Adjustable Robust Optimization

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CR03 - Robust combinatorial optimization, ENS-Lyon

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Little reminder

- We talked about "static" robust optimization problems:

$$\begin{aligned} \min \quad & \mathbf{c}^\top \mathbf{x} & \\ \text{s.t.} \quad & \mathbf{a}_i(\boldsymbol{\xi})^\top \mathbf{x} \geq b_i & \forall \boldsymbol{\xi} \in \underbrace{\Xi_i}_{\text{non-empty \& bounded}}, \forall i \in [m] \\ & \mathbf{x} \in \mathcal{X} & \end{aligned}$$

- "Static" refers to all decisions being taken before the realization of uncertainty ("here-and-now").
- Under this assumption many robust optimization problems may be reformulated as deterministic-equivalent problems.
- The reformulation often preserves the class of complexity of the deterministic problem.

Little reminder

$$\begin{aligned} \min \quad & \mathbf{c}^\top \mathbf{x} \\ \text{s.t.} \quad & \mathbf{a}_i(\xi)^\top \mathbf{x} \geq b_i & \forall \xi \in \Xi_i, \forall i \in [m] \\ & \mathbf{x} \in \mathcal{X} \end{aligned}$$

$\to \{\xi \in \mathbb{R}^{n_\xi} \mid D_i \xi \geq d_i\}$

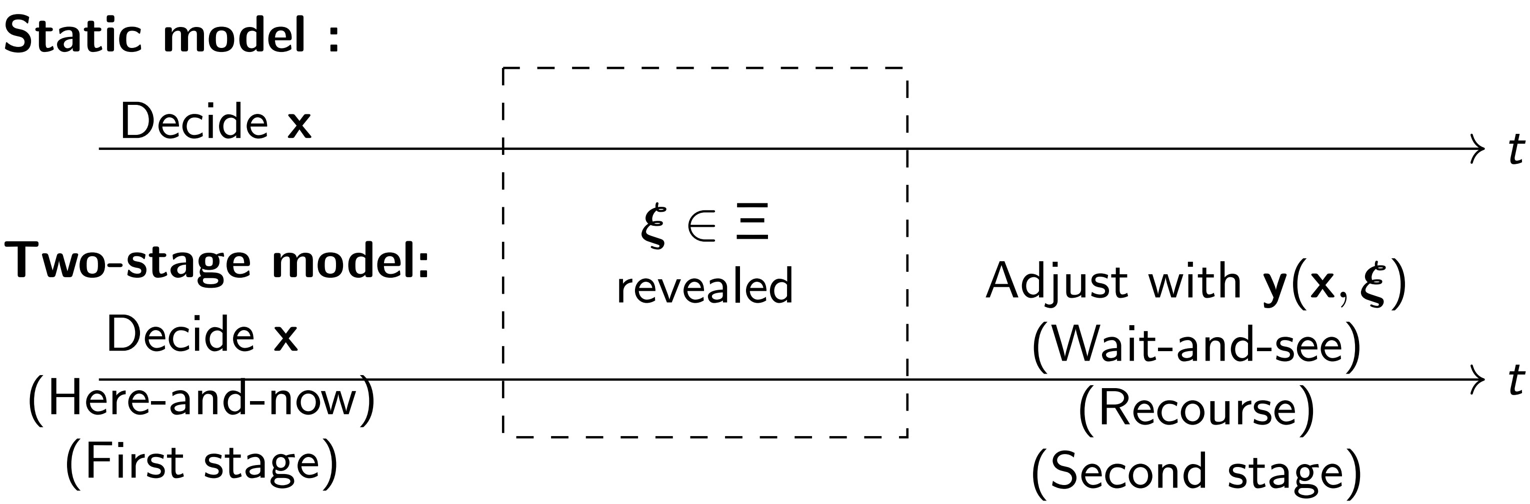
Write $\min_{\xi \in \Xi_i} a_i(\xi)^T x \geq b_i \quad \forall i \in [m]$ with $a_i(\xi) = A_i \xi$ then

$$\begin{aligned} \min \quad & x^T A_i \xi \\ \text{s.t.} \quad & \xi \in \Xi_i \end{aligned} = \max_{u \geq 0} \; d_i^T u \geq b_i, \quad D_i^T u = x^T A_i$$

drop the max

$$\xi^T A_i^T x \geq b \quad \forall \xi \in \Xi_i \longleftrightarrow \begin{aligned} & d_i^T u \geq b_i \\ & D_i^T u = x^T A_i \\ & u \geq 0 \end{aligned}$$

Today: Adjustable robust optimization



- The variables that can "adjust/adapt" to the realization of uncertainty are called "adjustable/adaptable" or "recourse" variables.

Today: Adjustable robust optimization

Sequential decision-making under uncertainty

$$x_0 \xrightarrow{\xi^1} x_1(\xi^1) \xrightarrow{\xi^2 | \xi_1} x_2(\xi^1, \xi^2) \xrightarrow{\xi^3 | \xi_1, \xi_2} x_3(\xi^1, \xi^2, \xi^3)$$

In the two-stage setting :

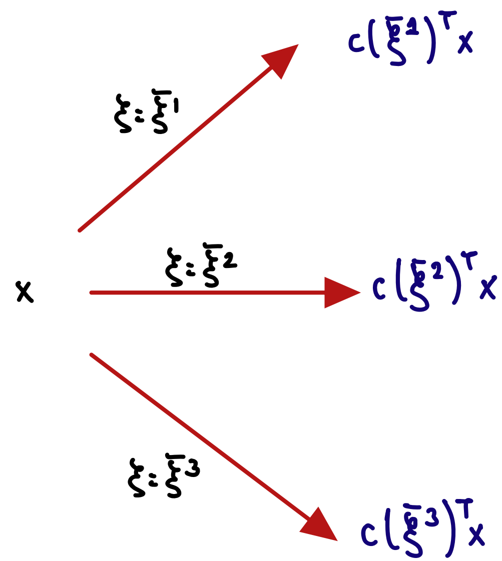
ξ (all uncertain parameters are revealed at once)

$$x \xrightarrow{\xi} y(\xi)$$

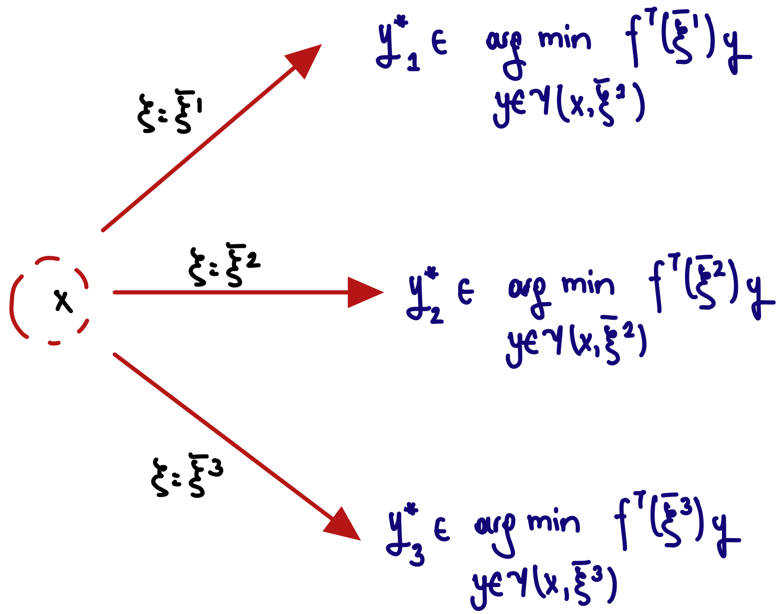
Today: Adjustable robust optimization

If we think about it with discrete uncertainty realisations say $\Xi = \{\bar{\xi}^1, \bar{\xi}^2, \bar{\xi}^3\}$

Static



Two-stage / Adjustable



Today: Adjustable robust optimization

How do we compare two solutions x^1 & x^2 ?

We would like to optimize some function $\rho_{\xi \in \Xi} \left(Q(x,\xi) \right)$

Expectation $\mathbb{E}^{\mathbb{P}}_{\xi \in \Xi} \left[Q(x,\xi) \right]$

CVaR $\mathrm{CVaR}^{\mathbb{P}}_{\alpha, \xi \in \Xi} \left(Q(x,\xi) \right) = \inf_{t \in \mathbb{R}} \; t + \frac{1}{1-\alpha} \mathbb{E} \left[Q(x,\xi) - t \right]_+$

→ Max (Robust) $\max_{\xi \in \Xi} \; Q(x,\xi)$

Max $\mathbb{E}$ over a family of dist. $\max_{\mathbb{P} \in \mathcal{P}} \; \mathbb{E}^{\mathbb{P}}_{\xi \in \Xi} \left[Q(x,\xi) \right]$ ~ distributionally robust

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Two-stage RO: Problem definition

- $\mathcal{X} \subseteq \mathbb{R}^{n_x^c} \times \mathbb{Z}^{n_x^d}$: first-stage feasible region
- $\boldsymbol{\xi}$: uncertain vector with support $\Xi \subseteq \mathbb{R}^{n_\xi}$
- $\Xi \subseteq \mathbb{R}^{n_\xi}$: uncertainty set, non-empty and compact
- $\mathcal{Y} \subseteq \mathbb{R}^{n_y^c} \times \mathbb{Z}^{n_y^d}$: recourse feasible region

$$\min_{\mathbf{x} \in \mathcal{X}} \quad \mathbf{c}^\top \mathbf{x} + \max_{\boldsymbol{\xi} \in \Xi} \mathcal{Q}(\mathbf{x}, \boldsymbol{\xi}) \tag{2ARO}$$

with

$$\begin{aligned} \mathcal{Q}(\mathbf{x}, \boldsymbol{\xi}) = \min \quad & \mathbf{f}(\boldsymbol{\xi})^\top \mathbf{y} \\ \text{s.t.} \quad & \mathbf{W}(\boldsymbol{\xi})\mathbf{y} \leq \mathbf{h}(\boldsymbol{\xi}) - \mathbf{T}(\boldsymbol{\xi})\mathbf{x} \\ & \mathbf{y} \in \mathcal{Y} \end{aligned}$$

linking constraints

Otherwise we should write

$$\min_{x \in X} \; c^\top x + \sup_{\xi \in \Xi} Q(x, \xi)$$

Remark

In writing (2ARO) *we assume that* $\mathcal{Q}(\mathbf{x}, \boldsymbol{\xi})$ *is an upper semi-continuous function in* $\boldsymbol{\xi} \in \Xi$.

Two-stage RO: Problem definition

Hypothesis and Notation

- $\mathcal{X}$ and $\mathcal{Y}$ are linearly constrained
- Ξ is polyhedral, *i.e.*, $\Xi = \{\boldsymbol{\xi} \in \mathbb{R}_{+}^{n_\xi} \mid \mathbf{D}\boldsymbol{\xi} \leq \mathbf{d}\}$
- Data is affine in $\boldsymbol{\xi}$:

can have
discrete or continuous
decision variables

$$\mathbf{h}(\boldsymbol{\xi}) = \begin{bmatrix} 1+\xi_1 \\ 1-\xi_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ \xi_1 \\ \xi_2 \end{bmatrix} = \mathbf{H}\tilde{\boldsymbol{\xi}}$$

$$\begin{aligned} \mathbf{T}(\boldsymbol{\xi}) &= \begin{bmatrix} 2+\xi_1-\xi_2 & 1-\xi_1+\xi_2 \\ 3-\xi_2 & 2-\xi_1 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix} \xi_1 + \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} \xi_2 \\ &= \mathbf{T_0} + \sum_{\mathbf{i=1}}^{\mathbf{2}} \mathbf{T_i}\boldsymbol{\xi}_\mathbf{i} \rightarrow \sum_{\mathbf{i=0}}^{\mathbf{2}} \mathbf{T_i}\boldsymbol{\xi}_\mathbf{i} \text{ with } \boldsymbol{\xi}_0 = 1 \end{aligned}$$

Two-stage RO: Formulations

Min - Max - Min formulation

$$\min_{x \in \mathcal{X}} \; \mathbf{c}^\top \mathbf{x} + \max_{\xi \in \Xi} \; \mathcal{Q}(\mathbf{x}, \xi) = \min_{x \in \mathcal{X}} \; \mathbf{c}^\top \mathbf{x} + \overbrace{\max_{\xi \in \Xi} \; \overbrace{\min_{y \in \mathcal{Y}} \; \mathbf{f}(\xi)^\top \mathbf{y}}^{Q(x,\xi)}}^{Q(x) = \max_{\xi \in \Xi} Q(x,\xi)}$$

$$\text{s.t.} \quad \mathbf{W}(\xi)\mathbf{y} \leq \mathbf{h}(\xi) - \mathbf{T}(\xi)\mathbf{x}$$

Overall problem can be seen as:

$$\min_{x \in X} \; c^\top x + Q(x)$$

- With the convention that if

$$\mathcal{Y}(\mathbf{x}, \bar{\xi}) = \{\mathbf{y} \in \mathcal{Y} \mid \mathbf{W}(\bar{\xi})\mathbf{y} \leq \mathbf{h}(\bar{\xi}) - \mathbf{T}(\bar{\xi})\mathbf{x}\} = \emptyset$$

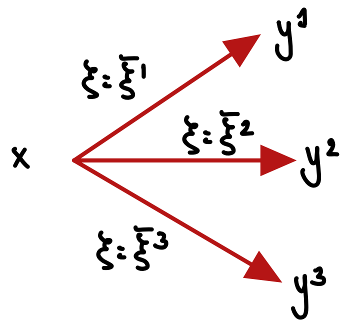
  for some $\bar{\xi} \in \Xi$ then

$$\min_{y \in \mathcal{Y}(x, \bar{\xi})} \mathbf{f}(\bar{\xi})^\top \mathbf{y} = +\infty.$$

  With this convention any $x \in X$ leading to infeasibility is not a candidate for being optimal

- Useful for exact solution schemes (more on this later).

Two-stage RO: Formulations



Write :

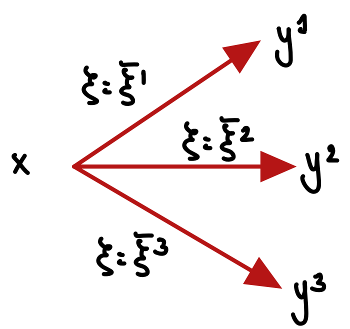
$$\min_{x \in X} \; c^T x + \max_{i=1,2,3} \; f_i^T y_i$$

$$\text{s.t} \quad T_i x + W_i y_i \leq h_i \qquad i = 1,2,3$$

$$y_i \in \mathcal{Y} \qquad i = 1,2,3$$

$f(\bar{\xi}^i) = f_i \qquad W(\bar{\xi}^i) = W_i$

$T(\bar{\xi}^i) = T_i \qquad h(\bar{\xi}^i) = h_i$



If ξ is continuous then y^{ξ} becomes a func.

$y(\cdot) : \Xi \to \mathcal{Y}$

$$\min_{x \in X} \; c^T x + \max_{\xi \in \Xi} \; f^T(\xi) y(\xi)$$

$$\left.\begin{array}{ll} \text{s.t} & T(\xi) x + W(\xi) y(\xi) \leq h(\xi) \\ & y(\xi) \in \mathcal{Y} \end{array}\right\} \; \forall \xi \in \Xi$$

Two-stage RO: Formulations

* See at the end of the slides for a proof of equivalence of models

$$
\begin{aligned}
\min_{\mathbf{x}\in\mathcal{X},\mathbf{y}(\cdot)} \quad & \mathbf{c}^\top \mathbf{x} + \max_{\boldsymbol{\xi}\in\Xi} \quad \mathbf{f}(\boldsymbol{\xi})^\top \mathbf{y}(\boldsymbol{\xi}) & \\
\text{s.t.} \quad & \mathbf{T}(\boldsymbol{\xi})\mathbf{x} + \mathbf{W}(\boldsymbol{\xi})\mathbf{y}(\boldsymbol{\xi}) \leq \mathbf{h}(\boldsymbol{\xi}) & \forall \boldsymbol{\xi} \in \Xi \\
& \mathbf{y}(\boldsymbol{\xi}) \in \mathcal{Y} & \forall \boldsymbol{\xi} \in \Xi
\end{aligned}
$$

- $\mathbf{y}(\cdot) : \Xi \to \mathcal{Y}$ are functionals to be optimized.
- Useful for approximation schemes known as "decision rules" (more on this later).

Remark

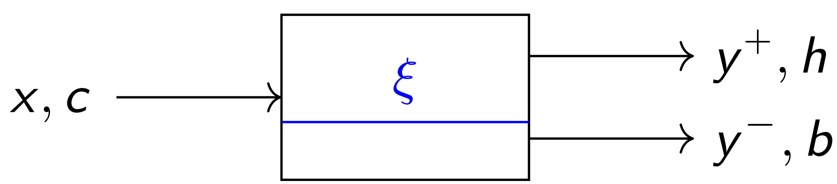
Any $\mathbf{x} \in \mathcal{X}$ for which there exists $\bar{\boldsymbol{\xi}} \in \Xi$ such that

$$\mathcal{Y}(\mathbf{x}, \bar{\boldsymbol{\xi}}) = \{\mathbf{y} \in \mathcal{Y} \mid \mathbf{W}(\bar{\boldsymbol{\xi}})\mathbf{y} \leq \mathbf{h}(\bar{\boldsymbol{\xi}}) - \mathbf{T}(\bar{\boldsymbol{\xi}})\mathbf{x}\} = \emptyset$$

cannot be a feasible solution to this model.

Two-stage RO: A newsvendor example

- Order quantity x
- Uncertain demand ξ with $\xi \in \Xi = \{|\xi - \bar{\xi}| \le \rho\bar{\xi}\}$ where $\bar{\xi} \ge 0$
- Excess quantity $y^+(\xi)$
- Shortage quantity $y^-(\xi)$
- Order cost c, return cost $h > 0$ and shortage cost $b > 0$



$$\xi \in \Xi = \{|\xi - \bar{\xi}| \le \rho\bar{\xi}\}$$

$(1-\rho)\bar{\xi} \le \xi \le (1+\rho)\bar{\xi}$

Two-stage RO: A newsvendor example

- Two-stage problem written in the min-max-min form:

$$\min_{x \geq 0} \quad cx + \max_{\xi \in \Xi} \quad \min_{y^+ \geq 0, y^- \geq 0} \quad hy^+ + by^-$$
$$y^+ - y^- = x - \xi$$

given x & ξ
either y^+ or $y^- > 0$
depending on
$x - \xi > 0$ or < 0

- Two-stage problem written in the functional form:

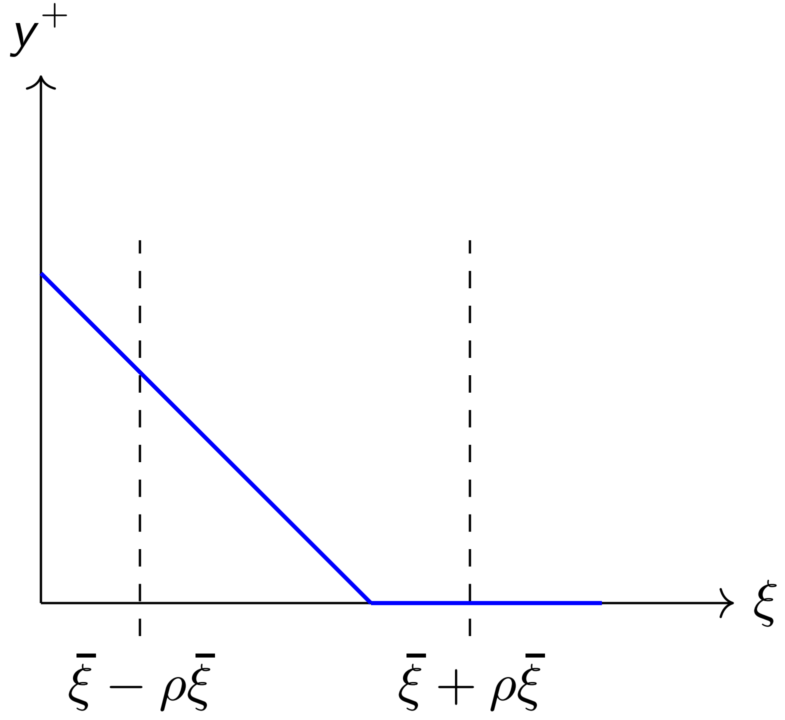
$$\min_{\substack{x \geq 0 \\ y^+(\cdot):\Xi \to \mathbb{R}_+ \\ y^-(\cdot):\Xi \to \mathbb{R}_+}} \quad cx + \max_{\xi \in \Xi} \quad hy^+(\xi) + by^-(\xi)$$
$$\text{s.t.} \quad y^+(\xi) - y^-(\xi) = x - \xi \qquad \forall \xi \in \Xi$$

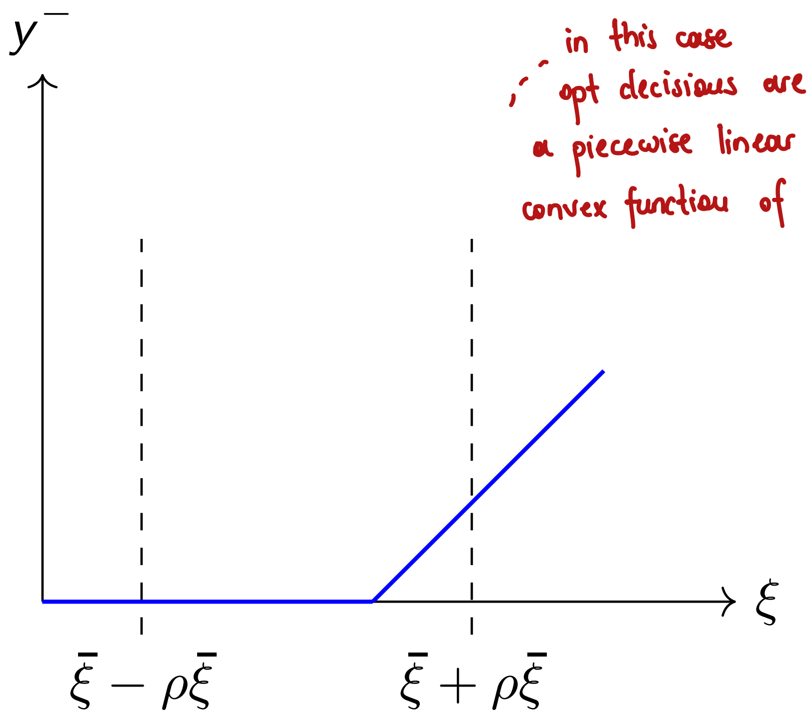
- For this problem, the optimal recourse can be written as:

$$y^+(\xi) = \max\{x - \xi, 0\}$$
$$y^-(\xi) = \max\{0, \xi - x\}$$

Two-stage RO: A newsvendor example

- Optimal recourse quantities as a function of ξ when $x = 10$:



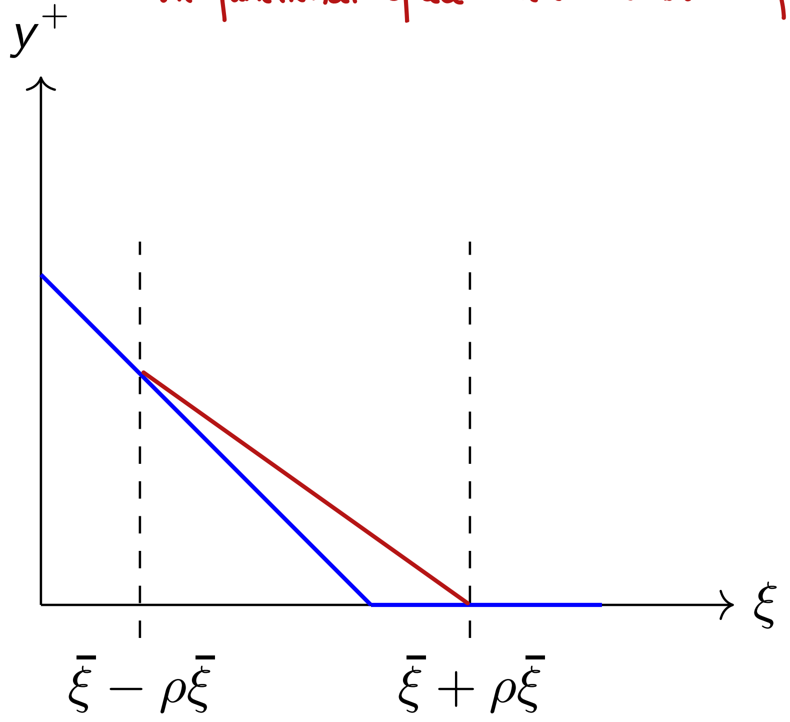


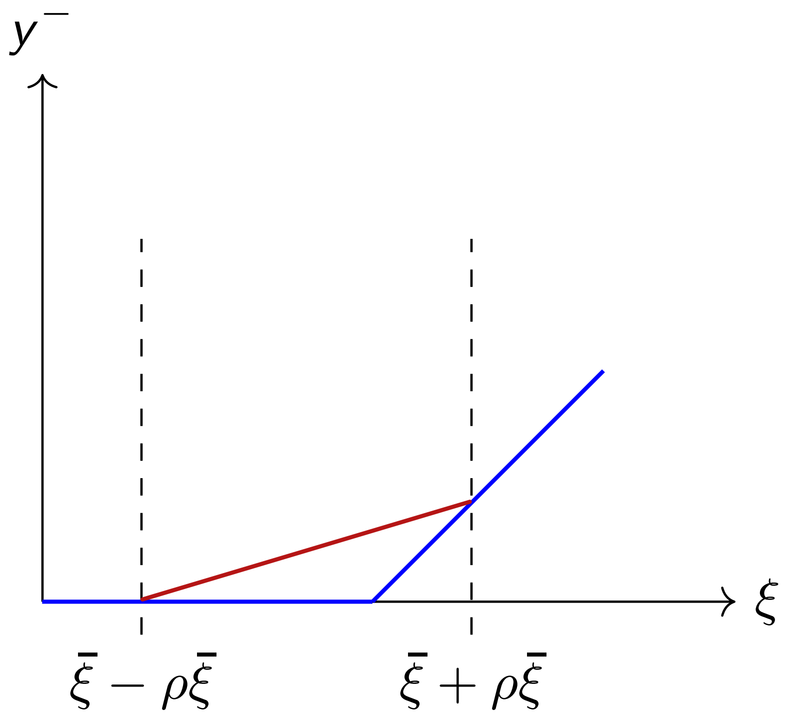
in this case opt decisions are a piecewise linear convex function of ξ

Two-stage RO: A newsvendor example

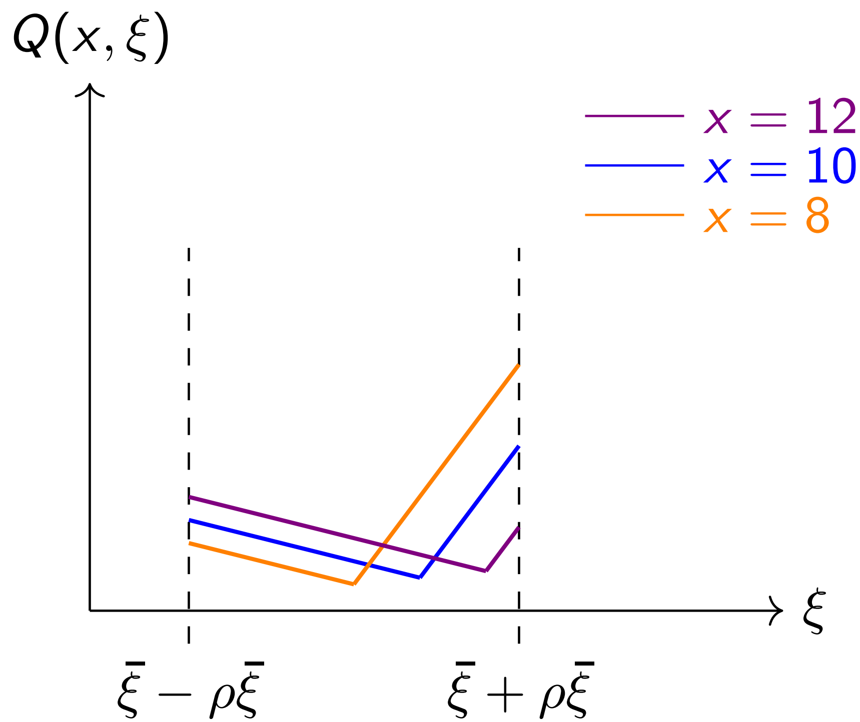
- Optimal recourse quantities as a function of ξ when $x = 10$:

In the functional space there are other opt. solutions





Two-stage RO: A newsvendor example



$Q(x,\xi)$ is a piecewise-linear convex function of ξ

- When x is low worst case is $\bar{\xi} + \rho\bar{\xi}$, otherwise worst case is $\bar{\xi} - \rho\bar{\xi}$.
- Optimality is achieved for x such that:

$$Q(x, \bar{\xi} + \rho\bar{\xi}) = Q(x, \bar{\xi} - \rho\bar{\xi})$$

find the balance between return & shortage costs

Two-stage RO: Relaxations and Restrictions

- Consider $\tilde{\Xi} \subset \Xi$:

$$\min_{\mathbf{x}\in\mathcal{X}} \quad \mathbf{c}^\top \mathbf{x} + \max_{\boldsymbol{\xi}\in\tilde{\Xi}} \quad \min_{\mathbf{y}\in\mathcal{Y}} \quad \mathbf{f}(\boldsymbol{\xi})^\top \mathbf{y}$$
$$\text{s.t.} \quad \mathbf{W}(\boldsymbol{\xi})\mathbf{y} \leq \mathbf{h}(\boldsymbol{\xi}) - \mathbf{T}(\boldsymbol{\xi})\mathbf{x}$$

  is a relaxation of (2RO) and therefore provides a lower bound.
- In particular, for a finite subset $\tilde{\Xi} = \{\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_N\} \subset \Xi$ we can write:

$$\begin{aligned} \min_{\mathbf{x}\in\mathcal{X}, \mathbf{y}_1,\ldots,\mathbf{y}_N\in\mathcal{Y}} \quad & \mathbf{c}^\top \mathbf{x} + \theta \\ \text{s.t.} \quad & \theta \geq \mathbf{f}_\ell^\top \mathbf{y}_\ell & \ell = 1, \ldots, N \\ & \mathbf{T}_\ell \mathbf{x} + \mathbf{W}_\ell \mathbf{y}_\ell \leq \mathbf{h}_\ell & \ell = 1, \ldots, N \end{aligned}$$

  and solve the problem as a (large-scale) (mixed-integer) linear program.

Two-stage RO: Relaxations and Restrictions

- Consider $\tilde{\mathcal{Y}} \subset \mathcal{Y}$:

$$\begin{aligned} \min_{\mathbf{x}\in\mathcal{X}} \quad \mathbf{c}^\top \mathbf{x} + \max_{\boldsymbol{\xi}\in\Xi} \quad \min_{\mathbf{y}\in\tilde{\mathcal{Y}}} \quad & \mathbf{f}(\boldsymbol{\xi})^\top \mathbf{y} \\ \text{s.t.} \quad & \mathbf{W}(\boldsymbol{\xi})\mathbf{y} \le \mathbf{h}(\boldsymbol{\xi}) - \mathbf{T}(\boldsymbol{\xi})\mathbf{x} \end{aligned}$$

  is a restriction of (2RO) and therefore provides an upper bound.

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Two-stage RO with continuous and fixed recourse

$$\begin{aligned}
\min_{\mathbf{x}\in\mathcal{X},\mathbf{y}(\cdot)} \quad & \mathbf{c}^\top\mathbf{x} + \max_{\boldsymbol{\xi}\in\Xi} \quad \mathbf{f}(\xi)^\top\mathbf{y}(\xi) \\
\text{s.t.} \quad & \mathbf{T}(\xi)\mathbf{x} + \mathbf{W}(\xi)\mathbf{y}(\xi) \leq \mathbf{h}(\xi) && \forall\boldsymbol{\xi}\in\Xi \\
& \mathbf{y}(\xi) \in \mathcal{Y} && \forall\boldsymbol{\xi}\in\Xi
\end{aligned}$$

Assume

- $\mathcal{Y} = \mathbb{R}_+^{n_y} \to$ continuous recourse
- $\mathbf{f}(\xi)$ and $\mathbf{W}(\xi)$ are deterministic $\to$ fixed recourse

Two-stage RO with continuous and fixed recourse

$$
\begin{aligned}
\min_{\mathbf{x}\in\mathcal{X},\mathbf{y}(\cdot)} \quad & \mathbf{c}^\top \mathbf{x} + \max_{\boldsymbol{\xi}\in\Xi} \quad \mathbf{f}^\top \mathbf{y}(\boldsymbol{\xi}) & \\
\text{s.t.} \quad & \mathbf{T}(\boldsymbol{\xi})\mathbf{x} + \mathbf{W}\mathbf{y}(\boldsymbol{\xi}) \leq \mathbf{h}(\boldsymbol{\xi}) & \forall \boldsymbol{\xi} \in \Xi \\
& \mathbf{y}(\boldsymbol{\xi}) \in \mathbb{R}_+^{n_y} & \forall \boldsymbol{\xi} \in \Xi
\end{aligned}
$$

Decision rule approximations

$$
\begin{aligned}
\min_{\mathbf{x}\in\mathcal{X},\mathbf{y}(\cdot),\theta\in\mathbb{R}} \quad & \mathbf{c}^\top \mathbf{x} + \theta \\
\text{s.t.} \quad & \theta \geq \mathbf{f}^\top \mathbf{y}(\boldsymbol{\xi}) && \forall \boldsymbol{\xi} \in \Xi \\
& \mathbf{T}(\boldsymbol{\xi})\mathbf{x} + \mathbf{W}\mathbf{y}(\boldsymbol{\xi}) \leq \mathbf{h}(\boldsymbol{\xi}) && \forall \boldsymbol{\xi} \in \Xi \\
& \mathbf{y}(\boldsymbol{\xi}) \in \mathbb{R}_+^{n_y} && \forall \boldsymbol{\xi} \in \Xi
\end{aligned}
$$

Idea

- Restrict the form of $\mathbf{y}(\boldsymbol{\xi})$ to a simple family of functions.
- Optimize within the chosen family.
- Most common restrictions: affine, piecewise affine/constant.

note: static robust optimization can be seen as a decision rule $y(\xi) = y \quad \forall \xi \in \Xi$

Attention!

- Being restrictions, decision rule approximations may be infeasible.
- If feasible, they provide feasible solutions (primal bounds) to (2ARO).
- They typically improve upon the static robust solution. (but not always)

in the case of min upper bounds

Affine decision rules[1]

$$\begin{aligned}
\min_{\mathbf{x}\in\mathcal{X},\mathbf{y}(\cdot)} \quad & \mathbf{c}^\top \mathbf{x} + \max_{\boldsymbol{\xi}\in\Xi} \quad \mathbf{f}^\top \mathbf{y}(\boldsymbol{\xi}) \\
\text{s.t.} \quad & \sum_{i=1}^{n_\xi} \mathbf{T_i}\xi_i \mathbf{x} + \mathbf{W}\mathbf{y}(\boldsymbol{\xi}) \leq \mathbf{H}\boldsymbol{\xi} & \forall \boldsymbol{\xi} \in \Xi \\
& \mathbf{y}(\boldsymbol{\xi}) \geq 0 & \forall \boldsymbol{\xi} \in \Xi
\end{aligned}$$

Idea

- Restrict $\mathbf{y}(\boldsymbol{\xi})$ to be an affine function of $\boldsymbol{\xi}$

$y(\xi) = \alpha_0 + A\xi$

$$\mathbf{y}_i(\boldsymbol{\xi}) = \alpha_{i0} + \boldsymbol{\alpha}_i^\top \boldsymbol{\xi} \quad \forall i = 1, \ldots, n_y \rightarrow \mathbf{y}(\boldsymbol{\xi}) = \mathbf{A}\boldsymbol{\xi}$$

$\bar{A} = \left[\alpha_0 \,|\, A \right] \quad \bar{\xi} = \begin{bmatrix} 1 \\ \xi \end{bmatrix}$

- Optimize $\mathbf{A} \in \mathbb{R}^{n_y \times (n_\xi + 1)}$ to obtain the best such approximation.

the parameters of the affine function become decision variables

[1] Ben-Tal et al., 2004

Affine decision rules[1]

$$
\begin{aligned}
\min_{\mathbf{x}\in\mathcal{X},\mathbf{A}\in\mathbb{R}^{n_y\times(n_\xi+1)},\theta\in\mathbb{R}} \quad & \mathbf{c}^\top\mathbf{x}+\theta & & \text{(Aff)}\\
\text{s.t.} \quad & \theta \geq \mathbf{f}^\top\mathbf{A}\boldsymbol{\xi} & \forall\boldsymbol{\xi}\in\Xi &\\
& \sum_{i=1}^{n_\xi}\mathbf{T_i}\xi_i\mathbf{x}+\mathbf{W}\mathbf{A}\boldsymbol{\xi}\leq\mathbf{H}\boldsymbol{\xi} & \forall\boldsymbol{\xi}\in\Xi &\\
& \mathbf{A}\boldsymbol{\xi}\geq 0 & \forall\boldsymbol{\xi}\in\Xi &
\end{aligned}
$$

we can now see why
we need f, W
constant in ξ

Every constraint is affine
in ξ with x, A, θ fixed

Remark

(Aff) is a static linear robust optimization problem with a polyhedral uncertainty set $\rightarrow$ reformulate into a deterministic equivalent problem through LP-duality.

[1] Ben-Tal et al., 2004

Affine decision rules[1]

Take for instance : $\theta \geq f^T A \xi \quad \forall \xi \in \Xi \quad \longleftrightarrow \quad \theta \geq \max_{\xi \in \mathbb{R}^{n_\xi}} f^T A \xi$

$$D\xi \leq d \quad (u_{obj})$$

$$\longleftrightarrow \quad \theta \geq \min_{s.t} \; d^T u_{obj}$$

$$D^T u_{obj} = f^T A$$

$$u_{obj} \geq 0$$

$$\longleftrightarrow \quad \theta \geq d^T u_{obj}$$

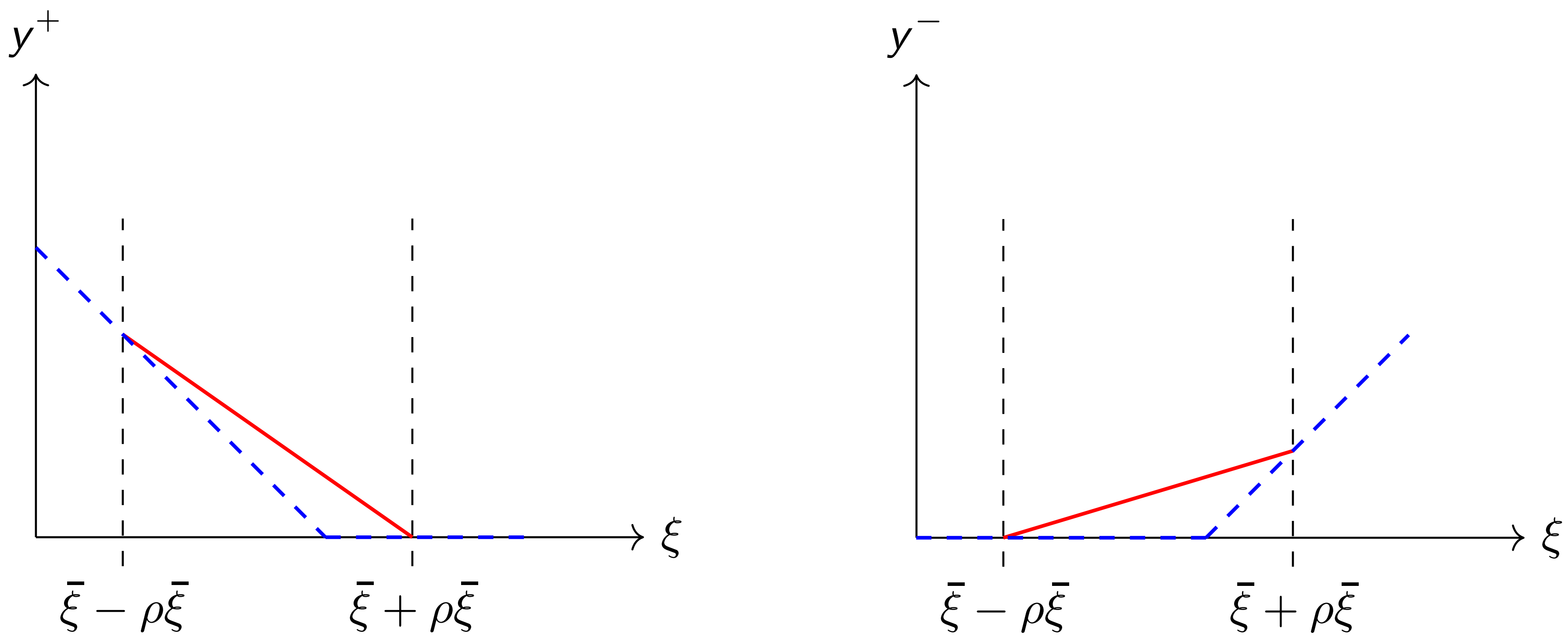
$$D^T u_{obj} = f^T A$$

$$u_{obj} \geq 0$$

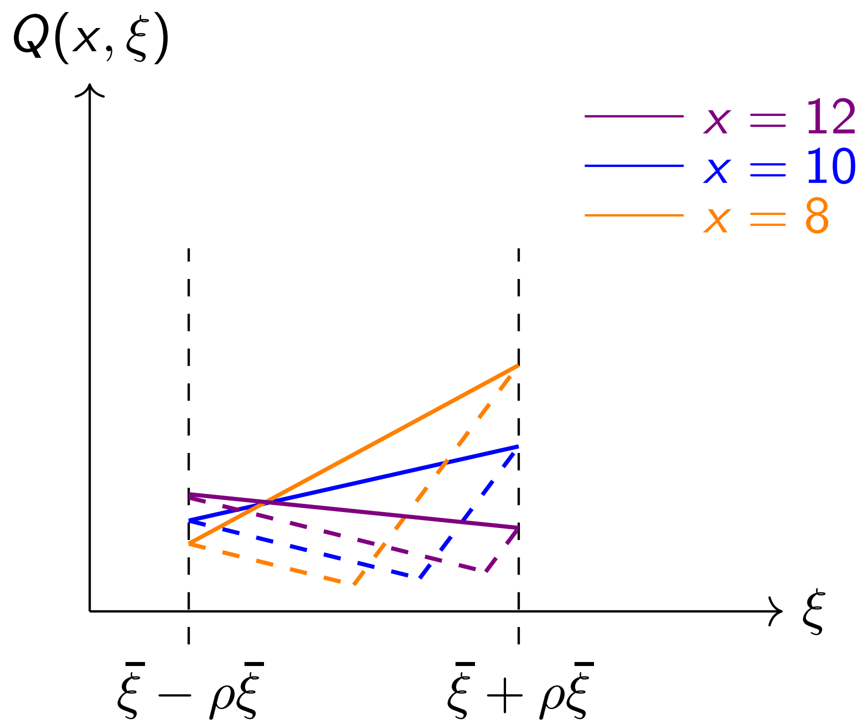
[1] Ben-Tal et al., 2004

Example: Newsvendor (Cont'd)

- Let $y^+ = \alpha_0^1 + \alpha_1^1 \xi$ and $y^- = \alpha_0^2 + \alpha_1^2 \xi$
- Optimal *linear* recourse quantities as a function of ξ when $x = 10$:



Example: Newsvendor (Cont'd)



- When x is low worst case is $\bar{\xi} + \rho\bar{\xi}$, otherwise worst case is $\bar{\xi} - \rho\bar{\xi}$.
- Optimality is achieved at equality.
- Optimal x value is the same as in the exact solution[2].

[2] proved more generally in Bertsimas et al., 2010.

On the quality of affine decision rules

- The quality of a decision rule is measured based on the relative gap:

$$100 \times \frac{|z_{\mathrm{AFF}} - z_{\mathrm{Dual}}|}{|z_{\mathrm{Dual}}|}$$

where z_{Dual} is a dual bound such that $z_{\mathrm{Dual}} \leq z_{\mathrm{2ARO}}$. since z_{2ARO} is not available

- Let $\hat{\Xi} \subseteq \Xi$ be a finite subset of realizations.
- Then the following relaxation provides a dual bound:

$$\begin{aligned}
\min_{\substack{\mathbf{x}\in\mathcal{X},\theta\in\mathbb{R}\\ \mathbf{y}^1,\dots,\mathbf{y}^{|\hat{\Xi}|}\in\mathbb{R}_+^{n_y}}} \quad & \mathbf{c}^\top \mathbf{x} + \theta \\
\text{s.t.} \quad & \theta \geq \mathbf{f}^\top \mathbf{y}^k && \forall k \in [|\hat{\Xi}|] \\
& \mathbf{T}(\xi^k)\mathbf{x} + \mathbf{W}\mathbf{y}^k \leq \mathbf{h}(\xi^k) && \forall k \in [|\hat{\Xi}|]
\end{aligned}$$

- But how do we choose $|\hat{\Xi}|$ in a meaningful way?

On the quality of affine decision rules

- The quality of a decision rule is measured based on the relative gap:

$$100 \times \frac{|z_{\mathrm{AFF}} - z_{\mathrm{Dual}}|}{|z_{\mathrm{Dual}}|}$$

where z_{Dual} is a dual bound such that $z_{\mathrm{Dual}} \leq z_{\mathrm{2ARO}}$.

Hadjiyiannis et al. (2011)

- Solve (Aff) to optimality, let $(\mathbf{x}^*, \mathbf{A}^*, \theta^*)$ be an optimal solution.
- Extract the “binding” scenarios by solving:

$$\max_{\boldsymbol{\xi} \in \Xi} \ \mathbf{f}^\top \mathbf{A}^* \boldsymbol{\xi} - \theta^*$$

$$\max_{\boldsymbol{\xi} \in \Xi} \ \mathbf{T}_i(\boldsymbol{\xi})\mathbf{x}^* + \mathbf{W}_i \mathbf{A}^* \boldsymbol{\xi} - \mathbf{h}_i(\boldsymbol{\xi}) \qquad \forall i \in [m]$$

- Constitute $\hat{\Xi}$ of binding scenarios.

Monolithic reformulations

$$\min_{\mathbf{x}\in\mathcal{X}} \quad \mathbf{c}^\top \mathbf{x} + \max_{\boldsymbol{\xi}\in\Xi} \quad \min_{\mathbf{y}\in\mathbb{R}_+^{n_y}} \quad \mathbf{f}^\top \mathbf{y}$$
$$\text{s.t.} \quad \mathbf{W}\mathbf{y} \le \mathbf{h}(\boldsymbol{\xi}) - \mathbf{T}(\boldsymbol{\xi})\mathbf{x}$$

Assume

- Relatively complete recourse, *i.e.*, $\mathcal{Y}(\mathbf{x},\boldsymbol{\xi}) \neq \emptyset$ for $\mathbf{x}\in\mathcal{X}, \boldsymbol{\xi}\in\Xi$

guarantees that any solution $x\in X$ has value $Q(x) < +\infty$

$\min_{y\in Y} f(\xi)^T y$
$W(\xi) y \le h(\xi) - T(\xi)x$ is feasible for any $x\in X$

Monolithic reformulations

$$\begin{aligned} \min_{\mathbf{x}\in\mathcal{X}} \quad \mathbf{c}^\top \mathbf{x} + \max_{\boldsymbol{\xi}\in\Xi} \quad \min_{\mathbf{y}\in\mathbb{R}_+^{n_y}} \quad & \mathbf{f}^\top \mathbf{y} \\ \text{s.t.} \quad & \mathbf{W}\mathbf{y} \leq \mathbf{h}(\boldsymbol{\xi}) - \mathbf{T}(\boldsymbol{\xi})\mathbf{x} \end{aligned}$$

- Start by writing the dual of the inner minimization problem for $\mathbf{x}$ and $\boldsymbol{\xi}$ given:

$$\begin{aligned} \max_{\mathbf{u}\in\mathbb{R}_-^{m_y}} \quad & \mathbf{u}^\top (\mathbf{h}(\boldsymbol{\xi}) - \mathbf{T}(\boldsymbol{\xi})\mathbf{x}) \\ \text{s.t.} \quad & \mathbf{W}^\top \mathbf{u} \leq \mathbf{f} \end{aligned}$$

- Since we assumed that the primal is feasible we have that the dual is bounded.
- We can also assume that the dual is feasible (why?).

Monolithic reformulations

$$Q(x,\xi) = \begin{array}{ll} \min\limits_{\mathbf{y}\in\mathbb{R}^{n_y}_+} & \mathbf{f}^\top\mathbf{y} \\ \text{s.t.} & \mathbf{W}\mathbf{y} \le \mathbf{h}(\xi) - \mathbf{T}(\xi)\mathbf{x} \end{array} \overset{\text{strong duality}}{=} \begin{array}{ll} \max\limits_{\mathbf{u}\in\mathbb{R}^{m_y}_-} & \mathbf{u}^\top(\mathbf{h}(\xi) - \mathbf{T}(\xi)\mathbf{x}) \\ \text{s.t.} & \mathbf{W}^\top\mathbf{u} \le \mathbf{f} \end{array}$$

$$D := \{ u \in \mathbb{R}^{m_y}_- \mid W^T u \le f \}$$

$$\begin{array}{ll} \max\limits_{\mathbf{u}\in\mathbb{R}^{m_y}_-} & \mathbf{u}^\top(\mathbf{h}(\xi) - \mathbf{T}(\xi)\mathbf{x}) \\ \text{s.t.} & \mathbf{W}^\top\mathbf{u} \le \mathbf{f} \end{array} = \max_{u\in D} u^T(h(\xi) - T(\xi)x)$$

linear programming problem

Let $ext(D)$ be the set of extreme points of D

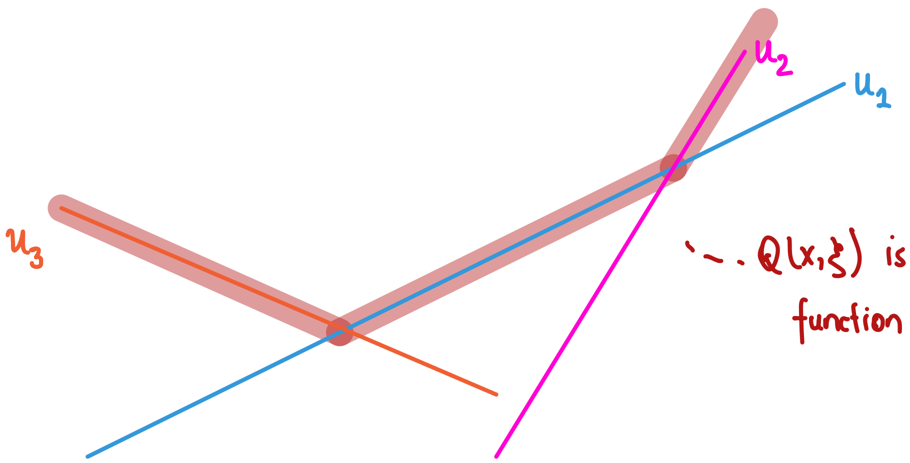
$$Q(x,\xi) = \max_{u\in D} u^T(h(\xi)-T(\xi)x) = \max_{i=1,\dots,|ext(D)|} u_i^T(h(\xi)-T(\xi)x)$$

Monolithic reformulations

if $h(\xi)$ & $T(\xi)$ are affine functions of ξ

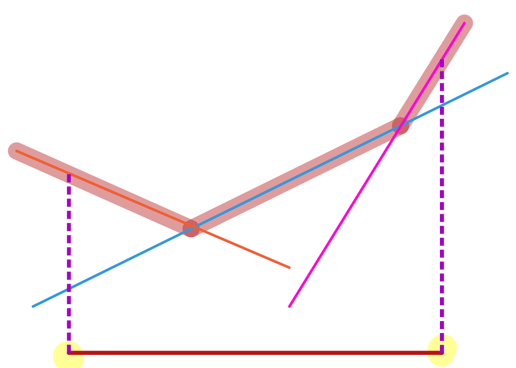
For x fixed $u_i^T(h(\xi)-T(\xi)x)$ defines an affine function of ξ for each $i=1,\dots,|ext(D)|$

$$Q(x,\xi) = \max_{i=1,\dots,|ext(D)|} u_i^T(h(\xi)-T(\xi)x) :$$



$Q(x,\xi)$ is a convex function of $\xi \in \Xi$

$$Q(x) = \max_{\xi \in \Xi} Q(x,\xi)$$



$\xi \in \Xi$

Monolithic reformulations

- Since $\mathcal{Q}(\mathbf{x}, \boldsymbol{\xi})$ is convex in $\boldsymbol{\xi}$, an optimal solution of $\max_{\boldsymbol{\xi} \in \Xi} \mathcal{Q}(\mathbf{x}, \boldsymbol{\xi})$ is an extreme point of Ξ.
- We may therefore write

$$\max_{\boldsymbol{\xi} \in \Xi} \mathcal{Q}(\mathbf{x}, \boldsymbol{\xi}) = \max_{\boldsymbol{\xi}_j \in \mathrm{ext}(\Xi)} \mathcal{Q}(\mathbf{x}, \boldsymbol{\xi}_j) = \max_{\boldsymbol{\xi}_j \in \mathrm{ext}(\Xi), \mathbf{u}_i \in \mathrm{ext}(\mathcal{D})} \mathbf{u}_i^\top (\mathbf{h}(\boldsymbol{\xi}_j) - \mathbf{T}(\boldsymbol{\xi}_j)\mathbf{x})$$

- This leads to a first exponential-sized reformulation of the problem:

$$\begin{aligned} \min_{\mathbf{x} \in \mathcal{X}, \theta \in \mathbb{R}} \quad & \mathbf{c}^\top \mathbf{x} + \theta \\ \text{s.t.} \quad & \theta \geq \mathbf{u}_i^\top (\mathbf{h}(\boldsymbol{\xi}_j) - \mathbf{T}(\boldsymbol{\xi}_j)\mathbf{x}) \qquad \forall \boldsymbol{\xi}_j \in \mathrm{ext}(\Xi), \mathbf{u}_i \in \mathrm{ext}(\mathcal{D}) \end{aligned}$$

- monolithic & linear
- exponential sized
- need to enumerate ext(Ξ), ext(D)

Monolithic reformulations

- For each $\boldsymbol{\xi}_j \in \mathrm{ext}(\Xi)$, there must be a recourse solution $\mathbf{y}_j^* \in \mathbb{R}_+^{n_y}$ such that

$$\mathbf{W}\mathbf{y}_j^* \leq \mathbf{h}(\boldsymbol{\xi}) - \mathbf{T}(\boldsymbol{\xi})\mathbf{x}$$

- We then have that

$$\max_{\boldsymbol{\xi}\in\Xi} \mathcal{Q}(\mathbf{x},\boldsymbol{\xi}) = \max_{\boldsymbol{\xi}_j\in\mathrm{ext}(\Xi)} \mathcal{Q}(\mathbf{x},\boldsymbol{\xi}_j) = \max_{j=1,\ldots,|\mathrm{ext}(\Xi)|} \mathbf{f}^\top \mathbf{y}_j^*$$

- Which leads to yet another exponential size formulation of the problem:

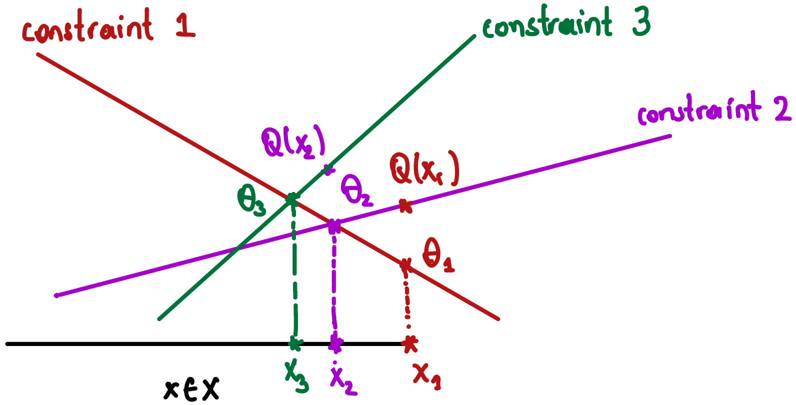
- monolithic & linear
- exponential sized
- need to enumerate ext(Ξ)

$$\begin{aligned}
\min_{\substack{\mathbf{x}\in\mathcal{X},\theta\in\mathbb{R}\\ \mathbf{y}_1,\ldots,\mathbf{y}_{|\mathrm{ext}(\Xi)|}\in\mathbb{R}_+^{n_y}}} \quad & \mathbf{c}^\top\mathbf{x} + \theta \\
\text{s.t.} \quad & \theta \geq \mathbf{f}^\top\mathbf{y}_j & j = 1,\ldots,|\mathrm{ext}(\Xi)| \\
& \mathbf{T}(\boldsymbol{\xi}_j)\mathbf{x} + \mathbf{W}\mathbf{y}_j \leq \mathbf{h}(\boldsymbol{\xi}_j) & j = 1,\ldots,|\mathrm{ext}(\Xi)|
\end{aligned}$$

y_j are decision var.s
& will take the necessary
values to push θ down

Decomposition algorithms

$$\min_{\mathbf{x}\in\mathcal{X}} \quad \mathbf{c}^\top \mathbf{x} + \theta$$
$$\text{s.t.} \quad \theta \geq \dots$$



Idea (Very high level)

- Solve a relaxation of the problem by including only a subset of the constraints on θ.
- Let $(\mathbf{x}^*, \theta^*)$ be an optimal relaxation solution.
- Solve $\max_{\boldsymbol{\xi}\in\Xi} \mathcal{Q}(\mathbf{x}^*, \boldsymbol{\xi})$ to calculate the worst-case value of $\mathbf{x}^*$.
- If $\theta^* < \max_{\boldsymbol{\xi}\in\Xi} \mathcal{Q}(\mathbf{x}^*, \boldsymbol{\xi})$ then add constraints on θ.
- Otherwise $(\mathbf{x}^*, \theta^*)$ is an optimal solution.

all necessary constraints are identified

Constraint-generation algorithm[3]

- Solve the relaxation:

$$\min_{\mathbf{x}\in\mathcal{X},\theta\in\mathbb{R}} \mathbf{c}^\top \mathbf{x} + \theta$$
$$\text{s.t.} \quad \theta \geq \mathbf{u}_\ell^\top (\mathbf{h}(\boldsymbol{\xi}_\ell) - \mathbf{T}(\boldsymbol{\xi}_\ell)\mathbf{x}) \qquad \ell = 1,\dots,N$$

- Let $(\mathbf{x}^*, \theta^*)$ be an optimal solution.
- Solve $\max_{\boldsymbol{\xi}\in\Xi} \mathcal{Q}(\mathbf{x}^*, \boldsymbol{\xi})$. must exist since otherwise $\theta^* = Q(x^*)$
- If $\theta^* < \mathcal{Q}(\mathbf{x}^*)$, let $(\mathbf{u}_{N+1}, \boldsymbol{\xi}_{N+1})$ define a violated constraint.
- Add the constraint:

$$\theta \geq \mathbf{u}_{N+1}^\top (\mathbf{h}(\boldsymbol{\xi}_{N+1}) - \mathbf{T}(\boldsymbol{\xi}_{N+1})\mathbf{x})$$

  to refine the relaxation.
- Otherwise $(\mathbf{x}^*, \theta^*)$ is an optimal solution.

at most $|ext(\Xi)| \times |ext(D)|$ iterations
(assuming (u, ξ) are extreme points of D & Ξ)

[3] Thiele, 2009, Bertsimas et al., 2013

Constraint-and-column generation (CCG) algorithm[4]

- Consider a discrete set $\tilde{\Xi} = \{\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_N\} \subset \mathrm{ext}(\Xi)$, and solve the relaxation:

$$\begin{aligned} \min_{\substack{\mathbf{x}\in\mathcal{X},\theta\in\mathbb{R} \\ \mathbf{y}_1,\ldots,\mathbf{y}_N\in\mathbb{R}_+^{n_y}}} \quad & \mathbf{c}^\top \mathbf{x} + \theta \\ \text{s.t.} \quad & \theta \geq \mathbf{f}^\top \mathbf{y}_\ell & \ell = 1, \ldots, N \\ & \mathbf{T}(\boldsymbol{\xi}_\ell)\mathbf{x} + \mathbf{W}\mathbf{y}_\ell \leq \mathbf{h}(\boldsymbol{\xi}_\ell) & \ell = 1, \ldots, N. \end{aligned}$$

- Let $(\mathbf{x}^*, \theta^*)$ be an optimal solution.
- Solve $\max_{\boldsymbol{\xi}\in\Xi} \mathcal{Q}(\mathbf{x}^*, \boldsymbol{\xi})$.
- If $\theta^* < \mathcal{Q}(\mathbf{x}^*)$, let $\boldsymbol{\xi}_{N+1}$ be a realization that needs to be added to $\tilde{\Xi}$.
- Add variables $\mathbf{y}_{N+1} \in \mathbb{R}_+^{n_y}$ and constraints:

$$\begin{aligned} & \theta \geq \mathbf{f}^\top \mathbf{y}_{N+1} \\ & \mathbf{T}(\boldsymbol{\xi}_{N+1})\mathbf{x} + \mathbf{W}\mathbf{y}_{N+1} \leq \mathbf{h}(\boldsymbol{\xi}_{N+1}) \end{aligned}$$

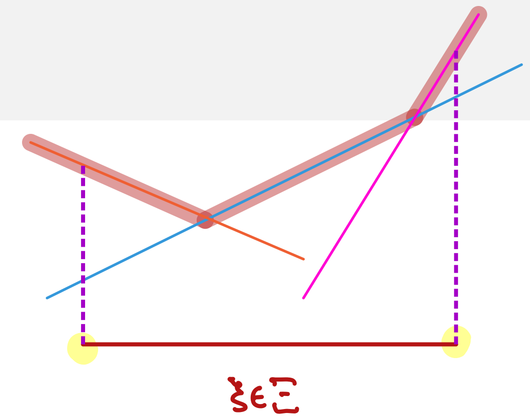
at most $|\Xi|$ iterations (assuming ξ are extreme points of Ξ)

- Otherwise $(\mathbf{x}^*, \theta^*)$ is an optimal solution.

[4] Zeng and Zhao, 2013

Separation problem

$\max_{\xi \in \Xi} Q(x, \xi)$:



- At each iteration of decomposition algorithms, we need to solve:

$$\max_{\boldsymbol{\xi} \in \Xi} \underbrace{\mathcal{Q}(\mathbf{x}^*, \boldsymbol{\xi})}_{\substack{\text{convex func. of } \xi \\ \text{not explicitly known}}} = \max_{\boldsymbol{\xi} \in \Xi} \min_{\mathbf{y} \in \mathbb{R}_+^{n_y}} \quad \mathbf{f}^\top \mathbf{y}$$
$$\text{s.t.} \quad \mathbf{W}\mathbf{y} \leq \mathbf{h}(\boldsymbol{\xi}) - \mathbf{T}(\boldsymbol{\xi})\mathbf{x}^*$$

  to evaluate the true worst-case value of a given first-stage solution $\mathbf{x}^*$.
- This is a difficult problem since it amounts to maximizing a convex function.
- Here we will talk about generic exact approaches based on reformulation as a MIP.

Separation problem

- Write the dual of the inner minimization problem for $\mathbf{x}^*$ and $\boldsymbol{\xi}$ given:

$$\mathcal{Q}(\mathbf{x}^*, \boldsymbol{\xi}) = \max_{\mathbf{u} \in \mathbb{R}_-^{m_y}} \quad \mathbf{u}^\top (\mathbf{h}(\boldsymbol{\xi}) - \mathbf{T}(\boldsymbol{\xi})\mathbf{x}^*)$$
$$\text{s.t.} \quad \mathbf{W}^\top \mathbf{u} \le \mathbf{f}$$

- Merge the two max problems to obtain the bilinear subproblem:

disjoint bilinear

$\xi^* \in ext(\Xi)$, $u^* \in ext(D)$

$$\max \quad -\mathbf{u}^\top (\mathbf{h}(\boldsymbol{\xi}) - \mathbf{T}(\boldsymbol{\xi})\mathbf{x}^*)$$
$$\text{s.t.} \quad -\mathbf{W}^\top \mathbf{u} \le \mathbf{f}$$
$$\mathbf{u} \in \mathbb{R}_+^{m_y}$$
$$\boldsymbol{\xi} \in \Xi := \{\boldsymbol{\xi} \in \mathbb{R}_+^{n_\xi} : \mathbf{D}\boldsymbol{\xi} \le \mathbf{d}\}.$$

$u^\top h(\xi) = u^\top H \xi = \sum_{i,j} h_{ij} u_i \xi_j$

bilinear

- Can be linearized if $\boldsymbol{\xi} \in \{0,1\}^{n_\xi}$ for any extreme point solution of Ξ.

Separation problem

- Assume that $\xi_j \in \{0,1\}$ for $j = 1, \dots, n_\xi$ and $0 \le u_i \le M_i$ for $i = 1, \dots, m_y$.
- Replace each bilinear term $u_i \times \xi_j$ with the auxiliary variable ζ_{ij}. $\rightarrow \zeta_{ij} = u_i \times \xi_j$
- Introduce the linearization constraints (McCormick envelope):

$$\left.\begin{aligned} \zeta_{ij} &\le u_i \\ \zeta_{ij} &\le M_i \xi_j \\ \zeta_{ij} &\ge u_i + M_i(\xi_j - 1) \\ \zeta_{ij} &\ge 0 \end{aligned}\right\}$$

$\xi_j = 1 \rightarrow \zeta_{ij} = u_i$

$\xi_j = 0 \rightarrow \zeta_{ij} = 0$

Remark

The extreme points of the budgeted uncertainty set

$$\Xi^\Gamma = \left\{ \boldsymbol{\xi} \in [0,1]^{n_\xi} \,\middle|\, \sum_{i=1}^{n_\xi} \xi_i \le \Gamma \right\}$$

are binary vectors, $\boldsymbol{\xi} \in \{0,1\}^{n_\xi}$*, when* Γ *is integer.*

$\sum_{i,j} h_{ij} u_i \xi_j \rightarrow \sum_{i,j} h_{ij} \zeta_{ij}$

$\zeta_{ij} \le u_i$

$\zeta_{ij} \le M_i \xi_j$

$\zeta_{ij} \ge u_i + M_i(\xi_j - 1)$

$\zeta_{ij} \ge 0$

Separation problem

1

When $\Gamma \notin \mathbb{Z}$, we can trasform Ξ^Γ as follows:

$$\tilde{\Xi}^\Gamma = \left\{ \xi^1, \xi^2 \in [0,1]^{n_\xi} \;\middle|\; \sum_{i=1}^{n_\xi} \xi_i^1 \leq \lfloor \Gamma \rfloor, \sum_{i=1}^{n_\xi} \xi_i^2 \leq 1, \; \xi_i^1 + \xi_i^2 \leq 1 \quad \forall i \in [n_\xi] \right\}$$

data is transformed as : $\xi_i \to \xi_i^1 + (\Gamma - \lfloor \Gamma \rfloor)\xi_i^2$

2 More generally: assume that we can write $\xi = Aw$ with $w \in \Omega \subseteq \mathbb{R}^{n_w}$ s.t extreme points of Ω are $\{0,1\}$.

$u^T \xi \to u^T A w$: $u_i w_j$ is linearized using $w_j \in \{0,1\}$

Separation problem (when extreme points of Ξ are not $\notin \{0,1\}^{\xi}$)

- Use the LP duality conditions on the inner problem to write:

$$
\begin{aligned}
\max \quad & \mathbf{f}^\top \mathbf{y} \\
\text{s.t.} \quad & \left.\begin{aligned} & \mathbf{W}\mathbf{y} \leq \mathbf{h}(\boldsymbol{\xi}) - \mathbf{T}(\boldsymbol{\xi})\mathbf{x}^* \\ & \mathbf{y} \in \mathbb{R}_+^{n_y} \end{aligned}\right\} \text{Primal feasibility} \\
& \left.\begin{aligned} & -\mathbf{W}^\top \mathbf{u} \leq \mathbf{f} \\ & \mathbf{u} \in \mathbb{R}_+^{m_y} \end{aligned}\right\} \text{Dual feasibility} \\
\text{CS} \; & \left\{\begin{aligned} & \mathbf{u}_i \times (\mathbf{h}(\boldsymbol{\xi}) - \mathbf{T}(\boldsymbol{\xi})\mathbf{x}^* - \mathbf{W}\mathbf{y})_i = 0 && \forall i = 1, \ldots, m_y \\ & (\mathbf{W}^\top \mathbf{u} + \mathbf{f})_j \times \mathbf{y}_j = 0 && \forall j = 1, \ldots, n_y \end{aligned}\right. \\
& \boldsymbol{\xi} \in \Xi := \{\boldsymbol{\xi} \in \mathbb{R}_+^{n_\xi} : \mathbf{D}\boldsymbol{\xi} \leq \mathbf{d}\}
\end{aligned}
$$

- To handle the bilinear CS constraints, linearize with big-M constraints.
- Introduce an auxiliary variable $\zeta_i \in \{0,1\}$ for $i = 1, \ldots, m_y$ and write:

$\zeta_i = 0 \rightarrow u_i = 0$

$\zeta_i = 1 \rightarrow (Wy - h(\xi) - T(\xi)x^*)_i = 0$

$$
\mathbf{u}_i = 0 \quad \vee \quad (\mathbf{h}(\boldsymbol{\xi}) - \mathbf{T}(\boldsymbol{\xi})\mathbf{x}^* - \mathbf{W}\mathbf{y})_i = 0
$$

$$
\Longleftrightarrow
$$

$$
\mathbf{u}_i \leq M\zeta_i \quad (\mathbf{h}(\boldsymbol{\xi}) - \mathbf{T}(\boldsymbol{\xi})\mathbf{x}^* - \mathbf{W}\mathbf{y})_i \leq M(1 - \zeta_i)
$$

Handling infeasibility in recourse

- Up to now we worked with a relatively complete recourse assumption.
- When this assumption is not satisfied it is possible for $\mathcal{Y}(\mathbf{x}, \boldsymbol{\xi})$ to be empty for some $\mathbf{x} \in \mathcal{X}$ and $\boldsymbol{\xi} \in \Xi$.
- Assume now that $\mathcal{Y}(\bar{\mathbf{x}}, \bar{\boldsymbol{\xi}}) = \emptyset$ for some $\bar{\mathbf{x}} \in \mathcal{X}$ and $\bar{\boldsymbol{\xi}} \in \Xi$.
- Then

$$\begin{aligned} \max_{\mathbf{u} \in \mathbb{R}_{-}^{m_y}} \quad & \mathbf{u}^\top (\mathbf{h}(\bar{\boldsymbol{\xi}}) - \mathbf{T}(\bar{\boldsymbol{\xi}})\bar{\mathbf{x}}) \\ \text{s.t.} \quad & \mathbf{W}^\top \mathbf{u} \le \mathbf{f} \end{aligned}$$

is unbounded. (why?)

How to cut off infeasible $\bar{x} \in \mathcal{X}$?

Handling infeasibility in recourse

$$\max_{\mathbf{u}\in\mathbb{R}_-^{m_y}} \quad \mathbf{u}^\top(\mathbf{h}(\bar{\xi}) - \mathbf{T}(\bar{\xi})\bar{\mathbf{x}})$$

$$\text{s.t.} \quad \mathbf{W}^\top\mathbf{u} \le \mathbf{f}$$

- For C&CG, simply add variable $y_{\bar{\xi}} \in \mathbb{R}_+^{n_y}$ and constraints

$$\theta \geq f^\top y_{\bar{\xi}}$$

$$T(\bar{\xi})x + W y_{\bar{\xi}} \leq h(\bar{\xi})$$

- For CG note that dual unbounded implies existence of $u \in \mathbb{R}_-^{m_y}$ s.t starting from $\bar{u} \in \mathbb{R}_-^{m_y}$, $W^\top \bar{u} \leq f$

$\bar{u} + \lambda u$ is feasible $\forall \lambda > 0$ &

$u^\top (h(\bar{\xi}) - T(\bar{\xi})\bar{x}) > 0$

Add the constraint: $u^\top (h(\bar{\xi}) - T(\bar{\xi})x) \leq 0$

Handling infeasibility in recourse

- MIP reformulations of the max-min problem are all based on bounding the dual variables.
- If the inner maximization problems are unbounded then dual variables cannot be bounded.
- How do we solve

$$\max_{\boldsymbol{\xi}\in\Xi} \mathcal{Q}(\mathbf{x}^*, \boldsymbol{\xi}) = \max_{\boldsymbol{\xi}\in\Xi} \min_{\mathbf{y}\in\mathbb{R}_+^{n_y}} \quad \mathbf{f}^\top \mathbf{y}$$
$$\text{s.t.} \quad \mathbf{W}\mathbf{y} \leq \mathbf{h}(\boldsymbol{\xi}) - \mathbf{T}(\boldsymbol{\xi})\mathbf{x}^*$$

in that case?

Handling infeasibility in recourse

- Consider the recourse problem written as a feasibility problem for given $(\bar{\mathbf{x}}, \bar{\theta})$:

① no feasible y s.t $f^\top y \le \bar{\theta}$

② $Wy \le h(\xi) - T(\xi)\bar{x}$ cannot be satisfied

$$\begin{aligned}
\min_{\mathbf{y}\in\mathbb{R}_+^{n_y}} \quad & 0 \\
\text{s.t.} \quad & \mathbf{f}^\top \mathbf{y} \le \bar{\theta} && (\sigma \in \mathbb{R}_-) \\
& \mathbf{W}\mathbf{y} \le \mathbf{h}(\bar{\xi}) - \mathbf{T}(\bar{\xi})\bar{\mathbf{x}} && (\mathbf{u} \in \mathbb{R}_-^{m_y})
\end{aligned}$$

- Its dual is then given as:

$$\begin{aligned}
\max_{\mathbf{u}\in\mathbb{R}_-^{m_y},\, \sigma\in\mathbb{R}_-} \quad & \sigma\bar{\theta} + \mathbf{u}^\top(\mathbf{h}(\bar{\xi}) - \mathbf{T}(\bar{\xi})\bar{\mathbf{x}}) \\
\text{s.t.} \quad & \mathbf{f}\sigma + \mathbf{W}\mathbf{u} \le 0
\end{aligned}$$

$(\sigma, u) = (0, 0)$ is a feasible solution

- The optimal value of the dual problem is greater than or equal to 0.
- The optimal value is equal to 0 only if the primal problem is feasible.

Handling infeasibility in recourse

- Further, any feasible solution $(\sigma, \mathbf{u})$ can be scaled to obtain another feasible solution.
- The dual variables can be bounded without changing the conclusion:

$$\begin{aligned} \max_{\mathbf{u}\in\mathbb{R}_-^{m_y},\ \sigma\in\mathbb{R}_-} \quad & \sigma\bar{\theta} + \mathbf{u}^\top(\mathbf{h}(\bar{\xi}) - \mathbf{T}(\bar{\xi})\bar{\mathbf{x}}) \\ \text{s.t.} \quad & \mathbf{f}\sigma + \mathbf{W}\mathbf{u} \leq 0 \\ & |\sigma| + \sum_{i=1}^{m_y} |u_i| \leq 1 \end{aligned} \leq 0$$

∴ either obj > 0 → dual unbd. primal infeas.

OR obj = 0 → primal/dual feas.

Handling infeasibility in recourse

- We can then write the separation problem:

$$\begin{aligned}\max_{\boldsymbol{\xi}\in\Xi,\mathbf{u}\in\mathbb{R}_-^{m_y},\ \sigma\in\mathbb{R}_-} \quad & \sigma\bar{\theta} + \mathbf{u}^\top(\mathbf{h}(\boldsymbol{\xi}) - \mathbf{T}(\boldsymbol{\xi})\bar{\mathbf{x}}) \\ \text{s.t.} \quad & \mathbf{f}\sigma + \mathbf{W}\mathbf{u} \leq 0 \\ & |\sigma| + \sum_{i=1}^{m_y} |u_i| \leq 1\end{aligned}$$

- Bilinear terms can be linearized if $\boldsymbol{\xi} \in \{0,1\}^{n_\xi}$ for any extreme point solution of Ξ.
- Dual variables can be bounded by 1 when necessary.
- If $\boldsymbol{\xi} \notin \{0,1\}^{n_\xi}$ for any extreme point solution of Ξ then we use KKT conditions after introducing artificial variables.

LP-optimality

Handling infeasibility in recourse

- When the optimal value of the separation problem is > 0 then we need to cut off the current solution $\bar{\mathbf{x}} \in \mathcal{X}$.
- For the constraint generation algorithm, we add:

$$\sigma^* \bar{\theta} + \mathbf{u}^{*\top}(\mathbf{h}(\boldsymbol{\xi}^*) - \mathbf{T}(\boldsymbol{\xi}^*)\bar{\mathbf{x}}) \leq 0$$

Two cases:

1. $\sigma = 0 \rightarrow u^T(h(\xi^*) - T(\xi^*)\bar{x}) \leq 0$
2. $\sigma < 0 \rightarrow \theta \geq \bar{u}^{*T}(h(\xi^*) - T(\xi^*)\bar{x})$

with $(\boldsymbol{\xi}^*, \mathbf{u}^*, \sigma^*)$ being an optimal solution of the separation problem.

- For the constraint-and-column generation algorithm, we add variables $\mathbf{y}_{\xi^*} \in \mathbb{R}_+^{n_y}$ and constraints:

$$\theta \geq \mathbf{f}^\top \mathbf{y}_{\xi^*}$$
$$\mathbf{T}(\boldsymbol{\xi}^*)\mathbf{x} + \mathbf{W}\mathbf{y}_{\xi^*} \leq \mathbf{h}(\boldsymbol{\xi}^*)$$

Two-stage RO: Why the sup?

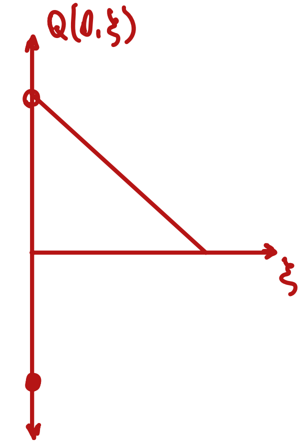
$x=0,\ \xi=0 \longrightarrow \min_{y\in\{0,1\}:\, y\geq 0} 2y-1 = -1$

$x=0,\ \xi>0 \longrightarrow \min_{y\in\{0,1\}:\, y\geq \xi} (\xi-1)(1-2y)$

$= 1-\xi$

- Consider[1]:

$$\min_{x\in\{0,1\}} \ \sup_{\xi\in[0,1]} \ \min_{y\in\{0,1\}\,:\, y\geq\xi, y\geq x} (\xi-1)(1-2y)$$



- Fix $x = 0$.
- If $\xi = 0$ then $y = 0$ is optimal with value -1.
- If $\xi > 0$ then $y = 1$ is optimal with value $1 - \xi$.
- The function $\mathcal{Q}(x = 0, \xi)$ is discontinuous at $\xi = 0$.

↳ $\max_{\xi\in\Xi} Q(x,\xi)$ does not have an optimal solution

Remark

In the following, we will assume that $\mathcal{Q}(\mathbf{x}, \boldsymbol{\xi})$ is an upper semi-continuous function in $\boldsymbol{\xi} \in \Xi$ and replace the sup *with a* max.

1. Continuous recourse and $f(\xi)=f$, $W(\xi)=W$ for all $\xi\in\Xi$

2. Mixed-integer recourse and only the objective function is uncertain. ($f(\xi)=f$, $W(\xi)=W$, $T(\xi)=T$ for all $\xi\in\Xi$)

[1] adapted from Subramanyam et al. (19)

$$\min_{x\in\mathcal{X}} \quad c^\top x + \max_{\xi\in\Xi} \mathcal{Q}(x,\xi)$$

$$\begin{aligned}\min_{x\in\mathcal{X},y(\cdot)} \quad & c^\top x + \max_{\xi\in\Xi} \quad f(\xi)^\top y(\xi) \\ \text{s.t.} \quad & T(\xi)x + W(\xi)y(\xi) \le h(\xi) && \forall \xi\in\Xi \\ & y(\xi)\in\mathcal{Y} && \forall \xi \in \Xi\end{aligned}$$

$$\begin{aligned}\mathcal{Q}(x,\xi) = \min \quad & f(\xi)^\top y \\ \text{s.t.} \quad & W(\xi)y \le h(\xi) - T(\xi)x \\ & y\in\mathcal{Y}\end{aligned}$$

are these two models equivalent?

① Feasibility:

In min-max-min if $\exists x\in X$ s.t $\{y\in Y \mid W(\xi)y \le h(\xi) - T(\xi)x\} \ne \emptyset \quad \forall \xi\in\Xi$

then $x\in X$ s.t $\exists \bar{\xi}\in\Xi$ s.t $\{y\in Y \mid W(\bar{\xi})y \le h(\bar{\xi}) - T(\bar{\xi})x\} = \emptyset$ cannot be optimal.

In functional for $x\in X$ to be feasible, we need

$$\exists y\in Y \quad \text{for } \forall\xi\in\Xi \quad \text{s.t} \quad T(\xi)x + W(\xi)y(\xi) \le h(\xi)$$

↳ Set of $x\in X$ s.t $Q(x) < +\infty$ in min-max-min & set of feasible $x\in X$ in functional are the same.

② Optimality:

Take an opt. sol of functional $(x, y(\cdot))$ value $= c^T x + \max_{\xi\in\Xi} f^T(\xi)y(\xi)$

If we take x part of this sol. in max-min do we have

$$c^T x + \max_{\xi\in\Xi} Q(x,\xi) = c^T x + \max_{\xi\in\Xi} f^T(\xi)y(\xi) \ ?$$

Suppose not then $\max_{\xi\in\Xi} Q(x,\xi) < \max_{\xi\in\Xi} f^T(\xi)y(\xi)$

implies $\forall\xi\in\Xi \quad Q(x,\xi) \le \max_{\xi\in\Xi} f^T(\xi)y(\xi)$ & for $\xi'\in \operatorname{argmax}_{\xi\in\Xi} f^T(\xi)y(\xi)$ we must have

that $\quad Q(x,\xi') < f^T(\xi')y(\xi')$

By replacing $y(\xi')$ with $\quad y'\in \operatorname{argmin}_{y\in Y(x,\xi')} Q(x,\xi')$ we still have $y(\cdot)$ feasible

and with smaller value.

✗ contradiction to optimality of $y(\cdot)$.

Similarly we can obtain an optimal policy $(x, y(\cdot))$ from an opt. solution x of min-max-min with the same value.

Note however that for the same x, there may exist many $y(\cdot)$ that are optimal but

not necessarily $\quad y(\xi) \in \underset{y \in Y(x,\xi)}{\operatorname{argmin}} \ f(\xi)^T y$