Mathematical Programming

(Mixed-Integer) Linear Programming and Algorithms

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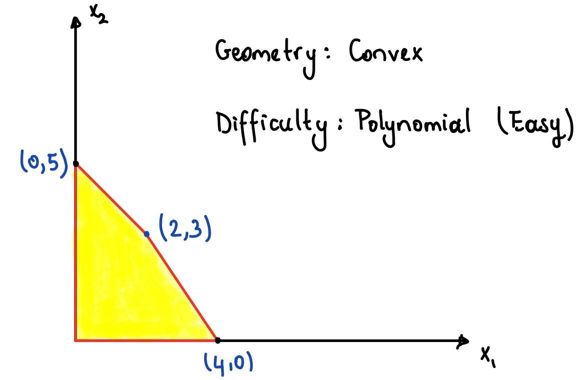
CR03 - Robust combinatorial optimization, ENS-Lyon

Little reminder

- Linear programming :

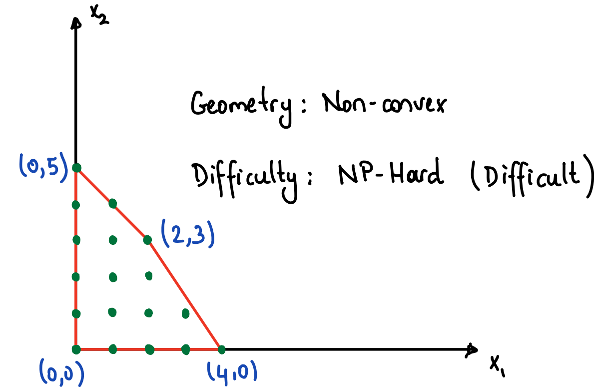
$$a_{11}x_1 + \cdots + a_{1n}x_n \geq b_1$$
$$\vdots$$
$$a_{m1}x_1 + \cdots + a_{mn}x_n \geq b_m$$

$$\begin{aligned} \min \quad & c^\top x \\ \text{s.t.} \quad & \boxed{Ax \geq b} \\ & x \in \mathbb{R}^n_+. \end{aligned}$$



- Mixed-integer linear programming :

$$\begin{aligned} \min \quad & c^\top x \\ \text{s.t.} \quad & Ax \geq b \\ & x \in \mathbb{R}^{n_c}_+ \times \mathbb{Z}^{n_d}_+. \end{aligned}$$



Little reminder

3 possibilities: (1) LP infeasible: $\{x \in \mathbb{R}^n_+ \mid Ax \geq b\} = \emptyset$

(2) LP is unbounded: $\exists \bar{x} \in \{x \in \mathbb{R}^n_+ \mid Ax \geq b\}$, $\bar{d} \in \mathbb{R}^n_+ \neq 0$ s.t

$\bar{x} + \lambda d \in \{x \in \mathbb{R}^n_+ \mid Ax \geq b\} \quad \forall \lambda \in \mathbb{R}_{\geq}$

(3) LP has at least one optimal solution

* Analogously for MIP

- Linear programming :

$$\begin{aligned} \min \quad & c^\top x \\ \text{s.t.} \quad & Ax \geq b \\ & x \in \mathbb{R}^n_+. \end{aligned}$$

- Mixed-integer linear programming :

$$\begin{aligned} \min \quad & c^\top x \\ \text{s.t.} \quad & Ax \geq b \\ & x \in \mathbb{R}^{n_c}_+ \times \mathbb{Z}^{n_d}_+. \end{aligned}$$

Duality in linear programming

- In linear programming each problem is intimately related to another problem called it *dual*.

$$(\mathbb{LP}) : \begin{cases} \min & c^\top x \\ \text{s.t.} & Ax \geq b \\ & x \in \mathbb{R}^n_+. \end{cases} \qquad (\mathbb{DP}) : \begin{cases} \max & b^\top \lambda \\ \text{s.t.} & A^\top \lambda \leq c \\ & \lambda \in \mathbb{R}^m_+. \end{cases}$$

- The dual problem can be seen as a bound calculating problem with the aim of certifying optimality.

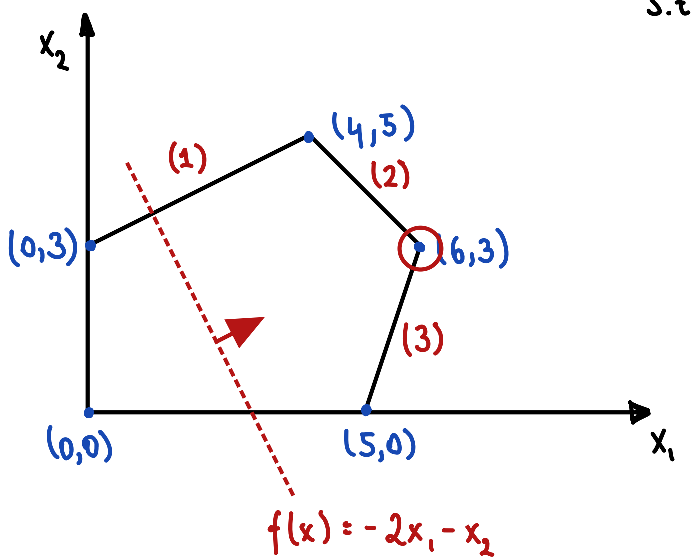
Definition (lower bound)

A real value z_{LB} is said to be a lower bound on the optimal value z^* of $(\mathbb{LP})$ if $\forall x \geq 0$ such that $Ax \geq b$ we have that $c^\top x \geq z_{\text{LB}}$, *i.e.*, $z^* \geq z_{\text{LB}}$.

Definition (upper bound)

Similarly a real value z_{UB} is said to be an upper bound on the optimal value z^* of $(\mathbb{LP})$ if $\forall x \geq 0$ such that $Ax \geq b$ we have that $c^\top x \leq z_{\text{UB}}$, *i.e.*, $z^* \geq z_{\text{UB}}$.

Obtaining the dual formulation



$$
\begin{aligned}
z^* = \min \quad & -2x_1 - x_2 & & \text{(P)}\\
\text{s.t} \quad & & &\\
& x_1 - 2x_2 \geq -6 & & (1)\\
& -x_1 - x_2 \geq -9 & & (2)\\
& -3x_1 + x_2 \geq -15 & & (3)\\
& x_1, x_2 \geq 0 & &
\end{aligned}
$$

LB_1 : $(1) * 1$: $x_1 - 2x_2 \geq -6$

$(3) * 1$: $-3x_1 + x_2 \geq -15$

$$\overline{-2x_1 - x_2 \geq -21}$$

$z^* = -2x_1 - x_2 \geq -21$

$z^* \geq -21$

LB_2 : $(2) * 2$: $-2x_1 - 2x_2 \geq -18$

$z^* = -2x_1 - x_2 \overset{?}{\geq} -2x_1 - 2x_2 \geq -18$

$x_1, x_2 \geq 0$

$z^* \geq -18$ $\qquad -1 \geq -2$

Obtaining the dual formulation

LB_3: (2) * 4/3 : $-4/3\, x_1 - 4/3\, x_2 \geq -12$

(3) * 1/3 $\quad -x_1 + 1/3\, x_2 \geq -5$

$-7/3\, x_1 - x_2 \geq -17$

$z^* = -2x_1 - x_2 \overset{?}{\geq} -7/3\, x_1 - x_2 \geq -17$

$x_1, x_2 \geq 0$

$-2 \geq -7/3$

$z^* \geq -17$

(2) * 5/4 : $-5/4\, x_1 - 5/4\, x_2 \geq -45/4$

(3) * 1/4 : $-3/4\, x_1 + 1/4\, x_2 \geq -15/4$

$-2x_1 - x_2 \geq -15$

$z^* = -2x_1 - x_2 \geq -15$

has the same value as the primal solution (6,3)

Obtaining the dual formulation

$$x_1 - 2x_2 \geqslant -6 \quad (\lambda_1 \geqslant 0) \qquad \lambda_1 x_1 - 2\lambda_1 x_2 \geqslant -6\lambda_1$$

$$-x_1 - x_2 \geqslant -9 \quad (\lambda_2 \geqslant 0) \qquad -\lambda_2 x_1 - \lambda_2 x_2 \geqslant -9\lambda_2$$

$$-3x_1 + x_2 \geqslant -15 \quad (\lambda_3 \geqslant 0) \qquad -3\lambda_3 x_1 + \lambda_3 x_2 \geqslant -15\lambda_3$$

$$\underbrace{(\lambda_1 - \lambda_2 - 3\lambda_3)}_{c_1} x_1 + \underbrace{(-2\lambda_1 - \lambda_2 + \lambda_3)}_{c_2} x_2 \geqslant -6\lambda_1 - 9\lambda_2 - 15\lambda_3$$

Q: when can we establish that:

$$z^* = -2x_1 - x_2 \geqslant c_1 x_1 + c_2 x_2 \geqslant -6\lambda_1 - 9\lambda_2 - 15\lambda_3 \quad \forall x_1, x_2 \geqslant 0 \quad ?$$

Obtaining the dual formulation

Let's write an optimization problem to calculate the best dual bound that we can obtain from the above procedure:

$$
\begin{aligned}
z^* = \min \quad & -2x_1 - x_2 \\
\text{s.t} \quad & x_1 - 2x_2 \geq -6 \quad (\lambda_1) \\
& -x_1 - x_2 \geq -9 \quad (\lambda_2) \\
& -3x_1 + x_2 \geq -15 \quad (\lambda_3) \\
& x_1, x_2 \geq 0
\end{aligned}
$$

$$
\begin{aligned}
\max \quad & -6\lambda_1 - 9\lambda_2 - 15\lambda_3 \\
\text{s.t} \quad & \lambda_1 - \lambda_2 - 3\lambda_3 \leq -2 \\
& -2\lambda_1 - \lambda_2 + \lambda_3 \leq -1 \\
& \lambda_1, \lambda_2, \lambda_3 \geq 0
\end{aligned}
$$

because $x_1, x_2 \geq 0$

because we don't need to change the sense of constraints

Obtaining the dual formulation

In general:

$$(P) \quad z_P = \min_{\text{s.t}} c^T x \qquad \underbrace{A}_{m\times n} x \geq b \in \mathbb{R}^m \qquad x \geq 0 \in \mathbb{R}^n$$

$$z_D = \max_{\text{s.t}} b^T \lambda \qquad \underbrace{A^T}_{n\times m} \lambda \leq c \in \mathbb{R}^n \qquad \lambda \geq 0 \in \mathbb{R}^m \quad (D)$$

More generally:

min $c^T x$ s.t		max $b^T\lambda$ s.t	
$Ax \geq b$	(λ)	$A^T\lambda \leq c$	(x)
$\leq$		$\geq$	
$=$		$=$	
$x \geq 0$		$\lambda \geq 0$	
≤ 0		$\leq$	
urs		urs	

coefficient obtained by the combination of constraints $(A^T\lambda)\, x \geq b^T\lambda$

coefficient in the obj. value

want to impose: $c_i x_i \geq a_i^T \lambda x_i$

sign of dual var.s determined by whether we want to inverse the sense of a constraint

⚠ If we write the dual of the dual problem, we should obtain the primal problem.

Obtaining the dual formulation

Table – Building the dual problem of a linear program

Primal	Dual
Objective function : $\min\ c^\top x$	Right hand side c
Right hand side b	Objective function : $\max\ b^\top \lambda$
Constraint matrix A	Constraint matrix $A^\top$
Constraint $A_i x \geq b_i$	Variable $\lambda_i \geq 0$
Constraint $A_i x \leq b_i$	Variable $\lambda_i \leq 0$
Constraint $A_i x = b_i$	Variable $\lambda_i \in \mathbb{R}$
Variable $x_j \geq 0$	Constraint : $(A^j)^\top \lambda \leq c_j$
Variable $x_j \leq 0$	Constraint : $(A^j)^\top \lambda \geq c_j$
Variable $x_j \in \mathbb{R}$	Constraint : $(A^j)^\top \lambda = c_j$

Weak duality theorem

All results of the fact that any feasible solution of the dual provides a LB on the optimal value of the primal (by construction)

Theorem (Weak duality)

1. *If* $(\mathbb{LP})$ *is unbounded then* $(\mathbb{DP})$ *is infeasible.* If (DP) is feasible then (LP) is bounded & conversely.
2. *If* $(\mathbb{DP})$ *is unbounded then* $(\mathbb{LP})$ *is infeasible.*
3. *If* $(\mathbb{LP})$ *is feasible and bounded, then* $z_{(\mathbb{DP})} \leq z_{(\mathbb{LP})}$. optimal value of primal

This also implies (DP) is feasible & bounded

optimal value of dual

Attention !

If $(\mathbb{LP})$ is infeasible then $(\mathbb{DP})$ can be infeasible or unbounded.

Attention !

It is possible for both $(\mathbb{LP})$ and $(\mathbb{DP})$ to be infeasible.

Strong duality theorem and complementary slackness

Theorem (Strong duality)

$(\mathbb{LP})$ *has an optimal solution* x^* *if and only if* $(\mathbb{DP})$ *has an optimal solution* λ^* *such that :*

$$c^\top x^* = b^\top \lambda^*.$$

Theorem (Complementary slackness)

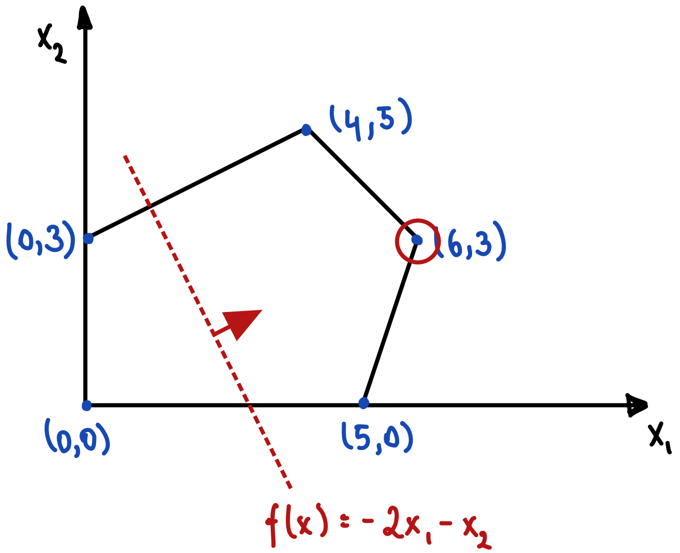
Let x *be a feasible solution of* $(\mathbb{LP})$ *and* λ *a feasible solution of* $(\mathbb{DP})$*, then* x *and* λ *are optimal solutions of* $(\mathbb{LP})$ *and* $(\mathbb{DP})$ *if and only if :*

$$\begin{aligned} x_j \times (c_j - \lambda^\top A_{\cdot,j}) &= 0 & \forall j \in [n] = n_x \\ \lambda_i \times (b_i - A_{i,\cdot} x) &= 0 & \forall i \in [m] = m_x = n_\lambda \end{aligned}$$

where $A_{\cdot,j}$ *is the column of* A *indexed by* j *and* $A_{i,\cdot}$ *the row of* A *indexed by* i*.*

Using complementary slackness it suffices to solve either $(\mathbb{LP})$ or $(\mathbb{DP})$.

Solution of LPs with Simplex : geometric view

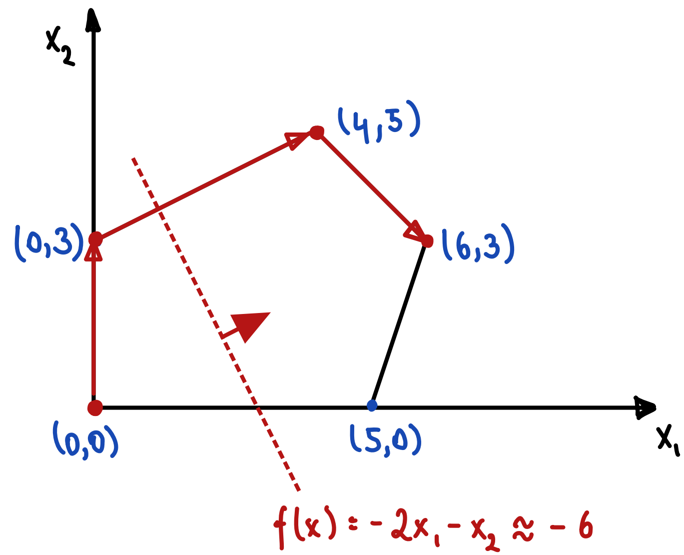


If LP has an optimal solution then LP has an extreme point that is an optimal solution!

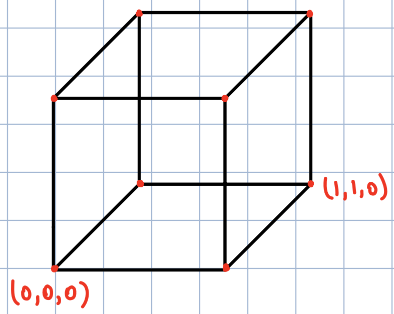
Idea : Pivot from one extreme point to another in the direction of improvement of the objective function.

(we assume feasibility)

$$
\begin{aligned}
z^* = \min \quad & -2x_1 - x_2 \\
\text{s.t} \quad & x_1 - 2x_2 \geq -6 \\
& -x_1 - x_2 \geq -9 \\
& -3x_1 + x_2 \geq -15 \\
& x_1, x_2 \geq 0
\end{aligned}
$$



Example 1:



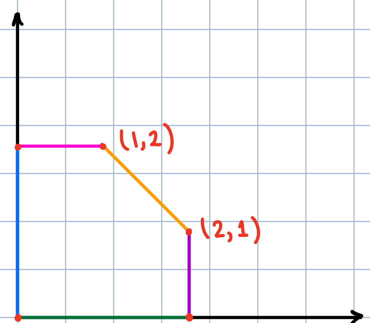
$$P_1 = \left\{ x \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} 0 \leq x_1 \leq 1 \\ 0 \leq x_2 \leq 1 \\ 0 \leq x_3 \leq 1 \end{array} \right\}$$

Each extreme point is an intersection of 3 hyperplanes:

$(0,0,0)$: $\quad x_1 = 0$, $x_2 = 0$, $x_3 = 0$

$(1,1,0)$: $\quad x_1 = 1$, $x_2 = 1$, $x_3 = 0$

Example 2:



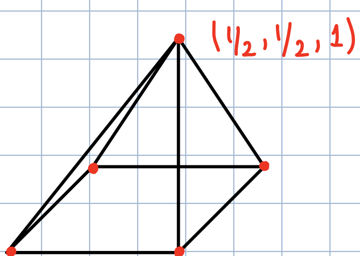
$$P_2 = \left\{ x \in \mathbb{R}^2 \;\middle|\; \begin{array}{l} x_1 + x_2 \leq 3 \\ 0 \leq x_1 \leq 2 \\ 0 \leq x_2 \leq 2 \end{array} \right\}$$

Each extreme point is an intersection of 2 hyperplanes:

$(1,2)$: $\quad x_1 + x_2 = 3$, $x_2 = 2$

$(2,1)$: $\quad x_1 + x_2 = 3$, $x_1 = 2$

Example 3:



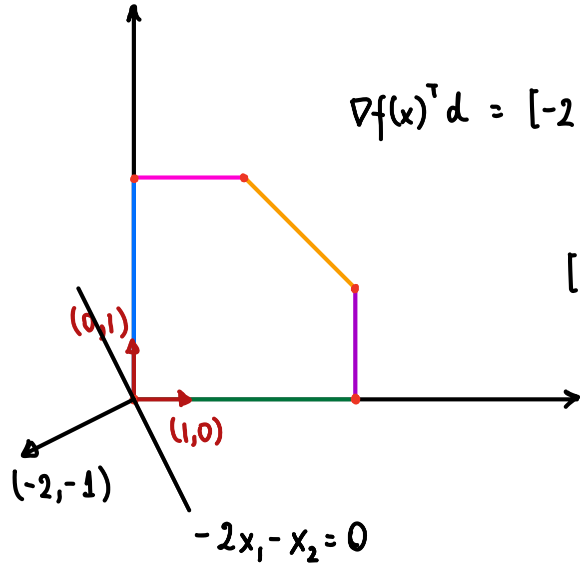
$$P_3 = \left\{ x \in \mathbb{R}^3_+ \;\middle|\; \begin{array}{l} x_1 - \frac{1}{2} x_3 \geq 0 \\ x_2 + \frac{1}{2} x_3 \leq 1 \\ -x_1 - \frac{1}{2} x_3 \geq -1 \\ -x_2 + \frac{1}{2} x_3 \leq 0 \end{array} \right\}$$

Each extreme point is an intersection of 3 hyperplanes, except for $(1/2, 1/2, 1)$!

$(1/2, 1/2, 1)$ is a degenerate extreme point, it can be represented in 4 different ways

Solution of LPs with Simplex : geometric view

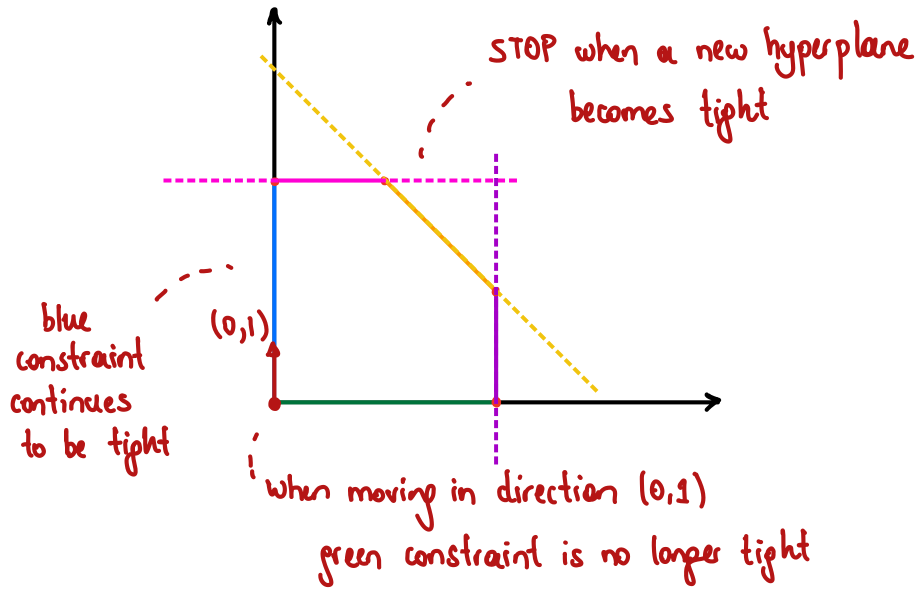
$$\begin{aligned} \min \quad & -2x_1 - x_2 \\ \text{s.t} \quad & x_1 + x_2 \leq 3 \\ & x_1 \leq 2 \\ & x_2 \leq 2 \\ & x_1, x_2 \in \mathbb{R}_+ \end{aligned}$$

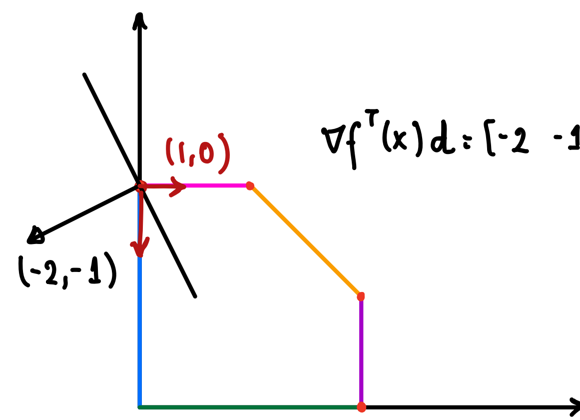


$$\left.\begin{aligned} \nabla f(x)^T d = [-2\ -1]\begin{bmatrix}1\\0\end{bmatrix} = -2 < 0 \\ [-2\ -1]\begin{bmatrix}0\\1\end{bmatrix} = -1 < 0 \end{aligned}\right\}$$

Both directions are improving

Assume that we choose to follow direction (0,1)





$$\nabla f^T(x)\, d = [-2\ -1]\begin{bmatrix}1\\0\end{bmatrix} = -2 < 0$$

Continue in direction (1,0)

Solution of LPs with Simplex : algebraic view

Idea: Treat the constraints of the problem as a system of linear equations.

↳ For this we first need to write constraints $Ax \geq b$ in equality form.

Standard form:

$$\begin{array}{ll} \min & c^T x \\ \text{s.t} & Ax \geq b \\ & x \geq 0 \end{array} = \begin{array}{ll} \min & c^T x \\ \text{s.t} & Ax - Is = b \\ & x, s \geq 0 \end{array} = \begin{array}{ll} \min & \tilde{c}^T \tilde{x} \\ \text{s.t} & \tilde{A}\tilde{x} = b \\ & \tilde{x} \geq 0 \end{array}$$

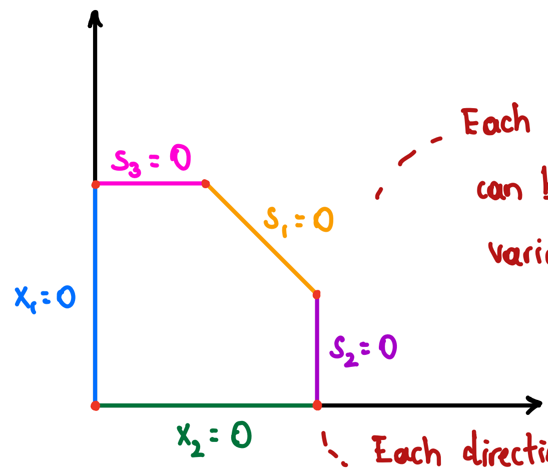
$\tilde{c}^T = [c \quad 0]$

with $\tilde{A} = [A \quad -I]$

$\tilde{x} = \begin{bmatrix} x \\ s \end{bmatrix}$

Connection to geometric form:

$$\begin{array}{llllll} \min & -2x_1 - x_2 & & & & \\ \text{s.t} & x_1 + x_2 & + s_1 & & & \leq 3 \\ & x_1 & & + s_2 & & \leq 2 \\ & x_2 & & & + s_3 & \leq 2 \\ & x_1, x_2 \in \mathbb{R}_+, & s_1, s_2, s_3 \in \mathbb{R}_+ & & & \end{array}$$



Each hyperplane can be associated with a variable being $= 0$

Each direction corresponds to increasing a single variable from 0 to be > 0

Solution of LPs with Simplex : algebraic view

$$\begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{21} & a_{22} & & a_{n2} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

m constraints, n variables

typically $n \geq m \rightarrow$ there are many x vectors satisfying the system

To obtain a unique solution we need a square system

Idea : Fix $n-m$ variables to 0, solve the remaining square system.

Solution of LPs with Simplex : algebraic view

- Let N denote the indices of variables fixed to 0 and B denote the remaining variables.

$$\begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{21} & a_{22} & & a_{n2} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \longrightarrow \quad A_B x_B + A_N x_N = b$$

$$A_B x_B = b - A_N \underbrace{x_N}_{=0}$$

If A_B is invertible we obtain $x_B = A_B^{-1} b$

If $A_B^{-1} b \geq 0$ then $x = [x_B, x_N]$ is an extreme point and a basic feasible solution.

Solution of LPs with Simplex : algebraic view

$$(\mathbb{LP}) : \begin{cases} \min & c^\top x \\ \text{s.t.} & Ax = b \\ & x \geq 0 \end{cases}$$

Definition (Basic feasible solutions)

The point $x^0 \in \mathbb{R}^n$ is a *basic feasible solution* of $(\mathbb{LP})$ if and only if there exists a partition $[A_B|A_N]$ of the columns of A, corresponding to the partition (B, N) of $[n]$, such that :

- $A_B \in \mathbb{R}^{m \times m}$ is non singular,
- $x_N^0 = 0$ and $x_B^0 = A_B^{-1} b \geq 0$.

The matrix A_B is commonly called the *basis* and the variables indexed by B and N are respectively the *basic* and *nonbasic* variables.

Solution of LPs with Simplex : algebraic view

$$(\mathbb{LP}) : \begin{cases} \min & c^\top x \\ \text{s.t.} & Ax = b \\ & x \geq 0 \end{cases}$$

Definition (Simplex dictionary)

From a basic feasible solution with basis A_B, we obtain the *simplex dictionary* :

$$\begin{aligned} z &= c_B^\top A_B^{-1} b + (c_N^\top - c_B^\top A_B^{-1} A_N) x_N \\ x_B &= A_B^{-1} b - A_B^{-1} A_N x_N \end{aligned}$$

- A dictionary (and the associated basic feasible solution) is optimal if :

$$(c_N^\top - c_B^\top A_B^{-1} A_N) \geq 0.$$

- The vector $(c_N^\top - c_B^\top A_B^{-1} A_N)$ is often referred to as the *reduced cost*.

$$z = \overbrace{c_B^\top A_B^{-1} b}^{\text{current objective value}} + \overbrace{(c_N^\top - c_B^\top A_B^{-1} A_N)}^{\text{effect of non-basic variables on the objective function}} x_N$$

$$x_B = \underbrace{A_B^{-1} b}_{\text{current solution}} - \underbrace{A_B^{-1} A_N x_N}_{\text{effect of non-basic variables on the current solution}}$$

direction: $\begin{array}{cl} \begin{bmatrix} e_i \\ -A_B^{-1} A_{\{i\}} \end{bmatrix} & \begin{array}{l} \text{in the non-basic space} \\ \text{in the basic space} \end{array} \end{array}$

directional derivative: $\nabla f^T(x) d = c^T \begin{bmatrix} e_i \\ -A_B^{-1} A_{\{i\}} \end{bmatrix} = c_i - c_B A_B^{-1} A_{\{i\}}$

By doing this for every $i \in N$, we obtain the reduced cost vector: $c_N - c_B A_B^{-1} A_N$

Solution of LPs with Simplex : algebraic view

$$(\mathbb{LP}) : \begin{cases} \min & c^\top x \\ \text{s.t.} & Ax = b \\ & x \geq 0 \end{cases}$$

Definition (Simplex dictionary)

From a basic feasible solution with basis A_B, we obtain the *simplex dictionary* :

$$z = c_B^\top A_B^{-1} b + (c_N^\top - c_B^\top A_B^{-1} A_N) x_N$$
$$x_B = A_B^{-1} b - A_B^{-1} A_N x_N$$

- If the dictionary is not optimal then $\exists$ at least one index $e \in N$ such that $(c_N^\top - c_B^\top A_B^{-1} A_N)_e < 0$.
- Then increasing the value of x_e from 0 creates an improved solution.
- As the value of x_e increases a variable x_l with $l \in B$ becomes 0 and leaves the basis (if $(\mathbb{LP})$ is bounded).
- This results in a *simplex pivot*.

Solution of LPs with Simplex : algebraic view

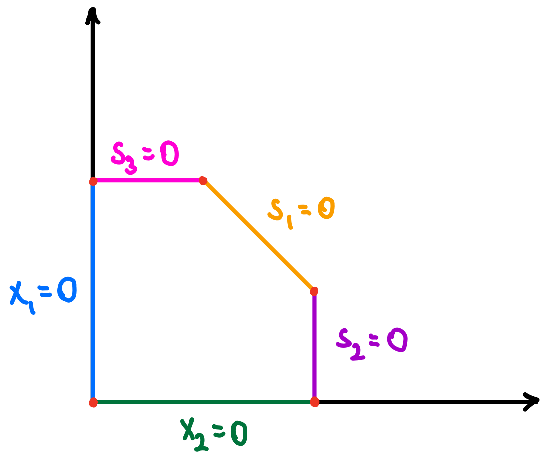
Input: A feasible LP in standard form $\min\{\mathbf{c}^T\mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0\}$;
An initial feasible basic solution with basis B.

$\left(\overline{\mathbf{A}}, \overline{\mathbf{b}}, \overline{\mathbf{c}}\right)$:= simplex dictionary corresponding to B;
while $\overline{\mathbf{c}} \ngeq 0$ **do**
  Choose $e \in \{j \in N : \bar{c}_j < 0\}$; `// entering variable: pivoting rule`
  if $\overline{\mathbf{A}}_{i,e} \leq 0, \forall i \in N$ **then**
    Return "Unbounded" ; `// no basic variable decreases when` x_e `increases`
  else
    $l \in \operatorname{argmin}_{i \in B} \left\{ \frac{\bar{b}_i}{\overline{\mathbf{A}}_{i,e}} : \overline{\mathbf{A}}_{i,e} > 0 \right\}$; `// leaving variable: ratio test`
  $B \leftarrow (B \setminus \{l\}) \cup \{e\}$; `// update the basis`
  $\left(\overline{\mathbf{A}}, \overline{\mathbf{b}}, \overline{\mathbf{c}}\right) \leftarrow$ simplex dictionary corresponding to $\mathbf{A}_B$;
$(\mathbf{x}^\star_B, \mathbf{x}^\star_N) = (\overline{\mathbf{b}}, 0)$;
Return the optimal basis indices B and the corresponding optimal basic solution $\mathbf{x}^\star$;

Algorithm 1: Primal simplex algorithm

At optimality, an optimal dual solution is obtained as $\lambda^* = c_B^T A_B^{-1}$

Example : Simplex in algebraic form



$$\min_{x \in P} \; -2x_1 - x_2$$

$$P_2 = \left\{ x \in \mathbb{R}^2 \;\middle|\; \begin{array}{l} x_1 + x_2 \leq 3 \\ 0 \leq x_1 \leq 2 \\ 0 \leq x_2 \leq 2 \end{array} \right\}$$

$$\begin{array}{lllllll} x_1 & + x_2 & + s_1 & & & = & 3 \\ x_1 & & & + s_2 & & = & 2 \\ & x_2 & & & + s_3 & = & 2 \\ x_1 & & & & & \geq & 0 \\ & x_2 & & & & \geq & 0 \\ & & s_1 & & & \geq & 0 \\ & & & s_2 & & \geq & 0 \\ & & & & s_3 & \geq & 0 \end{array}$$

$$A = \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & s_1 & s_2 & s_3 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Starting with the basis $B = \{s_1, s_2, s_3\}$ $\quad N = \{x_1, x_2\}$

$$A_N = \begin{array}{c} \begin{array}{cc} x_1 & x_2 \end{array} \\ \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array} \qquad A_B = \begin{array}{c} \begin{array}{ccc} s_1 & s_2 & s_3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

Dictionary :

$$\begin{aligned} z &= 0 - 2x_1 - x_2 \\ s_1 &= 3 - x_1 - x_2 \\ s_2 &= 2 - x_1 \\ s_3 &= 2 - x_2 \end{aligned}$$

Example : Simplex in algebraic form

Step 1: $B = \{s_1, s_2, s_3\}$ $N = \{x_1, x_2\}$

entering variable

Dictionary: $z = 0 - 2x_1 - x_2$

$s_1 = 3 - x_1 - x_2$

$s_2 = 2 - x_1$ → leaving variable

$s_3 = 2 \quad - x_2$

$x_1 = 2 - s_2$, substitute:

$z = 0 - 2(2 - s_2) - x_2$

$s_1 = 3 - (2 - s_2) - x_2$

$x_1 = 2 - s_2$

$s_3 = 2 \quad - x_2$

clean up:

$z = -4 + 2s_2 - x_2$

$s_1 = 1 + s_2 - x_2$

$x_1 = 2 - s_2$

$s_3 = 2 \quad - x_2$

Step 2: $B = \{s_1, x_1, s_3\}$ $N = \{s_2, x_2\}$

entering variable

Dictionary: $z = -4 + 2s_2 - x_2$

$s_1 = 1 \quad s_2 - x_2$ → leaving variable

$x_1 = 2 - \quad s_2$

$s_3 = 2 \quad - x_2$

$x_2 = 1 - s_1 + s_2$

$$A_N = \begin{array}{cc} s_2 & x_2 \end{array} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A_B = \begin{array}{ccc} s_1 & x_1 & s_3 \end{array} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example : Simplex in algebraic form

substitute :

$$
\begin{aligned}
z &= -4 + 2s_2 - (1 - s_1 + s_2) \\
x_2 &= 1 - s_1 + s_2 \\
x_1 &= 2 - s_2 \\
s_3 &= 2 - (1 - s_1 + s_2)
\end{aligned}
$$

clean up :

$$
\begin{aligned}
z &= -5 + s_1 + 3s_2 \\
x_2 &= 1 - s_1 + s_2 \\
x_1 &= 2 - s_2 \\
s_3 &= 1 + s_1 - s_2
\end{aligned}
$$

Step 3 : $B = \{x_1, x_2, s_3\}$ $N = \{s_1, s_2\}$

$$
\begin{aligned}
z &= -5 \boxed{+ s_1 + 3s_2} \\
x_2 &= 1 - s_1 + s_2 \\
x_1 &= 2 - s_2 \\
s_3 &= 1 + s_1 - s_2
\end{aligned}
$$

positive reduced cost vector

$$
A_N = \begin{array}{c} \begin{array}{cc} s_2 & s_1 \end{array} \\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \end{array}
\qquad
A_B = \begin{array}{c} \begin{array}{ccc} x_2 & x_1 & s_3 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \end{array}
$$

solution $(x_1, x_2) = (2, 1)$ with value -5 is optimal

Exercise 23:

a. Write the dual of

$$\max \ c^T x$$
s.t
$$Ax \le b$$

b. Write the dual of

$$\max \ x_1 + 2x_2$$
s.t
$$x_1 + x_2 \geq 4$$
$$x_1 + 5x_2 \leq 5$$
$$x_1 - 2x_2 = 7$$
$$x_1 \geq 0, \ x_2 \leq 0$$

Exercise 25:

$$\min \ -2x_1 - 3x_2 - 2x_3 - 3x_4$$
s.t
$$-2x_1 - x_2 - 3x_3 - 2x_4 \geq -8$$
$$3x_1 + 2x_2 + 2x_3 + x_4 \leq 7$$
$$x_1, x_2, x_3, x_4 \geq 0$$

a. Write the dual

b. Starting from the opt. solution $x_1 = 0, x_2 = 2, x_3 = 0, x_4 = 3$ find an optimal dual solution

c. Imagine now that there's a variable $x_5 \geq 0$ that we forgot to include in our initial model. The objective coefficient of this variable is c_5, its contribution to the first & second constraint is a_{15} & a_{25}, respectively. Is our solution still optimal?

$$\min \quad -2x_1 - 3x_2 - 2x_3 - 3x_4$$
s.t

$$-2x_1 - x_2 - 3x_3 - 2x_4 \geq -8 \quad (\lambda_1 \geq 0)$$

$$3x_1 + 2x_2 + 2x_3 + x_4 \leq 7 \quad (\lambda_2 \leq 0)$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$$\max \quad -8\lambda_1 + 7\lambda_2$$
s.t

$$-2\lambda_1 + 3\lambda_2 \leq -2$$
$$-\lambda_1 + 2\lambda_2 \leq -3$$
$$-3\lambda_1 + 2\lambda_2 \leq -2$$
$$-2\lambda_1 + \lambda_2 \leq -3$$
$$\lambda_1 \geq 0 \quad \lambda_2 \leq 0$$

standard form:

$$\min \quad -2x_1 - 3x_2 - 2x_3 - 3x_4$$
s.t

$$-2x_1 - x_2 - 3x_3 - 2x_4 - s_1 = -8 \quad \leftarrow s_1 = 0$$

$$3x_1 + 2x_2 + 2x_3 + x_4 + s_2 = 7 \quad \leftarrow s_2 = 0$$

$$x \geq 0, \; s \geq 0$$

$B = \{x_2, x_4\} \quad N = \{x_1, x_3, s_1, s_2\}$

$$A_B = \begin{bmatrix} -1 & -2 \\ 2 & 1 \end{bmatrix} \qquad c_B = [-3 \;\; -3]$$

$$c_B A_B^{-1} = 1/3 \, [-3 \;\; -3] \begin{bmatrix} 1 & 2 \\ -2 & -1 \end{bmatrix} = [1 \;\; -1] = \lambda^*$$

Reduced cost of variable x_5:

- Note that $B = \{x_2, x_4\}$ $N = \{x_1, x_3, x_5, s_1, s_2\}$ constitutes a basis.

- Is this basis optimal? $c_5 - \lambda^* \begin{bmatrix} a_{15} \\ a_{25} \end{bmatrix} = c_5 - [1 \;\; -1] \begin{bmatrix} a_{15} \\ a_{25} \end{bmatrix} \overset{?}{<} 0$

   ↳ If not then current sol. still opt.

   ↳ Otherwise, enter x_5 to the basis ...

Primal bounds

Definition (primal bound)

Let $(\mathbb{P}) : \min\{f(x) : x \in \mathcal{X}\}$ be an optimization problem such that $\mathcal{X} \neq \emptyset$. Let $\hat{x} \in \mathcal{X}$ and $\hat{z} = f(\hat{x})$. Then $\hat{z}$ is a primal bound of $(\mathbb{P})$.

obtained as the value of a feasible solution

Remark

In the case of a minimization problem a primal bound represents an upper bound.

Remark

For a mixed-integer program $(\mathbb{MIP}) : z^* = \min\{c^\top x : Ax \geq b, x \in \mathbb{R}^{n_x^c} \times \mathbb{Z}^{n_x^d}\}$. If $\hat{x} \in \mathbb{R}^{n_x^c} \times \mathbb{Z}^{n_x^d}$ satisfies $A\hat{x} \geq b$, then $\hat{z} = c^\top \hat{x}$ is a primal bound of $(\mathbb{MIP})$ and we have

$$\hat{z} \geq z^*.$$

Restrictions and primal bounds

Definition (restriction)

Let $(\mathbb{P}) : \min\{f(x) : x \in \mathcal{X}\}$ be an optimization problem. The optimization problem $(\widehat{\mathbb{P}}) : \min\{\widehat{f}(x) : x \in \widehat{\mathcal{X}}\}$ is a *restriction* of $(\mathbb{P})$ if and only if :

- $\widehat{\mathcal{X}} \subseteq \mathcal{X}$, and ← reduced feasible region
- $\widehat{f}(x) \geq f(x), \forall x \in \widehat{\mathcal{X}}$. ← pessimistic evaluation of solutions

Remark

Let $(\mathbb{P})$ be an optimization problem and $(\widehat{\mathbb{P}}) : \widehat{z} = \min\{\widehat{f}(x) : x \in \widehat{\mathcal{X}}\}$ be a restriction of $(\mathbb{P})$. If $\widehat{\mathcal{X}} \neq \emptyset$ then for all $x \in \widehat{\mathcal{X}}$, $\widehat{f}(x)$ is a primal bound of $(\mathbb{P})$. In particular $\widehat{z}$ is the smallest among these primal bounds.

any feasible solution of a restriction provides a primal bound

Remark

One way to obtain a restriction of an optimization problem is to add constraints to the definition of the problem.

It is also possible to remove variables

Dual bounds and the optimality gap

z_{UB} = primal

z^* = min

z_{LB} = dual

- A dual bound plays the opposite role to a primal bound and serves to certify the optimality of a given feasible solution.
- While the primal bound indicates that the value of a solution cannot get worse the dual bound indicates that it cannot get better.
- Using the primal and the dual bound, we can calculate an optimality gap :

$$100 \times \frac{|z_P - z_D|}{|z_P| + \epsilon}.$$

~ relative distance btw. the primal & the dual bound

- An optimal solution is found if the gap is zero.
- Most optimization algorithms will then improve primal and dual bounds iteratively until convergence.

Remark

One way to obtain a dual bound is through what is called a relaxation. There are also dual bounding problems like the linear programming dual. In the context of mixed-integer programming the most commonly used dual problem is the Lagrangean dual. However, no strong duality result exists for mixed-integer programs.

Relaxations and dual bounds

Definition (Relaxation)

The optimization problem $(\widetilde{\mathbb{P}}) : \min\{\widetilde{f}(x) : x \in \widetilde{\mathcal{X}}\}$ is a **relaxation** of $(\mathbb{P}) : \min\{f(x) : x \in \mathcal{X}\}$ if and only if :

1. $\mathcal{X} \subseteq \widetilde{\mathcal{X}}$; ← enlarged feasible region
2. $\widetilde{f}(x) \leq f(x)$, for all $x \in \mathcal{X}$. ← optimistic evaluation of solutions

Proposition

Let $(\widetilde{\mathbb{P}})$ be a relaxation of $(\mathbb{P})$.

1. If $(\widetilde{\mathbb{P}})$ is infeasible, so is $(\mathbb{P})$.
2. Otherwise, let $z^* = +\infty$ if $(\mathbb{P})$ is infeasible, and $z^* = \min\{f(x) : x \in \mathcal{X}\}$ otherwise. Then, we have that
$$\widetilde{z} \leq z^*.$$

Remark

Ideally it is easy to calculate.

It is clear that the optimal value of a relaxation represents a dual bound.

Relaxations and dual bounds

Definition (Relaxation)

The optimization problem $(\widetilde{\mathbb{P}}) : \min\{\widetilde{f}(x) : x \in \widetilde{\mathcal{X}}\}$ is a **relaxation** of $(\mathbb{P}) : \min\{f(x) : x \in \mathcal{X}\}$ if and only if :

1. $\mathcal{X} \subseteq \widetilde{\mathcal{X}}$;
2. $\widetilde{f}(x) \leq f(x)$, for all $x \in \mathcal{X}$.

Definition (LP relaxation)

For a mixed-integer program $(\mathbb{MIP}) : z^* = \min\{c^\top x : Ax \geq b, x \in \mathbb{R}^{n_x^c} \times \mathbb{Z}^{n_x^d}\}$. The problem

admits more solutions

$$z^*_{\mathrm{LR}} = \min\{c^\top x : Ax \geq b, x \in \mathbb{R}^{n_x}\}$$

is called its LP (continuous) relaxation.

Remark

If the relaxed solution is naturally integer then we are done!

Let x^*_{LR} be an optimal solution of the LP relaxation of $(\mathbb{MIP})$. If $x^*_{\mathrm{LR}} \in \mathbb{R}^{n_x^c} \times \mathbb{Z}^{n_x^d}$ then x^*_{LR} is an optimal solution of $(\mathbb{MIP})$.

Relaxations and dual bounds

Definition (Relaxation)

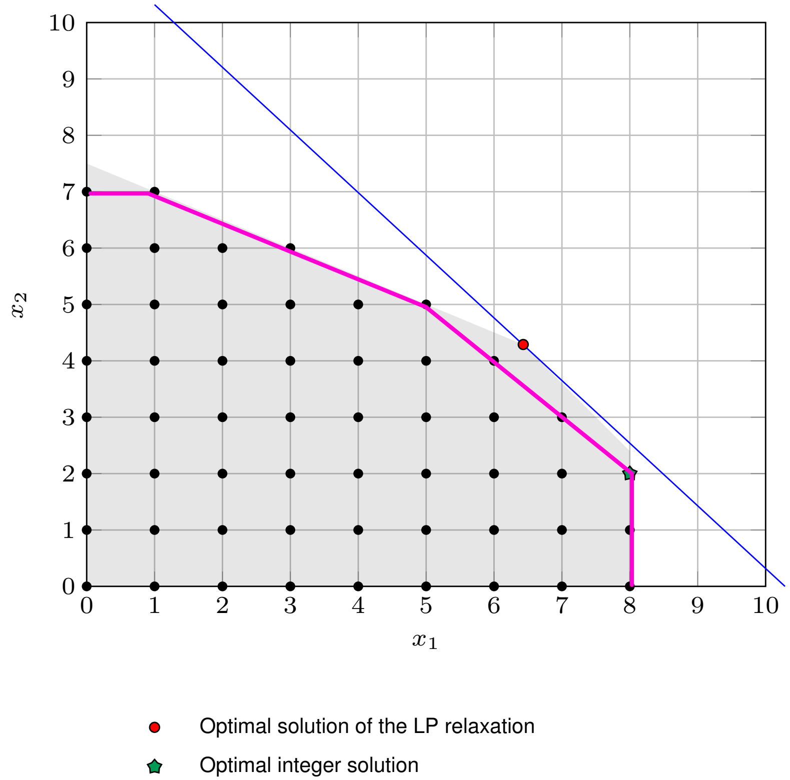
The optimization problem $(\widetilde{\mathbb{P}}) : \min\{\widetilde{f}(x) : x \in \widetilde{\mathcal{X}}\}$ is a **relaxation** of $(\mathbb{P}) : \min\{f(x) : x \in \mathcal{X}\}$ if and only if :

1. $\mathcal{X} \subseteq \widetilde{\mathcal{X}}$;
2. $\widetilde{f}(x) \leq f(x)$, for all $x \in \mathcal{X}$.

Remark

In general, the optimal solution $\tilde{x}^*$ of a relaxation $(\tilde{\mathbb{P}})$ is optimal for the original problem $(\mathbb{P})$ if $\tilde{x}^* \in \mathcal{X}$ and $\tilde{f}(\tilde{x}^*) = f(\tilde{x}^*)$.

Example : linear programming (continuous) relaxation



Solution of the linear relaxation

$$x_1 = 6 + \frac{3}{7}$$
$$x_2 = 4 + \frac{2}{7}$$
$$\text{value} = 5142 + \frac{6}{7}$$

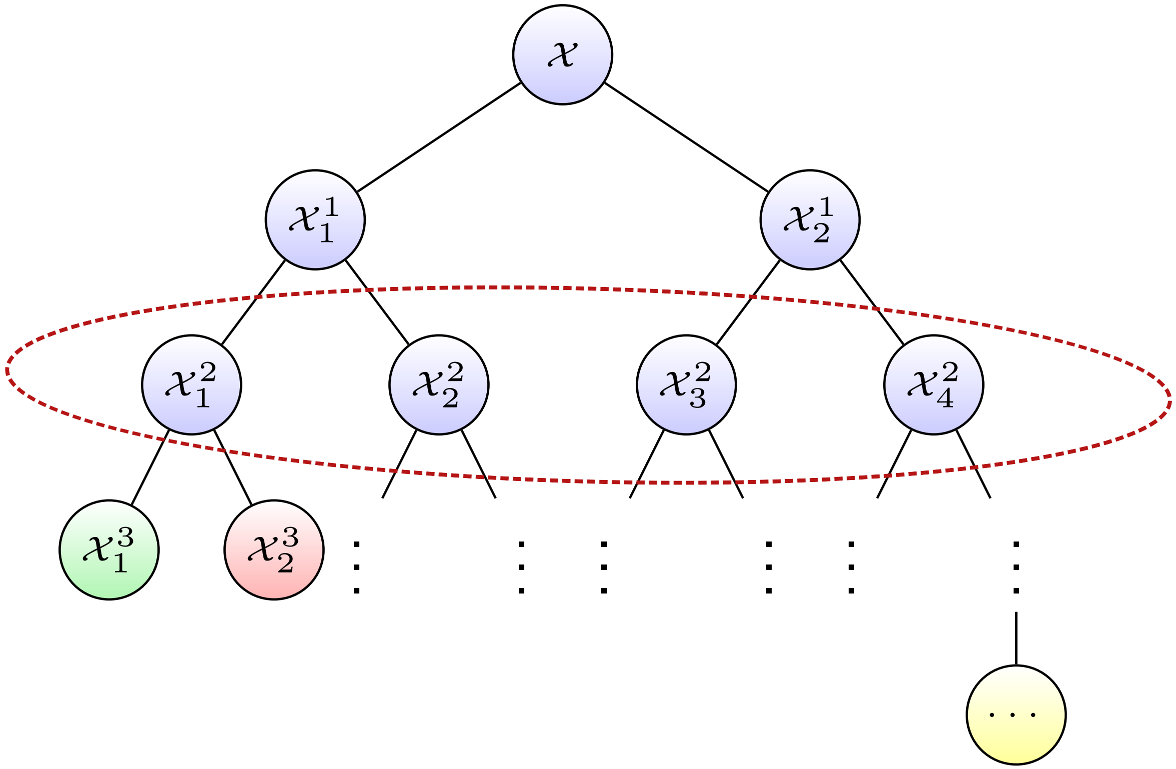
Optimal solution of the MIP

$$x_1 = 8$$
$$x_2 = 2$$
$$\text{value} = 4900$$

Exploration of the feasible set and the B&B tree

- Root of the tree :
  $(\mathbb{P}) : z^* := \min\{f(x) : x \in \mathcal{X}\}$
- Node :
  Restrictions of $(\mathbb{P})$ defined by

  $$z_k^l := \min\{f(x) : x \in \mathcal{X}_k^l\}$$
- Leaf :
  a restriction solved to optimality (green) or concluded to be suboptimal (either by bound (yellow) or by infeasibility (red))



Proposition

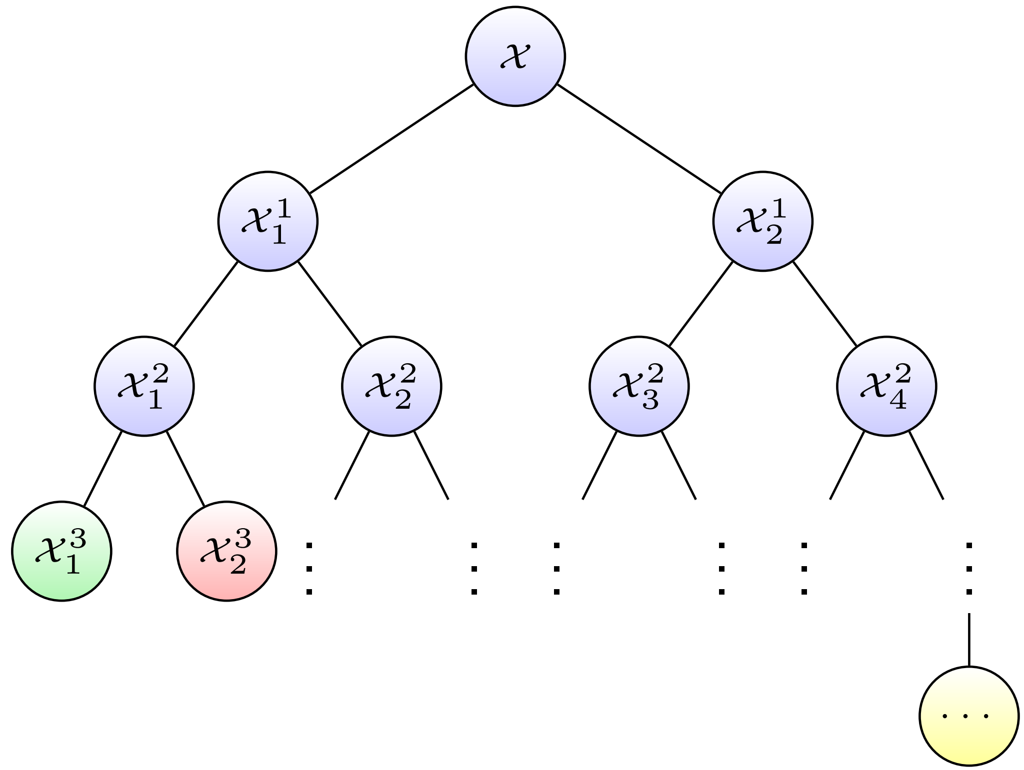
For $l \in [L]$ and $k \in [K_l]$ we have that $z_k^l \geq z^*$. Further, for $l \in [L]$, we have that :

$$z^* = \min_{k \in [K_l]} \{z_k^l\}.$$

This can be written more generally for any set of restrictions of which the union = $\mathcal{X}$

Exploration of the feasible set and the B&B tree

- Root of the tree :
  $(\mathbb{P}) : z^* := \min\{f(x) : x \in \mathcal{X}\}$
- Node :
  Restrictions of $(\mathbb{P})$ defined by
  $$z_k^l := \min\{f(x) : x \in \mathcal{X}_k^l\}$$
- Leaf :
  a restriction solved to optimality (green) or concluded to be suboptimal (either by bound (yellow) or by infeasibility (red))



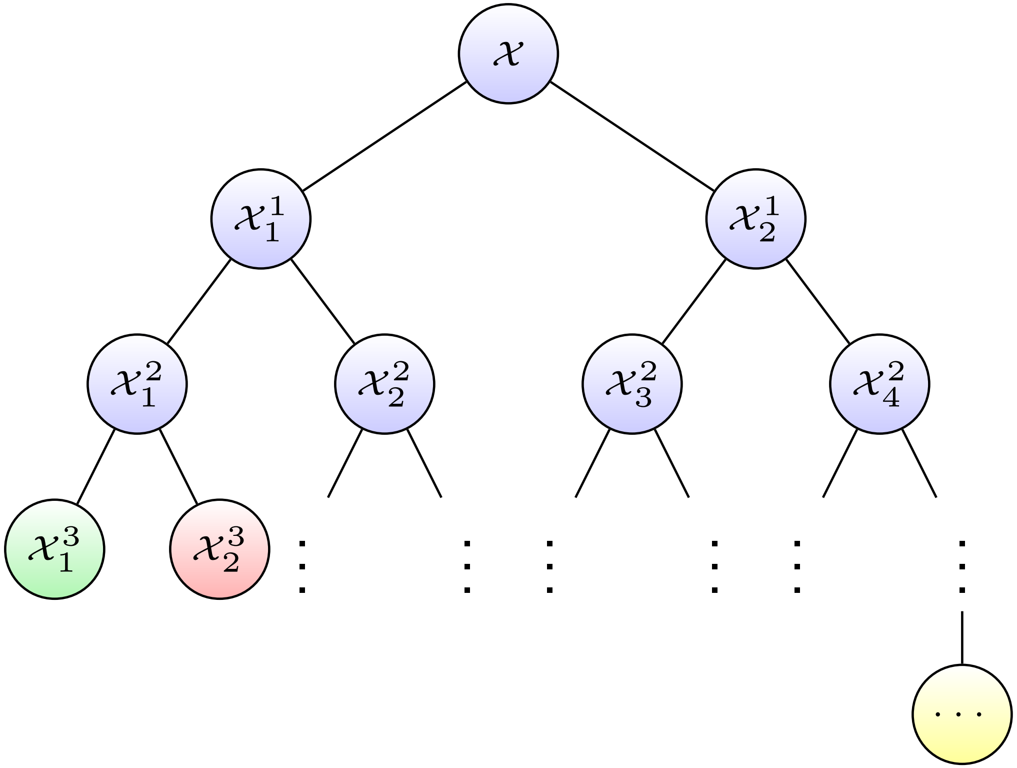
Remark

Often even the restrictions are not easy to solve to optimality. Therefore we typically rely on a dual bound for each restriction. Sometimes obtaining this dual bound comes with the added bonus of a restriction being solved to optimality or with a conclusion of infeasibility.

especially when we use a relaxation

Exploration of the feasible set and the B&B tree

- Root of the tree :
  $(\mathbb{P}) : z^* := \min\{f(x) : x \in \mathcal{X}\}$
- Node :
  Restrictions of $(\mathbb{P})$ defined by
  $$z_k^l := \min\{f(x) : x \in \mathcal{X}_k^l\}$$
- Leaf :
  a restriction solved to optimality (green) or concluded to be suboptimal (either by bound (yellow) or by infeasibility (red))



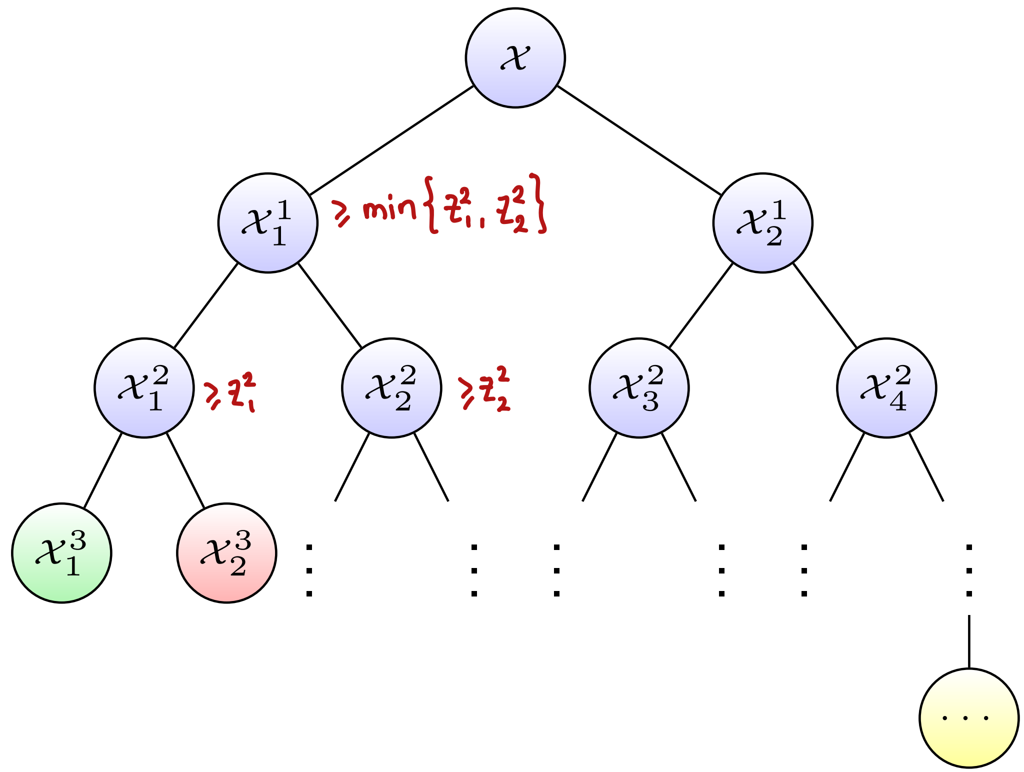
Proposition

Let $\hat{z}$ be a primal bound on the optimal value of $(\mathbb{P})$ and let $\tilde{z}_k^l$ be a dual bound on the optimal value of the restriction defined by $\mathcal{X}_k^l$ for some $l \in [L]$ and $k \in [K_l]$.
If $\tilde{z}_k^l \geq \hat{z}$ then the restriction $\mathcal{X}_k^l$ does not contain a solution whose value is better than $\hat{z}$.

This allows eliminating large parts of the feasible region from consideration.

Exploration of the feasible set and the B&B tree

- Root of the tree :
  $(\mathbb{P}) : z^* := \min\{f(x) : x \in \mathcal{X}\}$
- Node :
  Restrictions of $(\mathbb{P})$ defined by
  $$z_k^l := \min\{f(x) : x \in \mathcal{X}_k^l\}$$
- Leaf :
  a restriction solved to optimality (green) or concluded to be suboptimal (either by bound (yellow) or by infeasibility (red))



Proposition

Let $\hat{z}$ be a primal bound on the optimal value of $(\mathbb{P})$ and let $\tilde{z}_k^l$ be a dual bound on the optimal value of the restriction defined by $\mathcal{X}_k^l$ for $l \in [L]$ and $k \in [K_l]$.
If for $l \in [L]$, we have that $\tilde{z}_k^l \geq \hat{z}$ for $k \in [K_l]$ then $\hat{z} = z^*$.

This can be written more generally for any set of restrictions of which the union $= \mathcal{X}$

LP-based B&B algorithm for solving MILPs

$$(\mathbb{MIP}) : z^* := \min\{x \in \mathbb{R}_+^{n_c} \times \mathbb{Z}_+^{n_d} \mid Ax \geq b\}$$

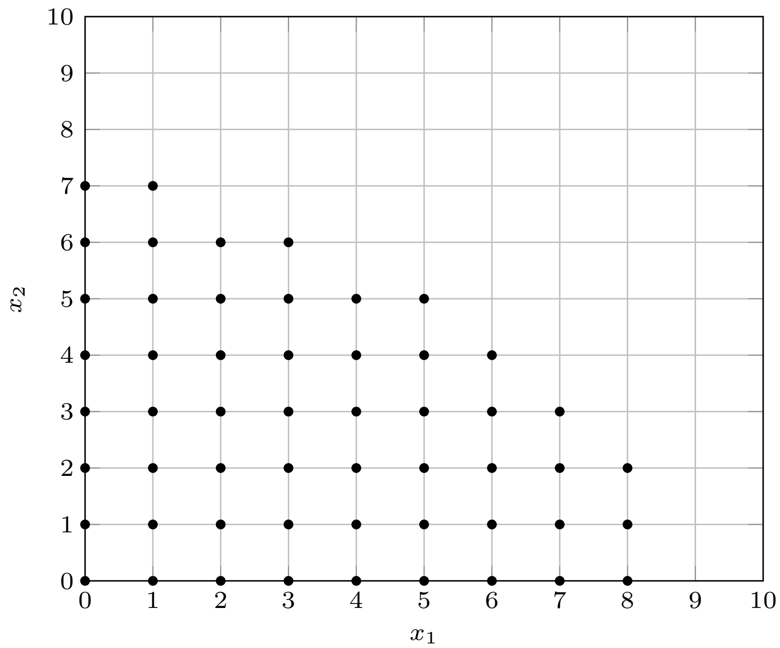
1. **Initialize :** $Q = \{(\mathbb{MIP})\}$, $k = 0$, $x^* = \emptyset$, $z_P = +\infty$
2. **Repeat while** $Q \neq \emptyset$ **:**
   1. Choose the next restriction to treat from Q, call it $(\mathbb{P}_k)$ and remove it from Q
   2. Solve the LP relaxation of $(\mathbb{P}_k)$, $k \leftarrow k+1$
   3. If the LP relaxation is feasible, let $LR(\mathbb{P}_k)$ be the optimal value of the LP relaxation and x_{LR}^k its optimal solution ~ Otherwise prune by infeasibility
   4. If $x_{LR}^k \in \mathbb{R}_+^{n_c} \times \mathbb{Z}_+^{n_d}$ then $(\mathbb{P}_k)$ is solved to optimality :
      - If $LR(\mathbb{P}_k) < z_P$ then $z_P \leftarrow LR(\mathbb{P}_k)$, $x^* \leftarrow x_{LR}^k$, ~ Update primal bound
   5. Else if $LR(\mathbb{P}_k) \lneq z_P$ (If the dual bound is not promising then ~ Prune by bound)
      - Identify $i \in \mathcal{D}$ such that $x_{LR,i}^k = f \notin \mathbb{Z}$
      - Create two restrictions : $(\mathbb{MIP}) + (x_i \leq \lfloor f \rfloor)$ and $(\mathbb{MIP}) + (x_i \geq \lceil f \rceil)$ ~ Partition
      - Add the restrictions to Q.
3. **Return :** x^*, z_P

— best primal bound

best feasible solution

Example 1

$$\begin{array}{rrcrl} \text{max} & 500x_1 & + & 450x_2 & \\ \text{s.t.} & 6x_1 & + & 5x_2 & \leq 60 \\ & 10x_1 & + & 20x_2 & \leq 150 \\ & x_1 & & & \leq 8 \\ & & & x_1, x_2 & \in \mathbb{Z}_+ \end{array}$$

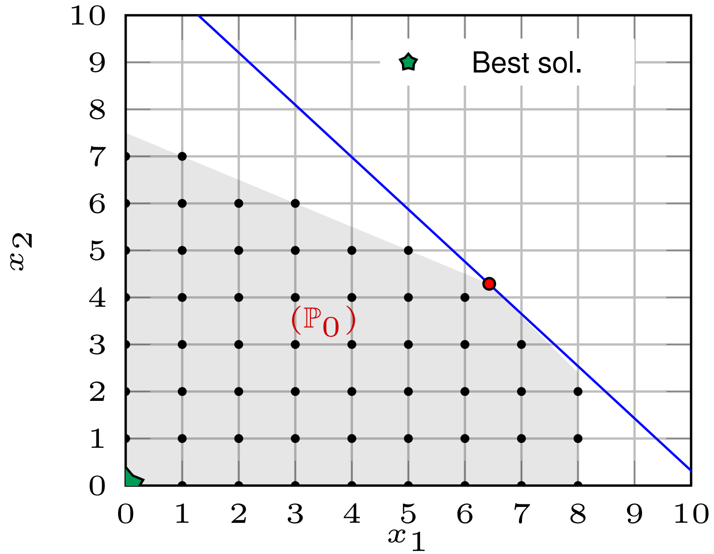


Example 1

$(\mathbb{P}_0)$

$(\mathbb{P}_0)$
$x_1 \approx 6.43, x_2 \approx 4.29$
$LR(\mathbb{P}_0) \approx 5142.85$

Best primal bound $z = 0$



Example 1

Best primal bound $z = 0$

$(\mathbb{P}_0)$

$x_1 \approx 6.43, x_2 \approx 4.29$

$LR(\mathbb{P}_0) \approx 5142.85$

$x_1 \leq \lfloor 6.43 \rfloor$

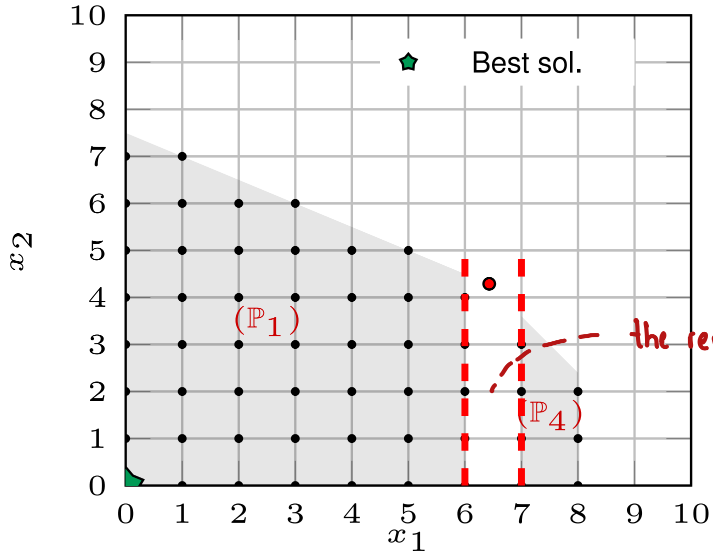
$x_1 \geq \lceil 6.43 \rceil$

$(\mathbb{P}_1) = (\mathbb{P}_0) \cap (x_1 \leq 6)$

Open

$(\mathbb{P}_4) = (\mathbb{P}_0) \cap (x_1 \geq 7)$

Open



the removed region does not contain any integer solution

Example 1

Best primal bound $z = 0$

$(\mathbb{P}_0)$

$x_1 \approx 6.43,\ x_2 \approx 4.29$

$LR(\mathbb{P}_0) \approx 5142.85$

$x_1 \leq \lfloor 6.43 \rfloor$ $\qquad$ $x_1 \geq \lceil 6.43 \rceil$

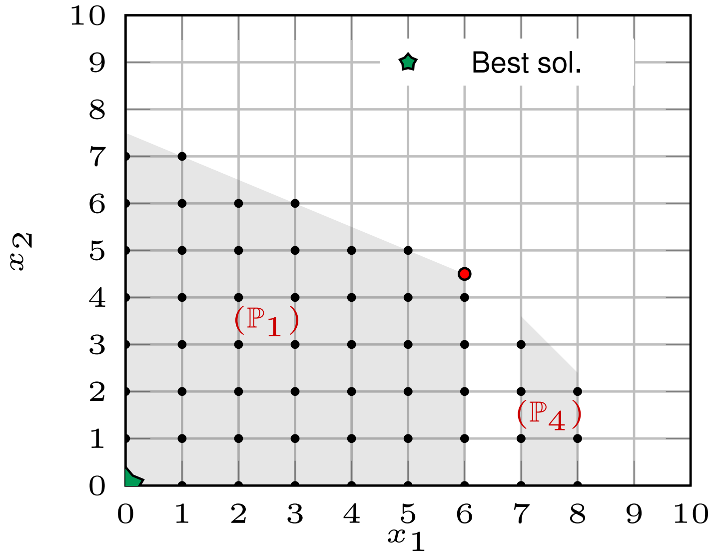
$(\mathbb{P}_1) = (\mathbb{P}_0) \cap (x_1 \leq 6)$

$x_1 = 6,\ x_2 = 4.5$

$LR(\mathbb{P}_1) = 5025$

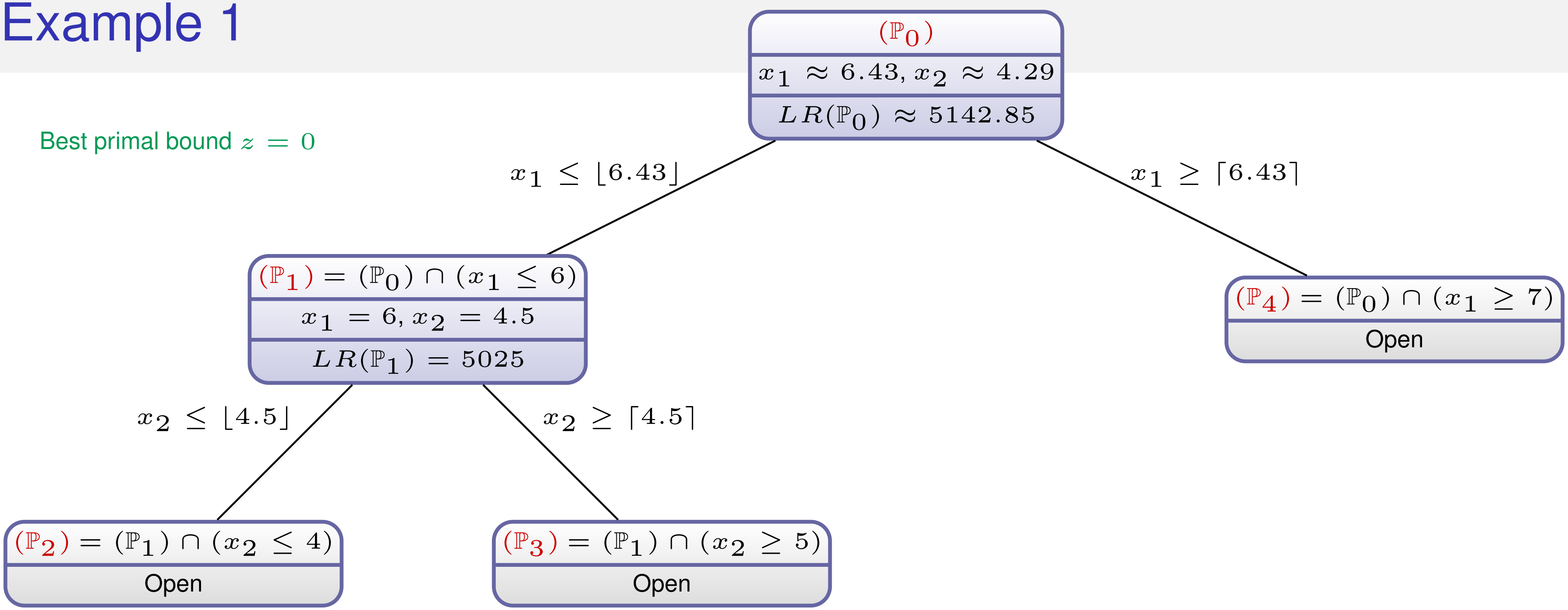
$(\mathbb{P}_4) = (\mathbb{P}_0) \cap (x_1 \geq 7)$

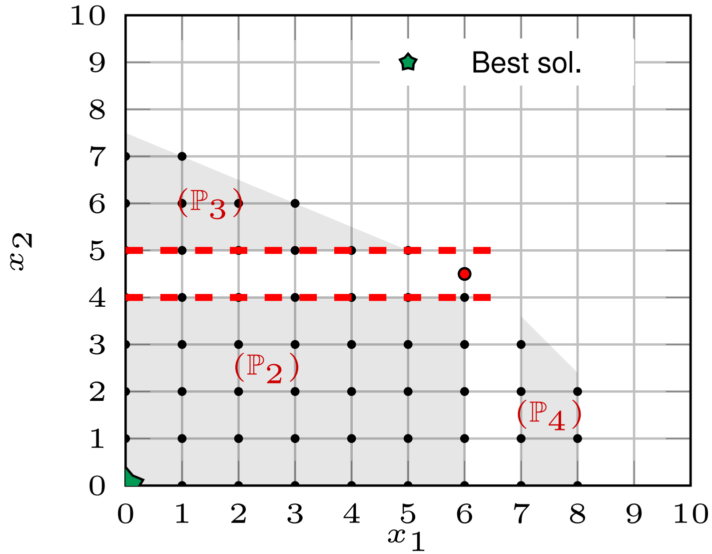
Open



Example 1

Best primal bound $z = 0$





Example 1

Best primal bound $z = 4800$

$(\mathbb{P}_0)$
$x_1 \approx 6.43, x_2 \approx 4.29$
$LR(\mathbb{P}_0) \approx 5142.85$

$x_1 \leq \lfloor 6.43 \rfloor$

$(\mathbb{P}_1) = (\mathbb{P}_0) \cap (x_1 \leq 6)$
$x_1 = 6, x_2 = 4.5$
$LR(\mathbb{P}_1) = 5025$

$x_2 \leq \lfloor 4.5 \rfloor$

$(\mathbb{P}_2) = (\mathbb{P}_1) \cap (x_2 \leq 4)$
$x_1 = 6, x_2 = 4$
$LR(\mathbb{P}_2) = 4800$

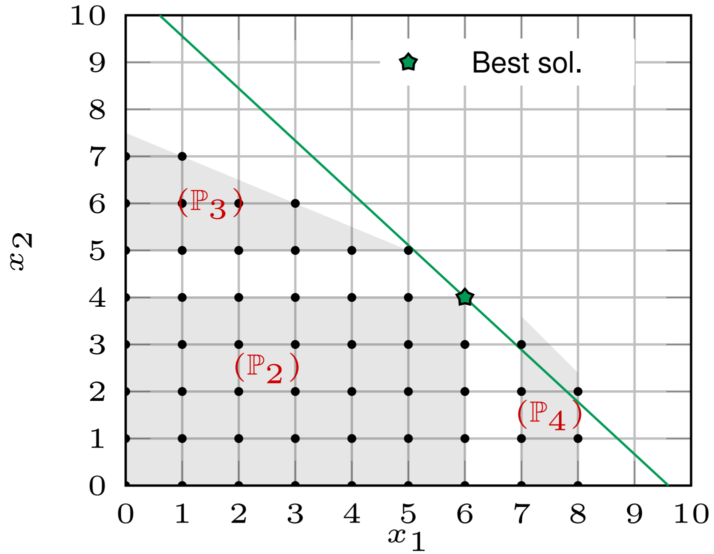
feasible sol. : update bound

$x_2 \geq \lceil 4.5 \rceil$

$(\mathbb{P}_3) = (\mathbb{P}_1) \cap (x_2 \geq 5)$
Open

$x_1 \geq \lceil 6.43 \rceil$

$(\mathbb{P}_4) = (\mathbb{P}_0) \cap (x_1 \geq 7)$
Open



Example 1

Best primal bound $z = 4800$

$x^* = (6, 4)$

$(\mathbb{P}_0)$
$x_1 \approx 6.43, x_2 \approx 4.29$
$LR(\mathbb{P}_0) \approx 5142.85$

$x_1 \leq \lfloor 6.43 \rfloor$

$(\mathbb{P}_1) = (\mathbb{P}_0) \cap (x_1 \leq 6)$
$x_1 = 6, x_2 = 4.5$
$LR(\mathbb{P}_1) = 5025$

$x_2 \leq \lfloor 4.5 \rfloor$

$(\mathbb{P}_2) = (\mathbb{P}_1) \cap (x_2 \leq 4)$
$x_1 = 6, x_2 = 4$
$LR(\mathbb{P}_2) = 4800$

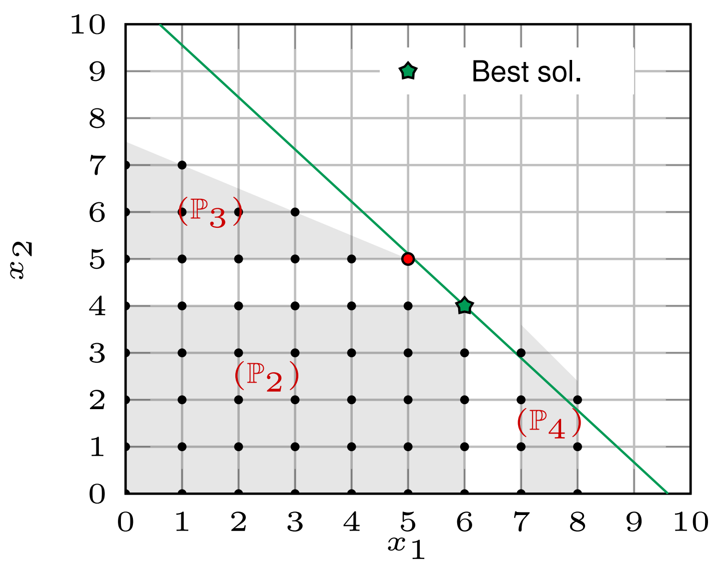
$x_2 \geq \lceil 4.5 \rceil$

$(\mathbb{P}_3) = (\mathbb{P}_1) \cap (x_2 \geq 5)$
$x_1 = 5, x_2 = 5$
$LR(\mathbb{P}_3) = 4750$

! primal bound is not updated

$x_1 \geq \lceil 6.43 \rceil$

$(\mathbb{P}_4) = (\mathbb{P}_0) \cap (x_1 \geq 7)$
Open



Example 1

Best primal bound $z = 4800$

$(\mathbb{P}_0)$

$x_1 \approx 6.43, x_2 \approx 4.29$

$LR(\mathbb{P}_0) \approx 5142.85$

$x_1 \leq \lfloor 6.43 \rfloor$

$(\mathbb{P}_1) = (\mathbb{P}_0) \cap (x_1 \leq 6)$

$x_1 = 6, x_2 = 4.5$

$LR(\mathbb{P}_1) = 5025$

$x_2 \leq \lfloor 4.5 \rfloor$

$(\mathbb{P}_2) = (\mathbb{P}_1) \cap (x_2 \leq 4)$

$x_1 = 6, x_2 = 4$

$LR(\mathbb{P}_2) = 4800$

$x_2 \geq \lceil 4.5 \rceil$

$(\mathbb{P}_3) = (\mathbb{P}_1) \cap (x_2 \geq 5)$

$x_1 = 5, x_2 = 5$

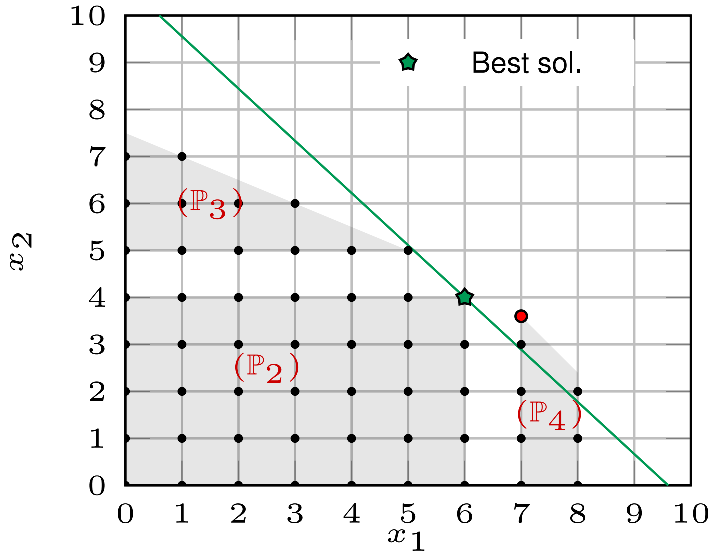
$LR(\mathbb{P}_3) = 4750$

$x_1 \geq \lceil 6.43 \rceil$

$(\mathbb{P}_4) = (\mathbb{P}_0) \cap (x_1 \geq 7)$

$x_1 = 7, x_2 = 3.6$

$LR(\mathbb{P}_4) = 5120$



Example 1

Best primal bound $z = 4800$

$(\mathbb{P}_0)$
$x_1 \approx 6.43, x_2 \approx 4.29$
$LR(\mathbb{P}_0) \approx 5142.85$

$x_1 \leq \lfloor 6.43 \rfloor$

$(\mathbb{P}_1) = (\mathbb{P}_0) \cap (x_1 \leq 6)$
$x_1 = 6, x_2 = 4.5$
$LR(\mathbb{P}_1) = 5025$

$x_2 \leq \lfloor 4.5 \rfloor$

$(\mathbb{P}_2) = (\mathbb{P}_1) \cap (x_2 \leq 4)$
$x_1 = 6, x_2 = 4$
$LR(\mathbb{P}_2) = 4800$

$x_2 \geq \lceil 4.5 \rceil$

$(\mathbb{P}_3) = (\mathbb{P}_1) \cap (x_2 \geq 5)$
$x_1 = 5, x_2 = 5$
$LR(\mathbb{P}_3) = 4750$

$x_1 \geq \lceil 6.43 \rceil$

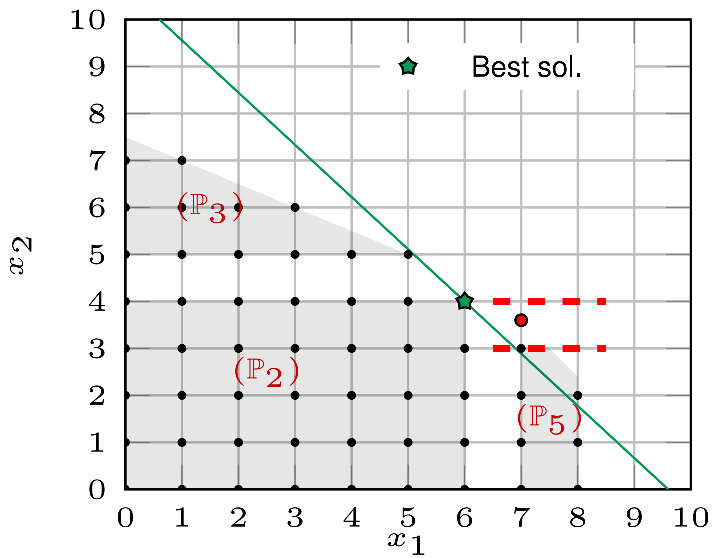
$(\mathbb{P}_4) = (\mathbb{P}_0) \cap (x_1 \geq 7)$
$x_1 = 7, x_2 = 3.6$
$LR(\mathbb{P}_4) = 5120$

$x_2 \leq \lfloor 3.6 \rfloor$

$(\mathbb{P}_5) = (\mathbb{P}_4) \cap (x_2 \leq 3)$
Open

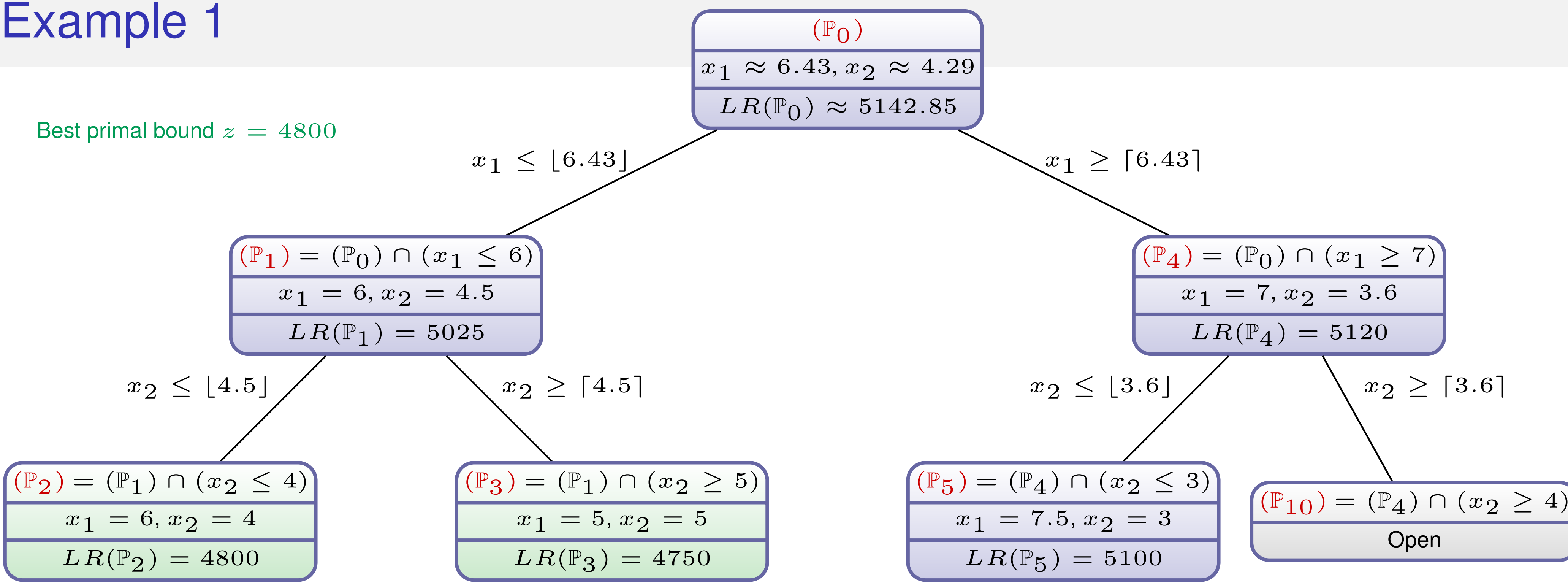
$x_2 \geq \lceil 3.6 \rceil$

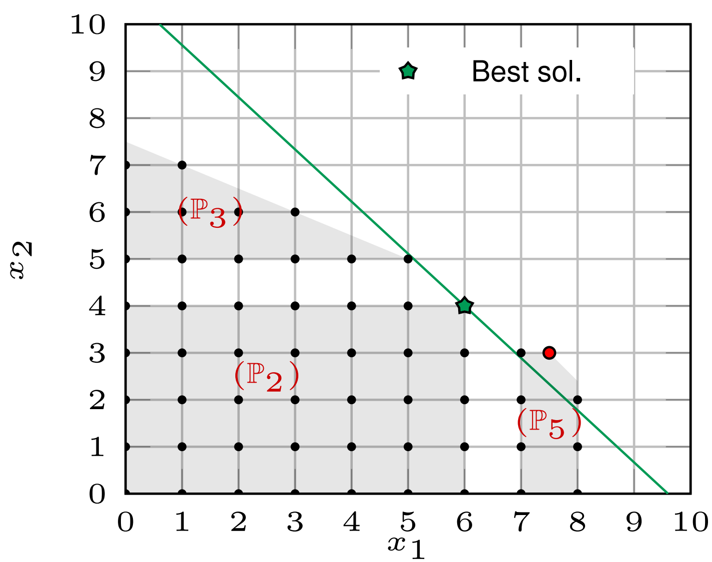
$(\mathbb{P}_{10}) = (\mathbb{P}_4) \cap (x_2 \geq 4)$
Open



Example 1

Best primal bound $z = 4800$





Example 1

Best primal bound $z = 4800$

$(\mathbb{P}_0)$
$x_1 \approx 6.43, x_2 \approx 4.29$
$LR(\mathbb{P}_0) \approx 5142.85$

$x_1 \leq \lfloor 6.43 \rfloor$

$(\mathbb{P}_1) = (\mathbb{P}_0) \cap (x_1 \leq 6)$
$x_1 = 6, x_2 = 4.5$
$LR(\mathbb{P}_1) = 5025$

$x_2 \leq \lfloor 4.5 \rfloor$

$(\mathbb{P}_2) = (\mathbb{P}_1) \cap (x_2 \leq 4)$
$x_1 = 6, x_2 = 4$
$LR(\mathbb{P}_2) = 4800$

$x_2 \geq \lceil 4.5 \rceil$

$(\mathbb{P}_3) = (\mathbb{P}_1) \cap (x_2 \geq 5)$
$x_1 = 5, x_2 = 5$
$LR(\mathbb{P}_3) = 4750$

$x_1 \geq \lceil 6.43 \rceil$

$(\mathbb{P}_4) = (\mathbb{P}_0) \cap (x_1 \geq 7)$
$x_1 = 7, x_2 = 3.6$
$LR(\mathbb{P}_4) = 5120$

$x_2 \leq \lfloor 3.6 \rfloor$

$(\mathbb{P}_5) = (\mathbb{P}_4) \cap (x_2 \leq 3)$
$x_1 = 7.5, x_2 = 3$
$LR(\mathbb{P}_5) = 5100$

$x_1 \leq \lfloor 7.5 \rfloor$

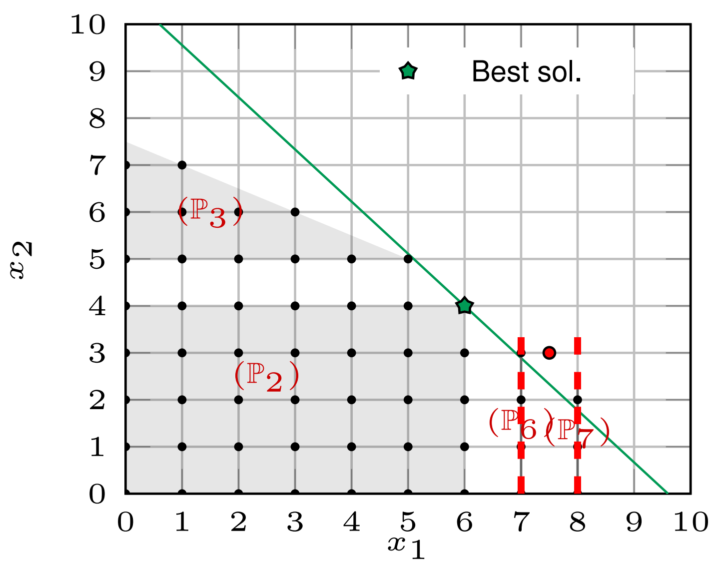
$(\mathbb{P}_6) = (\mathbb{P}_5) \cap (x_1 \leq 7)$
Open

$x_1 \geq \lceil 7.5 \rceil$

$(\mathbb{P}_7) = (\mathbb{P}_5) \cap (x_1 \geq 8)$
Open

$x_2 \geq \lceil 3.6 \rceil$

$(\mathbb{P}_{10}) = (\mathbb{P}_4) \cap (x_2 \geq 4)$
Open



Example 1

Best primal bound $z = 4850$

$x^* = (7,3)$

$(\mathbb{P}_0)$: $x_1 \approx 6.43,\ x_2 \approx 4.29$; $LR(\mathbb{P}_0) \approx 5142.85$

$x_1 \le \lfloor 6.43 \rfloor$ — $x_1 \ge \lceil 6.43 \rceil$

$(\mathbb{P}_1) = (\mathbb{P}_0) \cap (x_1 \le 6)$: $x_1 = 6,\ x_2 = 4.5$; $LR(\mathbb{P}_1) = 5025$

$(\mathbb{P}_4) = (\mathbb{P}_0) \cap (x_1 \ge 7)$: $x_1 = 7,\ x_2 = 3.6$; $LR(\mathbb{P}_4) = 5120$

$x_2 \le \lfloor 4.5 \rfloor$ — $x_2 \ge \lceil 4.5 \rceil$

$(\mathbb{P}_2) = (\mathbb{P}_1) \cap (x_2 \le 4)$: $x_1 = 6,\ x_2 = 4$; $LR(\mathbb{P}_2) = 4800$

$(\mathbb{P}_3) = (\mathbb{P}_1) \cap (x_2 \ge 5)$: $x_1 = 5,\ x_2 = 5$; $LR(\mathbb{P}_3) = 4750$

$x_2 \le \lfloor 3.6 \rfloor$ — $x_2 \ge \lceil 3.6 \rceil$

$(\mathbb{P}_5) = (\mathbb{P}_4) \cap (x_2 \le 3)$: $x_1 = 7.5,\ x_2 = 3$; $LR(\mathbb{P}_5) = 5100$

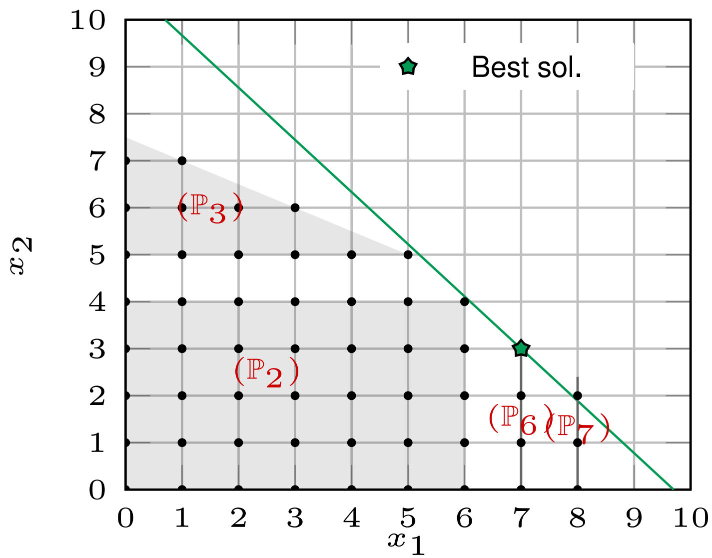
$(\mathbb{P}_{10}) = (\mathbb{P}_4) \cap (x_2 \ge 4)$: Open

$x_1 \le \lfloor 7.5 \rfloor$ — $x_1 \ge \lceil 7.5 \rceil$

$(\mathbb{P}_6) = (\mathbb{P}_5) \cap (x_1 \le 7)$: $x_1 = 7,\ x_2 = 3$; $LR(\mathbb{P}_6) = 4850$

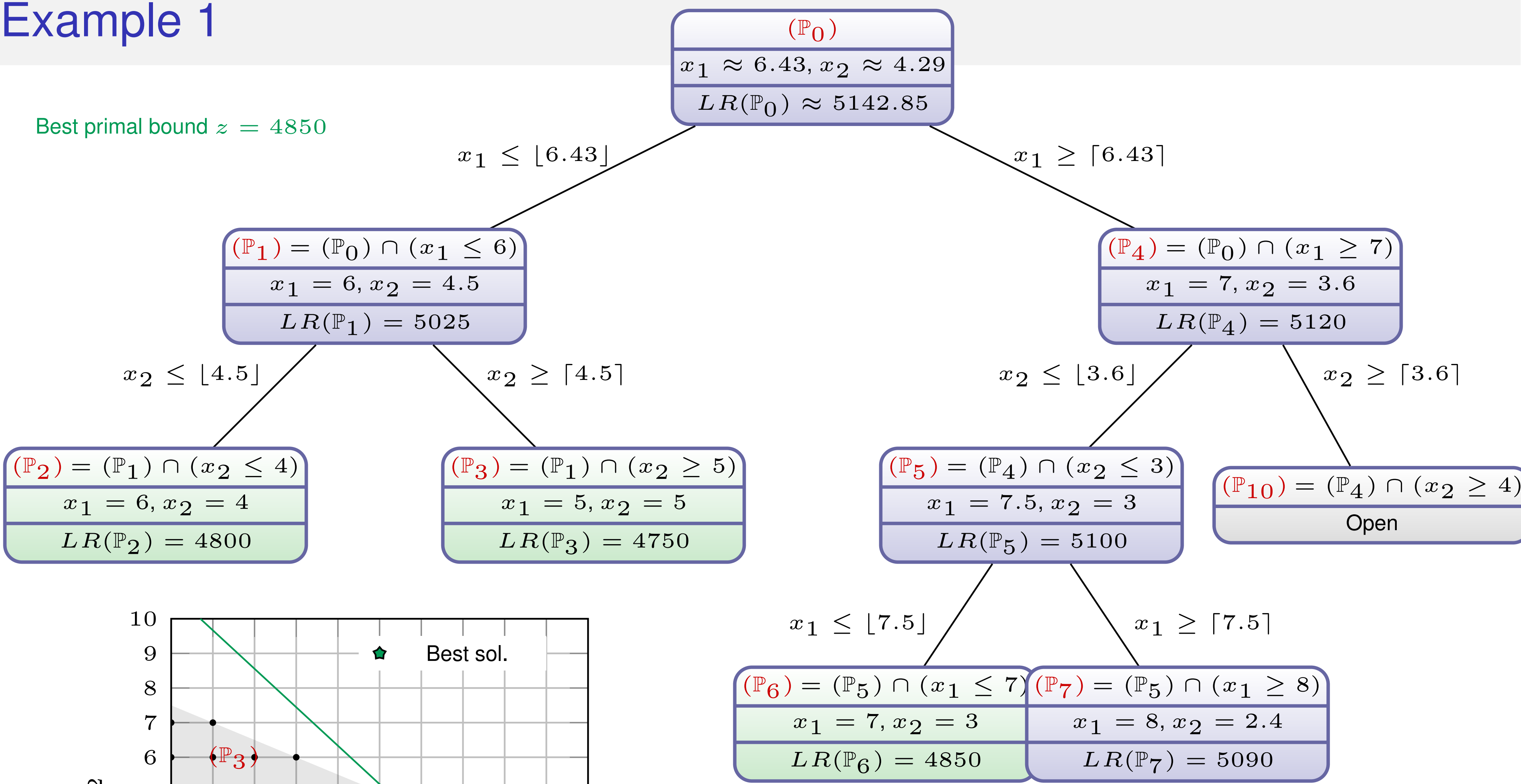
improving feasible sol.

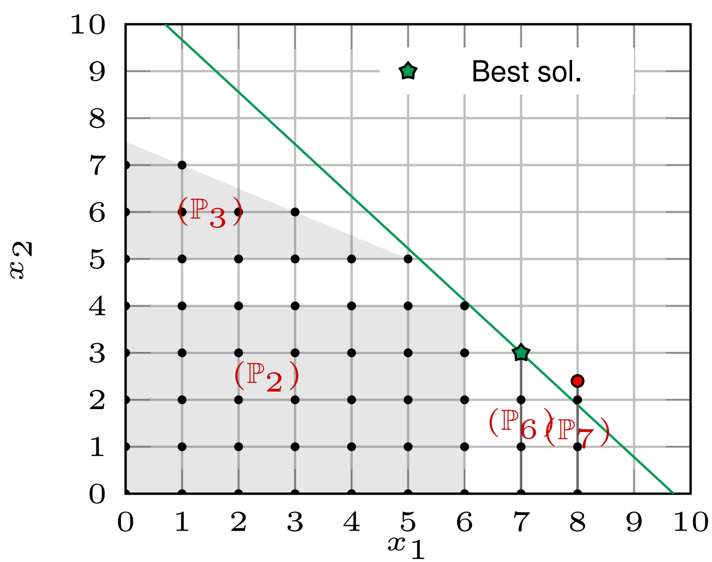
$(\mathbb{P}_7) = (\mathbb{P}_5) \cap (x_1 \ge 8)$: Open



Example 1

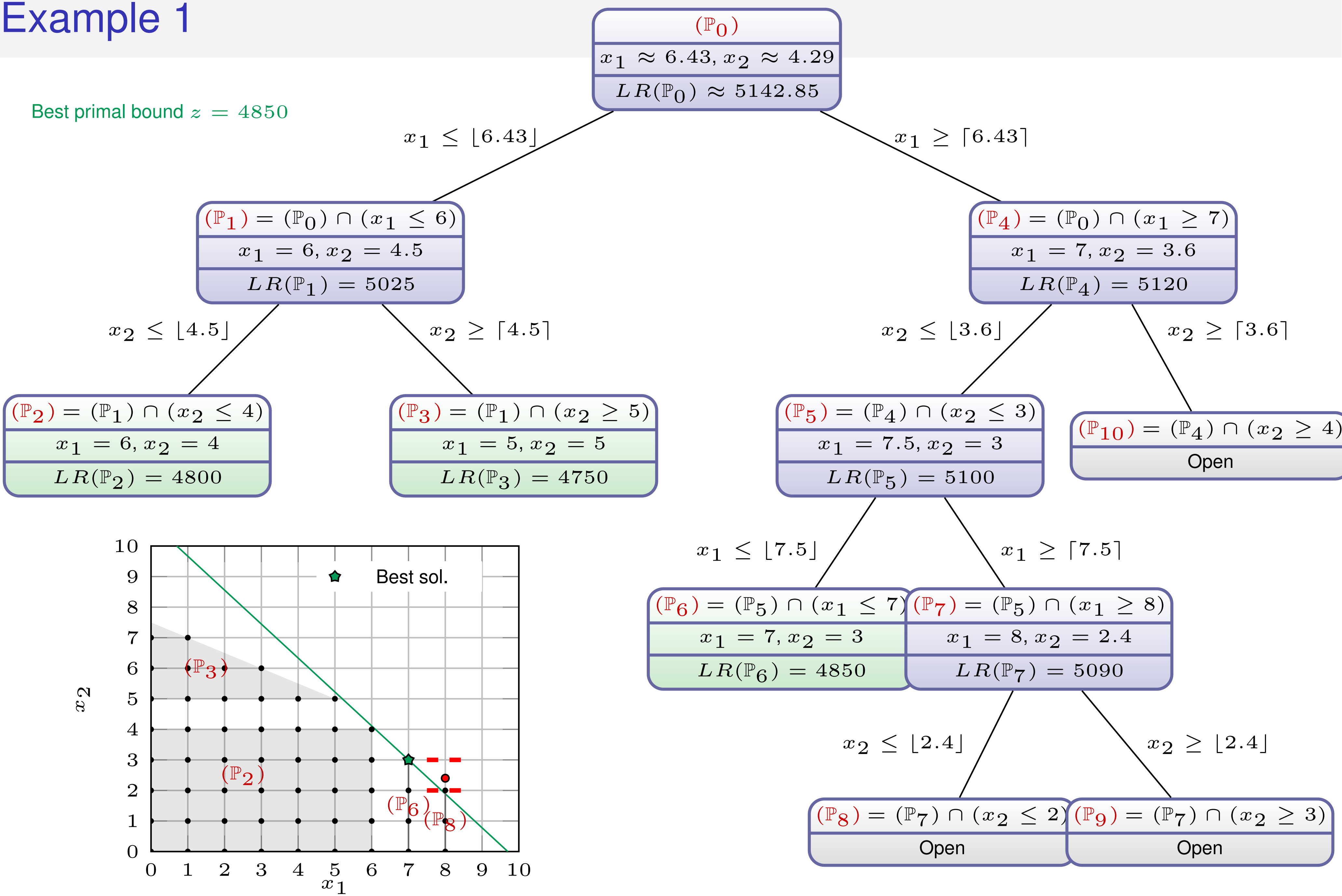
Best primal bound $z = 4850$





Example 1

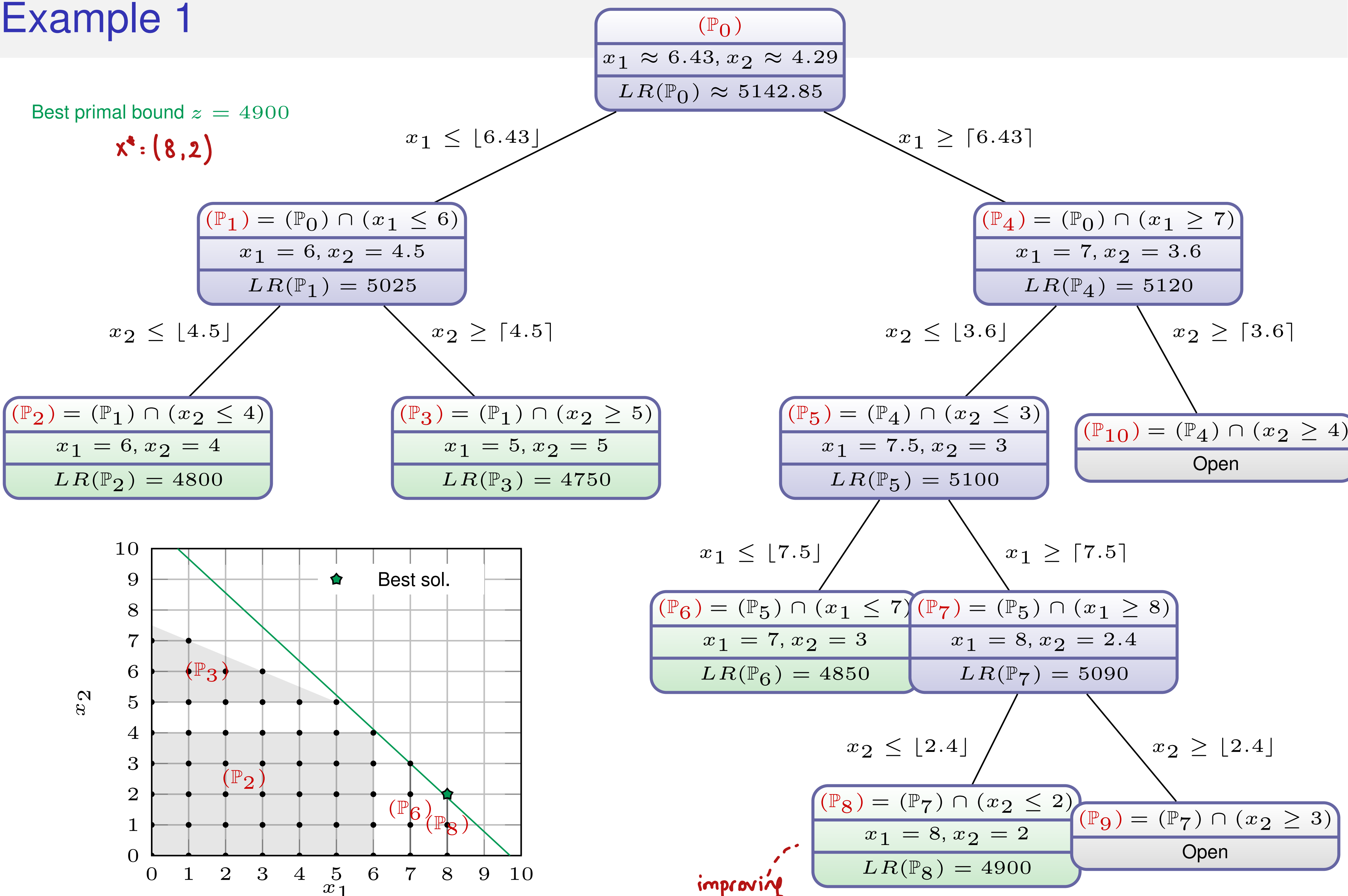
Best primal bound $z = 4850$



Example 1

Best primal bound $z = 4900$

$x^*: (8,2)$



improving feasible sol.

Example 1

Best primal bound $z = 4900$

$x^*: (8,2)$

$(\mathbb{P}_0)$: $x_1 \approx 6.43, x_2 \approx 4.29$; $LR(\mathbb{P}_0) \approx 5142.85$

$x_1 \leq \lfloor 6.43 \rfloor$ — $x_1 \geq \lceil 6.43 \rceil$

$(\mathbb{P}_1) = (\mathbb{P}_0) \cap (x_1 \leq 6)$: $x_1 = 6, x_2 = 4.5$; $LR(\mathbb{P}_1) = 5025$

$(\mathbb{P}_4) = (\mathbb{P}_0) \cap (x_1 \geq 7)$: $x_1 = 7, x_2 = 3.6$; $LR(\mathbb{P}_4) = 5120$

$x_2 \leq \lfloor 4.5 \rfloor$ — $x_2 \geq \lceil 4.5 \rceil$

$(\mathbb{P}_2) = (\mathbb{P}_1) \cap (x_2 \leq 4)$: $x_1 = 6, x_2 = 4$; $LR(\mathbb{P}_2) = 4800$

$(\mathbb{P}_3) = (\mathbb{P}_1) \cap (x_2 \geq 5)$: $x_1 = 5, x_2 = 5$; $LR(\mathbb{P}_3) = 4750$

$x_2 \leq \lfloor 3.6 \rfloor$ — $x_2 \geq \lceil 3.6 \rceil$

$(\mathbb{P}_5) = (\mathbb{P}_4) \cap (x_2 \leq 3)$: $x_1 = 7.5, x_2 = 3$; $LR(\mathbb{P}_5) = 5100$

$(\mathbb{P}_{10}) = (\mathbb{P}_4) \cap (x_2 \geq 4)$: Open

$x_1 \leq \lfloor 7.5 \rfloor$ — $x_1 \geq \lceil 7.5 \rceil$

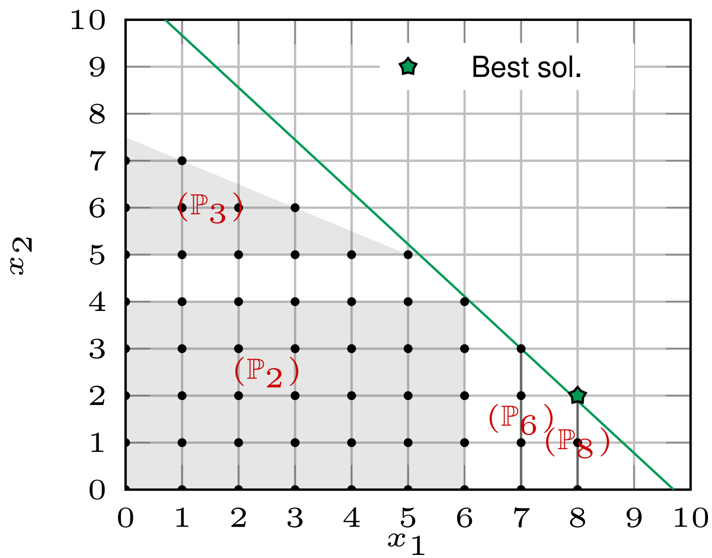
$(\mathbb{P}_6) = (\mathbb{P}_5) \cap (x_1 \leq 7)$: $x_1 = 7, x_2 = 3$; $LR(\mathbb{P}_6) = 4850$

$(\mathbb{P}_7) = (\mathbb{P}_5) \cap (x_1 \geq 8)$: $x_1 = 8, x_2 = 2.4$; $LR(\mathbb{P}_7) = 5090$

$x_2 \leq \lfloor 2.4 \rfloor$ — $x_2 \geq \lfloor 2.4 \rfloor$

$(\mathbb{P}_8) = (\mathbb{P}_7) \cap (x_2 \leq 2)$: $x_1 = 8, x_2 = 2$; $LR(\mathbb{P}_8) = 4900$

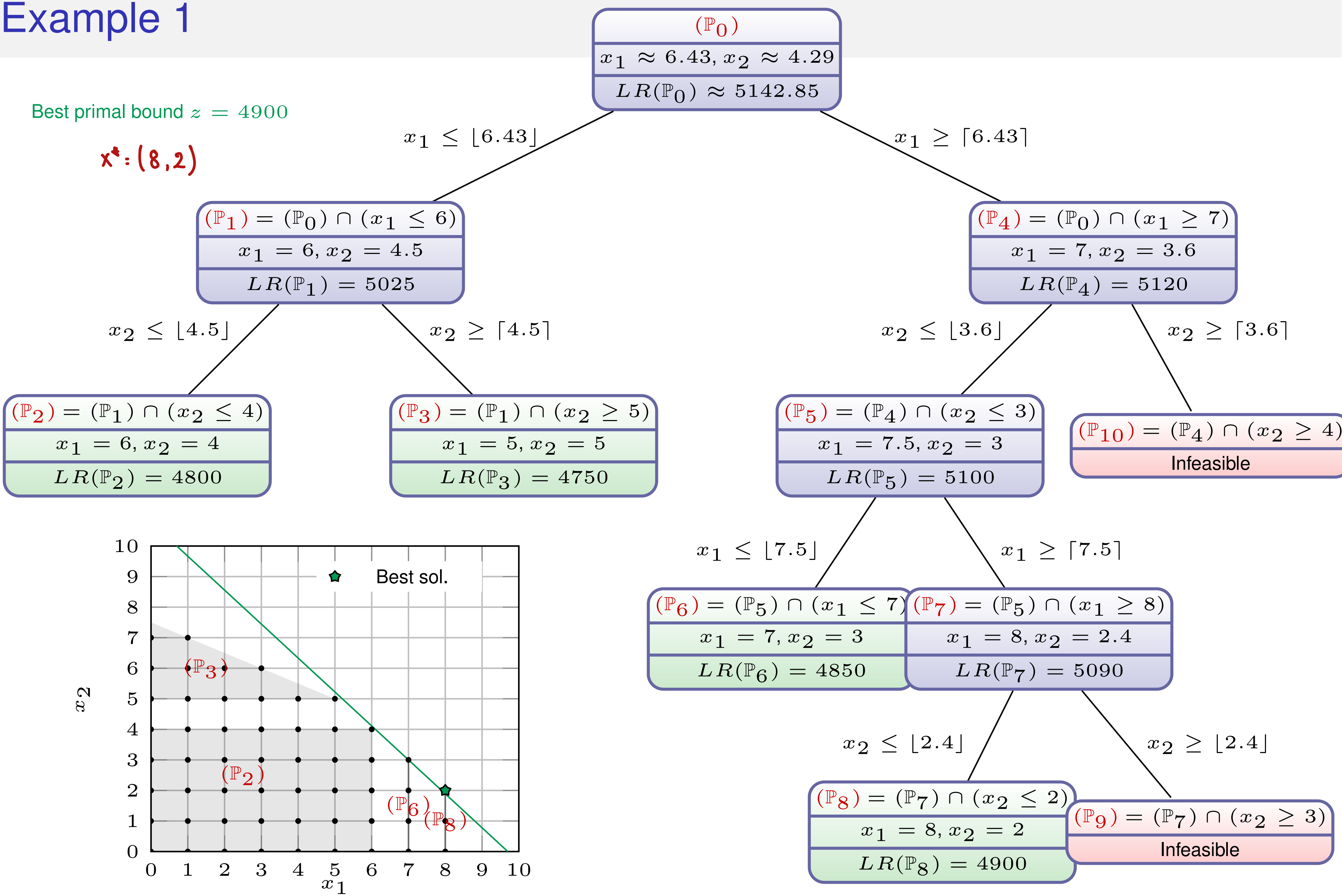
$(\mathbb{P}_9) = (\mathbb{P}_7) \cap (x_2 \geq 3)$: Infeasible



Example 1

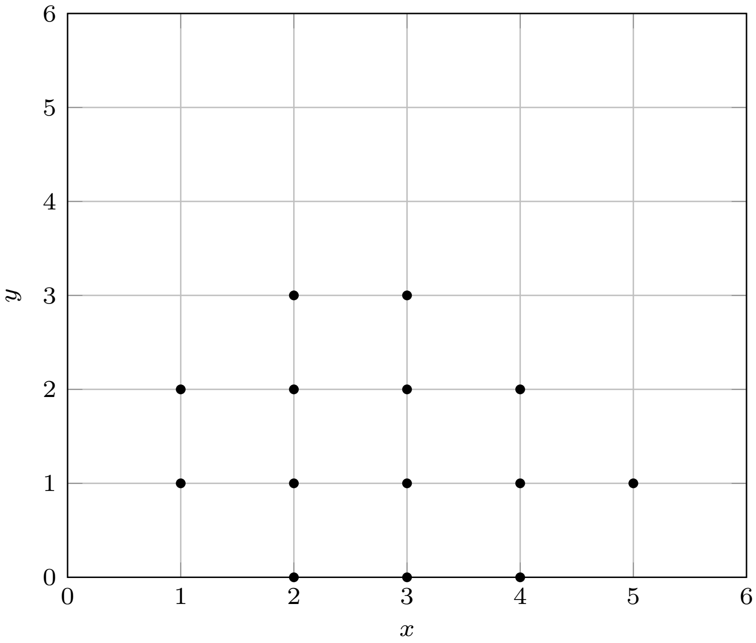
Best primal bound $z = 4900$

$x^*: (8,2)$



Example 2

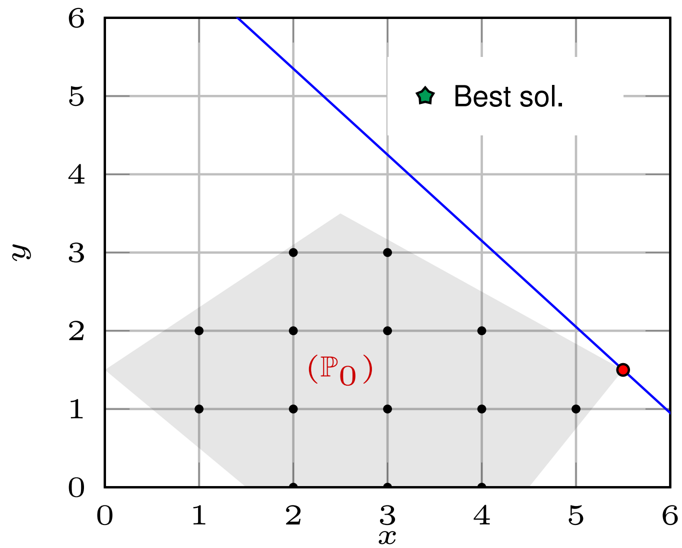
$$
\begin{array}{rrcrcl}
\text{max} & 11x & + & 10y & & \\
\text{s.c.} & 2x & + & 2y & \geq & 3 \\
 & -8x & + & 10y & \leq & 15 \\
 & 4x & + & 6y & \leq & 31 \\
 & -6x & + & 4y & \geq & -27 \\
 & & & x, y & \in & \mathbb{Z}_+
\end{array}
$$



Example 2

$(\mathbb{P}_0)$
$x = 5.5, y = 1.5$
$LR(\mathbb{P}_0) = 75.5$

Best primal bound $z = -\infty$



Example 2

Best primal bound $z = -\infty$

$(\mathbb{P}_0)$

$x = 5.5, y = 1.5$

$LR(\mathbb{P}_0) = 75.5$

$y \leq \lfloor 1.5 \rfloor$

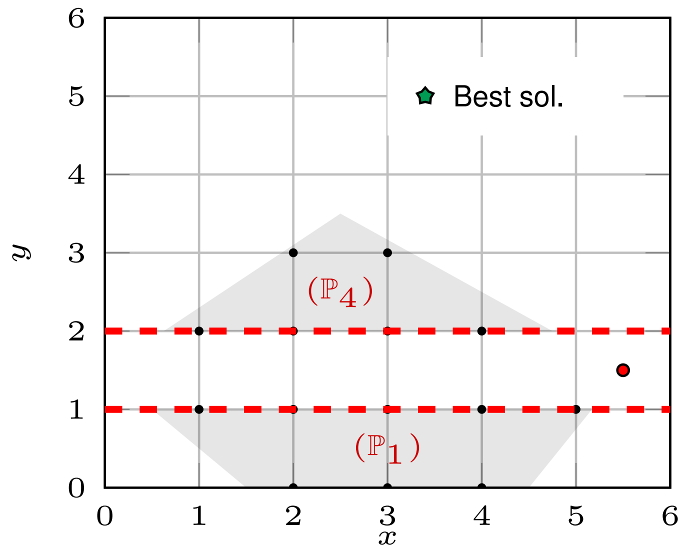
$y \geq \lceil 1.5 \rceil$

$(\mathbb{P}_1) = (\mathbb{P}_0) \cap (y \leq 1)$

Open

$(\mathbb{P}_4) = (\mathbb{P}_0) \cap (y \geq 2)$

Open



Example 2

Best primal bound $z = -\infty$

$(\mathbb{P}_0)$

$x = 5.5, y = 1.5$

$LR(\mathbb{P}_0) = 75.5$

$y \leq \lfloor 1.5 \rfloor$

$y \geq \lceil 1.5 \rceil$

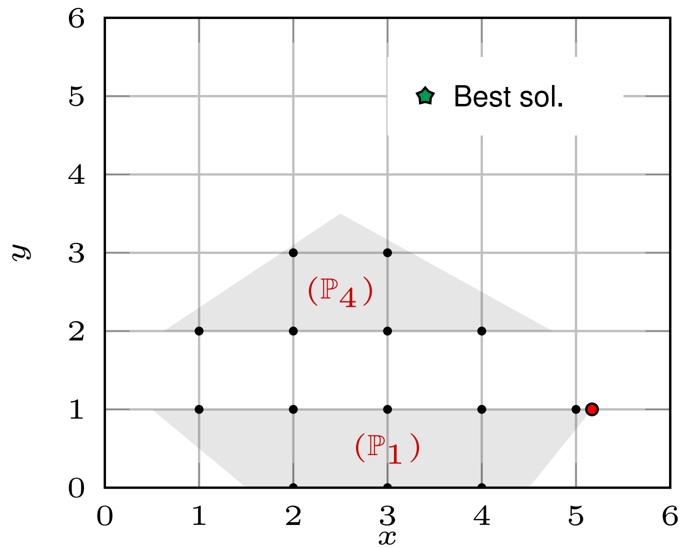
$(\mathbb{P}_1) = (\mathbb{P}_0) \cap (y \leq 1)$

$x \approx 5.17, y = 1$

$LR(\mathbb{P}_1) \approx 66.84$

$(\mathbb{P}_4) = (\mathbb{P}_0) \cap (y \geq 2)$

Open



Example 2

Best primal bound $z = -\infty$

$(\mathbb{P}_0)$
$x = 5.5, y = 1.5$
$LR(\mathbb{P}_0) = 75.5$

$y \leq \lfloor 1.5 \rfloor$

$(\mathbb{P}_1) = (\mathbb{P}_0) \cap (y \leq 1)$
$x \approx 5.17, y = 1$
$LR(\mathbb{P}_1) \approx 66.84$

$y \geq \lceil 1.5 \rceil$

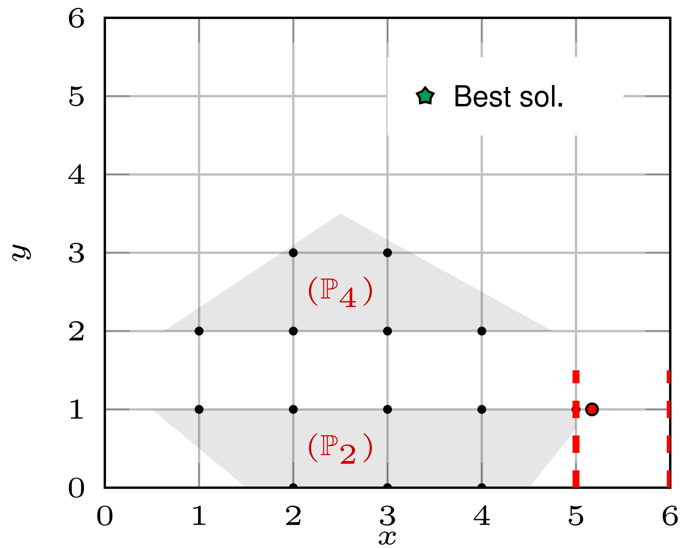
$(\mathbb{P}_4) = (\mathbb{P}_0) \cap (y \geq 2)$
Open

$x \leq \lfloor 5.17 \rfloor$

$(\mathbb{P}_2) = (\mathbb{P}_1) \cap (x \leq 5)$
Open

$x \geq \lceil 5.17 \rceil$

$(\mathbb{P}_3) = (\mathbb{P}_1) \cap (x \geq 6)$
Open



Example 2

Best primal bound $z = 65$

$x^* = (5, 1)$

$(\mathbb{P}_0)$
$x = 5.5,\ y = 1.5$
$LR(\mathbb{P}_0) = 75.5$

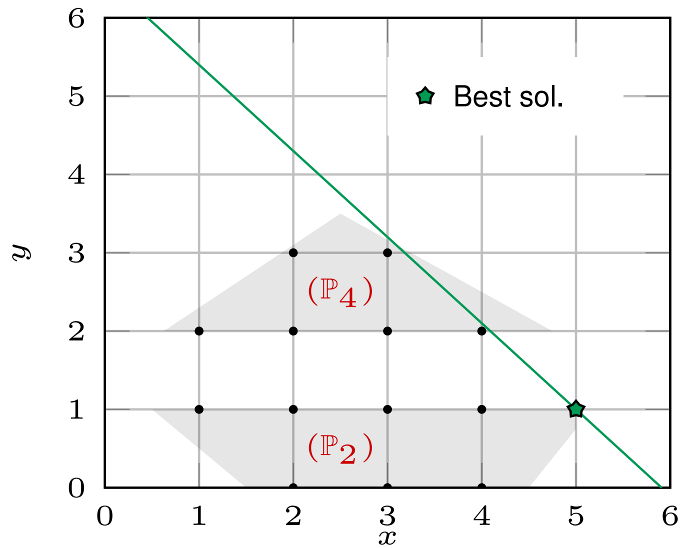
$y \leq \lfloor 1.5 \rfloor$: **$(\mathbb{P}_1) = (\mathbb{P}_0) \cap (y \leq 1)$**
$x \approx 5.17,\ y = 1$
$LR(\mathbb{P}_1) \approx 66.84$

$x \leq \lfloor 5.17 \rfloor$: **$(\mathbb{P}_2) = (\mathbb{P}_1) \cap (x \leq 5)$**
$x = 5,\ y = 1$
$LR(\mathbb{P}_2) = 65$

feasible sol. : update bound

$x \geq \lceil 5.17 \rceil$: **$(\mathbb{P}_3) = (\mathbb{P}_1) \cap (x \geq 6)$**
Open

$y \geq \lceil 1.5 \rceil$: **$(\mathbb{P}_4) = (\mathbb{P}_0) \cap (y \geq 2)$**
Open



Example 2

Best primal bound $z = 65$

$(\mathbb{P}_0)$
$x = 5.5, y = 1.5$
$LR(\mathbb{P}_0) = 75.5$

$y \leq \lfloor 1.5 \rfloor$

$(\mathbb{P}_1) = (\mathbb{P}_0) \cap (y \leq 1)$
$x \approx 5.17, y = 1$
$LR(\mathbb{P}_1) \approx 66.84$

$x \leq \lfloor 5.17 \rfloor$

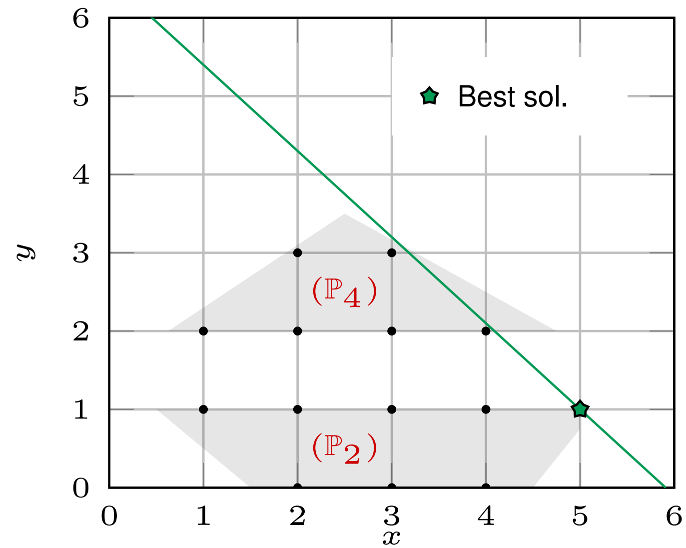
$(\mathbb{P}_2) = (\mathbb{P}_1) \cap (x \leq 5)$
$x = 5, y = 1$
$LR(\mathbb{P}_2) = 65$

$x \geq \lceil 5.17 \rceil$

$(\mathbb{P}_3) = (\mathbb{P}_1) \cap (x \geq 6)$
Infeasible

$y \geq \lceil 1.5 \rceil$

$(\mathbb{P}_4) = (\mathbb{P}_0) \cap (y \geq 2)$
Open



Example 2

Best primal bound $z = 65$

$(\mathbb{P}_0)$
$x = 5.5, y = 1.5$
$LR(\mathbb{P}_0) = 75.5$

$y \leq \lfloor 1.5 \rfloor$

$(\mathbb{P}_1) = (\mathbb{P}_0) \cap (y \leq 1)$
$x \approx 5.17, y = 1$
$LR(\mathbb{P}_1) \approx 66.84$

$x \leq \lfloor 5.17 \rfloor$

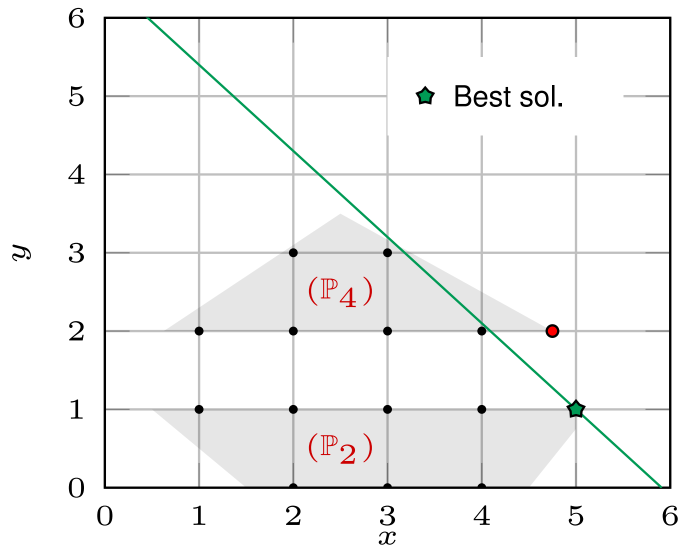
$(\mathbb{P}_2) = (\mathbb{P}_1) \cap (x \leq 5)$
$x = 5, y = 1$
$LR(\mathbb{P}_2) = 65$

$x \geq \lceil 5.17 \rceil$

$(\mathbb{P}_3) = (\mathbb{P}_1) \cap (x \geq 6)$
Infeasible

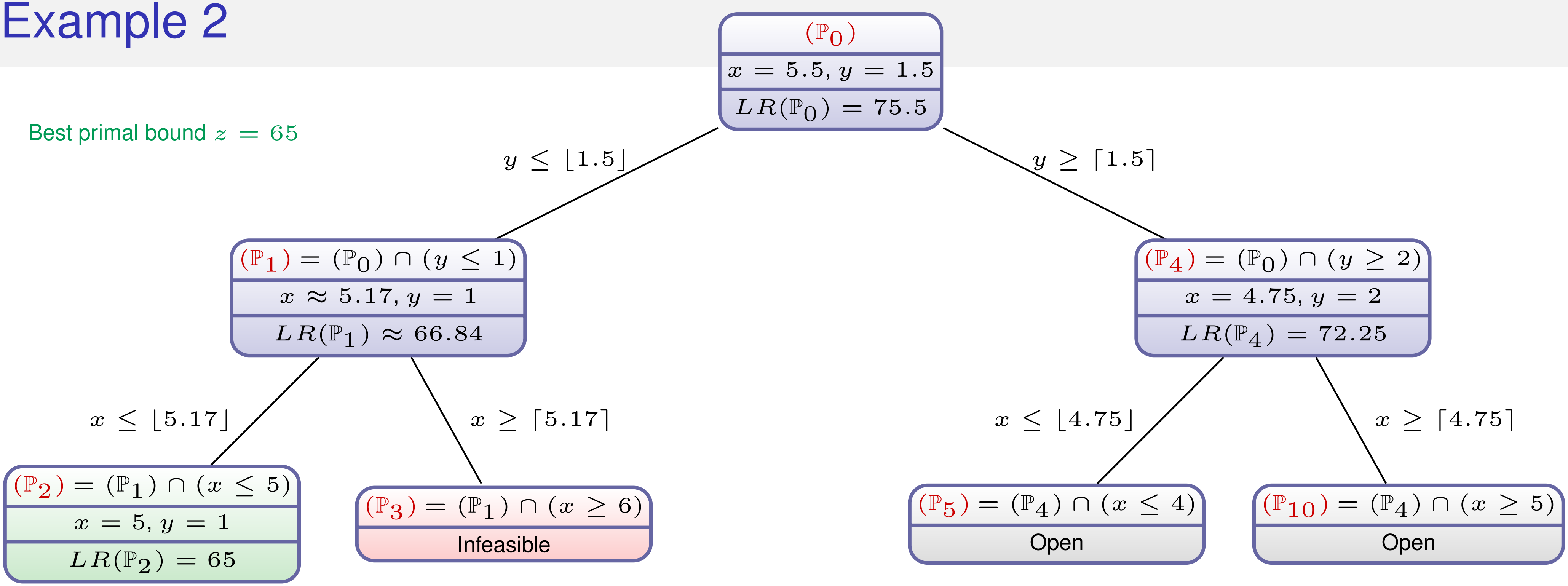
$y \geq \lceil 1.5 \rceil$

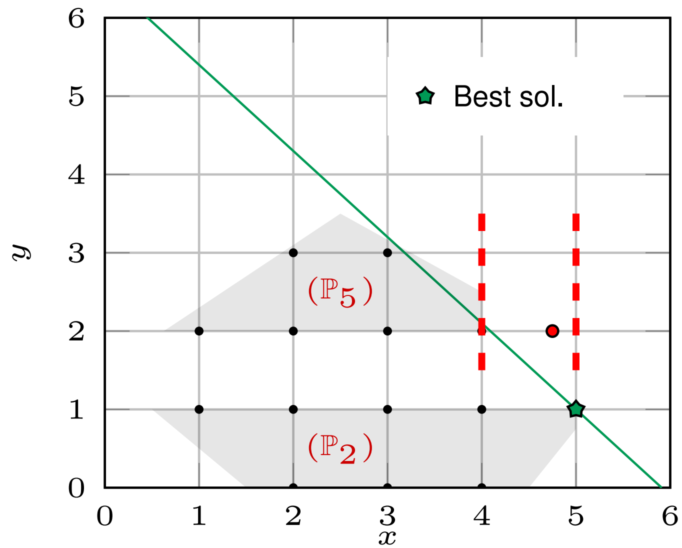
$(\mathbb{P}_4) = (\mathbb{P}_0) \cap (y \geq 2)$
$x = 4.75, y = 2$
$LR(\mathbb{P}_4) = 72.25$



Example 2

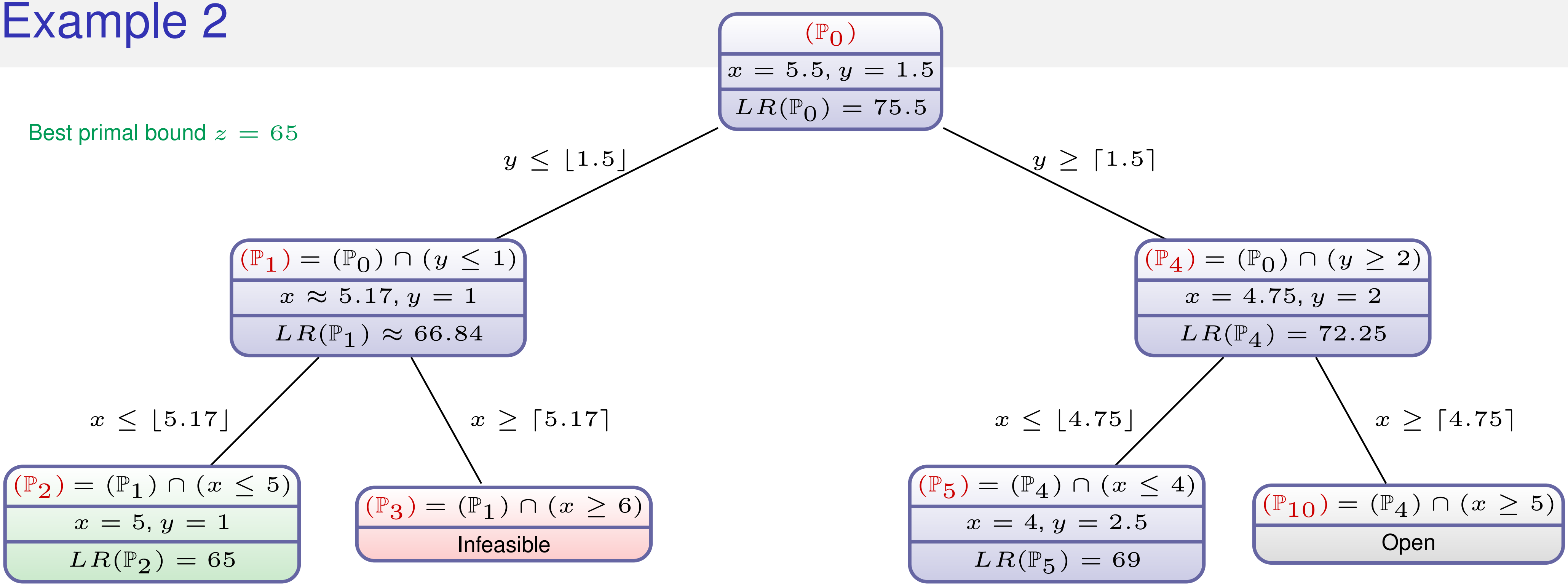
Best primal bound $z = 65$

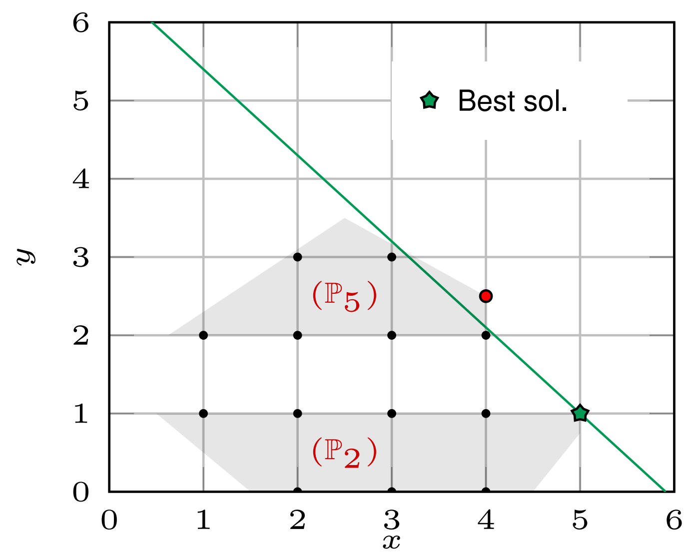




Example 2

Best primal bound $z = 65$





Example 2

Best primal bound $z = 65$

$(\mathbb{P}_0)$
$x = 5.5, y = 1.5$
$LR(\mathbb{P}_0) = 75.5$

$y \leq \lfloor 1.5 \rfloor$

$(\mathbb{P}_1) = (\mathbb{P}_0) \cap (y \leq 1)$
$x \approx 5.17, y = 1$
$LR(\mathbb{P}_1) \approx 66.84$

$x \leq \lfloor 5.17 \rfloor$

$(\mathbb{P}_2) = (\mathbb{P}_1) \cap (x \leq 5)$
$x = 5, y = 1$
$LR(\mathbb{P}_2) = 65$

$x \geq \lceil 5.17 \rceil$

$(\mathbb{P}_3) = (\mathbb{P}_1) \cap (x \geq 6)$
Infeasible

$y \geq \lceil 1.5 \rceil$

$(\mathbb{P}_4) = (\mathbb{P}_0) \cap (y \geq 2)$
$x = 4.75, y = 2$
$LR(\mathbb{P}_4) = 72.25$

$x \leq \lfloor 4.75 \rfloor$

$(\mathbb{P}_5) = (\mathbb{P}_4) \cap (x \leq 4)$
$x = 4, y = 2.5$
$LR(\mathbb{P}_5) = 69$

$y \leq \lfloor 2.5 \rfloor$

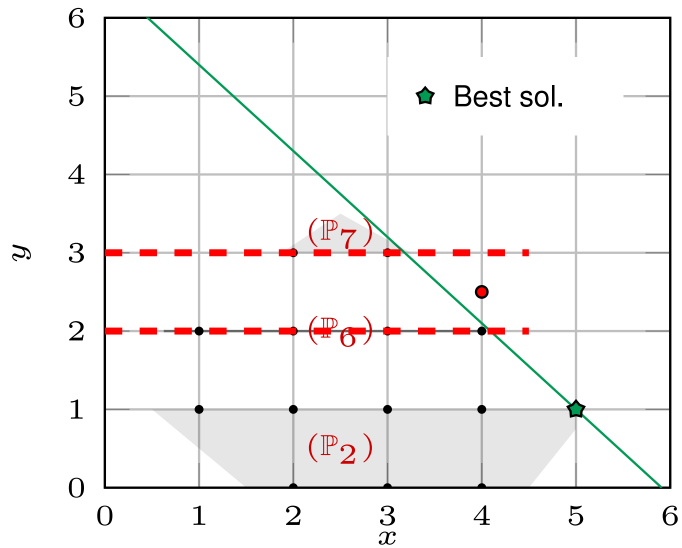
$(\mathbb{P}_6) = (\mathbb{P}_5) \cap (y \leq 2)$
Open

$y \geq \lceil 2.5 \rceil$

$(\mathbb{P}_7) = (\mathbb{P}_5) \cap (y \geq 3)$
Open

$x \geq \lceil 4.75 \rceil$

$(\mathbb{P}_{10}) = (\mathbb{P}_4) \cap (x \geq 5)$
Open



Example 2

Best primal bound $z = 65$

$(\mathbb{P}_0)$
$x = 5.5, y = 1.5$
$LR(\mathbb{P}_0) = 75.5$

$y \le \lfloor 1.5 \rfloor$ — $y \ge \lceil 1.5 \rceil$

$(\mathbb{P}_1) = (\mathbb{P}_0) \cap (y \le 1)$
$x \approx 5.17, y = 1$
$LR(\mathbb{P}_1) \approx 66.84$

$(\mathbb{P}_4) = (\mathbb{P}_0) \cap (y \ge 2)$
$x = 4.75, y = 2$
$LR(\mathbb{P}_4) = 72.25$

$x \le \lfloor 5.17 \rfloor$ — $x \ge \lceil 5.17 \rceil$

$(\mathbb{P}_2) = (\mathbb{P}_1) \cap (x \le 5)$
$x = 5, y = 1$
$LR(\mathbb{P}_2) = 65$

$(\mathbb{P}_3) = (\mathbb{P}_1) \cap (x \ge 6)$
Infeasible

$x \le \lfloor 4.75 \rfloor$ — $x \ge \lceil 4.75 \rceil$

$(\mathbb{P}_5) = (\mathbb{P}_4) \cap (x \le 4)$
$x = 4, y = 2.5$
$LR(\mathbb{P}_5) = 69$

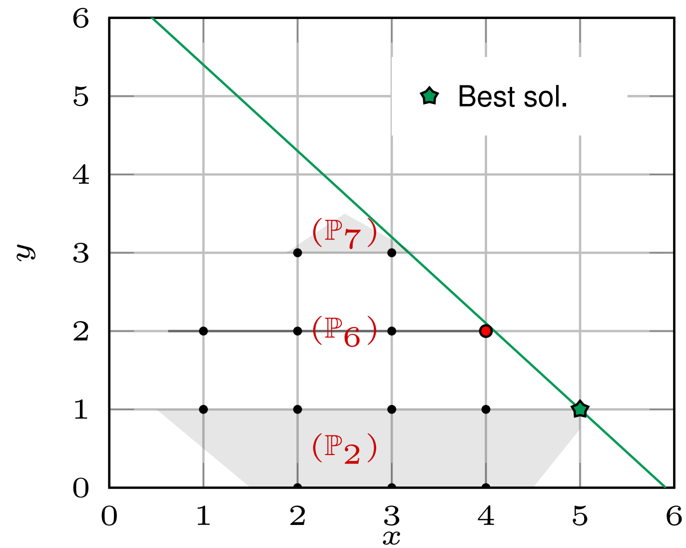
$(\mathbb{P}_{10}) = (\mathbb{P}_4) \cap (x \ge 5)$
Open

$y \le \lfloor 2.5 \rfloor$ — $y \ge \lceil 2.5 \rceil$

$(\mathbb{P}_6) = (\mathbb{P}_5) \cap (y \le 2)$
$x = 4, y = 2$
$LR(\mathbb{P}_6) = 64$

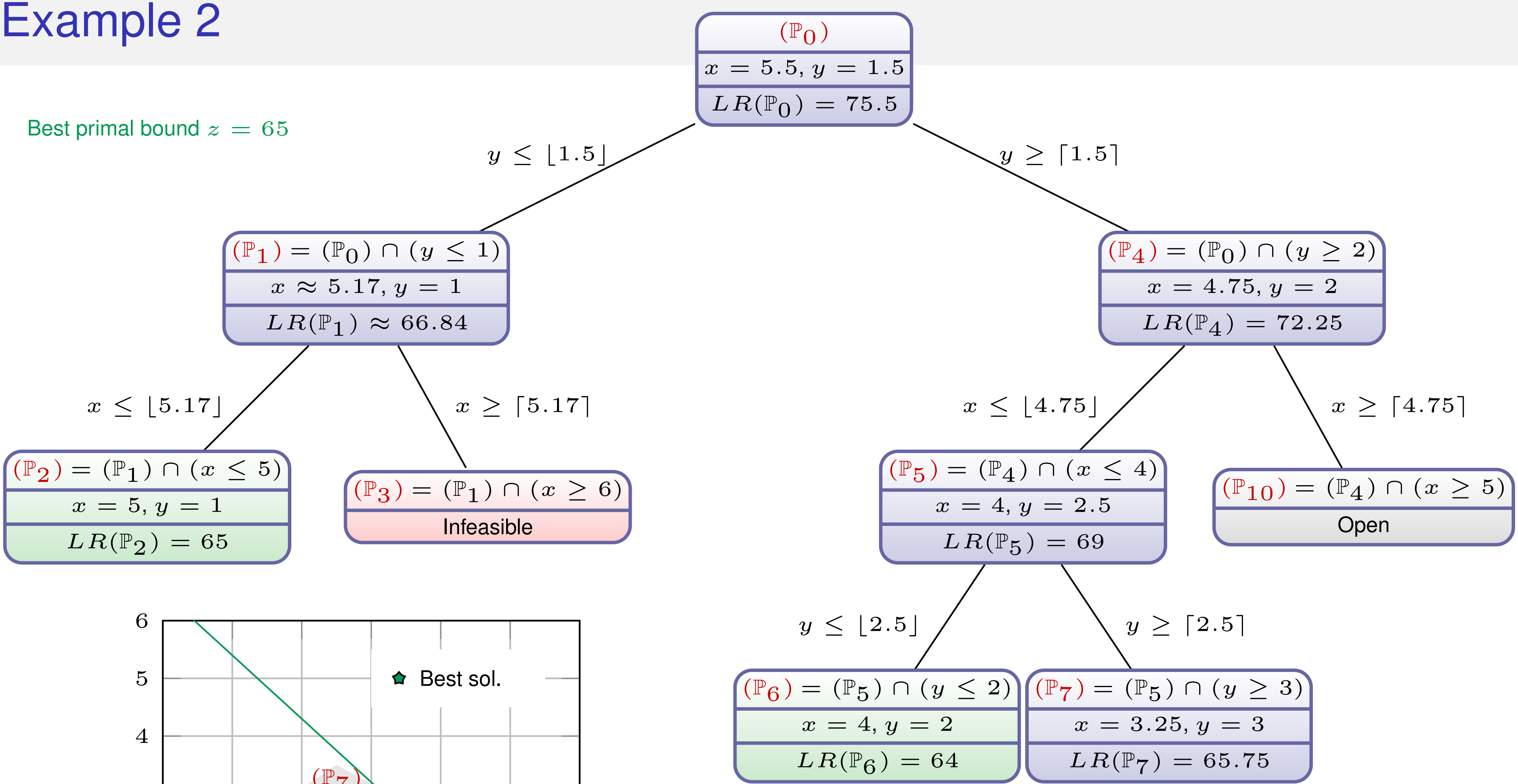
no bound update

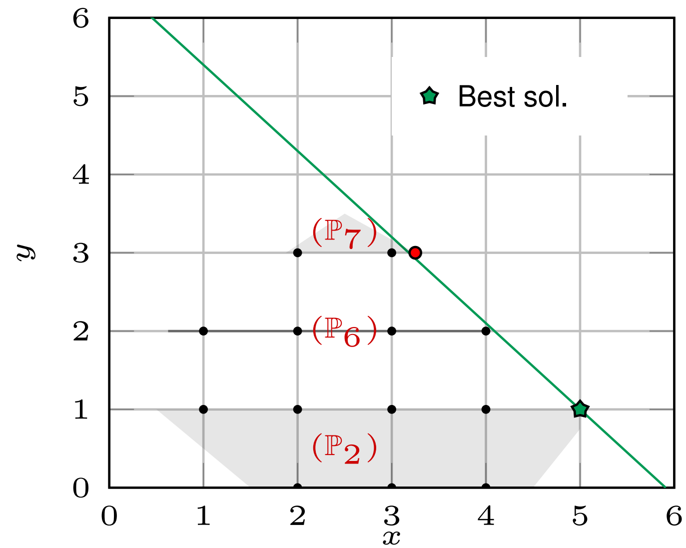
$(\mathbb{P}_7) = (\mathbb{P}_5) \cap (y \ge 3)$
Open



Example 2

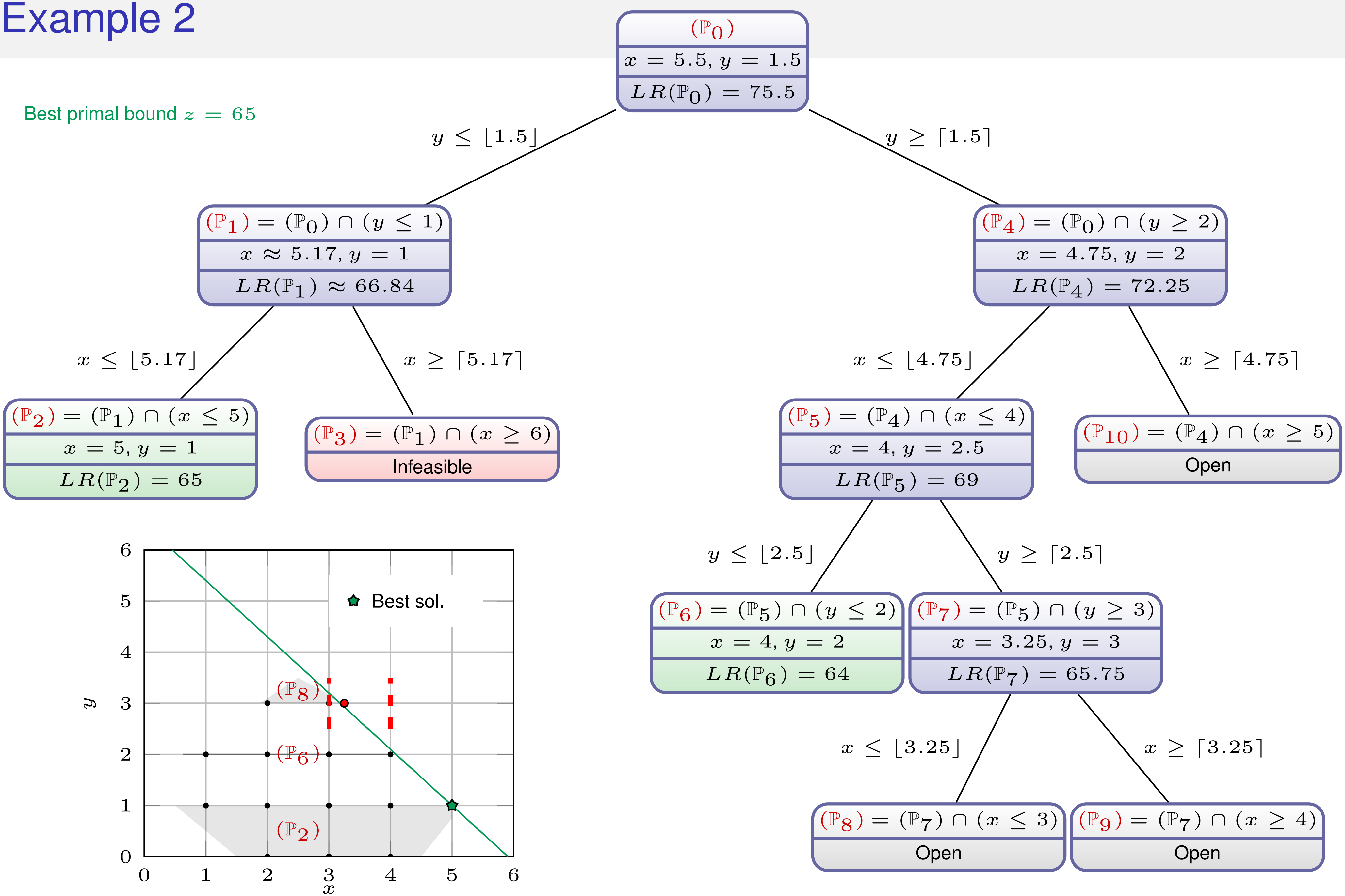
Best primal bound $z = 65$





Example 2

Best primal bound $z = 65$



Example 2

Best primal bound $z = 65$

$(\mathbb{P}_0)$
$x = 5.5, y = 1.5$
$LR(\mathbb{P}_0) = 75.5$

$y \leq \lfloor 1.5 \rfloor$

$(\mathbb{P}_1) = (\mathbb{P}_0) \cap (y \leq 1)$
$x \approx 5.17, y = 1$
$LR(\mathbb{P}_1) \approx 66.84$

$x \leq \lfloor 5.17 \rfloor$

$(\mathbb{P}_2) = (\mathbb{P}_1) \cap (x \leq 5)$
$x = 5, y = 1$
$LR(\mathbb{P}_2) = 65$

$x \geq \lceil 5.17 \rceil$

$(\mathbb{P}_3) = (\mathbb{P}_1) \cap (x \geq 6)$
Infeasible

$y \geq \lceil 1.5 \rceil$

$(\mathbb{P}_4) = (\mathbb{P}_0) \cap (y \geq 2)$
$x = 4.75, y = 2$
$LR(\mathbb{P}_4) = 72.25$

$x \leq \lfloor 4.75 \rfloor$

$(\mathbb{P}_5) = (\mathbb{P}_4) \cap (x \leq 4)$
$x = 4, y = 2.5$
$LR(\mathbb{P}_5) = 69$

$y \leq \lfloor 2.5 \rfloor$

$(\mathbb{P}_6) = (\mathbb{P}_5) \cap (y \leq 2)$
$x = 4, y = 2$
$LR(\mathbb{P}_6) = 64$

$y \geq \lceil 2.5 \rceil$

$(\mathbb{P}_7) = (\mathbb{P}_5) \cap (y \geq 3)$
$x = 3.25, y = 3$
$LR(\mathbb{P}_7) = 65.75$

$x \leq \lfloor 3.25 \rfloor$

$(\mathbb{P}_8) = (\mathbb{P}_7) \cap (x \leq 3)$
$x = 3, y \approx 3.17$
$LR(\mathbb{P}_8) \approx 64.67$

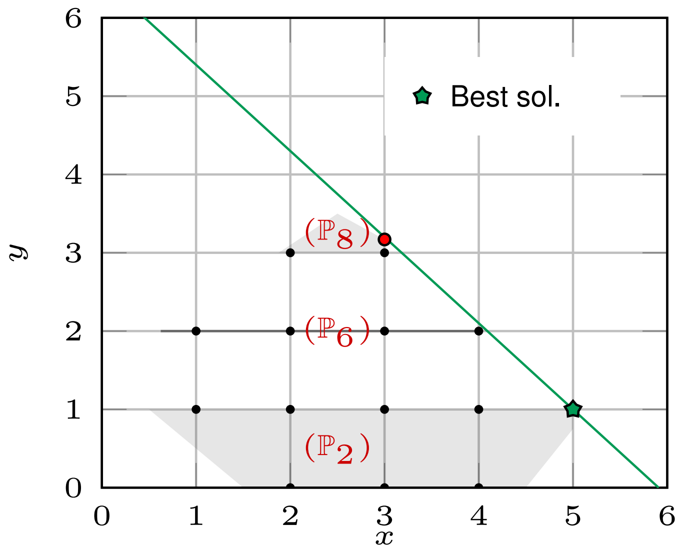
No solution with value higher than 64.67 can be found here → eliminate

$x \geq \lceil 3.25 \rceil$

$(\mathbb{P}_9) = (\mathbb{P}_7) \cap (x \geq 4)$
Open

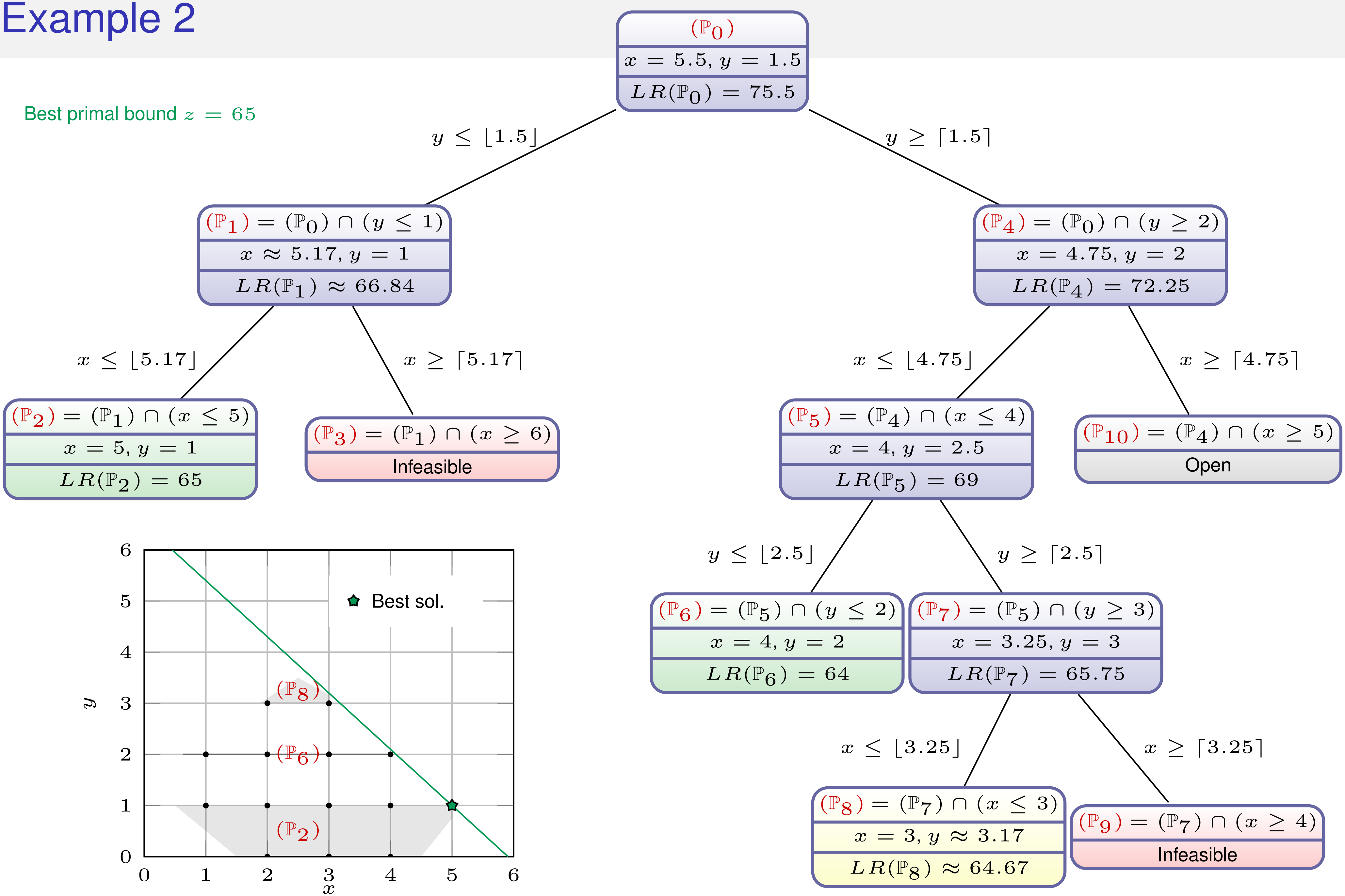
$x \geq \lceil 4.75 \rceil$

$(\mathbb{P}_{10}) = (\mathbb{P}_4) \cap (x \geq 5)$
Open



Example 2

Best primal bound $z = 65$



Example 2

Best primal bound $z = 65$

$x^*: (5,1)$

$(\mathbb{P}_0)$: $x = 5.5, y = 1.5$; $LR(\mathbb{P}_0) = 75.5$

- $y \leq \lfloor 1.5 \rfloor$: $(\mathbb{P}_1) = (\mathbb{P}_0) \cap (y \leq 1)$: $x \approx 5.17, y = 1$; $LR(\mathbb{P}_1) \approx 66.84$
  - $x \leq \lfloor 5.17 \rfloor$: $(\mathbb{P}_2) = (\mathbb{P}_1) \cap (x \leq 5)$: $x = 5, y = 1$; $LR(\mathbb{P}_2) = 65$
  - $x \geq \lceil 5.17 \rceil$: $(\mathbb{P}_3) = (\mathbb{P}_1) \cap (x \geq 6)$: Infeasible
- $y \geq \lceil 1.5 \rceil$: $(\mathbb{P}_4) = (\mathbb{P}_0) \cap (y \geq 2)$: $x = 4.75, y = 2$; $LR(\mathbb{P}_4) = 72.25$
  - $x \leq \lfloor 4.75 \rfloor$: $(\mathbb{P}_5) = (\mathbb{P}_4) \cap (x \leq 4)$: $x = 4, y = 2.5$; $LR(\mathbb{P}_5) = 69$
    - $y \leq \lfloor 2.5 \rfloor$: $(\mathbb{P}_6) = (\mathbb{P}_5) \cap (y \leq 2)$: $x = 4, y = 2$; $LR(\mathbb{P}_6) = 64$
    - $y \geq \lceil 2.5 \rceil$: $(\mathbb{P}_7) = (\mathbb{P}_5) \cap (y \geq 3)$: $x = 3.25, y = 3$; $LR(\mathbb{P}_7) = 65.75$
      - $x \leq \lfloor 3.25 \rfloor$: $(\mathbb{P}_8) = (\mathbb{P}_7) \cap (x \leq 3)$: $x = 3, y \approx 3.17$; $LR(\mathbb{P}_8) \approx 64.67$
      - $x \geq \lceil 3.25 \rceil$: $(\mathbb{P}_9) = (\mathbb{P}_7) \cap (x \geq 4)$: Infeasible
  - $x \geq \lceil 4.75 \rceil$: $(\mathbb{P}_{10}) = (\mathbb{P}_4) \cap (x \geq 5)$: Infeasible