A Quaternion-Based Unscented Kalman Filter for Orientation Estimation: Mathematical Foundations and Kinematics

Aysel Rajabli
ayselracabli01@gmail.com

May 5, 2025

Abstract

This paper presents a comprehensive mathematical formulation of a Quaternion-based Unscented Kalman Filter (UKF) for orientation estimation of a rigid body. Unit quaternions provide a singularity-free representation of orientation, while the UKF addresses the nonlinear dynamics and measurement models inherent in attitude estimation. We derive the quaternion kinematics, state propagation, sigma point generation, and measurement update steps, ensuring preservation of the quaternion's unit norm.

1 Introduction

Orientation estimation is a fundamental problem in robotics, aerospace, and navigation systems, requiring accurate representation of a rigid body's attitude. Traditional methods, such as Euler angles, encounter singularities (e.g., gimbal lock), and direction cosine matrices are computationally intensive. Unit quaternions offer a compact, singularity-free parameterization of the rotation group SO(3), making them well-suited for attitude estimation [1].

The Unscented Kalman Filter (UKF) is an effective approach for nonlinear state estimation, avoiding the linearization errors of the Extended Kalman Filter (EKF) [2]. By propagating a deterministic set of sigma points, the UKF captures the state distribution's mean and covariance with second-order accuracy. Applying the UKF to quaternion-based systems is challenging due to the non-Euclidean nature of unit quaternions, which lie on the manifold $\mathbb{S}^3$ and combine via multiplication rather than addition.

This paper develops a theoretical framework for a Quaternion-based UKF, focusing on the mathematical derivations and kinematics for orientation estimation. We consider a sensor suite of a three-axis accelerometer, gyroscope, and magnetometer (MARG sensors), commonly used in attitude and heading reference systems (AHRS). The solution is presented as a conceptual design, emphasizing the mathematical structure.

2 Mathematical Preliminaries

2.1 Quaternions

A unit quaternion $\mathbf{q} = [q_0, \mathbf{v}]^T \in \mathbb{S}^3 \subset \mathbb{R}^4$, where $q_0 \in \mathbb{R}$ is the scalar part and $\mathbf{v} = [q_1, q_2, q_3]^T \in \mathbb{R}^3$ is the vector part, represents a rotation in SO(3). The unit norm constraint is:

$$\mathbf{q}^T\mathbf{q} = q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1. \tag{1}$$

The quaternion product is defined as:

$$\mathbf{q}_1 \otimes \mathbf{q}_2 = \begin{bmatrix} q_{1,0}q_{2,0} - \mathbf{v}_1 \cdot \mathbf{v}_2 \\ q_{1,0}\mathbf{v}_2 + q_{2,0}\mathbf{v}_1 + \mathbf{v}_1 \times \mathbf{v}_2 \end{bmatrix}, \tag{2}$$

where $\cdot$ and $\times$ denote the dot and cross products, respectively. The conjugate $\mathbf{q}^* = [q_0, -\mathbf{v}]^T$ satisfies $\mathbf{q} \otimes \mathbf{q}^* = [1, 0, 0, 0]^T$. A vector $\mathbf{v} \in \mathbb{R}^3$ is rotated from the body frame to the inertial frame via:

$$\mathbf{v}_{\text{inertial}} = \mathbf{q} \otimes \begin{bmatrix} 0 \\ \mathbf{v} \end{bmatrix} \otimes \mathbf{q}^*. \tag{3}$$

2.2 Quaternion Kinematics

The time evolution of the orientation quaternion is driven by the angular velocity $\omega = [\omega_x, \omega_y, \omega_z]^T$, typically measured by a gyroscope:

$$\dot{\mathbf{q}}(t) = \frac{1}{2}\mathbf{q}(t) \otimes \begin{bmatrix} 0 \\ \omega(t) \end{bmatrix}. \tag{4}$$

In discrete time, with sampling period Δt, the quaternion update is:

$$\mathbf{q}_{k+1} = \exp\left(\frac{\Delta t}{2}\Omega(\omega_k)\right)\mathbf{q}_k, \tag{5}$$

where $\Omega(\omega) = \begin{bmatrix} 0 & -\omega^T \\ \omega & -\lfloor\omega\rfloor_\times \end{bmatrix}$, and $\lfloor\omega\rfloor_\times$ is the skew-symmetric matrix:

$$\lfloor\omega\rfloor_\times = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}. \tag{6}$$

2.2.1 Detailed Explanation of Quaternion Kinematics Equations

The kinematics equations govern how the orientation quaternion evolves as the rigid body rotates, driven by the angular velocity $\omega(t)$. Below, we provide a detailed explanation of each equation, including their derivation, physical interpretation, and role in the UKF.

Continuous-Time Kinematics (Equation 4) Equation 4, $\dot{\mathbf{q}}(t) = \frac{1}{2}\mathbf{q}(t) \otimes \begin{bmatrix} 0 \\ \omega(t) \end{bmatrix}$, is a differential equation describing the rate of change of the quaternion $\mathbf{q}(t) = [q_0(t), q_1(t), q_2(t), q_3(t)]^T$. The left-hand side, $\dot{\mathbf{q}}(t) = \frac{d\mathbf{q}(t)}{dt}$, represents the instantaneous change in orientation. The right-hand side computes this change based on the current quaternion $\mathbf{q}(t)$ and the angular velocity

$\omega(t) = [\omega_x(t), \omega_y(t), \omega_z(t)]^T$, expressed as a pure quaternion $\begin{bmatrix} 0 \\ \omega(t) \end{bmatrix} = [0, \omega_x(t), \omega_y(t), \omega_z(t)]^T$. The quaternion product $\mathbf{q}(t) \otimes \begin{bmatrix} 0 \\ \omega(t) \end{bmatrix}$ is computed using the quaternion multiplication rule. For $\mathbf{q}(t) = [q_0, \mathbf{v}]^T$ with $\mathbf{v} = [q_1, q_2, q_3]^T$, and $\begin{bmatrix} 0 \\ \omega(t) \end{bmatrix} = [0, \omega(t)]^T$, we have:

$$\mathbf{q}(t) \otimes \begin{bmatrix} 0 \\ \omega(t) \end{bmatrix} = \begin{bmatrix} q_0 \cdot 0 - \mathbf{v} \cdot \omega(t) \\ q_0\omega(t) + 0 \cdot \mathbf{v} + \mathbf{v} \times \omega(t) \end{bmatrix} = \begin{bmatrix} -\mathbf{v} \cdot \omega(t) \\ q_0\omega(t) + \mathbf{v} \times \omega(t) \end{bmatrix}.$$

Expanding this, with $\mathbf{v} = [q_1, q_2, q_3]^T$ and $\omega(t) = [\omega_x, \omega_y, \omega_z]^T$, the scalar part is:

$$-\mathbf{v} \cdot \omega(t) = -(q_1\omega_x + q_2\omega_y + q_3\omega_z),$$

and the vector part is:

$$q_0\omega(t) + \mathbf{v} \times \omega(t) = q_0[\omega_x, \omega_y, \omega_z]^T + \begin{bmatrix} q_2\omega_z - q_3\omega_y \\ q_3\omega_x - q_1\omega_z \\ q_1\omega_y - q_2\omega_x \end{bmatrix}.$$

Multiplying by $\frac{1}{2}$, the differential equation becomes:

$$\dot{\mathbf{q}}(t) = \frac{1}{2} \begin{bmatrix} -(q_1\omega_x + q_2\omega_y + q_3\omega_z) \\ q_0\omega_x + q_2\omega_z - q_3\omega_y \\ q_0\omega_y + q_3\omega_x - q_1\omega_z \\ q_0\omega_z + q_1\omega_y - q_2\omega_x \end{bmatrix}.$$

The factor $\frac{1}{2}$ accounts for the double-cover property of quaternions, where a 3D rotation by angle θ corresponds to a quaternion rotation by $\theta/2$. This equation models the instantaneous effect of the angular velocity on the quaternion, ensuring that the quaternion remains on $\mathbb{S}^3$ (since $\dot{\mathbf{q}}(t)$ is orthogonal to $\mathbf{q}(t)$, preserving the norm $\mathbf{q}^T\mathbf{q} = 1$).

The derivation follows from the incremental rotation over a small time dt. A small rotation by angle $\|\omega(t)\|dt$ around axis $\omega(t)/\|\omega(t)\|$ is represented by the quaternion:

$$\mathbf{q}_{\text{rot}} \approx \begin{bmatrix} \cos\left(\frac{\|\omega(t)\|dt}{2}\right) \\ \sin\left(\frac{\|\omega(t)\|dt}{2}\right) \frac{\omega(t)}{\|\omega(t)\|} \end{bmatrix} \approx \begin{bmatrix} 1 \\ \frac{1}{2}\omega(t)dt \end{bmatrix}.$$

The updated quaternion is $\mathbf{q}(t + dt) = \mathbf{q}(t) \otimes \mathbf{q}_{\text{rot}}$. Taking the limit as $dt \to 0$, the derivative is:

$$\dot{\mathbf{q}}(t) = \lim_{dt \to 0} \frac{\mathbf{q}(t) \otimes \begin{bmatrix} 1 \\ \frac{1}{2}\omega(t)dt \end{bmatrix} - \mathbf{q}(t)}{dt} = \frac{1}{2}\mathbf{q}(t) \otimes \begin{bmatrix} 0 \\ \omega(t) \end{bmatrix}.$$

In the UKF, this equation forms the basis for the state transition model, describing how orientation evolves continuously.

Discrete-Time Kinematics (Equation 5) Equation 5, $\mathbf{q}_{k+1} = \exp\left(\frac{\Delta t}{2}\Omega(\omega_k)\right)\mathbf{q}_k$, provides the discrete-time update rule for the quaternion over a sampling period Δt. The angular velocity ω_k

is assumed constant over $[t_k, t_k + \Delta t]$. The matrix exponential $\exp\left(\frac{\Delta t}{2}\Omega(\omega_k)\right)$ is a 4×4 matrix that maps $\mathbf{q}_k$ to $\mathbf{q}_{k+1}$, representing the rotation induced by ω_k.

The matrix $\Omega(\omega_k)$ (defined in Equation 6) linearizes the quaternion kinematics. The term $\frac{\Delta t}{2}\Omega(\omega_k)$ scales the rotation by the time step and the double-cover factor. The exponential $\exp\left(\frac{\Delta t}{2}\Omega(\omega_k)\right)$ approximates the quaternion rotation:

$$\mathbf{q}_{\text{rot}} = \begin{bmatrix} \cos\left(\frac{\|\omega_k\|\Delta t}{2}\right) \\ \sin\left(\frac{\|\omega_k\|\Delta t}{2}\right)\frac{\omega_k}{\|\omega_k\|} \end{bmatrix},$$

such that $\mathbf{q}_{k+1} = \mathbf{q}_k \otimes \mathbf{q}_{\text{rot}}$. The matrix form is used for numerical stability and compatibility with the UKF's state propagation. The derivation assumes that $\omega(t)$ is constant over Δt. Solving the differential equation $\dot{\mathbf{q}}(t) = \frac{1}{2}\Omega(\omega_k)\mathbf{q}(t)$ (an equivalent matrix form of Equation 4), the solution is:

$$\mathbf{q}(t_k + \Delta t) = \exp\left(\frac{\Delta t}{2}\Omega(\omega_k)\right)\mathbf{q}(t_k).$$

This equation is critical in the UKF's time update step, where the state (including $\mathbf{q}_k$) is propagated forward.

Skew-Symmetric Matrix (Equation 6) Equation 6 defines the skew-symmetric matrix $\lfloor\omega\rfloor_\times$, used in $\Omega(\omega) = \begin{bmatrix} 0 & -\omega^T \\ \omega & -\lfloor\omega\rfloor_\times \end{bmatrix}$. For $\omega = [\omega_x, \omega_y, \omega_z]^T$, we have:

$$\lfloor\omega\rfloor_\times = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}.$$

This matrix represents the cross-product operation, where $\lfloor\omega\rfloor_\times\mathbf{v} = \omega \times \mathbf{v}$. In $\Omega(\omega)$, it captures the vector part of the quaternion product, while $-\omega^T$ and ω handle the scalar part. The matrix $\Omega(\omega)$ enables a linear representation of the nonlinear quaternion kinematics, facilitating the matrix exponential in Equation 5. The matrix $\Omega(\omega)$ is derived from the quaternion product in Equation 4. Writing $\dot{\mathbf{q}}(t) = \frac{1}{2}\mathbf{q}(t) \otimes \begin{bmatrix} 0 \\ \omega(t) \end{bmatrix}$ in matrix form, we get:

$$\dot{\mathbf{q}}(t) = \frac{1}{2}\Omega(\omega(t))\mathbf{q}(t),$$

where $\Omega(\omega(t))$ is constructed to match the quaternion product's components. This form is essential for the discrete-time update and UKF sigma point propagation.

3 Quaternion-Based Unscented Kalman Filter

The UKF estimates the state vector $\mathbf{x}_k = [\mathbf{q}_k^T, \mathbf{b}_k^T]^T$, where $\mathbf{q}_k \in \mathbb{S}^3$ is the orientation quaternion and $\mathbf{b}_k \in \mathbb{R}^3$ is the gyroscope bias. The measurement vector $\mathbf{y}_k \in \mathbb{R}^6$ includes accelerometer and magnetometer readings, modeled to provide orientation information.

3.1 State and Measurement Models

The discrete-time state transition model is:

$$\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \omega_k, \mathbf{w}_k), \tag{7}$$

where $\mathbf{w}_k \sim \mathcal{N}(0, \mathbf{Q}_k)$ is the process noise, and the state $\mathbf{x}_k$ consists of the orientation quaternion $\mathbf{q}_k$ and gyroscope bias $\mathbf{b}_k$:

$$\mathbf{q}_{k+1} = \exp\left(\frac{\Delta t}{2}\Omega(\omega_k - \mathbf{b}_k)\right)\mathbf{q}_k, \tag{8}$$

$$\mathbf{b}_{k+1} = \mathbf{b}_k + \mathbf{w}_{b,k}. \tag{9}$$

> **Note:** The measured angular velocity ω_k from the gyroscope contains both the true angular velocity and a slowly-varying bias, i.e., $\omega_k = \omega_k^{\text{true}} + \mathbf{b}_k$. In order to propagate the quaternion based on the true physical rotation of the system, we must subtract the estimated bias $\mathbf{b}_k$ from the raw measurement. Failing to correct for this bias would result in accumulated orientation error over time, known as drift. The subtraction $\omega_k - \mathbf{b}_k$ ensures that the exponential map used to propagate the quaternion integrates the true angular velocity, not a biased one.

The measurement model is:

$$\mathbf{y}_k = \mathbf{h}(\mathbf{x}_k, \mathbf{v}_k), \tag{10}$$

where $\mathbf{v}_k \sim \mathcal{N}(0, \mathbf{R}_k)$ is the measurement noise, and:

$$\mathbf{h}(\mathbf{x}_k) = \begin{bmatrix} \mathbf{q}_k \otimes \begin{bmatrix} 0 \\ \mathbf{g} \end{bmatrix} \otimes \mathbf{q}_k^* \\ \mathbf{q}_k \otimes \begin{bmatrix} 0 \\ \mathbf{m} \end{bmatrix} \otimes \mathbf{q}_k^* \end{bmatrix}, \tag{11}$$

with $\mathbf{g}$ and $\mathbf{m}$ representing the known gravity and magnetic field vectors in the inertial frame. These measurements provide absolute orientation cues and are rotated into the body frame using quaternion rotation.

3.2 Sigma Point Generation

For a state vector of dimension $n = 7$ comprising a 4-dimensional unit quaternion $\mathbf{q}_k$ and a 3-dimensional gyroscope bias vector $\mathbf{b}_k$ the Unscented Kalman Filter (UKF) generates $2n+1 = 15$ sigma points to approximate the mean and covariance of the nonlinear system. To ensure the quaternion remains on the unit sphere $\mathbb{S}^3$, a *multiplicative error model* is used. Instead of adding small perturbations to the quaternion, these perturbations are mapped through the Lie algebra $\mathfrak{so}(3)$ to $\mathbb{S}^3$ using the exponential map, and then multiplied with the current quaternion. This maintains the unit norm constraint of the quaternion throughout the UKF prediction and update steps.

The state covariance matrix $\mathbf{P}_k \in \mathbb{R}^{7\times7}$ is decomposed using the Cholesky decomposition:

$$\mathbf{P}_k = \mathbf{S}_k\mathbf{S}_k^\top, \tag{12}$$

where $\mathbf{S}_k$ is a lower triangular matrix. Each column of $\mathbf{S}_k$ represents a direction and scale of uncertainty in the state space. Let $\mathbf{e}_i \in \mathbb{R}^7$ be the i-th standard basis (unit) vector. Then $\mathbf{S}_k\mathbf{e}_i$ is the i-th scaled perturbation vector in the state space.

The sigma points are generated as follows:

$$\mathcal{X}_{k,0} = \hat{\mathbf{x}}_k, \quad \text{(nominal state)} \tag{13}$$

$$\mathcal{X}_{k,i} = \hat{\mathbf{x}}_k + \begin{bmatrix} \exp\left(\frac{1}{2}\Omega(\delta\boldsymbol{\phi}_i)\right)\hat{\mathbf{q}}_k \\ (\delta\mathbf{b}_i) \end{bmatrix}, \quad i = 1, \ldots, n, \tag{14}$$

$$\mathcal{X}_{k,i+n} = \hat{\mathbf{x}}_k - \begin{bmatrix} \exp\left(\frac{1}{2}\Omega(\delta\boldsymbol{\phi}_i)\right)\hat{\mathbf{q}}_k \\ (\delta\mathbf{b}_i) \end{bmatrix}, \quad i = 1, \ldots, n, \tag{15}$$

where:

- $\delta\mathbf{x}_i = \mathbf{S}_k\mathbf{e}_i$ is the i-th perturbation direction,
- $\delta\boldsymbol{\phi}_i = (\delta\mathbf{x}_i)_{1:3}$ is the small rotation vector used to perturb the quaternion (from Lie algebra),
- $\delta\mathbf{b}_i = (\delta\mathbf{x}_i)_{4:6}$ is the perturbation for the gyroscope bias,
- $\exp\left(\frac{1}{2}\Omega(\cdot)\right)$ maps a small rotation vector into a unit quaternion using the quaternion exponential map,
- $\Omega(\delta\boldsymbol{\phi})$ is the quaternion skew-symmetric matrix form of the rotation vector.

This construction ensures that the quaternion part of the state is perturbed via a valid rotation on the manifold $\mathbb{S}^3$, while the bias is updated via standard vector addition in $\mathbb{R}^3$.

3.3 Time Update

Each sigma point $\mathcal{X}_{k,i}$ is propagated through the nonlinear state transition function using the current angular velocity ω_k:

$$\mathcal{X}_{k+1|k,i} = \mathbf{f}(\mathcal{X}_{k,i}, \omega_k, 0). \tag{16}$$

This function applies Equation (8) to propagate the orientation quaternion using the corrected angular velocity $\omega_k - \mathbf{b}_k$, while keeping the gyroscope bias constant in the prediction (process noise is temporarily ignored during propagation).

Quaternion Mean on $\mathbb{S}^3$ Since quaternions lie on the 3-sphere $\mathbb{S}^3$, their mean cannot be computed by regular vector averaging. Instead, the predicted quaternion mean $\hat{\mathbf{q}}_{k+1|k}$ is computed using a *barycentric (or intrinsic) mean*:

$$\hat{\mathbf{q}}_{k+1|k} = \arg\min_{\mathbf{q}\in\mathbb{S}^3} \sum_{i=0}^{2n} w_i \|\log(\mathbf{q}^* \otimes \mathcal{X}_{k+1|k,i,\mathbf{q}})\|^2, \tag{17}$$

where:

- $\mathcal{X}_{k+1|k,i,\mathbf{q}}$ is the quaternion component of the i-th predicted sigma point,
- $\mathbf{q}^*$ is the quaternion conjugate of the mean candidate,
- $\log(\cdot)$ maps a quaternion perturbation from the manifold $\mathbb{S}^3$ to the tangent space (Lie algebra),
- w_i are the UKF weights, e.g., $w_0 = \frac{\kappa}{n+\kappa}$ and $w_i = \frac{1}{2(n+\kappa)}$ for $i = 1, \ldots, 2n$.

This optimization finds the quaternion that is "closest on average" (in a geodesic sense) to the propagated quaternion sigma points.

Bias Prediction The gyroscope bias, being a vector in $\mathbb{R}^3$, can be averaged linearly:

$$\hat{\mathbf{b}}_{k+1|k} = \sum_{i=0}^{2n} w_i \, \mathcal{X}_{k+1|k,i,\mathbf{b}}, \tag{18}$$

where $\mathcal{X}_{k+1|k,i,\mathbf{b}}$ is the bias component of the i-th predicted sigma point.

Predicted Covariance The predicted state covariance $\mathbf{P}_{k+1|k}$ is computed as the weighted spread of the predicted sigma points around the mean:

$$\mathbf{P}_{k+1|k} = \sum_{i=0}^{2n} w_i (\mathcal{X}_{k+1|k,i} - \hat{\mathbf{x}}_{k+1|k})(\mathcal{X}_{k+1|k,i} - \hat{\mathbf{x}}_{k+1|k})^\top + \mathbf{Q}_k. \tag{19}$$

Here:

- $\hat{\mathbf{x}}_{k+1|k}$ is the predicted state composed of the quaternion mean $\hat{\mathbf{q}}_{k+1|k}$ and the bias mean $\hat{\mathbf{b}}_{k+1|k}$,
- $\mathbf{Q}_k$ is the process noise covariance,
- The quaternion difference is handled using the log map in practice, to ensure proper error representation on $\mathbb{S}^3$.

This step ensures the updated uncertainty accounts for the nonlinearity of the system as captured by the spread of the sigma points.

3.4 Measurement Update

Each predicted sigma point $\mathcal{X}_{k+1|k,i}$ is passed through the nonlinear measurement model $\mathbf{h}(\cdot)$ to generate the corresponding predicted measurement:

$$\mathcal{Y}_{k+1|k,i} = \mathbf{h}(\mathcal{X}_{k+1|k,i}, 0). \tag{20}$$

This transforms each sigma point from state space to measurement space, using the nominal (zero) measurement noise during propagation. The measurement function typically rotates the known gravity and magnetic field vectors from the inertial frame to the body frame using quaternion rotation (see Equation (11)).

Predicted Measurement Mean and Covariance The predicted measurement mean is computed as a weighted sum:

$$\hat{\mathbf{y}}_{k+1|k} = \sum_{i=0}^{2n} w_i \mathcal{Y}_{k+1|k,i}. \tag{21}$$

This gives the expected sensor reading given the predicted state distribution.

The measurement covariance captures the expected spread of measurements around the mean:

$$\mathbf{P}_{yy,k+1} = \sum_{i=0}^{2n} w_i (\mathcal{Y}_{k+1|k,i} - \hat{\mathbf{y}}_{k+1|k})(\mathcal{Y}_{k+1|k,i} - \hat{\mathbf{y}}_{k+1|k})^\top + \mathbf{R}_k, \tag{22}$$

where $\mathbf{R}_k$ is the measurement noise covariance matrix. This accounts for both the uncertainty from the state distribution and the sensor noise.

Cross-Covariance and Kalman Gain The cross-covariance between the state and measurement is:

$$\mathbf{P}_{xy,k+1} = \sum_{i=0}^{2n} w_i (\mathcal{X}_{k+1|k,i} - \hat{\mathbf{x}}_{k+1|k})(\mathcal{Y}_{k+1|k,i} - \hat{\mathbf{y}}_{k+1|k})^\top. \tag{23}$$

This captures how changes in the state affect the measurements. Using this, the Kalman gain is computed as:

$$\mathbf{K}_{k+1} = \mathbf{P}_{xy,k+1}\mathbf{P}_{yy,k+1}^{-1}. \tag{24}$$

This gain tells us how much to trust the measurement update relative to the prediction.

State Correction To update the orientation quaternion, we apply a multiplicative correction:

$$\hat{\mathbf{q}}_{k+1} = \exp\left(\frac{1}{2}\Omega(\mathbf{K}_{k+1}(\mathbf{y}_{k+1} - \hat{\mathbf{y}}_{k+1|k}))\right)\hat{\mathbf{q}}_{k+1|k}. \tag{25}$$

This ensures that the quaternion stays on the manifold $\mathbb{S}^3$. The innovation $\mathbf{y}_{k+1} - \hat{\mathbf{y}}_{k+1|k}$ is mapped into the Lie algebra using $\Omega(\cdot)$ and exponentiated into a unit quaternion for correction. The bias update is additive, since it lies in Euclidean space:

$$\hat{\mathbf{b}}_{k+1} = \hat{\mathbf{b}}_{k+1|k} + \mathbf{K}_{k+1}(\mathbf{y}_{k+1} - \hat{\mathbf{y}}_{k+1|k}). \tag{26}$$

Covariance Update Finally, the updated state covariance is given by:

$$\mathbf{P}_{k+1} = \mathbf{P}_{k+1|k} - \mathbf{K}_{k+1}\mathbf{P}_{yy,k+1}\mathbf{K}_{k+1}^\top. \tag{27}$$

This reduces uncertainty in directions where the measurements provide useful information.

4 Theoretical Considerations

To ensure the mathematical integrity of the solution, we address:

- Quaternion normalization: After each update, normalize $\hat{\mathbf{q}}_{k+1} \leftarrow \hat{\mathbf{q}}_{k+1}/\|\hat{\mathbf{q}}_{k+1}\|$ to enforce the unit norm constraint.
- Numerical stability: Use a high-precision matrix square root (e.g., Cholesky decomposition) for $\mathbf{S}_k$ to generate sigma points accurately.
- Noise covariance tuning: The process noise covariance $\mathbf{Q}_k$ and measurement noise covariance $\mathbf{R}_k$ must be carefully designed to reflect sensor characteristics, ensuring robust estimation.

The algorithm is summarized in Algorithm 1.

5 Conclusion

This paper developed a mathematical framework for a Quaternion-based Unscented Kalman Filter for orientation estimation. By leveraging unit quaternions to represent attitude and the UKF to handle nonlinear dynamics, the proposed solution provides a theoretically sound approach to attitude estimation. The derivations ensure that the quaternion's unit norm is preserved and that the non-Euclidean nature of $\mathbb{S}^3$ is respected. [3, 4].

Algorithm 1 Quaternion-Based Unscented Kalman Filter

Initialize $\hat{\mathbf{x}}_0$, $\mathbf{P}_0$, $\mathbf{Q}_k$, $\mathbf{R}_k$, and UKF parameters (w_i, κ).
for each time step k **do**
    Generate sigma points $\mathcal{X}_{k,i}$ using multiplicative quaternion errors.
    Propagate sigma points: $\mathcal{X}_{k+1|k,i} = \mathbf{f}(\mathcal{X}_{k,i}, \omega_k, 0)$.
    Compute predicted mean $\hat{\mathbf{x}}_{k+1|k}$ and covariance $\mathbf{P}_{k+1|k}$.
    Transform sigma points through measurement model: $\mathcal{Y}_{k+1|k,i}$.
    Compute measurement mean $\hat{\mathbf{y}}_{k+1|k}$, covariance $\mathbf{P}_{yy,k+1}$, and cross-covariance $\mathbf{P}_{xy,k+1}$.
    Calculate Kalman gain $\mathbf{K}_{k+1}$.
    Update state $\hat{\mathbf{x}}_{k+1}$ and covariance $\mathbf{P}_{k+1}$.
    Normalize quaternion: $\hat{\mathbf{q}}_{k+1} \leftarrow \hat{\mathbf{q}}_{k+1}/\|\hat{\mathbf{q}}_{k+1}\|$.
end for

References

[1] E. Kraft, "A quaternion-based unscented kalman filter for orientation tracking," *Proceedings of the 6th International Conference on Information Fusion*, pp. 900–905, 2003.

[2] E. A. Wan and R. Van Der Merwe, "The unscented kalman filter for nonlinear estimation," *Proceedings of the IEEE Adaptive Systems for Signal Processing, Communications, and Control Symposium*, pp. 153–158, 2000.

[3] A. C. B. Chiella, B. O. Teixeira, and G. A. S. Pereira, "Quaternion-based robust attitude estimation using an adaptive unscented kalman filter," *Sensors*, vol. 19, no. 10, p. 2372, 2019.

[4] K. Ghanizadegan and H. A. Hashim, "Quaternion-based unscented kalman filter for 6-dof vision-based inertial navigation in gps-denied regions," *arXiv preprint arXiv:2412.02768*, 2024.