Adjustable Robust Optimization: Approximate Solution via Decision Rules

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CR03 - Robust combinatorial optimization, ENS-Lyon

Little reminder

- We are interested in the solution of adjustable robust optimization problems of the form:

$$\begin{aligned}
\min_{\mathbf{x}\in\mathcal{X},\mathbf{y}(\cdot)} \quad & \mathbf{c}^\top \mathbf{x} + \max_{\boldsymbol{\xi}\in\Xi} \quad \mathbf{f}(\boldsymbol{\xi})^\top \mathbf{y}(\boldsymbol{\xi}) & & \text{(2ARO)}\\
\text{s.t.} \quad & \mathbf{T}(\boldsymbol{\xi})\mathbf{x} + \mathbf{W}(\boldsymbol{\xi})\mathbf{y}(\boldsymbol{\xi}) \le \mathbf{h}(\boldsymbol{\xi}) & \forall \boldsymbol{\xi}\in\Xi &\\
& \mathbf{y}(\boldsymbol{\xi}) \in \mathcal{Y} & \forall \boldsymbol{\xi}\in\Xi &
\end{aligned}$$

- $\mathbf{y}(\cdot) : \Xi \to \mathcal{Y}$ are functionals to be optimized.

Remark

- Can be solved to exact optimality when $\mathcal{Y} = \mathbb{R}^{n_y}_+$ and $\mathbf{f}(\boldsymbol{\xi}) = \mathbf{f}$ and $\mathbf{W}(\boldsymbol{\xi}) = \mathbf{W}$ for $\boldsymbol{\xi} \in \Xi$.
- Resulting algorithms have exponential worst-case complexity and require solution of difficult optimization problems at each iteration.

due to $Q(x) = \max_{\xi\in\Xi} \min_{y\in\mathbb{R}^{n_y}_+} f^\top y$ $\quad Wy \le h(\xi) - Tx$

Decision rule approximations

$$
\begin{aligned}
\min_{\mathbf{x}\in\mathcal{X},\mathbf{y}(\cdot),\theta\in\mathbb{R}} \quad & \mathbf{c}^\top \mathbf{x} + \theta \\
\text{s.t.} \quad & \theta \geq \mathbf{f}(\boldsymbol{\xi})^\top \mathbf{y}(\boldsymbol{\xi}) && \forall \boldsymbol{\xi} \in \Xi \\
& \mathbf{T}(\boldsymbol{\xi})\mathbf{x} + \mathbf{W}(\boldsymbol{\xi})\mathbf{y}(\boldsymbol{\xi}) \leq \mathbf{h}(\boldsymbol{\xi}) && \forall \boldsymbol{\xi} \in \Xi \\
& \mathbf{y}(\boldsymbol{\xi}) \in \mathcal{Y} && \forall \boldsymbol{\xi} \in \Xi
\end{aligned}
$$

Idea

- Restrict the form of $\mathbf{y}(\boldsymbol{\xi})$ to a simple family of functions.
- Optimize within the chosen family.
- Most common restrictions: affine, piecewise affine/constant.

Attention!

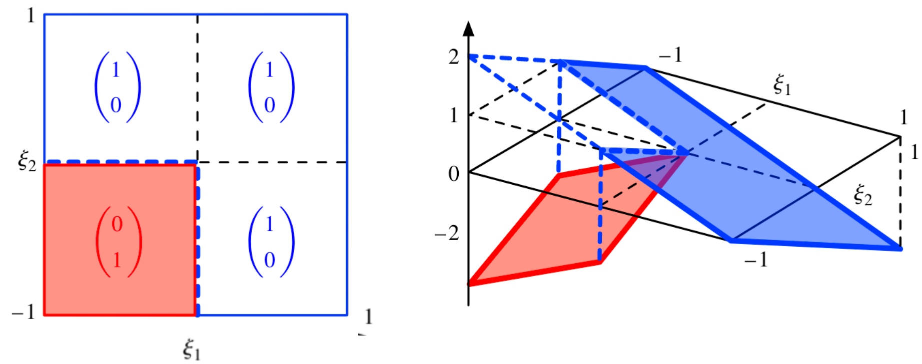
- Being restrictions, decision rule approximations may be infeasible.
- If feasible, they provide feasible solutions (primal bounds) to (2ARO).
- They typically improve upon the static robust solution.

Decision rule approximations

$$\sup_{\boldsymbol{\xi}\in\mathbb{R}^2 \mid -1\le\boldsymbol{\xi}\le 1} \quad \min_{\mathbf{y}\in\{0,1\}^2} \quad (\xi_1+\xi_2)(y_2-y_1)$$

$$\text{s.t.} \quad y_1 + y_2 = 1$$
$$y_1 \ge \xi_1$$
$$y_1 \ge \xi_2$$

$y^*(\xi)$ $\qquad$ $Q^*(\xi)$



either ξ_1 or $\xi_2 > 0$ forces $y_1 = 1$:

$\binom{1}{0}$ is the only feasible sol. in that case

when $\xi_1, \xi_2 \le 0$ we can choose btw. :

$\binom{1}{0}, \binom{0}{1}$

$(1,0)$ gives $-(\xi_1+\xi_2)$ in the obj. function

$\rightarrow$ $(0,1)$ gives $(\xi_1+\xi_2)$ " "

Set of realizations $\xi \in \Xi$ for which $\binom{1}{0}$ is feasible & optimal is non-convex & open

$$\sup_{\xi\in\Xi} Q(\xi) = \sup\left\{ \sup_{\substack{\xi\in\Xi \mid \xi_1>0 \\ \text{or } \xi_2>0}} -(\xi_1+\xi_2) \, , \sup_{\xi\in\Xi \mid \xi_1,\xi_2\le 0} (\xi_1+\xi_2) \right\}$$

$$= 1 \quad \left(\begin{array}{l} \xi_2 = -1 \text{ and } \xi_1 \to 0 \\ \text{or } \xi_1 = -1 \quad \xi_2 \to 0 \end{array} \right)$$

Affine decision rules[1]

$$\begin{aligned} \min_{\mathbf{x}\in\mathcal{X},\mathbf{y}(\cdot)} \quad & \mathbf{c}^\top \mathbf{x} + \max_{\boldsymbol{\xi}\in\Xi} \quad \mathbf{f}(\boldsymbol{\xi})^\top \mathbf{y}(\boldsymbol{\xi}) & \\ \text{s.t.} \quad & \mathbf{T}(\boldsymbol{\xi})\mathbf{x} + \mathbf{W}(\boldsymbol{\xi})\mathbf{y}(\boldsymbol{\xi}) \leq \mathbf{h}(\boldsymbol{\xi}) & \forall \boldsymbol{\xi} \in \Xi \\ & \mathbf{y}(\boldsymbol{\xi}) \in \mathcal{Y} & \forall \boldsymbol{\xi} \in \Xi \end{aligned}$$

Assume

- $\mathcal{Y} = \mathbb{R}_+^{n_y}$ $\rightarrow$ continuous recourse
- $\mathbf{f}(\boldsymbol{\xi})$ and $\mathbf{W}(\boldsymbol{\xi})$ are deterministic $\rightarrow$ fixed recourse

Not always necessary but very helpful

[1] Ben-Tal et al., 2004

Affine decision rules[1]

$$\min_{\mathbf{x}\in\mathcal{X},\mathbf{y}(\cdot)} \quad \mathbf{c}^\top \mathbf{x} + \max_{\boldsymbol{\xi}\in\Xi} \quad \mathbf{f}^\top \mathbf{y}(\boldsymbol{\xi})$$

$$\text{s.t.} \quad \sum_{i=1}^{n_\xi} \mathbf{T_i}\xi_i \mathbf{x} + \mathbf{W}\mathbf{y}(\xi) \leq \mathbf{H}\xi \qquad \forall \boldsymbol{\xi} \in \Xi$$

$$\mathbf{y}(\xi) \geq 0 \qquad \forall \boldsymbol{\xi} \in \Xi$$

Idea

- Restrict $\mathbf{y}(\boldsymbol{\xi})$ to be an affine function of $\boldsymbol{\xi}$

$$\mathbf{y}_i(\boldsymbol{\xi}) = \alpha_{i0} + \boldsymbol{\alpha}_i^\top \boldsymbol{\xi} \quad \forall i = 1, \ldots, n_y \rightarrow \mathbf{y}(\boldsymbol{\xi}) = \mathbf{A}\boldsymbol{\xi}$$

- Optimize $\mathbf{A} \in \mathbb{R}^{n_y \times (n_\xi + 1)}$ to obtain the best such approximation.

[1] Ben-Tal et al., 2004

Affine decision rules[1]

$$
\begin{aligned}
\min_{\mathbf{x}\in\mathcal{X},\mathbf{A}\in\mathbb{R}^{n_y\times(n_\xi+1)}} \quad & \mathbf{c}^\top\mathbf{x}+\theta && && \text{(Aff)}\\
\text{s.t.} \quad & \theta \geq \boxed{\mathbf{f}^\top}\mathbf{A}\boldsymbol{\xi} && \forall\boldsymbol{\xi}\in\Xi\\
& \sum_{i=1}^{n_\xi}\mathbf{T}_{\mathbf{i}}\xi_i\mathbf{x}+\boxed{\mathbf{W}}\mathbf{A}\boldsymbol{\xi}\leq\mathbf{H}\boldsymbol{\xi} && \forall\boldsymbol{\xi}\in\Xi\\
& \mathbf{A}\boldsymbol{\xi}\geq 0 && \forall\boldsymbol{\xi}\in\Xi
\end{aligned}
$$

Fixed recourse ensures that we maintain linear functions in ξ

Remark

- (Aff) is a static linear robust optimization problem with a polyhedral uncertainty set.
- It can be reformulated into a deterministic equivalent problem through LP-duality.

Remark

Reformulations require only a polynomial number of additional variables and constraints.

[1] Ben-Tal et al., 2004

Affine decision rules[1]

$$
\begin{aligned}
\min_{\mathbf{x}\in\mathcal{X},\mathbf{A}\in\mathbb{R}^{n_y\times(n_\xi+1)}} \quad & \mathbf{c}^\top\mathbf{x}+\theta && \text{(Aff)}\\
\text{s.t.} \quad & \theta \geq \mathbf{f}^\top\mathbf{A}\boldsymbol{\xi} && \forall\boldsymbol{\xi}\in\Xi\\
& \sum_{i=1}^{n_\xi}\mathbf{T}_i\xi_i\mathbf{x}+\mathbf{W}\mathbf{A}\boldsymbol{\xi}\leq\mathbf{H}\boldsymbol{\xi} && \forall\boldsymbol{\xi}\in\Xi\\
& \mathbf{A}\boldsymbol{\xi}\geq 0 && \forall\boldsymbol{\xi}\in\Xi
\end{aligned}
$$

Remark

If the fixed recourse assumption is not satisfied quadratic terms in $\boldsymbol{\xi}$ appear in semi-infinite constraints. In that case, reformulation through strong duality arguments is restricted to few special cases.

for instance Ξ is ellipsoidal

[1] Ben-Tal et al., 2004

Weaknesses of affine decision rules

- They are only effective when the recourse variables are continuous. Indeed, imposing

$$\mathbf{y}(\boldsymbol{\xi}) = \alpha_0 + \boldsymbol{\alpha}^\top \boldsymbol{\xi} \in \mathbb{Z} \qquad \forall \boldsymbol{\xi} \in \Xi$$

  implies that only α_0 can take a strictly positive value. This is equivalent to a static robust approach in which $\mathbf{y}(\boldsymbol{\xi}) = \bar{\mathbf{y}}$ for $\boldsymbol{\xi} \in \Xi$.
- They are most effective under the fixed recourse assumption.
- They can be highly suboptimal in which case piecewise affine decision rules can be useful. (For certain problems the gap scales with $\min(\sqrt{n_\xi}, \sqrt{m})$ where m is the # of constraints)
- To handle the cases of random recourse and integer recourse piecewise constant decisions are more appropriate.

Piecewise affine decision rules

$$
\begin{aligned}
\min_{\mathbf{x}\in\mathcal{X},\mathbf{y}(\cdot),\theta\in\mathbb{R}} \quad & \mathbf{c}^\top \mathbf{x} + \theta \\
\text{s.t.} \quad & \theta \geq \mathbf{f}(\boldsymbol{\xi})^\top \mathbf{y}(\boldsymbol{\xi}) && \forall \boldsymbol{\xi} \in \Xi \\
& \mathbf{T}(\boldsymbol{\xi})\mathbf{x} + \mathbf{W}(\boldsymbol{\xi})\mathbf{y}(\boldsymbol{\xi}) \leq \mathbf{h}(\boldsymbol{\xi}) && \forall \boldsymbol{\xi} \in \Xi \\
& \mathbf{y}(\boldsymbol{\xi}) \in \mathcal{Y} && \forall \boldsymbol{\xi} \in \Xi
\end{aligned}
$$

Assume

- $\mathcal{Y} = \mathbb{R}_+^{n_y}$ $\rightarrow$ continuous recourse
- $\mathbf{f}(\boldsymbol{\xi})$ and $\mathbf{W}(\boldsymbol{\xi})$ are deterministic $\rightarrow$ fixed recourse

Piecewise affine decision rules

$$
\begin{aligned}
\min_{\mathbf{x}\in\mathcal{X},\mathbf{y}(\cdot),\theta\in\mathbb{R}} \quad & \mathbf{c}^\top \mathbf{x} + \theta \\
\text{s.t.} \quad & \theta \geq \mathbf{f}^\top \mathbf{y}(\boldsymbol{\xi}) && \forall \boldsymbol{\xi} \in \Xi \\
& \mathbf{T}(\boldsymbol{\xi})\mathbf{x} + \mathbf{W}\mathbf{y}(\boldsymbol{\xi}) \leq \mathbf{h}(\boldsymbol{\xi}) && \forall \boldsymbol{\xi} \in \Xi \\
& \mathbf{y}(\boldsymbol{\xi}) \geq 0 && \forall \boldsymbol{\xi} \in \Xi
\end{aligned}
$$

Idea

- Partition the uncertainty set into K subsets, *i.e.*,

$$\Xi = \bigcup_{k=1}^{K} \Xi_k.$$

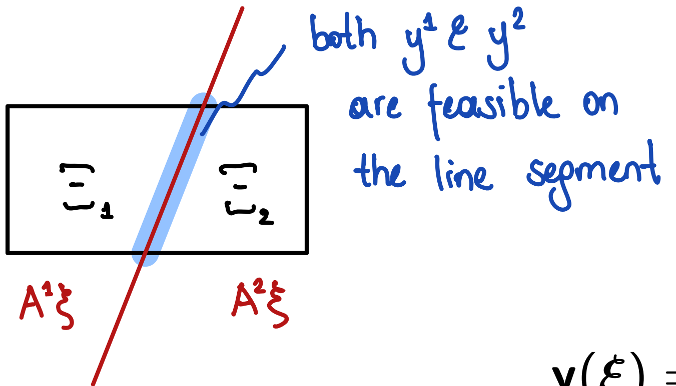
For the moment we do not assume Ξ_k to be closed or convex neither disjoint.

- Define one affine policy over each subset

$$\mathbf{y}(\boldsymbol{\xi}) = \mathbf{A}^k \boldsymbol{\xi} \quad \forall k \in [K], \boldsymbol{\xi} \in \Xi_k.$$

Assign at least one feasible affine policy to each realization

Piecewise affine decision rules



$$
\mathbf{y}(\xi) = \begin{cases} \mathbf{A}^1\xi & \xi \in \Xi_1 \\ \mathbf{A}^2\xi & \xi \in \Xi_2 \\ \vdots & \\ \mathbf{A}^K\xi & \xi \in \Xi_K \end{cases}
$$

Remark

- How many subsets do we create? Do the subsets have a specific form?
- Is the number of subsets fixed? Does it evolve throughout the solution process?
- Are subsets chosen a priori to optimization or not? (implicit/explicit design)?
- How do we handle realizations at the intersection of subsets?

If multiple solutions are feasible for a realization & they have diff. costs then we may choose the best or the worst

Piecewise affine rules by uncertainty set partitioning[2]

$$
\begin{aligned}
\min_{\mathbf{x}\in\mathcal{X},\mathbf{y}(\cdot),\theta\in\mathbb{R}} \quad & \mathbf{c}^\top \mathbf{x} + \theta \\
\text{s.t.} \quad & \theta \geq \mathbf{f}^\top \mathbf{y}(\boldsymbol{\xi}) && \forall \boldsymbol{\xi} \in \Xi \\
& \mathbf{W}\mathbf{y}(\boldsymbol{\xi}) \leq \mathbf{h}(\boldsymbol{\xi}) - \mathbf{T}(\boldsymbol{\xi})\mathbf{x} && \forall \boldsymbol{\xi} \in \Xi \\
& \mathbf{y}(\boldsymbol{\xi}) \geq 0 && \forall \boldsymbol{\xi} \in \Xi
\end{aligned}
$$

Idea

- Iteratively partition the uncertainty set into polyhedral subsets.
- Define one affine recourse policy over each subset (note the fixed recourse assumption).
- At the intersection of subsets assume that the policy with worst objective value is implemented.

[2] Postek and Den Hertog, 2016 and Bertsimas and Dunning, 2016

Piecewise affine rules by uncertainty set partitioning[2]

- When there is no partitioning $\mathbf{y}(\boldsymbol{\xi}) = \mathbf{A}\boldsymbol{\xi}$ for $\boldsymbol{\xi} \in \Xi$ $\rightarrow$ affine decision rule:

we hide the constraints
$A\xi \geq 0$ in this system

$$\begin{aligned} \min_{\mathbf{x}\in\mathcal{X},\mathbf{A}\in\mathbb{R}^{n_y \times n_\xi},\theta\in\mathbb{R}} \quad & \mathbf{c}^\top \mathbf{x} + \theta \\ \text{s.t.} \quad & \theta \geq \mathbf{f}^\top \mathbf{A}\boldsymbol{\xi} && \forall \boldsymbol{\xi} \in \Xi \\ & \mathbf{T}(\boldsymbol{\xi})\mathbf{x} + \mathbf{W}\mathbf{A}\boldsymbol{\xi} \leq \mathbf{h}(\boldsymbol{\xi}) && \forall \boldsymbol{\xi} \in \Xi \end{aligned} \tag{AFF}$$

- Let $(\mathbf{x}^*, \mathbf{A}^*, \theta^*)$ be the optimal affine decision rule solution.
- Extract the "binding" scenarios:

$$\begin{aligned} \hat{\boldsymbol{\xi}}_0 &\in \operatorname{argmax}_{\boldsymbol{\xi}\in\Xi} \ \mathbf{f}^\top \mathbf{A}^* \boldsymbol{\xi} - \theta^* \\ \hat{\boldsymbol{\xi}}_i &\in \operatorname{argmax}_{\boldsymbol{\xi}\in\Xi} \ \mathbf{T}_i(\boldsymbol{\xi})\mathbf{x}^* + \mathbf{W}\mathbf{A}^*\boldsymbol{\xi} - \mathbf{h}_i(\boldsymbol{\xi}) && \forall i \in [m] \end{aligned}$$

- Let $\hat{\Xi} = \bigcup_{i=0}^m \{\hat{\boldsymbol{\xi}}_i\}$.

set of scenarios bounding the current solution

[2] Postek and Den Hertog, 2016 and Bertsimas and Dunning, 2016

Piecewise affine rules by uncertainty set partitioning[2]

- Let $\Xi_1 = \{\boldsymbol{\xi} \in \Xi \mid \boldsymbol{\beta}^\top \boldsymbol{\xi} \leq \beta\}$ and $\Xi_2 = \{\boldsymbol{\xi} \in \Xi \mid \boldsymbol{\beta}^\top \boldsymbol{\xi} \geq \beta\}$.
- The partitioned problem becomes:

$$\begin{aligned} \min_{\mathbf{x}\in\mathcal{X},\mathbf{A}_1,\mathbf{A}_2\in\mathbb{R}^{n_y \times n_\xi}} \quad & \mathbf{c}^\top \mathbf{x} + \theta & & \text{(2-AFF)} \\ \text{s.t.} \quad & \theta \geq \mathbf{f}^\top \mathbf{A}_s \boldsymbol{\xi} & \forall s = 1,2, \boldsymbol{\xi} \in \Xi_s & \\ & \mathbf{T}(\boldsymbol{\xi})\mathbf{x} + \mathbf{W}\mathbf{A}_s\boldsymbol{\xi} \leq \mathbf{h}(\boldsymbol{\xi}) & \forall s = 1,2, \boldsymbol{\xi} \in \Xi_s & \end{aligned}$$

Proposition

If $\hat{\Xi} \subseteq \Xi_1$ or $\hat{\Xi} \subseteq \Xi_2$ then $z^{2-\mathrm{AFF}} = z^{\mathrm{AFF}}$. Otherwise, we have that $z^{2-\mathrm{AFF}} \leq z^{\mathrm{AFF}}$.

Indeed all realisations bounding the solution (x^*, A^*, θ^*) are present in one of the partitions. The value that we obtain from this partition cannot be better than θ^*.

[2] Postek and Den Hertog, 2016 and Bertsimas and Dunning, 2016

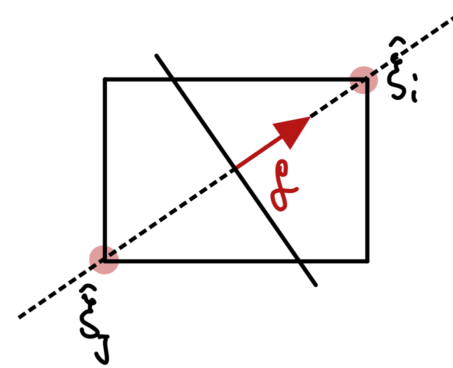
Piecewise affine rules by uncertainty set partitioning[2]

- Postek and Den Hertog propose separating with a hyperplane $\boldsymbol{g}^\top \boldsymbol{\xi} = h$ such that at least one element of $\hat{\Xi}$ is on either side, *i.e.*,

$$\exists \hat{\boldsymbol{\xi}}_i, \hat{\boldsymbol{\xi}}_j \in \hat{\Xi} \text{ s.t. } \boldsymbol{g}^\top \hat{\boldsymbol{\xi}}_i \leq h \text{ and } \boldsymbol{g}^\top \hat{\boldsymbol{\xi}}_j \geq h.$$

- Heuristic: identify two elements in $\hat{\Xi}$ that are farthest from each other and partition with a hyperplane that separates them strongly.
- Let, $\hat{\boldsymbol{\xi}}_{i^*}$ and $\hat{\boldsymbol{\xi}}_{j^*}$ represent the two vectors that maximize $||\hat{\boldsymbol{\xi}}_i - \hat{\boldsymbol{\xi}}_j||$ for $\hat{\boldsymbol{\xi}}_i, \hat{\boldsymbol{\xi}}_j \in \hat{\Xi}$.
- Then the hyperplane $\boldsymbol{g}^\top \boldsymbol{\xi} \leq h$ with

$$\boldsymbol{g} = \hat{\boldsymbol{\xi}}_{i^*} - \hat{\boldsymbol{\xi}}_{j^*} \text{ and } h = \frac{\boldsymbol{g}^\top (\hat{\boldsymbol{\xi}}_i + \hat{\boldsymbol{\xi}}_j)}{2}$$



achieves the desired result.

Remark

Two subsets will be created as a result of this approach.

[2] Postek and Den Hertog, 2016 and Bertsimas and Dunning, 2016

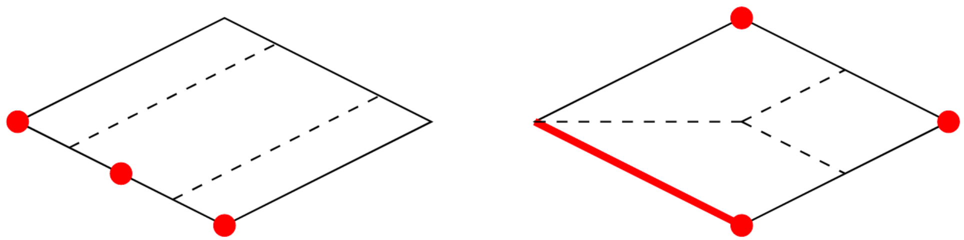
Piecewise affine rules by uncertainty set partitioning[2]

- Bertsimas and Dunning propose Voronoi diagrams.
- For each $\hat{\xi}_i \in \hat{\Xi}$:

The set of realisations in Ξ that are closer to $\hat{\xi}_i$ than any other $\hat{\xi}_j \in \hat{\Xi}$

$$\Xi(\hat{\xi}_i) = \{\xi \in \Xi \mid ||\hat{\xi}_i - \xi|| \le ||\hat{\xi}_j - \xi|| \quad \forall \hat{\xi}_j \in \hat{\Xi}\}$$

$$(\hat{\xi}_i - \xi)^T(\hat{\xi}_i - \xi) \le (\hat{\xi}_j - \xi)^T(\hat{\xi}_j - \xi)$$



$$\hat{\xi}_i^T\hat{\xi}_i - 2\hat{\xi}_i^T\xi + \cancel{\xi^T\xi}$$

$$\le \hat{\xi}_j^T\hat{\xi}_j - 2\hat{\xi}_j^T\xi + \cancel{\xi^T\xi}$$

$$2\xi^T(\hat{\xi}_j - \hat{\xi}_i) \le \hat{\xi}_j^T\hat{\xi}_j - \hat{\xi}_i^T\hat{\xi}_i$$

- This yields for $\hat{\xi}_i \in \hat{\Xi}$:

$$\Xi(\hat{\xi}_i) = \Xi \cap \{\xi \mid \xi^\top(\hat{\xi}_j - \hat{\xi}_i) \le \frac{1}{2}(\hat{\xi}_j + \hat{\xi}_i)^\top(\hat{\xi}_j - \hat{\xi}_i) \quad \forall \hat{\xi}_j \in \hat{\Xi}, j \ne i\}.$$

constraints linear in ξ

- This is a polyhedron with $|\hat{\Xi}| - 1$ additional linear constraints.

Remark

$|\hat{\Xi}|$ subsets will be created as a result of this approach.

If Ξ is a polyhedron then subsets are polyhedra as well.

[2] Postek and Den Hertog, 2016 and Bertsimas and Dunning, 2016

Piecewise affine rules by uncertainty set partitioning[2]

- Let at iteration r:

$$\Xi = \bigcup_{k \in N_r} \Xi_{rk}.$$

Each of these are defined as a polyhedron + linear constraints added through the iterations

- We solve:

$$\begin{aligned}
\min_{\substack{\mathbf{x} \in \mathcal{X}, \theta \in \mathbb{R} \\ \mathbf{A}^{(r,1)}, \dots, \mathbf{A}^{(r,N_r)} \in \mathbb{R}^{n_y \times n_\xi}}} \quad & \mathbf{c}^\top \mathbf{x} + \theta && \text{(Part-r)} \\
\text{s.t.} \quad & \theta \geq \mathbf{f}^\top \mathbf{A}^{(r,k)} \boldsymbol{\xi} && \forall k \in [N_r], \boldsymbol{\xi} \in \Xi_{rk} \\
& \mathbf{T}(\boldsymbol{\xi})\mathbf{x} + \mathbf{W}\mathbf{A}^{(r,k)} \boldsymbol{\xi} \leq \mathbf{h}(\boldsymbol{\xi}) && \forall k \in [N_r], \boldsymbol{\xi} \in \Xi_{rk}
\end{aligned}$$

Remark

- (Part-r) is a static robust optimization problem, it is semi-infinite.
- Under the polyhedral subsets assumption it can be reformulated as a monolithic LP/MILP.

[2] Postek and Den Hertog, 2016 and Bertsimas and Dunning, 2016

Piecewise affine rules by uncertainty set partitioning[2]

- Let $(\mathbf{x}^{r*}, \mathbf{A}^{*(r,1)}, \ldots, \mathbf{A}^{*(r,N_r)}, \theta^{r*})$ be an optimal solution.
- Extract the worst realizations for each constraint by solving, for $k \in [N_r]$:

realisations constraining the k^{th} solution

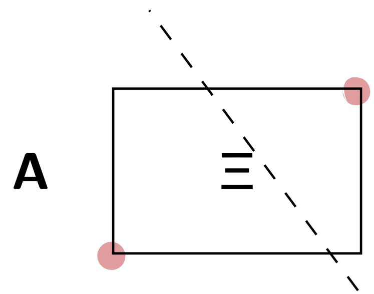
$$\begin{cases} \hat{\xi}_{k0} \in \operatorname{argmax}_{\xi \in \Xi_{rk}} & \mathbf{f}^\top \mathbf{A}^{*(r,k)} \xi - \theta^* \\ \hat{\xi}_{ki} \in \operatorname{argmax}_{\xi \in \Xi_{rk}} & \mathbf{T}_i(\xi)\mathbf{x}^{r*} + \mathbf{W}_i \mathbf{A}^{*(r,k)} \xi - \mathbf{h}_i(\xi) \qquad \forall i \in [m_y] \end{cases}$$

- Let $\hat{\Xi}_{rk} = \bigcup_{i=0}^{m_y} \{\hat{\xi}_{ki}\}$ and partition each Ξ_{rk} based on the realizations in $\hat{\Xi}_{rk}$.
- Let $\hat{\Xi} = \bigcup_{r'=1}^{r} \bigcup_{k \in [N_{r'}]} \hat{\Xi}_{r'k}$ be the set of realizations identified in iterations $1, \ldots, r$.
- Calculate a dual bound by solving:

$$\begin{aligned} z^{\mathrm{Dual}} := \min_{\substack{\mathbf{x} \in \mathcal{X}, \theta \in \mathbb{R} \\ \mathbf{y}^1, \ldots, \mathbf{y}^{|\hat{\Xi}|} \in \mathbb{R}_+^{n_y}}} \quad & \mathbf{c}^\top \mathbf{x} + \theta \\ \text{s.t.} \quad & \theta \geq \mathbf{f}^\top \mathbf{y}^i & \forall i \in [|\hat{\Xi}|] \\ & \mathbf{T}(\xi_i)\mathbf{x} + \mathbf{W}\mathbf{y}^i \leq \mathbf{h}(\xi_i) & \forall i \in [|\hat{\Xi}|]. \end{aligned}$$

[2] Postek and Den Hertog, 2016 and Bertsimas and Dunning, 2016

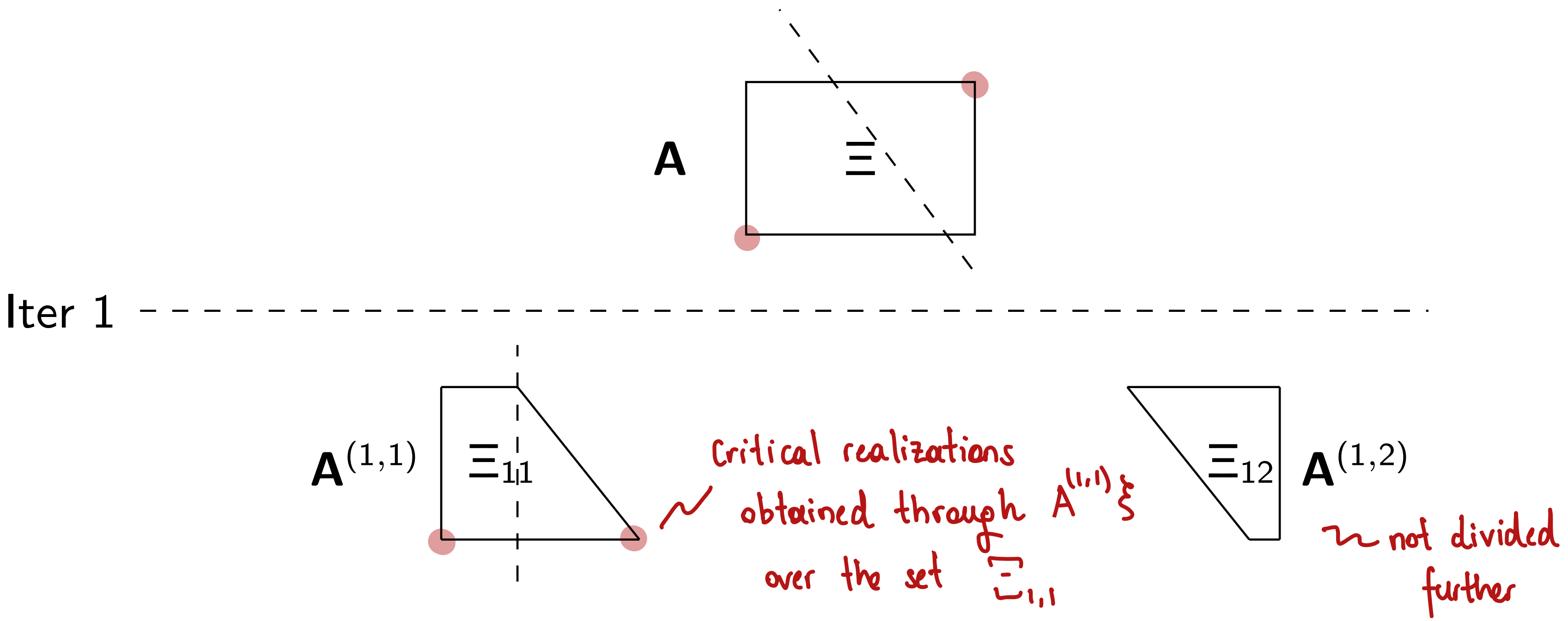
Piecewise affine rules by uncertainty set partitioning[2]



Critical realizations obtained through $y(\xi) = A\xi$ over the set Ξ

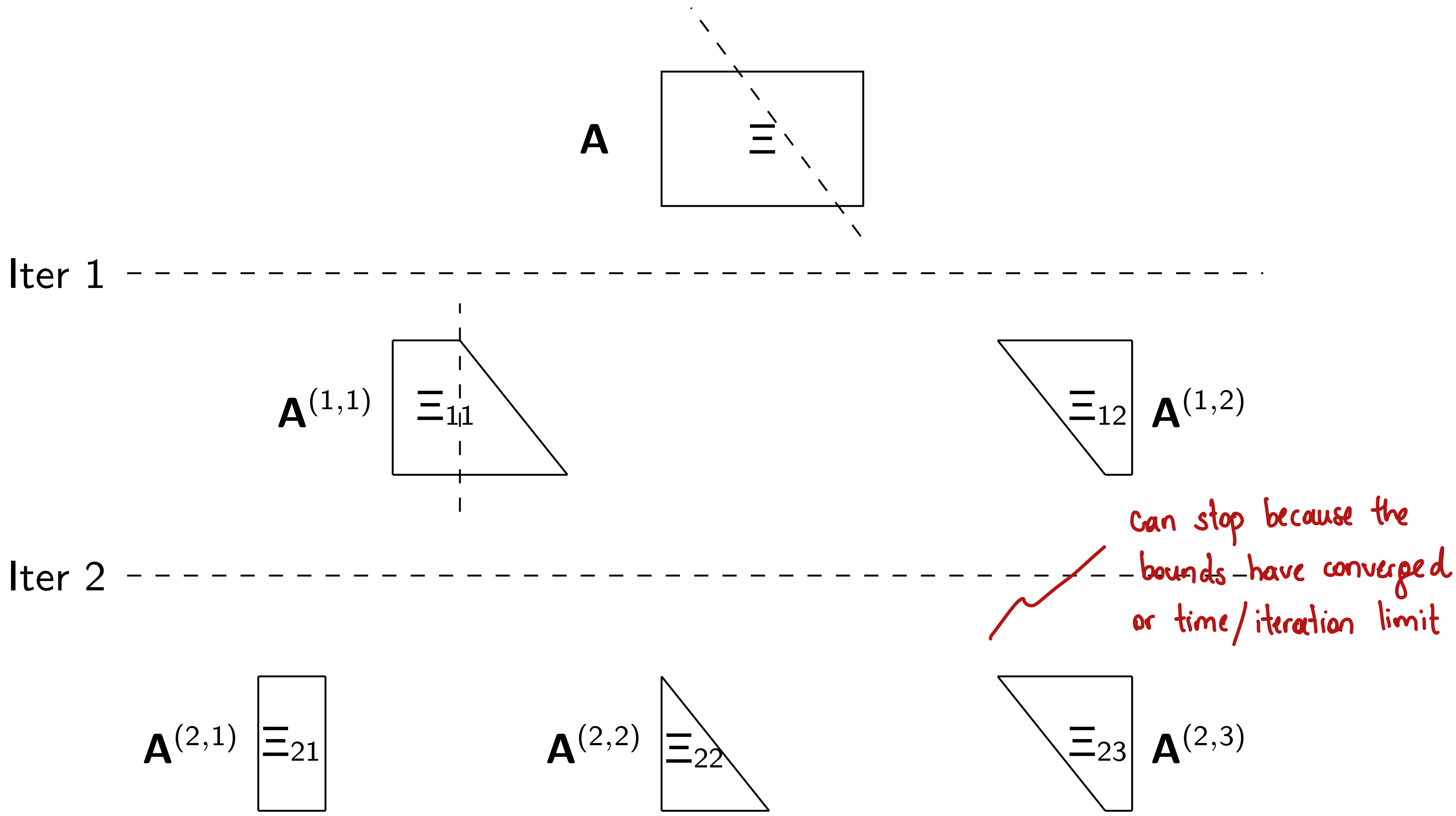
[2] Postek and Den Hertog, 2016 and Bertsimas and Dunning, 2016

Piecewise affine rules by uncertainty set partitioning[2]



[2] Postek and Den Hertog, 2016 and Bertsimas and Dunning, 2016

Piecewise affine rules by uncertainty set partitioning[2]

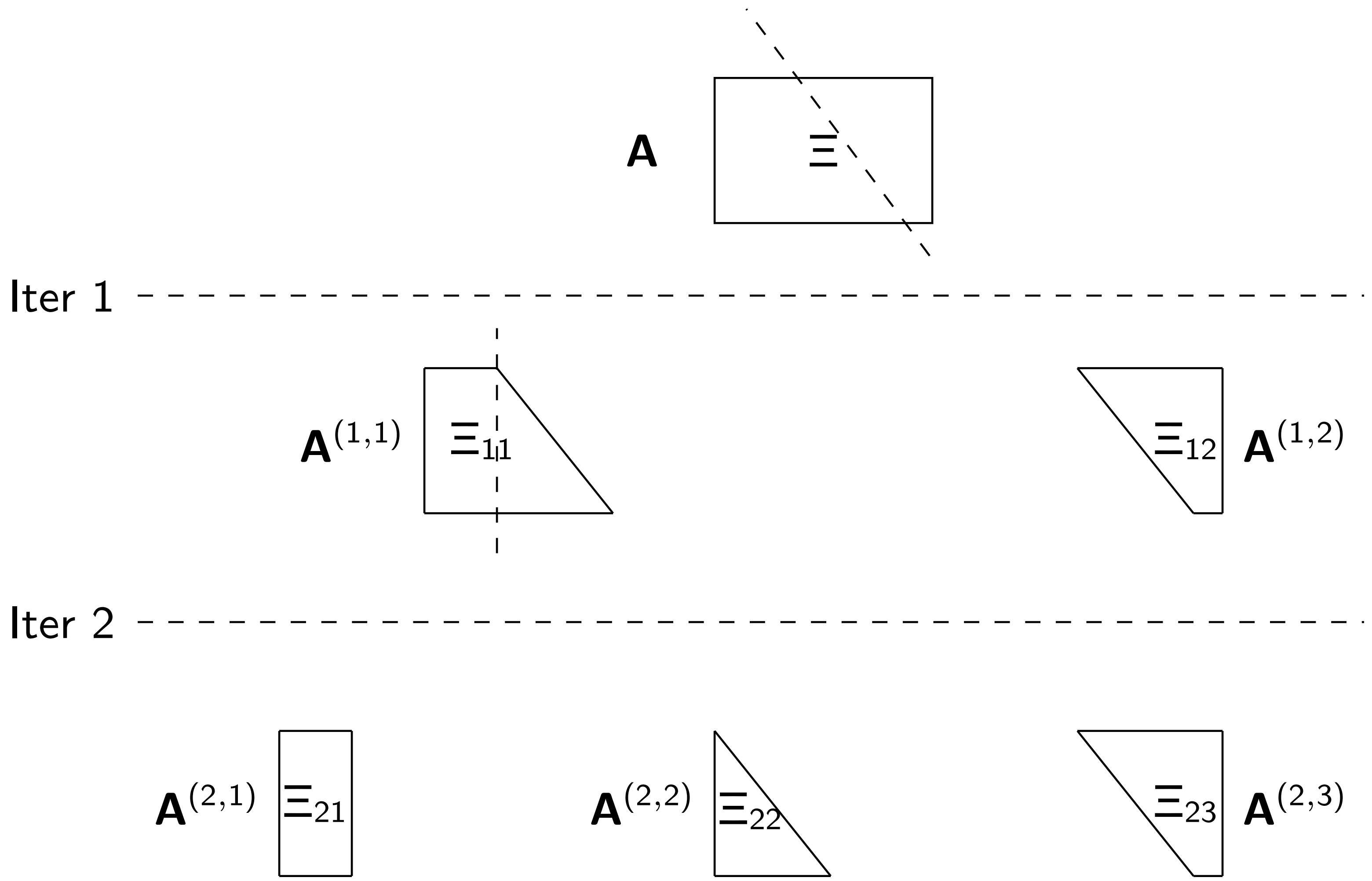


Remark

- Illustration of the algorithm with the partitioning approach of Postek and Den Hertog (2016).

[2] Postek and Den Hertog, 2016 and Bertsimas and Dunning, 2016

Piecewise affine rules by uncertainty set partitioning[2]



Remark

Convergence criterion can be established for purely continuous problems.

[2] Postek and Den Hertog, 2016 and Bertsimas and Dunning, 2016

Piecewise constant decision rules

$$
\begin{aligned}
\min_{\mathbf{x}\in\mathcal{X},\mathbf{y}(\cdot),\theta\in\mathbb{R}} \quad & \mathbf{c}^\top \mathbf{x} + \theta \\
\text{s.t.} \quad & \theta \geq \mathbf{f}(\boldsymbol{\xi})^\top \mathbf{y}(\boldsymbol{\xi}) && \forall \boldsymbol{\xi} \in \Xi \\
& \mathbf{W}(\boldsymbol{\xi})\mathbf{y}(\boldsymbol{\xi}) \leq \mathbf{h}(\boldsymbol{\xi}) - \mathbf{T}(\boldsymbol{\xi})\mathbf{x} && \forall \boldsymbol{\xi} \in \Xi \\
& \mathbf{y}(\boldsymbol{\xi}) \in \mathcal{Y} && \forall \boldsymbol{\xi} \in \Xi
\end{aligned}
$$

Idea

- Partition the uncertainty set into K subsets, *i.e.*,

$$\Xi = \bigcup_{k=1}^{K} \Xi_k.$$

- Define one recourse policy over each partition

$$\mathbf{y}(\boldsymbol{\xi}) = \mathbf{y}^k \quad \forall \boldsymbol{\xi} \in \Xi_k.$$

Attention!

- $\mathcal{Y}$ can contain continuous and integer variables.
- This allows handling $\mathbf{f}$ and $\mathbf{W}$ as affine functions of $\boldsymbol{\xi}$.

Since y^k is assumed constant over Ξ_k no quadratic term in ξ appears

Piecewise constant rules by uncertainty set partitioning[3]

$$\begin{aligned}
\min_{\mathbf{x}\in\mathcal{X},\mathbf{y}(\cdot),\theta\in\mathbb{R}} \quad & \mathbf{c}^\top \mathbf{x} + \theta \\
\text{s.t.} \quad & \theta \geq \mathbf{f}(\boldsymbol{\xi})^\top \mathbf{y}(\boldsymbol{\xi}) && \forall \boldsymbol{\xi} \in \Xi \\
& \mathbf{W}(\boldsymbol{\xi})\mathbf{y}(\boldsymbol{\xi}) \leq \mathbf{h}(\boldsymbol{\xi}) - \mathbf{T}(\boldsymbol{\xi})\mathbf{x} && \forall \boldsymbol{\xi} \in \Xi \\
& \mathbf{y}(\boldsymbol{\xi}) \in \mathcal{Y} && \forall \boldsymbol{\xi} \in \Xi
\end{aligned}$$

Idea

- Iteratively partition the uncertainty set into polyhedral subsets.
- Define one recourse policy over each subset.

[3] Postek and Den Hertog, 2016 and Bertsimas and Dunning, 2016

Piecewise constant rules by uncertainty set partitioning[3]

- Let at iteration r:

$$\Xi = \bigcup_{k \in N_r} \Xi_{rk}.$$

- We solve:

$$\begin{aligned}
\min_{\substack{\mathbf{x}\in\mathcal{X},\theta\in\mathbb{R}\\ \mathbf{y}^{(r,1)},\ldots,\mathbf{y}^{(r,N_r)}\in\mathcal{Y}}} \quad & \mathbf{c}^\top \mathbf{x} + \theta & & \text{(Part-r)}\\
\text{s.t.} \quad & \theta \geq \mathbf{f}(\boldsymbol{\xi})^\top \mathbf{y}^{(r,k)} & \forall k \in [N_r], \boldsymbol{\xi} \in \Xi_{rk} &\\
& \mathbf{T}(\boldsymbol{\xi})\mathbf{x} + \mathbf{W}(\boldsymbol{\xi})\mathbf{y}^{(r,k)} \leq \mathbf{h}(\boldsymbol{\xi}) & \forall k \in [N_r], \boldsymbol{\xi} \in \Xi_{rk} &
\end{aligned}$$

Remark

- (Part-r) is a static robust optimization problem, it is semi-infinite.
- Under the polyhedral subsets assumption it can be reformulated as a monolithic LP/MILP.
- Rest of the algorithm follows similarly.

[3] Postek and Den Hertog, 2016 and Bertsimas and Dunning, 2016

Piecewise constant rules by uncertainty set partitioning[3]

- When mixed-integer recourse is considered the numerical burden of the algorithm becomes more important.
- This burden increases with the number of subsets in the partition.
- We described a nested partitioning approach and partitioned each subset at each iteration.
- One can instead partition the original uncertainty set from scratch or choose not to further partition certain subsets. In particular:

try to push each θ_k as low as possible

$$
\begin{aligned}
\min_{\substack{\mathbf{x}\in\mathcal{X},\theta\in\mathbb{R}\\ \mathbf{y}^{(r,1)},\ldots,\mathbf{y}^{(r,N_r)}\in\mathcal{Y}}} \quad & \mathbf{c}^\top \mathbf{x} + \theta + \epsilon \sum_{k\in[N_r]} \theta_k && \text{(Part-r)}\\
\text{s.t.} \quad & \theta \geq \theta_k && \forall k \in [N_r]\\
& \theta_k \geq \mathbf{f}(\boldsymbol{\xi})^\top \mathbf{y}^{(r,k)} && \forall k \in [N_r], \boldsymbol{\xi} \in \Xi_{rk}\\
& \mathbf{T}(\boldsymbol{\xi})\mathbf{x} + \mathbf{W}(\boldsymbol{\xi})\mathbf{y}^{(r,k)} \leq \mathbf{h}(\boldsymbol{\xi}) && \forall k \in [N_r], \boldsymbol{\xi} \in \Xi_{rk}
\end{aligned}
$$

if $\theta_k \neq \theta$ then no need to further partition the subset k. $\sim$ Partition only the subsets blocking the objective function

[3] Postek and Den Hertog, 2016 and Bertsimas and Dunning, 2016

An example [4]

- Consider for $\epsilon \in (0,1)$:

$$
\begin{aligned}
z(\epsilon) = \min_{\theta \in \mathbb{R}, y_1 \in \{0,1\}, y_2 \in \{0,1\}} \quad & \theta \\
\text{s.t.} \quad & \theta \geq y_1(\xi) + y_2(\xi) && \forall \xi \in [0,1] \\
& y_1(\xi) \geq \epsilon - \xi && \forall \xi \in [0,1] \\
& y_2(\xi) \geq -\epsilon + \xi && \forall \xi \in [0,1]
\end{aligned}
$$

- Optimal static solution $\theta = 2$ with $y_1, y_2 = 1$.
- Optimal adjustable solution $\theta = 1$ with policy:

$$
y_1(\xi) = \begin{cases} 1 & 0 \leq \xi < \epsilon \\ 0 & \epsilon \leq \xi \leq 1 \end{cases} \qquad y_2(\xi) = \begin{cases} 0 & 0 \leq \xi \leq \epsilon \\ 1 & \epsilon < \xi \leq 1 \end{cases}
$$

Attention!

- Partitioning methods will never find an optimal solution or decrease the bound gap unless a subset $[a, b]$ such that $a = \epsilon$ or $b = \epsilon$ is created.

[4] from Bertsimas and Dunning, 2016

An example [4]

for $y_1(\xi) \geq \epsilon - \xi$

for $y_2(\xi) \geq \xi - \epsilon$

- Starting from the static solution, $\xi = 0$ and $\xi = 1$ are identified as the worst case realizations.
- Then partitioning $\Xi = [0, 1]$ based on these realizations will create $\Xi_1 = [0, 0.5]$ and $\Xi_2 = [0.5, 1]$.
- The resulting optimization problem that needs to be solved is then:

$$
\begin{aligned}
z(\epsilon) := \min_{\substack{\theta \in \mathbb{R},\, y_1^{(1,1)}, y_2^{(1,1)} \in \{0,1\} \\ y_1^{(1,2)}, y_2^{(1,2)} \in \{0,1\}}} \quad & \theta \\
\text{s.t.} \quad & \theta \geq y_1^{(1,1)}(\xi) + y_2^{(1,1)}(\xi) && \forall \xi \in [0, 0.5] \\
& \theta \geq y_1^{(1,2)}(\xi) + y_2^{(1,2)}(\xi) && \forall \xi \in [0.5, 1] \\
& y_1^{(1,1)}(\xi) \geq \epsilon - \xi \quad : \epsilon > 0 && \forall \xi \in [0, 0.5] \\
& y_2^{(1,1)}(\xi) \geq -\epsilon + \xi \quad : 0.5 - \epsilon && \forall \xi \in [0, 0.5] \\
& y_1^{(1,2)}(\xi) \geq \epsilon - \xi \quad : \epsilon - 0.5 && \forall \xi \in [0.5, 1] \\
& y_2^{(1,2)}(\xi) \geq -\epsilon + \xi \quad : 1 - \epsilon > 0 && \forall \xi \in [0.5, 1].
\end{aligned}
$$

one of these is > 0 unless $\epsilon = 0.5$

- Optimal solution: $\theta = 2$, $y_1^{(1,1)}(\xi) = 1$, $y_2^{(1,1)}(\xi) = 1$, $y_1^{(1,2)}(\xi) = 1$, $y_2^{(1,2)}(\xi) = 1$.

[4] from Bertsimas and Dunning, 2016

An example [4]

- At each iteration, the algorithm will indentify the extreme points of each sub-interval and create new sub-intervals based on these extreme points.
- If one of these sub-intervals, say $[a, b]$, is such that $a < \epsilon < b$, then:

$$
\begin{aligned}
y_1(\xi) &\geq \epsilon - \xi > 0 && \xi \in [a, b] \\
y_2(\xi) &\geq \xi - \epsilon > 0 && \xi \in [a, b]
\end{aligned}
$$

- This implies that θ will be lower bounded by $y_1(\xi) + y_2(\xi) = 2$.
- Indeed, given a finite number of iterations, the uncertainty set partitioning algorithms will never find a solution of value 1 or decrease the bound gap unless a subset $[a, b]$ such that $a = \epsilon$ or $b = \epsilon$ is created.
- Assuming that each iteration a sub-interval is divided into two from the middle of the sub-interval, such a sub-interval will only be created if $\epsilon = \frac{q}{2^p}$ for some $q, p \in \mathbb{Z}_+$.

[4] from Bertsimas and Dunning, 2016

Combining piecewise affine and constant rules[5]

$$
\begin{array}{lll}
\min\limits_{\substack{\mathbf{x}\in\mathcal{X},\theta\in\mathbb{R}\\ \mathbf{y}^C(\cdot):\Xi\to\mathbb{R}^{n_y^c}\\ \mathbf{y}^D(\cdot):\Xi\to\mathbb{Z}^{n_y^d}}} & \mathbf{c}^\top\mathbf{x}+\theta & \\
\text{s.t.} & \theta \geq \mathbf{f}_C(\boldsymbol{\xi})^\top\mathbf{y}^C(\boldsymbol{\xi})+\mathbf{f}_D(\boldsymbol{\xi})^\top\mathbf{y}^D(\boldsymbol{\xi}) & \forall\boldsymbol{\xi}\in\Xi \\
& \mathbf{T}(\boldsymbol{\xi})\mathbf{x}+\mathbf{W}_C(\boldsymbol{\xi})\mathbf{y}^C(\boldsymbol{\xi})+\mathbf{W}_D(\boldsymbol{\xi})\mathbf{y}^D(\boldsymbol{\xi})\leq\mathbf{h}(\boldsymbol{\xi}) & \forall\boldsymbol{\xi}\in\Xi \\
& (\mathbf{y}^C(\boldsymbol{\xi}),\mathbf{y}^D(\boldsymbol{\xi}))\in\mathcal{Y}\subseteq\mathbb{R}^{n_y^c}\times\mathbb{Z}^{n_y^d} & \forall\boldsymbol{\xi}\in\Xi
\end{array}
$$

Assumptions

- $\mathbf{f}_C$ and $\mathbf{W}_C$ are deterministic

Idea

- Let $\mathbf{y}^C(\cdot)$ be piecewise affine and $\mathbf{y}^D(\cdot)$ be piecewise constant.
- Iteratively partition the uncertainty set into polyhedral subsets.
- Define one recourse policy over each subset.

[5] Postek and Den Hertog, 2016 and Bertsimas and Dunning, 2016

Combining piecewise affine and constant rules[5]

- Let at iteration r:

$$\Xi = \bigcup_{k \in N_r} \Xi_{rk}.$$

- We solve:

$$\begin{aligned}
\min_{\substack{\mathbf{x}\in\mathcal{X},\theta\in\mathbb{R} \\ \mathbf{y}^{(r,1)},\dots,\mathbf{y}^{(r,N_r)}\in\mathbb{Z}^{n_y^d} \\ \mathbf{A}^{(r,1)},\dots,\mathbf{A}^{(r,N_r)}\in\mathbb{R}^{n_y^c\times n_\xi}}} \quad & \mathbf{c}^\top \mathbf{x} + \theta && \text{(Part-r)} \\
\text{s.t.} \quad & \theta \geq \mathbf{f}_C^\top \mathbf{A}^{(r,k)}\boldsymbol{\xi} + \mathbf{f}_D(\boldsymbol{\xi})^\top \mathbf{y}^{(r,k)} && \forall k \in [N_r], \boldsymbol{\xi} \in \Xi_{rk} \\
& \mathbf{T}(\boldsymbol{\xi})\mathbf{x} + \mathbf{W}_C \mathbf{A}^{(r,k)}\boldsymbol{\xi} + \mathbf{W}_D(\boldsymbol{\xi})\mathbf{y}^{(r,k)} \leq \mathbf{h}(\boldsymbol{\xi}) && \forall k \in [N_r], \boldsymbol{\xi} \in \Xi_{rk} \\
& (\mathbf{A}^{(r,k)}\boldsymbol{\xi}, \mathbf{y}^{(r,k)}) \in \mathcal{Y} && \forall k \in [N_r], \boldsymbol{\xi} \in \Xi_{rk}
\end{aligned}$$

Remark

- (Part-r) is a static robust optimization problem, it is semi-infinite.
- Under the polyhedral subsets assumption it can be reformulated as a monolithic MILP.

[5] Postek and Den Hertog, 2016 and Bertsimas and Dunning, 2016