Adjustable Robust Optimization

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CR03 - Robust combinatorial optimization, ENS-Lyon

Little reminder

Write $\min_{\xi \in \Xi_i} a_i(\xi)^\top x \geq b_i \quad \forall i \in [m]$ with $a_i(\xi) = A_i \xi$ then

$$\begin{array}{ll} \min & x^\top A_i \xi \\ \text{s.t} & \xi \in \Xi_i \end{array} = \begin{array}{ll} \max_{u \geq 0} & d_i^\top u \\ & D_i^\top u = x^\top A_i \end{array} \geq b_i$$

drop the max

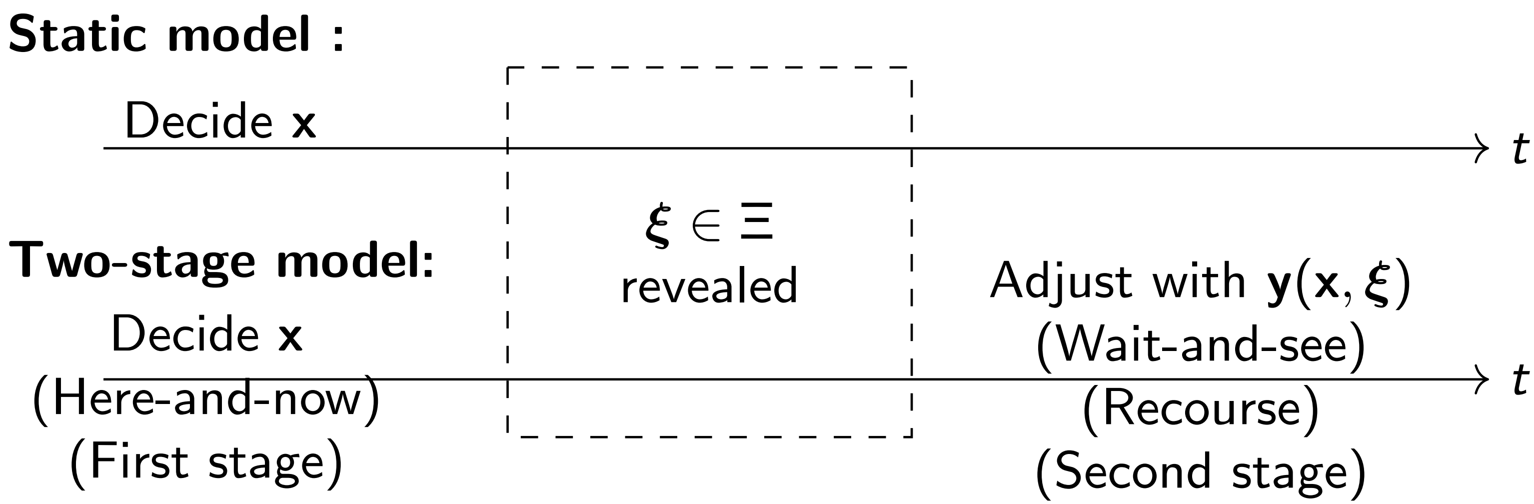
- We talked about "static" robust optimization problems:

$$\begin{array}{lll} \min & \mathbf{c}^\top \mathbf{x} & \\ \text{s.t.} & \mathbf{a}_i(\boldsymbol{\xi})^\top \mathbf{x} \geq b_i & \forall \boldsymbol{\xi} \in \Xi_i, \forall i \in [m] \\ & \mathbf{x} \in \mathcal{X} & \end{array}$$

$\xi \in \mathbb{R}^{n_\xi} \mid D_i \xi \geq d_i$

- Here, the word "static" refers to the fact that all decision variables are decided before the realization of uncertainty or are "here-and-now".
- Under this assumption many robust optimization problems may be reformulated as deterministic-equivalent problems.
- The reformulation often preserves the class of complexity of the deterministic problem.

Today: Adjustable robust optimization



- The variables that can "adjust/adapt" to the realization of uncertainty will then be called "adjustable/adaptable" or "recourse" variables.
- When all adjustable variables are continuous we will call it "continuous recourse".
- When some adjustable variables are required to be integer we will call it "mixed-integer recourse".

Today: Adjustable robust optimization

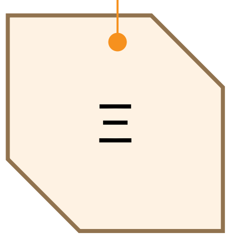
Locate healthcare centers and allocate population

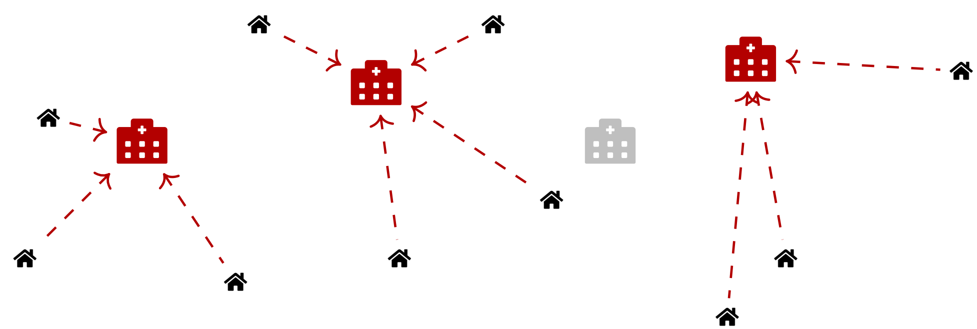
$\mathbf{x}$

$f(\xi, \mathbf{x})$ = number of untreated patients

in anticipation of possible disasters

ξ





Static model

$$\min_{\mathbf{x}\in\mathcal{X}} \max_{\xi\in\Xi} \; f(\xi, \mathbf{x})$$

Today: Adjustable robust optimization

First-stage decision

Recourse decision

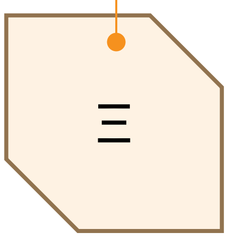
Locate healthcare centers

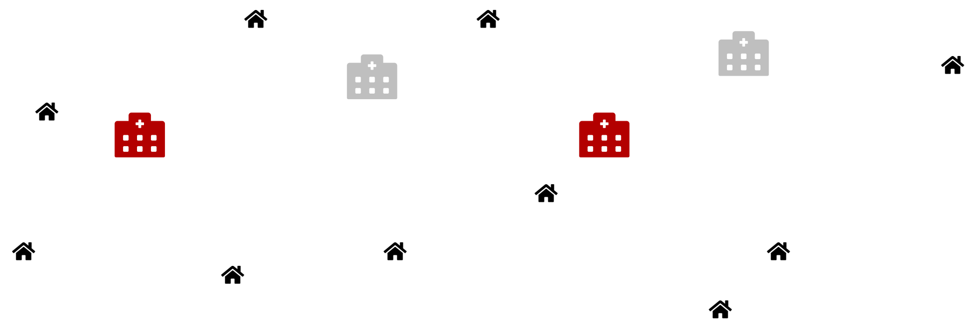
$\mathbf{x}$

Locate emergency centers and allocate population

in anticipation of possible disasters

ξ





Today: Adjustable robust optimization

First-stage decision

Locate healthcare centers

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Recourse decision

Locate emergency centers and allocate population

$\mathcal{Q}(\mathbf{x}, \xi)$ = number of untreated patients

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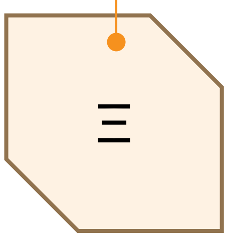


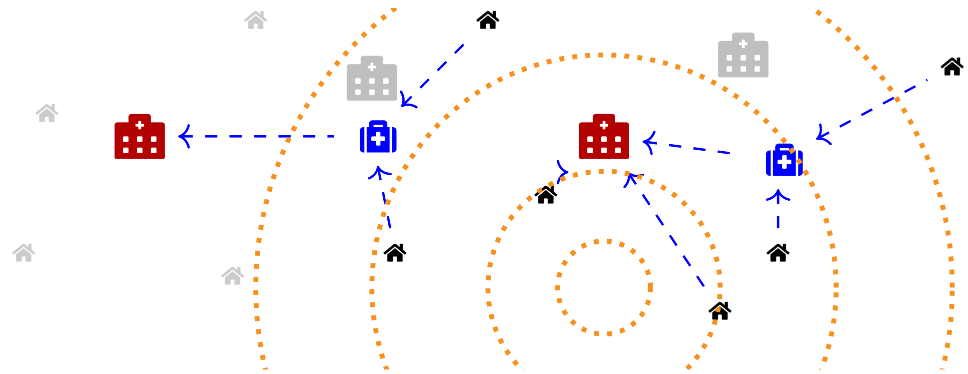
Recourse problem

$$\mathcal{Q}(\mathbf{x}, \xi) = \min \{ f(\xi, \mathbf{x}, \mathbf{y}) \mid \mathbf{y} \in \mathcal{Y}(\mathbf{x}, \xi) \}$$

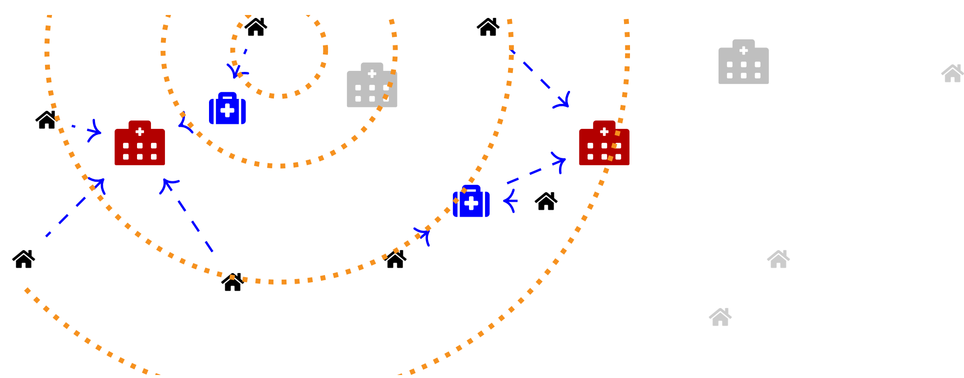
An optimization problem needs to be solved to determine the best recourse actions

Decisions in blue are made given a location decision x and a disaster realization ξ

Scenario 1



Scenario 2



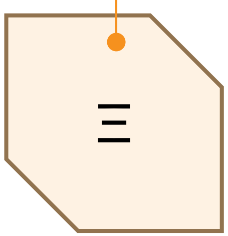
Today: Adjustable robust optimization

First-stage decision

Locate healthcare centers

in anticipation of possible disasters

ξ



$\mathbf{x}$

Recourse decision

Locate emergency centers and allocate population

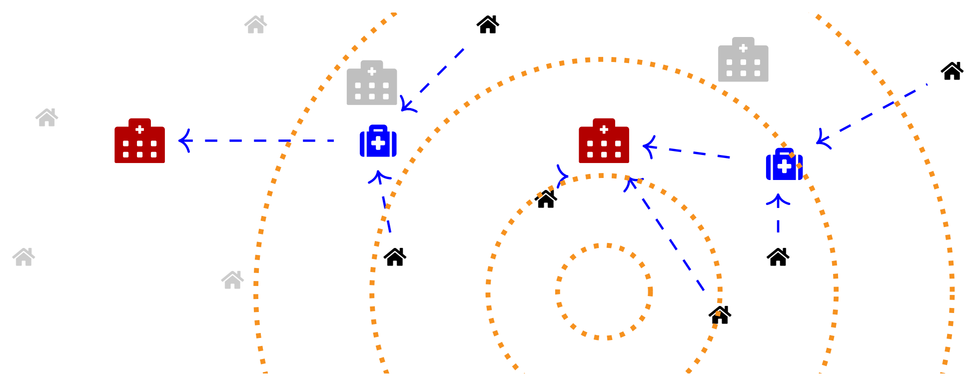
$\underbrace{\text{}}_{}$ $\mathcal{Q}(\mathbf{x}, \xi)$ = number of untreated patients

Two-stage model

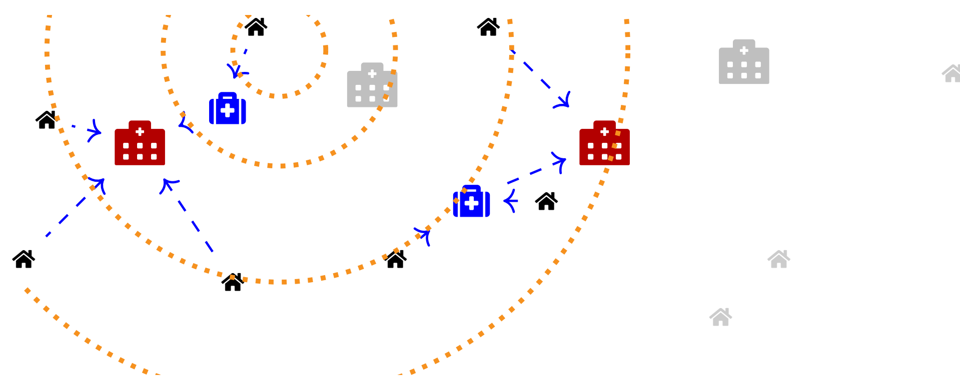
$$\min_{\mathbf{x} \in \mathcal{X}} \max_{\xi \in \Xi} \; \mathcal{Q}(\mathbf{x}, \xi)$$

↳ determine best first-stage action x to minimize worst-case future cost under optimal recourse

Scenario 1



Scenario 2



Sequential decision-making under uncertainty

$$x_0 \xrightarrow{\xi^1} x_1(\xi^1) \xrightarrow{\xi^2 \mid \xi_1} x_2(\xi^2) \xrightarrow{\xi^3 \mid \xi_1, \xi_2} x_3(\xi^3) \quad \cdots\cdots$$

In the two-stage setting :

$$x \xrightarrow{\xi \text{ (all uncertain parameters are revealed at once)}} y(\xi)$$

If we think about it with discrete uncertainty realisations say $\Xi = \{\bar{\xi}^1, \bar{\xi}^2, \bar{\xi}^3\}$

$$x \xrightarrow{\xi = \bar{\xi}^1} y_1^* = \underset{y \in \mathcal{Y}(x, \bar{\xi}^1)}{\operatorname{argmin}} Q(x, \bar{\xi}^1) := \min_{y \in \mathcal{Y}} f^T(\bar{\xi}^1) y \quad W(\bar{\xi}^1) y \leq h(\bar{\xi}^1) - T(\bar{\xi}^1) x$$

$$x \xrightarrow{\xi = \bar{\xi}^2} y_2^* = \underset{y \in \mathcal{Y}(x, \bar{\xi}^2)}{\operatorname{argmin}} Q(x, \bar{\xi}^2) = \min_{y \in \mathcal{Y}} f^T(\bar{\xi}^2) y \quad W(\bar{\xi}^2) y \leq h(\bar{\xi}^2) - T(\bar{\xi}^2) x$$

$$x \xrightarrow{\xi = \bar{\xi}^3} y_3^* = \underset{y \in \mathcal{Y}(x, \bar{\xi}^3)}{\operatorname{argmin}} Q(x, \bar{\xi}^3) = \min_{y \in \mathcal{Y}} f^T(\bar{\xi}^3) y \quad W(\bar{\xi}^3) y \leq h(\bar{\xi}^3) - T(\bar{\xi}^3) x$$

How do we compare two solutions x^1 & x^2 ?

we would like to optimize some function $\rho_{\xi \in \Xi}\left(Q(x, \xi)\right)$

Expectation	$\mathbb{E}^{\mathbb{P}}_{\xi \in \Xi}\left[Q(x, \xi)\right]$
CVaR	$\mathrm{CVaR}^{\mathbb{P}}_{\xi \in \Xi}\left(Q(x, \xi)\right) = \inf_{t \in \mathbb{R}} \; t + \frac{1}{1-\alpha} \mathbb{E}\left[Q(x, \xi) - t\right]_+$
Max (Robust)	$\max_{\xi \in \Xi} Q(x, \xi)$
Max over a family of dist.	$\max_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^{\mathbb{P}}_{\xi \in \Xi}\left[Q(x, \xi)\right] \sim$ distributionally robust

Two-stage RO: Problem definition

- $\mathcal{X} \subseteq \mathbb{R}^{n_x^c} \times \mathbb{Z}^{n_x^d}$: first-stage feasible region
- $\boldsymbol{\xi}$: uncertain vector with support $\Xi \subseteq \mathbb{R}^{n_\xi}$
- $\Xi \subseteq \mathbb{R}^{n_\xi}$: uncertainty set, non-empty and compact
- $\mathcal{Y} \subseteq \mathbb{R}^{n_y^c} \times \mathbb{Z}^{n_y^d}$: recourse feasible region

$$\min_{\mathbf{x}\in\mathcal{X}} \quad \mathbf{c}^\top \mathbf{x} + \max_{\boldsymbol{\xi}\in\Xi} \mathcal{Q}(\mathbf{x}, \boldsymbol{\xi}) \tag{2ARO}$$

with

$$\begin{aligned} \mathcal{Q}(\mathbf{x}, \boldsymbol{\xi}) = \min \quad & \mathbf{f}(\boldsymbol{\xi})^\top \mathbf{y} \\ \text{s.t.} \quad & \mathbf{W}(\boldsymbol{\xi})\mathbf{y} \leq \mathbf{h}(\boldsymbol{\xi}) - \mathbf{T}(\boldsymbol{\xi})\mathbf{x} \\ & \mathbf{y} \in \mathcal{Y} \end{aligned}$$

we look at only linear problems

Otherwise we should write

$$\min_{x\in X} \quad c^\top x + \sup_{\xi\in\Xi} \mathcal{Q}(x,\xi)$$

Remark

In writing (2ARO) *we assume that* $\mathcal{Q}(\mathbf{x}, \boldsymbol{\xi})$ *is an upper semi-continuous function in* $\boldsymbol{\xi} \in \Xi$.

Two-stage RO: Problem definition

Hypothesis and Notation

- $\mathcal{X}$ and $\mathcal{Y}$ are linearly constrained

can have
discrete or continuous
decision variables

- Ξ is polyhedral, *i.e.*, $\Xi = \{\boldsymbol{\xi} \in \mathbb{R}_+^{n_\xi} \mid \mathbf{D}\boldsymbol{\xi} \leq \mathbf{d}\}$
- Data is affine in $\boldsymbol{\xi}$:

$$\mathbf{h}(\boldsymbol{\xi}) = \begin{bmatrix} 1+\xi_1 \\ 1-\xi_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ \xi_1 \\ \xi_2 \end{bmatrix} = \mathbf{H}\tilde{\boldsymbol{\xi}}$$

$$\begin{aligned} \mathbf{T}(\boldsymbol{\xi}) &= \begin{bmatrix} 2+\xi_1-\xi_2 & 1-\xi_1+\xi_2 \\ 3-\xi_2 & 2-\xi_1 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix} \xi_1 + \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} \xi_2 \\ &= \mathbf{T_0} + \sum_{i=1}^{2} \mathbf{T_i}\boldsymbol{\xi_i} \rightarrow \sum_{i=0}^{2} \mathbf{T_i}\boldsymbol{\xi_i} \text{ with } \boldsymbol{\xi}_0 = 1 \end{aligned}$$

Two-stage RO: Formulations

Min - Max - Min formulation

$$\min_{x \in \mathcal{X}} \quad \mathbf{c}^\top \mathbf{x} + \max_{\xi \in \Xi} \; \mathcal{Q}(\mathbf{x}, \xi) = \min_{x \in \mathcal{X}} \quad \mathbf{c}^\top \mathbf{x} + \underbrace{\max_{\xi \in \Xi} \; \underbrace{\min_{y \in \mathcal{Y}} \; \mathbf{f}(\xi)^\top \mathbf{y} \quad \text{s.t.} \; \mathbf{W}(\xi)\mathbf{y} \leq \mathbf{h}(\xi) - \mathbf{T}(\xi)\mathbf{x}}_{Q(x,\xi)}}_{Q(x) = \max_{\xi \in \Xi} Q(x,\xi)}$$

Overall problem can be seen as:

$$\min_{x \in \mathcal{X}} c^\top x + Q(x)$$

- With the convention that if

$$\mathcal{Y}(\mathbf{x}, \bar{\xi}) = \{\mathbf{y} \in \mathcal{Y} \mid \mathbf{W}(\bar{\xi})\mathbf{y} \leq \mathbf{h}(\bar{\xi}) - \mathbf{T}(\bar{\xi})\mathbf{x}\} = \emptyset$$

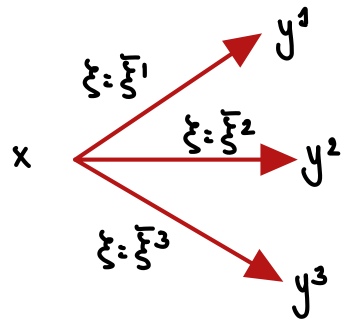
  for some $\bar{\xi} \in \Xi$ then

$$\min_{y \in \mathcal{Y}(x, \bar{\xi})} \mathbf{f}(\bar{\xi})^\top \mathbf{y} = +\infty.$$

  With this convention any $x \in \mathcal{X}$ leading to infeasibility is not a candidate for being optimal

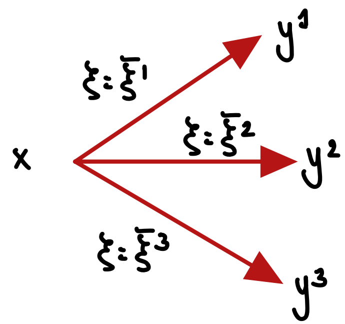
- Useful for exact solution schemes (more on this later).

Two-stage RO: Formulations



Write :

$$
\begin{aligned}
\min_{x \in X} \quad & c^T x + \max_{i=1,2,3} f_i^T y_i \\
\text{s.t} \quad & T_i x + W_i y_i \leq h_i \qquad i=1,2,3 \\
& y_i \in \mathcal{Y} \qquad\qquad\quad\;\; i=1,2,3
\end{aligned}
$$



If ξ is continuous then y^{ξ} becomes a func.

$y(\cdot): \Xi \to \mathcal{Y}$

$$
\begin{aligned}
\min_{x \in X} \quad & c^T x + \max_{\xi \in \Xi} f^T(\xi) y(\xi) \\
\text{s.t} \quad & \left.\begin{aligned} & T(\xi) x + W(\xi) y(\xi) \leq h(\xi) \\ & y(\xi) \in \mathcal{Y} \end{aligned}\right\} \quad \forall \xi \in \Xi
\end{aligned}
$$

Two-stage RO: Formulations

For polyhedral Ξ :

- infinite # of variables
- infinite # of constraints

$$
\begin{aligned}
\min_{\mathbf{x}\in\mathcal{X},\mathbf{y}(\cdot)} \quad & \mathbf{c}^\top \mathbf{x} + \max_{\boldsymbol{\xi}\in\Xi} \quad \mathbf{f}(\boldsymbol{\xi})^\top \mathbf{y}(\boldsymbol{\xi}) \\
\text{s.t.} \quad & \mathbf{T}(\boldsymbol{\xi})\mathbf{x} + \mathbf{W}(\boldsymbol{\xi})\mathbf{y}(\boldsymbol{\xi}) \leq \mathbf{h}(\boldsymbol{\xi}) && \forall \boldsymbol{\xi} \in \Xi \\
& \mathbf{y}(\boldsymbol{\xi}) \in \mathcal{Y} && \forall \boldsymbol{\xi} \in \Xi
\end{aligned}
$$

- $\mathbf{y}(\cdot) : \Xi \to \mathcal{Y}$ are functionals to be optimized.
- Useful for approximation schemes known as "decision rules" (more on this later).

Remark

Any $\mathbf{x} \in \mathcal{X}$ for which there exists $\bar{\boldsymbol{\xi}} \in \Xi$ such that

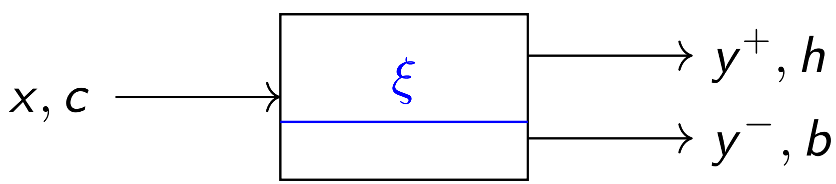
$$
\mathcal{Y}(\mathbf{x}, \bar{\boldsymbol{\xi}}) = \{\mathbf{y} \in \mathcal{Y} \mid \mathbf{W}(\bar{\boldsymbol{\xi}})\mathbf{y} \leq \mathbf{h}(\bar{\boldsymbol{\xi}}) - \mathbf{T}(\bar{\boldsymbol{\xi}})\mathbf{x}\} = \emptyset
$$

cannot be a feasible solution to this model.

Two-stage RO: A newsvendor example

$\bar{\xi} - \rho\bar{\xi} \leq \xi \leq \bar{\xi} + \rho\bar{\xi}$

- Order quantity x
- Uncertain demand ξ with $\xi \in \Xi = \{|\xi - \bar{\xi}| \leq \rho\bar{\xi}\}$ where $\bar{\xi} \geq 0$
- Excess quantity $y^+(\xi)$
- Shortage quantity $y^-(\xi)$
- Order cost c, return cost $h > 0$ and shortage cost $b > 0$



$$\xi \in \Xi = \{|\xi - \bar{\xi}| \leq \rho\bar{\xi}\}$$

Two-stage RO: A newsvendor example

- Two-stage problem written in the min-max-min form:

$$\min_{x\geq 0} \quad cx + \max_{\xi\in\Xi} \quad \min_{y^+\geq 0, y^-\geq 0} \quad hy^+ + by^-$$
$$y^+ - y^- = x - \xi$$

given x & ξ
either y^+ or $y^- > 0$
depending on
$x-\xi > 0$ or < 0

- Two-stage problem written in the functional form:

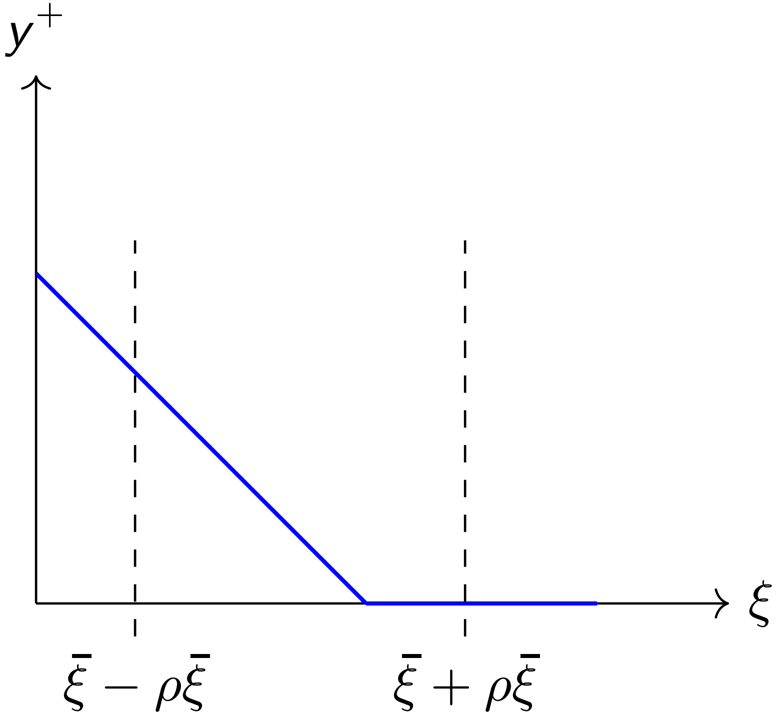
$$\min_{\substack{x\geq 0\\ y^+(\cdot):\Xi\to\mathbb{R}_+\\ y^-(\cdot):\Xi\to\mathbb{R}_+}} \quad cx + \max_{\xi\in\Xi} \quad hy^+(\xi) + by^-(\xi)$$
$$\text{s.t.} \quad y^+(\xi) - y^-(\xi) = x - \xi \qquad \forall \xi \in \Xi$$

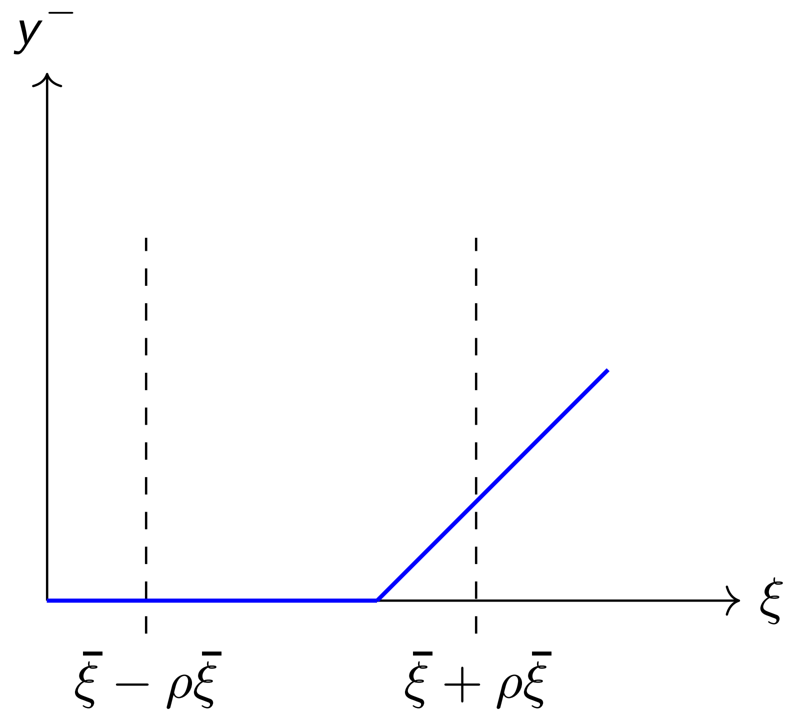
- For this problem, the optimal recourse can be written as:

$$y^+ = \max\{x-\xi, 0\}$$
$$y^- = \max\{0, \xi - x\}$$

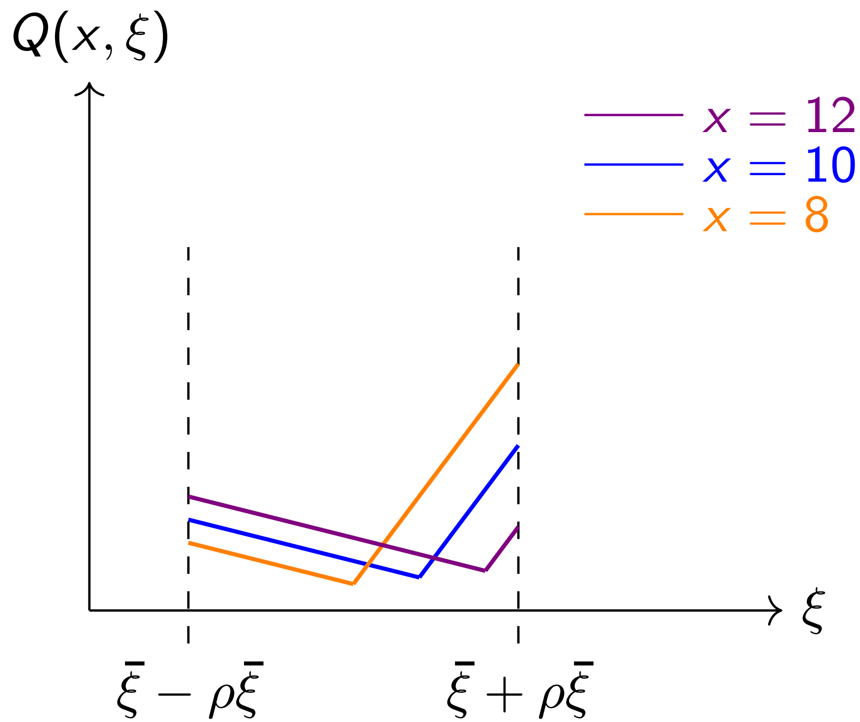
Two-stage RO: A newsvendor example

- Optimal recourse quantities as a function of ξ when $x = 10$:





Two-stage RO: A newsvendor example



- When x is low worst case is $\bar{\xi} + \rho\bar{\xi}$, otherwise worst case is $\bar{\xi} - \rho\bar{\xi}$.
- Optimality is achieved for x such that:

$$Q(x, \bar{\xi} + \rho\bar{\xi}) = Q(x, \bar{\xi} - \rho\bar{\xi})$$

find the balance
between return & shortage costs

Two-stage RO: Relaxations and Restrictions

- Consider $\tilde{\Xi} \subset \Xi$:

$$\min_{\mathbf{x}\in\mathcal{X}} \quad \mathbf{c}^\top \mathbf{x} + \max_{\boldsymbol{\xi}\in\tilde{\Xi}} \quad \min_{\mathbf{y}\in\mathcal{Y}} \quad \mathbf{f}(\boldsymbol{\xi})^\top \mathbf{y}$$
$$\text{s.t.} \quad \mathbf{W}(\boldsymbol{\xi})\mathbf{y} \leq \mathbf{h}(\boldsymbol{\xi}) - \mathbf{T}(\boldsymbol{\xi})\mathbf{x}$$

  is a relaxation of (2RO) and therefore provides a lower bound.

- In particular, for a finite subset $\tilde{\Xi} = \{\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_N\} \subset \Xi$ we can write:

$$\begin{aligned}
\min_{\mathbf{x}\in\mathcal{X},\mathbf{y}_1,\ldots,\mathbf{y}_N\in\mathcal{Y}} \quad & \mathbf{c}^\top \mathbf{x} + \theta \\
\text{s.t.} \quad & \theta \geq \mathbf{f}_\ell^\top \mathbf{y}_\ell && \ell = 1, \ldots, N \\
& \mathbf{T}_\ell \mathbf{x} + \mathbf{W}_\ell \mathbf{y}_\ell \leq \mathbf{h}_\ell && \ell = 1, \ldots, N
\end{aligned}$$

But how to choose $\tilde{\Xi}$ to get a good bound?

  and solve the problem as a (large-scale) (mixed-integer) linear program.

Two-stage RO: Relaxations and Restrictions

- Consider $\tilde{\mathcal{Y}} \subset \mathcal{Y}$:

$$\min_{\mathbf{x}\in\mathcal{X}} \quad \mathbf{c}^\top \mathbf{x} + \max_{\boldsymbol{\xi}\in\Xi} \quad \min_{\mathbf{y}\in\tilde{\mathcal{Y}}} \quad \mathbf{f}(\xi)^\top \mathbf{y}$$
$$\text{s.t.} \quad \mathbf{W}(\xi)\mathbf{y} \leq \mathbf{h}(\xi) - \mathbf{T}(\xi)\mathbf{x}$$

But how to restrict $\mathcal{Y}$ in a meaningful way?

is a restriction of (2RO) and therefore provides an upper bound.

Two-stage continuous RO: Exact algorithms

$$\min_{\mathbf{x}\in\mathcal{X}} \quad \mathbf{c}^\top \mathbf{x} + \max_{\boldsymbol{\xi}\in\Xi} \quad \min_{\mathbf{y}\in\mathcal{Y}} \quad \mathbf{f}(\xi)^\top \mathbf{y}$$

$$\text{s.t.} \quad \mathbf{W}(\xi)\mathbf{y} \le \mathbf{h}(\xi) - \mathbf{T}(\xi)\mathbf{x}$$

Assume

- $\mathcal{Y} = \mathbb{R}_+^{n_y} \rightarrow$ continuous recourse
- $\mathbf{f}(\xi)$ and $\mathbf{W}(\xi)$ are deterministic (this is called fixed recourse in the literature)
- Relatively complete recourse, *i.e.*, $\mathcal{Y}(\mathbf{x}, \xi) \neq \emptyset$ for $\mathbf{x} \in \mathcal{X}, \boldsymbol{\xi} \in \Xi$

guarantees that any solution sent to the second stage has value $Q(x) < +\infty$

$\min_{y\in Y} \; f(\xi)^T y$ is feasible for any $x \in X$
$W(\xi)\, y \le h(\xi) - T(\xi)\, x$

Two-stage continuous RO: Exact algorithms

$$\begin{aligned} \min_{\mathbf{x}\in\mathcal{X}} \quad \mathbf{c}^\top \mathbf{x} + \max_{\boldsymbol{\xi}\in\Xi} \quad \min_{\mathbf{y}\in\mathbb{R}^{n_y}_+} \quad & \mathbf{f}^\top \mathbf{y} \\ \text{s.t.} \quad & \mathbf{W}\mathbf{y} \leq \mathbf{h}(\boldsymbol{\xi}) - \mathbf{T}(\boldsymbol{\xi})\mathbf{x} \end{aligned}$$

- Start by writing the dual of the inner minimization problem for $\mathbf{x}$ and $\boldsymbol{\xi}$ given:

$$\begin{aligned} \max_{\mathbf{u}\in\mathbb{R}^{m_y}_-} \quad & \mathbf{u}^\top(\mathbf{h}(\boldsymbol{\xi}) - \mathbf{T}(\boldsymbol{\xi})\mathbf{x}) \\ \text{s.t.} \quad & \mathbf{W}^\top \mathbf{u} \leq \mathbf{f} \end{aligned}$$

- Since we assumed that the primal is feasible we have that the dual is bounded.
- We can also assume that the dual is feasible (why?).

Two-stage continuous RO: Exact algorithms

$$Q(x,\xi) = \max_{\mathbf{u}\in\mathbb{R}_-^{m_y}} \quad \mathbf{u}^\top(\mathbf{h}(\xi) - \mathbf{T}(\xi)\mathbf{x}) \qquad = \max_{u\in D} \; u^T(h(\xi)-T(\xi)x)$$

$$\text{s.t.} \quad \mathbf{W}^\top\mathbf{u} \leq \mathbf{f}$$

linear programming problem

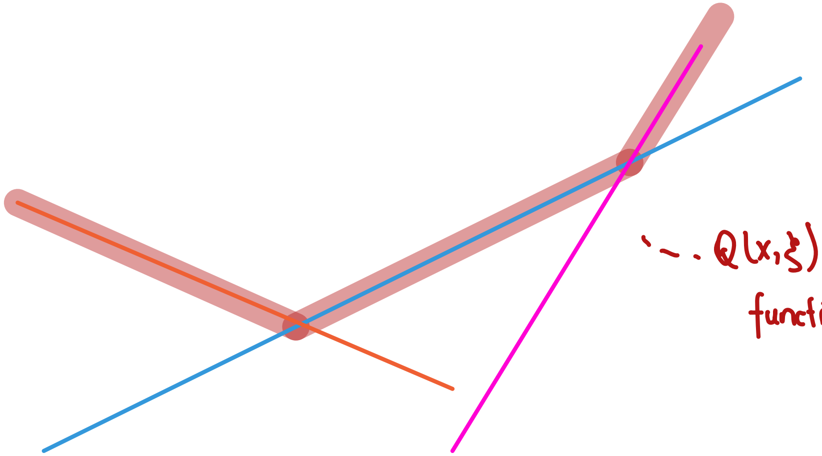
$$D := \{ u \in \mathbb{R}_-^{m_y} \mid W^T u \leq f \}$$

Let $ext(D)$ be the set of extreme points of D

$$\max_{u\in D} u^T(h(\xi)-T(\xi)x) = \max_{i=1,\dots,|ext(D)|} u_i^T(h(\xi)-T(\xi)x)$$

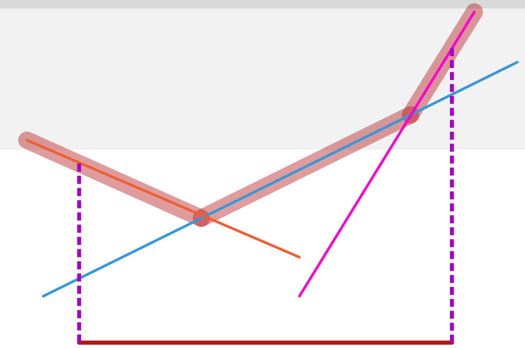
For x fixed $u_i^T(h(\xi)-T(\xi)x)$ defines an affine function of ξ for $i=1,\dots,|ext(D)|$

$$\max_{i=1,\dots,|ext(D)|} u_i^T(h(\xi)-T(\xi)x) \quad :$$



$Q(x,\xi)$ is a convex function of $\xi\in\Xi$

Two-stage continuous RO: Exact algorithms



- Since $\mathcal{Q}(\mathbf{x}, \boldsymbol{\xi})$ is convex, we know that an optimal solution of $\max_{\boldsymbol{\xi} \in \Xi} \mathcal{Q}(\mathbf{x}, \boldsymbol{\xi})$ is an extreme point of Ξ.
- We may therefore write

$$\max_{\boldsymbol{\xi} \in \Xi} \mathcal{Q}(\mathbf{x}, \boldsymbol{\xi}) = \max_{\boldsymbol{\xi}_j \in \mathrm{ext}(\Xi)} \mathcal{Q}(\mathbf{x}, \boldsymbol{\xi}_j) = \max_{\boldsymbol{\xi}_j \in \mathrm{ext}(\Xi), \mathbf{u}_i \in \mathrm{ext}(\mathcal{D})} \mathbf{u}_i^\top (\mathbf{h}(\boldsymbol{\xi}_j) - \mathbf{T}(\boldsymbol{\xi}_j)\mathbf{x})$$

- This leads to a first exponential-sized reformulation of the problem:

$$\begin{aligned} \min_{\mathbf{x} \in \boldsymbol{\mathcal{X}}, \theta \in \mathbb{R}} \quad & \mathbf{c}^\top \mathbf{x} + \theta \\ \text{s.t.} \quad & \theta \geq \mathbf{u}_i^\top (\mathbf{h}(\boldsymbol{\xi}_j) - \mathbf{T}(\boldsymbol{\xi}_j)\mathbf{x}) \qquad \forall \boldsymbol{\xi}_j \in \mathrm{ext}(\Xi), \mathbf{u}_i \in \mathrm{ext}(\mathcal{D}) \end{aligned}$$

- monolithic & linear
- exponential sized
- need to enumerate ext(Ξ), ext($\mathcal{D}$)

Two-stage continuous RO: Exact algorithms

- A second formulation can be obtained by realizing that for each $\boldsymbol{\xi}_j \in \mathrm{ext}(\Xi)$, there is a dedicated optimal recourse solution $\mathbf{y}_j^* \in \mathbb{R}_+^{n_y}$ such that

$$\mathbf{W}\mathbf{y}_j^* \leq \mathbf{h}(\boldsymbol{\xi}) - \mathbf{T}(\boldsymbol{\xi})\mathbf{x}$$

- We then have that

$$\max_{\boldsymbol{\xi}\in\Xi} \mathcal{Q}(\mathbf{x},\boldsymbol{\xi}) = \max_{\boldsymbol{\xi}_j\in\mathrm{ext}(\Xi)} \mathcal{Q}(\mathbf{x},\boldsymbol{\xi}_j) = \max_{j=1,\dots,|\mathrm{ext}(\Xi)|} \mathbf{f}^\top \mathbf{y}_j^*$$

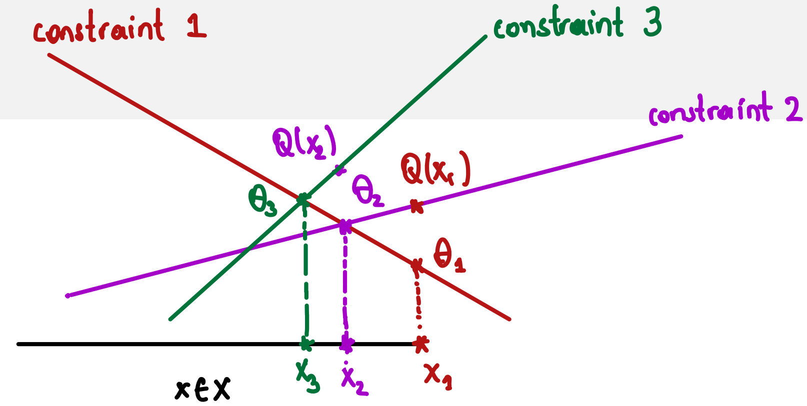
- Which leads to yet another exponential size formulation of the problem:

- monolithic & linear
- exponential sized
- need to enumerate ext(Ξ)

$$\begin{array}{lll} \displaystyle\min_{\substack{\mathbf{x}\in\mathcal{X},\theta\in\mathbb{R}\\ \mathbf{y}_1,\dots,\mathbf{y}_{|\mathrm{ext}(\Xi)|}\in\mathbb{R}_+^{n_y}}} & \mathbf{c}^\top\mathbf{x} + \theta & \\ \text{s.t.} & \theta \geq \mathbf{f}^\top\mathbf{y}_j & j = 1,\dots,|\mathrm{ext}(\Xi)| \\ & \mathbf{T}(\boldsymbol{\xi}_j)\mathbf{x} + \mathbf{W}\mathbf{y}_j \leq \mathbf{h}(\boldsymbol{\xi}_j) & j = 1,\dots,|\mathrm{ext}(\Xi)| \end{array}$$

y_j are decision var.s
& will take the necessary
values to push θ down

Two-stage continuous RO: Exact algorithms



$$\min_{\mathbf{x}\in\mathcal{X}} \quad \mathbf{c}^\top \mathbf{x} + \underbrace{\max_{\boldsymbol{\xi}\in\Xi} \quad \min_{\mathbf{y}\in\mathbb{R}^{n_y}_+} \quad \mathbf{f}^\top \mathbf{y} \quad \text{s.t.} \quad \mathbf{W}\mathbf{y} \leq \mathbf{h}(\boldsymbol{\xi}) - \mathbf{T}(\boldsymbol{\xi})\mathbf{x}}_{\theta}$$

Idea (Very high level)

- Solve a relaxation of the problem by including only a subset of the constraints on θ.
- Let $(\mathbf{x}^*, \theta^*)$ be an optimal relaxation solution.
- Solve $\max_{\boldsymbol{\xi}\in\Xi} \mathcal{Q}(\mathbf{x}^*, \boldsymbol{\xi})$ to calculate the worst-case value of $\mathbf{x}^*$.
- If $\theta^* < \max_{\boldsymbol{\xi}\in\Xi} \mathcal{Q}(\mathbf{x}^*, \boldsymbol{\xi})$ then add constraints on θ.
- Otherwise $(\mathbf{x}^*, \theta^*)$ is an optimal solution.

all necessary constraints are identified

Two-stage continous RO: Constraint-generation algorithm[1]

- Solve the relaxation:

$$\begin{aligned} \min_{\mathbf{x}\in\mathcal{X},\theta\in\mathbb{R}} \quad & \mathbf{c}^\top \mathbf{x} + \theta \\ \text{s.t.} \quad & \theta \geq \mathbf{u}_\ell^\top(\mathbf{h}(\boldsymbol{\xi}_\ell) - \mathbf{T}(\boldsymbol{\xi}_\ell)\mathbf{x}) \qquad \ell = 1, \ldots, N \end{aligned}$$

- Let $(\mathbf{x}^*, \theta^*)$ be an optimal solution.
- Solve $\max_{\boldsymbol{\xi}\in\Xi} \mathcal{Q}(\mathbf{x}^*, \boldsymbol{\xi})$.

must exist since otherwise $\theta^* = Q(x^*)$

- If $\theta^* < \mathcal{Q}(\mathbf{x}^*)$, let $(\mathbf{u}_{N+1}, \boldsymbol{\xi}_{N+1})$ define a violated constraint.
- Add the constraint:

$$\theta \geq \mathbf{u}_{N+1}^\top(\mathbf{h}(\boldsymbol{\xi}_{N+1}) - \mathbf{T}(\boldsymbol{\xi}_{N+1})\mathbf{x})$$

  to refine the relaxation.
- Otherwise $(\mathbf{x}^*, \theta^*)$ is an optimal solution.

at most $|\text{ext}(\Xi)| \times |\text{ext}(D)|$ iterations

[1] Thiele, 2009, Bertsimas et al., 2013

Two-stage continuous RO: Constraint-and-column generation (CCG) algorithm[2]

- Consider a discrete set $\tilde{\Xi} = \{\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_N\} \subset \mathrm{ext}(\Xi)$, and solve the relaxation:

$$\begin{aligned} \min_{\substack{\mathbf{x}\in\mathcal{X},\theta\in\mathbb{R} \\ \mathbf{y}_1,\ldots,\mathbf{y}_N\in\mathbb{R}_+^{n_y}}} \quad & \mathbf{c}^\top \mathbf{x} + \theta \\ \text{s.t.} \quad & \theta \geq \mathbf{f}^\top \mathbf{y}_\ell && \ell = 1, \ldots, N \\ & \mathbf{T}(\boldsymbol{\xi}_\ell)\mathbf{x} + \mathbf{W}\mathbf{y}_\ell \leq \mathbf{h}(\boldsymbol{\xi}_\ell) && \ell = 1, \ldots, N. \end{aligned}$$

- Let $(\mathbf{x}^*, \theta^*)$ be an optimal solution.
- Solve $\max_{\boldsymbol{\xi}\in\Xi} \mathcal{Q}(\mathbf{x}^*, \boldsymbol{\xi})$.
- If $\theta^* < \mathcal{Q}(\mathbf{x}^*)$, let $\boldsymbol{\xi}_{N+1}$ be a realization that needs to be added to $\tilde{\Xi}$.
- Add variables $\mathbf{y}_{N+1} \in \mathbb{R}_+^{n_y}$ and constraints:

$$\begin{aligned} & \theta \geq \mathbf{f}^\top \mathbf{y}_{N+1} \\ & \mathbf{T}(\boldsymbol{\xi}_{N+1})\mathbf{x} + \mathbf{W}\mathbf{y}_{N+1} \leq \mathbf{h}(\boldsymbol{\xi}_{N+1}) \end{aligned}$$

at most $|\Xi|$ iterations

- Otherwise $(\mathbf{x}^*, \theta^*)$ is an optimal solution.

[2] Zeng and Zhao, 2013

Two-stage continous RO: Separation problem

- At each iteration of Benders' and CCG algorithms, we need to solve:

$$\max_{\boldsymbol{\xi}\in\Xi} \underbrace{\mathcal{Q}(\mathbf{x}^*,\boldsymbol{\xi})}_{\text{convex func. of } \xi \text{ not explicitly known}} = \max_{\boldsymbol{\xi}\in\Xi} \min_{\mathbf{y}\in\mathbb{R}_+^{n_y}} \quad \mathbf{f}^\top\mathbf{y}$$
$$\text{s.t.} \quad \mathbf{W}\mathbf{y} \leq \mathbf{h}(\boldsymbol{\xi}) - \mathbf{T}(\boldsymbol{\xi})\mathbf{x}^*$$

written like this
not an optimization problem
we can give to a solver

  to evaluate the true worst-case value of a given first-stage solution $\mathbf{x}^*$.
- This is a difficult problem since it amounts to maximizing a convex function.
- Here we will talk about generic exact approaches based on reformulation as a MIP.

Two-stage continous RO: Separation problem

- Write the dual of the inner minimization problem for $\mathbf{x}^*$ and $\boldsymbol{\xi}$ given:

$$\begin{aligned} \mathcal{Q}(\mathbf{x}^*, \boldsymbol{\xi}) = \max_{\mathbf{u} \in \mathbb{R}_-^{m_y}} \quad & \mathbf{u}^\top (\mathbf{h}(\boldsymbol{\xi}) - \mathbf{T}(\boldsymbol{\xi})\mathbf{x}^*) \\ \text{s.t.} \quad & \mathbf{W}^\top \mathbf{u} \le \mathbf{f} \end{aligned}$$

- Merge the two max problems to obtain the bilinear subproblem:

$$\begin{aligned} \max \quad & \mathbf{u}^\top (\mathbf{h}(\boldsymbol{\xi}) - \mathbf{T}(\boldsymbol{\xi})\mathbf{x}^*) \\ \text{s.t.} \quad & \mathbf{W}^\top \mathbf{u} \le \mathbf{f} \\ & \mathbf{u} \in \mathbb{R}_-^{m_y} \\ & \boldsymbol{\xi} \in \Xi := \{\boldsymbol{\xi} \in \mathbb{R}_+^{n_\xi} : \mathbf{D}\boldsymbol{\xi} \le \mathbf{d}\}. \end{aligned}$$

$u^T h(\xi) = u^T H \xi = \sum_{i,j} h_{ij} u_i \xi_j$

linearize

- Can be linearized if $\boldsymbol{\xi} \in \{0,1\}^{n_\xi}$ for any extreme point solution of Ξ.

Two-stage continous RO: Separation problem

- Assume that $\xi_j \in \{0,1\}$ for $j = 1, \ldots, n_\xi$ and $-M_i \leq u_i \leq 0$ for $i = 1, \ldots, m_y$.
- Replace each bilinear term $u_i \times \xi_j$ with the auxiliary variable ζ_{ij}.
- Introduce the linearization constraints (McCormick envelope):

$$\begin{aligned}
\zeta_{ij} &\geq u_i & \qquad & \xi_j = 1 \rightarrow \zeta_{ij} = u_i \\
\zeta_{ij} &\leq 0 & & \xi_j = 0 \rightarrow \zeta_{ij} = 0 \\
\zeta_{ij} &\leq u_i + M_i(1 - \xi_j) \\
\zeta_{ij} &\geq -M_i \xi_j
\end{aligned}$$

Remark

The extreme points of the budgeted uncertainty set

$$\Xi^\Gamma = \left\{ \boldsymbol{\xi} \in [0,1]^{n_\xi} \middle| \sum_{i=1}^{n_\xi} \xi_i \leq \Gamma \right\}$$

are binary vectors, $\boldsymbol{\xi} \in \{0,1\}^{n_\xi}$, *when* Γ *is integer.*

Two-stage continous RO: Separation problem

When $\Gamma \notin \mathbb{Z}$, we can trasform Ξ^{Γ} as follows:

$$\Xi^{\Gamma} = \left\{ \xi^1, \xi^2 \in [0,1]^{n_\xi} \;\middle|\; \sum_{i=1}^{n_\xi} \xi^1_i \leq \lfloor \Gamma \rfloor, \sum_{i=1}^{n_\xi} \xi^2_i \leq 1, \; \xi^1_i + \xi^2_i \leq 1 \quad \forall i \in [n_\xi] \right\}$$

data is transformed as : $\xi_i \rightarrow \xi^1_i + (\Gamma - \lfloor \Gamma \rfloor) \xi^2_i$

comes at the cost of n_ω

In a more generalized way assume that we can write $\xi = A\omega$ with $\omega \in \Omega \subseteq \mathbb{R}^{n_\omega}$ s.t extreme points of Ω are 0-1. In that case we can carry out the linearization.

Two-stage continous RO: Separation problem

- Use the KKT conditions on the inner problem to write:

$$
\begin{array}{lll}
\max & \mathbf{f}^\top \mathbf{y} & \\
\text{s.t.} & \left.\begin{array}{l} \mathbf{W}\mathbf{y} \leq \mathbf{h}(\boldsymbol{\xi}) - \mathbf{T}(\boldsymbol{\xi})\mathbf{x}^* \\ \mathbf{y} \in \mathbb{R}_+^{n_y} \end{array}\right\} \text{Primal feasibility} & \\
& \left.\begin{array}{l} \mathbf{W}^\top \mathbf{u} \leq \mathbf{f} \\ \mathbf{u} \in \mathbb{R}_-^{m_y} \end{array}\right\} \text{Dual feasibility} & \\
\text{CS} & \left\{\begin{array}{l} \mathbf{u}_i \times (\mathbf{h}(\boldsymbol{\xi}) - \mathbf{T}(\boldsymbol{\xi})\mathbf{x}^* - \mathbf{W}\mathbf{y})_i = 0 \\ (\mathbf{W}^\top \mathbf{u} - \mathbf{f})_j \times \mathbf{y}_j = 0 \end{array}\right. & \begin{array}{l} \forall i = 1, \dots, m_y \\ \forall j = 1, \dots, n_y \end{array} \\
& \boldsymbol{\xi} \in \Xi := \{\boldsymbol{\xi} \in \mathbb{R}_+^{n_\xi} : \mathbf{D}\boldsymbol{\xi} \leq \mathbf{d}\} &
\end{array}
$$

- To handle the bilinear CS constraints, linearize with big-M constraints.
- Introduce an auxiliary variable $\zeta_i \in \{0, 1\}$ for $i = 1, \dots, m_y$ and write:

$$
\begin{gathered}
\mathbf{u}_i = 0 \quad \vee \quad (\mathbf{h}(\boldsymbol{\xi}) - \mathbf{T}(\boldsymbol{\xi})\mathbf{x}^* - \mathbf{W}\mathbf{y})_i = 0 \\
\Longleftrightarrow \\
\mathbf{u}_i \geq -M\zeta_i \quad (\mathbf{h}(\boldsymbol{\xi}) - \mathbf{T}(\boldsymbol{\xi})\mathbf{x}^* - \mathbf{W}\mathbf{y})_i \leq M(1 - \zeta_i)
\end{gathered}
$$

$\zeta_i = 0 \rightarrow u_i = 0$

$\zeta_i = 1 \rightarrow (Wy - h(\xi) - T(\xi)x^*)_i = 0$

Handling infeasibility in recourse

- Up to now we worked with a relatively complete recourse assumption.
- When this assumption is not satisfied it is possible for $\mathcal{Y}(\mathbf{x}, \boldsymbol{\xi})$ to be empty for some $\mathbf{x} \in \mathcal{X}$ and $\boldsymbol{\xi} \in \Xi$.
- Assume now that $\mathcal{Y}(\bar{\mathbf{x}}, \bar{\boldsymbol{\xi}}) = \emptyset$ for some $\bar{\mathbf{x}} \in \mathcal{X}$ and $\bar{\boldsymbol{\xi}} \in \Xi$.
- Then

$$\begin{aligned} \max_{\mathbf{u} \in \mathbb{R}_-^{m_y}} \quad & \mathbf{u}^\top (\mathbf{h}(\bar{\boldsymbol{\xi}}) - \mathbf{T}(\bar{\boldsymbol{\xi}})\bar{\mathbf{x}}) \\ \text{s.t.} \quad & \mathbf{W}^\top \mathbf{u} \leq \mathbf{f} \end{aligned}$$

is unbounded. (why?)

How to cut off infeasible $\bar{x} \in \mathcal{X}$?

Handling infeasibility in recourse

$$\max_{\mathbf{u}\in\mathbb{R}^{m_y}_{-}} \quad \mathbf{u}^\top(\mathbf{h}(\bar{\xi}) - \mathbf{T}(\bar{\xi})\bar{\mathbf{x}})$$

$$\text{s.t.} \quad \mathbf{W}^\top \mathbf{u} \leq \mathbf{f}$$

- For C&CG, simply add variable $y_{\bar{\xi}} \in \mathbb{R}^{n_y}_{+}$ and constraints

$$\theta \geq f^T y_{\bar{\xi}}$$

$$T(\bar{\xi})x + W y_{\bar{\xi}} \leq h(\bar{\xi})$$

- For CG note that dual unbounded implies existence of $u \in \mathbb{R}^{m_y}_{-}$ s.t starting from $\bar{u} \in \mathbb{R}^{m_y}_{-}$, $W^T\bar{u} \leq f$

  $\bar{u} + \lambda u$ is feasible $\forall \lambda > 0$ &

  $u^T(h(\bar{\xi}) - T(\bar{\xi})\bar{x}) > 0$

  Add the constraint: $u^T(h(\bar{\xi}) - T(\bar{\xi})x) \leq 0$

Handling infeasibility in recourse

- MIP reformulations of the max-min problem are all based on bounding the dual variables.
- If the inner minimization problems are unbounded then dual variables cannot be bounded.
- How do we solve

$$\max_{\boldsymbol{\xi} \in \Xi} \mathcal{Q}(\mathbf{x}^*, \boldsymbol{\xi}) = \max_{\boldsymbol{\xi} \in \Xi} \min_{\mathbf{y} \in \mathbb{R}_+^{n_y}} \quad \mathbf{f}^\top \mathbf{y}$$
$$\text{s.t.} \quad \mathbf{W}\mathbf{y} \leq \mathbf{h}(\boldsymbol{\xi}) - \mathbf{T}(\boldsymbol{\xi})\mathbf{x}^*$$

in that case?

Handling infeasibility in recourse

- Consider the recourse problem written as a feasibility problem for given $(\bar{\mathbf{x}}, \bar{\theta})$:

① no feasible y s.t $f^\top y \le \bar{\theta}$

② $Wy \le h(\xi) - T(\xi)\bar{x}$ cannot be satisfied

$$\begin{aligned} \min_{\mathbf{y} \in \mathbb{R}_+^{n_y}} \quad & 0 \\ \text{s.t.} \quad & \mathbf{f}^\top \mathbf{y} \le \bar{\theta} && (\sigma \in \mathbb{R}_-) \\ & \mathbf{W}\mathbf{y} \le \mathbf{h}(\bar{\xi}) - \mathbf{T}(\bar{\xi})\bar{\mathbf{x}} && (\mathbf{u} \in \mathbb{R}_-^{m_y}) \end{aligned}$$

- Its dual is then given as:

$$\begin{aligned} \max_{\mathbf{u} \in \mathbb{R}_-^{m_y},\, \sigma \in \mathbb{R}_-} \quad & \sigma\bar{\theta} + \mathbf{u}^\top(\mathbf{h}(\bar{\xi}) - \mathbf{T}(\bar{\xi})\bar{\mathbf{x}}) \\ \text{s.t.} \quad & \mathbf{f}\sigma + \mathbf{W}\mathbf{u} \le 0 \end{aligned}$$

$(\sigma, u) = (0, 0)$ is a feasible solution

- The optimal value of the dual problem is greater than or equal to 0.
- The optimal value is equal to 0 only if the primal problem is feasible.

Handling infeasibility in recourse

- Further, any feasible solution $(\sigma, \mathbf{u})$ can be scaled to obtain another feasible solution.
- The dual variables can be bounded without changing the conclusion:

$$
\begin{aligned}
\max_{\mathbf{u}\in\mathbb{R}_-^{m_y},\ \sigma\in\mathbb{R}_-} \quad & \sigma\bar{\theta} + \mathbf{u}^\top(\mathbf{h}(\bar{\xi}) - \mathbf{T}(\bar{\xi})\bar{\mathbf{x}}) \\
\text{s.t.} \quad & \mathbf{f}\sigma + \mathbf{W}\mathbf{u} \le 0 \\
& |\sigma| + \sum_{i=1}^{m_y} |u_i| \le 1
\end{aligned}
$$

either obj > 0 → dual unbd. primal infeas.

OR obj = 0 → primal / dual feas.

- We can then solve the separation problem:

$$
\begin{aligned}
\max_{\boldsymbol{\xi}\in\Xi,\mathbf{u}\in\mathbb{R}_-^{m_y},\ \sigma\in\mathbb{R}_-} \quad & \sigma\bar{\theta} + \mathbf{u}^\top(\mathbf{h}(\xi) - \mathbf{T}(\xi)\bar{\mathbf{x}}) \\
\text{s.t.} \quad & \mathbf{f}\sigma + \mathbf{W}\mathbf{u} \le 0 \\
& |\sigma| + \sum_{i=1}^{m_y} |u_i| \le 1
\end{aligned}
$$

ξ becomes a variable

bounding dual variables by 1 when necessary.

Handling infeasibility in recourse

- When the optimal value of the separation problem is >0 then we need to cut off the current solution $\bar{\mathbf{x}} \in \mathcal{X}$.
- For the constraint generation algorithm, we add:

Two cases:

① $\sigma = 0 \to u^{\top}(h(\xi^*) - T(\xi^*)\bar{x}) \leq 0$

$$\sigma^* \bar{\theta} + \mathbf{u}^{*\top}(\mathbf{h}(\xi^*) - \mathbf{T}(\xi^*)\bar{\mathbf{x}}) \leq 0$$

② $\sigma < 0 \to$

$\theta \geq \bar{u}^{*\top}(h(\xi^*) - T(\xi^*)\bar{x})$

with $(\boldsymbol{\xi}^*, \mathbf{u}^*, \sigma^*)$ being an optimal solution of the separation problem.

- For the constraint-and-column generation algorithm, we add variables $\mathbf{y}_{\xi^*} \in \mathbb{R}_+^{n_y}$ and constraints:

$$\theta \geq \mathbf{f}^\top \mathbf{y}_{\xi^*}$$
$$\mathbf{T}(\xi^*)\mathbf{x} + \mathbf{W}\mathbf{y}_{\xi^*} \leq \mathbf{h}(\xi^*)$$

Decision rule approximations

$$\begin{aligned}
\min_{\mathbf{x}\in\mathcal{X},\mathbf{y}(\cdot),\theta\in\mathbb{R}} \quad & \mathbf{c}^\top \mathbf{x} + \theta \\
\text{s.t.} \quad & \theta \geq \mathbf{f}(\boldsymbol{\xi})^\top \mathbf{y}(\boldsymbol{\xi}) && \forall \boldsymbol{\xi} \in \Xi \\
& \mathbf{T}(\boldsymbol{\xi})\mathbf{x} + \mathbf{W}(\boldsymbol{\xi})\mathbf{y}(\boldsymbol{\xi}) \leq \mathbf{h}(\boldsymbol{\xi}) && \forall \boldsymbol{\xi} \in \Xi \\
& \mathbf{y}(\boldsymbol{\xi}) \in \mathcal{Y} && \forall \boldsymbol{\xi} \in \Xi
\end{aligned}$$

Idea

- Restrict the form of $\mathbf{y}(\boldsymbol{\xi})$ to a simple family of functions. note static robust optimization can be seen as a decision rule.
- Optimize within the chosen family.
- Most common restrictions: affine, piecewise affine/constant.

Attention!

- Being restrictions, decision rule approximations may be infeasible.
- If feasible, they provide feasible solutions (primal bounds) to (2ARO). in the case of min upper bounds
- They typically improve upon the static robust solution. (but not always)

Two-stage continuous RO: Affine decision rules[3]

$$
\begin{aligned}
\min_{\mathbf{x}\in\mathcal{X},\mathbf{y}(\cdot)} \quad & \mathbf{c}^\top \mathbf{x} + \max_{\boldsymbol{\xi}\in\Xi} \quad \mathbf{f}(\boldsymbol{\xi})^\top \mathbf{y}(\boldsymbol{\xi}) \\
\text{s.t.} \quad & \mathbf{T}(\boldsymbol{\xi})\mathbf{x} + \mathbf{W}(\boldsymbol{\xi})\mathbf{y}(\boldsymbol{\xi}) \leq \mathbf{h}(\boldsymbol{\xi}) && \forall \boldsymbol{\xi} \in \Xi \\
& \mathbf{y}(\boldsymbol{\xi}) \in \mathcal{Y} && \forall \boldsymbol{\xi} \in \Xi
\end{aligned}
$$

Assume

- $\mathcal{Y} = \mathbb{R}_+^{n_y} \rightarrow$ continuous recourse
- $\mathbf{f}(\boldsymbol{\xi})$ and $\mathbf{W}(\boldsymbol{\xi})$ are deterministic (fixed recourse)

Same context as the exact algorithms

[3] Ben-Tal et al., 2004

Two-stage continuous RO: Affine decision rules[3]

$$\min_{\mathbf{x}\in\mathcal{X},\mathbf{y}(\cdot)} \quad \mathbf{c}^\top \mathbf{x} + \max_{\boldsymbol{\xi}\in\Xi} \quad \mathbf{f}^\top \mathbf{y}(\boldsymbol{\xi})$$

$$\text{s.t.} \quad \sum_{i=1}^{n_\xi} \mathbf{T_i}\xi_i \mathbf{x} + \mathbf{W}\mathbf{y}(\boldsymbol{\xi}) \leq \mathbf{H}\boldsymbol{\xi} \qquad \forall \boldsymbol{\xi} \in \Xi$$

$$\mathbf{y}(\boldsymbol{\xi}) \geq 0 \qquad \forall \boldsymbol{\xi} \in \Xi$$

Idea

- Restrict $\mathbf{y}(\boldsymbol{\xi})$ to be an affine function of $\boldsymbol{\xi}$

$\alpha_0 + A\xi = \bar{A}\bar{\xi}$

$\bar{A} = [\alpha_0 \mid A]$ $\bar{\xi} = \begin{bmatrix} 1 \\ \xi \end{bmatrix}$

$$\mathbf{y}_i(\boldsymbol{\xi}) = \alpha_{i0} + \boldsymbol{\alpha}_i^\top \boldsymbol{\xi} \quad \forall i = 1, \ldots, n_y \rightarrow \mathbf{y}(\boldsymbol{\xi}) = \mathbf{A}\boldsymbol{\xi}$$

- Optimize $\mathbf{A} \in \mathbb{R}^{n_y \times (n_\xi + 1)}$ to obtain the best such approximation.

the parameters of the affine function become decision variables

[3] Ben-Tal et al., 2004

Two-stage continuous RO: Affine decision rules[3]

$$
\begin{aligned}
\min_{\mathbf{x}\in\mathcal{X},\mathbf{A}\in\mathbb{R}^{n_y\times(n_\xi+1)}} \quad & \mathbf{c}^\top\mathbf{x}+\theta && && \text{(Aff)}\\
\text{s.t.} \quad & \theta \geq \mathbf{f}^\top\mathbf{A}\boldsymbol{\xi} && \forall\boldsymbol{\xi}\in\Xi\\
& \sum_{i=1}^{n_\xi}\mathbf{T_i}\xi_i\mathbf{x}+\mathbf{W}\mathbf{A}\boldsymbol{\xi}\leq\mathbf{H}\boldsymbol{\xi} && \forall\boldsymbol{\xi}\in\Xi\\
& \mathbf{A}\boldsymbol{\xi}\geq 0 && \forall\boldsymbol{\xi}\in\Xi
\end{aligned}
$$

we can now see why we need f, W constant in ξ

Every constraint is affine in ξ with x, A, θ fixed

Remark

(Aff) is a static linear robust optimization problem with a polyhedral uncertainty set $\rightarrow$ reformulate into a deterministic equivalent problem through LP-duality.

[3] Ben-Tal et al., 2004

Two-stage continuous RO: Affine decision rules[3]

Take for instance : $\theta \geq f^T A \xi \quad \forall \xi \in \Xi \quad \longleftrightarrow \quad \max_{\xi \in \mathbb{R}^{n_\xi}} f^T A \xi \leq \theta$

$$D\xi \leq d \qquad (u_{obj})$$

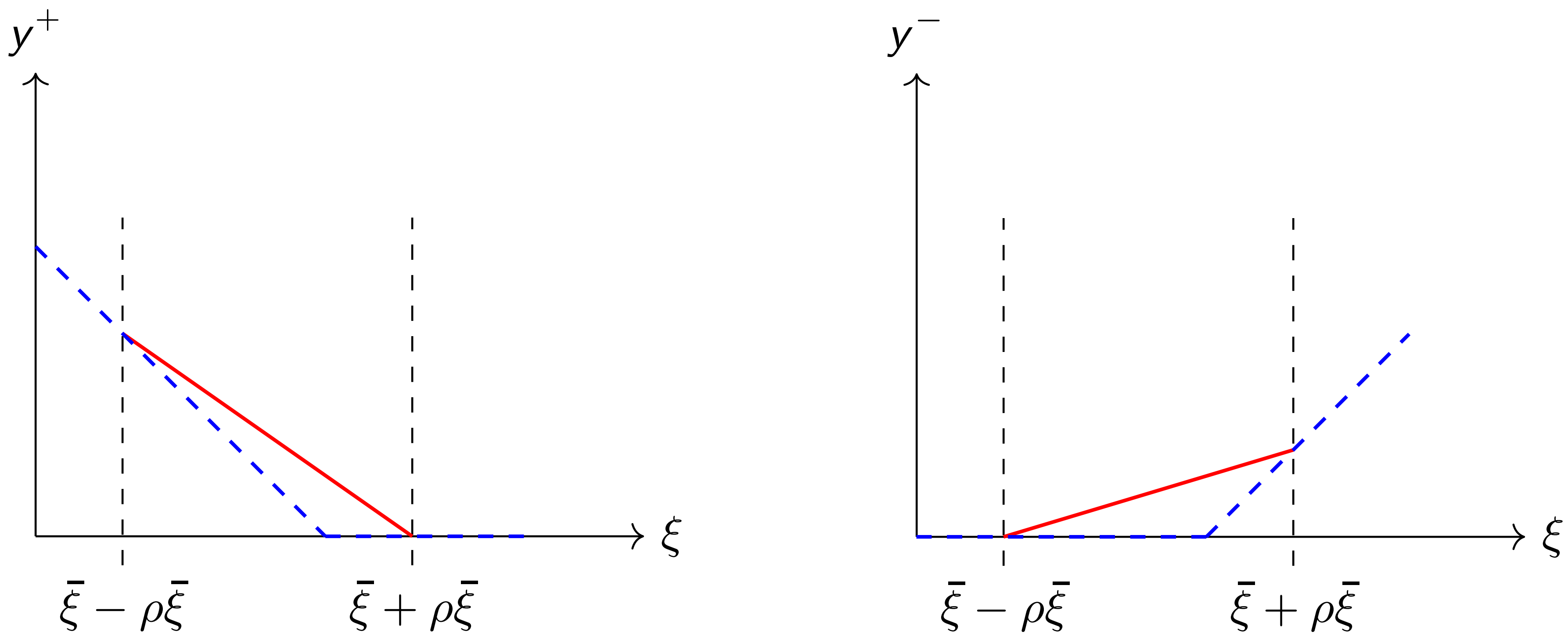
$$\longleftrightarrow \quad \min \; d^T u_{obj} \leq \theta$$
$$\text{s.t.} \quad D^T u_{obj} = f^T A$$
$$u_{obj} \geq 0$$

$$\longleftrightarrow \quad d^T u_{obj} \leq \theta$$
$$D^T u_{obj} = f^T A$$
$$u_{obj} \geq 0$$

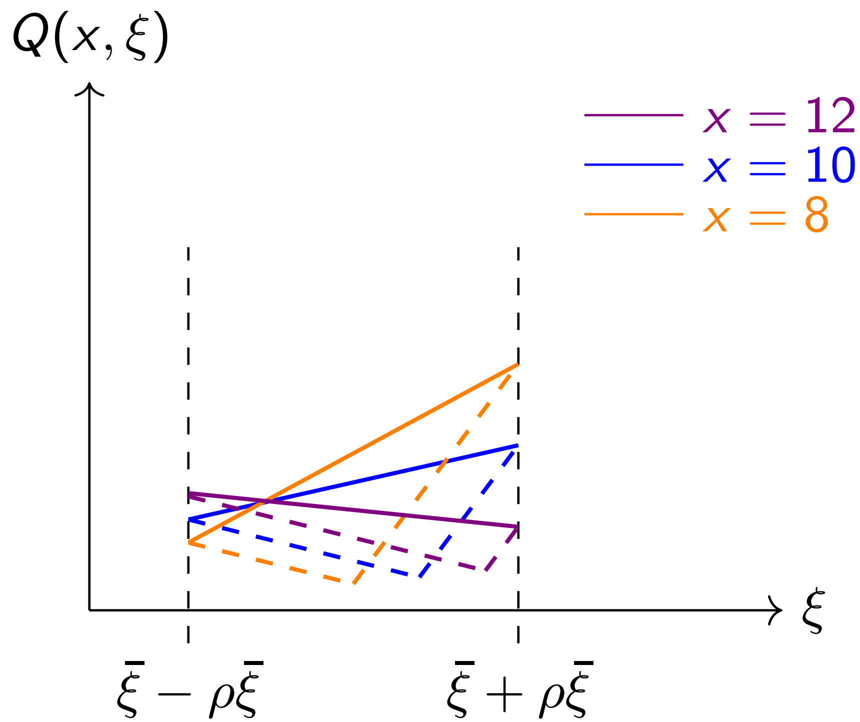
[3] Ben-Tal et al., 2004

Example: Newsvendor (Cont'd)

- Let $y^+ = \alpha_0^1 + \alpha_1^1 \xi$ and $y^- = \alpha_0^2 + \alpha_1^2 \xi$
- Optimal *linear* recourse quantities as a function of $\cancel{d}$ ξ when $x = 10$:



Example: Newsvendor (Cont'd)



- When x is low worst case is $\bar{\xi} + \rho\bar{\xi}$, otherwise worst case is $\bar{\xi} - \rho\bar{\xi}$.
- Optimality is achieved at equality.
- Optimal x value is the same as in the exact solution[4].

[4] proved more generally in Bertsimas et al., 2010.

On the quality of affine decision rules

- The quality of a decision rule is measured based on the relative gap:

$$100 \times \frac{|z_{\text{AFF}} - z_{\text{Dual}}|}{|z_{\text{Dual}}|}$$

where z_{Dual} is a dual bound such that $z_{\text{Dual}} \leq z_{\text{2ARO}}$. since z_{2ARO} is not available

- Let $\hat{\Xi} \subseteq \Xi$ be a finite subset of realizations.
- Then the following relaxation provides a dual bound:

$$\begin{aligned}
\min_{\substack{\mathbf{x}\in\mathcal{X},\theta\in\mathbb{R} \\ \mathbf{y}^1,\dots,\mathbf{y}^{|\hat{\Xi}|}\in\mathbb{R}_+^{n_y}}} \quad & \mathbf{c}^\top \mathbf{x} + \theta \\
\text{s.t.} \quad & \theta \geq \mathbf{f}^\top \mathbf{y}^k && \forall k \in [|\hat{\Xi}|] \\
& \mathbf{T}(\boldsymbol{\xi}^k)\mathbf{x} + \mathbf{W}\mathbf{y}^k \leq \mathbf{h}(\boldsymbol{\xi}^k) && \forall k \in [|\hat{\Xi}|]
\end{aligned}$$

- But how do we choose $|\hat{\Xi}|$ in a meaningful way?

On the quality of affine decision rules

- The quality of a decision rule is measured based on the relative gap:

$$100 \times \frac{|z_{\mathrm{AFF}} - z_{\mathrm{Dual}}|}{|z_{\mathrm{Dual}}|}$$

where z_{Dual} is a dual bound such that $z_{\mathrm{Dual}} \leq z_{\mathrm{2ARO}}$.

Hadjiyiannis et al. (2011)

- Solve (Aff) to optimality, let $(\mathbf{x}^*, \mathbf{A}^*, \theta^*)$ be an optimal solution.
- Extract the "binding" scenarios by solving:

$$\max_{\boldsymbol{\xi} \in \Xi} \; \mathbf{f}^\top \mathbf{A}^* \boldsymbol{\xi} - \theta^*$$

$$\max_{\boldsymbol{\xi} \in \Xi} \; \mathbf{T}_i(\boldsymbol{\xi})\mathbf{x}^* + \mathbf{W}_i \mathbf{A}^* \boldsymbol{\xi} - \mathbf{h}_i(\boldsymbol{\xi}) \qquad \forall i \in [m]$$

- Constitute $\hat{\Xi}$ of binding scenarios.

Remark

This can also be a good warm-start strategy for exact methods such as CCG.