On the Polyhedral Structure of Two-Level Lot-Sizing Problems with Supplier Selection

Ayşe N. Arslan, Jean-Philippe P. Richard, Yongpei Guan

Industrial and Systems Engineering, University of Florida, 303 Weil Hall P.O. Box 116595, Gainesville, Florida 32611

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Abstract: In this article, we study a two-level lot-sizing problem with supplier selection (LSS), which is an *NP-hard* problem arising in different production planning and supply chain management applications. After presenting various formulations for LSS, and computationally comparing their strengths, we explore the polyhedral structure of one of these formulations. For this formulation, we derive several families of strong valid inequalities, and provide conditions under which they are facet-defining. We show numerically that incorporating these valid inequalities within a branch-and-cut framework leads to significant improvements in computation. Naval Research Logistics 63: 647–666, 2017

Keywords: Cutting planes; lot-sizing; polyhedral study; supplier selection

1. INTRODUCTION

We consider a supplier with P production plants and a single customer whose (positive) deterministic demand for each time period is known. Customer demand may be produced in any of the plants but must be satisfied without backlogging. In this setting, it is possible to carry inventory at both the customer location and at production plants. The supplier has control over the timing of shipments which leads to the coordination of production and transportation decisions. We seek to determine an optimal production and transportation schedule that minimizes fixed and variable production costs, as well as inventory holding and transportation costs. We assume that production and shipment capacities in each period are sufficiently large to satisfy the entire demand. We also assume that beginning and ending inventory levels are zero across the supply chain. The above problem, which we refer to as *lot-sizing with supplier selection* (LSS), arises in vendor managed inventory (VMI), requirements planning with substitutions, and multi-item multi-facility supply chain planning. We elaborate on these applications next.

VMI is a supply chain initiative where the supplier is authorized to manage the inventory of agreed-on SKUs at retail locations. This approach, where inventory across multiple echelons is managed collectively, has long attracted the interest of the supply chain research community; see [6]. Successful practical applications of this idea have shown that (i) economies of scale can be achieved through shipment and production consolidation, (ii) bullwhip effect can be reduced through integration of information systems, (iii) stockout frequency can be reduced, and (iv) cost savings for the entire system are possible.

Requirements planning with substitutions is a production planning problem that arises from the desire to exploit flexibility in bill of materials. Specifically, the problem is a two-stage production planning problem where it is possible to use substitute components or subassemblies produced by an upstream stage to meet demand in each period at the downstream stage; see [4]. Requirements planning with substitutions reduces to LSS, if we view substitute parts as different suppliers, and assume that a single part is required for each end product.

Cross-facility capacity management is crucial in high technology industries with high capital investment and short life cycles. The problem can be formulated as a multi-commodity network flow model for multi-product, multi-facility production planning; see [18]. The single-product multi-facility substructure, which corresponds to LSS, is identified in Ref. [18] as a main source of difficulty in solving the problem. Similarly, LSS arises as a substructure of virtually all multi-item multi-facility deterministic production planning problems.

In this article, we study LSS from a polyhedral perspective. Because strong inequalities generated for this common (sub)structure can be used as cutting planes for all models that contain it, our results can be used in the solution of other multi-level multi-facility lot-sizing problems. We next review

Correspondence to: Ayse N. Arslan (nur.arslan@ufl.edu)

lot-sizing literature, focusing on polyhedral studies that are most related to our work.

Many variants of the lot-sizing problem have been studied in the past. Most early studies of lot-sizing focus on single echelon problems. Building on an extended-formulation for facility location given by Ref. [10], Ref. [5] gives a linear description of the convex hull of the single-item uncapacitated lot-sizing problem. Convex hull descriptions or strong valid inequalities have been derived for variants of lot-sizing that allow backlogging [11, 14], capacities [2, 3, 15] and stochastic demand [8, 9], among others. We refer interested readers to Ref. [16] for an extensive discussion.

Polyhedral studies of multi-echelon lot-sizing problems have received comparably less interest in the literature. Valid inequalities for a variant of the problem where assembly structures are complex are given in Ref. [7]. Recent results on serial assembly are obtained in Ref. [13]. In particular, the authors give a convex hull description for the two-echelon case in a higher dimension through the use of dynamic programming. This work is extended by incorporating intermediate demand and considering multiple echelons in Ref. [19]. The authors derive a family of facet-defining inequalities for the problem and propose a hierarchy of formulations whose strengths are compared both theoretically and numerically. Another two-level variant where there are multiple items at the lower level facing independent demand, and a single item at the upper level facing dependent demand is studied in Ref. [17]. Several relaxations are studied, along with a proof that a stock-dominant relaxation solves the problem under consideration to optimality under a non-speculative cost assumption. A supplier selection extension of the traditional lot-sizing problem where at each period a lot-size and a subset of suppliers must be selected is studied in Ref. [20]. The authors provide a convex hull description for the uncapacitated case. Algorithms and valid inequalities for LSS for the special case where there are two plants are studied in Ref. [12]. In particular, a dynamic programming algorithm is presented and valid inequalities are introduced. The results we derive in this article are for the extension of the problem where P plants are available for production. Further, a more comprehensive study of formulations, and families of facet-defining inequalities is undertaken.

The remainder of this article is organized as follows: In Section 2, we introduce the problem, prove that it is *NP-hard*, and describe several mixed integer programming formulations for it. In Section 3, we study the polyhedral structure of a formulation of the problem with traditional variables. We identify several families of facet-defining inequalities. In Section 4, we describe polynomial time separation algorithms and evaluate the effectiveness of these inequalities inside of a branch-and-cut algorithm. We conclude the article in Section 5.

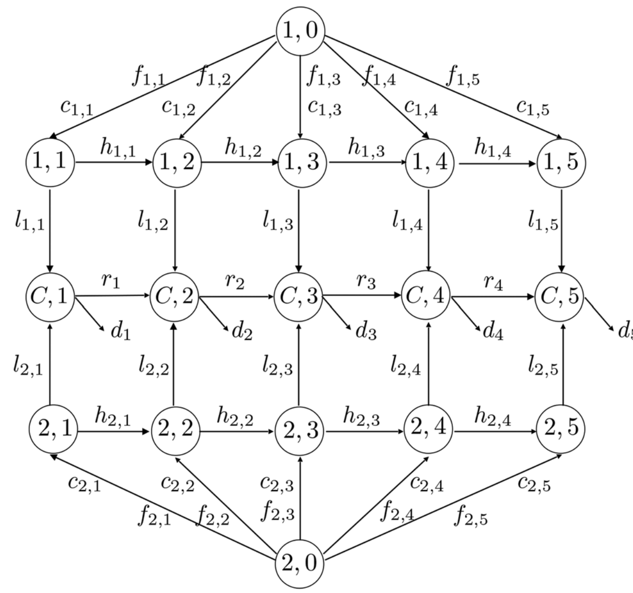

Figure 1. Instance of LSS with $P = 2$ and $T = 5$.

2. PROBLEM DEFINITION AND MATHEMATICAL FORMULATIONS

In this section, we formally introduce LSS and present several formulations for it.

For $p \in \mathcal{P} := \{1, \ldots, P\}$ and $t \in \mathcal{T} := \{1, \ldots, T\}$, we let $f_{p,t}$ and $c_{p,t}$ be the fixed and variable cost of production at plant p in period t, respectively. We also let $h_{p,t}$ be the variable inventory holding cost, $l_{p,t}$ be the variable transportation cost for plant p in period t, and r_t be the variable inventory holding cost at the customer level in period t. A network representation of an instance where $P = 2$ and $T = 5$ is given in Fig. 1. Let $\mathcal{T}_0 = \mathcal{T} \cup \{0\}$. In this representation, we include nodes (p, t) for $p \in \mathcal{P}$ and $t \in \mathcal{T}_0$ as well as nodes (C, t) for $t \in \mathcal{T}$. An arc from node $(p, 0)$ to any other node (p, t) represents a production decision at plant p in period t. Moreover, an arc from node (p, t) to node (C, t) represents a shipment decision from plant p in period t. The remaining arcs represent inventory holding decisions.

A formulation of the problem is given by

$$\min \quad \sum_{p\in\mathcal{P}} \sum_{t\in\mathcal{T}} (f_{p,t} y_{p,t} + c_{p,t} x_{p,t} + h_{p,t} s_{p,t} + l_{p,t} v_{p,t}) + \sum_{t\in\mathcal{T}} r_t \sigma_t \tag{1}$$

$$s.t. \quad s_{p,t-1} + x_{p,t} = v_{p,t} + s_{p,t} \quad \forall p \in \mathcal{P}, \forall t \in \mathcal{T} \tag{2}$$

$$\sigma_{t-1} + \sum_{p\in\mathcal{P}} v_{p,t} = d_t + \sigma_t \quad \forall t \in \mathcal{T} \tag{3}$$

$$x_{p,t} \le d_{tT} y_{p,t} \quad \forall p \in \mathcal{P}, \forall t \in \mathcal{T} \tag{4}$$

$$y_{p,t} \le 1 \quad \forall p \in \mathcal{P}, \forall t \in \mathcal{T} \tag{5}$$

$$\sigma \in \mathbb{R}_+^T, s, v, x \in \mathbb{R}_+^{PT}, y \in \mathbb{Z}_+^{PT}, \tag{6}$$

where we define $d_{ij} = \sum_{t=i}^{j} d_t$ for $i, j \in \mathcal{T}$. For $p \in \mathcal{P}$ and $t \in \mathcal{T}$, $x_{p,t}$ is the quantity produced at plant p in period t, $y_{p,t}$ is the corresponding setup variable, $v_{p,t}$ is the quantity shipped from plant p in period t, and σ_t and $s_{p,t}$ are the levels of inventory at the end of period t for the customer and plant p, respectively. Here σ_0 and $s_{p,0}$ are the initial inventory for the customer and plant p, respectively, and are taken to be zero by assumption.

We can eliminate the variables σ_t and $s_{p,t}$ by using the relations $\sigma_t = (\sum_{p\in\mathcal{P}} \sum_{i=1}^{t} v_{p,i}) - d_{1t}$ and $s_{p,t} = \sum_{i=1}^{t}(x_{p,i} - v_{p,i})$. After performing these substitutions, (1)–(6) can be written as

$$\min \quad \sum_{p\in\mathcal{P}} \sum_{t\in\mathcal{T}} (f_{p,t} y_{p,t} + \gamma_{p,t} x_{p,t} + \delta_{p,t} v_{p,t}) - C \tag{7}$$

$$s.t. \quad \sum_{i=1}^{t} x_{p,i} \ge \sum_{i=1}^{t} v_{p,i} \quad \forall p \in \mathcal{P}, \forall t \in \mathcal{T} \setminus \{T\} \tag{8}$$

$$\sum_{i\in\mathcal{T}} x_{p,i} = \sum_{i\in\mathcal{T}} v_{p,i} \quad \forall p \in \mathcal{P} \tag{9}$$

$$\sum_{p\in\mathcal{P}} \sum_{i=1}^{t} v_{p,i} \ge d_{1t} \quad \forall t \in \mathcal{T} \setminus \{T\} \tag{10}$$

$$\sum_{p\in\mathcal{P}} \sum_{i\in\mathcal{T}} v_{p,i} = d_{1T} \tag{11}$$

$$x_{p,t} \le d_{tT} y_{p,t} \quad \forall p \in \mathcal{P}, \forall t \in \mathcal{T} \tag{12}$$

$$y_{p,t} \le 1 \quad \forall p \in \mathcal{P}, \forall t \in \mathcal{T} \tag{13}$$

$$v, x \in \mathbb{R}_+^{PT}, y \in \mathbb{Z}_+^{PT}, \tag{14}$$

where $C = \sum_{t\in\mathcal{T}} r_t d_{1t}$, $\gamma_{p,t} = c_{p,t} + \sum_{i=t}^{T} h_{p,i}$ and $\delta_{p,t} = l_{p,t} + \sum_{i=t}^{T}(r_i - h_{p,i})$, for $p \in \mathcal{P}$ and for $t \in \mathcal{T}$. In the remainder of this article, we refer to (7)–(14) as the *natural formulation* of LSS.

We next argue that we should not expect to find a polynomial time algorithm to solve LSS as this problem is *NP-hard*. The single-item multi-facility supply chain planning problem is proven to be *NP-hard* in Ref. [18]. Their proof involves a reduction from the facility location problem which is known to be *NP-hard*. The only difference between LSS and the single-item multi-facility supply chain planning problem given in Ref. [18] is that, in LSS, it is possible to hold inventory at the customer level. We therefore adapt the proof given in Ref. [18] to show that LSS is *NP-hard* by reduction from the facility location problem.

PROPOSITION 2.1: LSS is NP-hard.

PROOF: Consider an instance of the facility location problem with a set $\mathcal{P}$ of facilities and a set $\mathcal{T}$ of customers. We denote the fixed cost of opening facility p by K_p and the cost of assigning customer t to facility p by $a_{p,t}$ for $p \in \mathcal{P}$ and $t \in \mathcal{T}$. We next reduce this instance to one of LSS. This LSS instance has a set $\mathcal{P}$ of plants, and a set $\mathcal{T}$ of time periods. We let $f_{p,t} = K_p$, $l_{p,t} = a_{p,t}$, $h_{p,t} = 0$ for $p \in \mathcal{P}$ and $t \in \mathcal{T}$, $d_t = 1$, $r_t = \max_{p\in\mathcal{P},\tau\in\mathcal{T}}\{a_{p,\tau}\} + 1$ for $t \in \mathcal{T}$, $c_{p,1} = 0$ and $c_{p,t} = 1$ for $p \in \mathcal{P}$ and $t \in \mathcal{T} \setminus \{1\}$. Consider any feasible solution to the facility location problem where facility p is assigned to the subset of customers τ_p of $\mathcal{T}$. Then a feasible solution of same objective value can be obtained for LSS by producing the demand of periods τ_p in plant p in period 1, and by shipping each demand when it is due. Similarly, take any optimal solution to the LSS instance. It is easy to see that it must be such that $x_{p,t} = 0$ for $p \in \mathcal{P}$ and $t \in \mathcal{T} \setminus \{1\}$. It follows that $x_{p,1} = \sum_{i\in\tau_p} d_i$ for some $\tau_p \subseteq \mathcal{T}$ and $v_{p,i} = d_i$ for $i \in \tau_p$. It can be readily verified that the facility location solution where customers τ_p are assigned to plant p, has the same objective value as this solution. This shows that LSS is *NP-hard*. □

An $O(PT^{P+1})$ dynamic programming algorithm to solve LSS is presented in Ref. [1]. A similar algorithm to solve LSS with nonnegative setup costs is given in Ref. [4] with running time $O(P(2T)^{P+1})$. Additionally, Ref. [12] describes an algorithm for the special case where $P = 2$ that has a running time $O(T^4)$, while Ref. [18] describes an algorithm with running time $O(P^{T+1} T \log T)$. The dynamic programming recursions presented in Ref. [1] are used to create the convex hull of solutions to LSS in an extended space with an exponential number of variables. We next investigate other extended formulations.

Defining $\phi_{p,t,t',t''}$ as the amount produced at plant p in period t shipped in period t' to satisfy the demand in period t'', we write

$$\min \quad \sum_{p\in\mathcal{P}} \left[\sum_{t\in\mathcal{T}} f_{p,t} y_{p,t} + \sum_{t\in\mathcal{T}} \sum_{t'=t}^{T} \sum_{t''=t'}^{T} (\gamma_{p,t} + \delta_{p,t'}) \phi_{p,t,t',t''} \right] - C \tag{15}$$

$$s.t. \quad \sum_{p\in\mathcal{P}} \sum_{t=1}^{t''} \sum_{t'=t}^{t''} \phi_{p,t,t't''} = d_{t''} \quad \forall t'' \in \mathcal{T} \tag{16}$$

$$\sum_{t'=t}^{t''} \phi_{p,t,t',t''} \le d_{t''} y_{p,t} \quad \forall p \in \mathcal{P}, \forall t'' \in \mathcal{T}, \forall t \le t'' \tag{17}$$

$$y_{p,t} \le 1 \quad \forall p \in \mathcal{P}, \forall t \in \mathcal{T} \tag{18}$$

$$\phi \in \mathbb{R}_+^{\frac{PT^3+3PT^2+2PT}{6}}, y \in \mathbb{Z}_+^{PT}. \tag{19}$$

Formulation (15)–(19) has $O(PT^3)$ variables. However, for the problem we study, we do not need to keep track of the time period at which shipments occur since they can be deduced from the time period of production and the time period at which demand is used by solving a shortest path problem. This observation allows us to derive a smaller formulation that we describe next. Let $\phi_{p,t,t'}$ be the amount produced at plant p in period t to satisfy the demand in period t'. Define $H^p(j,t)$ as the length of the shortest path between nodes (p, j) and (C, t) in the network described in section [2]. Using this optimal shipment cost, which can be precomputed in time $O(PT^2)$, we can formulate LSS as

$$\min \left\{ \sum_{p\in\mathcal{P}} \left[\sum_{t\in\mathcal{T}} f_{p,t} y_{p,t} + \sum_{t'\in\mathcal{T}} \sum_{t=1}^{t'} (\gamma_{p,t} + H^p(t,t'))\phi_{p,t,t'} \right] - C \middle| (\phi, y) \in Q^{MC}_{\mathcal{P},\mathcal{T}}[\mathbf{d}] \right\}, \tag{20}$$

where $Q^{\mathrm{MC}}_{\mathcal{P},\mathcal{T}}[\mathbf{d}] = R^{\mathrm{MC}}_{\mathcal{P},\mathcal{T}}[\mathbf{d}] \cap \left(\mathbb{R}_+^{\frac{PT^2+PT}{2}} \times \mathbb{Z}^{PT}\right)$ and

$$R^{\mathrm{MC}}_{\mathcal{P},\mathcal{T}}[\mathbf{d}] = \left\{ (\phi; y) \in \mathbb{R}_+^{\frac{PT^2+PT}{2}} \times \mathbb{R}^{PT} \;\middle|\; \begin{array}{lll} \sum_{p\in\mathcal{P}} \sum_{t=1}^{t'} \phi_{p,t,t'} = d_{t'} & \forall t' \in \mathcal{T} & \text{(21a)} \\ \phi_{p,t,t'} \le d_{t'} y_{p,t} & \forall p \in \mathcal{P}, \forall t' \in \mathcal{T}, \forall t \le t' & \text{(21b)} \\ y_{p,t} \le 1 & \forall p \in \mathcal{P}, \forall t \in \mathcal{T} & \text{(21c)} \end{array} \right\}.$$

In the above formulation, we avoid the introduction of the extra index t'', thereby reducing the number of variables from $O(PT^3)$ to $O(PT^2)$. This reduction is based on the assumption that it is always optimal to implement a least variable cost shipment decision. We remark that if there are fixed costs and capacities related to shipment variables then (21a)–(21c) cannot be used. Conversely, using the equalities $x_{p,t} = \sum_{t'=t}^{t''} \sum_{t''=t}^{T} \phi_{p,t,t',t''}$ and $v_{p,t} = \sum_{t'=1}^{t} \sum_{t''=t}^{T} \phi_{p,t',t,t''}$ for $p \in \mathcal{P}$ and $t \in \mathcal{T}$, capacity and fixed charge restrictions can easily be imposed using the formulation (15)–(19).

PROPOSITION 2.2: Let $z_{\mathrm{LP}}^{\mathrm{MC4}}$ and $z_{\mathrm{LP}}^{\mathrm{MC3}}$ denote the LP-relaxation objective values of (15)–(19) and (20), respectively. Then $z_{\mathrm{LP}}^{\mathrm{MC4}} = z_{\mathrm{LP}}^{\mathrm{MC3}}$.

PROOF: We first show that $z_{\mathrm{LP}}^{\mathrm{MC3}} \le z_{\mathrm{LP}}^{\mathrm{MC4}}$. Let $(\bar{\phi}, \bar{y})$ be an optimal solution to the LP-relaxation of (15)–(19), and $\bar{z}_{\mathrm{LP}}^{\mathrm{MC4}}$ be its corresponding objective value. Let $\hat{\phi}_{p,t,t''} = \sum_{t'=t}^{t''} \bar{\phi}_{p,t,t',t''}$ for $p \in \mathcal{P}$, $t \in \mathcal{T}$, and $t'' \in \{t, \ldots, T\}$ and $\hat{y}_{p,t} = \bar{y}_{p,t}$ for $p \in \mathcal{P}$ and $t \in \mathcal{T}$. Clearly $(\hat{\phi}, \hat{y})$ is a feasible solution to (21a)–(21c). Further, its objective value is less than $\bar{z}_{\mathrm{LP}}^{\mathrm{MC4}}$ since $H^p(t,t'')\hat{\phi}_{p,t,t''} = H^p(t,t'') \sum_{t'=t}^{t''} \bar{\phi}_{p,t,t',t''} \le \sum_{t'=t}^{t''} \delta_{p,t'} \bar{\phi}_{p,t,t',t''}$ for all $p \in \mathcal{P}$, $t \in \mathcal{T}$ and $t'' \in \{t, \ldots, T\}$, where the inequality follows by the definition of $H^p(t,t'')$. It follows that $z_{\mathrm{LP}}^{\mathrm{MC3}} \le z_{\mathrm{LP}}^{\mathrm{MC4}}$.

We next show that $z_{\mathrm{LP}}^{\mathrm{MC4}} \le z_{\mathrm{LP}}^{\mathrm{MC3}}$. Let $(\hat{\phi}, \hat{y})$ be an optimal solution to the LP-relaxation of (21a)–(21c), and $\hat{z}_{\mathrm{LP}}^{\mathrm{MC3}}$ be its corresponding objective value. For each $p \in \mathcal{P}$, $t \in \mathcal{T}$, and $t'' \in \{t, \ldots, T\}$ such that $\hat{\phi}_{p,t,t''} > 0$ we select $t' \in \mathrm{argmin}_{\tau\in\{t,\ldots,t''\}} \delta_{p,\tau}$, and set $\bar{\phi}_{p,t,t',t''} = \hat{\phi}_{p,t,t''}$ while setting all other ϕ variables to zero. We also set $\bar{y}_{p,t} = \hat{y}_{p,t}$ for $p \in \mathcal{P}$ and $t \in \mathcal{T}$. Clearly, $(\bar{\phi}, \bar{y})$ is a feasible solution to (15)–(19) and has the same objective value as $\hat{z}_{\mathrm{LP}}^{\mathrm{MC3}}$. It then follows that $z_{\mathrm{LP}}^{\mathrm{MC4}} \le z_{\mathrm{LP}}^{\mathrm{MC3}}$. □

It is clear from Proposition 2.2 that (20) should be preferred to (15)–(19) when solving LSS instances since it has a smaller number of variables and constraints while providing the same relaxation value as (15)–(19). In contrast, when extensions of LSS with shipment capacities and fixed charges are considered (20) cannot be used and we must resort to (15)–(19).

We next show with an example that (21a)–(21c) has vertices where y is fractional. A similar derivation shows that (15)–(19) also has vertices where y is fractional. In the ensuing discussion, we denote the convex hull of integer solutions of (21a)–(21c) as $Q^{I,\mathrm{MC}}_{\mathcal{P},\mathcal{T}}[\mathbf{d}] = \mathrm{conv}(Q^{\mathrm{MC}}_{\mathcal{P},\mathcal{T}}[\mathbf{d}])$. When sets $\mathcal{P}$, $\mathcal{T}$ and the vector $\mathbf{d}$ are clear from the context, we use the shorthand notation $Q^{I,\mathrm{MC}}$.

EXAMPLE 2.1: Consider an instance with two plants and three periods for which the costs $f_{p,t}$, $\gamma_{p,t}$ and $\delta_{p,t}$ for $p \in \mathcal{P}$ and $t \in \mathcal{T}$, and demand d_t for $t \in \mathcal{T}$ are presented in Fig. 2.

The solution $(\bar{\phi}, \bar{y})$ where $\bar{y}_{1,1} = \bar{y}_{1,2} = \bar{y}_{2,1} = \frac{1}{2}$ and $\bar{\phi}_{1,1,1} = \bar{\phi}_{1,1,3} = \bar{\phi}_{1,2,2} = \bar{\phi}_{1,2,3} = \bar{\phi}_{2,1,1} = \bar{\phi}_{2,1,2} = \frac{1}{2}$ and all other variables are set to 0 is a feasible solution to $R^{\mathrm{MC}}_{\mathcal{P},\mathcal{T}}[\mathbf{d}]$. We next consider the dual problem. We associate variables u_t for $t \in \mathcal{T}$ to (21a), $\omega_{p,t,t'}$ for $p \in \mathcal{P}$, $t \in \mathcal{T}$ and $t' \in \{t, \ldots, T\}$ to (21b), and $v_{p,t}$ for $p \in \mathcal{P}$ and $t \in \mathcal{T}$ to (21c). The solution $(\bar{u}, \bar{\omega}, \bar{v})$ where $\bar{u}_1 = 2.75$, $\bar{u}_2 = 1.25$, $\bar{u}_3 = 0.75$, $\bar{\omega}_{1,1,1} = \bar{\omega}_{1,2,3} = \bar{\omega}_{2,1,1} = 0.75$, $\bar{\omega}_{1,1,3} = \bar{\omega}_{1,2,2} = \bar{\omega}_{2,1,2} = \bar{\omega}_{2,2,2} = \bar{\omega}_{2,2,3} = 0.25$, $\bar{\omega}_{1,3,3} = \bar{\omega}_{2,3,3} = 5$ and where all other variables are set to 0 is dual feasible. Moreover, for every tight constraint in the primal (resp. dual) the corresponding dual (resp. primal) variable is strictly positive. This shows that $(\bar{\phi}, \bar{y})$ and $(\bar{u}, \bar{\omega}, \bar{v})$ are (strictly) complementary and that $(\bar{\phi}, \bar{y})$ is a fractional extreme point of $R^{\mathrm{MC}}_{\mathcal{P},\mathcal{T}}[\mathbf{d}]$.

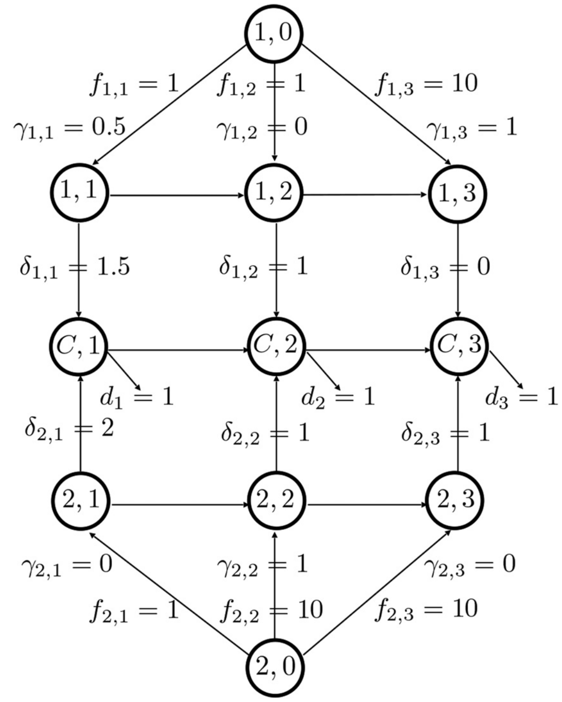

Figure 2. Problem with modified costs $\gamma_{p,t}$ and $\delta_{p,t}$ and minimum shipment costs $H^p(j,t)$.

The fractional extreme point of Example 2.1 is cut off by the valid inequality

$$\frac{\phi_{1,2,2} + \phi_{2,1,2}}{d_2} + \frac{\phi_{1,1,3} + \phi_{1,2,3}}{d_3} \le y_{1,1} + y_{1,2} + y_{2,1}. \quad (22)$$

We next introduce a family of valid inequalities for $Q^{I,MC}$ that generalize (22). For each element i of a subset $\mathcal{I}$ of periods $\mathcal{I} \subseteq \mathcal{T} \setminus \{1\}$, we select a nonempty subset V_i of plant-period pairs that can be used to produce d_i, *i.e.*, $V_i \neq \emptyset$ and $V_i \subseteq \{(p,t) | p \in \mathcal{P}, t \in \{1,\ldots,i\}\}$. Given these sets, we let $\mathcal{I}_{p,1}$ be the subset of $\mathcal{I}$ for which $(p,1)$ is selected among the plant-period pairs of V_i, *i.e.*, $\mathcal{I}_{p,1} = \{i \in \mathcal{I} \mid (p,1) \in V_i\}$. We say that $U \subseteq \{(p,t) \in V | t \ge 2\}$ is a *covering* of a subset $\mathcal{S}$ of $\mathcal{I}$ if $U \cap V_i \neq \emptyset$ for all $i \in \mathcal{S}$, where $V = \cup_{i \in \mathcal{I}} V_i$. In the remainder, we assume that there exists a covering of $\mathcal{I}$. In other words, we assume for each $i \in \mathcal{I}$ that $V_i \supseteq \{(p,t)\}$ for some $p \in \mathcal{P}$ and $t \in \mathcal{T} \setminus \{1\}$.

Given these sets, we define the *covering inequality* as

$$\sum_{i \in \mathcal{I}} \sum_{(p,t) \in V_i} \frac{\phi_{p,t,i}}{d_i} \le \sum_{(p,t) \in V | t \ge 2} y_{p,t} + \sum_{p \in \mathcal{P}} |\mathcal{I}_{p,1}| y_{p,1}. \quad (23)$$

In particular, (22) is of the form (23) with $\mathcal{I} = \{2,3\}$, $V_2 = \{(1,2),(2,1)\}$, $V_3 = \{(1,1),(1,2)\}$, $V = \{(1,1),(2,1),(1,2)\}$, $\mathcal{I}_{1,1} = \{3\}$, and $\mathcal{I}_{2,1} = \{2\}$.

THEOREM 2.1: Covering inequality (23) is valid for $Q^{I,\mathrm{MC}}$ if

(i) The size of a minimal covering of $\mathcal{I}$ is at least $|\mathcal{I}| - \min_{p \in \mathcal{P}} |\mathcal{I}_{p,1}|$.
(ii) For all $p \in \mathcal{P}$ the size of a minimal covering of $\mathcal{I} \setminus \mathcal{I}_{p,1}$ is at least $|\mathcal{I}| - |\mathcal{I}_{p,1}|$.

PROOF: We first argue that condition (i) implies that for any non-empty subset $\mathcal{S}$ of $\mathcal{I}$ such that $|\mathcal{S}| > \min_{p \in \mathcal{P}} |\mathcal{I}_{p,1}|$, the size of a minimal covering of $\mathcal{S}$ should be at least $|\mathcal{S}| - \min_{p \in \mathcal{P}} |\mathcal{I}_{p,1}|$. Assume not, then there exists a covering U of $\mathcal{S}$, with $|U| \le |\mathcal{S}| - \min_{p \in \mathcal{P}} |\mathcal{I}_{p,1}| - 1$. For each $i \in \mathcal{I} \setminus \mathcal{S}$, select $(p,t) \in V_i$ such that $t \ge 2$, and include it in U. Such element exists since we assume that there exists a covering of $\mathcal{I}$. The set obtained is a covering of $\mathcal{I}$ of size at most $|\mathcal{I}| - \min_{p \in \mathcal{P}} |\mathcal{I}_{p,1}| - 1$, which is the desired contradiction. A similar argument shows that, for each $p \in \mathcal{P}$, condition (ii) implies that for any nonempty subset $\mathcal{S}$ of $\mathcal{I}$, the size of a minimal covering of $\mathcal{S} \setminus \mathcal{I}_{p,1}$ should be at least $|\mathcal{S}| - |\mathcal{I}_{p,1} \cap \mathcal{S}|$.

Consider any feasible solution (y,ϕ) of LSS. Let $\mathcal{I}^* = \{i \in \mathcal{I} | \phi_{p,t,i} > 0 \text{ for some } (p,t) \in V_i\}$. Clearly, the left-hand-side of (23) is less than or equal to $|\mathcal{I}^*|$ since (21a) implies that $\sum_{(p,t) \in V_i} \frac{\phi_{p,t,i}}{d_i} \le 1$ for each $i \in \mathcal{I}$. Moreover, for each $i \in \mathcal{I}^*$ there exists $(p,t) \in V_i$ such that $y_{p,t} = 1$. Let $U = \{(p,t) \in V | y_{p,t} = 1\}$ and $\overline{\mathcal{P}} = \{p \in \mathcal{P} | (p,1) \in U\}$. We consider two cases.

First, assume that $\overline{\mathcal{P}} \neq \emptyset$. If $|\mathcal{I}^*| \le \sum_{p \in \overline{\mathcal{P}}} |\mathcal{I}_{p,1}|$ then (23) is valid since $y_{p,1} = 1$ for $p \in \overline{\mathcal{P}}$. Therefore, we assume that $|\mathcal{I}^*| > \sum_{p \in \overline{\mathcal{P}}} |\mathcal{I}_{p,1}|$. For some $\bar{p} \in \overline{\mathcal{P}}$, let $U_{\bar{p}} = U \setminus \bigcup_{p \in \overline{\mathcal{P}} \setminus \{\bar{p}\}} \{(p,1)\}$. Then, $U^* = U_{\bar{p}} \setminus \{(\bar{p},1)\}$ is nonempty and is a covering of $(\mathcal{I}^* \setminus \bigcup_{p \in \overline{\mathcal{P}} \setminus \{\bar{p}\}} \mathcal{I}_{p,1}) \setminus \mathcal{I}_{\bar{p},1}$. By condition (ii), we must have that $|U^*| \ge |\mathcal{I}^* \setminus \bigcup_{p \in \overline{\mathcal{P}} \setminus \{(\bar{p},1)\}} \mathcal{I}_{p,1}| - |(\mathcal{I}^* \setminus \bigcup_{p \in \overline{\mathcal{P}} \setminus \{\bar{p}\}} \mathcal{I}_{p,1}) \cap \mathcal{I}_{\bar{p},1}|$. Therefore, we have that $\sum_{(p,t) \in V | t \ge 2} y_{p,t} + \sum_{p \in \mathcal{P}} |\mathcal{I}_{p,1}| y_{p,1} \ge |\mathcal{I}^* \setminus \bigcup_{p \in \overline{\mathcal{P}} \setminus \{(\bar{p},1)\}} \mathcal{I}_{p,1}| - |(\mathcal{I}^* \setminus \bigcup_{p \in \overline{\mathcal{P}} \setminus \{\bar{p}\}} \mathcal{I}_{p,1}) \cap \mathcal{I}_{\bar{p},1}| + \sum_{p \in \overline{\mathcal{P}}} |\mathcal{I}_{p,1}| \ge |\mathcal{I}^*| - \sum_{p \in \overline{\mathcal{P}} \setminus \{\bar{p}\}} |\mathcal{I}_{p,1}| - |\mathcal{I}_{\bar{p},1}| + \sum_{p \in \overline{\mathcal{P}}} |\mathcal{I}_{p,1}| \ge |\mathcal{I}^*|$, where the first inequality holds because $y_{p,1} = 1$ for $p \in \overline{\mathcal{P}}$, and the second holds because $(\mathcal{I}^* \cap \mathcal{I}_{p,1}) \subseteq \mathcal{I}_{p,1}$ for $p \in \overline{\mathcal{P}}$.

Second, assume that $\overline{\mathcal{P}} = \emptyset$. Then U is a covering of $\mathcal{I}^*$. We assume that $|U| \le |\mathcal{I}^*|$ since otherwise (23) is valid as $y_{p,t} = 1$ for $(p,t) \in U$. Moreover, we assume that $|\mathcal{I}^*| > \min_{p \in \mathcal{P}} |\mathcal{I}_{p,1}|$ since otherwise (23) is valid as combining (21a) for $t' = 1$ and (21b) for $t' = 1$ and $p \in \mathcal{P}$ implies that $\sum_{p \in \mathcal{P}} y_{p,1} \ge 1$, and therefore $\sum_{p \in \mathcal{P}} |\mathcal{I}_{p,1}| y_{p,1} \ge \min_{p \in \mathcal{P}} |\mathcal{I}_{p,1}| \ge |\mathcal{I}^*|$. Then condition (i) implies that the size of U is at least $|\mathcal{I}^*| - \min_{p \in \mathcal{P}} |\mathcal{I}_{p,1}|$. Therefore, we have that $\sum_{(p,t) \in V | t \ge 2} y_{p,t} + \sum_{p \in \mathcal{P}} |\mathcal{I}_{p,1}| y_{p,1} \ge |\mathcal{I}^*| - \min_{p \in \mathcal{P}} |\mathcal{I}_{p,1}| + \sum_{p \in \mathcal{P}} |\mathcal{I}_{p,1}| y_{p,1} \ge |\mathcal{I}^*|$, where the last inequality holds because $\sum_{p \in \mathcal{P}} y_{p,1} \ge 1$ as explained previously. $\square$

Theorem 2.1 implies that (21) is valid for the set of Example 2.1, as $U_1 = \{(1,2)\}$ is a covering of $\mathcal{I}$, $U_2 = \{(1,2)\}$ is a covering of $|\mathcal{I}| \setminus \{3\}$, $U_3 = \{(1,2)\}$ is a covering of $|\mathcal{I}| \setminus \{2\}$, and $|\mathcal{I}_{1,1}| = |\mathcal{I}_{2,1}| = 1$.

Verifying that conditions (i)–(ii) hold is not trivial since finding a minimal covering is in general a difficult optimization problem. We next introduce a subfamily of inequalities whose construction guarantees that conditions (i)–(ii) hold. In the set of covering inequalities defined above, consider sets $\mathcal{I}$ of cardinality 2, and restrict sets V_i for $i \in \mathcal{I}$ to share all elements except for $\bigcup_{p\in\mathcal{P}}\{(p,1)\}$.

More precisely, let $\mathcal{I} = \{i, j\} \subseteq \mathcal{T} \setminus \{1\}$, where we assume without loss of generality that $i<j$. Let $V_i \subseteq \{(p,t)|p \in \mathcal{P}, t \in \{1,\ldots,i\}\}$ and $V_j \subseteq \{(p,t)|p \in \mathcal{P}, t \in \{1,\ldots,j\}\}$ be such that $V_i \setminus \bigcup_{p\in\mathcal{P}}\{(p,1)\} = V_j \setminus \bigcup_{p\in\mathcal{P}}\{(p,1)\} = W \neq \emptyset$. Note that $W \cap \{(p,T)\} = \emptyset$ for $p \in \mathcal{P}$, since only d_T can be produced in period T. We then write the *two-covering* inequality as

$$\sum_{(p,t)\in V_i} \frac{\phi_{p,t,i}}{d_i} + \sum_{(p,t)\in V_j} \frac{\phi_{p,t,j}}{d_j} \leq \sum_{(p,t)\in W} y_{p,t} + \sum_{p\in\mathcal{P}} |\mathcal{I}_{p,1}| y_{p,1}, \tag{24}$$

where $\mathcal{I}_{p,1} \subseteq \{i, j\}$ for $p \in \mathcal{P}$. By Theorem 2.1, (24) is valid for LSS, if $|\mathcal{I}_{p,1}| \geq 1$ for $p \in \mathcal{P}$, since $U = \{(p,t)\}$ for any $(p,t) \in W$ is a covering of $\mathcal{I}$. The following result regarding the conditions under which (24) is facet-defining is proven in Ref. [1].

THEOREM 2.2: A valid two-covering inequality (24) is facet-defining for $Q^{I,\text{MC}}$ if $|\mathcal{I}_{p,1}| = 1$ for $p \in \mathcal{P}$ and $\bigcup_{p\in\mathcal{P}} \mathcal{I}_{p,1} = \{i, j\}$.

In particular, Theorem 2.2 shows that (22) defines a facet of $Q^{\text{I,MC}}$ when $P = 2$ and $T \geq 3$.

3. POLYHEDRAL STUDY OF THE NATURAL FORMULATION OF LSS

In this section, we derive several families of facet-defining inequalities for the convex hull of the natural formulation of LSS.

3.1. Preliminaries

For $\mathbf{d} \in \mathbb{R}^T_+$, we define $R^{\text{NF}}_{\mathcal{P},\mathcal{T}}[\mathbf{d}]$ as

$$\left\{ (x;v;y) \in \mathbb{R}^{3PT} \;\middle|\; \begin{array}{lll} \sum_{i=1}^t x_{p,i} \geq \sum_{i=1}^t v_{p,i} & \forall p \in \mathcal{P}, \forall t \in \mathcal{T}\setminus\{T\} & \text{(25a)} \\ \sum_{i\in\mathcal{T}} x_{p,i} = \sum_{i\in\mathcal{T}} v_{p,i} & \forall p \in \mathcal{P} & \text{(25b)} \\ \sum_{p\in\mathcal{P}} \sum_{i=1}^t v_{p,i} \geq d_{1t} & \forall t \in \mathcal{T}\setminus\{T\} & \text{(25c)} \\ \sum_{p\in\mathcal{P}} \sum_{i\in\mathcal{T}} v_{p,i} = d_{1T} & & \text{(25d)} \\ x_{p,t} \leq d_{tT} y_{p,t} & \forall p \in \mathcal{P}, \forall t \in \mathcal{T} & \text{(25e)} \\ \sum_{p\in\mathcal{P}} y_{p,1} \geq 1 & & \text{(25f)} \\ y_{p,t} \leq 1 & \forall p \in \mathcal{P}, \forall t \in \mathcal{T} & \text{(25g)} \\ x_{p,t} \geq 0 & \forall p \in \mathcal{P}, \forall t \in \mathcal{T} & \text{(25h)} \\ v_{p,t} \geq 0 & \forall p \in \mathcal{P}, \forall t \in \mathcal{T} & \text{(25i)} \end{array} \right\}.$$

We do not include the constraints $y_{p,t} \geq 0$ in the formulation of R as they are implied by (25e) and (25h). We are interested in the set of mixed integer solutions of $R^{\text{NF}}_{\mathcal{P},\mathcal{T}}[\mathbf{d}]$. For this reason, we define $Q^{\text{NF}}_{\mathcal{P},\mathcal{T}}[\mathbf{d}] = R^{\text{NF}}_{\mathcal{P},\mathcal{T}}[\mathbf{d}] \cap (\mathbb{R}^{2PT} \times \mathbb{Z}^{PT})$ and $Q^{I,\text{NF}}_{\mathcal{P},\mathcal{T}}[\mathbf{d}] = \text{conv}(Q^{\text{NF}}_{\mathcal{P},\mathcal{T}}[\mathbf{d}])$. When the sets $\mathcal{P}$, $\mathcal{T}$ and the vector $\mathbf{d}$ are clear from the context, we use the simpler notation R, Q, and Q^I. It is clear that Q^I is a polyhedron.

In the remainder of this section, we establish some basic polyhedral results about Q^I. We use $(x;v;y)$ as a shorthand notation for the vector $(x_1, x_2, \ldots, x_P; v_1, v_2, \ldots, v_P; y_1, y_2, \ldots, y_P)$ in Q^I (or Q) where vectors x_p, v_p, and y_p are the production, shipment and setup variables of plant p, for $p \in \mathcal{P}$. Components of these vectors are the variables $x_{p,t}$, $v_{p,t}$ and $y_{p,t}$ corresponding to time period t, for $t \in \mathcal{T}$. Finally, given a vector $z \in \mathbb{R}^n$, we use the notation $z_{\looparrowright_{[z_i=\theta]}}$ to represent the vector obtained by replacing its component z_i (if it exists) by the value θ. To streamline notation, we write $(z_{\looparrowright_{[z_i=\theta]}})_{\looparrowright_{[z_j=\gamma]}}$ as $z_{\looparrowright_{[z_i=\theta,\, z_j=\gamma]}}$. We impose the following assumption in the remainder of this article.

ASSUMPTION 3.1: All demands are strictly positive, i.e., $\mathbf{d} \in \mathbb{R}^T_{++}$.

This assumption is common in the study of lot-sizing problems [5, 19] and streamlines proofs.

We next establish the dimension of Q^I. A proof of this result can be found in Ref. [1]. Knowing the dimension of Q^I when $P = 1$ is useful in subsequent derivations.

PROPOSITION 3.1: When $P \geq 2$, $\dim(Q^I) = 3PT - (P+1)$. When P = 1, $\dim(Q^I) = 3T - 3$.

When sets $\mathcal{P}$ and $\mathcal{T}$ are clear from the context, we often make use of the set $\mathbb{Q}_{\mathbb{PT}}[\mathbf{d}]$ defined as

$$\left\{(x;v;y) \in Q[\mathbf{d} \circ \mathbf{1}_{\mathbb{T}}] \;\middle|\; \begin{array}{ll} x_{p,t}=0,\ v_{p,t}=0,\ y_{p,t}=0 & \forall t \in \overline{\mathbb{T}}, \forall p \in \mathcal{P} \\ x_{p,t}=0,\ v_{p,t}=0,\ y_{p,t}=0 & \forall t \in \mathcal{T}, \forall p \in \overline{\mathbb{P}} \end{array}\right\}$$

where $\mathbb{P} \subseteq \mathcal{P}$, $\mathbb{T} \subseteq \mathcal{T}$, $\mathbf{d} \in \mathbb{R}^T_+$, $\mathbf{1}_{\mathbb{T}}$ is the indicator vector of $\mathbb{T}$, $a \circ b$ represents the componentwise product of conforming vectors a and b, $\overline{\mathbb{T}} = \mathcal{T} \setminus \mathbb{T}$ and $\overline{\mathbb{P}} = \mathcal{P} \setminus \mathbb{P}$. We also define $\mathbb{Q}^I_{\mathbb{P},\mathbb{T}}[\mathbf{d}] = \text{conv}(\mathbb{Q}_{\mathbb{PT}}[\mathbf{d}])$. It is clear that $\mathbb{Q}^I_{\mathcal{P},\mathcal{T}}[\mathbf{d}] = Q^I[\mathbf{d}]$. To illustrate the use of the notation introduced before, consider $(x^1;y^1;v^1)$ to be a solution of $\mathbb{Q}_{\{1\},\{1,\ldots,t\}}[\mathbf{d}]$ and $(x^2;y^2;v^2)$ to be a solution of $\mathbb{Q}_{\mathcal{P}\setminus\{1\},\{t+1,\ldots,T\}}[\mathbf{d}]$ for some $t \in \mathcal{T}$. Then $(x^1;y^1;v^1) + (x^2;y^2;v^2)$ is a solution of Q in which the demand for the first t periods is satisfied by plant 1, while the demand of later periods is satisfied by the other plants.

Observe that, after removing from $\mathbb{Q}_{\mathbb{P},\mathbb{T}}[\mathbf{d}]$ all variables that are fixed to zero, we obtain a problem of the same form as Q, with a possibly different number of plants and time periods. This observation, combined with Theorem 3.1, leads to the following result.

COROLLARY 3.1: Assume that $\mathbf{d}_{\mathbb{T}}$, the vector obtained from $\mathbf{d}$ by only keeping those components with indices in $\mathbb{T}$, is strictly positive. Then $\dim(\mathbb{Q}_{\mathbb{P},\mathbb{T}}[\mathbf{d}]) = 3|\mathbb{P}||\mathbb{T}| - (|\mathbb{P}|+1)$ if $|\mathbb{P}| \geq 2$ and $\dim(\mathbb{Q}_{\mathbb{P},\mathbb{T}}[\mathbf{d}]) = 3|\mathbb{T}| - 3$ if $|\mathbb{P}| = 1$.

For the remainder of this section we pose

ASSUMPTION 3.2: $T \geq 2$.

This assumption is without loss of generality as we give a minimal linear description of $Q^I_{\mathcal{P},\mathcal{T}}$ for the case where $T = 1$ in section "Linear Description of $Q^I_{\mathcal{P},\mathcal{T}}$ When $T = 1$" of Appendix. We next present a result that shows that the inequalities in the description of R are, for the most part, facet-defining for Q^I. This result is proven in Ref. [1].

THEOREM 3.1:

- (i) For $p \in \mathcal{P}$ and $t \in \mathcal{T} \setminus \{T\}$, (25a) defines a facet of Q^I.
- (ii) For $t \in \mathcal{T} \setminus \{T\}$, (25c) defines a facet of Q^I.
- (iii) For $p \in \mathcal{P}$ and $t \in \mathcal{T}$, (25e) defines a facet of Q^I if and only if (i) $P \geq 2$ or (ii) P = 1 and $t \neq 1$.
- (iv) Inequality (25f) defines a facet of Q^I if and only if $P \geq 2$.
- (v) For $p \in \mathcal{P}$ and $t \in \mathcal{T}$, (25g) defines a facet of Q^I if and only if (i) $P \geq 2$ or (ii) P = 1 and $t \geq 2$.
- (vi) For $p \in \mathcal{P}$ and $t \in \mathcal{T}$, (25h) defines a facet of Q^I if and only if $t \geq 2$.
- (vii) For $p \in \mathcal{P}$ and $t \in \mathcal{T}$, (25i) defines a facet of Q^I if and only if (a) $2 \leq t \leq T-1$ or (b) $P \geq 3$ and t = 1.

We mention that, in the traditional uncapacitated lot-sizing polytope, inequality $x_1 \leq d_{1T} y_1$ is not facet-defining. This characteristic remains when $P = 1$ but does not hold anymore when $P \geq 2$.

3.2. Families of Non-Trivial Facet-Defining Inequalities

In this section, we present families of non-trivial facet-defining inequalities for Q^I. We first give in Theorem 3.2, a family of inequalities that bound from above the amount of production occurring at a plant in successive periods. To explain these inequalities, we observe that valid inequalities $\sum_{i=t}^T v_{p,i} \geq \sum_{i=t}^T x_{p,i}$ and $d_{(j+1)T} \geq \sum_{i=j+1}^T v_{p,i}$ can be, respectively, obtained by subtracting (25b) from (25a) written for $t-1$, and by relaxing to a single plant the inequality obtained by subtracting (25c) with $t = j$ from (25d). Summing these two inequalities (in the case where $j \geq t$) yields the valid inequality

$$\sum_{i=t}^{T} x_{p,i} \leq \sum_{i=t}^{j} v_{p,i} + d_{(j+1)T}, \tag{25}$$

which is the basis for ensuing strengthened inequality (27).

The following lemma is useful in proving Theorem 3.2.

LEMMA 3.1: Let $P \geq 2$. Define $\mathcal{H} = \{(x;v;y) \in \mathbb{R}^{3PT} | y_{2,1} = 1\}$. Then $\dim(\mathbb{Q}^I_{\mathcal{P}\setminus\{1\},\mathcal{T}} \cap \mathcal{H}) = 3PT - (P+1) - 3T$.

PROOF: When $P = 2$, the result follows directly from Corollary 3.1 as $|\mathcal{P} \setminus \{1\}| = 1$ and $y_{2,1} = 1$ in all feasible solutions of $\mathbb{Q}^I_{\mathcal{P}\setminus\{1\},\mathcal{T}}[\mathbf{d}]$. When $P \geq 3$, Corollary 3.1 shows that we can construct affinely independent points $(\dot{x}^r;\dot{v}^r;\dot{y}^r)$ of $\mathbb{Q}^I_{\mathcal{P}\setminus\{1\},\mathcal{T}}[\mathbf{d}]$ for $r = 1,\ldots,\rho := 3PT - 3T - P + 1$. Consider now the collection of points $(\dot{x}^r;\dot{v}^r;\dot{y}^r)_{\dashv\!\to_{[y_{2,1}=1]}}$. These points clearly belong to $\mathbb{Q}^I_{\mathcal{P}\setminus\{1\},\mathcal{T}}[\mathbf{d}]$ and $\mathcal{H}$. Because a single component was modified, this family of vectors must contain a subcollection of at least $\rho - 1$ affinely independent vectors. Further, because $\mathbb{Q}^I_{\mathcal{P}\setminus\{1\},\mathcal{T}}[\mathbf{d}]$ is not contained in $\mathcal{H}$ as $|\mathcal{P} \setminus \{1\}| \geq 2$, every subcollection of affinely independent vectors in $\mathcal{H}$ cannot have more than $\rho - 1$ members. □

THEOREM 3.2: The production upper bound inequality

$$\sum_{i=t}^{k} x_{p,i} \leq \sum_{i=t}^{j} v_{p,i} + d_{(j+1)T} \sum_{i=t}^{k} y_{p,i} \tag{27}$$

is valid for Q^I when $1 \le t \le k \le j \le T-1$. Furthermore, such a valid inequality is facet-defining for Q^I if and only if $P \ge 2$ or $\mathrm{P}=1$ and $t \ge 2$.

PROOF: We assume without loss of generality that $p=1$. Consider $t \in \mathcal{T}\setminus\{T\}$, $k \in \{t,\dots,T-1\}$ and $j \in \{k,\dots,T-1\}$. We first argue that (27) is valid for Q^I under these assumptions. When $y_{1,i}=0$ for all $i \in \{t,\dots,k\}$, (27) reduces to $0 \le \sum_{i=t}^{j} v_{1,i}$, which is valid for Q^I as $Q^I \subseteq \mathbb{R}_+^{3PT}$. If $y_{1,i}=1$ for some $i \in \{t,\dots,k\}$ then (27) is dominated by (25). We denote by F the face of Q^I that (27) induces. Clearly, $F \ne Q^I$ as the vector $(\mathring{x};\mathring{v};\mathring{y})$ with nonzero components $(\mathring{x}_1,\mathring{v}_1,\mathring{y}_1)=(\mathbf{0},\mathbf{0},e_t)$; $(\mathring{x}_2,\mathring{v}_2,\mathring{y}_2)=(d_{1T}e_1,d_{1T}e_1,e_1)$ when $P\ge 2$ and $(\mathring{x}_1,\mathring{v}_1,\mathring{y}_1)=(d_{1T}e_1,d_{1T}e_1,\sum_{i=1}^{t}e_i)$ when $P=1$, belongs to Q^I but not to F.

Next we prove that (27) is facet-defining for Q^I under the stated assumptions. There are three cases.

Case 1: Assume $P \ge 2$ and $t>1$. Then using Lemma 3.1, there are affinely independent solutions

(i) $(\tilde{x}^i;\tilde{v}^i;\tilde{y}^i)$

for $i=1,\dots,3PT-P-3T$ in $\mathbb{Q}^I_{\mathcal{P}\setminus\{1\},\mathcal{T}}[\mathbf{d}]$ for which $\tilde{y}^i_{2,1}=1$. We next define $\mathbf{d}'=\mathbf{d}_{\looparrowright_{[d_{t-1}=d_{(t-1)j}]}}$. By Corollary 3.1, there are affinely independent points $(\acute{x}^s;\acute{v}^s;\acute{y}^s)$ for $s=1,\dots,3T-3(j-t)-5$ in $\mathbb{Q}_{1,\mathcal{T}\setminus\{t,\dots,j\}}[\mathbf{d}']$. We construct

(ii) $(\dot{x}^s;\dot{v}^s;\dot{y}^s)=(\acute{x}^s;\acute{v}^s;\acute{y}^s)_{\looparrowright_{[y_{2,1}=1]}}$.

These $3PT-P-3(j-t)-5$ solutions belong to F as $x_{1,i}=v_{1,i}=y_{1,i}=0$ for $i=t,\dots,j$. Next we construct for $i=t,\dots,j$ and $r=t+1,\dots,j$, vectors $(\check{x}^i;\check{v}^i;\check{y}^i)$, and $(\bar{x}^r;\bar{v}^r;\bar{y}^r)$, with non-zero components

(iii) $(\check{x}^i_1,\check{v}^i_1,\check{y}^i_1)=(d_{1(t-1)}e_1+d_{tT}e_t,d_{1(t-1)}e_1+\sum_{q=t}^{i-1}d_qe_q+d_{ij}e_i+d_{(j+1)T}e_{j+1},e_1+e_t)$, $(\check{x}^i_2,\check{v}^i_2,\check{y}^i_2)=(\mathbf{0},\mathbf{0},e_1)$,
(iv) $(\bar{x}^r_1,\bar{v}^r_1,\bar{y}^r_1)=(d_{1j}e_1+d_{(j+1)T}e_r,d_{1j}e_1+d_{(j+1)T}e_{j+1},e_1+e_r)$, $(\bar{x}^r_2,\bar{v}^r_2,\bar{y}^r_2)=(\mathbf{0},\mathbf{0},e_1)$.

These $2(j-t)+1$ vectors are affinely independent from each other and from those of families (i) and (ii) since each added vector has a nonzero component where all previous points had a zero component. Additionally, we construct vectors $(\check{x}^m;\check{v}^m;\check{y}^m)$ and $(\hat{x}^n;\hat{v}^n;\hat{y}^n)$ for $m=t+1,\dots,k$, and $n=k+1,\dots,j$, and vectors $(\dddot{x};\dddot{v};\dddot{y})$, $(\ddot{x};\ddot{v};\ddot{y})$ and $(\grave{x};\grave{v};\grave{y})$ with nonzero components

(v) $(\check{x}^m_1,\check{v}^m_1,\check{y}^m_1)=(d_{1(m-1)}e_1+d_{mT}e_m,d_{1(m-1)}e_1+d_{mj}e_m+d_{(j+1)T}e_{j+1},e_1+e_m)$, $(\check{x}^m_2,\check{v}^m_2,\check{y}^m_2)=(\mathbf{0},\mathbf{0},e_1)$,
(vi) $(\hat{x}^n_1,\hat{v}^n_1,\hat{y}^n_1)=(\check{x}^t_1,\check{v}^t_1,\check{y}^t_1)+(0,0,e_n)$, $(\hat{x}^n_2,\hat{v}^n_2,\hat{y}^n_2)=(\mathbf{0},\mathbf{0},e_1)$,
(vii) $(\dddot{x}_1,\dddot{v}_1,\dddot{y}_1)=(d_{1j}e_1+d_{(j+1)T}e_t,d_{1j}e_1+d_{(j+1)T}e_{j+1},e_1+e_t)$, $(\dddot{x}_2,\dddot{v}_2,\dddot{y}_2)=(\mathbf{0},\mathbf{0},e_1)$,
(viii) $(\ddot{x}_1,\ddot{v}_1,\ddot{y}_1)=(d_{tT}e_t,d_{tj}e_t+d_{(j+1)T}e_{j+1},e_t)$, $(\ddot{x}_2,\ddot{v}_2,\ddot{y}_2)=(d_{1(t-1)}e_1,d_{1(t-1)}e_1,e_1)$,
(ix) $(\grave{x}_1,\grave{v}_1,\grave{y}_1)=(d_{1T}e_1,d_{1T}e_1,e_1)$.

Observe that, all vectors in families (i)–(iv) satisfy the equalities $x_{1,i}=d_{(j+1)T}y_{1,i}$ for $i=t+1,\dots,j$, $x_{1,t}=d_{tT}y_{1,t}$, $\sum_{i=1}^{T}x_{1,i}=d_{1T}y_{1,1}$ and $y_{2,1}=1$. Vectors (v)–(ix) satisfy all but one of these equalities and each equality is violated by only one such point. Therefore, we conclude that these $(j-t)+3$ vectors are affinely independent from vectors (i)–(iv), and from each other.

Case 2: Assume that $P \ge 2$ and $t=1$. Similar to above, there exists affinely independent solutions $(\tilde{x}^i;\tilde{v}^i;\tilde{y}^i)$ for $i=1,\dots,3PT-P-3T$ in $\mathbb{Q}^I_{\mathcal{P}\setminus\{1\},\mathcal{T}}[\mathbf{d}]$ for which $\tilde{y}^i_{2,1}=1$. We construct the vectors

(i) $(\overline{\overline{x}}^i;\overline{\overline{v}}^i;\overline{\overline{y}}^i)=(\tilde{x}^i;\tilde{v}^i;\tilde{y}^i)_{\looparrowright_{[y_{1,j+1}=1]}}$.

By Corollary 3.1, there are affinely independent solutions $(\acute{x}^s;\acute{v}^s;\acute{y}^s)$ for $s=1,\dots,3(T-j)-2$ in $\mathbb{Q}_{\{1\},\{j+1,\dots,T\}}[\mathbf{d}]$. Then we construct the solutions

(ii) $(\hat{x};\hat{v};\hat{y})+(\acute{x}^s;\acute{v}^s;\acute{y}^s)$

where $(\hat{x};\hat{v};\hat{y})$ is a vector with nonzero components $(\hat{x}_2,\hat{v}_2,\hat{y}_2)=(d_{1j}e_1,d_{1j}e_1,e_1)$. These $3PT-P-3j-2$ solutions belong to F as $x_{1,i}=v_{1,i}=y_{1,i}=0$ for $i=1,\dots,j$. Next we construct for $i=1,\dots,j$, and $r=2,\dots,j$, the vectors $(\check{x}^i;\check{v}^i;\check{y}^i)$, and $(\dot{x}^r;\dot{v}^r;\dot{y}^r)$ with nonzero components

(iii) $(\check{x}^i_1,\check{v}^i_1,\check{y}^i_1)=(d_{1T}e_1,\sum_{q=1}^{i-1}d_qe_q+d_{ij}e_i+d_{(j+1)T}e_{j+1},e_1+e_{j+1})$, $(\check{x}^i_2,\check{v}^i_2,\check{y}^i_2)=(\mathbf{0},\mathbf{0},e_1)$,
(iv) $(\dot{x}^r_1,\dot{v}^r_1,\dot{y}^r_1)=(d_{(j+1)T}e_r,d_{(j+1)T}e_{j+1},e_r+e_{j+1})$, $(\dot{x}^r_2,\dot{v}^r_2,\dot{y}^r_2)=(d_{1j}e_1,d_{1j}e_1,e_1)$.

The $(2j-1)$ vectors in families (iii) and (iv) are affinely independent from each other, and from previously described vectors since each of them has a nonzero component where previous vectors have a zero component. Next, we construct for $m=2,\dots,k$ and $n=k+1,\dots,j$ vectors $(\bar{x}^m;\bar{v}^m;\bar{y}^m)$, and $(\grave{x}^n;\grave{v}^n;\grave{y}^n)$ and vectors $(\dddot{x};\dddot{v};\dddot{y})$, $(\check{x};\check{v};\check{y})$, and $(\ddot{x};\ddot{v};\ddot{y})$ with nonzero components

(v) $(\bar{x}^m_1,\bar{v}^m_1,\bar{y}^m_1)=(d_{mT}e_m,d_{mj}e_m+d_{(j+1)T}e_{j+1},e_m+e_{j+1})$, $(\bar{x}^m_2,\bar{v}^m_2,\bar{y}^m_2)=(d_{1(m-1)}e_1,d_{1(m-1)}e_1,e_1)$
(vi) $(\grave{x}^n_1,\grave{v}^n_1,\grave{y}^n_1)=(\check{x}^1_1,\check{v}^1_1,\check{y}^1_1)+(\mathbf{0},\mathbf{0},e_n)$, $(\grave{x}^n_2,\grave{v}^n_2,\grave{y}^n_2)=(\mathbf{0},\mathbf{0},e_1)$,

(vii) $(\ddot{\ddot{x}}_1, \ddot{\ddot{v}}_1, \ddot{\ddot{y}}_1) = (d_{(j+1)T}e_1, d_{(j+1)T}e_{j+1}, e_1 + e_{j+1})$, $(\ddot{\ddot{x}}_2, \ddot{\ddot{v}}_2, \ddot{\ddot{y}}_2) = (d_{1j}e_1, d_{1j}e_1, e_1)$,

(viii) $(\check{x}_1, \check{v}_1, \check{y}_1) = (d_{1T}e_1, d_{1j}e_1 + d_{(j+1)T}e_{j+1}, e_1 + e_{j+1})$,

(ix) $(\ddot{x}_1, \ddot{v}_1, \ddot{y}_1) = (d_{1T}e_1, d_{1j}e_1 + d_{(j+1)T}e_{j+1}, e_1)$, $(\ddot{x}_2, \ddot{v}_2, \ddot{y}_2) = (\mathbf{0}, \mathbf{0}, e_1)$.

Observe that, every solution in families (i)–(iv) satisfy the equalities $x_{1,i} = d_{(j+1)T}y_{1,i}$ for $i = 2, \ldots, j$, $x_{1,1} = d_{1T}y_{1,1}$, $y_{2,1} = 1$, and $y_{1,j+1} = 1$. The $(j+2)$ vectors (v)–(ix) satisfy all but one of these equalities and each equality is violated exactly once by one of vectors (v)–(ix). Therefore, we conclude that vectors (i)–(ix) are affinely independent.

Case 3: Assume that $P = 1$ and $t \geq 2$, we construct vectors by removing all components corresponding to plant 2 from the points created for the case where $P \geq 2$ and $t \geq 2$. In particular, we retain $(\dot{x}^s; \dot{v}^s; \dot{y}^s)$ for $s = 1, \ldots, 3T - 3(j-t) - 5$, $(\check{x}^i; \check{v}^i; \check{y}^i)$ for $i = t, \ldots, j$, $(\bar{x}^r; \bar{v}^r; \bar{y}^r)$ for $r = t+1, \ldots, j$, $(\check{x}^m; \check{v}^m; \check{y}^m)$ for $m = t+1, \ldots, k$, $(\hat{x}^n; \hat{v}^n; \hat{y}^n)$ for $n = k+1, \ldots, j$, and $(\ddot{\ddot{x}}; \ddot{\ddot{v}}; \ddot{\ddot{y}})$. These $3T - 3$ points belong to F and are affinely independent thereby showing that F is a facet.

Finally, when $P = 1$ and $t = 1$, (27) is not facet-defining for Q^I. In fact, every solution in the face of Q^I defined by (27) must satisfy $\sum_{i=1}^k x_{1,i} = d_{1T}$ as $\sum_{i=1}^k x_{1,i} = \sum_{i=1}^j v_{1,i} + d_{j+1T}\sum_{i=1}^k y_{1,i}$, $\sum_{i=1}^k y_{1,i} \geq 1$, and $\sum_{i=1}^j v_{1,i} \geq d_{1j}$. This concludes the proof. □

One might conjecture that the requirement $k \in \{t, \ldots, j\}$ in the definition of (27) is too stringent and could be replaced with $k \in \{t, \ldots, T\}$. It follows from an argument similar to that used in the proof of Theorem 3.2 that (27) is indeed valid when $k \in \{j+1, \ldots, T\}$. However, when $k > j$, the corresponding inequality is not facet-defining for Q^I. In fact, the inequality obtained as an equal weight conic combination of (27) when $k = j$ and (25e) for $i = j+1, \ldots, k$ dominates it as $d_{(j+1)T} \geq d_{iT}$ for $i = j+1, \ldots, k$.

EXAMPLE 3.1: Consider an instance of LSS where $P = 2$, $T = 5$, and $(d_1, d_2, d_3, d_4, d_5) = (5, 7, 8, 9, 4)$. It follows from Theorem 3.2 that

$$x_{1,1} \leq v_{1,1} + 28y_{1,1}, \quad \text{where } (p,t,j,k) = (1,1,1,1)$$
$$x_{2,2} + x_{2,3} \leq v_{2,2} + v_{2,3} + 13(y_{2,2} + y_{2,3}), \quad \text{where } (p,t,j,k) = (2,2,3,3)$$
$$x_{1,1} \leq v_{1,1} + v_{1,2} + 21y_{1,1}, \quad \text{where } (p,t,j,k) = (1,1,2,1)$$
$$x_{2,2} + x_{2,3} \leq v_{2,2} + v_{2,3} + v_{2,4} + 4(y_{2,2} + y_{2,3}), \quad \text{where } (p,t,j,k) = (2,2,4,3)$$

are facet-defining for Q^I.

Inequality (27) states that, if production occurs at plant p during the time window $\{t, \ldots, k\}$, then the quantity produced is smaller than the total shipment from plant p during time window $\{t, \ldots, j\}$, plus the total demand leftover after time period j.

The non-trivial inequalities presented above involve a single plant. Next, we present facet-defining inequalities for Q^I that are similar to (l, S) inequalities [5] and involve all plants. To describe them, we first define $L = \{1, \ldots, l\}$ for $l \in \mathcal{T}$. For $p \in \mathcal{P}$, we let $(S^p \bigcup \bar{S}^p \bigcup V^p \cup \bar{V}^p)$ be a partition of L. For each such partition and for $i \in \{1, \ldots, l+1\}$, we define

$$\sigma_p(i) = \begin{cases} \max\{j \mid \{i, i+1, \ldots, j\} \subseteq (V^p \cup \bar{V}^p)\} & \text{if } i \in \bar{V}^p, \\ i - 1 & \text{otherwise.} \end{cases}$$

In particular, $\sigma_p(l+1) = l$.

Given this notation, we refer to inequalities of the form

$$\sum_{p \in \mathcal{P}} \left[\sum_{i \in S^p} x_{p,i} + \sum_{i \in \bar{S}^p} d_{il} y_{p,i} + \sum_{i \in V^p} v_{p,i} + \sum_{i \in \bar{V}^p} (v_{p,i} + d_{(\sigma_p(i)+1)l} y_{p,i}) \right] \geq d_{1l} \tag{28}$$

as (l, S, V) inequalities. In (28) we refer to the variables associated with V^p and $\bar{V}^p$ as *shipment terms* and those associated with S^p and $\bar{S}^p$ as *production terms*. Observe that for $i \in \bar{V}^p$, $\sigma_p(i)$ refers to the largest index j for which all terms from period i to period j are shipment terms.

For $i \in \bar{V}^p$, if $\sigma_p(i) = l$, then $v_{p,i} + d_{(\sigma_p(i)+1)l} y_{p,i} = v_{p,i}$ as we assume that $d_{(l+1)l} = 0$. Therefore, including i in V^p would yield the same inequality, and we may pose the following assumption.

ASSUMPTION 3.3: $\sigma_p(i) < l$ for $i \in \bar{V}^p$ and $p \in \mathcal{P}$.

We next argue that, under some mild assumptions, (28) is valid for Q^I. Lemma 3.2 helps in establishing this result. It uses the ensuing notation.

For $(x; v; y) \in Q$ and $p \in \mathcal{P}$, we define

$$I^p[y] = \{j \in L \mid j \in (\bar{S}^p \cup \bar{V}^p) \text{ and } y_{p,j} = 1\}$$

and

$$t^p[y] = \begin{cases} l + 1 & \text{if } \ I^p[y] = \emptyset, \\ \min\{j \mid j \in I^p[y]\} & \text{if } \ I^p[y] \neq \emptyset. \end{cases}$$

We also define, $\ell^p(x; v; y) = \sum_{i \in S^p} x_{p,i} + \sum_{i \in \bar{S}^p} d_{il} y_{p,i} + \sum_{i \in V^p} v_{p,i} + \sum_{i \in \bar{V}^p} (v_{p,i} + d_{(\sigma_p(i)+1)l} y_{p,i})$, which is the part of (28) related to plant p. Finally, we define $LHS(x; v; y)$, or simply LHS, to be $\sum_{p \in \mathcal{P}} \ell^p(x; v; y)$, i.e., to be the left-hand-side value of (28) for point $(x; v; y)$.

LEMMA 3.2: Let $(x; v; y)$ be a feasible solution to Q and let $p \in \mathcal{P}$. Define $s_p = \sigma_p(t^p[y])$.

(a) If there exists $\kappa^p \in \{0, \dots, l\}$ such that $\{1, \dots, \kappa^p\} \subseteq (S^p \cup \bar{S}^p \cup \bar{V}^p)$ and $\{\kappa^p + 1, \dots, l\} \subseteq (V^p \cup \bar{V}^p)$, then

$$\ell^p(x; v; y) \geq \sum_{i=1}^{s_p} v_{p,i} + d_{(s_p+1)l}.$$

(b) If there exists $s \leq s_p$ and $\kappa^p \in \{0, \dots, s\}$ such that $\{1, \dots, \kappa^p\} \subseteq (S^p \cup \bar{S}^p \cup \bar{V}^p)$ and $\{\kappa^p + 1, \dots, s\} \subseteq (V^p \cup \bar{V}^p)$, then

$$\ell^p(x; v; y) \geq \sum_{i=1}^{s} v_{p,i} + d_{(s_p+1)l} y_{p,t^p[y]},$$

where $y_{p,t^p[y]}$ is taken to be zero if $I^p[y] = \emptyset$.

PROOF: We start with the proof of (a). First we observe that, for $i \leq t^p[y] - 1$

(i) $d_{il} y_{p,i} = x_{p,i}$ if $i \in \bar{S}^p$,
(ii) $v_{p,i} + d_{(\sigma_p(i)+1)l} y_{p,i} \geq x_{p,i}$ if $i \in \bar{V}^p$,
(iii) $v_{p,i} + d_{(\sigma_p(i)+1)l} y_{p,i} = v_{p,i}$ if $i \in \bar{V}^p$,

since $y_{p,i} = 0 = x_{p,i}$ under this assumption. There are several cases.

Case 1: Assume that $I^p[y] = \emptyset$. In this case $t^p[y] = l + 1$ and $s^p = l$. We write that

$$\ell^p(x; v; y) \geq \sum_{i=1}^{\kappa^p} x_{p,i} + \sum_{i=\kappa^p+1}^{l} v_{p,i} \geq \sum_{i=1}^{l} v_{p,i}$$

where the first inequality is obtained by applying (i) for $i \in \bar{S}^p$, (ii) for $i \in \bar{V}^p$ with $i \leq \kappa^p$, and (iii) for $i \in \bar{V}^p$ with $i \geq \kappa^p + 1$, while the second inequality holds because of (25a) or (25b). This proves the results as $d_{(s_p+1)l} = 0$ in this case.

Case 2: Assume that $I^p[y] \neq \emptyset$.

Case 2.1: If $t^p[y] \in \bar{S}^p$, then $s_p = t^p[y] - 1$ and $\kappa^p \geq s_p + 1$. We write that

$$\ell^p(x; v; y) \geq \sum_{i=1}^{s^p} x_{p,i} + d_{(s_p+1)l} y_{p,(s_p+1)} \geq \sum_{i=1}^{s_p} v_{p,i} + d_{(s_p+1)l},$$

where the first inequality is obtained by applying (i) for $i \in \bar{S}^p$ with $i \leq s_p$, (ii) for $i \in \bar{V}^p$ with $i \leq s_p$, and by lower bounding all terms with $i \geq s_p + 2$ by zero, while the second inequality follows from (25a) or (25b) and the fact that $y_{p,(s_p+1)} = 1$.

Case 2.2: If $t^p[y] \in \bar{V}^p$, then $s_p \geq t^p[y]$ and $\{t^p[y], \dots, s_p\} \subseteq (V^p \cup \bar{V}^p)$. Using Assumption 3.3, we have that $\kappa_p \geq s_p + 1$ since $\{t^p[y], \dots, s_p\} \in (V^p \cup \bar{V}^p)$. We write that

$$\ell^p(x; v; y) \geq \sum_{i=1}^{t^p[y]-1} x_{p,i} + v_{p,t^p[y]} + d_{(s_p+1)l} y_{p,t^p[y]} + \sum_{i=t^p[y]+1}^{s_p} v_{p,i} \geq \sum_{i=1}^{s_p} v_{p,i} + d_{(s_p+1)l}$$

where the first inequality is obtained by applying (i) for $i \in \bar{S}^p$ with $i \leq t^p[y] - 1$, (ii) for $i \in \bar{V}^p$ with $i \leq t^p[y] - 1$, lower bounding all $i \in (V^p \cup \bar{V}^p)$ with $t^p[y] + 1 \leq i \leq s_p$ by $v_{p,i}$, and lower bounding all terms with $i \geq s_p + 1$ by 0, while the second inequality follows from (25a) or (25b) and the fact that $y_{p,t^p[y]} = 1$.

We next prove (b). There are two cases.

Case 1: Assume that $t^p[y] \geq s$. Then $y_{p,i} = 0$ for $i \in (\bar{S}^p \cup \bar{V}^p)$ such that $i \leq s$. As $\kappa^p \leq s$, we write

$$\ell^p(x; v; y) \geq \sum_{i=1}^{\kappa^p} x_{p,i} + \sum_{i=\kappa_p+1}^{s} v_{p,i} + d_{(s_p+1)l} y_{p,t^p[y]} \geq \sum_{i=1}^{s} v_{p,i} + d_{(s_p+1)l} y_{p,t^p[y]}$$

where the first inequality holds by applying (i) for $i \in \bar{S}^p$ with $i \leq \kappa^p$, (ii) for $i \in \bar{V}^p$ with $i \leq \kappa^p$, and (iii) for $i \in \bar{V}^p$ with $\kappa^p + 1 \leq i \leq s$, and by lower bounding all terms involving $i \geq s+1$ with $i \neq t^p[y]$ by zero, while the second inequality holds because of (25a) or (25b).

Case 2: Assume that $t^p[y] \leq s$. Clearly $t^p[y] \in \bar{V}^p$. Assume not, then $t^p[y] \in \bar{S}^p$. This would then imply that $s \leq s_p = t^p[y] - 1 \leq s - 1$, a contradiction. It follows that $\{t^p[y], \dots, s\} \subseteq (V^p \cup \bar{V}^p)$ and therefore, we may assume that $\kappa^p \leq t^p[y] - 1$. We write

$$\ell^p(x; v; y) \geq \sum_{i=1}^{\kappa^p} x_{p,i} + \sum_{i=\kappa^p+1}^{t^p[y]-1} v_{p,i} + \sum_{i=t^p[y]}^{s} v_{p,i} + d_{(s_p+1)l} y_{p,t^p[y]} \geq \sum_{i=1}^{s} v_{p,i} + d_{(s_p+1)l} y_{p,t^p[y]},$$

where the first inequality holds by applying (i) for $i \in \bar{S}^p$ with $i \leq \kappa^p$, (ii) for $i \in \bar{V}^p$ with $i \leq \kappa^p$, and (iii) for $i \in \bar{V}^p$ with $\kappa^p + 1 \leq i \leq t^p[y] - 1$, by lower bounding all terms $i \in \bar{V}^p$ with $t^p[y]+1 \leq i \leq s$ by $v_{p,i}$, and by lower bounding all terms with $i \geq s + 1$ by zero, while the second inequality holds because of (25a) or (25b). □

We next give necessary and sufficient conditions for (*l, S, V*) inequalities (28) to be valid for Q^I.

THEOREM 3.3: An (l, S, V) inequality (28) is valid for Q^I if and only if one of the following conditions is satisfied

(a) $1 \in \bigcup_{p\in\mathcal{P}}(S^p \bigcup V^p)$ and, for each $p \in \mathcal{P}$, there exists $k^p \in \{0,\dots,l\}$ such that $\{1,\dots,k^p\} \subseteq (S^p \bigcup \bar{S}^p \bigcup \bar{V}^p)$ and $\{k^p+1,\dots,l\} \subseteq (V^p \bigcup \bar{V}^p)$,
(b) $1 \in \bigcap_{p\in\mathcal{P}}(\bar{S}^p \cup \bar{V}^p)$ and, for each $p \in \mathcal{P}$, there exists $k^p \in \{0,\dots,\sigma'\}$ such that $\{1,\dots,k^p\} \subseteq (S^p \cup \bar{S}^p \cup \bar{V}^p)$ and $\{k^p+1,\dots,\sigma'\} \subseteq (V^p \cap \bar{V}^p)$, where $\sigma' = \max_{p\in\mathcal{P}}\{\sigma_p(1)\}$.

PROOF: Let $(x;v;y)$ be any feasible solution to Q. As before, define $s_p = \sigma_p(t^p[y])$ for $p \in \mathcal{P}$. Let $s^* = \min_{p\in\mathcal{P}} s_p$ and let $p^* \in \mathcal{P}$ be such that $s^* = s_{p^*}$.

We start by showing that (28) is valid under (a). We write

$$\begin{aligned} LHS(x;v;y) &= \sum_{p\in\mathcal{P}} \ell^p(x;v;y) \\ &\geq \sum_{p\in\mathcal{P}}\sum_{i=1}^{s_p} v_{p,i} + \sum_{p\in\mathcal{P}} d_{(s_p+1)l} \\ &\geq \sum_{p\in\mathcal{P}}\sum_{i=1}^{s^*} v_{p,i} + d_{(s^*+1)l} \geq d_{1l}, \end{aligned}$$

where the first inequality follows from Lemma 3.2(a) using $\kappa^p = k^p$, the second is obtained by lower bounding terms $v_{p,i}$ by zero for $p \neq p^*$ and $i \in \{s^*+1,\dots,s_p\}$ and by lower bounding $d_{(s_p+1)l}$ by zero for $p \neq p^*$, while the third holds because of (25c) or (25d).

We now show that (28) is valid under assumption (b). Since $1 \in \cap_{p\in\mathcal{P}}(\bar{S}^p \cup \bar{V}^p)$, then $I^p[y] \neq \emptyset$ and $t^p[y] = 1$ for some $p \in \mathcal{P}$ as $\sum_{p\in\mathcal{P}} y_{p,1} \geq 1$. It follows that $s^* \leq \max_{p\in\mathcal{P}}\sigma_p(1) = \sigma'$. We write that

$$\begin{aligned} LHS(x;v;y) &= \sum_{p\in\mathcal{P}} \ell^p(x;v;y) \\ &\geq \sum_{p\in\mathcal{P}}\sum_{i=1}^{s^*} v_{p,i} + \sum_{p\in\mathcal{P}} d_{(s_p+1)l}\, y_{p,t^p[y]} \\ &\geq d_{1s^*} + d_{(s_{p^*}+1)l} \geq d_{1l} \end{aligned} \tag{29}$$

where the first inequality is obtained by applying Lemma 3.2(b) with $s = s^* \leq s_p$ for each $p \in \mathcal{P}$ and $\kappa^p = \min\{k^p, s\}$, the second inequality holds because of (25c) and the fact that $I^{p^*}[y] \neq \emptyset$, and the third because $s_{p^*} = s^*$.

Next, we prove that if (a) or (b) are not satisfied then it is possible to construct a feasible solution of Q that violates (28). Let π be an index of $\mathcal{P}$ such that $j < i$ where $j \in V^\pi$ and $i \in (S^\pi \bigcup \bar{S}^\pi)$ and, in the case of (b), $i, j \leq \sigma'$. It the latter case, it must then be that $\sigma' > 1$. For condition (a), let ρ be an index of $\mathcal{P}$ such that $1 \in (S^\rho \bigcup V^\rho)$. Then if $\pi \neq \rho$, we construct the vector $(\tilde{x};\tilde{v};\tilde{y})$ with nonzero components $(\tilde{x}_\pi, \tilde{v}_\pi, \tilde{y}_\pi) = (d_{jl}e_j, (d_{jl}-d_i)e_j + d_i e_i, e_j)$ and $(\tilde{x}_\rho, \tilde{v}_\rho, \tilde{y}_\rho) = (d_{1(j-1)}e_1 + d_{(l+1)T}e_{l+1}, d_{1(j-1)}e_1 + d_{(l+1)T}e_{l+1}, e_1 + e_{l+1})$. If $\pi = \rho = p$, we construct $(\hat{x};\hat{v};\hat{y})$ where $\hat{x}_p = \tilde{x}_\pi + \tilde{x}_\rho$, $\hat{v}_p = \tilde{v}_\pi + \tilde{v}_\rho$, and $\hat{y}_p = \tilde{y}_\pi + \tilde{y}_\rho$ if $j \geq 2$, and $\hat{y}_p = \tilde{y}_\rho$ if $j = 1$. For condition (b), we choose index $\rho \in \mathcal{P}$ such that $\sigma_\rho(1) = \sigma'$. In this case, it must be that $\pi \neq \rho$, since $\{1,\dots,\sigma'\} \subseteq \bar{V}^\rho$ for plant ρ and $j \in V^\pi$ and $j \leq \sigma'$ for plant π. We then construct the vector $(\bar{x};\bar{v};\bar{y})$ with nonzero components $(\bar{x}_\pi, \bar{v}_\pi, \bar{y}_\pi) = (d_{j\sigma'}e_j, (d_{j\sigma'}-d_i)e_j + d_i e_i, e_j)$ and $(\bar{x}_\rho, \bar{v}_\rho, \bar{y}_\rho) = ((d_{1j-1} + d_{(\sigma'+1)T})e_1, d_{1j-1}e_1 + d_{(\sigma'+1)T}e_{\sigma'+1}, e_1)$. For all aforementioned points, the left-hand-side of (28) is equal to $d_{1l} - d_i$ whereas the right-hand-side is d_{1l}, proving the result. □

When $1 \in \bigcup_{p\in\mathcal{P}}(S^p \bigcup V^p)$, the condition of Theorem 3.3 specifies that, in order for (28) to be valid, all production terms must occur before terms in V^p, for each plant $p \in \mathcal{P}$. Assumption 3.3 then implies that valid *(l, S, V)* inequalities can always be written in a way that terms in V^p only show up in periods $\{k^p+1,\dots,l\}$ for a suitable $k^p \in \{0,\dots,l\}$. In the remainder of this article, we assume that *(l, S, V)* inequalities are written in this way.

When $1 \in \bigcap_{p\in\mathcal{P}}(\bar{S}^p \bigcup \bar{V}^p)$, the validity condition of Theorem 3.3 is less restrictive. It states that, for (28) to be valid, it suffices that all production terms occur before terms in V^p, for each plant $p \in \mathcal{P}$ within the time window $\{1,\dots,\sigma'\}$. In particular, the ordering of production and shipment terms after σ' is arbitrary. Moreover, when $1 \in \cap_{p\in\mathcal{P}}(\bar{S}^p \cup \bar{V}^p)$ we have that $\sigma_p(1) < l$ for $p \in \mathcal{P}$ because of Assumption 3.3. We therefore assume without loss of generality that $\sigma' < l$ in the remainder of this section.

EXAMPLE 3.2: Consider an instance of LSS where $P=2$, $T=5$, and $(d_1,d_2,d_3,d_4,d_5) = (5,7,8,9,4)$. It follows from Theorem 3.3 that

$$5 \leq x_{1,1} + v_{2,1} \qquad l=1, S^1=\{1\}, V^2=\{1\}, \tag{30}$$

$$12 \leq x_{1,1} + 7y_{1,2} + (v_{2,1} + 7y_{2,1}) + 7y_{2,2} \qquad l=2, S^1=\{1\}, \bar{S}^1=\{2\}, \bar{S}^2=\{2\}, \bar{V}^2=\{1\}, \tag{31}$$

$$29 \leq 29y_{1,1} + (v_{1,2} + 9y_{1,2}) + (v_{1,3} + 9y_{1,3}) + x_{1,4} + \sum_{i=1}^{4} v_{2,i} \qquad l=4, S^1=\{4\}, \bar{S}^1=\{1\}, \bar{V}^1=\{2,3\}, V^2=L, \tag{32}$$

$$33 \leq (v_{1,1} + 28y_{1,1}) + x_{1,2} + x_{1,3} + 13y_{1,4} + x_{1,5} + \sum_{i=1}^{5} v_{2,i} \qquad l=5, S^1=\{2,3,5\}, \bar{S}^1=\{4\}, \bar{V}^1=\{1\}, V^2=L, \tag{33}$$

are valid inequalities for Q^I.

Inequalities (31) and (32) are facet-defining while (30) and (33) are not. Inequality (30) is dominated by $5 \leq v_{1,1} + v_{2,1}$

which is of the form (25c). Also, (33) is dominated by an equal weight conic combination of the inequalities $x_{1,4} \leq 13y_{1,4}$ and $x_{1,1} \leq v_{1,1} + 28y_{1,1}$, which are shown to be facet-defining in Theorems 3.1 (iii) and 3.2, and equalities $\sum_{i\in\mathcal{T}} v_{1,i} = \sum_{i\in\mathcal{T}} x_{1,i}$ and $33 = \sum_{p\in\mathcal{P}}\sum_{i\in\mathcal{T}} v_{p,i}$.

We next present necessary and sufficient conditions for (28) to be facet-defining for Q^I. The proof of this result consists in exposing a sufficient number of affinely independent tight points on the face defined by (28). This proof can be found in the Appendix.

THEOREM 3.4: For $P \geq 2$, let (28) be a valid inequality for Q^I that does not define the same face of Q^I as (25f). Then (28) is facet-defining for Q^I if and only if the following conditions are satisfied:

(a) $1 \in \underset{p\in\mathcal{P}}{\cup}(S^p \bigcup V^p)$,
(b) $l<T$,
(c) $V^p = L$ or $(\bar{S}^p \cup \bar{V}^p) \neq \emptyset$ for each $p \in \mathcal{P}$.

Theorem 3.4 shows that inequalities (31)–(32) are facet-defining for Q^I. Inequality (30) is not facet-defining for Q^I since it violates condition (c) for $p=1$. Similarly, (33) is not facet-defining for Q^I since it violates condition (b).

When (28) involves only production terms, we obtain inequalities

$$\sum_{p\in\mathcal{P}}\left[\sum_{i\in S^p} x_{p,i} + \sum_{i\in L\setminus S^p} d_{il}y_{p,i}\right] \geq d_{1l}, \tag{34}$$

which are similar in structure to the traditional (*l, S*) inequalities of Ref. [5]. Theorem 3.4 shows that inequalities (34) are facet-defining for Q^I if and only if $1 \in \underset{p\in\mathcal{P}}{\cup} S^p$, $l<T$ and $\bar{S}^p \neq \emptyset$ for $p \in \mathcal{P}$. These conditions are similar to those given in Ref. [5].

Theorem 3.5 provides a minimal linear description of $Q^I_{\mathcal{P},\mathcal{T}}$ for the case where $T=1$. Its proof is presented in the Appendix.

THEOREM 3.5: The convex hull of $Q_{P,1}[\mathbf{d}]$, for $\mathbf{d} \in \mathbb{R}_{++}$ is given by

$$Q^* = \left\{(x;v;y) \in \mathbb{R}^P_+ \times \mathbb{R}^{2P} \;\middle|\; \begin{array}{lll} x_{p,1} = v_{p,1} & \forall p \in \mathcal{P} & (35a) \\ \sum_{p\in\mathcal{P}} x_{p,1} = d_1 & & (35b) \\ x_{p,1} \leq d_1 y_{p,1} & \forall p \in \mathcal{P} & (35c) \\ y_{p,1} \leq 1 & \forall p \in \mathcal{P} & (35d) \end{array}\right\}.$$

Theorem 3.6 shows that the families of inequalities derived in this section are sufficient to describe the convex hull of LSS when $P = T = 2$. A proof is given in Ref. [1].

THEOREM 3.6: The convex hull of $Q_{2,2}[\mathbf{d}]$, for $\mathbf{d} \in \mathbb{R}^2_{++}$ is given by inequalities (25a)–(25i), production upper bound inequalities $x_{p,1} \leq v_{p,1} + d_2y_{p,1}$ for $p \in \mathcal{P}$ and (l, S, V) inequalities $d_1 \leq v_{1,1} + d_1y_{2,1}$, and $d_1 \leq v_{2,1} + d_1y_{1,1}$.

4. COMPUTATIONAL RESULTS

In this section, we investigate the effectiveness of formulations and families of inequalities we derived in the solution of instances of LSS.

4.1. Separation Algorithms

First, we note that the number of production inequalities (27) is bounded above by $O(PT^3)$. They can therefore be trivially separated in polynomial time. For this reason, we focus only on the separation of the (*l, S, V*) inequalities (28) which are exponential in number. In particular, we present an $O(PT^4)$ algorithm that, given a fractional solution $(x^*;v^*;y^*)$, outputs a violated (*l, S, V*) inequality if one exists. When validity conditions of Theorem 3.3 are applied, (*l, S, V*) inequalities can be written in the form

$$\sum_{p\in\mathcal{P}}\left[\sum_{i\in S^p} x_{p,i} + \sum_{i\in\bar{S}^p} d_{il}y_{p,i} + \sum_{i\in\bar{V}^p}(v_{p,i} + d_{(\sigma_p(i)+1)l}y_{p,i}) + \sum_{i=k_p}^{l} v_{p,i}\right] \geq d_{1l}, \tag{35}$$

where $k_p \leq l+1$ and S^p, $\bar{S}^p$, and $\bar{V}^p$ form a partition of $\{1,\ldots,k_p\}$ for $p \in \mathcal{P}$. Therefore, we must decide for each $i \in L = \{1,\ldots,l\}$ and each plant $p \in \mathcal{P}$, whether i belongs to S^p, $\bar{S}^p$, $\bar{V}^p$ or $V^p = \{k_p,\ldots,l\}$. Similar to the separation of (*l, S*) inequalities, see [5], we minimize for each plant $p \in \mathcal{P}$ and $l \in \mathcal{T}$ the left-hand-side of (35). To do so, we solve a collection of shortest path problems on networks $G^p_l = (N^p_l, A^p_l)$ for $p \in \mathcal{P}$ and for $l \in \{2,\ldots,T-1\}$. A graphical representation of one such network is given in Fig. 3.

To construct network G^p_l, we create nodes (*i, j*) for all $i \in \{1,\ldots,l\}$ and $j \in \{i,\ldots,l\}$ together with nodes (0, 0) and $(l+1, l+1)$. To streamline notation, we refer to node (0, 0) as s and to node $(l+1, l+1)$ as t. Node $(i,j) \in N^p_l \setminus \{s,t\}$ represents the decision to include the entire time window $[i,j]$ in one of the sets S^p, $\bar{S}^p$, or $\bar{V}^p$. We include an arc from each node $(i,j) \in N^p_l \setminus \{t\}$ to any other node $(j+1,k)$ to represent the decision of assigning time periods $[i,j]$ to a common set among $S^p, \bar{S}^p$ and $\bar{V}^p$ while assigning periods $[j+1,k]$ to a (possibly different) common set among $S^p, \bar{S}^p$ and $\bar{V}^p$. The cost of such an arc is equal to $\min\{\sum_{i=j+1}^{k} x^*_{p,i}, \sum_{i=j+1}^{k} d_{il}y^*_{p,i}, \sum_{i=j+1}^{k}(v^*_{p,i} + d_{(k+1)l}y^*_{p,i})\}$. Moreover, we connect each node $(i,j) \in N^p_l \setminus \{t\}$ to node t to model the decision of assigning the entire

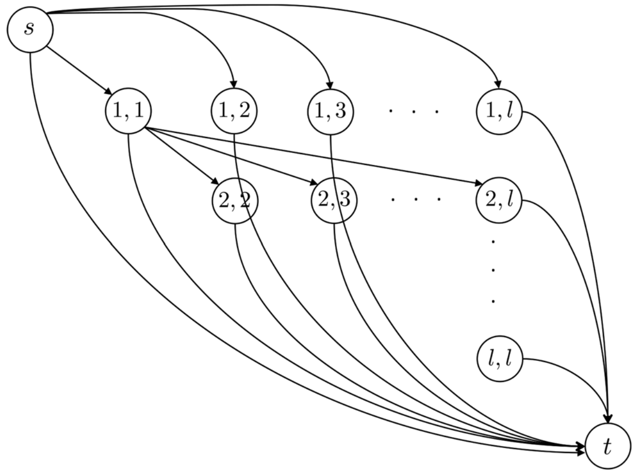

Figure 3. Network for separation of (*l, S, V*) inequalities.

time window $[j+1, l]$ to V^p. The cost of the corresponding arc is $\sum_{i=j+1}^{l} v^*_{p,i}$. It is easy to verify that G_l^p is acyclic.

It can now be easily verified that any path from s to t in network G_l^p represents a partition of L into subintervals. Further, the shortest path from s to t computes such a partition for which the component of the left-hand-side of (35) related to plant p is computed exactly and is minimized. Repeating the shortest path computation for each $p \in \mathcal{P}$ allows us to determine whether an (*l, S, V*) inequality with $L = \{1, \ldots, l\}$ is violated. To obtain a maximally violated (*l, S, V*) inequality given a fractional solution $(x^*; v^*; y^*)$, it suffices to enumerate all values of $l \in \{2, \ldots, T-1\}$. This leads to Algorithm 1.

Algorithm 1 Separation algorithm for (*l, S, V*) inequalities

for $l = 2, \ldots, T-1$ **do**
  for $p \in \mathcal{P}$ **do**
    Create network $G_l^p = (N_l^p, A_l^p)$.
    Find a shortest path on G_l^p. Let α_l^p be the length of this shortest path.
  end for
  if $\sum_{p\in\mathcal{P}} \alpha_l^p < d_{1l}$ **then** output $(S^p, \bar{S}^p, V^p, \bar{V}^p)$ for $p \in \mathcal{P}$.
  end if
end for

We next analyze the running time of Algorithm 1. First recall that the shortest path problem on an acyclic graph can be solved in time $O(m+n)$ where m is the number of arcs in the network and n is the number of nodes. Since $G_l^p = (N_l^p, A_l^p)$ is acyclic and $|N_l^p| \le |A_l^p|$, it suffices to derive a bound on the cardinality of A_l^p to estimate the worst-case running time of the algorithm. For fixed l, the number of nodes in G_l^p is bounded above by $O(l^2)$. Moreover, the degree of each node is bounded above by $O(l)$. It follows that the total number of arcs in G_l^p is $O(l^3)$. Finding a shortest path on G_l^p therefore requires $O(l^3)$ computation. Since the solution of a shortest path problem is required for each $l \in \{2, \ldots, T-1\}$ and for each $p \in \mathcal{P}$, we conclude that the amount of computation required by the separation algorithm is bounded above by $O(PT^4)$.

REMARK 1: The arc cost calculations required for creating G_l^p need to be performed efficiently to achieve the $O(PT^4)$ bound. In particular, before creating G_l^p, we define $D_t = \sum_{i=1}^{t} d_i$, $Y_{p,t} = \sum_{i=1}^{t} y^*_{p,i}$ and $V_{p,t} = \sum_{i=1}^{t} v^*_{p,i}$. Because $D_t = D_{t-1} + d_t$, $Y_{p,t} = Y_{p,t-1} + y^*_{p,t}$ and $V_{p,t} = V_{p,t-1} + v^*_{p,t}$ for $t > 1$, computing values D_t for $t \in \mathcal{T}$ and values $Y_{p,t}$ and $V_{p,t}$ for $p \in \mathcal{P}$ and for $t \in \mathcal{T}$ requires an amount of computation that is bounded above by $O(PT)$. Once these values are calculated, the cost of arc $((i,j),(j+1,k))$ can be computed from the cost of arc $((i,j),(j+1,k-1))$ by adding $x^*_{p,k}$ to the first term, $(D_l - D_{k-1})y^*_{p,k}$ to the second term, and $v^*_{p,k} + (D_l - D_k)y^*_{p,k} - d_k(Y_{p,k-1} - Y_{p,j})$ to the third term. Moreover, the cost of arc $((i,j),t)$ can be obtained as $(V_{p,l} - V_{p,j})$. We conclude that each arc cost can be calculated using an amount of computation that is bounded above by $O(1)$, resulting in a total effort of $O(l^3)$.

Although Algorithm 1 is polynomial, it is relatively expensive to run in practice. For this reason, we implement a heuristic based on the following observation.

REMARK 2: If we lower bound $\sigma_p(i)$ by i for $p \in \mathcal{P}$ and for $i \in \bar{V}^p$, (*l, S, V*) inequalities become

$$\sum_{p\in\mathcal{P}} \left[\sum_{i\in S^p} x_{p,i} + \sum_{i\in\overline{S^p}} d_{il} y_{p,i} + \sum_{i\in\bar{V}^p} (v_{p,i} + d_{(i+1)l} y_{p,i}) + \sum_{i=k_p}^{l} v_{p,i} \right] \ge d_{1l}. \qquad (36)$$

Because $d_{(\sigma_p(i)+1)l} \le d_{(i+1)l}$, (2) is valid but not necessarily facet-defining for LSS. When applying Algorithm 1 to separate (36), the amount of computation required can be bounded above by $O(PT^2)$. This is due to the fact that when building G_l^p, it is sufficient to include nodes (*i, i*) for $i \in L \cup \{0, l+1\}$. Therefore the shortest path cost on G_l^p can be calculated in $O(l)$ for fixed $l \in L$ and $p \in \mathcal{P}$ since the degree of each node in the graph is bounded above by 2. Since $|\mathcal{P}| = P$ and $|L| = O(T)$, we obtain the desired result. If a violated inequality is found using this heuristic, it is easy to verify that the (*l, S, V*) inequality based on $(S^p, \bar{S}^p, V^p, \bar{V}^p)$ is also valid and violated. This provides us with a faster procedure to heuristically separate (*l, S, V*) inequalities.

4.2. Experimental Setup and Instance Generation

We present two sets of computational experiments in Section 4.3. These instances are available at "http://www.ise.ufl.edu/guan/publications/" for downloading. The first set is composed of uncapacitated LSS instances with $T = 25$. We use these instances to evaluate the strength of the different formulations introduced in this article and of the facet-defining inequalities we developed. The second set is comprised of capacitated instances of LSS with number of periods T chosen randomly in the interval [70, 100]. We use these instances to evaluate the strength of (l, S, V) inequalities inside of a branch-and-cut framework. Together these experiments show that our results provide computational improvements for both LSS instances as well as instances of other problems for which LSS is a relaxation.

We generate random instances as follows. We use the ratio of maximum fixed production cost to maximum variable production cost, δ, as the main lever to produce instances with different characteristics. We set the maximum production cost, c_{max}, to 100 and then calculate the maximum fixed cost as $f_{max} = \delta c_{max}$. To create fixed and variable production costs we first generate a random number $\alpha_{p,t}$ uniformly in the interval $[0, 1]$, and then calculate $c_{p,t} = \alpha_{p,t} c_{max}$ and $f_{p,t} = (1 - \alpha_{p,t}) f_{max}$. This choice of parameters tends to generate harder instances by creating less obvious tradeoffs between periods with small fixed cost and large production cost and periods with large fixed cost and small production cost. Customer demand and inventory cost for each period are generated using a discrete uniform distribution in the intervals [50, 100] and [5, 20], respectively. For each plant, we generate transportation and inventory holding costs using a discrete uniform distribution with values in the interval [5, 20].

For uncapacitated instances, we set the production and shipment capacity of each plant in each time period to $\sum_{t \in \mathcal{T}} d_t$. For capacitated instances, we generate a common production and shipment capacity for all plants and time periods. This number is randomly selected in the interval $[1.5\bar{d}, 1.75\bar{d}]$ where $\bar{d} = \frac{\sum_{t \in \mathcal{T}} d_t}{T}$ is the average demand. In our experiments, all instances generated in this fashion were feasible, although there is no guarantee that it would always be so. An a priori feasibility test could be performed to verify that the cumulative capacity across all plants is sufficient to supply the cumulative demand up to period t. If instances where the demand is greater than the capacity were encountered, they would have been eliminated from further consideration.

4.3. Results

In this section, we present the results of our computational experiments. We perform all computations using IBM ILOG CPLEX version 12.5, on a Windows machine with Windows 7 Professional running a 64-bit x86 processor equipped with 2.83 GHz Intel Core2 Quad Q9500 chips and 4GB RAM.

In our first set of experiments, we consider the relative gap at the root node right before branching. The gap reported is calculated as $100\frac{z^* - \bar{z}}{z^*}$ where z^* is objective function value of the optimal solution and $\bar{z}$ is the optimal value of the LP relaxation at the the root node right before branching. We summarize our results in Table 1. We investigate the following: the natural formulation with default CPLEX cuts (*CPX*), the natural formulation strengthened with (l, S, V) inequalities separated using the heuristic separation algorithm and CPLEX cuts (*LSVH*), the natural formulation strengthened with (l, S, V) inequalities separated using the exact separation algorithm and CPLEX cuts (*LSVE*), the natural formulation strengthened with production upper bound inequalities and CPLEX cuts (*PUB*), and the multi-commodity formulation with three-index variables (*MC3*). For this experiment, we create 10 uncapacitated instances of LSS for each combination of the parameters $P \in \{2, 3, 4, 5\}$ and $\delta \in \{25, 50, 75, 100\}$. The initial average relative gap of the relaxation for each choice of parameters is reported under the header *RGap*. The average relative gap at the root node right before branching and the number of cuts added to the relaxation are reported under the headers *Gap* and *Cuts* (when applicable, the number of user cuts is reported under U while the number of CPLEX cuts is reported under C), respectively. Table 1 shows that the natural formulation strengthened with (l, S, V) cuts separated using the exact separation algorithm solved LSS at the root node for most instances considered. This suggests that (l, S, V) cuts provide a good approximation of the convex hull of LSS. Using the exact separation algorithm in practice might however be too time-consuming. Table 1 shows that (l, S, V) cuts separated using the heuristic separation algorithm are effective at reducing the initial relative gap. In fact, on average, heuristically separated (l, S, V) inequalities close 87% of the gap that cannot be closed using CPLEX cuts alone. For all of the instances presented in Table 1 the multi-commodity formulation (*MC3*) has zero initial relative gap. This is the reason we do not report the number of cuts generated for the multi-commodity formulation. This suggests that multi-commodity formulations are strong in practice. In fact, our computational experience is that the multi-commodity formulation is effective in solving uncapacitated LSS instances even as the number of plants increases. Its use however becomes less appealing as additional restrictions, such as capacities or fixed charges on shipment arcs are introduced.

We next explore the effectiveness of (l, S, V) cuts in solving capacitated instances of LSS. Similar to before, we create 10 capacitated instances of LSS for each choice of the parameters $P \in \{2, 3, 4, 5\}$ and $\delta \in \{125, 250\}$. In Table 2, we report average results across these 10 instances. We compare the solution performance of the natural formulation with default CPLEX cuts (*CPX*), to the natural formulation strengthened

Table 1. Strength of different formulations on instance set 1

		CPX			LSVH				LSVE				PUB				MC3	
						Cuts				Cuts				Cuts				
P.δ	*RGap*	*Gap*	*Cuts*	*Time*	*Gap*	*U*	*C*	*Time*	*Gap*	*U*	*C*	*Time*	*Gap*	*U*	*C*	*Time*	*Gap*	*Time*
2.25	20.54	0.4	23	0.07	0.0	37	18	0.07	0.0	34	19	0.52	1.3	5	23	0.42	0.0	0.03
2.50	36.33	1.1	32	0.13	0.0	91	18	0.15	0.0	80	20	1.06	2.9	13	32	0.37	0.0	0.03
2.75	40.67	2.4	33	0.10	0.0	141	18	0.25	0.0	110	20	1.47	4.8	17	33	0.33	0.0	0.03
2.100	51.31	4.4	36	0.15	0.1	225	11	0.37	0.0	185	12	2.16	9.7	16	35	0.46	0.0	0.02
3.25	31.41	0.8	43	0.15	0.1	113	24	0.25	0.0	100	26	2.20	2.6	17	40	0.48	0.0	0.05
3.50	45.27	2.7	41	0.22	0.3	193	15	0.44	0.0	177	15	3.62	7.0	22	44	0.74	0.0	0.04
3.75	47.59	4.9	48	0.28	0.9	239	16	0.57	0.1	209	17	3.81	9.1	28	46	0.78	0.0	0.05
3.100	56.81	7.5	50	0.32	0.9	334	11	0.92	0.3	296	11	6.33	12.3	30	49	0.84	0.0	0.04
4.25	34.71	1.5	46	0.19	0.1	149	25	0.43	0.0	123	23	3.68	3.7	21	45	0.66	0.0	0.06
4.50	46.63	4.8	54	0.35	0.9	282	20	0.90	0.3	250	20	7.35	8.9	33	54	1.00	0.0	0.05
4.75	55.40	8.9	59	0.43	1.8	345	13	1.27	0.8	375	12	9.81	11.8	38	56	1.40	0.0	0.05
4.100	57.96	9.0	63	0.49	2.8	362	18	1.38	1.0	415	17	9.65	13.1	37	61	1.37	0.0	0.05
5.25	37.44	1.7	47	0.26	0.1	179	23	0.66	0.4	134	28	6.11	3.9	18	46	0.90	0.0	0.06
5.50	49.67	4.9	64	0.46	0.7	313	17	1.28	0.8	251	20	8.66	8.5	32	59	1.45	0.0	0.07
5.75	58.25	10.7	69	0.61	3.1	384	19	1.80	2.5	380	18	10.34	14.3	36	69	1.74	0.0	0.07
5.100	63.09	13.5	71	0.70	2.6	523	14	3.02	3.1	473	14	12.01	16.6	51	66	2.04	0.0	0.07

Table 2. Branch-and-cut performance for instance set 2

		$\delta = 125$			$\delta = 250$		
		CPX	*LSVH*	*MC4*	*CPX*	*LSVH*	*MC4*
$P = 2$	*RGap*	2.21	1.27	2.59	1.44	1.13	2.01
	Time	1241	609	3625	10	6	2788
	NNodes	2,803,693	1,220,881	68,670	17,497	5152	50,186
	Gap	0.11	0.10	0.58	0.10	0.10	0.23
	LCuts	0	20	0	0	10	0
	CCuts	111	92	496	62	47	429
	#Opt	8	9	0	10	10	3
$P = 3$	*RGap*	1.81	1.23	3.84	1.24	0.88	1.78
	Time	1064	378	3649	62	53	2737
	NNodes	1,483,977	491,653	44,444	61,332	53,707	28,730
	Gap	0.11	0.10	0.53	0.10	0.10	0.25
	LCuts	0	16	0	0	7	0
	CCuts	123	108	659	72	58	588
	#Opt	9	10	0	10	10	5
$P = 4$	*RGap*	1.61	1.19	3.79	1.42	1.04	5.23
	Time	1477	691	3699	34	24	3404
	NNodes	1,267,324	481,330	32,115	24,588	17,493	25,804
	Gap	0.11	0.10	0.48	0.10	0.10	0.42
	LCuts	0	17	0	0	11	0
	CCuts	140	121	818	78	76	731
	#Opt	7	9	0	10	10	1
$P = 5$	*RGap*	1.60	1.09	6.36	1.47	0.91	11.37
	Time	2449	1629	3604	148	222	3000
	NNodes	1,947,505	1,142,436	24,299	1,096,02	1,742,53	19,755
	Gap	0.16	0.13	0.47	0.10	0.10	0.34
	LCuts	0	19	0	0	13	0
	CCuts	146	139	992	85	80	817
	#Opt	5	7	1	10	10	3

with both heuristically separated (*l, S, V*) cuts and default CPLEX cuts (*LSVH*), and the multi-commodity formulation with default CPLEX cuts (*MC4*) adjusted to incorporate capacities as described in Section 2. We do not use the multi-commodity formulation with three-index variables (*MC3*) since it cannot be used in the presence of shipment capacities. Moreover, we choose not to investigate the effectiveness of production upper bound cuts since our results in Table 1 suggest that they are not as useful as default CPLEX cuts.

In our implementation, we first derive a set of violated (*l, S, V*) cuts by using the callback functionalities of CPLEX. In this framework, our polynomial-time heuristic separation routine of Section 4.1 is invoked each time a fractional solution of the root relaxation is found. If it returns violated cuts, up to 10 of them are added to the cut pool and the relaxation is solved again. This procedure continues until the improvement in gap becomes less than 0.1% over three consecutive iterations or when the separation routine is visited more than 50 times. When the callback routine is terminated, we calculate the slack of each cut discovered. We then create a new model containing the initial formulation and the cuts we generated that have zero slack. We solve the resulting model with CPLEX in its default settings.

In Table 2, we report the results obtained by solving each model using CPLEX in its default settings with a time limit of 3600 s. We set the relative optimality gap tolerance to 0.1%. In Table 2, we denote the average relative optimality gap reported by CPLEX at the root node right before branching as *RGap*, the average total solution time (including time spent in the separation routine and the time spent building the model) in seconds as *Time*, the average number of nodes in the branch-and-bound tree as *NNodes*, the average relative optimality gap when CPLEX stops as *Gap*, the average number of (*l, S, V*) cuts added to the formulation as *LCuts*, the average number of CPLEX cuts added to the formulation as *CCuts*, and the number of instances that are solved to 0.1% optimality as *#Opt*.

Table 2 shows that when $\delta = 125$, adding (*l, S, V*) cuts to the natural formulation reduced the number of nodes by 54% on average, the solution time by 51% on average, and the strengthened formulation was able to solve more instances to optimality within the time limit. These results illustrate the computational potential of using (*l, S, V*) cuts in solving capacitated LSS instances.

Our results also show that the multi-commodity formulation was unable to solve most of the instances considered within the time limit when $\delta = 125$. It was able to solve less than 50% of the instances within the time limit when $\delta = 250$. When $\delta = 250$, adding (*l, S, V*) cuts to the natural formulation reduced the number of nodes in most instances. The difference in solution times was small, however, because the solution time for these instances using default CPLEX cuts is under 180 s on average.

We conclude from these experiments that for uncapacitated instances both the natural formulation with (*l, S, V*) cuts and multi-commodity formulations provide strong relaxations of LSS. Our results in Table 2 show that when capacitated instances are considered the natural formulation strengthened with (*l, S, V*) cuts outperforms both the multi-commodity formulation and the natural formulation strengthened with default CPLEX cuts only.

5. CONCLUSION

In this article, we studied a two-level lot-sizing problem with supplier selection (LSS). This *NP-hard* problem arises, either naturally or as a relaxation, in different production planning and supply chain management applications. We described various formulations of the problem and compared their strengths computationally. We proposed two new families of facet-defining inequalities for a formulation of the problem with traditional variables and described separation algorithms for these inequalities that can be executed in polynomial time. We demonstrated that these inequalities can be successfully used to reduce the solution times of branch-and-cut algorithms.

APPENDIX

Proof of Theorem 3.4

THEOREM 3.4: For $P \geq 2$, let (28) be a valid inequality for Q^I that does not define the same face of Q^I as (25f). Then (28) is facet-defining for Q^I if and only if the following conditions are satisfied:

(a) $1 \in \underset{p \in \mathcal{P}}{\cup} (S^p \cup V^p)$,
(b) $1 < T$,
(c) $V^p = L$ or $(\bar{S}^p \cup \bar{V}^p) \neq \emptyset$ for each $p \in \mathcal{P}$.

PROOF: We denote by F the face of Q^I that (28) defines. For the direct implication, assume that (28) is facet-defining for Q^I.

First, assume by contradiction that $1 \notin \bigcup_{p \in \mathcal{P}} (S^p \bigcup V^p)$, i.e., $1 \in \bigcap_{p \in \mathcal{P}} (\bar{S}^p \cup \bar{V}^p)$. Let F' be the face of Q^I that (25f) induces. We claim that $F' \supseteq F$, thereby showing that (28) is either not facet-defining or induces that same face as (25f). Assume for a contradiction that $F' \not\supseteq F$. Then there exists $(x; v; y) \in F$ such that $\sum_{p \in \mathcal{P}} y_{p,1} \geq 2$. Let $p^* \in \text{argmin}_{p \in \mathcal{P}} \sigma_p(t^p[y])$. There exists $\bar{p} \in \mathcal{P} \setminus \{p^*\}$ such that $y_{\bar{p},1} = 1$. Inequality (29) then implies that $LHS(x; v; y) \geq d_{1l} + d_{(\sigma_{\bar{p}}(1)+1)l}$. Since $\sigma_{\bar{p}}(1) < l$ by Assumption 3.3, we conclude that $(x; v; y) \notin F$, the desired contradiction.

Second, assume for a contradiction that $l = T$. Because of Assumption 3.3, and the validity assumption (a) of Theorem 3.3, we must have that $\{i, \ldots, \sigma_p(i)\} \in \bar{V}^p$ with $\sigma_p(i) < l$ for each $i \in \bar{V}^p$. Define for $p \in \mathcal{P}$, $\mathcal{I}^p = \{i \in \bar{V}^p | i - 1 \notin \bar{V}^p\}$ and $k^p = \max\{i | i \notin V^p\}$. In the preceding definition, we take $k^p = 0$ if $V^p = L$. Then (28) can be expressed as an equal weight conic combination of the valid inequalities (25e) for $i \in \bar{S}^p$ and $p \in \mathcal{P}$, (27) with $t = i$ and $j = k = \sigma_p(i)$, for $p \in \mathcal{P}$ and $i \in \mathcal{I}^p$, (25a) with $t = k^p$ for $p \in \mathcal{P}$, and (25d), after exchanging the right-hand-side and left-hand-side terms of (25a) and (25d). If $k^p \geq 1$ for some $p \in \mathcal{P}$, then the above derivation uses at least one of (25e) or (27), and (25a). Because the corresponding inequalities define distinct facets of Q^I, we conclude that (28) is not facet-defining for Q^I. If $k^p = 0$ for all $p \in \mathcal{P}$, then (27) reduces to $\sum_{p\in\mathcal{P}} \sum_{i=1}^{T} v_{p,i} \geq d_{1T}$. This inequality is always satisfied at equality as shown by (25d).

Finally, assume that $V^\pi \neq L$ and $(\bar{S}^\pi \cup \bar{V}^\pi) = \emptyset$ for some index $\pi \in \mathcal{P}$. Then using the validity condition (a) of Theorem 3.3, the contribution of plant π to the left-hand-side of (28) is $\sum_{i=1}^{k^\pi} x_{\pi,i} + \sum_{i=k^\pi+1}^{l} v_{\pi,i}$ for some $k^\pi \in \{1,\ldots,l\}$. Consider now the inequality (28) obtained by changing the partition $(S^\pi, \bar{S}^\pi, V^\pi, \bar{V}^\pi)$ of L where $\bar{S}^\pi = \bar{V}^\pi = \emptyset$ to $(\emptyset, \emptyset, L, \emptyset)$. The original inequality can be obtained as an equal weight conic combination of this new inequality and (25a) with $t = k^\pi$. Since these inequalities define distinct faces of Q^I, then (28) is not facet-defining.

We next show that conditions (a)–(c) are sufficient for (28) to be facet-defining for Q^I. To this end, we present $3PT - (P+1)$ affinely independent solutions in $(F \cap Q)$.

For $p \in \mathcal{P}$, define $m_p = \min\{t | t \in (\bar{S}^p \bigcup \bar{V}^p)\}$ where we assume $m_p = l+1$ if $(\bar{S}^p \cup \bar{V}^p) = \emptyset$. Further, let $m = \min_{p\in\mathcal{P}} m_p$. For the remainder of this proof, we select π to be an index in $\mathcal{P}$ for which $1 \in (S^\pi \cup V^\pi)$. Such index exists because of (a). In the solutions presented below, if $p = \pi$, we sum the solution vectors presented for plants p and π.

First, assume that $m = l+1$. Then, by assumption (c), $V^p = L$ for all $p \in \mathcal{P}$. Then (28) reduces to $\sum_{p\in\mathcal{P}} \sum_{i=1}^{l} v_{p,i} \geq d_{1l}$, which is shown to be facet-defining in Theorem 3.1 (ii).

Second, assume that $1 < m < l+1$. For this case we select π to be equal to p, although we express the points as a function of π, to help streamline notation when discussing the case where $m = 1$. We define $\mathbf{d}' = \mathbf{d}_{\hookrightarrow [d_{m-1} = d_{(m-1)l}]}$. By Theorem 3.1 and Corollary 3.1, there are affinely independent solutions $(\hat{x}^s; \hat{v}^s; \hat{y}^s)$ for $s = 1,\ldots,3P(m-1) - P$ in $\mathbb{Q}^I_{\mathcal{P},\{1,\ldots,m-1\}}[\mathbf{d}']$ and $(\acute{x}^r; \acute{v}^r; \acute{y}^r)$ for $r = 1,\ldots,3P(T-l) - P$ in $\mathbb{Q}^I_{\mathcal{P},\{l+1,\ldots,T\}}[\mathbf{d}]$. We construct the affinely independent vectors $(\overline{\overline{x}}^s; \overline{\overline{v}}^s; \overline{\overline{y}}^s)$ and $(\tilde{\tilde{x}}^r; \tilde{\tilde{v}}^r; \tilde{\tilde{y}}^r)$ for $s \in \{1,\ldots,3P(m-1) - P\}$ and $r \in \{2,\ldots,3P(T-l) - P\}$ defined as

(i) $(\overline{\overline{x}}^s; \overline{\overline{v}}^s; \overline{\overline{y}}^s) = (\hat{x}^s; \hat{v}^s; \hat{y}^s) + (\acute{x}^1; \acute{v}^1; \acute{y}^1)$,
(ii) $(\tilde{\tilde{x}}^r; \tilde{\tilde{v}}^r; \tilde{\tilde{y}}^r) = (\hat{x}^1; \hat{v}^1; \hat{y}^1) + (\acute{x}^r; \acute{v}^r; \acute{y}^r)$.

Note that these $3PT - 3P(l-m+1) - 2P - 1$ points belong to F as $\{1,\ldots,m-1\}$ is either completely included in S^p or completely included in V^p for each $p \in \mathcal{P}$ by condition (c), and the validity assumption (a) of Theorem 3.3.

Next, we proceed in two steps.

Step 1: We construct $3P(l-m+1)$ points that are obtained by creating three affinely independent solutions for each $i \in \{m,\ldots,l\}$ and for each $p \in \mathcal{P}$.

First, for each $i \in \bar{V}^p \cap \{m,\ldots,l\}$, we construct vectors $(\check{x}^{p,i}; \check{v}^{p,i}; \check{y}^{p,i})$, $(\mathring{x}^{p,i}; \mathring{v}^{p,i}; \mathring{y}^{p,i})$, and $(\dddot{x}^{p,i}; \dddot{v}^{p,i}; \dddot{y}^{p,i})$ with nonzero components

(iii) $(\check{x}^{p,i}_p, \check{v}^{p,i}_p, \check{y}^{p,i}_p) = (d_{il}e_i + d_{(l+1)T}e_{l+1}, d_{i\sigma_p(i)}e_i + d_{(\sigma_p(i)+1)l}e_{\sigma_p(i)+1} + d_{(l+1)T}e_{l+1}, e_i + e_{l+1})$, $(\check{x}^{p,i}_\pi, \check{v}^{p,i}_\pi, \check{y}^{p,i}_\pi) = (d_{1(i-1)}e_1, d_{1(i-1)}e_1, e_1)$,
(iv) $(\mathring{x}^{p,i}_p, \mathring{v}^{p,i}_p, \mathring{y}^{p,i}_p) = (d_{iT}e_i, d_{i\sigma_p(i)}e_i + d_{(\sigma_p(i)+1)T}e_{\sigma_p(i)+1}, e_i)$,
$(\mathring{x}^{p,i}_\pi, \mathring{v}^{p,i}_\pi, \mathring{y}^{p,i}_\pi) = (d_{1(i-1)}e_1, d_{1(i-1)}e_1, e_1)$,
(v) $(\dddot{x}^{p,i}_p, \dddot{v}^{p,i}_p, \dddot{y}^{p,i}_p) = (d_{(\sigma_p(i)+1)l}e_i + d_{(l+1)T}e_{l+1}, d_{(\sigma_p(i)+1)l}e_{\sigma_p(i)+1} + d_{(l+1)T}e_{l+1}, e_i + e_{l+1})$, $(\dddot{x}^{p,i}_\pi, \dddot{v}^{p,i}_\pi, \dddot{y}^{p,i}_\pi) = (d_{1\sigma_p(i)}e_1, d_{1\sigma_p(i)}e_1, e_1)$

Vectors (iii)–(v) belong to F by definition of $\sigma_p(i)$. They satisfy all but one of the equalities $\frac{x_{p,i}}{d_{(\sigma_p(i)+1)l}} + (1 - \frac{d_{iT}}{d_{(\sigma_p(i)+1)l}})\frac{v_{p,i}}{d_{i\sigma_p(i)}} = y_{p,i}$, $x_{p,i} = v_{p,i} + d_{(\sigma_p(i)+1)l}y_{p,i}$, and $v_{p,i} = d_{i\sigma_p(i)}y_{p,i}$ that are satisfied by all solutions in families (i)–(ii) as well as other points of the form (iii)–(v). Because $l < T$ by condition (b), each equality is violated by exactly one of these vectors, showing that they are affinely independent from each other and from all previous solutions.

Second, for each $i \in S^p \cap \{m,\ldots,l\}$, we construct vectors $(\dot{x}^{p,i}; \dot{v}^{p,i}; \dot{y}^{p,i})$, $(\hat{x}^{p,i}; \hat{v}^{p,i}; \hat{y}^{p,i})$, and $(\tilde{x}^{p,i}; \tilde{v}^{p,i}; \tilde{y}^{p,i})$ with nonzero components

(vi) $(\dot{x}^{p,i}_p, \dot{v}^{p,i}_p, \dot{y}^{p,i}_p) = (d_{(l+1)T}e_{l+1}, d_{(l+1)T}e_{l+1}, e_i + e_{l+1})$,
$(\dot{x}^{p,i}_\pi, \dot{v}^{p,i}_\pi, \dot{y}^{p,i}_\pi) = (d_{1l}e_1, d_{1l}e_1, e_1)$,
(vii) $(\hat{x}^{p,i}_p, \hat{v}^{p,i}_p, \hat{y}^{p,i}_p) = (d_{1l}e_1 + d_{(l+1)T}e_{l+1}, d_{1(i-1)}e_1 + d_{il}e_i + d_{(l+1)T}e_{l+1}, e_1 + e_i + e_{l+1})$,
(viii) $(\tilde{x}^{p,i}_p, \tilde{v}^{p,i}_p, \tilde{y}^{p,i}_p) = (d_{il}e_i + d_{(l+1)T}e_{l+1}, d_{il}e_i + d_{(l+1)T}e_{l+1}, e_i + e_{l+1})$, $(\tilde{x}^{p,i}_\pi, \tilde{v}^{p,i}_\pi, \tilde{y}^{p,i}_\pi) = (d_{1(i-1)}e_1, d_{1(i-1)}e_1, e_1)$

Vectors (vi)–(viii) belong to F since $i \geq m > 1$ and $i \in S^p$ imply that $1 \in S^p$ because of assumption (a) of Theorem 3.3 and the definition of m. If $(i-1) \in \bar{V}^p$, we define $r = \min\{k \in \{m,\ldots,l\} | \{k,\ldots,i-1\} \in \bar{V}^p\}$. Otherwise, we let $r = i$. Vectors (vi)–(viii) satisfy all but one of the equalities $v_{p,i} = d_{il}y_{p,i} + \sum_{j=r}^{i-1}(x_{p,j} - v_{p,j})$, $\sum_{j=r}^{i} x_{p,j} = \sum_{j=r}^{i} v_{p,j}$, and $x_{p,i} = 0$ that are satisfied by all previous solutions, as well as other points of the form (vi)–(viii). Moreover, each equality is violated by exactly one of these vectors, showing that they are affinely independent from each other and from all vectors presented earlier.

Third, for each $i \in \bar{S}^p \cap \{m,\ldots,l\}$, we construct vectors $(\check{x}^{p,i}; \check{v}^{p,i}; \check{y}^{p,i})$, $(\ddot{x}^{p,i}; \ddot{v}^{p,i}; \ddot{y}^{p,i})$ and $(\tilde{x}^{p,i}; \tilde{v}^{p,i}; \tilde{y}^{p,i})$ with nonzero components

(ix) $(\check{x}^{p,i}_p, \check{v}^{p,i}_p, \check{y}^{p,i}_p) = (d_{iT}e_i, d_{iT}e_i, e_i)$,
$(\check{x}^{p,i}_\pi, \check{v}^{p,i}_\pi, \check{y}^{p,i}_\pi) = (d_{1(i-1)}e_1, d_{1(i-1)}e_1, e_1)$,
(x) $(\ddot{x}^{p,i}_p, \ddot{v}^{p,i}_p, \ddot{y}^{p,i}_p) = (d_{iT}e_i, d_{il}e_i + d_{(l+1)T}e_{l+1}, e_i)$,
$(\ddot{x}^{p,i}_\pi, \ddot{v}^{p,i}_\pi, \ddot{y}^{p,i}_\pi) = (d_{1(i-1)}e_1, d_{1(i-1)}e_1, e_1)$,
(xi) $(\tilde{x}^{p,i}_p, \tilde{v}^{p,i}_p, \tilde{y}^{p,i}_p) = (d_{il}e_i + d_{(l+1)T}e_{l+1}, d_{il}e_i + d_{(l+1)T}e_{l+1}, e_i + e_{l+1})$, $(\tilde{x}^{p,i}_\pi, \tilde{v}^{p,i}_\pi, \tilde{y}^{p,i}_\pi) = (d_{1(i-1)}e_1, d_{1(i-1)}e_1, e_1)$

For r defined as in the case where $i \in S^p$, each of these vectors satisfy all but one of the equalities $v_{p,i} = d_{il}y_{p,i} + \sum_{j=r}^{i-1}(x_{p,j} - v_{p,j})$, $\sum_{j=r}^{i} x_{p,j} = \sum_{j=r}^{i} v_{p,j}$ and $x_{p,i} = d_{iT}y_{p,i}$ that are satisfied by all previous solutions, as well as other points of the form (ix)–(xi), because $l < T$ by condition (b). This shows that these vectors are affinely independent from each other and from all solutions presented earlier.

Fourth, for each $i \in V^p \cap \{m,\ldots,l\}$, we construct vectors $(\acute{x}^{p,i}, \acute{v}^{p,i}, \acute{y}^{p,i})$, $(\ddot{x}^{p,i}; \ddot{v}^{p,i}; \ddot{y}^{p,i})$, and $(\tilde{x}^{p,i}; \tilde{v}^{p,i}; \tilde{y}^{p,i})$ with nonzero components

(xii) $(\acute{x}^{p,i}_p; \acute{v}^{p,i}_p; \acute{y}^{p,i}_p) = (d_{(l+1)T}e_{l+1}, d_{(l+1)T}e_{l+1}, e_i + e_{l+1})$,
$(\acute{x}^{p,i}_\pi; \acute{v}^{p,i}_\pi; \acute{y}^{p,i}_\pi) = (d_{1l}e_1, d_{1l}e_1, e_1)$,
(xiii) $(\ddot{x}^{p,i}_p, \ddot{v}^{p,i}_p, \ddot{y}^{p,i}_p) = (d_{iT}e_i, d_{il}e_i + d_{(l+1)T}e_{l+1}, e_i)$,
$(\ddot{x}^{p,i}_\pi, \ddot{v}^{p,i}_\pi, \ddot{y}^{p,i}_\pi) = (d_{1(i-1)}e_1, d_{1(i-1)}e_1, e_1)$,
(xiv) $(\tilde{x}^{p,i}_p, \tilde{v}^{p,i}_p, \tilde{y}^{p,i}_p) = (d_{il}e_i + d_{(l+1)T}e_{l+1}, d_{il}e_i + d_{(l+1)T}e_{l+1}, e_i + e_{l+1})$, $(\tilde{x}^{p,i}_\pi, \tilde{v}^{p,i}_\pi, \tilde{y}^{p,i}_\pi) = (d_{1(i-1)}e_1, d_{1(i-1)}e_1, e_1)$

These vectors satisfy all but one of the equalities $v_{p,i} = d_{il}y_{p,i}$, $x_{p,i} = v_{p,i}$ and $d_{il}x_{p,i} = d_{iT}v_{p,i}$ that are satisfied by all previous solutions, as well as other points of the form (xii)–(xiv). Since $l < T$ by condition (b), each equality is violated by exactly one of these vectors, showing that they are affinely independent from each other and from all solutions presented earlier.

Step 2: We obtain the remaining P points by constructing one additional point for each plant $p \in \mathcal{P}$. We observe that all previous solutions satisfy, for each $p \in \mathcal{P}$, the equality $\sum_{i\in\bar{S}^p\cap\{m,\ldots,l\}} \sum_{j=r_i}^{i}(x_{p,j} - v_{p,j}) + \sum_{i\in V^p\cap\{m,\ldots,l\}}(x_{p,i} - v_{p,i}) + \sum_{i=l+1}^{T}(x_{p,i} - v_{p,i}) = 0$ where

$r_i = \min\{k \in \{m, \dots, l\} | \{k, \dots, i-1\} \in \bar{V}^p\}$ if $(i-1) \in \bar{V}^p$ and $i \notin \bar{V}^p$ and $r_i = i$, otherwise. We next present a family of affinely independent vectors that satisfy all but one of these equalities.

Case 1: Assume that $1 \in S^p$. We consider two subcases.

Case 1.1: If $\bar{S}^p = \emptyset$ then we let j^p be any index in $\bar{V}^p$. Such index exists because of condition (c). We construct $(\bar{x}^p; \bar{v}^p; \bar{y}^p)$ with nonzero components

(xv) $(\bar{x}^p_p, \bar{v}^p_p, \bar{y}^p_p) = (d_{1(j^p-1)}e_1 + d_{j^pT}e_{j^p}, d_{1(j^p-1)}e_1 + d_{j^p\sigma_p(j^p)}e_{j^p} + d_{(\sigma_p(j^p)+1)l}e_{\sigma_p(j^p)+1} + d_{(l+1)T}e_{l+1}, e_1 + e_{j^p})$

Case 1.2: If $\bar{S}^p \neq \emptyset$, we let j^p be any index in $\bar{S}^p$ and define $(\bar{x}^p; \bar{v}^p; \bar{y}^p)$ as follows

(xv′) $(\bar{x}^p_p, \bar{v}^p_p, \bar{y}^p_p) = (d_{1l}e_1 + d_{(l+1)T}e_{l+1}, d_{1(j^p-1)}e_1 + d_{j^pl}e_{j^p} + d_{(l+1)T}e_{l+1}, e_1 + e_{l+1})$.

Case 2: Assume that $1 \in V^p$, we construct $(\grave{x}^p, \grave{v}^p, \grave{y}^p)$ with nonzero components

(xv") $(\grave{x}^p_p, \grave{v}^p_p, \grave{y}^p_p) = (d_{1T}e_1, d_{1l}e_1 + d_{(l+1)T}e_{l+1}, e_1)$.

This concludes the proof for the case where $m > 1$.

Finally, assume that $m = 1$. By Theorem 3.1, there are affinely independent solutions $(\acute{x}^r; \acute{v}^r; \acute{y}^r)$ for $r = 1, \dots, 3P(T-l) - P$ in $\mathbb{Q}^I_{\mathcal{P},\{l+1,\dots,T\}}[\mathbf{d}]$. Then, we construct the vectors

(i) $(\bar{\bar{x}}^r, \bar{\bar{v}}^r, \bar{\bar{y}}^r) = (\tilde{x}; \tilde{v}; \tilde{y}) + (\acute{x}^r; \acute{v}^r; \acute{y}^r)$

where $(\tilde{x}; \tilde{v}; \tilde{y})$ has nonzero components $(\tilde{x}_\pi, \tilde{v}_\pi, \tilde{y}_\pi) = (d_{1l}e_1, d_{1l}e_1, e_1)$ where $\pi \in \mathcal{P}$ is an index for which $1 \in (S^\pi \cup V^\pi)$. By using the same construction as in the case where $m > 1$, three affinely independent solutions can be obtained for all periods $i \in \{2, \dots, l\}$ and for all plants $p \in \mathcal{P}$, yielding $3P(l-1)$ solutions in addition to those given in (i). In constructing these solutions, if $1 \in \bar{V}^p$, we modify point (vii) by replacing its shipment component by $\hat{v}_p = d_{1\sigma_p(1)}e_1 + d_{(\sigma_p(1)+1)(i-1)}e_{\sigma_p(1)+1} + d_{il}e_i + d_{(l+1)T}e_{l+1}$ if $i > \sigma_p(1)+1$ (no change is necessary when $i = \sigma_p(1)+1$). These solutions belong to F by definition of $\sigma_p(1)$. Moreover, for $i \in (S^p \cup \bar{S}^p)$, we redefine $r = \min\{k \in \{2, \dots, l\} | \{k, \dots, i-1\} \in \bar{V}^p\}$ if $(i-1) \in \bar{V}^p$ and $r = i$ otherwise. After these modifications, it is easy to verify that the points described above are still independent, as equalities used to argue independence do not involve the first period, and therefore when $\pi = p$ they continue to hold.

Below, we present three additional solutions for each plant $p \in \mathcal{P} \setminus \{\pi\}$ (giving a total of $3(P-1)$ points). We distinguish four cases depending on whether $1 \in S^p, \bar{S}^p, V^p$, or $\bar{V}^p$. Note that because these solutions pertain to different plants, and the proposed equalities are homogenous and involve a single plant, it is sufficient to verify independence from points (i)–(xiv) of the previous case.

First, if $1 \in S^p$, we construct vectors $(\tilde{x}^p; \tilde{v}^p; \tilde{y}^p)$, and $(\dot{x}^p; \dot{v}^p; \dot{y}^p)$ with nonzero components

(ii) $(\tilde{x}^p_p, \tilde{v}^p_p, \tilde{y}^p_p) = (d_{1l}e_1 + d_{(l+1)T}e_{l+1}, d_{1l}e_1 + d_{(l+1)T}e_{l+1}, e_1 + e_{l+1})$,

(iii) $(\dot{x}^p_p, \dot{v}^p_p, \dot{y}^p_p) = (d_{(l+1)T}e_{l+1}, d_{(l+1)T}e_{l+1}, e_1 + e_{l+1})$,
$(\dot{x}^p_\pi, \dot{v}^p_\pi, \dot{y}^p_\pi) = (d_{1l}e_1, d_{1l}e_1, e_1)$

We observe that (ii) violates the equality $(x_{p,1} - d_{1T}y_{p,1}) + \sum_{i \in S^p \cap \{2,\dots,l\}} \frac{d_{(l+1)T}}{d_{il}} \sum_{j=r_i}^{i} (v_{p,j} - x_{p,j}) = 0$ that is satisfied by all previous solutions. Moreover, (iii) violates the equality $x_{p,1} = d_{1l}y_{p,1}$ that is satisfied by all previous solutions and (ii). We next consider two cases to construct a third point for plant p.

Case 1: If $\bar{S}^p = \emptyset$ then we let $j^p = \min\{i | i \in \bar{V}^p\}$, which exists because of condition (c). We construct $(\bar{x}^p; \bar{v}^p; \bar{y}^p)$ with nonzero components

(iv) $(\bar{x}^p_p, \bar{v}^p_p, \bar{y}^p_p) = (d_{1(j^p-1)}e_1 + d_{j^pT}e_{j^p}, d_{1(j^p-1)}e_1 + d_{j^p\sigma_p(j^p)}e_{j^p} + d_{(\sigma_p(j^p)+1)l}e_{\sigma_p(j^p)+1} + d_{(l+1)T}e_{l+1}, e_1 + e_{j^p})$

Case 2: If $\bar{S}^p \neq \emptyset$, we let j^p be an index in $\bar{S}^p$ and modify $(\bar{x}^p; \bar{v}^p; \bar{y}^p)$ as follows

(iv′) $(\bar{x}^p_p, \bar{v}^p_p, \bar{y}^p_p) = (d_{1l}e_1 + d_{(l+1)T}e_{l+1}, d_{1(j^p-1)}e_1 + d_{j^pl}e_{j^p} + d_{(l+1)T}e_{l+1}, e_1 + e_{l+1})$.

We observe that all solutions presented previously satisfy the equality $\sum_{i \in \bar{S}^p \cap \{2,\dots,l\}} \sum_{j=r_i}^{i} (x_{p,j} - v_{p,j}) + \sum_{i \in V^p \cap \{2,\dots,l\}} (x_{p,i} - v_{p,i}) + \sum_{i=l+1}^{T} (x_{p,i} - v_{p,i}) = 0$ which is violated by (iv) and (iv′). Therefore, all points presented for plant p are affinely independent from each other and from previous solutions.

Second, if $1 \in \bar{S}^p$, we construct the vectors $(\tilde{x}^p; \tilde{v}^p; \tilde{y}^p)$, $(\check{x}^p; \check{v}^p; \check{y}^p)$, and $(\check{x}^p; \check{v}^p; \check{y}^p)$ with nonzero components

(v) $(\tilde{x}^p_p, \tilde{v}^p_p, \tilde{y}^p_p) = (d_{1l}e_1 + d_{(l+1)T}e_{l+1}, d_{1l}e_1 + d_{(l+1)T}e_{l+1}, e_1 + e_{l+1})$,

(vi) $(\check{x}^p_p, \check{v}^p_p, \check{y}^p_p) = (d_{1T}e_1, d_{1T}e_1, e_1)$,

(vii) $(\check{x}^p_p, \check{v}^p_p, \check{y}^p_p) = (d_{1T}e_1, d_{1l}e_1 + d_{(l+1)T}e_{l+1}, e_1)$.

Vector (v) violates the equality $(x_{p,1} - d_{1T}y_{p,1}) + \sum_{i \in S^p \cap \{2,\dots,l\}} \frac{d_{(l+1)T}}{d_{il}} \sum_{j=r_i}^{i} (v_{p,j} - x_{p,j}) = 0$ and vector (vi) violates the equality $x_{p,1} = d_{1l}y_{p,1}$ that are satisfied by all previous solutions. Finally, vector (vii) violates the equality $\sum_{i \in \bar{S}^p \cap \{2,\dots,l\}} \sum_{j=r_i}^{i} (x_{p,j} - v_{p,j}) + \sum_{i \in V^p \cap \{2,\dots,l\}} (x_{p,i} - v_{p,i}) + \sum_{i=l+1}^{T} (x_{p,i} - v_{p,i}) = 0$ that is satisfied by all previous solutions as well as (v) and (vi). Therefore, all points presented for plant p are affinely independent from each other and from previous solutions.

Third, if $1 \in V^p$ then $\{1, \dots, l\} \in V^p$ by condition (c). We construct the vectors $(\dot{x}^p; \dot{v}^p; \dot{y}^p)$, $(\tilde{x}^p; \tilde{v}^p; \tilde{y}^p)$, and $(\check{x}^p; \check{v}^p; \check{y}^p)$ with nonzero components

(viii) $(\dot{x}^p_p; \dot{v}^p_p; \dot{y}^p_p) = (d_{(l+1)T}e_{l+1}, d_{(l+1)T}e_{l+1}, e_1 + e_{l+1})$,
$(\dot{x}^p_\pi; \dot{v}^p_\pi; \dot{y}^p_\pi) = (d_{1l}e_1, d_{1l}e_1, e_1)$,

(ix) $(\tilde{x}^p_p, \tilde{v}^p_p, \tilde{y}^p_p) = (d_{1l}e_1 + d_{(l+1)T}e_{l+1}, d_{1l}e_1 + d_{(l+1)T}e_{l+1}, e_1 + e_{l+1})$,

(x) $(\check{x}^p_p, \check{v}^p_p, \check{y}^p_p) = (d_{1T}e_1, d_{1l}e_1 + d_{(l+1)T}e_{l+1}, e_1)$.

Vector (iii) violates the equality $v_{p,1} = d_{1l}y_{p,1}$ whereas (ix) violates the equality $x_{p,1} = 0$ that are satisfied by all previous solutions. Moreover, vector (x) violates the equality $x_{p,1} = v_{p,1}$ that is satisfied by all previous solutions, showing that these solutions are affinely independent from each other and from all solutions presented earlier.

Fourth, if $1 \in \bar{V}^p$, we construct the vectors $(\bar{x}^p, \bar{v}^p, \bar{y}^p)$, and $(\mathring{x}^p; \mathring{v}^p; \mathring{y}^p)$ with nonzero components

(xi) $(\bar{x}^p_p, \bar{v}^p_p, \bar{y}^p_p) = (d_{1T}e_1, d_{1\sigma_p(1)}e_1 + d_{(\sigma_p(1)+1)l}e_{\sigma_p(1)+1} + d_{(l+1)T}e_{l+1}, e_1)$,

(xii) $(\mathring{x}^p_p, \mathring{v}^p_p, \mathring{y}^p_p) = (d_{(\sigma_p(1)+1)l}e_1 + d_{(l+1)T}e_{l+1}, d_{(\sigma_p(1)+1)l}e_{\sigma_p(1)+1} + d_{(l+1)T}e_{l+1}, e_1 + e_{l+1})$, $(\mathring{x}^p_\pi, \mathring{v}^p_\pi, \mathring{y}^p_\pi) = (d_{1\sigma_p(1)}e_1, d_{1\sigma_p(1)}e_1, e_1)$

These points belong to F by definition of $\sigma_p(1)$. Vector (xi) violates the equality $x_{p,1} = v_{p,1} + d_{(\sigma_p(1)+1)l}y_{p,1}$, and vector (xii) violates $v_{p,1} = d_{1\sigma_p(1)}y_{p,1}$ that are satisfied by all previous solutions. Finally, we construct one additional vector based on two cases.

Case 1: If $S^p = \emptyset$, we construct the vector $(\dddot{x}^p, \dddot{v}^p, \dddot{y}^p)$ with nonzero components

(xiii) $(\dddot{x}^p_p, \dddot{v}^p_p, \dddot{y}^p_p) = (d_{1l}e_1 + d_{(l+1)T}e_{l+1}, d_{1\sigma_p(1)}e_1 + d_{(\sigma_p(1)+1)l} e_{\sigma_p(1)+1} + d_{(l+1)T}e_{l+1}, e_1 + e_{l+1})$.

This vector violates the equality $\frac{x_{p,1}}{d_{(\sigma_p(1)+1)l}} + (1 - \frac{d_{1T}}{d_{(\sigma_p(1)+1)l}})\frac{v_{p,1}}{d_{1\sigma_p(1)}} = y_{p,1}$, that is satisfied by all previous solutions.

Case 2: If $S^p \neq \emptyset$, we consider two subcases.

Case 2.1: If $(\sigma_p(1) + 1) \in S^p$, we construct the vector $(\ddot{x}^p, \ddot{v}^p, \ddot{y}^p)$ with nonzero components

(xiii′) $(\ddot{x}^p_p, \ddot{v}^p_p, \ddot{y}^p_p) = (d_{1T}e_1, d_{1\sigma_p(1)}e_1 + d_{(\sigma_p(1)+1)T}e_{\sigma_p(1)+1}, e_1)$.

We observe that this vector violates the equality $\sum_{i\in\bar{S}^p\cap\{2,\ldots,l\}}\sum_{j=r_i}^{i}(x_{p,j} - v_{p,j}) + \sum_{i=l+1}^{T}(x_{p,i} - v_{p,i}) + \sum_{i\in V^p\cap\{2,\ldots,l\}}(x_{p,i} - v_{p,i}) + (x_{p,1} - v_{p,1} - d_{(\sigma_p(1)+1)l}\,y_{p,1}) = 0$, that is satisfied by all previous solutions.

Case 2.2: If $(\sigma_p(1) + 1) \in \bar{S}^p$, we let $j^p = \min\{i | i \in S^p\} > 1$, we construct the vector $(\ddot{x}^p, \ddot{v}^p, \ddot{y}^p)$ with nonzero components

(xiii″) $(\ddot{x}^p_p, \ddot{v}^p_p, \ddot{y}^p_p) = (d_{1T}e_1, d_{1\sigma_p(1)}e_1 + d_{(\sigma_p(1)+1)(j^p-1)}e_{\sigma_p(1)+1} + d_{j^pT}e_{j^p}, e_1)$.

We observe that the equality $x_{p,1} - (1 + \frac{d_{(l+1)T}}{d_{1\sigma_p(1)}})v_{p,1} - d_{(\sigma_p(1)+1)l}\,y_{p,1} + \sum_{i\in S^p}\frac{d_{(l+1)T}}{d_{il}}\sum_{j=r_i}^{i}(v_{p,i} - x_{p,i}) = 0$ that is satisfied by all previous solutions is violated by this solution. Therefore, all points presented for plant p are affinely independent from each other and from previous solutions.

Finally, we present two more solutions for plant π. We let ρ be an index in $\mathcal{P} \setminus \{\pi\}$. Then, we construct the point $(\grave{x}; \grave{v}; \grave{y})$ with nonzero components

(xiv) $(\grave{x}_\pi, \grave{v}_\pi, \grave{y}_\pi) = (d_{(l+1)T}e_{l+1}, d_{(l+1)T}e_{l+1}, e_1 + e_{l+1})$,
$(\grave{x}_\rho, \grave{v}_\rho, \grave{y}_\rho) = (d_{1l}e_1, d_{1l}e_1, e_1)$

When $1 \in \bar{V}^\rho$, we modify $(\grave{x}; \grave{v}; \grave{y})$ by setting $\grave{v}_\rho = d_{1\sigma_\rho(1)}e_1 + d_{(\sigma_\rho(1)+1)l}e_{\sigma_\rho(1)+1}$. We let $\mathcal{P}_1 = \{p \in \mathcal{P} | 1 \in \bar{V}^p\}$, $\mathcal{P}_2 = \{p \in \mathcal{P} \setminus \{\pi\} | 1 \in S^p \text{ and } \bar{S}^p \neq \emptyset\}$, $\mathcal{P}'_2 = \{p \in \mathcal{P} \setminus \{\pi\} | 1 \in S^p \text{ and } \bar{S}^p = \emptyset\}$, $\mathcal{P}_3 = \{p \in \mathcal{P} | 1 \in \bar{S}^p\}$, and $\mathcal{P}_4 = \{p \in \mathcal{P}\setminus\{\pi\} | 1 \in V^p\}$. Vector (xiv) violates the equality $\sum_{p\in\mathcal{P}_1}\frac{v_{p,1}}{d_{1\sigma_p(1)}} + \sum_{p\in\mathcal{P}_2}\frac{x_{p,1}}{d_{1l}} + \sum_{p\in\mathcal{P}'_2}(\frac{x_{p,1}}{d_{1l}} + \frac{d_{j^pl}}{d_{(l+1)T}d_{1l}}(\sum_{i\in V^p}(v_{p,i} - x_{p,i}) + \sum_{i=l+1}^{T}(v_{p,i} - x_{p,i}))) + \sum_{p\in\mathcal{P}_3} y_{p,1} + \sum_{p\in\mathcal{P}_4}\frac{v_{p,1}}{d_{1l}} + y_{\pi,1} = 1$, where $j^p = \min\{i | i \in \bar{V}^p\}$ for $p \in \mathcal{P}'_2$. This equality is satisfied by all solutions presented previously.

To construct the final additional point, we consider three cases as $1 \in (S^\pi \cup V^\pi)$.

Case 1: If $1 \in S^\pi$ and $\bar{S}^\pi \neq \emptyset$, we define $j^\pi = \min\{q | q \in \bar{S}^\pi\}$. Then we construct the point $(\hat{x}; \hat{v}; \hat{y})$ with nonzero components

(xv) $(\hat{x}_\pi, \hat{v}_\pi, \hat{y}_\pi) = (d_{1l}e_1 + d_{(l+1)T}e_{l+1}, d_{1(j^\pi-1)}e_1 + d_{j^\pi l}e_{j^\pi} + d_{(l+1)T}e_{l+1}, e_1 + e_{l+1})$.

Case 2: If $1 \in S^\pi$ and $\bar{S}^\pi = \emptyset$ then we define $j^\pi = \min\{q | q \in \bar{V}^\pi\}$, we construct $(\hat{x}; \hat{v}; \hat{y})$ as follows

(xv′) $(\hat{x}_\pi, \hat{v}_\pi, \hat{y}_\pi) = (d_{1(j^\pi-1)}e_1 + d_{j^\pi T}e_{j^\pi}, d_{1(j^\pi-1)}e_1 + d_{j^\pi\sigma_\pi(j^\pi)}e_{j^\pi} + d_{(\sigma_\pi(j^\pi)+1)l}e_{\sigma_\pi(j^\pi)+1} + d_{(l+1)T}e_{l+1}, e_1 + e_{j^\pi})$.

Index $j^\pi \in (\bar{S}^p \cup \bar{V}^p)$ exists because of (c).

Case 3: If $1 \in V^\pi$ then we construct the point $(\ddddot{x}; \ddddot{v}; \ddddot{y})$ with nonzero components

(xv″) $(\ddddot{x}_\pi, \ddddot{v}_\pi, \ddddot{y}_\pi) = (d_{1T}e_1, d_{1l}e_1 + d_{(l+1)T}e_{l+1}, e_1)$.

Vectors (xv),(xv ′), and (xv ″) violate the equality $\sum_{i\in V^\pi\cap\{2,\ldots,l\}}(x_{\pi,i} - v_{\pi,i}) + \sum_{i=l+1}^{T}(x_{\pi,i} - v_{\pi,i}) + \sum_{i\in\bar{S}^\pi\cap\{2,\ldots,l\}}\sum_{j=r_i}^{i}(x_{\pi,j} - v_{\pi,j}) = 0$ that is satisfied by all solutions presented previously. This concludes the proof. □

Linear Description of $Q^I_{\mathcal{P},\mathcal{T}}$ When $T = 1$

In this section, we provide a minimal description of the convex hull of LSS when $T = 1$. When $P = 1$, the set reduces to the single point $(x_{1,1}, v_{1,1}, y_{1,1}) = (d_1, d_1, 1)$. For this reason, we next consider sets where $P \geq 2$.

THEOREM 3.5: The convex hull of $Q_{P,1}[\mathbf{d}]$, for $\mathbf{d} \in \mathbb{R}_{++}$ is given by

$$Q^* = \left\{(x; v; y) \in \mathbb{R}^P_+ \times \mathbb{R}^{2P} \;\middle|\; \begin{array}{lll} x_{p,1} = v_{p,1} & \forall p \in \mathcal{P} & (35a) \\ \sum_{p\in\mathcal{P}} x_{p,1} = d_1 & & (35b) \\ x_{p,1} \leq d_1 y_{p,1} & \forall p \in \mathcal{P} & (35c) \\ y_{p,1} \leq 1 & \forall p \in \mathcal{P} & (35d) \end{array}\right\}.$$

PROOF: It is clear that $Q_{P,1}[\mathbf{d}] \subseteq Q^*$. It follows that $Q^I_{P,1}[\mathbf{d}] \subseteq Q^*$ since Q^* is a polyhedron. To prove the reverse inclusion, we verify that the extreme points of Q^* belong to $Q_{P,1}[\mathbf{d}]$. Consider the linear programs

$$(\mathcal{LP}) \qquad \max\left\{\sum_{p\in\mathcal{P}}(\alpha_{p,1}x_{p,1} + \gamma_{p,1}v_{p,1} + \beta_{p,1}y_{p,1}) \;\middle|\; (x; v; y) \in Q^*\right\}.$$

Because of (35a), we may assume that $\gamma_{p,1} = 0$ for all $p \in \mathcal{P}$. We show next that $(\mathcal{LP})$ has an optimal solution that belongs to $Q_{P,1}[\mathbf{d}]$ for all choices of $\alpha_{p,1}$ and $\beta_{p,1}$. To this end, we present both a primal feasible solution $(x^*; x^*; y^*)$ and a dual feasible solution $(s^*; t^*; u^*; w^*)$ with identical objective values. In the expression $(s^*; t^*; u^*; w^*)$, s^*, t^*, u^*, and w^* are dual variables to (35a)–(35d), respectively, and therefore, $s^* \in \mathbb{R}^P$, $t^* \in \mathbb{R}$, while u^* and $w^* \in \mathbb{R}^P_+$. To describe these points, we define $\mathcal{P}^+ = \{p \in \mathcal{P} | \beta_{p,1} \geq 0\}$. We consider two cases.

Case 1: Assume first that $\mathcal{P}^+ \neq \emptyset$. In this case, select $\pi \in \operatorname{argmax}\{\alpha_{p,1} | p \in \mathcal{P}^+\}$, and define $\mathcal{P}^> = \{p \in \mathcal{P} \setminus \mathcal{P}^+ | \alpha_{p,1}d_1 + \beta_{p,1} > \alpha_\pi d_1\}$. There are two subcases.

Case 1.1: Assume first that $\mathcal{P}^> = \emptyset$. In this case, we let $x^*_{\pi,1} = d_1$, $x^*_{p,1} = 0$ for $p \in \mathcal{P} \setminus \{\pi\}$, $y^*_{p,1} = 1$ for $p \in \mathcal{P}^+$ and $y^*_{p,1} = 0$ for $\mathcal{P} \setminus \mathcal{P}^+$. This solution of $Q_{P,1}[\mathbf{d}]$ has objective value $z^1 = \alpha_{\pi,1}d_1 + \sum_{p\in\mathcal{P}^+}\beta_{p,1}$. For the dual solution, we let $s^*_p = 0$ for $p \in \mathcal{P}$, $t^* = \alpha_\pi$, $u^*_p = 0$ for $p \in \mathcal{P}^+$, $u^*_p = -\frac{\beta_{p,1}}{d_1}$ for $p \in \mathcal{P} \setminus \mathcal{P}^+$, $w^*_p = \beta_{p,1}$ for $p \in \mathcal{P}^+$ and $w^*_p = 0$ for $p \in \mathcal{P} \setminus \mathcal{P}^+$. This solution can be verified to be dual feasible with objective value z^1.

Case 1.2: Assume second that $\mathcal{P}^> \neq \emptyset$. Let $\rho \in \operatorname{argmax}\{\alpha_{p,1}d_1 + \beta_{p,1} | p \in \mathcal{P} \setminus \mathcal{P}^+\}$. In this case, we let $x^*_{\rho,1} = d_1$, $x^*_{p,1} = 0$ for $p \in \mathcal{P} \setminus \{\rho\}$, $y^*_{p,1} = 1$ for $p \in \mathcal{P}^+ \cup \{\rho\}$ and $y^*_{p,1} = 0$ for $p \in \mathcal{P} \setminus (\mathcal{P}^+ \cup \{\rho\})$. This solution of $Q_{P,1}[\mathbf{d}]$ has objective value $z^2 = \alpha_{\rho,1}d_1 + \sum_{p\in\mathcal{P}^+\cup\{\rho\}}\beta_{p,1}$. For the dual solution, we let $s^*_p = 0$ for $p \in \mathcal{P}$, $t^* = \alpha_{\rho,1} + \frac{\beta_{\rho,1}}{d_1}$, $u^*_p = 0$ for $p \in \mathcal{P}^+$, $u^*_p = -\frac{\beta_{p,1}}{d_1}$ for $p \in \mathcal{P} \setminus \mathcal{P}^+$, $w^*_p = \beta_{p,1}$ for $p \in \mathcal{P}^+$, and $w^*_p = 0$ for $p \in \mathcal{P} \setminus \mathcal{P}^+$. This solution can be verified to be dual feasible with objective value z^2.

Case 2: Assume finally that $\mathcal{P}^+ = \emptyset$. In this case, select $\pi \in \operatorname{argmax}\{\alpha_{p,1}d_1 + \beta_{p,1} | p \in \mathcal{P}\}$. We let $x^*_{\pi,1} = d_1$, $x^*_{p,1} = 0$ for $p \in \mathcal{P} \setminus \{\pi\}$, $y^*_{\pi,1} = 1$, and $y^*_{p,1} = 0$ for $p \in \mathcal{P} \setminus \{\pi\}$. This solution of $Q_{P,1}[\mathbf{d}]$ has objective value $z^3 = \alpha_{\pi,1}d_1 + \beta_{\pi,1}$. For the dual solution, we let $s^*_p = 0$ for $p \in \mathcal{P}$, $t^* = \alpha_{\pi,1} + \frac{\beta_{\pi,1}}{d_1}$, $u^*_p = -\frac{\beta_{p,1}}{d_1}$ for $p \in \mathcal{P}$, $w^*_p = 0$ for $p \in \mathcal{P}$. This solution can be verified to be dual feasible with objective value z^3. □

We next describe which inequalities, among those given in Theorem 3.5 are facet-defining. Clearly the dimension of Q^* is $2P-1$ by Theorem 3.1. To prove that (35c) is facet-defining for Q^*, we construct $2P-1$ affinely independent vectors in the corresponding face of Q^*. These vectors are $(\bar{x}^\pi; \bar{v}^\pi; \bar{y}^\pi) = (d_1 e_\pi; d_1 e_\pi; e_\pi)$ for $\pi \in \mathcal{P}$, and $(\tilde{x}^\pi; \tilde{v}^\pi; \tilde{y}^\pi) = (d_1 e_p; d_1 e_p; e_p + e_\pi)$ for $\pi \in \mathcal{P} \setminus \{p\}$. Next, we argue that (35d) is facet-defining for Q^*. We construct the vectors $(\hat{x}^\pi; \hat{v}^\pi; \hat{y}^\pi) = (d_1 e_\pi; d_1 e_\pi; e_\pi + e_p)$, and $(\dot{x}^\pi; \dot{v}^\pi; \dot{y}^\pi) = (d_1 e_p; d_1 e_p; e_p + e_\pi)$ for each $\pi \in \mathcal{P} \setminus \{p\}$ together with $(\check{x}; \check{v}; \check{y}) = (d_1 e_p; d_1 e_p; e_p)$.

Finally, an argument similar to that of Theorem 3.1 (vi) shows that the non-negativity constraint $x_{p,1} \geq 0$ is not facet-defining when $P \leq 2$. For $P \geq 3$, we first construct the points $(\ddot{x}^\pi; \ddot{v}^\pi; \ddot{y}^\pi) = (d_1 e_\pi; d_1 e_\pi; e_\pi)$ for each $\pi \in \mathcal{P} \setminus \{p\}$. We then let $\rho \neq p$ be an index of $\mathcal{P}$. We construct the points $(\acute{x}^\pi; \acute{v}^\pi; \acute{y}^\pi) = (d_1 e_\rho; d_1 e_\rho; e_\rho + e_\pi)$ for each $\pi \in \mathcal{P} \setminus \{\rho\}$, and the point $(\mathring{x}; \mathring{v}; \mathring{y}) = (d_1 e_\sigma; d_1 e_\sigma; e_\sigma + e_\rho + e_p)$ for an index σ in $\mathcal{P} \setminus \{\rho, p\}$.

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