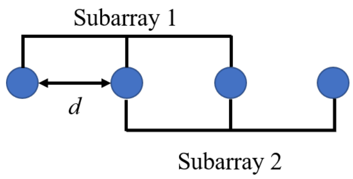

Fig. 6. Example of ESPRIT subarrays formation

where $\Phi = \text{diag}\{\exp^{j\frac{2\pi}{\lambda}d\sin\theta_1}, \ldots, \exp^{j\frac{2\pi}{\lambda}d\sin\theta_N}\}$ is an $N \times N$ diagonal matrix relating the signals received by the two subarrays, named the rotational operator [2]. The DOA estimates are given by [2]

$$\theta_k = \arcsin\left(\frac{\lambda \arg(\phi_k)}{2\pi d}\right), \tag{19}$$

where ϕ_k is the kth eigenvalue of subspace rotational operator and d is the spacing between the sensors. The pseudocode of the ESPRIT algorithm is provided in Algorithm 4.

Algorithm 4 ESPRIT Algorithm

1: $\mathbf{R} = \frac{1}{K}\sum_{t=1}^{K}\mathbf{x}(t)\mathbf{x}(t)^H$
2: $\mathbf{R} = \mathbf{U\Lambda U}^H$ {Eigendecomposition}
3: $\mathbf{\Psi} = \mathbf{T\Phi T}^H$ {Eigendecomposition}
4: $\mathbf{E}_1\mathbf{\Psi} = \mathbf{E}_2$
5: Solve ϕ
6: $\theta_k = \arcsin\left(\frac{\lambda\arg(\phi_k)}{2\pi d}\right)$

Fig. 7 is the output of the ESPRIT beamformer when two sources are located at $\theta_1 = -10°$ and $\theta_2 = 10°$.

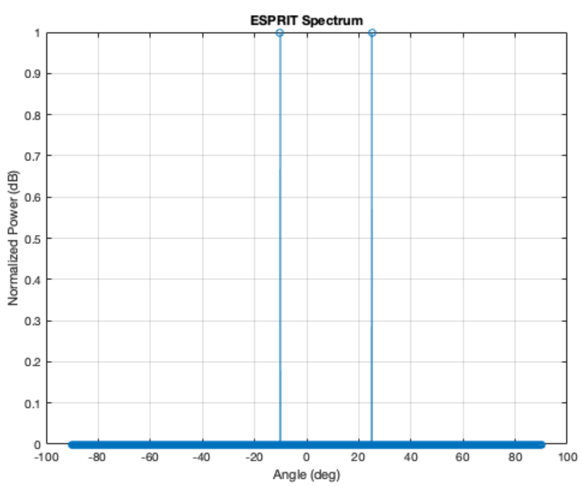

Fig. 7. DOA of two sources at $-10°$ and $10°$, M=4, K=200

V. Comparison of Beamforming Algorithms

Fig. 8 shows the root mean square of error (RMSE) of different beamformers we discussed. Two sources are located at $\theta_1 = 0°$ and $\theta_2 = 25°$. The RMSE is given by

$$RMSE = \frac{\sqrt{E(\theta_1 - \hat{\theta_1})^2} + \sqrt{E(\theta_2 - \hat{\theta_2})^2}}{2} \tag{20}$$

Since the two sources are within the beamwidth, CBF cannot resolve out the sources. Also, for the CBF, the RMSE stays the same regardless of SNR. This is because the weighting of elements is predefined as the steering vector. We can observe that subspace based estimation methods are more accurate than spectral based estimation methods.

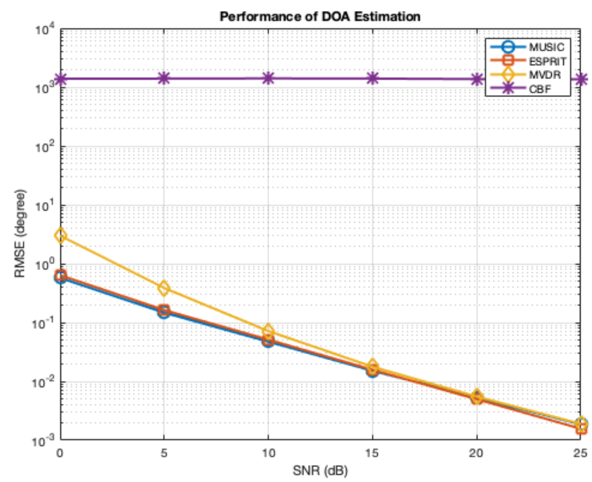

Fig. 8. RMSE of each algorithm, M=4, K=200

VI. Conclusions

In this paper, different direction of arrival (DOA) algorithms are discussed. The conventional delay and sum beamformer (CBF) cannot resolve out sources within the beamwidth. The beamwidth can be reduced by the number of sensors, but this increment of sensors will increase the cost of beamformer. Minimum Variance Distortionless Response Beamformer (MVDR) has a higher resolution than CBF because of output power minimization subject to the constraint. MUSIC algorithm has a much higher resolution than spectral based algorithms. ESPRIT also has a high resolution and it is computationally very efficient because it does not require an exhaustive search through all possible steering vectors.

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