Illuminating Inference:

A Comparative Study of Photon Flux Analysis --Bayesian and Frequentist Approaches

MSc Data Science

Thompson Rivers University

Professor: Dr. Jabed Tomal

Prepared by Ayisha C O K

Abstract

This Study conducts a detailed comparative analysis of Frequentist and Bayesian statistical methodologies applied to photon flux estimation, concentrating on scenarios with both single and variable photon counts (NASA, n.d.). Firstly, the investigation discerns negligible disparity between Frequentist and Bayesian methods in the context of single photon analysis, attesting to the solidity and dependability of both methods under stable photon count conditions. Secondly, in the examination of variable photon scenarios, traditional Frequentist methods display deficiencies, primarily when addressing the complexity of correlated parameters, thus underscoring the necessity for more nuanced techniques or assumptions to effectively manage the intricacies presented by variable photon data. Lastly, the Bayesian approach is shown to be particularly efficacious in scenarios with low photon counts (single photons), providing more precise estimations. The research evaluates this effectiveness via the implementation of various Bayesian priors—flat, informative, and Jeffreys' prior. The strategic selection and updating of these priors result in enhanced accuracy of estimations, thus highlighting the flexibility and proficiency of Bayesian methods in such specialized circumstances. This study contributes to the nuanced understanding of statistical applications in quantum mechanics and serves as a foundation for methodological advancements in photon flux estimation.

Keywords: Frequentist; confidence interval; Bayesian; credible interval; prior; posterior; Photons; Photon flux

1. Introduction:

Statistics and physics (Prosper, 2016), while both grounded in solid principles, approach theoretical disputes differently. Physics resolves conflicts based on empirical evidence from nature, whereas statistics relies on a collective discourse within its community to affirm or refute theories. This distinction underscores the unique challenges and methods each discipline employs, especially when addressing complex phenomena.

The exploration of statistical methodologies in photon analysis is a crucial and forward-thinking research area, especially due to the intricate behaviors of photons under varying conditions. Existing studies within the field of physics have delineated both Bayesian and frequentist approaches, especially as they pertain to statistical analysis and hypothesis testing in complex systems as applied in the Higgs boson (Prosper, 2016). Such comparative studies shed light on the nuances of statistical inference as it applies to quantum mechanics, where photon behavior exhibits considerable variability.

2. Objectives

The primary objective of this project is to conduct a comprehensive comparative analysis of Frequentist and Bayesian methodologies for estimating photon flux. This analysis will cover a range of scenarios, from straightforward single-photon instances to more complex cases involving fluctuating photon counts, with a focus on how each approach uniquely manages parameter variability. The secondary goal is to improve the accuracy of photon flux estimation in low-photon-count situations, often encountered in single-photon detection. This will involve the development and evaluation of various Bayesian models, integrating prior knowledge for a more precise estimation of average photon flux. Additionally, this phase of the project will also incorporate Frequentist techniques, aiming to enhance the precision of photon flux measurements under these specific conditions. The overarching aim is to blend the strengths of both Bayesian and Frequentist approaches to achieve a more accurate and robust understanding of photon flux dynamics across different photon detection scenarios using the simulation technique Markov Chain Monte Carlo (MCMC) Metropolis-Hastings.

3. Methodology

The methodology revolves around a comprehensive comparison between two fundamental statistical paradigms: frequentist and Bayesian methodologies. In particular, the study will delve into the intricacies of the Markov Chain Monte Carlo (MCMC) Metropolis-Hastings technique, a pivotal Bayesian computational approach.

3.1. Single Photons and varying photons

Photons, the elementary particles of light, play a pivotal role in the tapestry of physical phenomena, spanning macroscopic astrophysical scales to the quantum intricacies of subatomic particles (Lista, 2016). As quantized packets of electromagnetic radiation, photons exhibit both particle and wave-like properties, underpinning the foundations of quantum mechanics. This project covers photon flux estimation in the case of single and variable photons.

3.1.1. Single Photons

In quantum optics and photonics, single photons play a vital role due to their distinct quantum characteristics. These photons are known for their emission or detection within specific time intervals, which is crucial in experiments involving low levels of light. In practical terms, working with single photons entails generating, manipulating, and measuring extremely weak light to enable the precise counting of individual photons. This capability is fundamental for the progress of quantum information processing, quantum communication, and other fields reliant on the quantum properties of light.

In single-photon experiments, crucial for quantum optics and information processing, the generation and analysis of photon count data are key. The foundation of analyzing these photon events lies in employing the Poisson distribution, a statistical model that effectively predicts the likelihood of observing a certain number of photon occurrences within a specific time or spatial interval. This distribution is particularly useful for modeling scenarios with a constant mean rate of photon events. The probability mass function (PMF) of the Poisson distribution is given by:

$$P(X = f) = \frac{e^{-\mu}\mu^{f}}{f!} \tag{1}$$

Where:
- X is the random variable representing the number of events (photons in this case),
- f is a non-negative integer representing the observed number of photons,
- μ is the average rate of events (mean of the distribution)

This Poissonian model captures the intrinsic randomness inherent in photon counting, a crucial aspect of precise and accurate quantum measurements. However, when delving into more comprehensive analyses, particularly with extensive data sets, the transition to a Gaussian error assumption becomes essential. The rationale behind this shift is that the Poisson distribution approaches a normal (Gaussian) distribution as the mean value increases. Gaussian distributions are preferred in such scenarios due to their mathematical simplicity and the extensive understanding of their properties in statistical analysis. The probability distribution (Frequentism and Bayesianism: A Practical Introduction, n.d.) of measurements under the Gaussian error assumption is defined as:

$$P(D_i|F_{\text{true}}) = \frac{1}{\sqrt{2\pi e_i^2}} e^{-\frac{(F_i - F_{\text{true}})^2}{2e_i^2}} \tag{2}$$

In practical terms, this framework is utilized to generate photon count datasets. For example, a simulation might involve 50 photon count measurements with an assumed "true" photon flux of 1000. This chosen value is arbitrary but serves as a foundational assumption for the simulation, replicating real experimental conditions. Within an experimental context, each measurement is represented as $D_i=(F_i, e_i)$

F_i= is the i^{th} photon flux, and e_i its corresponding error. These measurements enable the estimation of the true photon flux. Visualizing this distribution, along with the presumed true value, elucidates the pattern of photon occurrences and assists in comprehending the complexities of photon behavior in quantum experiments.

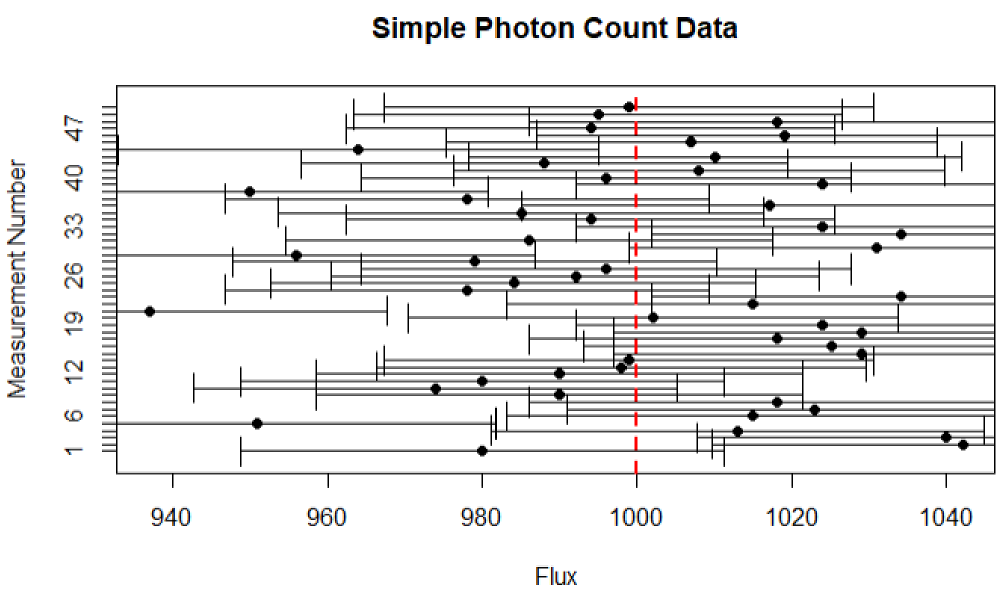

Figure 1: Single photons count

3.1.2. Varying Photons

Varying photons (Lyons, 2013; Prosper, 2016) refers to a scenario in which the number of detected photons fluctuates across different measurements or observations. In other words, the observed photon count is not constant but varies, and these variations can occur over time, introducing a dynamic element into the experimental data. Researchers often encounter varying photons in classical optics experiments and must consider this variability when interpreting results and conducting statistical analyses. Each measurement captures a different quantity of photons, introducing variability in the observed data. The distribution utilized for simulating varying photons involves two key components:

Stochastic True Flux Model:

This model employs a normal distribution to represent the true flux, which signifies the actual signal or phenomenon being observed. The parameters of this normal distribution are:

Mean (μ_{true}): 1000 and Standard Deviation (σ_{true}): 15 (moderate level of variability).

These parameters introduce stochastic variation in the true flux, reflecting the natural variability often observed in real-world phenomena (Frequentism and Bayesianism: A Practical Introduction, n.d.).

The true flux model is defined as: $F_{\text{true}} \sim N(\mu_{\text{true}}, \sigma_{\text{true}})$

$$f(x; \mu_{\text{true}}, \sigma_{\text{true}}) = \frac{1}{\sigma_{\text{true}}\sqrt{2\pi}} \exp\left(-\frac{(x-\mu_{\text{true}})^2}{2\sigma_{\text{true}}^2}\right) \tag{3}$$

Poisson Noise for Observed Flux:

To simulate the process of photon detection and counting, Poisson noise is added to the true flux. Each observation (F) is generated by adding Poisson-distributed noise to the corresponding true flux value.

$$F \sim \text{Poisson}(\lambda = F_{\text{true}})$$

$$P(X = f; \lambda) = \frac{e^{-\lambda}\lambda^f}{f!} \tag{4}$$

F represents the concept of observed photon counts as a variable, while f is used to refer to a specific number of photons within the context of that variable. This model accounts for the inherent variability in photon detection, aligning with the discrete and random nature of photon behavior as observed in practical scenarios. The combination of these components results in a dataset (F) that simulates varying photon counts across different measurements.

In the context of varying photons, the project explores the challenges of parameter estimation when dealing with objects exhibiting stochastic variation, such as quasars. The complex relationship between the mean and standard deviation in the Gaussian model necessitates sophisticated statistical methods, such as numerical optimization, for accurate parameter estimation. The study utilizes a dataset comprising 50 photon count measurements, assuming a true flux of 1000 photons. This dataset helps visualize the fluctuations in photon counts around the assumed true flux, thereby underscoring the nuanced nature of parameter estimation in scenarios characterized by varying photon counts. This approach not only enhances the understanding of varying photon counts but also highlights the need for advanced statistical techniques in interpreting complex and dynamic data in quantum optics and other photonics related fields

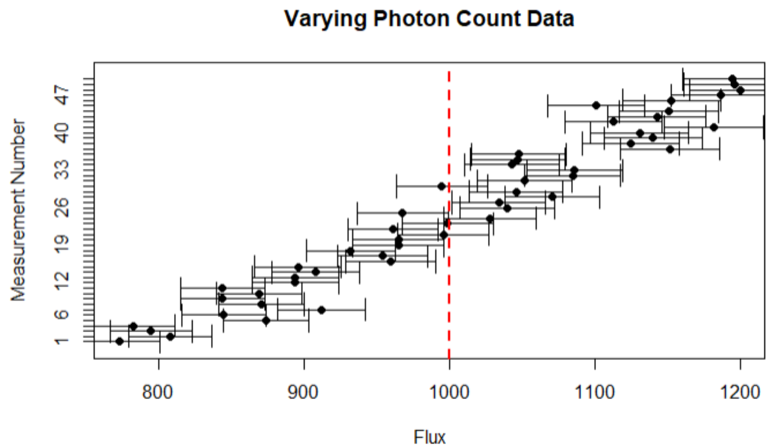

Figure 2: Varying photon count

This visualization helps interpret the varying photon count data, showing how the measurements fluctuate around the true flux value.

3.2. Frequentist Approach

The foundation of frequentist statistics (Wagenmakers et al.) is the idea that long-term frequencies are related to probability. According to this approach, sampling error is the main source of uncertainty, and the dataset under analysis is seen as one of many possible datasets addressing the same research issue. The foundation of frequentist inference is the idea that **probability is a limiting frequency**, which suggests that it is acceptable to provide probabilities to recurring events when uncertainty results from randomness. Since uncertainty about a statistical process's parameters is regarded as epistemic, the frequentist viewpoint refrains from making probability statements about them. As per the frequentist paradigm, probabilities are expressed as frequencies, and a probability becomes meaningful only in the context of hypothetically repeated experiments. In the Frequentist approach, the likelihood function represents the probability of observing the given data (photon counts) for different values of the parameter (photon flux), assuming a fixed and known value for the true parameter. In simpler terms, it quantifies how likely the observed data are under different hypothetical values of the parameter. The likelihood function is denoted by $L(\theta|x)$, where L is the likelihood function,θ represents the parameter(s) of the statistical model,x is the observed data.

Single Photons	Varying Photons :
The likelihood of observing f events (photon counts) for a given mean rate Ftrue $$L\left({}^{D}/_{F_{true}}\right) = \prod_{i=}^{N} P\left({}^{Di}/_{F_{true}}\right)$$ $$\log L = -\frac{N}{2}\log(2\pi e_i^2) - \frac{(F_i - F_{\text{true}})^2}{2e_i^2}$$	The likelihood function for a set of observed photon fluxes {Fi} assuming a true underlying flux $$L(\mu_{\text{true}}, \sigma_{\text{true}} \vert \{F_i\}) = \prod_i \int f(F_i; F_{\text{true}}) \times P(F_i; F_{\text{true}})\, dF_{\text{true}}$$ $$\propto L\left({}^{D}/_{\mu, \sigma}\right) = \prod_{i=}^{N} \frac{1}{\sqrt{2\pi(\sigma^2+e_i{}^2)}} e^{-\frac{(F_i-\mu)^2}{2(\sigma^2+e_i{}^2)}}$$ (Frequentism and Bayesianism: A Practical Introduction, n.d.)
	F_i - represents each observed photon flux value in the dataset $f(F_i;$ Ftrue)- is the probability density function (pdf) reflecting the likelihood of observing $P(F_i;$ Ftrue)- is the probability distribution function for the observed flux.

The primary goal of the Frequentist approach to photon flux estimation includes Maximum Likelihood Estimation (MLE) and Frequentist statistics often use confidence intervals to quantify the uncertainty around point estimates.

Maximum Likelihood Estimator (MLE):

Maximum Likelihood Estimation (MLE) is a crucial statistical technique utilized in academic research to estimate the parameters of a statistical model rigorously. It relies on the likelihood function, which quantifies how effectively the model's parameters account for the observed data. The fundamental concept of MLE revolves around identifying parameter values that maximize the likelihood of observing the provided data.

- F_{true} value for single photons can be estimated by equating the log likelihood function to zero which gives the estimated flux to be $F_{\text{est}} = \frac{1}{N}\sum F_i$ and $MLE = \text{F}_{\text{est}}$(if all the errors are equal)
- For varying photons, Maximum Likelihood Estimators(MLE) are utilized to estimate the parameter. However, those estimators are not the true value since it does not allow the correlation between the parameters. In the frequentist approach, traditional techniques often rely on assumptions of independence between parameters. When dealing with correlated parameters, especially in scenarios with varying photons, the estimation process becomes intricate. The correlation introduces complexities that traditional frequentist methods may struggle to handle effectively.

3.3. Bayesian Approach

Contrary to the frequentist paradigm, Bayesian statisticians (Hoff, 2009) adopt a perspective where probabilities are construed as **degrees of belief**, representing a quantification of our certainty regarding a specific statement or hypothesis. In the Bayesian framework:

- Observed data are not inherently perceived as random variables; rather, they are regarded as fixed and actualized outcomes.
- Model parameters, crucial determinants of the underlying statistical model, are treated as uncertain quantities and, as such, are amenable to probabilistic characterization.

The Bayesian approach to photon flux estimation employs a series of methodologies centered on adjusting beliefs regarding parameters by integrating both prior information and observed data. Diverging from the Frequentist approach, Bayesian statistics encompass pre-existing knowledge about the parameter (photon flux) through a prior distribution. The likelihood function is employed in Bayesian statistics to revise the prior distribution based on observed data, a crucial step in deriving the posterior distribution. The application of Bayes' Theorem merges the prior distribution with the likelihood function, culminating in the posterior distribution, which signifies the current understanding of the parameter. Bayesian point estimates and credible intervals offer concise summaries derived from the posterior distribution.

3.3.1. **Bayes theorem**

Bayes' Theorem forms the foundation of Bayesian statistics and is expressed as follows:

$$P\left({}^{F_{true}}/_{D}\right) = \frac{P\left({}^{D}/_{F_{true}}\right)P(F_{true})}{P(D)} \tag{5}$$

where D is the given observation with error, (F_i, e_i)

Likelihood: The probability of observing the data given specific values of the parameters, $P\left({}^{D}/_{F_{true}}\right)$

Prior: The initial belief or probability distribution representing existing knowledge about the parameters, $P(F_{true})$.

Posterior: The updated probability of the parameters given the observed data, $P\left({}^{F_{true}}/_{D}\right)$

3.3.2. Bayesian Approach to Single (low-photon-count)

The Bayesian analysis involves combining the prior beliefs with observed data through Bayes' Theorem to obtain the posterior distribution, which provides an updated and refined understanding of the parameters especially when the sample size is small as in the case of single photons which produces low-photon-count.

3.3.3. The choice of prior

The initial step in Bayesian analysis (Hoff, 2009) involves the selection of prior probability distributions for the unknown model parameters. In instances where no prior information is available regarding the parameters of interest, "uninformative" prior probability distributions are commonly employed. While these priors are obligatory in the Bayesian approach, their impact on the data analysis tends to be minimal. However, there are situations where the preference is for informative prior distributions, ideally derived objectively through data analysis. When objective derivation is impractical, informative priors may be drawn from expert opinions or existing scientific literature.

- Non-Informative Prior: A prior that does not incorporate specific information, allowing the data to strongly influence the posterior. Jeffreys proposed the following prior distribution:

Jeffreys prior g(θ):

Rule A: If Ω = (-∞, ∞), take g(θ) to be a constant (flat prior) i.e., θ is assumed to be uniformly distributed.
Rule B: If Ω = (0, ∞), take g(θ) $\propto 1/\theta$ i.e., log(θ) is assumed to be uniformly distributed.
These rules define the Jeffreys prior for different parameter spaces (Ω) and provide a way to specify the prior distribution for θ based on uniformity assumptions (Jeffreys, 1998)
The prior g(θ) is proportional to the square root of the determinant of the Fisher Information matrix

$$g(\theta) \propto \sqrt{|\mathbf{I}(\theta)|} \tag{6}$$

This formulation of Jeffreys's prior is particularly useful in scenarios where the natural parameterization of the problem is not obvious or subjective priors are hard to justify

- Informative Prior: A prior that includes specific information or beliefs about the parameters, influencing the posterior based on existing knowledge. An informative prior is used when you have specific prior knowledge about the parameter of interest. This contrasts with non-informative or weakly informative priors, which represent a lack of prior knowledge or only vague ideas about the parameter values. Incorporating an informative prior in Bayesian estimation can increase the precision of posterior distributions. This is particularly relevant in photon counting, where the precision of parameter estimates (like the mean or variance of photon counts) can be crucial for accurate modeling and interpretation of experimental data.

The choice between non-informative and informative priors depends on the available information and the researcher's confidence in prior beliefs.

3.3.4. Posterior

The posterior distribution encapsulates comprehensive information derived from the Bayesian analysis concerning a specific parameter. From this posterior distribution, various summary statistics can be computed, including but not limited to the mean, median, mode, and standard error. These statistics offer valuable insights into the characteristics and properties of the parameter, enhancing the overall understanding gained from the Bayesian analysis.
Posterior ∝ Likelihood × Prior
P (parameter given data) ∝ P(data given parameter) x P (parameter) (7)
Model 1 - Uniform Prior:
The MCMC algorithm is run with a uniform prior, updating the belief about F based on the observed data.
Model 2 - Updated Prior:
The posterior distribution from Model 1 becomes the prior for Model 2. The MCMC algorithm is run again, refining the estimate of F using updated prior information.
Model 3 - Jeffreys Prior:

The Jeffreys prior is used in combination with the likelihood function. The MCMC algorithm is applied to obtain the posterior distribution.

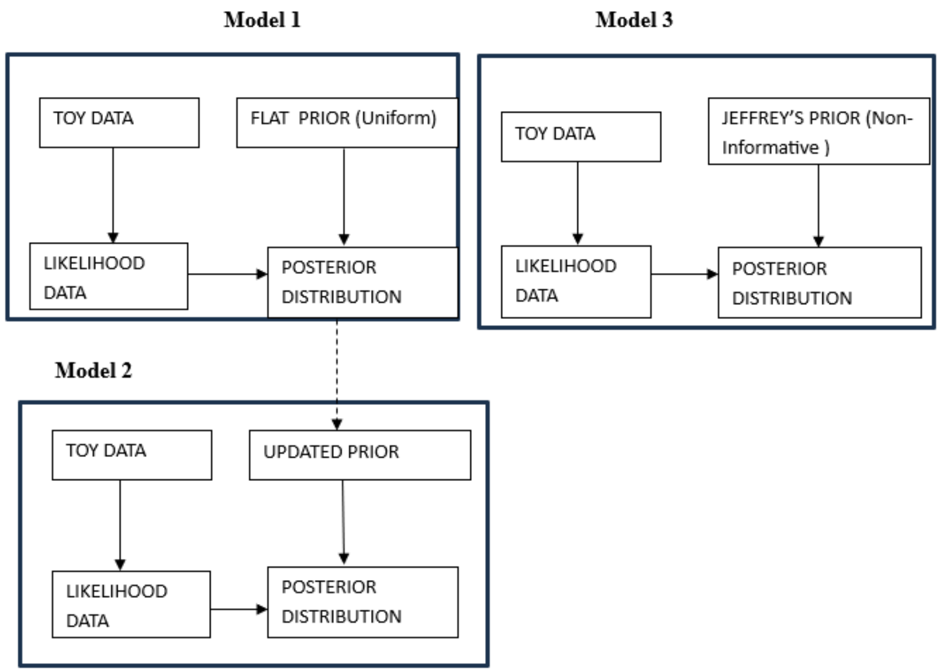

Figure 3: Models

3.3.5.Metropolis-Hastings Markov Chain Monte Carlo Algorithm

The Markov Chain Monte Carlo effectively combines the key principles of Monte Carlo methods and Markov chain theory. MCMC (Metropolis, et al., 1953) method, a cornerstone in the field of statistical physics, owes its inception to the seminal work of Metropolis and colleagues. Initially presented in the Journal of Chemical Physics, the Metropolis algorithm emerged from the collaborative efforts of a group of scientists, predominantly mathematical physicists, involved in significant projects including the atomic bomb. The primary aim of this algorithm was to compute high-dimensional integrals, a challenge deemed too complex for traditional numerical methods at that time. The development of this algorithm was intimately linked to the 'MANIAC,' the second computer created under Metropolis's guidance, who was proficient in both physics and mathematics. The Metropolis algorithm was later refined and expanded by Hastings, enhancing its utility for statistical simulations in high-dimensional spaces (Hastings, 1970).

A Metropolis-Hastings algorithm, as described by Metropolis et al. and Hastings, initiates with an initial value θ_t based on the previous value in the sequence, $\theta_{(t-1)}$ (Hastings, 1970; Metropolis, et al., 1953; Lin). This procedure involves a proposal density that generates a candidate value $\theta*$ and the computation of an acceptance probability P, which determines the likelihood of accepting the candidate value as the next value in the sequence.

Algorithm:

Step 1: Generate a candidate value $\theta*$ from a proposal density $p(\theta*|\theta_{(t-1)})$.

Step 2: Calculate the ratio $R = \frac{g(\theta^*)p(\theta_{t-1}|\theta^*)}{g(\theta_{t-1})p(\theta^*|\theta_{t-1})}$

Step 3: Compute the acceptance probability P = min(R, 1).
Step 4: Sample a value θ_t, where θ_t = θ∗ with probability P, or θ_t = θ_(t-1) otherwise.
Step 5: Repeat steps 1 to 4 until the desired number of samples is obtained.
When the proposal density is symmetric around the origin, the ratio simplifies to $R = \frac{g(\theta^*)}{g(\theta_{t-1})}$ Typically, a normal density is chosen as the proposal density. Then, the ratio of posterior densities is determined as $R = \frac{g(\theta^*)}{g(\theta_{t-1})}$ To decide whether to accept or reject the sampled observation θ*, generate an observation U from Uniform(0,1).
Step 6: If U < R, θ_t = θ∗; otherwise, θ_t = θ_(t-1).
The algorithm produces a Markov chain—a sequence of parameter values.
Step 7: Discard initial iterations (burn-in period) until the chain stabilizes.
Use the remaining samples to estimate posterior characteristics (means, standard deviations) and construct credible intervals.

3.3.6. Bayesian Approach to Varying Photons

On the other hand, Bayesian approaches, particularly those employing Markov Chain Monte Carlo (MCMC) methods, excel in handling correlated parameters. Bayesian methods naturally incorporate the uncertainty and correlation between parameters, allowing for a more flexible and accurate estimation process. In Bayesian analysis, the posterior distribution is explored through sampling methods like MCMC. This enables the exploration of the parameter space, considering dependencies and correlations between parameters.

A Bayesian model for varying photons is

Likelihood function :

$$L \quad \left({}^{D}/_{\mu,\sigma}\right) = \prod_{i=}^{N} \frac{1}{\sqrt{2\pi(\sigma^2+e_i{}^2)}} e^{-\frac{(F_i-\mu)^2}{2(\sigma^2+e_i{}^2)}} \qquad (8)$$

Prior: For σ --assumes a broad positive prior for sigma (standard deviation) using a uniform distribution

For μ -- assumes a broad prior for mu (mean) using a normal distribution

Posterior - The posterior function combines the likelihood and prior to calculate the posterior probability of the parameters given the observed data.

3.4. Confidence Intervals and Credible intervals

Confidence intervals and credible intervals serve the common purpose of estimating ranges for unknown parameters, yet they operate within distinct statistical frameworks. In frequentist statistics, a confidence interval is determined, offering a range of values for an unknown parameter, accompanied by a predetermined confidence level, derived from repeated sampling. In contrast, Bayesian statistics involve the calculation of a credible interval, which signifies a range of plausible values for an unknown parameter, supported by a specified level of belief, frequently influenced by prior knowledge and observed data.

4. Simulation Study

In this section, we undertake a Monte Carlo simulation (Witmer , 2017) study employing the Metropolis-Hastings algorithm to investigate and compare the performance of point estimators and confidence intervals. The Metropolis-Hastings algorithm stands as a preeminent Markov Chain Monte Carlo (MCMC) method, esteemed for several key attributes. Firstly, it proffers a pragmatic solution for the sampling of intricate probability distributions, rendering it indispensable within the realm of Bayesian inference, where analytical solutions frequently remain elusive. Secondly, the algorithm exhibits the capability to adapt its proposal distribution to the characteristics of the target distribution, thereby augmenting the efficiency of the sampling process. Consequently, the Metropolis-Hastings algorithm has acquired a pivotal role in the domain of scientific research, affording a robust and adaptable framework for the approximation of posterior distributions. Its utility extends to the execution of intricate simulations across diverse fields, encompassing domains such as physics and machine learning.

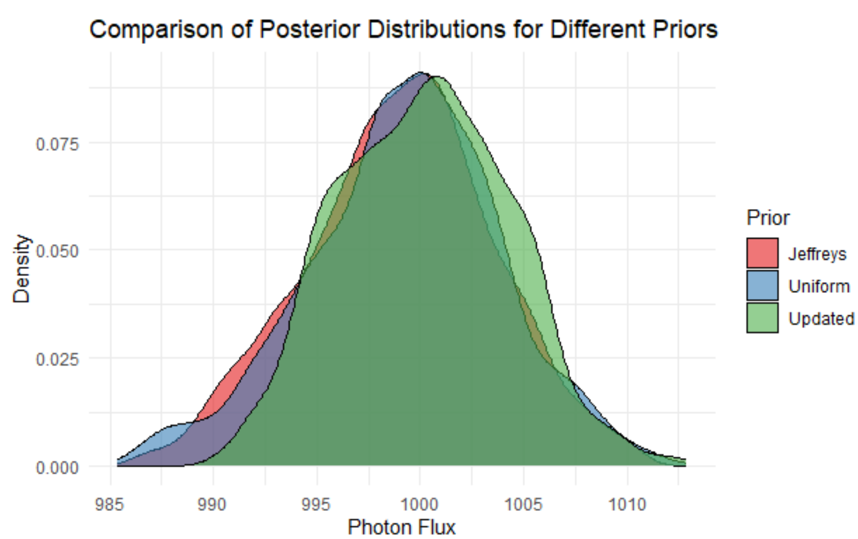

Figure 4:Comparison of Posterior Distributions

The central tendency of each curve in the graph signifies the photon flux estimate most likely according to its corresponding prior.A more pronounced peak denotes a stronger conviction in the associated photon flux value.The breadth of the curves reflects the level of certainty in the photon flux estimates; broader curves imply higher uncertainty, whereas slimmer ones signify tighter precision. Where these curves intersect, it indicates a consensus among the varying priors regarding the estimated value of the photon flux.

5. Results:

5.1. Comparison of Frequentist and Bayesian approach to single photons

<table>
<tr><td>Frequentist Approach</td><td>Bayesian Approach</td></tr>
<tr><td>

```
MLE Estimate of Photon Flux: 999.9982
Standard Error of MLE Estimate: 4.472132
95% Confidence Interval: 991.2328 1008.764
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Flat (Uniform) Prior Method:
Posterior Mean: 999.5077
Posterior Standard Deviation: 4.539805
95% Credible Interval: 989.541 1008.573
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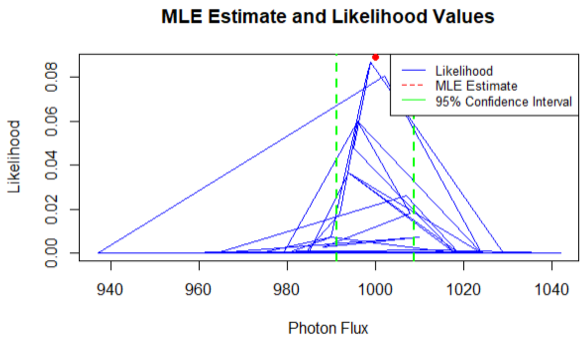

- The blue line shows the MLE estimate of Photon Flux, which is the value that gives the highest likelihood.
- The green dashed line shows the 95% confidence interval of Photon Flux, which is the range of values that contain the true value with 95% probability.
- The red dashed line shows the likelihood function, which is the probability of observing the data given a model parameter.
- The graph shows that the MLE estimate of Photon Flux is around 980, and the likelihood function has a peak at around 0.08. The confidence interval is widest at around 960 and narrows as Photon Flux increases. This means that there is more uncertainty about the true value of Photon Flux when it is close to the MLE estimate, and less uncertainty when it is far from it.

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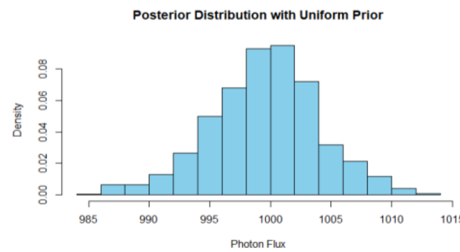
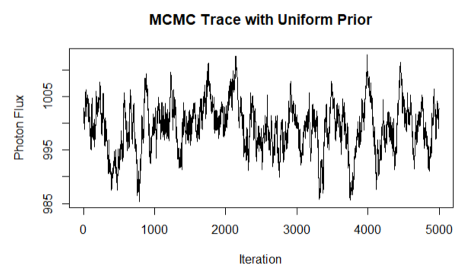


- The distribution reflects a uniform (flat) prior, indicating impartiality across the parameter space(Model 1). The posterior shape is primarily driven by the likelihood function, representing plausible parameter values based solely on observed data with no strong prior biases.



- A line graph that shows the trace of a Markov Chain Monte Carlo (MCMC) simulation with a uniform prior. The plot shows how the parameter estimate of the photon flux changes over the iterations of the MCMC algorithm.

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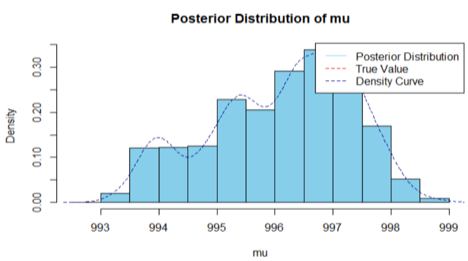
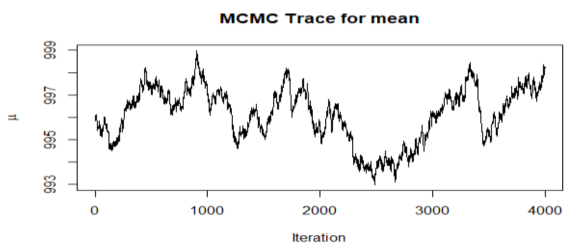
5.2. Comparison of Frequentist and Bayesian approach to varying photons

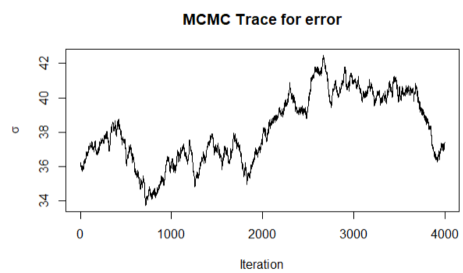
Frequentist Approach

In scenarios where photon counts vary and underlying parameters exhibit correlation, a frequentist approach must contend with the challenge of parameter non-identifiability and the potential for non-convexity in the likelihood function. In response, frequentist methodology typically extends beyond standard analytic solutions to encompass optimization techniques that are robust to such complexities. This might include the use of generalized estimating equations (GEEs), which can accommodate correlated response variables often encountered in repeated measures or cluster samples. Moreover, frequentist methods might implement simulation-based estimations like bootstrapping to derive confidence intervals that reflect the parameter correlations. However, these approaches are predicated on large sample assumptions that underpin the asymptotic properties of the estimators used, thus requiring careful consideration of sample size and model specification to ensure valid inference.

Bayesian Approach







- Histogram provides a visual representation of the posterior distribution of the parameter photon flux , allows to assess its shape, center, and uncertainty.
- MCMC trace for mean shows the values of the mean parameter at each iteration of the algorithm. The trace starts at around 993 and ends at around 997.The trace has a lot of fluctuations, but overall it seems to be converging towards a value of around 997.
- MCMC trace for error plot suggests that the algorithm has converged to a value of error(σ) around 38, which is the most likely value given the data and the model.
-

```
Posterior Mean: 996.1243 38.15956
Posterior Standard Deviation: 1.2198 2.027568
95% Credible Interval (Lower): 993.6948 34.4866
95% Credible Interval (Upper): 998.0399 41.54272
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5.3. Bayesian Approach to low-photon-count(single photons)

<table>
<tr><th>SINGLE PHOTONS</th><th>MODELS</th></tr>
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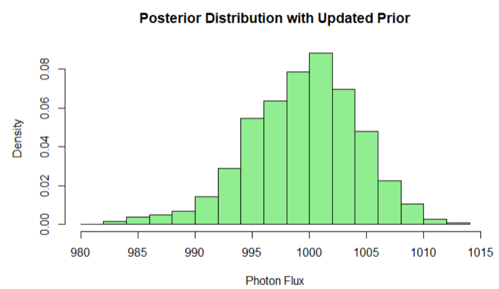

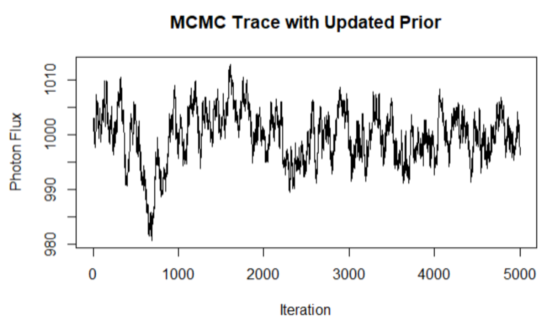

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- In the Model 2, the distribution is an iterative update, using the previous posterior as the current prior. This incorporates more data-derived information over successive analyses.
- The plot shows the photon flux (y-axis) as a function of the iteration number (x-axis). The photon flux appears to fluctuate around a mean value of approximately 1000.
-

```
Updating Prior Method:
Posterior Mean: 1000.375
Posterior Standard Deviation: 4.110877
95% Credible Interval: 992.9374 1008.206
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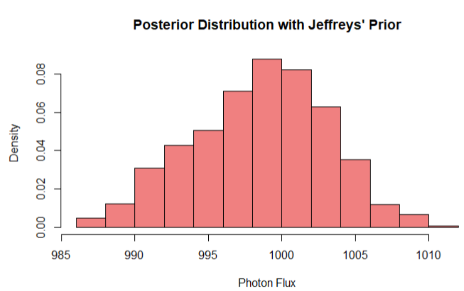

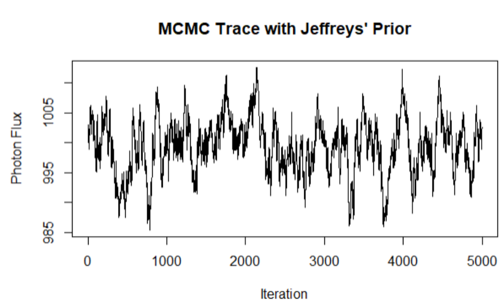

</td><td>

- In Model 3 distribution reflects Jeffreys' non-informative prior, proportional to the square root of Fisher information. This allows data to shape the posterior, emphasizing regions with more informative data.
- The trace plot looks like a random walk around a constant value, without any trends or patterns. This indicates that the MCMC algorithm has reached a stable state and the trace has become stationary.
-

```
Jeffreys' Prior Method:
Posterior Mean: 999.0303
Posterior Standard Deviation: 4.54447
95% Credible Interval: 989.954 1007.859
```

</td></tr>
</table>

6. Conclusion:

In summary, the challenges posed by correlated parameters in the frequentist approach underscore the limitations of traditional methods in capturing the nuances of complex systems. Bayesian approaches, with their flexibility and ability to account for dependencies, offer a more robust alternative in scenarios where traditional frequentist methods encounter difficulties. The Metropolis-Hastings algorithm is valuable for complex Bayesian analyses and provide a coherent framework for handling complex relationships within the data, making them particularly well-suited for scenarios with correlated parameters. The Updated prior appears to strike a balance between the Jeffreys and Uniform priors, offering a moderate level of confidence and a reasonable estimate of photon flux that takes into account new data. It doesn't lean too much towards the high certainty yet possibly biased estimate of the Jeffreys prior, nor does it spread too widely like the Uniform prior, which suggests a larger uncertainty. Thus, if we equate best with a compromise between certainty and adaptability to new information, the Updated prior could be considered the optimal choice among the three presented even by looking at the Confidence interval and estimated value even though the final decision always depends upon the situation

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R codes:

Github Link