Brief Papers

Bifurcation in Vehicle Dynamics and Robust Front Wheel Steering Control

Eiichi Ono, Shigeyuki Hosoe, Hoang D. Tuan, and Shun'ichi Doi

***Abstract*—A control strategy is proposed for designing a steering control for automotive vehicles to protect the vehicle from spin and to realize the improved cornering performance. The vehicle unstabilization is shown to be caused by a saddle-node bifurcation which depends heavily on a rear tire side force saturation. Based on this observation, the saturation characteristics of rear tires are modeled by a linear function with uncertainty terms of a special structure. The nonlinearity of the saturation is included in the part of uncertainty since it changes considerably due to road environments, whose rigorous identification is usually very difficult. Then a result from the linear H^∞ control theory seems to be appropriate to design a front wheel steering controller which compensates the instability against the nonlinear uncertainty. The designed controller is shown to work quite well for nonlinear systems in achieving robust stability and protecting the vehicle from spin. Furthermore, the computer simulations show that the control improves cornering performance in critical motions. The motion realized by the controller resembles the one known as a counter steering which skillful drivers often use.**

***Index Terms*— H^∞ control, road vehicle control, robustness, stability, state feedback.**

I. Introduction

A steering control of automotive vehicle has attracted considerable attention in recent years [1]–[5], and to the market an active four-wheel-steering system with feedback control has been introduced [6]. This control is introduced to improve the stability and maneuverability of the vehicle under all the driving circumstances. Especially, the vehicle steering control should improve the safety in critical cornering motions in an emergency. However, because of the limitation of the theoretical development in analysis of nonlinear systems, most of the designing had to be performed based on a linearized model of the vehicle dynamics in moderate cornering motions. Therefore, at least from theoretic view points, the designed control system may not be enough to compensate the stability in critical cornering motions. When the driving condition changes from moderate cornering on a dry road to hard cornering on a low-friction road, the dynamics may sometimes represent unstable characteristics. In the worst case, the vehicle will fall into spin.

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The purpose of this paper is to present a control strategy of the active front wheel steering system to protect the vehicle from spin and to realize the improved cornering performance. By using phase-plane analysis [7], it is shown in the first half of the paper that the vehicle unstabilization is caused by a saddle-node bifurcation which depends heavily on a rear tire side force saturation.

Based on such observation, in the latter half the design problem is considered. The saturation characteristics of rear tires are modeled by a linear function with uncertainty terms of a special structure. The nonlinearity of the saturation is considered as a part of uncertainty since it changes considerably due to road environments, whose rigorous identification is usually very difficult. Then a result from the linear H^∞ control theory seems to be appropriate to design a front wheel steering controller which compensates the instability against the nonlinear uncertainty. It should be noted that though the problem considered is nonlinear in its character, the linear designing technique is used. This helps to simplify designing fairly well. Moreover, the designed controller is shown to work quite well for nonlinear systems in achieving robust stability and protecting the vehicle from spin. Furthermore, the computer simulations show that the control improves cornering performance in critical motions. It is also interesting to point out that the motion realized by the controller resembles the one known as a counter steering which skillful drivers often use.

II. Nonlinear Modeling of Vehicle Dynamics and Its Analysis

A. Mathematical Model

Assuming that vehicle has a constant velocity, the two-dimensional model with nonlinear tire force characteristics of the vehicle behavior can be described by the differential equations

$$mv \cdot \left(\frac{d}{dt}\beta + r\right) = F_f + F_r \tag{1}$$

$$I_z \cdot \frac{d}{dt} r = (a_f F_f - a_r F_r) \cdot \cos\beta \tag{2}$$

or

$$\frac{d}{dt}\begin{pmatrix}\beta \\ r\end{pmatrix} = \begin{pmatrix} \dfrac{F_f + F_r}{mv} - r \\ \dfrac{a_f F_f - a_r F_r}{I_z} \cos\beta \end{pmatrix} \tag{3}$$

where β: is the sideslip angle, r is the yaw velocity, F_f is the cornering force of front two tires, F_r is the cornering force

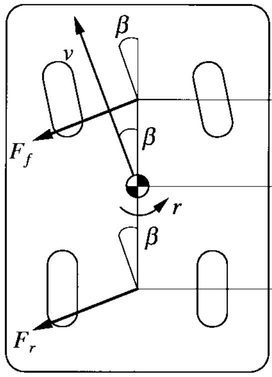


Fig. 1. Vechicle model.

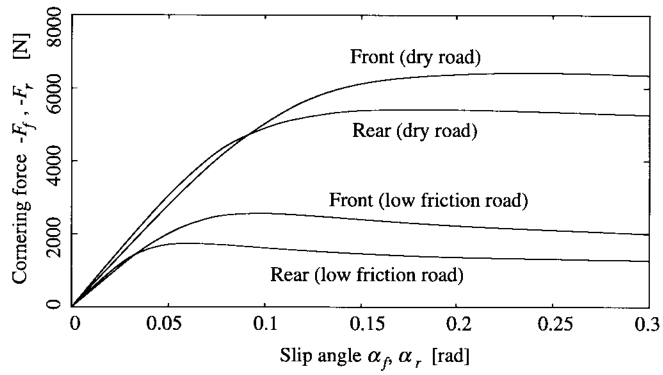


Fig. 2. Equivalent cornering characteristics.

of rear two tires, a_f (=1.2 m) is the distance from front axle to center of gravity (c.g.), a_r (=1.3 m) is the distance from rear axle to c.g., I_z (=3000 kgm^2) is the yaw moment of inertia, m (=1500 kg) is the vehicle mass, v (=20 m/s) is the vehicle velocity. The vehicle model is shown in Fig. 1. There are many models to describe its cornering force characteristic [8]. However, in order to avoid complication of the control design, the following simple mathematical model [9] is used in our study (see Fig. 2). First of all, the cornering forces F_f, F_r are modeled as functions of tire slip angles

$$\begin{aligned} F_f =& D_f \sin[C_f \tan^{-1}\{B_f(1-E_f)\alpha_f \\ &+ E_f \tan^{-1}(B_f\alpha_f)\}] \end{aligned} \tag{4}$$

$$\begin{aligned} F_r =& D_r \sin[C_r \tan^{-1}\{B_r(1-E_r)\alpha_r \\ &+ E_r \tan^{-1}(B_r\alpha_r)\}] \end{aligned} \tag{5}$$

$$\alpha_f = \beta + \tan^{-1}\left(\frac{a_f}{v}\cdot r\cos\beta\right) - \delta_f \tag{6}$$

$$\alpha_r = \beta - \tan^{-1}\left(\frac{a_r}{v}\cdot r\cos\beta\right) \tag{7}$$

where α_f is the slip angle of front tires, α_r is the slip angle of rear tires, and δ_f is the front steer angle. The coefficients B_j, C_j, D_j, E_j ($j = f, r$) of the models are given in Table I. To match the purpose of the present paper, the coefficients of the low-friction roads are chosen so that the vehicle goes easily into spin, which correspond to the driving condition traveling on a down sloping road at constant velocity with equivalent braking effect while throttle off. Observe that only δ_f is the manipulatorable variable. To stress this in the sequel, the right-hand side of (3) will be expressed as function $\boldsymbol{f}(\beta, r, \delta_f)$.

B. Simulation Results

Fig. 4 shows vehicle state trajectories traveling on a low-friction road during the first 2 s from initial states (+). There are one stable equilibrium point and two unstable equilibrium points for $-0.015 \leq \delta_f \leq 0.015$ [rad]. The equilibrium points (β_0, r_0) are such that

$$\boldsymbol{f}(\beta_0, r_0, \delta_f) = 0. \tag{8}$$

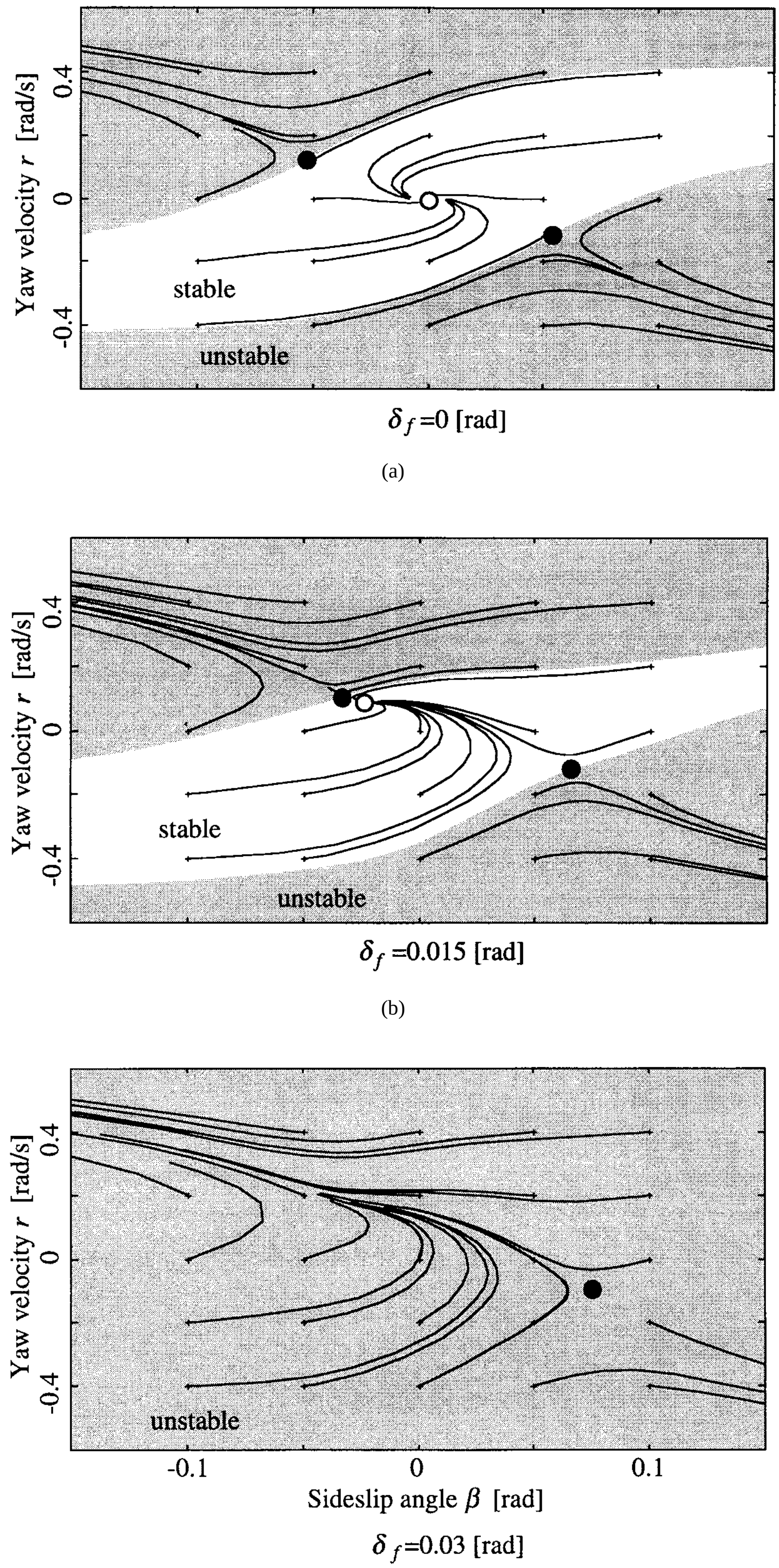
The vehicles from initial states in shadowed areas indicate divergence in sideslip angle representing vehicle spin. When $\delta_f = 0$ [rad], the stable equilibrium point coincides with the origin as shown in Fig. 3(a). This implies that the vehicles from initial states in white area converge to straight line traveling. When $\delta_f = 0.015$ [rad], yaw velocity of the stable equilibrium represents positive value $r > 0$ as shown in Fig. 3(b). This implies that left cornering corresponds to left steering ($\delta_f = 0.015$ [rad]). At $\delta_f \cong 0.015$ [rad], the two equilibria coalesce into one, and at $\delta_f = 0.03$ [rad], the stable equilibrium disappears as shown in Fig. 3(c). The vehicles from arbitrary initial states fall into spin at $\delta_f = 0.03$ [rad].

TABLE I
COEFFICIENTS OF THE TIRE FORCE MODELS

		B_f, B_r	C_f, C_r	D_f, D_r	E_f, E_r
High friction road	Front	6.7651	1.3000	-6436.8	-1.9990
	Rear	9.0051	1.3000	-5430.0	-1.7908
Low friction road	Front	11.275	1.5600	-2574.7	-1.9990
	Rear	18.631	1.5600	-1749.7	-1.7908

C. Analysis of Stability

It has been revealed that the qualitative structure of the vector field describing the dynamics changes at $\delta_f \cong \pm 0.015$ [rad]. This phenomenon is called by the “saddle-node bifurcation” [10]. Fig. 4 shows the bifurcation diagrams [10] of the vehicle traveling on a low-friction road. When driver’s steering angle is small from the center position, the vehicle state shifts along the solid curve in Fig. 4, i.e., the vehicle changes its movement from straight line traveling to the cornering corresponding to the steer angle. On the other hand, for steering angle that is larger than bifurcation points O, $\triangle$, the stable equilibrium point disappears, and the vehicle goes into divergent state and falls into spin. This implies that without any control it is impossible to get larger steady yaw velocity than the one at the bifurcation points. In order to get larger value of the steady yaw without vehicle spin we need control and with control (more precisely with dynamic control) it is actually possible to realize it. This point is examined as follows.



(c)

Fig. 3. State trajectories of the vehicle without control on a low-friction road (∘: stable equilibrium point and •: unstable equilibrium points).

Let us consider an active front wheel controlled system whose compensator is represented by

$$\frac{d}{dt}\boldsymbol{z} = \boldsymbol{f}_c(\boldsymbol{z}, \beta, r, \delta_{sw}) \tag{9}$$

$$\delta_f = g_c(\boldsymbol{z}, \beta, r, \delta_{sw}) \tag{10}$$

where $\boldsymbol{z}$ and δ_{sw} are the state of the compensator and the steering wheel angle, respectively. In the controlled system (9), (10), δ_f is the output of the controller and hence cannot be manipulated directly by drivers. The drivers can change δ_{sw} by manipulating the steering wheel corresponding to the objective yaw velocity. Notice that δ_{sw} is made independent of states, since we discuss the stability of vehicle system not including driver. This contrasts with [11] where δ_{sw} is a function of states when the driver-vehicle system is discussed there. With the above compensator, the closed-loop system is described by (9), (10), and

$$\frac{d}{dt}\begin{bmatrix}\beta \\ r\end{bmatrix} = \boldsymbol{f}(\beta, r, \delta_f).$$

Thus the equilibrium point $(\beta, r, \boldsymbol{z})$ of the controlled system must satisfy

$$\boldsymbol{f}(\beta, r, \delta_f) = 0, \ \boldsymbol{f}_c(z, \beta, r, \delta_{sw}) = 0$$
$$\delta_f = g_c(\boldsymbol{z}, \beta, r, \delta_{sw}). \tag{11}$$

On the other hand, the equilibrium point (β, r) of the uncontrolled system satisfies

$$\boldsymbol{f}(\beta, r, \delta_f) = 0. \tag{12}$$

Now, suppose that the compensator (9), (10) is constructed so that

1) for any δ_{sw}, (11) gives a unique equilibrium point $(\beta, r, \boldsymbol{z})$;
2) $(\beta, r, \boldsymbol{z})$ is asymptotically stable in the large;
3) for any β, r, and δ_f satisfying (12), there exist $\boldsymbol{z}$; δ_{sw} that satisfy (11).

Then, as a result, any equilibrium point (β, r), regardless being stable or not of the uncontrolled system, becomes the asymptotically stable equilibrium point $(\beta, r, \boldsymbol{z})$ of the controlled system (9), (10) for some $\boldsymbol{z}$. Thus we can get larger steady yaw velocity than the one at the bifurcation points. In the next section we will show that it is really possible to construct a compensator with the above conditions 1)–3).

At this point it is of interest to make the following observation. From Fig. 4, it can be seen that larger steady yaw velocity than the one at the bifurcation points O, $\triangle$ can be obtained by making smaller the value of δ_f, i.e., to get sharp cornering we must steer a wheel to the opposite direction to the cornering direction. This driving technique is known as a "counter steering" and often used by skillful drivers. It will be shown in our designed system that the same phenomenon will occur.

Next, let us discuss stability of equilibrium of (3) and examine what factor will be most influensive. Mention that the stability of an equilibrium point of system (3) can be examined by using a linearized model as it is stated in the following fact.

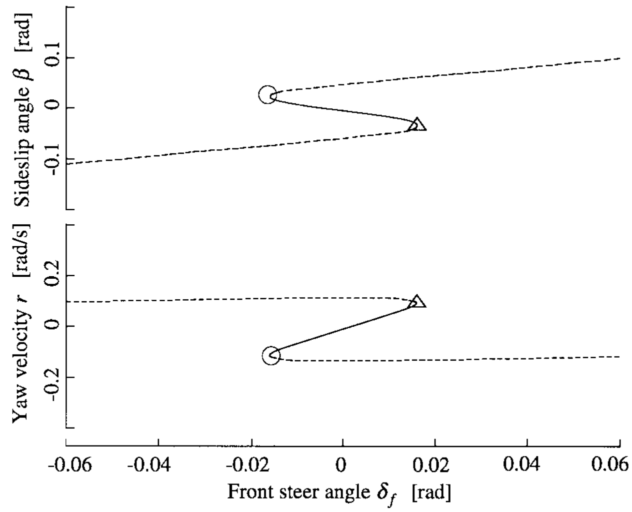


Fig. 4. Bifurcation diagrams of the vehicle traveling on a low-friction road (solid: stable equilibrium point, dashed: unstable equilibrium point).

Fact 2.1. [10]: Let $\boldsymbol{f}$ be a C_1 function. If all the eigenvalues of the Jacobian matrix of $\boldsymbol{f}$ at a equilibrium point have negative real parts, then this equilibrium point is asymptotically stable of system (3). If at least one of the eigenvalues of the Jacobian matrix of $\boldsymbol{f}$ at an equilibrium point has a positive real part, then this equilibrium point is unstable of system (3).

To use the above fact, by approximating $\cos\beta$ by one, the Jacobian matrix of $\boldsymbol{f}$ at equilibrium point $\boldsymbol{x}_0$ is approximated by the matrix

$$A_0 = \begin{bmatrix} -\dfrac{c_f^* + c_r^*}{mv} & -1 - \dfrac{a_f c_f^* - a_r c_r^*}{mv^2} \\ -\dfrac{a_f c_f^* - a_r c_r^*}{I_z} & -\dfrac{a_f^2 c_f^* + a_r^2 c_r^*}{I_z v} \end{bmatrix} \tag{13}$$

where c_f^* and c_r^*, respectively, are the tangents to slopes of front and the rear side force characteristics at the equilibrium point $\boldsymbol{x}_0$. The characteristic equation of matrix A_0 is

$$s^2 + p \cdot s + q = 0 \tag{14}$$

with

$$p = \frac{c_f^* + c_r^*}{mv} + \frac{a_f^2 c_f^* + a_r^2 c_r^*}{I_z v} \tag{15}$$

$$q = \frac{(a_f + a_r)^2 c_f^* c_r^*}{m I_z v^2} - \frac{a_f c_f^* - a_r c_r^*}{I_z}. \tag{16}$$

From (15) and (16) one can compute p and q along the equilibrium point in Fig. 4. Fig. 5 shows that p is always positive and then the stability of the equilibrium point is solely determined by the signature of q. In Fig. 5, the solid and dashed lines express stable and unstable equilibrium points, respectively. With increasing sideslip angle, q becomes to be negative that makes the equilibrium point unstable. Note that the case of $q < 0$ corresponds to

$$c_r^* < \frac{mv^2 a_f c_f^*}{mv^2 a_r + (a_f + a_r)^2 c_f^*} \tag{17}$$

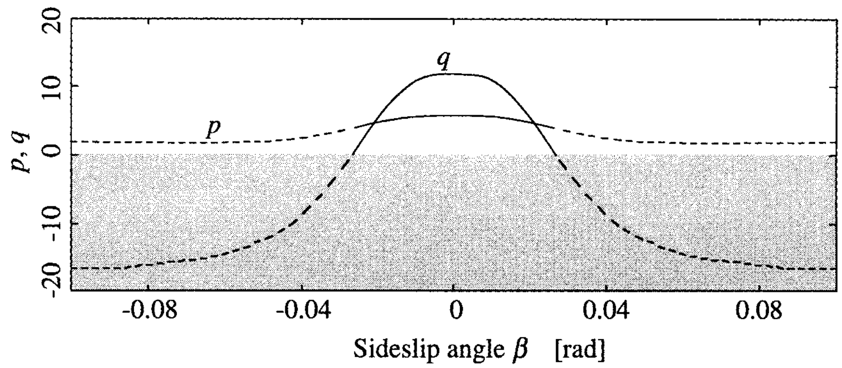


Fig. 5. Coefficients of characteristic equation $s^2 + p \cdot s + q = 0$.

showing that vehicle instability can be well interpreted in terms of saturation of rear side force characteristics concerned with road environments, i.e., vehicle instability is occurred when the slope of the rear side force c_r^* is bounded by the right-hand side of (17).

III. Robust Control of Front Wheel Steering

In this section, the vehicle model will be more simplified for control system design purpose and a front steering controller which permits rear tire side force saturation and protects from vehicle spin is designed based on this simplified model and the technique of H^∞ control. Here, we assume that vehicle state can be measured. Sideslip angle may be measured by optical sensor or may be estimated by using observer [12].

A. Controlling Model with Uncertainty

Of course, the simplified model should keep fundamental characteristics that makes the vehicle fall into spin. The previous section shows that saturation characteristics of rear tire force is the main cause for the vehicle spin. However, to know the exact points of saturation while driving is almost impossible. Thus a model which depends explicitly on the saturation points seems to be not appropriate. Hence we describe rear cornering force characteristics on arbitrary road environments by

$$F_r = -c_r(1 + W\Delta) \cdot \alpha_r \qquad (-1 \le \Delta \le 1) \tag{18}$$

where Δ is a function of α_r, representing the nonlinear part of F_r. Since Δ is considered as uncertainty term of the plant, it can be normalized so that $-1 \le \Delta(\alpha_r) \le 1$. Also, in (18) c_r is a cornering stiffness of a nominal model. Further, W is a weight used for normalization for which $c_r(1+W)$ and $c_r(1-W)$ are upper and lower bounds of the slopes of rear side force characteristics, respectively, i.e.,

$$c_r(1 - W) \le -\frac{\partial}{\partial \alpha_r} F_r \le c_r(1 + W)$$

or equivalently

$$\left| \frac{\partial}{\partial \alpha_r} [\Delta(\alpha_r) \cdot \alpha_r] \right| \le 1. \tag{19}$$

The relation (19) defines the special structure for the uncertainty term $\Delta(\alpha_r)$ and would play an very important role in our development. It is also assumed that the front cornering force F_f can be described approximately as

$$F_f = -c_f \cdot \alpha_f \tag{20}$$

where c_f is the slope of front side force characteristics at the origin on a dry road, which is called by front cornering stiffness. Finally, setting

$$\cos\beta \cong 1,\ \alpha_f \cong \beta + \frac{a_f}{v} r + \delta_f,\ \alpha_r \cong \beta - \frac{a_f}{v} r$$

the vehicle model can be described in the state-space form as

$$\frac{d}{dt}\boldsymbol{x} = A\boldsymbol{x} + B_1 \Delta \alpha_r + B_2 \delta_f \tag{21}$$

$$\alpha_r = C\boldsymbol{x} \tag{22}$$

where

$$\begin{aligned}
\boldsymbol{x} &= [\beta \quad r]^T \\
A &= \begin{bmatrix} -\dfrac{c_f + c_r}{mv} & -1 - \dfrac{a_f c_f - a_r c_r}{mv^2} \\ -\dfrac{a_f c_f - a_r c_r}{I_z} & -\dfrac{a_f^2 c_f + a_r^2 c_r}{I_z v} \end{bmatrix} \\
B_1 &= -W \begin{bmatrix} \dfrac{c_r}{mv} \\ -\dfrac{a_r c_r}{I_z} \end{bmatrix},\ B_2 = \begin{bmatrix} \dfrac{c_f}{mv} \\ \dfrac{a_f c_f}{I_z} \end{bmatrix} \\
C &= \begin{bmatrix} 1 & -\dfrac{a_r}{v} \end{bmatrix}.
\end{aligned}$$

Regarding $\Delta \cdot \alpha_r$ as w, the simplified vehicle model (21), (22) is expressed as a linear model G with nonlinear uncertainty Δ shown in Fig. 6. Since Δ is a nonlinear function, the bifurcation can happen in the simplified vehicle dynamics. Fig. 7 shows bifurcation diagrams of simplified vehicle model (21), (22), when Δ is a nonlinear function which agrees (18) with the characteristics of the rear cornering force on a low-friction road in Fig. 2. The equilibrium points are such $\boldsymbol{x}_0$ satisfying

$$A\boldsymbol{x}_0 + B_1 \Delta[\alpha_r(\boldsymbol{x}_0)] C\boldsymbol{x}_0 + B_2 \delta_f = 0. \tag{23}$$

A very important point we want to mention here is that the states of equilibria at bifurcation points in Fig. 7 approximately

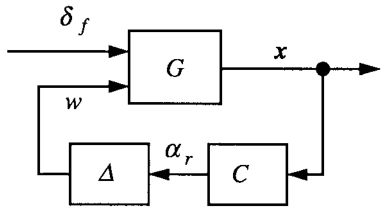


Fig. 6. Block diagram of (21) and (22).

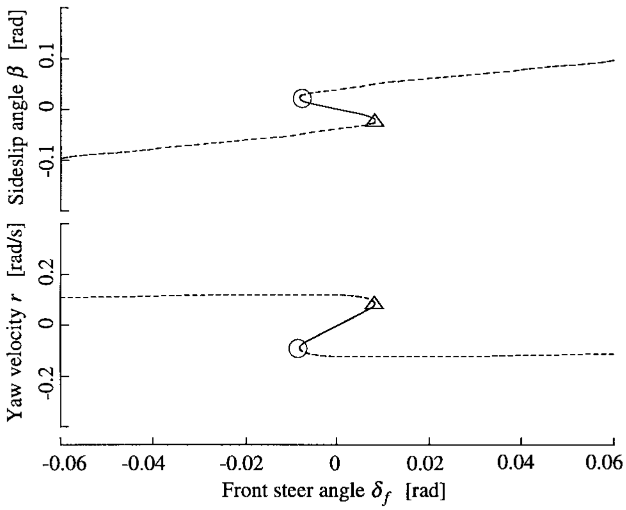


Fig. 7. Bifurcation diagrams of the simplified vehicle model traveling on a low-friction road (solid: stable equilibrium point, dashed: unstable equilibrium point).

coincide with the states of equilibria at bifurcation points in Fig. 4. This justifies our idea to approximate the vehicle dynamics model (3) by the simplified vehicle model (21), (22).

B. Control System Design

To achieve a desirable cornering characteristics corresponding to the steering wheel angle, the model following control structure shown in Fig. 8 is adopted. Here, G_0 represents a ideal model, $\boldsymbol{x}_{ref}$ is the reference value of the state, and δ_{sw} is the steering wheel angle. In this paper, G_0 is taken to be a linear model of the vehicle with normal cornering stiffness on a dry road. A state feedback gain K is designed by applying linear H^∞ control [13] to the augmented system in Fig. 9, whose output is given as

$$\boldsymbol{z} = \begin{bmatrix} \alpha_r \\ Du \end{bmatrix}. \tag{24}$$

The output Du is introduced for evaluating the control input consumption. Control input u to the vehicle directly affects the lateral acceleration, therefore, the comfort of the passengers [14]. The constant D is used to adjust the ride comfort. State feedback gain K is sought so that it satisfies

$$\sup_{w} \frac{||\boldsymbol{z}||_2}{||w||_2} < 1. \tag{25}$$

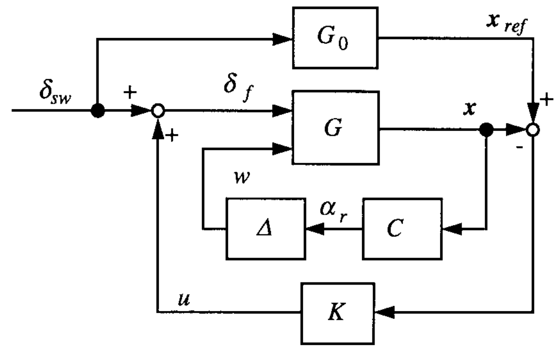


Fig. 8. Control system structure.

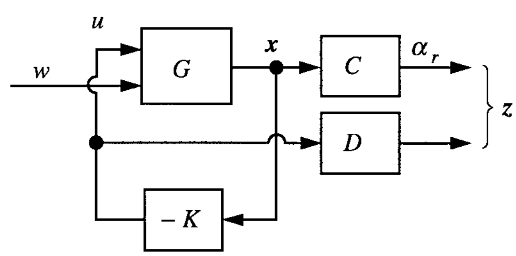


Fig. 9. Augmented system for designing the controller.

In the Appendix, it is shown that K (if it exist) can be given by

$$K = \frac{1}{D^2} B_2^T P \tag{26}$$

where P is a positive definite solution of the Riccati inequality

$$PA + A^T P + P\left(B_1 B_1^T - \frac{1}{D^2} B_2 B_2^T\right) P + C^T C < 0. \tag{27}$$

When the driver's steering $\delta_{sw} = 0$, the robust stability of the origin of the system in Fig. 8 is guaranteed by the small gain theorem [15]. In the Appendix, we show that there is an unique equilibrium of the closed system for any constant δ_{sw} in Fig. 8, and it is globally asymptotically stable.

The above analysis implies the following our main result on sufficient condition for protecting the vehicle from spin.

Antispin Condition: The front wheel steering control (26) protects the vehicle from spin caused by rear tire force saturations when it satisfies the L_2 gain < 1 for the augmented system shown in Fig. 9.

C. Simulation Results

The control effects are evaluated by computer simulations using the vehicle model (1)–(7) and control system in Fig. 8 with the weighting constant $D = 0.6$ for the control input consumption. Fig. 10 shows the state trajectories of the vehicle traveling on a low-friction road during first 2 s from initial states (+) with the front wheel steering control. The equilibrium point is globally stable, so that every trajectories are attracted to the equilibrium point. At $\delta_{sw} = 0$ [rad], every trajectories are attracted to the origin which corresponds to straight line traveling as shown in Fig. 10a). Fig. 10(b) shows that driver's left steering ($\delta_{sw} = 0.03$ [rad]) causes left

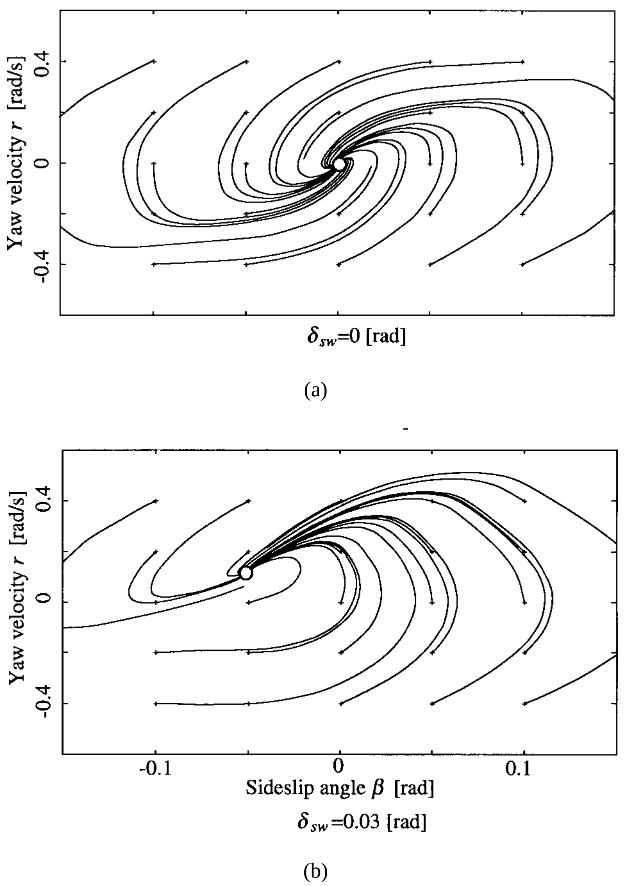


Fig. 10. State trajectories of the vehicle with control on a low-friction road (○: stable equilibrium point).

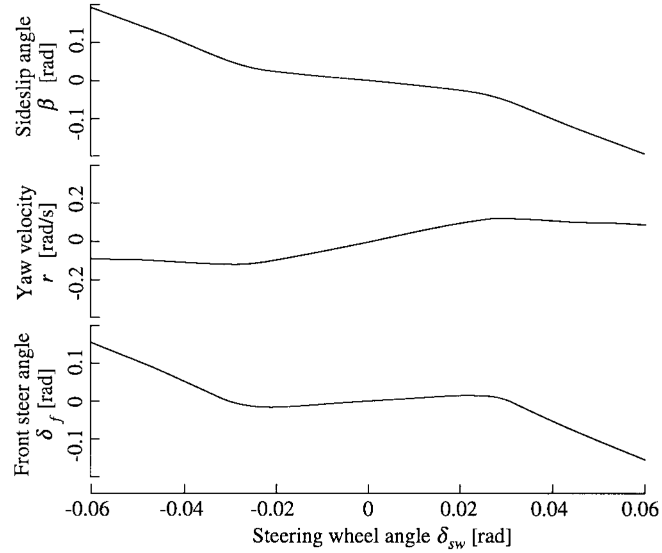


Fig. 11. Relation between steering wheel angle and state of equilibrium of the controlled vehicle traveling on a low-friction road.

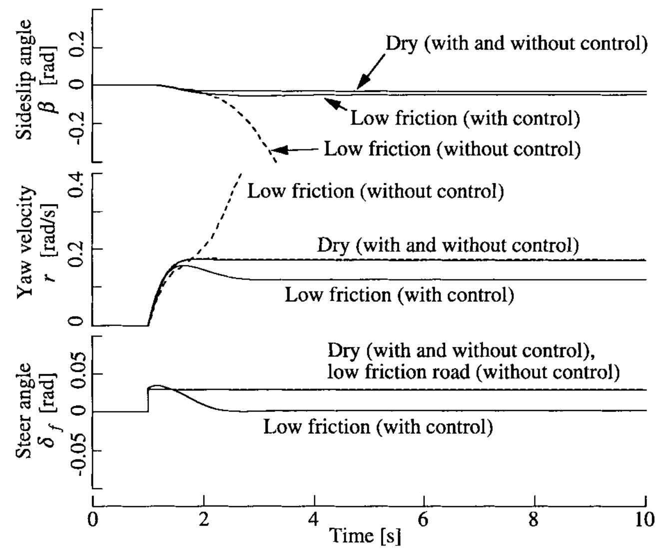


Fig. 12. Step steer responses on a dry road and on a low-friction road (solid: with control, dashed: without control, $\delta_{sw} = 0.03$ [rad]).

cornering. The equilibrium point which is located according to the driver's intended direction is globally stabilized by the front wheel steering control, while the vehicle without control has no stable equilibrium point [see Fig. 3(c)]. Fig. 11 shows the relation between the steering wheel angle δ_{sw} and the states of equilibrium. The qualitative structures of the vector field do not change in these figures, and the control protects the vehicle from spin even if the steering wheel angle exceeds limit of tire force. The equilibrium $[\beta_0, r_0, \delta_{sw} + K(\boldsymbol{x}_{ref} - \boldsymbol{x}_0)]$ is on the solid or dashed curve shown in Fig. 4. Furthermore, maximum value in yaw velocity of equilibrium in Fig. 11 coincides with maximum value in yaw velocity of unstable equilibrium in Fig. 4. This implies that the controller accomplishes critical cornering performance which was only obtained by "counter steer."

Step steer responses of $\delta_{sw} = 0.03$ [rad] on a dry road and on a low-friction road are shown in Fig. 12. The solid and dashed lines represent the vehicle with and without control, respectively. When the vehicles are traveling on a dry road, each tire causes cornering force linearly on this simulation condition. When errors between states and reference states are small then controls work little. When the vehicles are traveling on a low-friction road, the steering wheel angle exceeds limit of tire forces then the vehicle without control falls into spin. The change of road environments as plant perturbation makes the vehicle dynamics unstable. On the other hand, the controlled vehicle results in stable cornering behavior. This means that the controller designed by H^∞ control compensates robust stability against the plant perturbation which makes the plant unstable.

IV. Conclusions

The phenomena of vehicle spin is shown to be caused by a saddle-node bifurcation which depends heavily on rear side force saturation. The saturation characteristics which changes largely depending on change of road environments as plant perturbations and design method is proposed for a front wheel steering control system which permits the perturbations by H^∞ control. This control robustly stabilizes the system against the uncertainty which make the plant unstable. Some computer simulations has revealed control effects. The front wheel steering control is the "counter steer" and protects from vehicle spin.

Appendix

We will show that there is an unique equilibrium of the closed system with constant driver's steering δ_{sw} in Fig. 8, and it is globally asymptotically stable. First, the closed-loop system in Fig. 8 can be written as

$$\frac{d}{dt}\boldsymbol{x} = (A - B_2K)\boldsymbol{x} + B_1 \cdot \Delta[\alpha_r(\boldsymbol{x})] \cdot C\boldsymbol{x} + B_2(\delta_{sw} + K\boldsymbol{x}_{\rm ref}). \tag{28}$$

Since the robust stability of the origin under the condition $\delta_{sw} = 0$ is guaranteed, the matrix $A - B_2K + B_1\Delta C$ is asymptotically stable for any $\Delta \in [-1, 1]$ and so it is also invertible. Hence an equilibrium $\boldsymbol{x}_0$ of system (28) must satisfy the following equation:

$$\boldsymbol{x}_0 = -\{A - B_2K + B_1 \cdot \Delta[\alpha_r(\boldsymbol{x}_0)] \cdot C\}^{-1} \cdot B_2(\delta_{sw} + K\boldsymbol{x}_{\rm ref}^0) \tag{29}$$

where $\boldsymbol{x}_{\rm ref}^0$ is an equilibrium of system G_0. To show the existence of $\boldsymbol{x}_0$, define the nonlinear function $\boldsymbol{X}$ of variable Δ_x by setting

$$\boldsymbol{X}(\Delta_x) = -(A - B_2K + B_1 \cdot \Delta_x \cdot C)^{-1}B_2(\delta_{sw} + K\boldsymbol{x}_{ref}).$$

Obviously, function $\boldsymbol{X}$ is continuous and consequently, function maps from compact set to itself. Hence by Brouwer's fixed point theorem, the equation

$$\Delta_x = \Delta\{\alpha_r[\boldsymbol{X}(\Delta_x)]\}$$

has a solution Δ_0. Now, for every uncertainty function $\Delta(\alpha_r)$, it is already easy to check that

$$\boldsymbol{x}_0 = \boldsymbol{X}(\Delta_0)$$

satisfies (29), i.e., $\boldsymbol{x}_0$ is an equilibrium of (28). Moreover, this point would be unique if it is globally asymptotically stable as we will show in the next step to complete the proof. Rearrange (29) as

$$\frac{d}{dt}\xi = (A - B_2K)\xi + B_1 \cdot \{\Delta[\alpha_r(x)]\alpha_r(x) - \Delta[\alpha_r(\boldsymbol{x}_0)]\alpha_r(\boldsymbol{x}_0)\} \tag{30}$$

where

$$\xi = \boldsymbol{x} - \boldsymbol{x}_0.$$

Furthermore, by (19) that defines the special structure for $\Delta(\alpha_r)$, the function

$$f_\Delta(\alpha_r) = \Delta(\alpha_r) \cdot \alpha_r \tag{31}$$

satisfies sector condition

$$|f_\Delta(\alpha_r) - f_\Delta(\alpha_{r0})| \le \alpha_r - \alpha_{r0} \tag{32}$$

and then (30) can be rewritten as

$$\frac{d}{dt}\xi = (A - B_2K)\xi + B_1 \cdot \Delta_\xi(\xi, \boldsymbol{x}_0) \cdot C\xi \tag{33}$$

where

$$|\Delta_\xi(\xi, \boldsymbol{x}_0)| \le 1.$$

Now, take $V(\xi) = \xi^TP\xi$ as Lyapunov function of the closed-loop system (33). Since $|\Delta| \le 1$, from (27) there is $\sigma > 0$ such that

$$\begin{aligned}\frac{dV(t)}{dt} &= \xi^T(t)\left[\left(A - \frac{1}{D^2}B_2B_2^TP\right)^T P + P\left(A - \frac{1}{D^2}B_2B_2^TP\right)\right]\xi(t) + 2\xi^T(t)PB_1\Delta C_1\xi(t)\\ &\le \xi^T(t)\left[\left(A - \frac{1}{D^2}B_2B_2^TP\right)^T P + P\left(A - \frac{1}{D^2}B_2B_2^TP\right)\right]\xi(t) + \xi^T(t)PB_1B_1^TP\xi(t) + \xi^T(t)C_1^TC_1\xi(t)\\ &\le -\sigma||\xi(t)||^2\end{aligned} \tag{34}$$

showing the global exponential stability of the closed-loop system (33).

Let V_1 be a Lyapunov function of system G_0 such that

$$\left.\frac{dV_1(t)}{dt}\right|_{G_0} \le -\frac{4||PB_1||^2}{\sigma} \cdot ||\boldsymbol{x}_{ref} - \boldsymbol{x}_{ref}^0||^2$$

where

$$\left.\frac{dV_1(t)}{dt}\right|_{G_0}$$

denotes the derivative of V_1 along the solution of G_0.

Taking $V + V_1$ as Lyapunov function of the closed-loop system (33) $-G_0$ and using (34) we have

$$\begin{aligned}\frac{d[V(t) + V_1(t)]}{dt} &= \left.\frac{dV(t)}{dt}\right|_{(33)} + \left.\frac{dV_1(t)}{dt}\right|_{G_0}\\ &\le -\sigma||\xi||^2 + 2\xi^TPB_1(\boldsymbol{x}_{\rm ref} - \boldsymbol{x}_{\rm ref}^0) - \frac{4||PB_1||^2}{\sigma} \cdot ||\boldsymbol{x}_{\rm ref} - \boldsymbol{x}_{\rm ref}^0||^2\\ &\le -\sigma||\xi||^2 + \frac{\sigma}{2}||\xi||^2 + \frac{2||PB_1||^2}{\sigma} \cdot ||\boldsymbol{x}_{\rm ref} - \boldsymbol{x}_{\rm ref}^0||^2 - \frac{4||PB_1||^2}{\sigma} \cdot ||\boldsymbol{x}_{\rm ref} - \boldsymbol{x}_{\rm ref}^0||^2\\ &\le -\frac{\sigma}{2}||\xi||^2 - \frac{2||PB_1||^2}{\sigma} \cdot ||\boldsymbol{x}_{\rm ref} - \boldsymbol{x}_{\rm ref}^0||^2\end{aligned}$$

implying (0, $\boldsymbol{x}_{ref}^0$) is the globally asymptotically stable equilibrium of the closed-loop system (33)—G_0. Therefore $\boldsymbol{x}_0$ is the globally asymptotically stable equilibrium of (28).

The feedback control (26) already meets requirements 1) and 2) of our design purpose. So the last step is to show that the last requirement 3) is also fulfilled, i.e., for every equilibrium ($\boldsymbol{x}_0$, δ_f^0) of (21) there is (δ_{sw}^0, $\boldsymbol{x}_{\rm ref}^0$) such that

$(\boldsymbol{x}_0, \delta_f^0, \delta_{sw}^0, \boldsymbol{x}_{\rm ref}^0)$ is an equilibrium of the closed-loop system in Fig. 8. In fact, assume that the state-space form of G_0 is

$$\frac{d}{dt}\boldsymbol{x}_{\rm ref} = A_{\rm ref}\boldsymbol{x}_{\rm ref} + B_{\rm ref}\delta_{sw}$$

where $A_{\rm ref}$ is an asymptotically stable matrix. Then every equilibrium $\boldsymbol{x}_{ref}^0$ of G_0 satisfies

$$\boldsymbol{x}_{\rm ref}^0 = -A_{\rm ref}^{-1}B_{\rm ref}\delta_{sw}$$

for some δ_{sw}. Hence, from definition of δ_f, to show the existence of $(\delta_{sw}^0, \boldsymbol{x}_{\rm ref}^0)$ it suffices to show that the linear equation

$$(1 - KA_{\rm ref}^{-1}B_{\rm ref})\delta_{sw} = \delta_f^0 + K\boldsymbol{x}_0$$

always has a solution δ_{sw}^0. But it is trivial since we always can arrange

$$(1 - KA_{\rm ref}^{-1}B_{\rm ref}) \neq 0.$$

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