Efficient Core Maintenance in Dynamic Uncertain Graphs

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ABSTRACT

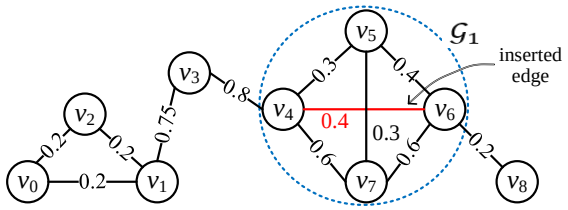
In an uncertain graph $\mathcal{G}$, a (k, η)-core is a maximal subgraph of $\mathcal{G}$ such that the probability that each vertex's degree in the subgraph is no less than k equals or exceeds η. Core decomposition in uncertain graphs, which is to calculate the η-threshold of each vertex for all possible k, is a fundamental problem for graphs analysis. While existing studies on core decomposition primarily address static uncertain graphs in the literature, many real-world scenarios involve highly dynamic uncertain graphs. It is costly to recompute all η-thresholds from the scratch whenever the uncertain graphs update, e.g., edge insertion and deletion, and edge probability increase and decrease. Hence, in this paper, we introduce efficient core maintenance algorithms tailored for dynamic uncertain graphs. Firstly, we investigate the impact of edge insertion and deletion on η-thresholds. Building upon this analysis, we introduce core maintenance algorithms designed to adjust the η-thresholds for edge insertions or deletions within the uncertain graphs. Our approaches involve identifying a compact subgraph encompassing all vertices necessitating η-threshold updates, followed by an iterative process of vertex deletion to complete the η-threshold updates. To improve the algorithms' efficiency, we devise three optimizations to further reduce the number of candidate η-thresholds requiring adjustment. Moreover, we extend proposed algorithms to handle the core maintenance for edge probability change. Finally, we conduct extensive experiments on both real and synthetic datasets to demonstrate the efficiency of the proposed algorithms. The results reveal that our proposed algorithms consistently outperform the baseline, exhibiting improvements ranging from at least three orders of magnitude to as high as six orders of magnitude.

PVLDB Reference Format:
Yu Chen, Qing Liu, and Yunjun Gao. Efficient Core Maintenance in Dynamic Uncertain Graphs. PVLDB, 14(1): XXX-XXX, 2020.
doi:XX.XX/XXX.XX

PVLDB Artifact Availability:
The source code, data, and/or other artifacts have been made available at https://github.com/ZJU-DAILY/TASK/UnMaintenance.

1 INTRODUCTION

Uncertainty is ubiquitous in graph data due to a variety of reasons, such as noisy measurements [2], inference and prediction models [28], and privacy concerns [8]. In these cases, the graph is modeled as an uncertain graph [24], whose edges are assigned with a probability of existence. Some uncertain graph examples include protein-protein interaction (PPI) networks with experimentally inferred links [4], social networks with inferred influence [36], and sensor networks with uncertain connectivity links [23]. In the literature, many cohesive subgraph models have been studied for uncertain graphs. In this paper, we focus on (k, η)-core [9].



(a) An example uncertain graph $\mathcal{G}$

k	v_0	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8
1	0.36	0.75	0.36	0.8	0.8	0.706	0.706	0.706	0.2
2	0.04	0.04	0.04	0.04	0.18	0.18	0.18	0.18	-

(b) Core decomposition result of $\mathcal{G}$

k	v_0	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8
1	0.36	0.75	0.36	0.8	0.8	0.706	0.76	0.76	0.2
2	0.04	0.04	0.04	0.04	0.258	0.258	0.258	0.258	-
3	-	-	-	-	0.036	0.036	0.036	0.036	-

(c) Core decomposition after the insertion of (v_4, v_6, 0.4)

Figure 1: A running example

Specifically, a (k, η)-core is a maximal subgraph of an uncertain graph $\mathcal{G}$ such that the probability that each vertex's degree in the subgraph is no less than k equals or exceeds η. Given an integer k, the η-threshold of a vertex u is the maximum η among all (k, η)-cores containing u. Core decomposition in an uncertain graph $\mathcal{G}$ involves calculating the η-threshold for each vertex w.r.t. all possible k values. We use Figure 1 as an example. Figure 1(a) depicts an uncertain graph $\mathcal{G}$. The subgraph $\mathcal{G}_1$ induced by vertices $\{v_4, v_5, v_6, v_7\}$ is a $(2, 0.18)$-core. Figure 1(b) shows the core decomposition results of $\mathcal{G}$. According to Figure 1(b), when $k = 1$, the η-threshold of v_4 is 0.8, meaning v_4 is in $(1, 0.8)$-core, but not in any $(1, \eta)$-cores with $\eta > 0.8$.

Core decomposition of uncertain graphs has a wide range of applications, such as describing biological functions of proteins in PPI networks [25], identifying influential spreaders in social networks [46], analyzing the topological structure of sensor networks [14], and community search [35, 39]. For example, in a collaboration uncertain graph $\mathcal{G}$, vertices represent individuals and edges associated with a probability denote the likelihood of collaboration between two individuals. The (k, η)-core decomposition of $\mathcal{G}$ can

Proceedings of the VLDB Endowment, Vol. 14, No. 1 ISSN 2150-8097.
doi:XX.XX/XXX.XX

facilitate the identification of a cohesive team, wherein individuals are highly probable to collaborate [9].

Motivations. In real-world applications, uncertain graphs are usually dynamic. For example, in PPI networks, we may find new protein interactions or revise some existing protein interactions, and the protein interaction likelihood changes under different conditions. In social networks, the friendship and influence between users are constantly changing. In sensor networks, nodes may be added or removed, and the connectivity probability between two nodes may change due to transmission issues. These changes involve the vertices/edges insertion and deletion, as well as adjustments to edge probability, in the uncertain graph, and may further affect some downstream applications, which require the latest η-thresholds. For instance, in PPI networks, the latest η-thresholds can help to discover new protein complexes [32, 37, 38]. In social networks, the η-threshold reflects the current probability of action or information propagation between users in a community. Timely η-thresholds updating enables accurate prediction and control of information flow [1, 28, 54]. In sensor networks, maintaining η-thresholds allows for real-time tracking of changes in network topology to ensure network security [7, 17]. Therefore, a key issue is how to efficiently maintain the η-thresholds for dynamic uncertain networks.

A naive approach to maintain the η-thresholds is to perform the core decomposition from the scratch by employing the peeling algorithm without update technology whenever the uncertain graph changes. The corresponding time complexity is $O(k_{max} n \log n + k_{max}^2 m d_{max})$, where m, n, k_{max}, and d_{max} denote the number of edges, the number of vertices, the maximum core number, and the maximum degree of the uncertain graph, respectively. It is time consuming to perform core decomposition from the scratch whenever the uncertain graph changes. For example, in the case of the uncertain graph *Orkut* with 117 million edges, the peeling algorithm fails to complete within a month. Similarly, for the uncertain graph *Comlj* with 34 million edges, the peeling algorithm requires over four days to finish. Hence, it is imperative to develop efficient algorithms for core maintenance in uncertain graphs. Motivated by the above reasons, for the first time, we study the problem of core maintenance in dynamic uncertain graphs, i.e., to update η-thresholds when the uncertain graph changes.

Challenges and Solutions. By now, many techniques have been proposed for core maintenance in different types of graphs [27, 30, 33, 45, 60]. However, these existing techniques can not be applied to the core maintenance in uncertain graphs due to different maintenance objects. Overall, the core maintenance in uncertain graphs faces two challenges. (1) The update of uncertain graphs influences not only the core number but also the η-thresholds, making core maintenance in uncertain graphs more complex compared to other types of graphs. For instance, in Figure 1, after inserting the edge $(v_4, v_6, 0.4)$ into $\mathcal{G}$, the core numbers of vertices v_4, v_5, v_6, and v_7 increase from 2 to 3, leading to the introduction of new η-thresholds for $k = 3$. (2) Updates in uncertain graphs can induce substantial alterations in η-thresholds, presenting challenges in pinpointing the affected thresholds. Notably, different k values necessitate updates to different vertices' η-thresholds. Hence, distinct candidate vertex sets should be identified for different k values to facilitate η-threshold updates. For instance, in Figure 1, different η-thresholds should be identified for updating when $k = 1$ and $k = 2$.

Faced with these challenges, we propose efficient algorithms for core maintenance in uncertain graphs. Our approaches encompasses handling edge insertions and deletions, as well as variations in edge probabilities. Note that this paper does not delve into vertex insertions and deletions, as they can be treated as sequences of edge insertions and deletions. To efficiently maintain the cores in uncertain graphs, it is crucial to identify the candidate η-thresholds that may require updates. To this end, we first determine the range $\mathcal{R}^+$ and $\mathcal{R}^-$ of k for edge insertions and deletions, respectively. Subsequently, for each k in $\mathcal{R}^+/\mathcal{R}^-$, we further determine the range of η-threshold. If a vertex's η-threshold falls in this range, its η-threshold should be examined for update. Based on this, we traverse the uncertain graph from u/v in a BFS manner to identify the candidate vertex set, whose η-threshold may change. Then, we iteratively delete the candidate vertex with the minimum probability that the vertex's degree is no less than k to update its η-threshold. As mentioned in the first challenge, the edge insertion (resp. deletion) may introduce a new coreness (delete an old coreness) for some vertices. Finally, we update the η-thresholds for these special cases. Specifically, for the newly added coreness, we should add new η-thresholds; for the deleted coreness, we should delete the corresponding η-thresholds as well.

To improve the efficiency of our algorithm, we develop three optimizations. The first optimization introduces the concepts of restricted vertices and potential set for edge insertions and deletions, respectively, to narrow down the range of η-threshold for examinaion. The second optimization utilizes the vertices visited in the BFS traversal to prune the vertices whose η-threshold does not need update. The third optimization utilizes lazy computing to minimize redundant calculations. Moreover, for edge probability change, we extend the above proposed algorithms and optimizations to handle the core maintenance.

In summary, our principal contributions are as follows.

- We formally define the core maintenance problem in uncertain graphs. To the best of our knowledge, it is the first time to explore this problem.
- We propose basic core maintenance algorithms for edge insertion and deletion. Furthermore, we devise three optimizations to further improve the efficiency of algorithms.
- We extend the proposed algorithms to handle the core maintenance for edge probability change.
- We conduct extensive experiments on both real and synthetic uncertain graphs to demonstrate the efficiency and effectiveness of our proposed algorithms and optimizations.

Outline. Section 2 reviews the related works. Section 3 formally defines the problem. Sections 4 and 5 introduce core maintenance algorithms for edge insertion and deletion, respectively. Section 6 presents the core maintenance algorithms for the edge probability change. Section 7 reports the experimental results for out proposed algorithms. Finally, Section 8 concludes the paper.

2 RELATED WORK

Cohesive subgraphs in uncertain graphs. Many cohesive subgraph models have been proposed for uncertain graphs, including core-based models (e.g., (k, η)-core [9], (α, β, η)-core [58], and (k, θ)-core [39, 42]), truss-based models (e.g., k-Bitruss [59] and

(k,γ)-truss [21, 62]), and clique-based models (e.g., (k,τ)-clique [11, 26], α-clique [41, 61], and (α,γ)-quasi-clique [40]).

In this work, we primarily focus on the (k,η)-core model. To be specific, Bonchi et al. [9] firstly extended the k-core model to uncertain graphs and proposed a peeling algorithm with dynamical η-degree updating technique for core decomposition. Here, η-degree of a vertex u is the maximum k such that the probability that u's degree is no less than k equals or exceeds η. Then, Yang et al. [53] proposed a UCF-Index, which store the core decomposition results, to facilitate a single (k,η)-core search. In some cases, the graph may be too large to be loaded into memory, and Wen et al. [50] extended the UCF-Index to UCEF-Index, which is constructed in external memory. Moreover, Dai et al. [12] discovered that employing the peeling algorithm with the dynamical η-degree updating technique to compute the (k,η)-core is incorrect, due to recursive floating-point number division operations. To compute the (k,η)-core correctrly, Dai et al. [12] proposed bottom-up and top-down algorithms. It is worth mentioning that existing works of (k,η)-core consider static uncertain graphs, which can not be applied for efficient (k,η)-core maintenance.

Core maintenance. In the literature, many efforts have been devoted to the study of core maintenance, which is to maintain the core number of each vertex for dynamic graphs. [27, 45] primarily focused on single edge operations to maintain vertex core number. Zhang et al. [57] proposed an order-based approach to reduce the time cost by maintaining a k-order. Then, [5, 6, 18, 19, 22, 47] have investigated core maintenance with multiple edge operations. Since the connections among different k-cores in the hierarchy are not considered in above works, [29] studied the hierarchical core maintenance which is to compute the k-core hierarchy incrementally. Moreover, [3, 51] explored the core maintenance in distributed environment. In addition, researchers have also studied the core maintenance problem for other core models, including colorful h-star core [16] and distance-generalized core [31], and various types of dynamic graphs, including weighted graphs [48, 52, 55, 60], bipartite graphs [30, 34, 49] and hypergraphs [15, 20, 33]. Nevertheless, due to the different semantics of core models, existing core maintenance works cannot be extended to handle the core maintenance in dynamic uncertain graphs.

3 PRELIMINARIES

We consider an undirected uncertain graph $\mathcal{G} = (V_\mathcal{G}, E_\mathcal{G}, p)$, where $V_\mathcal{G}$ is the set of vertices, $E_\mathcal{G}$ is the set of edges, and p: $E_\mathcal{G} \rightarrow (0,1]$ is a function assigning an existence probability to each edge. Given a vertex $u \in V_\mathcal{G}$, the degree of u is $deg(u,\mathcal{G}) = |\{v \in V_\mathcal{G} | (u,v) \in E_\mathcal{G}\}|$. For a vertex set $V_C \subseteq V_\mathcal{G}$, the induced subgraph of V_C is denoted by $\mathcal{G}[V_C] = (V_C, \{(u,v) | \forall u,v \in V_C \wedge (u,v) \in E_\mathcal{G}\}, p)$. We denote the probability of an edge $e \in E_\mathcal{G}$ by p_e, and assume that the existence probability of each edge is independent of each other [9, 39, 50, 53]. In this paper, we adopt the well-known possible-world semantics [44] for uncertain graph analysis. Specifically, let $G = (V_G, E_G) \sqsubseteq \mathcal{G}$ be a possible world of $\mathcal{G}$, where $V_G = V_\mathcal{G}$ and $E_G \subseteq E_\mathcal{G}$. The probability of G, denoted by $Pr(G)$, is:

$$Pr(G) = \prod_{e \in E_G} p_e \prod_{e \in E_\mathcal{G} \setminus E_G} (1 - p_e) \quad (1)$$

Note that each possible world is a deterministic graph, and, totally, there are $2^{|E_\mathcal{G}|}$ possible worlds for an uncertain graph $\mathcal{G}$. Let $\mathcal{G}^u_{\geq k}$ be the set of all possible worlds of $\mathcal{G}$, where u's degree is not less than k, i.e., $\mathcal{G}^u_{\geq k} = \{G \sqsubseteq \mathcal{G} | deg(u,G) \geq k\}$. Based on it, we give the definition of k-probability as follows.

Definition 3.1. **(k-probability)** *[53] Given an uncertain graph $\mathcal{G}$ and an integer k, the k-probability of a vertex $u \in V_\mathcal{G}$, denoted by $k\text{-}prob(u,\mathcal{G})$, is the probability that u's degree is not less than k, i.e., $k\text{-}prob(u,\mathcal{G}) = Pr[deg(u,\mathcal{G}) \geq k] = \sum_{G \subseteq \mathcal{G}^u_{\geq k}} Pr(G)$.*

Take the vertex v_3 in Figure 1(a) as an example. $1\text{-}prob(v_3,\mathcal{G}) = 1 - 0.25 * 0.2 = 0.95$ and $2\text{-}prob(v_3,\mathcal{G}) = 0.75 * 0.8 = 0.6$. We use $k\text{-}prob(u)$ instead of k-prob$(u,\mathcal{G})$ when the context is clear. Next, we give the core definition for uncertain graphs.

Definition 3.2. **((k,η)-core)** *[9] Given an uncertain graph $\mathcal{G}$, an integer k, and a probabilistic threshold $\eta \in [0,1]$, a (k,η)-core is a maximal subgraph $\mathcal{H} = (V_\mathcal{H}, E_\mathcal{H}, p) \subseteq \mathcal{G}$ satisfying: for each vertex $u \in V_\mathcal{H}$, the probability that u has degree no less than k in $\mathcal{H}$ is not less than η, i.e., $\forall u \in V_\mathcal{H}$, $k\text{-}prob(u,\mathcal{H}) \geq \eta$.*

For example, in Figure 1(a), let $k = 2$ and $\eta = 0.1$. The subgraph $\mathcal{G}_1$ is a (2, 0.1)-core since (i) 2-prob$(v_4,\mathcal{G}_1)$ = 0.18, 2-prob$(v_5,\mathcal{G}_1)$ = 0.258, 2-prob$(v_6,\mathcal{G}_1)$ = 0.24, and 2-prob$(v_7,\mathcal{G}_1)$ = 0.504, and (ii) $\forall v \in \mathcal{G} - \mathcal{G}_1$, $\forall \mathcal{G}' \subseteq \mathcal{G}$, 2-prob$(v,\mathcal{G}') < 0.1$. We would like to highlight three points for the (k,η)-core. (1) When $\eta = 0$, the (k,η)-core of an uncertain graph $\mathcal{G}$ corresponds to the k-core of the corresponding deterministic graph by ignoring the edge probabilities of $\mathcal{G}$. Hence, we assume $\eta > 0$ in this paper. (2) For a deterministic graph, the core number of a vertex u, denoted by $c(u)$, is the maximum k such that there exists a non-empty k-core containing u. For a vertex $u \in V_\mathcal{G}$, if $k > c(u)$, there does not exist a (k,η)-core with $\eta > 0$ containing u. (3) Like k-core, the (k,η)-core also has the nesting property. Specifically, given an integer k and two probabilistic thresholds η_1 and η_2 with $0 \leq \eta_1 \leq \eta_2 \leq 1$, it holds that (k,η_1)-core contains (k,η_2)-core. Based on the nesting property of the (k,η)-core, η-threshold is defined as follows.

Definition 3.3. **(η-threshold)** *[53] Given an uncertain graph $\mathcal{G}$, an integer k, and a vertex $u \in V_\mathcal{G}$, the η-threshold of u w.r.t. k, denoted by $\eta(k,u)$, is the maximum η such that there is a non-empty (k,η)-core containing u.*

In a word, the η-threshold of u w.r.t. k represents the maximum probability of u in the k-core. For example, in Figure 1(a), let $k = 2$, we have $\eta(2,v_4) = 0.18$ since there does not exist a $(2,\eta)$-core with $\eta > 0.18$ containing v_4. By employing the η-threshold, the (k,η)-core also can be defined as follows: the (k,η)-core consists of all vertices u with $\eta(k,u) \geq \eta$. Based on the η-threshold, the core decomposition and maintenance in uncertain graphs are defined as follows.

Definition 3.4. **(Core decomposition in uncertain graphs)** *[53] Given an uncertain graph $\mathcal{G}$, the core decomposition of $\mathcal{G}$ is to calculate $\eta(k,u)$ for each vertex $u \in V_\mathcal{G}$ w.r.t. each value of $k \in [1, c(u)]$.*

Problem 1. **(Core maintenance in uncertain graphs)** *Given an uncertain graph $\mathcal{G}$ and its core decomposition result, when $\mathcal{G}$ updates, the core maintenance of $\mathcal{G}$ is to update $\eta(k,u)$ for $\forall u \in V_\mathcal{G}$ w.r.t. $\forall k \in [1, c(u)]$.*

For example, in Figure 1(b), the core decomposition results of $\mathcal{G}$ are depicted, while Figure 1(c) displays the core decomposition results after the insertion of the edge $(v_4, v_6, 0.4)$ into $\mathcal{G}$. As analyzed in Introduction, when the uncertain graph updates, it is costly to perform core decomposition from scratch. Therefore, we propose efficient maintenance algorithms in the following three sections.

4 THE BASIC ALGORITHMS

In this section, we present the basic core maintenance algorithms for the edge insertion and deletion, respectively. Note that, due to space limitation, some proofs are removed to technical report [10].

4.1 Edge Insertion

According to Problem 1, we should update the η-threshold $\eta(k,u)$ for all vertices and k's. To avoid recomputing all $\eta(k,u)$'s, we should identify the candidate η-thresholds that need to be updated.

Li et al. [27] has demonstrated that, for deterministic graphs, when an edge (u,v) is inserted or deleted with $c(u) \leq c(v)$, it holds that (1) only vertices with the same core number as u require to update of their core numbers, and (2) core numbers increase or decrease by at most 1. Based on it, we have the following theorem.

Theorem 4.1. *Assume that an edge $(u,v,p_{(u,v)})$ is inserted into an uncertain graph $\mathcal{G}$. Let $k_0 = min\{c(u), c(v)\}$. For a vertex $w \in V_{\mathcal{G}}$, if $k \in \mathcal{R}^+ = [1, k_0+1]$, $\eta(k,w)$ needs to be updated. Otherwise, $\eta(k,w)$ does not change.*

Proof. We prove Theorem 4.1 from three cases. (1) If $k \in [1, k_0]$, an edge is added to the initial $(k, \eta(k,w))$-core, causing an change in $\eta(k,w)$. (2) If $k = k_0 + 1$, the core number of certain vertices may increase by 1 to $k_0 + 1$. These vertices are added to the initial $(k_0+1, \eta(k_0+1,w))$-core, causing a change in $\eta(k_0+1,w)$. (3) If $k \notin [1, k_0+1]$, no new vertices or edges are added to the original $(k, \eta(k,w))$-core. Thus, $\eta(k,w)$ remains unchanged. □

Theorem 4.1 indicates the range of k values to determine which $\eta(k,w)$ should be updated. Based on this, for each $k \in \mathcal{R}^+$, we first propose a connectivity theorem.

Theorem 4.2. *Assume that an edge $(u,v,p_{(u,v)})$ is inserted into an uncertain graph $\mathcal{G}$. Given an integer $k \in \mathcal{R}^+$, all the vertices whose $\eta(k,w)$ values have changed should form a connected subgraph $\mathcal{G}' \subseteq \mathcal{G} \cup (u,v,p_{(u,v)})$.*

Since $\eta(k,w)$ has another parameter related to k, i.e., the vertex, in what follows, we introduce how to identify the candidate vertex, whose $\eta(k,w)$ should be updated. To facilitate clarity, we introduce the concept of induced η-threshold subgraph.

Definition 4.1. **(Induced η-threshold subgraph)** *Given an uncertain graph $\mathcal{G}$ and an integer k, the induced η-threshold subgraph for a vertex $w \in V_{\mathcal{G}}$, denoted by $\mathcal{G}_k^w$, is the $(k, \eta(k,w))$-core containing w.*

According to the definition of (k,η)-core, the probability that w has a degree of at least k in $\mathcal{G}_k^w$ is not less than $\eta(k,w)$, i.e., $k\text{-prob}(w, \mathcal{G}_k^w) \geq \eta(k,w)$. We use $N(w, \mathcal{G}_k^w)$, simplified to N_k^w, to represent the neighbor set of w in $\mathcal{G}_k^w$.

To further identify the impact of different k values on vertices' η-thresholds, we initially focus on a fixed $k \in \mathcal{R}^+$. When an edge $(u,v,p_{(u,v)})$ is inserted, there are three distinct cases: (1) $\eta(k,u) < \eta(k,v)$, (2) $\eta(k,u) = \eta(k,v)$, (3) $\eta(k,u) > \eta(k,v)$. Below, we provide a detailed analysis of case-1. After the insertion, u acquires a new neighbor v, which has a higher $\eta(k,v)$. Thus, the edge (u,v) becomes part of the updated η-threshold subgraph of u, denoted as $\mathcal{G}_k^{u+}$. The updated neighbor set of u, denoted as $N_k^{u+} = N_k^u \cup \{v\}$, leads to an increase in the k-probability of u. In this case, the inequality $k\text{-prob}(u, \mathcal{G}_k^{u+}) > k\text{-prob}(u, \mathcal{G}_k^u)$ must hold. For convenience, $k\text{-}prob(u, \mathcal{G}_k^u)$ and $k\text{-}prob(u, \mathcal{G}_k^{u+})$ are simplified as $k\text{-}prob(u)$ and $k\text{-}prob^+(u)$, respectively. Based on the above definition, we propose the following theorem.

Theorem 4.3. *Assume that an edge $(u,v,p_{(u,v)})$ with $\eta(k,u) < \eta(k,v)$ is inserted into an uncertain graph $\mathcal{G}$. Given an integer $k \in \mathcal{R}^+$ and a vertex $w \in V_{\mathcal{G}}$, we have (1) if $\eta(k,w) < \eta(k,u)$, $\eta(k,w)$ does not change; and (2) if $\eta(k,w)$ changes, it increases at most to $k\text{-}prob^+(u)$.*

Proof. First, we prove conclusion (1). If $\eta(k,w) < \eta(k,u) < \eta(k,v)$, the initial $(k,\eta(k,w))$-core has contained both u and v. The inserted edge does not add new vertices to $(k,\eta(w))$-core. Thus, $\eta(k,w)$ remains unchanged. Second, we prove conclusion (2) by contradiction. Assume that the updated η-threshold of u, denoted as $\eta'(k,u)$, is greater than $k\text{-}prob^+(u)$. It follows that $k\text{-}prob(u, \mathcal{G}_k^{u'}) > k\text{-}prob(u, \mathcal{G}_k^{u+})$, where $\mathcal{G}_k^{u'}$ represents the induced η-threshold subgraph of u after the insertion. That is, $\mathcal{G}_k^{u+}$ is a proper subgraph of $\mathcal{G}_k^{u'}$ ($\mathcal{G}_k^{u+} \subset \mathcal{G}_k^{u'}$). However, $\mathcal{G}_k^{u+}$ has contained $\mathcal{G}_k^u$ and the inserted edge (u,v). Based on conclusion (1), $\eta(k,x)$ remains unchanged if $\eta(k,x) < \eta(k,u)$, so that x cannot be part of $\mathcal{G}_k^{u+}$. As a result, $\mathcal{G}_k^{u'}$ is at most $\mathcal{G}_k^{u+}$, which is a contradiction. We conclude that $\eta(k,u)$ increases to at most $k\text{-}prob^+(u)$. Similarly, assume that $\eta(k,w)$ is increased to $\eta'(k,w)$ and $\eta'(k,w) > k\text{-}prob^+(u)$. It must be true that $(u,v) \in \mathcal{G}_k^{w'}$, as otherwise $\mathcal{G}_k^{w'}$ forms a seed $(k, \eta'_k(w))$-core before the insertion as well, which is a contradiction. However, we have proved that $\eta(k,u)$ increases to at most $k\text{-}prob^+(u)$. Thus, edge (u,v) cannot be part of $\mathcal{G}_k^{w'}$, which is again contradiction. □

Based on Theorem 4.3, we formally present the core update theorem for the insertion of an edge in an uncertain graph, which contains three different cases.

Theorem 4.4. *Assume that an edge $(u,v,p_{(u,v)})$ is inserted into an uncertain graph $\mathcal{G}$. Given an integer $k \in \mathcal{R}^+$ and a vertex $w \in V_{\mathcal{G}}$, we have the following results.*

- *if $\eta(k,u) < \eta(k,v)$ and $\eta(k,u) \leq \eta(k,w) < k\text{-}prob^+(u)$, $\eta(k,w)$ may require updating and u is assigned as the root.*
- *if $\eta(k,u) = \eta(k,v)$ and $\eta(k,u) \leq \eta(k,w) < min\{k\text{-}prob^+(u), k\text{-}prob^+(v)\}$, $\eta(k,w)$ may require updating and both u and v can be assigned as the root.*
- *if $\eta(k,u) > \eta(k,v)$ and $\eta(k,v) \leq \eta(k,w) < k\text{-}prob^+(v)$, $\eta(k,w)$ may require updating and v is assigned as the root.*

According to Theorem 4.2, w must be connected to the root through a path containing only vertices within the η-threshold range. Each range contains a lower-bound (lb) but not an upper-bound (ub).

Proof. We prove three cases respectively. (1) if $\eta(k,u) < \eta(k,v)$, Theorem 4.3 demonstrates that only $\eta(k,w) \in [\eta(k,u), k\text{-}prob^+(u))$

Algorithm 1: Edge Insertion (EI)

Input: $\mathcal{G}$, an edge $(u, v, p_{(u,v)})$ to be inserted
Output: the updated $\eta(k, w)$ for $w \in V_{\mathcal{G}}$ at each possible k

1 $\mathcal{G}' \leftarrow \mathcal{G} \cup (u, v, p_{(u,v)})$;
2 $\mathcal{R}' \leftarrow [1, min\{c(u), c(v)\}]$;
3 **foreach** $k \in \mathcal{R}'$ **do**
4 determine lb, ub and the root using Theorem 4.4;
5 $C \leftarrow$ *FindInsert* $(\mathcal{G}', root, lb, ub, k)$;
6 *UpdateInsert* (k, C);
7 k++; // *critical* $\mathcal{R}^+$ *processing*
8 $V_c \leftarrow$ *Insertion* (G, u, v); //Maintaining the core number of each vertex using Algorithm proposed in [27].
9 **if** $V_c = \varnothing$ **then**
10 **return** ;
11 **if** $k > max_{i \in V_{\mathcal{G}} \setminus V_c}\{c(i)\}$ **then**
12 *UpdateInsert* (k, V_c);
13 **else**
14 determine the root using Theorem 4.4;
15 $lb = 0$; $ub = k\text{-}prob(root, \mathcal{G}_k^{root+} \cup V_c)$;
16 $C \leftarrow$ *FindInsert* $(\mathcal{G}', root, lb, ub, k)$;
17 *UpdateInsert* (k, C);

may need to be updated. Furthermore, if the $\eta(k, w)$ value of any vertex $w \in \mathcal{G}$ increases due to the insertion, then $\eta(k, u)$ must have increased as well. This arises from the recursive argument in the proof of Theorem 4.2. Otherwise, no vertices would have their $\eta(k, w)$ values changed. We know that all updated vertices form a connected subgraph $\mathcal{G}'$ (Theorem 4.2). As u also belongs to $\mathcal{G}'$, the proof is complete. (2) if $\eta(k, u) > \eta(k, v)$, similar arguments can be used to prove this case and are omitted here. (3) if $\eta(k, u) = \eta(k, v)$, we perform the above calculations for both u and v. The updated η-threshold of u and v, denoted as $\eta'(k, u)$ and $\eta'(k, v)$, are less than or equal to $k\text{-}prob^+(u)$ and $k\text{-}prob^+(v)$ respectively. According to Definition 3.2, u would be unable to join the $(k, k\text{-}prob^+(v))$-core if $k\text{-}prob^+(u) < k\text{-}prob^+(v)$. The case is same for v. However, u and v may join the same (k, η')-core, where η' is the minimum between $k\text{-}prob^+(u)$ and $k\text{-}prob^+(v)$. Thus, $min\{k\text{-}prob^+(u), k\text{-}prob^+(v)\}$ serves as ub for both u and v. □

Next, we formally propose the Edge Insertion (EI, Algorithm 1) algorithm for core maintenance in uncertain graphs. Firstly, Theorem 4.1 determines the range $\mathcal{R}^+$ of k for edge insertions. We divide $\mathcal{R}^+$ into two parts: $\mathcal{R}^+ = \mathcal{R}' \cup \{k_0 + 1\}$, with $\mathcal{R}' = [1, k_0]$. For each $k \in \mathcal{R}'$, lines 3-6 correctly maintain the η-threshold updates. Theorem 4.4 determines the η-threshold range and the root at an integer k (line 4). Then, Algorithm 2 identifies the candidate set C (line 5) and Algorithm 3 updates the $\eta(k, w)$ value for each vertex $w \in C$ (line 6). Since the edge insertion may introduce a new coreness for some vertices, we should add new η-thresholds. Finally, critical $\mathcal{R}^+$ processing (lines 7-17) is used to handle this special case. Detailed descriptions for the related algorithms are provided below.

Algorithm 2, named **FindInsert**, performs a breadth-first search (BFS) traversal to identify all candidates reachable from the root through vertices within the η-threshold range. Initially, the candidate set C is empty and the queue Q stores the vertices to be processed, starting with the addition of the root(lines 1-2). Each

Algorithm 2: FindInsert

Input: updated graph: $\mathcal{G}$, root: r, lb, ub, k
Output: candidate set C

1 $C \leftarrow$ empty set; $visited[] \leftarrow$ **false**;
2 $Q \leftarrow$ empty queue; Q.push(r); $visited[r] \leftarrow$ **true**;
3 **while** $Q \neq \varnothing$ **do**
4 $u \leftarrow Q$.pop(); C.push(u)
5 **foreach** $(u, v) \in E$ **do**
6 **if** $lb \leq \eta(k, v) < ub \wedge \neg visited[v]$ **then**
7 Q.push(v); $visited[v] \leftarrow$ **true**;
8 **return** C ;

Algorithm 3: UpdateInsert

Input: k-value: k, candidate set: C
Output: the updated $\eta(k, w)$ for each vertex $w \in C$

1 $curThres \leftarrow 0$;
2 compute $k\text{-}prob(v)$ for each $v \in C$; // DP Algorithm [9]
3 **while** $C \neq \varnothing$ **do**
4 $u \leftarrow arg\ min_{v \in C} k\text{-}prob(v)$;
5 $curThres \leftarrow max\{k\text{-}prob(u), curThres\}$;
6 $\eta(k, u) = curThres$;
7 $C \leftarrow C \setminus \{u\}$
8 **foreach** $(u, v) \in E_{\mathcal{G}} \wedge v \in C$ **do**
9 update $k\text{-}prob(v)$;

vertex in Q is processed individually, and all its neighbors within the η-threshold range are identified and added to Q. This iterative process continues until Q is empty (lines 3-7). Finally, the candidate set C containing all candidates is returned.

Algorithm 3, named **UpdateInsert**, updates the $\eta(k, w)$ value of each vertex $w \in C$ by iteratively removing the vertice with the minimum k-probability. Initially, the DP algorithm [9] initializes the k-probability of each candidate (line 2). Then, a vertex u with the minimum k-probability is selected and its $\eta(k, u)$ value is saved as curThres (lines 4-5). We update the $\eta(k, u)$ and remove u from the candidate set C (lines 6-7). Finally, DP algorithm recalculates the k-probability for each neighbor of u (lines 8-10). Once the algorithm terminates, all vertices in C have been updated at the current k.

The detailed pseudocode for critical $\mathcal{R}^+$ processing can be found in lines 7-17 of Algorithm 1. Initially, we use the core maintenance algorithm proposed for deterministic graph edge insertion [27] to identify and update the vertices that require an increment of 1 in their core numbers. The set V_c represent these vertices. Let u be one such vertex, and $c'(u)$ denote its updated core number. In the following discussion, we will address three distinct cases.

Case-1: V_c is empty (lines 9-10). This indicates that the core numbers of all vertices remain unchanged. Consequently, the initial subgraph of calculating $\eta(k_0 + 1, w)$ for each $w \in V_{\mathcal{G}}$, both before and after the edge insertion, remains the same. Therefore, each $\eta(k_0 + 1, w)$ value remains unaffected.

Case-2: V_c is not empty and $c'(u)$ exceeds the core numbers of all other vertices, which remain unchanged (lines 11-12). Prior to the edge insertion, there is no $(c'(u), \eta)$-core ($\eta > 0$). After the edge insertion, such a core will emerge. The candidate set at this stage

is V_c. Next, UpdateInsert is employed to update the $\eta(c'(u), w)$ of each candidate $w \in V_c$.

Case-3: V_c is not empty and $c'(u)$ is less than or equal to the core number of other vertices (lines 15-18). After the insertion, the new $\eta(c'(u), u)$ is added, so that the lower-bound is set to 0 (line 15). Since a vertex with an increased core number must be connected to u, the updated neighbor set of u may be $N_u^+ \subseteq (N_u \cup V_c)$. Thus, we can find the upper-bound and the root within the subgraph $(\mathcal{G}_k^{u+} \cup V_c)$ by Theorem 4.4 (lines 14-15). Then, FindInsert and UpdateInsert maintain the $\eta(c'(u), w)$ for each vertex $w \in$ C.

Example 4.1. *In Figure 1(b), let $(v_4, v_6, 0.4)$ be the edge to be inserted. Since $c(v_4) = c(v_6) = 2$, we determine $\mathcal{R}^+ = [1, 3]$. Each $k \in \mathcal{R}^+$ will be processed sequentially. When $k = 1$, the original $\eta(1, v_4)$ and $\eta(1, v_6)$ are 0.8 and 0.706, respectively. According to Theorem 4.4, $lb = 0.706$, $root = v_6$, and $ub = k\text{-}prob(v_6, \mathcal{G}_1^{v_6+}) = 0.856$. Next, a BFS traversal is performed starting from v_6. The candidate set $C = \{v_6, v_4, v_5, v_7, v_3, v_1\}$ can be identified and updated. When $k = 2$, the execution steps are the same as above. When $k = 3$, the Insertion algorithm returns $V_c = \{v_4, \ldots, v_7\}$. This corresponds to case-2. We directly determine and update the candidate set $C = V_c$.*

4.2 Edge Deletion

Firstly, Theorem 4.5 analyzes the range of k values to determine which $\eta(k, w)$ may be affected by edge deletion. Then, for each possible k, we first propose a connectivity theorem.

Theorem 4.5. *Assume that an edge $(u, v, p_{(u,v)})$ is deleted from an uncertain graph $\mathcal{G}$. Let $k_0 = min\{c(u), c(v)\}$. For a vertex $w \in V_{\mathcal{G}}$, if $k \in \mathcal{R}^- = [1, k_0]$, $\eta(k, w)$ needs to be updated. Otherwise, $\eta(k, w)$ does not change.*

Theorem 4.6. *Assume that an edge $(u, v, p_{(u,v)})$ is deleted from an uncertain graph $\mathcal{G}$. Given an integer $k \in \mathcal{R}^-$, the vertices whose $\eta(k, w)$ have changed should form a connected subgraph $\mathcal{G}'' \subseteq \mathcal{G}$.*

Similarly, $\eta(k, w)$ has another parameter related to k, i.e., the vertex. We introduce how to identify the candidate vertex, whose $\eta(k, w)$ may be updated. Given an integer $k \in \mathcal{R}^-$, if an edge $(u, v, p_{(u,v)})$ is deleted, there are also three distinct cases. We primarily consider the $\eta(k, u) < \eta(k, v)$ case. After the deletion, u loses one neighbor v, which has a higher $\eta(k, v)$. The updated η-threshold subgraph of u, denoted as $\mathcal{G}_k^{u-}$, leads to a decrease in the k-probability of u. Thus, the inequality $k\text{-}prob(u, \mathcal{G}_k^{u-}) < k\text{-}prob(u, \mathcal{G}_k^u)$ holds in this case. For convenience, we simplify k-prob$(u, \mathcal{G}_k^{u-})$ to k-prob$^-(u)$. Based on this, we propose the following theorem.

Theorem 4.7. *Assume that an edge $(u, v, p_{(u,v)})$ with $\eta(k, u) < \eta(k, v)$ is deleted from an uncertain graph $\mathcal{G}$. Given an integer $k \in \mathcal{R}^-$ and a vertex $w \in V_{\mathcal{G}}$, we have (1) if $\eta(k, w) > \eta(k, u)$, $\eta(k, w)$ does not change; and (2) if $\eta(k, w)$ changes, it decreases at least to $k\text{-}prob^-(u)$.*

proof. First, we prove conclusion (1). If $\eta(k, u) < \eta(k, w)$, the initial $(k, \eta(k, w))$-core does not include u. Thus, the deletion does not affect the $(k, \eta(k, w))$-core, and $\eta(k, w)$ remains unchanged. Second, we prove conclusion (2). We denote the updated η-threshold of vertex w at integer k as $\eta'(k, w)$. Let x be a neighbor of u and $\eta(k, x) = \eta(k, u)$. There are two different cases: (i) if $k\text{-}prob(u) \leq k\text{-}prob(x)$, the equation $\eta(k, x) = \eta(k, u) = k\text{-}prob(u)$ must exist. After the deletion, there is $k\text{-}prob^-(u) < k\text{-}prob(x)$. This corresponds to case-i, and we easily get $\eta'(k, u) \leq \eta'(k, x)$. (ii) if $k\text{-}prob(x) < k\text{-}prob(u)$, the equation $\eta(k, x) = \eta(k, u) = k\text{-}prob(x)$ must exist. If $k\text{-}prob^-(u) < k\text{-}prob(x)$, this is the same as described above. If $k\text{-}prob(x) \leq k\text{-}prob^-(u)$, both $(k, \eta(k, u))$-core and $(k, \eta(k, x))$-core remain unchanged. Thus, $\eta'(k, x) = \eta'(k, u) = k\text{-}prob(x)$. In both cases, the inequality $\eta'(k, x) \geq \eta'(k, u)$ holds.

Next, we prove $\eta'(k, u)$ decreases at least to $k\text{-}prob^-(u)$ by contradiction. Assume that $\eta'(k, u)$ is less than $k\text{-}prob^-(u)$. According to the above proof, if $\eta(k, y) \geq \eta(k, u)$, we have $\eta'(k, y) \geq \eta'(k, u)$. These vertices and u can form a (k, η)-core, where u haing the smallest η-threshold and $\eta = prob^-(u)$. According to Definition 3.3, $k\text{-}prob^-(u)$ will be the updated η-threshold of u, which is a contradiction. Based on this, we consider the one-hop neighbor w of u. If $\mathcal{G}_k^w$ contains u before the deletion, then $\eta'(k, w)$ decreases to at least $k\text{-}prob^-(u)$ due to u. By recursively considering the one-hop neighbors of w, we can conclude that all updated $\eta'(k, w)$ decreases to at least $k\text{-}prob^-(u)$. This completes the proof. □

Based on Theorem 4.7, we formally present the core update theorem for the deletion of an edge in an uncertain graph.

Theorem 4.8. *Assume that an edge $(u, v, p_{(u,v)})$ is deleted from an uncertain graph $\mathcal{G}$. Given an integer $k \in \mathcal{R}^-$ and a vertex $w \in V_{\mathcal{G}}$, we have the following results.*

- *if $\eta(k, u) < \eta(k, v)$ and $k\text{-}prob^-(u) < \eta(k, w) \leq \eta(k, u)$, $\eta(k, w)$ may require updating and u is assigned as the root.*
- *if $\eta(k, u) = \eta(k, v)$ and $min\{k\text{-}prob^-(u), k\text{-}prob^-(v)\} < \eta(k, w) \leq \eta(k, u)$, $\eta(k, w)$ may require updating and both u and v are assigned as the root.*
- *if $\eta(k, u) > \eta(k, v)$ and $k\text{-}prob^-(v) < \eta(k, w) \leq \eta(k, v)$, $\eta(k, w)$ may require updating and v is assigned as the root.*

According to Theorem 4.6, w must be connected to the root through a path containing only the vertices within the η-threshold range. Each range contains an upper-bound (ub) but not a lower-bound (lb).

Proof. We prove three cases respectively. (1) if $\eta(k, u) < \eta(k, v)$, Theorem 4.7 has proved that only $\eta(k, w) \in (k\text{-}prob^-(u), \eta(k, u)]$ may need to be updated. Similarly, if the $\eta(k, w)$ value of any vertex $w \in \mathcal{G}$ decreases due to the deletion, then $\eta(k, u)$ must have decreased as well. Theorem 4.6 shows that all those vertices form a connected subgraph $\mathcal{G}''$. As u belongs to $\mathcal{G}''$, the proof is complete. (2) if $\eta(k, u) > \eta(k, v)$, similar arguments can be used to prove this case and are omitted here. (3) if $\eta(k, u) = \eta(k, v)$, we first perform the above calculations for both u and v. The updated η-thresholds of u and v are denoted as $\eta'(k, u)$ and $\eta'(k, v)$, respectively. Let w be the common neighbor of u and v, and $\eta(k, w) = \eta(k, u) = \eta(k, v)$. Assuming $k\text{-}prob^-(u) < k\text{-}prob^-(v)$, $\eta'(k, u)$ is initially determined as $k\text{-}prob^-(u)$. If k neighbors of vertex w are dependent on u, we update $\eta'(k, w)$ to $k\text{-}prob^-(u)$. Next, $\eta'(k, v)$ will be determined. In the peeling algorithm, it is necessary that $\eta'(k, v) \geq \eta'(k, w)$. However, according to the proof of Theorem 4.7, we have obtained that $\eta'(k, w) \geq \eta'(k, v)$, which is a contradiction. Thus, the equation $\eta'(k, v) = \eta'(k, w) = k\text{-}prob^-(u)$ must hold. That is, vertices u, v and w may join the same (k, η')-core, where η' is the minimum between $k\text{-}prob^-(u)$ and $k\text{-}prob^-(v)$. Thus, $min\{k\text{-}prob^-(u), k\text{-}prob^-(v)\}$ serves as lb for both u and v. □

Algorithm 4: Edge Deletion (ED)

Input: $\mathcal{G}$, an edge $(u, v, p_{(u,v)})$ to be deleted
Output: the updated $\eta(k, w)$ for $w \in V_{\mathcal{G}}$ at each possible k

1 $\mathcal{G}' \leftarrow \mathcal{G} \setminus (u, v, p_{(u,v)})$;
2 $\mathcal{R}' \leftarrow [1, min\{c(u), c(v)\} - 1]$;
3 **foreach** $k \in \mathcal{R}'$ **do**
4 determine lb, ub and the roots ($r[]$) using Theorem 4.8;
5 $C \leftarrow$ *FindDelete* ($\mathcal{G}'$, $\varnothing$, $r[]$, lb, ub, k);
6 *UpdateInsert* (k, C);
7 k++; // critical $\mathcal{R}^-$ processing
8 $V_c \leftarrow$ *Deletion* (G, u, v); // [27]
9 **if** $V_c = \varnothing$ **then**
10 run lines 4-6;
11 **else**
12 $lb = 0$; $ub \leftarrow \max_{i \in V_c}\{\eta(k, i)\}$; $r[] \leftarrow \emptyset$;
13 **if** $c'(u) < k \wedge c'(v) \geq k$ **then**
14 $r[] \leftarrow v$ **if** $\eta(k, v) \leq ub$;
15 **else if** $c'(u) \geq k \wedge c'(v) < k$ **then**
16 $r[] \leftarrow u$ **if** $\eta(k, u) \leq ub$;
17 **else**
18 **if** $c(i) < k$ *for each vertex* $i \in V_{\mathcal{G}}$ **then**
19 **return** ;
20 $C \leftarrow$ *FindDelete* ($\mathcal{G}'$, V_c, $r[]$, lb, ub, k);
21 *UpdateInsert* (k, C);

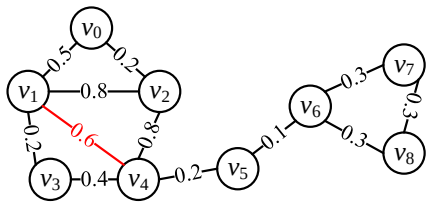
Next, we formally propose the Edge Deletion (ED, Algorithm 4) algorithm for core maintenance in uncertain graphs. Firstly, Theorem 4.8 determines the range $\mathcal{R}^-$ of k for edge deletions. Similarly, $\mathcal{R}^-$ is divided into two parts: $\mathcal{R}^- = \mathcal{R}' \cup \{k_0\}$, where $\mathcal{R}' = [1, k_0 - 1]$. For each $k \in \mathcal{R}'$, all vertices requiring updates can be correctly maintained through lines 4-6. Theorem 4.8 determines the η-threshold range and the root at an integer k. Especially, if $c(u) = c(v)$, ED handles two different cases: (1) if u can reach v, all candidates can be found using a single root (either u or v). (2) if u cannot reach v, neither u nor v can find each other. Thus, both u and v are used as roots and added to the array r, which can be either one-dimensional or two-dimensional. Then, the algorithm **FindDelete** identifies the candidate set C (line 5) and the same algorithm **UpdateInsert** updates the $\eta(k, w)$ for each vertex $w \in C$ (line 6). Since the edge deletion may decrease an old coreness for some vertices, we should delete the corresponding η-thresholds. Finally, critical $\mathcal{R}^-$ processing (lines 8-21) is used to handle this special case. We begin by providing a detailed description of critical $\mathcal{R}^-$ processing and then highlight the differences of FindDelete.

In the critical $\mathcal{R}^-$ processing, we initially utilize the core maintenance algorithm in [27] to update the vertices whose core numbers need to be decremented by 1. Let V_c denote the set of these vertices (line 8). Next, we will discuss two distinct cases.

Case-1: V_c is empty (lines 9-10). It indicates that the core numbers of all vertices remain unchanged. Thus, the initial subgraph used to calculate the $\eta(k_0, w)$ for each $w \in V_{\mathcal{G}}$ after the edge deletion only loses edge $(u, v, p_{(u,v)})$, compared with the initial subgraph before the edge deletion. This is similar to the case when $k \in \mathcal{R}'$, so that all vertices will be maintained through lines 4-6.

Case-2: V_c is not empty (lines 11-21). There are vertices whose core numbers decrease by 1. Let u be such vertex, and $c'(u)$ denotes its updated core number. Since $\eta(k_0, u)$ no longer exists, zero serves as the lb. Furthermore, the initial subgraph $\mathcal{G}_1$, used to calculate $\eta(k_0, w)$ for each vertex $w \in V_{\mathcal{G}}$, excludes V_c and its adjacent edges after the edge deletion. However, the initial subgraph $\mathcal{G}_1$ must include the subgraph $\mathcal{G}_2$, which appears after determining the $\eta(k_0, w)$ of each vertex $w \in V_c$. As the vertices with the smallest k-probability are successively removed, $\mathcal{G}_1$ inevitably evolves into $\mathcal{G}_2$. In other words, the (k_0, η')-core is determined, where $\eta' = max_{i \in V_c}\{\eta_k(i)\}$ and it is set to ub (line 12). After the deletion, each vertex $w \in V_c$ is excluded from the (k_0, η)-core, which directly affects the neighboring vertices of w. Thus, all vertices in V_c become roots. Next, we consider three different cases. (i) $u \in V_c$ and $v \notin V_c$ (lines 13-14). If the $\eta(k_0, v)$ is less than or equal to ub, v may be affected as well. It is uncertain whether v can reach V_c. Therefore, v is recorded as a separate root. (ii) $u \notin V_c$ and $v \in V_c$ (lines 15-16). This situation is similar to (i). (iii) $u \in V_c$ and $v \in V_c$ (lines 17-19). V_c can be directly used to find all candidates. Furthermore, if the core number of each vertex is below k_0, there is no (k_0, η)-core. Hence, the related maintenance can be skipped.

Finally, the **FindDelete** algorithm follows a similar approach as Algorithm 2. Detailed pseudocode is omitted here due to space constraints. After the deletion, multiple roots (denoted as V_c and r) may exist. The neighbor x affected by $w \in V_c$, whose $\eta(k, x)$ belong to the η-threshold range, is added to the queue Q. If a root in the array r has not been added to Q, it is directly added to Q. The initialization of Q is now complete. The subsequent steps follow the same approach as Algorithm 2. In particular, the η-threshold range in Theorem 4.8 contains an upper-bound but not a lower-bound.



(a) An example uncertain graph and (v_1, v_4, 0.6) is the edge to be deleted.

k	v_0	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8
1	0.6	0.92	0.92	0.52	0.92	0.28	0.51	0.51	0.51
2	0.1	0.48	0.48	0.08	0.48	0.02	0.09	0.09	0.09

(b) Core decomposition before the deletion of (v_1, v_4, 0.6)

Figure 2: An example uncertain graph before edge deletion.

Example 4.2. *In Figure 2, let* $(v_1, v_4, 0.6)$ *be the edge to be deleted. Since* $c(v_1) = c(v_4) = 2$, *we determine* $\mathcal{R}^- = [1, 2]$. *Next, each* $k \in \mathcal{R}^-$ *will be processed sequentially. For* $k = 1$, *the original* $\eta(1, v_1)$ *and* $\eta(1, v_4)$ *are both 0.92. Using Theorem 4.8, we find* $ub = 0.92$, $r = \{v_1, v_4\}$, *and* $lb = k\text{-}prob(v_1, \mathcal{G}_1^{v_6-}) = k\text{-}prob(v_4, \mathcal{G}_1^{v_4-}) = 0.8$. *A BFS traversal is performed starting from* v_1 *and* v_4 *and the candidate set* $C = \{v_1, v_2, v_4\}$ *can be identified and updated. When* $k = 2$, *the Deletion algorithm returns* $V_c = \emptyset$, *which corresponds to case-1. Similarly, We can find* $ub = 0.48$, $r = \{v_1, v_4\}$, *and* $lb = 0$. *Correspondingly, candidate set* $C = \{v_0, \ldots, v_8\}$ *is found and updated.*

4.3 Theoretical Analysis

Complexity Analysis. For each $k \in \mathcal{R}^+/\mathcal{R}^-$, if $\eta(k,u) < \eta(k,v)$, the total time complexity of Algorithm 1 or Algorithm 4 is $O(k_0 \cdot (|C|\log|C| + |V_c| \sum_{u \in V_c} deg(u)) + k_0^2 \cdot \sum_{u \in C} deg^2(u))$. Due to space constraints, detailed complexity analysis will be provided in the appendix. Here, V_c is bounded by C and k_0 is bounded by the minimum core number of u and v. Also, the size of C is bounded by the maximum subgraph. Each vertex w in the subgraph are connected to the root, and its $\eta(k,w)$ within the η-threshold range. Therefore, in practice, the maintenance algorithm is very fast because the visited subgraph during the search process is usually much smaller than $\mathcal{G}$, and k_0 is usually smaller than the largest core number in the deterministic graph.

Boundedness Analysis. For the proposed (k,η)-core maintenance problem, we represent the set of vertices whose η-threshold changes as CNG. If an incremental algorithm $\mathcal{A}$ is *bounded* [43], its cost should be a polynomial function of the size of CNG's c-hop neighbors, denoted as $||\text{CNG}||_c$, where c is a positive integer. Based on this, we propose Theorem 4.9.

Theorem 4.9. *The (k,η)-core maintenance algorithm is unbounded for both edge insertion and edge deletion in uncertain graphs.*

Next, we conduct a separate analysis of the two cases. Firstly, we focus on edge deletions. The removal algorithm for core maintenance in [45] is a bounded algorithm. Its effectiveness stems from the fact that in an undirected and unweighted graph, we can efficiently prune the BFS process by directly computing the maximum-core degree and purecore degree of u's neighbors through a simple decrement operation. However, in uncertain graphs, the DP algorithm needs to be employed to update the k-probability values for both the root u and its neighbors. The frequent utilization of DP during the BFS process may not yield significant pruning benefits compared to the additional computational overhead. Hence, our algorithm considers all vertices that are connected to the root and fall within the η-threshold range, without relying on a polynomial function of $||\text{CNG}||_c$. Secondly, we focus on edge insertions. Previous studies have demonstrated that both k-truss maintenance and k-core maintenance exhibit asymmetry concerning edge insertions and deletions [56]. While no maintenance algorithm with complexity dependent on CNG is currently known for edge insertions, the complexity of the (k,η)-core definition poses a challenge in devising a linear time maintenance algorithm.

5 OPTIMIZATIONS

This section introduces three optimizations to improve the efficiency of our basic core maintenance algorithm. Firstly, we contract the η-threshold range at an integer k in Theorem 4.4 and 4.8. Secondly, we employ the vertices visited during the BFS traversal to facilitate additional pruning. Thirdly, we employ a lazy computing approach to minimize redundant calculations.

5.1 Bounds Contraction

Upper-bound Contraction in Edge Insertion. We begin by revisiting the core update Theorem 4.4, which applies to the insertion of an edge $(u,v,p_{(u,v)})$ in uncertain graphs. Given an integer k, if $\eta(k,u) < \eta(k,v)$, the upper-bound $k\text{-}prob^+(u) = k\text{-}prob(u,\mathcal{G}_k^{u+})$ can be determined. Here, $\mathcal{G}_k^{u+}$ simultaneously considers the initial induced η-threshold subgraph of vertex u and the inserted edge. By reducing the size of $\mathcal{G}_k^{u+}$, the upper-bound diminishes, resulting in a smaller η-threshold range. Following the iterative principle of the peeling algorithm, we prioritize the vertex with the minimum k-probability and provide the subsequent definition.

Definition 5.1. **(Restricted Vertex)** *Assume that an edge $(u,v,p_{(u,v)})$ with $\eta(k,u) < \eta(k,v)$ is inserted into an uncertain graph $\mathcal{G}$. Given an integer k, a restricted vertex refers to a neighbor w of vertex u, satisfying $\eta(k,w) = \eta(k,u)$ and $k\text{-}prob(w,\mathcal{G}_k^w) = \eta(k,w)$.*

Considering an integer k, the insertion of an edge $(u,v,p_{(u,v)})$ does not introduce a new neighbor to w. Thus, $k\text{-}prob(w,\mathcal{G}_k^{w+})$ remains the same as $k\text{-}prob(w,\mathcal{G}_k^w)$, where $k\text{-}prob(w,\mathcal{G}_k^{w+})$ serves as the upper-bound for the updated $\eta(k,w)$. If $k\text{-}prob(w,\mathcal{G}_k^w) = \eta(k,w)$, then $\eta(k,w)$ remains unchanged. If $\eta(k,u)$ increases, w cannot be included in the updated η-threshold subgraph of u, denoted as $\mathcal{G}_k^{u+}$. Based on this, we present Lemma 5.1.

Lemma 5.1. *Assume that an edge $(u,v,p_{(u,v)})$ is inserted into an uncertain graph $\mathcal{G}$. Given an integer k, the $\eta(k,w)$ value of the restricted vertex w remains unchanged.*

Therefore, it is crucial to identify and exclude such restricted vertices, which reduces the size of $\mathcal{G}_k^{u+}$ and diminishes the upper-bound. This reduction leads to a smaller η-threshold range, enhancing the efficiency of Algorithm 1. Let us reconsider Example 4.1. When $k=1$, v_6 becomes the root. v_5 and v_7 are neighbors of v_6 with $\eta(1,v_5) = \eta(1,v_7) = \eta(1,v_6)$. By computing $k\text{-}prob(v5,\mathcal{G}_1^{v_5+}) = 0.706 = \eta(1,v_5)$ and $k\text{-}prob(v7,\mathcal{G}_1^{v_7+}) = 0.888 > \eta(1,v_7)$, we identify v_5 as a restricted vertex. Then, $ub = k\text{-}prob(v_6,\mathcal{G}_1^{v_6+}\backslash\{v_5\}) = 0.76$ can be computed, which determines the η-threshold range $[0.706, 0.76)$. This range is a subset of the initial range $[0.706, 0.856)$. By conducting a BFS traversal starting from v_6, we identify the candidate set $C = \{v_6, v_5, v_7\}$. Finally, $\eta(1,v_6)$ and $\eta(1,v_7)$ are also updated to 0.76. This is just an example when $k=1$.

In addition, the lower-bound (lb) and upper-bound (ub) of the η-threshold range must satisfy $lb < ub$. If $lb \geq ub$, it implies that u cannot join a (k,η')-core ($\eta' > \eta(k,u)$) without the help of restricted vertices. Thus, it is possible to identify opportunities for early termination of the current k processing.

Lower-bound Contraction in Edge Deletion. We now revisit the core update Theorem 4.8, which applies to the deletion of an edge $(u,v,p_{(u,v)})$ in uncertain graphs. Given an integer k, if $\eta(k,u) < \eta(k,v)$, the lower-bound $k\text{-}prob^-(u) = k\text{-}prob(u,\mathcal{G}_k^{u-})$ can be identified. Here, $\mathcal{G}_k^{u-}$ represents the updated η-threshold subgraph of u, and N_k^{u-} denotes the updated neighbor set of u in $\mathcal{G}_k^{u-}$. We have two situations to consider: (i) $|N_k^{u-}| \geq k$. It means $k\text{-}prob^-(u)$ is non-zero and can be served as the lower-bound (lb). (ii) $|N_k^{u-}| < k$. It means $k\text{-}prob^-(u)$ is zero. Using zero as the lb would result in a large number of candidates. By expanding the size of $\mathcal{G}_k^{u-}$, it is possible to increase the lower-bound, thereby leading to a smaller η-threshold range. Hence, our objective is to introduce additional neighbors w to $\mathcal{G}_k^{u-}$ with the highest $\eta(k,w)$. Before introducing the optimization of lower-bound contraction, we first give the following definition.

Definition 5.2. **(Potential Set)** *Assume that an edge $(u, v, p_{(u,v)})$ with $\eta(k,u) < \eta(k,v)$ is deleted from an uncertain graph $\mathcal{G}$. Given an integer k, the potential set of vertex u, denoted as P_k^u, contains neighbors w of u, satisfying the conditions $w \notin N_k^{u-}$ and $c(w) \geq k$.*

If $|N_k^{u-}| < k$, We can expand $\mathcal{G}_k^{u-}$ through a greedy algorithm. The vertices in P_k^u are sorted in a non-increasing order based on their η-thresholds. Each vertex $w_i \in P_k^u$ is sequentially added to $\mathcal{G}_k^{u-}$, and $k\text{-}prob(u, \mathcal{G}_k^{u-} \cup w_{1,\ldots,i})$ is updated accordingly. Next, Lemma 5.2 gives the stopping condition for the algorithm.

Lemma 5.2. *Given an integer k, when we add the vertices w_i in the non-increasing set P_k^u to $\mathcal{G}_k^{u-}$, the $\eta(k, w_i)$ value of the vertex w_i remains unchanged if $k\text{-}prob(u, \mathcal{G}_k^{u-} \cup w_{1,\ldots,i}) \geq \eta(k, w_i)$.*

Obviously, some vertices in P_k^u might belong to the same (k,η)-core as u. Once $k\text{-}prob(u, \mathcal{G}_k^{u-} \cup w_{1,\ldots,i})$ becomes greater than or equal to $\eta(k, w_i)$, it signifies that $N_k^{u-} \cup w_{1,\ldots,i}$ is at least a $(k, \eta(k, w_i))$-core. Consequently, $\eta(k, w_i)$ remains unchanged and serves as the lower-bound (lb). If all vertices in P_k^u have been processed without satisfying Lemma 5.2, then lb remains zero. Let us reconsider Example 4.2. When $k = 2$, v_1 and v_4 become the roots. The updated neighbor set $N_2^{v_1-} = N_2^{v_4-} = \{v_2\}$, both having sizes smaller than 2. The potential set of v_1 is $P_2^{v_1} = \{v_0, v_3\}$. We add v_0 to $\mathcal{G}_2^{v_1-}$ and compute $k\text{-}prob(v_1, \mathcal{G}_2^{v_1-} \cup v_0) = 0.4 > \eta(2, v_0)$. Similarly, the potential set of v_4 is $P_2^{v_4} = \{v_3, v_5\}$. We add v_3 to $\mathcal{G}_2^{v_4-}$ and compute $k\text{-}prob(v_4, \mathcal{G}_2^{v_4-} \cup v_3) = 0.32 > \eta(2, v_3)$. According to Theorem 4.8, The smaller value of $\eta(2, v_0)$ and $\eta(2, v_3)$ is taken as lb. Consequently, the determined η-threshold range $(0.08, 0.48]$ is a a subset of the initial range $(0, 0.48]$. The candidate set $C = \{v_0, v_1, v_2, v_4\}$ is found and updated.

In addition, if P_k^u contains a large number of vertices, updating $k\text{-}prob^-(u)$ multiple times can create additional computational overhead. To mitigate this, we can use Equation 16 in [13] for incremental updates. Assuming $\eta(k, w_i)$ becomes the lb, it implies w_i will not be added to C. The initial $k\text{-}prob(u)$ calculated in line 2 of Algorithm 3 may be smaller than $\eta(k, w_i)$. However, in this case, w_i and u must belong to at least the $(k, \eta(k, w_i))$-core, which is contradictory. Hence, it is crucial to set the initial value of *low* in Algorithm 3 to $\eta(k, w_i)$.

5.2 Dynamic Update

Incremental Update in Edge Insertion. Algorithm 2 performs a BFS traversal to identify all candidates reachable from the root through vertices within the η-threshold range, and returns the candidate set C. Given an integer k, let c be a vertex in the candidate set C, indicating that the insertion of an edge may potentially increase the $\eta(k, c)$ value. Now, assuming w is a neighbor of c, and $\eta(k, w) < \eta(k, c)$. Since w does not add a new neighbor, the equality $k\text{-}prob(w, \mathcal{G}_k^{w+}) = k\text{-}prob(w, \mathcal{G}_k^{w})$ still holds. Based on the basic iterative steps of the peeling algorithm, vertex w must be processed prior to c, ensuring that $\eta(k, w)$ remains unchanged. Consequently, we formally propose Lemma 5.3 as follows.

Lemma 5.3. *Assume that an edge is inserted into an uncertain graph $\mathcal{G}$. Given an integer k, a pre-determined η-threshold range $[lb, ub)$ and the root, a candidate must be connected to the root through a non-increasing path containing only vertices within $[lb, ub)$.*

Based on Lemma 5.3, when processing a candidate vertex c at an integer k, it is only necessary to consider its neighbors w with $\eta(k, w) \geq \eta(k, c)$. In order to reduce the size of candidate set C, we dynamically update the lower-bound (lb) to the $\eta(k, c)$ value of the currently processed vertex c. Let us reconsider Example 4.1, where the update range $[0.706, 0.856)$ is determined. A BFS traversal starts from v_6 with $lb = 0.706$. Then, v_4, v_5 and v_7 are identified and added to the candidate set C. Upon processing v_4, lb is updated to 0.8, and v_3 is identified. When v_3 is processed, $lb = 0.8$ prevents v_1 from being discovered and added to C. Finally, $C = \{v_6, v_4, v_5, v_7, v_3\}$ is determined and C is a subset of the initial candidate set.

Decremental Update in Edge Deletion. Given an integer k, let c be a vertex in the candidate set C, indicating that the deletion of an edge might decrease the $\eta(k, c)$ value. If vertex w is a neighbor of c and $\eta(k, w) > \eta(k, c)$, the initial η-threshold subgraph of w, denoted as $\mathcal{G}_k^w$, does not include c. Deleting an edge cannot decrease the $\eta(k, w)$ value. Consequently, c cannot be part of the updated η-threshold subgraph of w, denoted as $\mathcal{G}_k^{w'}$. In other words, the $\eta(k, w)$ value will not be affected by c. Consequently, we formally propose Lemma 5.4 as follows.

Lemma 5.4. *Assume that an edge is deleted from an uncertain graph $\mathcal{G}$. Given an integer k, a pre-determined η-threshold range $(lb, ub]$ and the root, a candidate must be connected to the root through a non-decreasing path containing only vertices within $(lb, ub]$.*

Based on Lemma 5.4, when processing a candidate vertex c at an integer k, we only need to consider its neighbors w with $\eta(k, w) \leq \eta(k, c)$. It is important to dynamically update the upper-bound (ub) to the $\eta(k, c)$ value of the vertex c being processed. This decremental update helps obtain a smaller candidate set C, thereby enhancing the efficiency of Algorithm 4. Let us reconsider Example 4.1. When $k = 2$, the η-threshold range $(0, 0.48]$ has been determined. Starting with v_1 and $ub = 0.48$, a BFS traversal identifies v_0, v_2 and v_3, which are then added to the candidate set C. Processing v_2 leads to the inclusion of v_4 in set C. Similarly, processing v_4 adds v_5 to C. However, v_6 is not discovered and added to C due to the updated $ub = 0.02$ when processing v_5. Furthermore, another root v_4 has been processed. Finally, $C = \{v_0, \ldots, v_5\}$ is determined, which is a subset of the initial candidate set.

5.3 Lazy Computation

Algorithm 3 sequentially processes the vertex with the minimum k-probability. However, if two vertices v_1 and v_2 are processed successively, their common neighbor needs to be recalculated twice, which results in additional computational overhead. To address this issue, we propose a lazy computation optimization in Algorithm 5.

The main objective of Algorithm 5 is to identify each vertex v with determinable $\eta(k, v)$ and add v to queue D initially (line 5). If D is empty, only the vertex with the smallest k-probability is found (lines 6-7). During the removal phase, the two mentioned cases are handled separately. When D is non-empty, all vertices in D are updated in batches, and their neighbors are pushed into the set S in which each vertex appears only once (lines 9-12). If D is empty, the vertex with the smallest k-probability is processed, similar to Algorithm 3. Obviously, Algorithm 5 recalculates the k-probability of a common neighbor vertex at most once, significantly

Algorithm 5: LazyUpdate

Input: k-value: k, candidate set: C
Output: the updated $\eta(k, w)$ for each vertex $w \in C$

```
1  D ← empty queue; S ← empty set; curThres ← 0;
2  compute k-prob(v) for each v ∈ C;
3  while C ≠ ∅ do
4      if D = ∅ then
5          D ← {v ∈ C | k-prob(v) ≤ curThres};
6          if D = ∅ then
7              u ← arg min_{v∈C} k-prob(v);
8      if D ≠ ∅ then
9          S ← ⋃_{v∈D} {u ∈ (N_v ∩ C) | k-prob(u) > curThres};
10         η(k, v) = curThres for all v ∈ D;
11         C ← C \ D;    D ← ∅;
12         foreach v ∈ S do
13             update k-prob(v);
14             if k-prob(v) ≤ curThres then
15                 D.push(v);
16     else
17         lines 5-9 of Algorithm 3, and add above judgments
           (lines 14-15);
```

improving the overall execution efficiency of EI (Algorithm 1) and DI (Algorithm 4). Furthermore, a self-generation mechanism is employed (lines 15-16, 18), where any neighbor vertex with a k-probability smaller than the current *curThres* is added to D. Then, all vertices $w \in D$ have a determinable $\eta(k, w)$ and can be processed in batch, thereby further enhancing the execution efficiency. It should be emphasized that if EI utilizes the optimization of lower-bound contraction, the initial value of the *lb* must be set accordingly.

6 PROBABILITY CHANGE

In this section, we provide a brief description of the core maintenance algorithms for increasing and decreasing probability values in uncertain graphs respectively.

6.1 Probability Increase

Assuming the probability of edge (u, v) increases to $p^+_{(u,v)}$ (where $p^+_{(u,v)} \leq 1$), then the core number of all vertex remains unaffected. The influence range of k, denoted as $\mathcal{R}' = [1, k_0]$, can be determined, where $k_0 = min\{c(u), c(v)\}$. A similar proof can be found in Theorem 4.1 and is omitted here. Given an integer $k \in \mathcal{R}'$ and $\eta(k, u) < \eta(k, v)$, $\mathcal{G}^u_k$ represent the induced η-threshold subgraph of vertex u. After increasing the probability, the updated subgraph of u can still be represented by $\mathcal{G}^{u+}_k$. It is evident that $k\text{-}prob^+(u) > k\text{-}prob(u)$ also holds. Consequently, in this case, Theorem 4.4 also applies to the core maintenance of uncertain graphs. The only difference lies in modifying $\mathcal{R}^+$ to $\mathcal{R}'$.

Recall that algorithm EI divided $\mathcal{R}^+$ into two parts, denoted as $\mathcal{R}^+ = \mathcal{R}' \cup \{k_0 + 1\}$. When $k = k_0 + 1$, the $\eta(k, w)$ value of each vertex w retains unchanged, eliminating the need for critical $\mathcal{R}^+$ processing. For each k belonging to $\mathcal{R}'$, the $\eta(k, w)$ of each vertex $w \in V_{\mathcal{G}}$ can be correctly maintained through lines 4-6 in Algorithm 1. Thus, after increasing the edge probability, a simplified maintenance algorithm **PI** can be used for uncertain graphs. In the worst case, the total time complexity of PI is $O(k_0 \cdot |C| \log |C| + k_0^2 \cdot \sum_{u \in C} deg^2(u))$. In addition, the three optimizations described in Section 5 can also be applied to speed up this basic algorithm.

6.2 Probability Decrease

Assume that the probability of edge (u, v) decreases to $p^-_{(u,v)}$ (where $p^-_{(u,v)} \geq 0$) and $\eta(k, u) < \eta(k, v)$. The influence range of k can be determined as $\mathcal{R}^- = [1, k_0]$, where $k_0 = \{c(u), c(v)\}$. A similar proof can be found in Theorem 4.5 and is omitted here. Given an integer $k \in \mathcal{R}^-$, $\mathcal{G}^{u-}_k$ can still represent the updated η-threshold subgraph of u. It is evident that $k\text{-}prob^-(u) < k\text{-}prob(u)$ holds as well. Consequently, Theorem 4.8 also applies to the core maintenance of uncertain graphs when decreasing the edge probability.

Recall that algorithm DI divided $\mathcal{R}^-$ into two parts: $\mathcal{R}^- = \mathcal{R}' \cup \{k_0\}$, where $\mathcal{R}' = [1, k_0 - 1]$. However, the core number of the vertex remains unchanged. When $k = k_0$, the only requirement is to maintain the $\eta(k, w)$ of each vertex $w \in V_{\mathcal{G}}$. In summary, for each k belonging to $\mathcal{R}^-$, the core maintenance of the uncertain graph can be accomplished through lines 1-6 in Algorithm 4. The only difference is to replace $\mathcal{R}'$ in Algorithm 4 with $\mathcal{R}^-$. This basic maintenance algorithm is referred to as **PD**. In the worst case, the total time complexity of PD is $O(k_0 \cdot |C| \log |C| + k_0^2 \cdot \sum_{u \in C} deg^2(u))$. We can also use the three optimizations in Section 5 to speed up this basic algorithm.

7 EXPERIMENTS

This section evaluates the performance of our proposed algorithms. All algorithms are implemented in C++, and compiled by G++ with -O3 optimization. The experiments are conducted on CentOS Linux 7 Core with Intel Xeon 2.40GHz and 125G main memory.

7.1 Experimental Settings

Datasets: We employ eight real-world graphs in experiments, which are summarized in Table 1. Fruit-Fly[1] and Biomine[2] are protein-protein interaction networks, where the probability of an edge represents the possibility that the interaction of two proteins exists. Flickr[3] is an online community for sharing photos, where the probability of an edge is set to the Jaccard coefficient of the interest communities shared by the two users. DBLP[4] is a co-authorship network, where the probability of an edge is determined by an exponential function to the number of collaborations. Google is a web graph that was released in 2002 by Google. Patents is a citation network spanning 37 years (1963 to 1999), consisting of 3,923,922 utility patents granted during that period. Comlj and Orkut are online communities that provide integrated functionalities such as forums, blogs, and more. Google, Patents, Comlj, and Orkut are from SNAP[5], and their edge probabilities are generated uniformly at the range [0, 1] as with [12, 35].

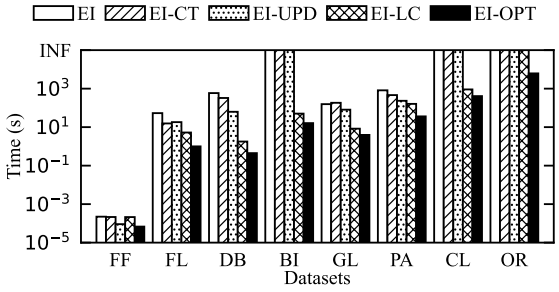
Algorithms: We evaluate a set of algorithms in experiments.

- **BASE**: The core decomposition algorithms, i.e., the peeling algorithm without dynamic update technology [12, 53].

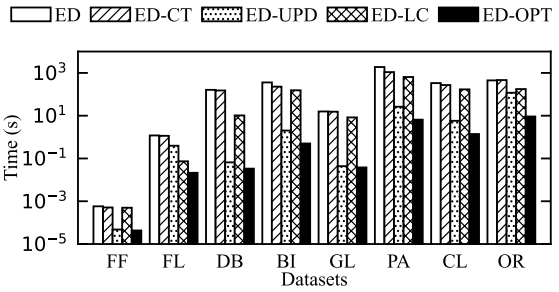
[1] http://thebiogrid.org
[2] https://biomine.ijs.si/downloads/
[3] https://www.flickr.com
[4] https://dblp.uni-trier.de
[5] https://snap.stanford.edu/data/index.html

Table 1: Uncertain graphs (d_{max}: maximum degree; k_{max}: maximum core number)

Dataset (Abbr.)	$\|V_{\mathcal{G}}\|$	$\|E_{\mathcal{G}}\|$	d_{max}	k_{max}
Fruit-fly (*FF*)	3,752	3,692	27	4
Flickr (*FK*)	24,125	300,836	546	225
DBLP (*DB*)	684,911	2,284,991	611	114
Biomine (*BI*)	1,008,201	6,722,513	139624	448
Google (*GL*)	875,713	5,105,039	6332	44
Patents (*PA*)	3,774,768	16,518,948	793	64
Comlj (*CL*)	3,997,962	34,681,189	14815	360
Orkut (*OR*)	3,072,441	117,185,083	33313	253



(a) Edge insertion



(b) Edge deletion

Figure 3: Evaluation of optimizations

- **EI** and **ED**: Our proposed basic algorithms for edge insertion and deletion, respectively.
- **PI** and **PD**: Our proposed basic algorithms for edge probability increase and decrease, respectively.
- **Three optimizations**. **-CT**: contraction optimization. **-UPD**: update optimization. **-LC**: lazy computation optimization. **-OPT**: all three optimizations.

Parameters and Metrics: The evaluated parameters include the number of updated edges, the change in probability values, and the uncertain graph size. In experiments, we randomly select 500 edges to simulate uncertain graphs update and report the average running time. The algorithm terminates if it does not complete within 10^4 seconds. Due to space limitations and similar trends over different uncertain graphs, we report experimental results of partial datasets in this paper.

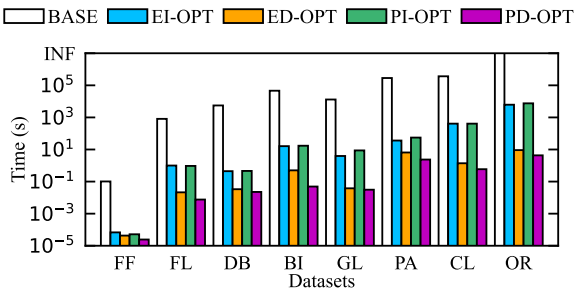
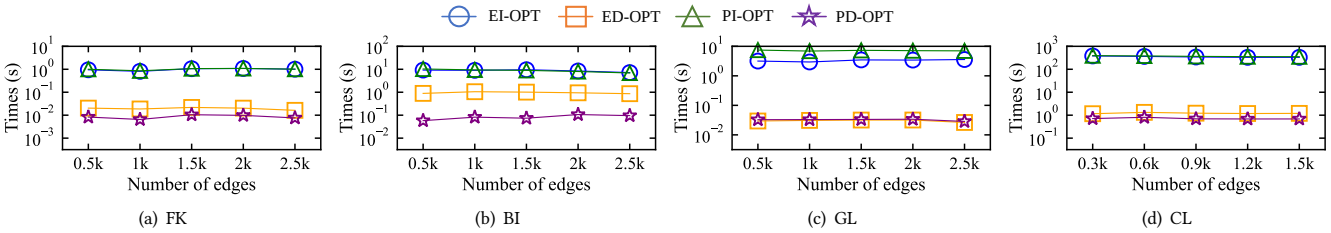


Figure 4: Evaluation of optimal algorithms.

7.2 Experimental Results

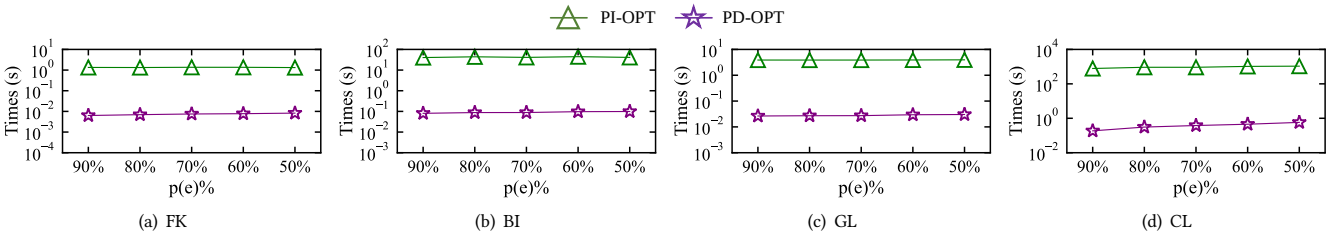
EXP-1: Evaluation of Optimizations. Firstly, we evaluate the efficiency of the three optimizations, i.e., -CT, -UPD, and -LC. To this end, we integrate the optimizations into four basic core maintenance algorithms, i.e., EI, ED, PI, and PD. Since PI and PD have similar results with EI and ED. The results of PI and PD are removed to our technical report due to space limitation. Figure 8 shows the results. We can observe that the three optimizations have different improvements over basic algorithms. Specifically, Figure 3(a) depicts the time of edge insertion on different datasets. EI-LC demonstrates the best performance on all datasets except FF, followed by EI-UPD, EI-CT, and EI. For instance, for DB dataset, EI-LC, EI-UPD, and EI-CT are 329.3, 9.4, and 1.8 times faster than EI, respectively. The lazy computation optimization significantly improves the efficiency of EI. Figure 3(b) reports the time for edge deletion, where ED-UPD has a greater performance improvement for ED than the other two optimizations. For example, for DB dataset, ED-UPD, ED-LC, and ED-CT are 2295.6, 15.7, and 1.1 times faster than ED, respectively. In BFS traversal, the η-threshold of the currently visited vertex can be used to further prune the vertices whose η-threshold do not need updates, improving the efficiency of ED. Moreover, the three optimizations together also make basic core maintenance algorithms EI and ED more efficient. For instance, for DB dataset, EI-OPT and ED-OPT exhibit significantly improved efficiency compared to EI and ED, with speedups of 1317.4 and 4829.8 times, respectively. Note that, in the remainder of the experiments, we only report our proposed algorithms with all optimizations.

EXP-2: Base VS. Our Proposed Optimized Algorithms. In this experiment, we compare our proposed optimized algorithms with baseline, which is to perform core decomposition from the scratch when the uncertain graphs update. The results are shown in Figure 4. It is evident that all algorithms we proposed exhibit a significant improvement across all datasets in terms of running time compared to BASE. Specially, EI-OPT and PI-OPT are at least three orders of magnitude faster than BASE. ED-OPT and PD-OPT outperform BASE by at least four orders of magnitude. In the best case, our proposed optimized algorithms are up to six orders of magnitude faster than BASE. For example, for GL dataset, ED-OPT can process an edge deletion in 0.04 second, while BASE takes 13,080 second. The reason for this significant speed improvement is that BASE should handle the whole uncertain graph size for each update while our proposed algorithms only need to process the induced η-threshold subgraph, which is small.



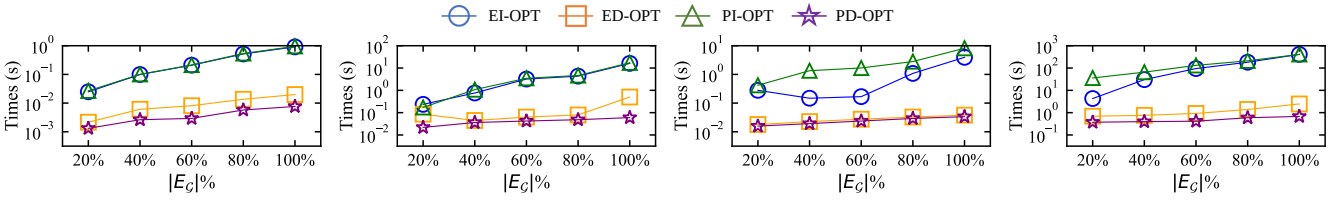
(a) FK (b) BI (c) GL (d) CL

Figure 5: Effect of the number of updated edges (average time)



(a) FK (b) BI (c) GL (d) CL

Figure 6: Effect of $p(e)$



(a) FK (b) BI (c) GL (d) CL

Figure 7: Scalability evaluation

EXP-3: Effect of the Number of Updated Edges. To test the effect of the number of updated edges on processing time, a sequence of r updates is performed. For most graphs, r is selected from the set $\{500, 1000, 1500, 2000, 2500\}$, while for CL and OR, the set $\{300, 600, 900, 1200, 1500\}$ is used. The experimental results, shown in Figure 5, indicate that as the number of updated edges increases, the average update time does not increase significantly. The main reasons are as follows. The proposed maintenance algorithms incorporate two distinct parameters, namely, the vertex and k, which may exhibit a negative correlation. For example, if the core number k' is relatively large, only a few vertices may be included in a (k', η)-core, which results in a smaller number of vertices needing updates. Thus, in a broader range of $\mathcal{R}^+/\mathcal{R}^-$, processing larger k values incurs minimal update costs compared to processing smaller k values, which ensures the stability of our update algorithm.

EXP-4: Effect of $p(e)$. We study the effect of $p(e)$ on PI-OPT and PD-OPT by varying $p(e)$ from $p(e)\cdot 50\%$ to $p(e)\cdot 90\%$. The experimental results, shown in Figure 6, indicate that when $\Delta p(e)$ increases, the average update time grows by a small amount and remains almost stable. The main reason is as follows. Larger $\Delta p(e)$ results in larger upper-bounds and smaller lower-bounds for PI and PD, respectively. Thus, the η-threshold update range becomes larger, requiring updates larger subgraph.

EXP-5: Scalability Evaluation. Finally, we evaluate the scalability of our proposed algorithms. To this end, we randomly select 20%, 40%, 60%, 80%, and 100% of edges from origianl uncertain graphs, and then get the corresponding subgraphs induced by these edges. The experimental results are shown in Figure 7. As expected, the running time of all algorithms grows when the graphs become larger. It is because the increase in graph size leads to higher vertex degrees, resulting in increased computational cost for k-probability and BFS traversal cost. Additionally, vertices in dense graphs may have larger core numbers, leading to a larger $\mathcal{R}^+/\mathcal{R}^-$ and more running time. Moreover, we can observe that the performance of ED-OPT and PD-OPT degrade more slightly than that of EI-OPT and PI-OPT. The reason behind is that the higher degree contributes to an increase in both $k\text{-}prob^+$ and $k\text{-}prob^-$. For ED-OPT and PD-OPT, the increased $k\text{-}prob^-$ corresponds to a smaller η-threshold range, thereby reducing the increase in update time. However, the conclusion is the opposite for ED-OPT and PD-OPT.

8 CONCLUSIONS

In this paper, we study the problem of core maintenance in uncertain graphs. We propose basic maintenance algorithms for four update operations specific to uncertain graphs. To further improve efficiency, we propose three optimization techniques. Extensive experimental evaluations demonstrate the efficiency of our proposed algorithms. In our future work, we will investigate parallel maintenance algorithms in large uncertain graphs to further enhance efficiency. We will also study the maintenance of other cohesive subgraphs (e.g., k-truss) in uncertain graphs.

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APPENDIX

A THE PROOF OF THEOREM 4.2

We prove it by contradiction. Assume that the updated vertices do not form a connected subgraph, thus leading to the presence of two or more separate subgraphs. Now, we consider two separate subgraphs, denoted as S_1 and S_2. Since there is only one edge insertion, only one of these subgraphs, say S_1, can contain a vertex that acquires a new neighbor in $\mathcal{G}$. In contrast, none of the vertices in S_2 undergo modifications in their adjacency. This is a contradiction. If a vertex has its η-threshold increased, then it must have either gained a new neighbor or at least one of its existing neighbors whose η-threshold are also increased. Applying this recursively, we must identify a vertex whose increased η-threshold stem form the addition of a new neighbor. However, for S_2, such a vertex does not exist. This also constitutes a contradiction.

B THE PROOF OF THEOREM 4.5

Proof. We prove Theorem 4.5 from two distinct cases. (1) If $k \in [1, k_0]$, an edge or a vertex is deleted from the initial $(k, \eta(k, w))$-core, causing an change in $\eta(k, w)$. (2) If $k \notin [1, k_0]$, neither vertices nor edges are subjected to removal from the pre-existing $(k, \eta(k, w))$-core, thereby maintaining the stasis of the $\eta(k, w)$. □

C THE PROOF OF THEOREM 4.6

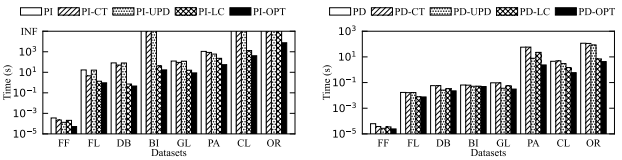
We prove it by contradiction. Assume that the updated vertices do not form a connected subgraph, thus leading to the presence of two or more separate subgraphs. Now, we consider two separate subgraphs, denoted as S_1 and S_2. Since there is only one edge deletion, it follows that merely one of the subgraphs, say S_1, can comprise a vertex who loss an old neighbor in $\mathcal{G}$. In contrast, none of the vertices in S_2 undergo modifications in their adjacency. This is a contradiction. If a vertex has its η-threshold decreased, then it must have either lost a neighbor or at least one of its existing neighbors whose η-threshold are also decreased. Applying this recursively, we should be able to trace back to a vertex whose lower η-threshold stem form the loss of an old neighbor. However, for S_2, such a vertex does not exist. This also constitutes a contradiction.

D THE FINDDELETE ALGORITHM

Algorithm 6: FindDelete

Input: updated graph: $\mathcal{G}$, set: V_c, the root set: r, low-bound: lb, upper-bound: ub, k-value: k
Output: candidate set C and its induced subgraph $\mathcal{H}$

initialize C, $\mathcal{H}$, $visited[]$, Q as lines 1-3 in Algorithm 2;
if $V_c \neq \emptyset$ **then**
  foreach $u \in V_c$ **do**
    $\eta(k, u) = 0$;
    foreach $(u, v) \in E$ **do**
      if $\eta(k, v) \in (lb, ub] \wedge v \notin V_c \wedge \neg visited[v]$ **then**
        Q.push(v); $visited[v] \leftarrow$ **true**; V'.push(v);
foreach $u \in r \wedge \neg visited[u]$ **do**
  Q.push(u); $visited[u] \leftarrow$ **true**; V'.push(u);
run lines 5-11 in Algorithm 2 by modifying the judgment symbols $\geq$ and $<$ to $>$ and $\leq$ respectively;
return *C and* $\mathcal{H}$ *;*



(a) Edge probability increase (b) Edge probability decrease

Figure 8: Evaluation of optimizations

E COMPLEXITY ANALYSIS IN SUBSECTION 4.3

Given an integer $k \in \mathcal{R}^+/\mathcal{R}^-$ and $\eta(k, u) < \eta(k, v)$, Algorithm 1 and Algorithm 4 have the following time complexities. Firstly, the DP algorithm for calculating the k-probability of a given vertex u requires $O(k \cdot deg(u))$ time. Secondly, Algorithm 2 takes $O(deg(u))$ time for each $u \in C$ to identify all candidates. Thirdly, Algorithm 3 uses a minimum priority queue to maintain all vertices in C, where their k-probabilities serve as keys. It totally takes $O(|C| \cdot \log|C|)$ time to remove the vertex with the minimum k-probability. Consequently, the total calculation and update times are $O(\sum_{u \in C} k \cdot deg(u))$ and $O(\sum_{u \in C} deg(u) \cdot k \cdot deg(u))$ respectively. In the worst scenario, the time complexity of the above three steps is $O(|C| \cdot \log|C| + \sum_{u \in C} k \cdot deg^2(u))$. In addition, the core maintenance algorithm for deterministic graph has a worst-case time complexity of $O(|V_c| \cdot \sum_{u \in V_c} deg(u))$. It should be noted that $V_c \subseteq C$ must be true here. Therefore, considering each k, the total time complexity of Algorithm 1 or Algorithm 4 is $O(k_0 \cdot (|C| \log|C| + |V_c| \sum_{u \in V_c} deg(u)) + k_0^2 \cdot \sum_{u \in C} deg^2(u))$. Here, V_c is bounded by C and k_0 is bounded by the minimum core number of u and v. Also, the size of C is bounded by the maximum subgraph. Each vertex w in the subgraph are connected to the root, and its $\eta(k, w)$ within the η-threshold range. Therefore, in practice, the maintenance algorithm is very fast because the visited subgraph during the search process is usually much smaller than $\mathcal{G}$, and k_0 is usually smaller than the largest core number in the deterministic graph.

The Experimental Results of EI/ED

F THE EXPERIMENTAL RESULTS OF EI/ED