Quantum Animations Theorem of Asymptotics

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1 Gaussian

Start with the initial state:

$$\psi_0(x) = \frac{1}{(2\pi\sigma^2)^{\frac{1}{4}}} e^{-\frac{(x-\mu)^2}{4\sigma^2}} e^{ipx} \tag{1}$$

This statisfies

$$\int_{-\infty}^{\infty} \|\psi_0(x)\|^2 \, dx = 1 \tag{2}$$

Its Fourier Transform is

$$\psi_0(k) = \left(\frac{2\sigma^2}{\pi}\right)^{\frac{1}{4}} e^{i\mu p} e^{-i\mu k} e^{-(k-p)^2\sigma^2} \tag{3}$$

The convention used is

$$\psi_0(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \psi_0(x) \, dx \tag{4}$$

This is also normalized.

$$\int_{-\infty}^{\infty} \|\psi_0(k)\|^2 \, dk = 1 \tag{5}$$

The time evolved state is found using

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} e^{-\frac{ik^2t}{2}} \psi_0(k) \, dk \tag{6}$$

Doing the calculation, we get

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}} \left(\frac{2\sigma^2}{\pi}\right)^{\frac{1}{4}} \sqrt{\frac{\pi}{\sigma^2 + \frac{it}{2}}} e^{i\mu p} e^{-p^2\sigma^2} e^{a^2\left(\sigma^2 + \frac{it}{2}\right)} \tag{7}$$

where a is given by

$$a = \frac{2\sigma^2 p + i(x-\mu)}{2\sigma^2 + it} \tag{8}$$

To confirm that it is correct, just verify that $\psi(x, t = 0) = \psi_0(x)$! So now all norms are preserved.