TD - PL/PLNE

CR03-Robust combinatorial optimization
ENS - Lyon
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Exercice 1. Alternative systems

Lemma 1 (Farkas' Lemma) *Let $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. The linear system :*

$$\begin{aligned} Ax &= b \\ x &\geq 0 \end{aligned}$$

has a solution if and only if $y^\top b \geq 0$ for all $y \in \mathbb{R}^m$ such that $y^\top A \geq 0$.

Theorem 1 (Fredholm's theorem) *Let $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. The linear system :*

$$Ax = b$$

has a solution if and only if $y^\top b = 0$ for all $y \in \mathbb{R}^m$ such that $y^\top A = 0$.

Question 1. Prove Farkas's lemma using linear programming duality.

Question 2. Prove Fredholm's alternatives theorem using Farkas' lemma.

Exercice 2. Benders' reformulation

Consider the following mixed-integer linear program :

$$\begin{aligned} \min_{x\in\mathbb{Z}_+^{n_x}, y\in\mathbb{R}_+^{n_y}} \quad & c^\top x + f^\top y \\ \text{s.t.} \quad & Tx + Wy \geq h \end{aligned}$$

Show that this problem can equivalently be written as :

$$\begin{aligned} \min_{x\in\mathbb{Z}_+^{n_x}, \theta\in\mathbb{R}} \quad & c^\top x + \theta \\ \text{s.t.} \quad & \theta \geq \pi_i^\top (h - Tx) && \pi_i \in ext(\Pi) \\ & \pi_i^\top (h - Tx) \leq 0 && \pi_j \in ray(\Pi) \end{aligned}$$

where $\Pi = \{\pi \geq 0 \mid \pi^\top W \leq f\}$ and $ext(\Pi)$ and $ray(\Pi)$ are, respectively, its extreme points and exreme rays.

Hint : Start by writing the problem as $\min_{x\in\mathbb{Z}_+^{n_x}} c^\top x + Q(x)$ where $Q(x)$ is defined as :

$$\begin{aligned} Q(x) := \min \quad & f^\top y \\ \text{s.t.} \quad & Wy \geq h - Tx \end{aligned}$$

then use duality theory.

Exercice 3. Dual of the shortest path problem

Let $G = (V, A)$ be a directed graph with costs c_{ij} for $(i, j) \in A$ and consider the shortest path problem on this graph from $s \in V$ to $t \in V$.

Let x_{ij} for $(i, j) \in A$ be decision variables such that $x_{ij} = 1$ if arc (i, j) is chosen as part of the shortest path and 0 otherwise. A linear programming formulation for this problem can be written as :

$$
\begin{aligned}
\min \quad & \sum_{(i,j)\in A} c_{ij}x_{ij} \\
\text{s.t.} \quad & \sum_{j\in\delta^-(i)} x_{ji} - \sum_{j\in\delta^+(i)} x_{ij} = d_i && \forall i \in V \\
& x_{ij} \geq 0 && \forall (i, j) \in A
\end{aligned}
$$

where $\delta^+(i)/\delta^-(i)$ are, respectively, the forward and backward star of $j \in V$ and $d_i = -1$ for $i = s$ and $d_i = 1$ for $i = t$ and $d_i = 0$, otherwise.

Question 1. Write the linear programming dual of this formulation.

Exercice 4. Branch-and-bound algorithm

Consider the following instance of the (binary) knapsack problem where the knapsack capacity is given as $W = 6$ and the item profits u_i and item weights w_i are summarized in the following table :

i	1	2	3	4	5
u_i	30	28	27	34	7
w_i	2	2	3	4	1

Question 1. Apply the branch-and-bound algorithm to this instance :

— Dual bounds can be calculated by solving linear programming relaxations.

— Choice of branching variables and nodes to treat can be done randomly (or according to a rule of your choosing).

Exercice 5. Lagrangean dual

Consider the following optimization problem :

$$\begin{aligned} \min_{x\in\{0,1\}^n} \quad & c^\top x \\ \text{s.t.} \quad & x \in X \\ & a^\top x \leq b \end{aligned} \tag{P}$$

where X is a given polyhedron.

Question 1. Show that the problem :

$$\begin{aligned} L(\lambda) := \min_{x\in\{0,1\}^n} \quad & c^\top x + \lambda(a^\top x - b) \\ \text{s.t.} \quad & x \in X \end{aligned}$$

is a relaxation of (P) for all $\lambda \geq 0$.

Question 2. Show that the problem :

$$\max_{\lambda\geq 0} L(\lambda)$$

provides a dual bound on the optimal value of P. (This is called a Lagrangean dual problem.)

Question 3. We next propose to show that the Lagrangean dual problem may provide a better dual bound for (P) compared to its continuous (LP) relaxation.

1. Show that for given λ, $L(\lambda)$ is equal to the solution of the following optimization problem :

$$\begin{aligned} \max \quad & \theta \\ \text{s.t.} \quad & \theta \leq c^\top x^i + \lambda(a^\top x^i - b) \qquad \forall x^i \in (X \cap \{0,1\}^n) \end{aligned}$$

2. Based on the previous expression provide a linear programming formulation for the Lagrangean dual problem.
3. Write the dual of the Lagrangean dual problem written as a linear program. Show that it is equivalent to :

$$\begin{aligned} \min \quad & c^\top x \\ \text{s.t.} \quad & x \in \mathrm{conv}(X \cap \{0,1\}^n) \\ & a^\top x \leq b \end{aligned}$$

   where $\mathrm{conv}(X \cap \{0,1\}^n)$ is the convex hull of all feasible solutions.
4. What can we say about the Lagrangean dual bound compared to the dual bound obtained from :

$$\begin{aligned} \min \quad & c^\top x \\ \text{s.t.} \quad & x \in (X \cap [0,1]^n) \\ & a^\top x \leq b \end{aligned}$$