Optimisation robuste

Feuille de TD : modélisation, équivalence de formulations, matrices totalement unimodulaires

Modélisation

Exercice 1. Suppose that you are interested in choosing a set of investments $\{1, ..., 7\}$ using boolean variables. Model the following constraints :

1. You cannot invest in all of them.
2. You must choose at least one of them.
3. Investment 1 cannot be chosen if investment 3 is chosen.
4. Investment 4 can be chosen only if investment 2 is also chosen.
5. You must choose either both investments 1 and 5 or neither.
6. You must choose either at least one of the investments 1,2,3 or at least two from 2,4,5,6.

Exercice 2. Problème de clustering

Soit $\mathcal{O}$ un ensemble d'objets et D_{ij} une mesure de dissimilarité pour chaque paire d'objets $(i, j) \in \mathcal{O} \times \mathcal{O}$. On désire séparer cet ensemble d'objet en deux sous-ensembles tout en minimisant la dissimilarité des objets au sein de chaque sous-ensemble.

1. Donner une première formulation non-linéaire de ce problème en utilisant uniquement les variables binaires u_i et v_i indiquant respectivement si l'objet $i \in \mathcal{O}$ appartient au premier ou au second sous-ensemble.
2. Linéariser cette formulation en introduisant les variables binaires x_{ij} et y_{ij} représentant respectivement les produits $u_i \times u_j$ et $v_i \times v_j$.
3. Est-il possible de donner une formulation utilisant uniquement les variables u_i et x_{ij} ?

Exercice 3. Disjonction Modéliser les contraintes suivantes par une conjonction de contraintes faisant intervenir des variables binaires :

$$\begin{pmatrix} x_1 + x_2 & \leq 12 \\ 3x_1 - x_2 & \leq 15 \end{pmatrix} \vee \begin{pmatrix} x_1 + x_2 & \geq 8 \\ 3x_1 - x_2 & \geq 15 \end{pmatrix}$$
$$0 \leq x_1 \leq 9, 0 \leq x_2 \leq 5$$

Le modèle proposé fera intervenir une constante M dont on donnera la plus petite valeur nécessaire pour que la modélisation soit valide.

Exercice 4. Fonction objectif convexe Formuler les problèmes d'optimisation suivants comme des programmes linéaires :

$$\begin{cases} \min & f(x_1, x_2) \\ & 0 \leq x_1, x_2 \leq 10 \end{cases}$$

1. $f(x_1, x_2) = \max\{x_1, x_2\}$
2. $f(x_1, x_2) = |x_1 - x_2|$

Exercice 5. Fonction objectif linéaire par morceaux On considère le problème d'optimisation suivant :

$$\begin{cases} \min & f(x_1 + x_2) \\ & x_1 - x_2 \leq 3 \\ & 2x_1 + x_2 \leq 5 \\ & 0 \leq x_1, x_2 \leq 10 \end{cases}$$

où la fonction $x \mapsto f(x)$ est une fonction linéaire par morceaux non convexe définie sur $\mathbb{R}^+$ par

$$f(x) = \begin{cases} -x + 3 & \text{si } 0 \leq x < 1 \\ \frac{1}{2}x + \frac{3}{2} & \text{si } 1 \leq x < 3 \\ -x + 6 & \text{si } 3 \leq x < 5 \\ 2x - 9 & \text{si } x \geq 5 \end{cases}$$

1. Représenter graphiquement la fonction de coût.
2. Modéliser le problème à l'aide d'un programme linéaire en nombres entiers.
3. Donner une solution optimale du problème.

Relaxations, formulations alternatives

Exercice 6. Let

$$P_1 = \left\{ x \in \mathbb{B}^4 : \quad 97\,x_1 + 32\,x_2 + 25\,x_3 + 20\,x_4 \leq 139 \right\}$$

$$P_2 = \left\{ x \in \mathbb{B}^4 : \quad 2\,x_1 + x_2 + x_3 + x_4 \leq 3 \right\}$$

$$P_3 = \left\{ x \in \mathbb{B}^4 : \quad \begin{aligned} & x_1 + x_2 + x_3 \leq 2 \\ & x_1 + x_2 + x_4 \leq 2 \\ & x_1 + x_3 + x_4 \leq 2 \end{aligned} \right\}$$

Show that $P_1 = P_2 = P_3$.

Exercice 7. Rappelons le problème de localisation suivant. Une compagnie veut construire des entrepôts afin d'approvisionner un ensemble de magasins dénoté $I = \{1, \dots, n\}$. Après prospection, la compagnie a retenu un ensemble de sites potentiels dénoté $J = \{1, \dots, m\}$. Construire un entrepôt sur le site $j \in J$ engage un coût k_j tandis qu'approvisionner un magasin $i \in I$ à partir d'un entrepôt situé en $j \in J$ entraîne un coût fixe c_{ij}.

Deux formulations classiques existent pour le problème. Ces formulations utilisent la variable binaire y_i pour décrire si l'entrepôt $j \in J$ est ouvert et la variable dinaire x_{ij} pour décrire si l'entrepôt j déssert le client $i \in I$. On introduit les deux formulations suivantes :

$$(WF) \begin{cases} \min & \sum_{j \in J} k_j y_j + \sum_{i \in I, j \in J} c_{ij} x_{ij} \\ & \sum_{j \in J} x_{ij} = 1, \quad \forall i \in I \\ & \sum_{i \in I} x_{ij} \leq |I| y_j, \quad \forall j \in J \\ & x, y \text{ binaires} \end{cases} \qquad (SF) \begin{cases} \min & \sum_{j \in J} k_j y_j + \sum_{i \in I, j \in J} c_{ij} x_{ij} \\ & \sum_{j \in J} x_{ij} = 1, \quad \forall i \in I \\ & x_{ij} \leq y_j, \quad \forall i \in I, j \in J \\ & x, y \text{ binaires} \end{cases}$$

Soient P_{WF} et P_{SF} les polytopes obtenus pour les deux formulations ci-dessous en relâchant l'intégralité des variables. Montrer que la formulation de (SF) est plus forte que celle de (WF), càd, que $P_{SF} \subseteq P_{WF}$.

Exercice 8. Consider the following 0-1 IP

$$(P_1) \quad \max \left\{ cx : \sum_{j=1}^{n} a_{ij}\, x_j = b_i \quad \text{for } i = 1, \dots, m \quad x \in \mathbb{B}^n \right\}$$

and the following 0-1 knapsack problem :

$$(P_2) \quad \max \left\{ cx : \sum_{j=1}^{n} \left(\sum_{i=1}^{m} u_i\, a_{ij} \right) x_j = \sum_{i=1}^{m} u_i\, b_i \quad x \in \mathbb{B}^n \right\}$$

where $u \in \mathbb{R}^m$.
Show that P_2 is a relaxation of P_1, i.e., that $P_1 \subseteq P_2$.

Matrices totalement unimodulaires

Exercice 9. Definition. *A matrix A is totally unimodular (TU) if every square submatrix has determinant +1, -1 or 0.*

Sufficient Condition. *A matrix A is TU if :*

1. $a_{ij} \in \{1, 0, -1\}$ *for all i, j;*
2. *Each column of A contains at most two nonzero coefficients*
$$\sum_{i=1}^{m} |a_{ij}| \leq 2.$$
3. *There exists a partition (M_1, M_2) of the set of rows of matrix A such that each column j containing two nonzero coefficients satisfies*
$$\sum_{i \in M_1} a_{ij} - \sum_{i \in M_2} a_{ij} = 0.$$

Are the following matrices totally unimodular or not ?

$$A_1 = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix} \qquad A_2 = \begin{pmatrix} -1 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 \end{pmatrix}$$

Exercice 10. Prove that the polyhedron $P = \{(x_1, \dots, x_m, y) \in \mathbb{R}_+^{m+1} : y \leq 1,\ x_i \leq y \text{ for } i = 1, \dots, m\}$ has integer vertices.

Modélisation - optimisation robuste

Remark 1. exercices marked with $\star$ are to be realized later after having seen the dualization theorem.

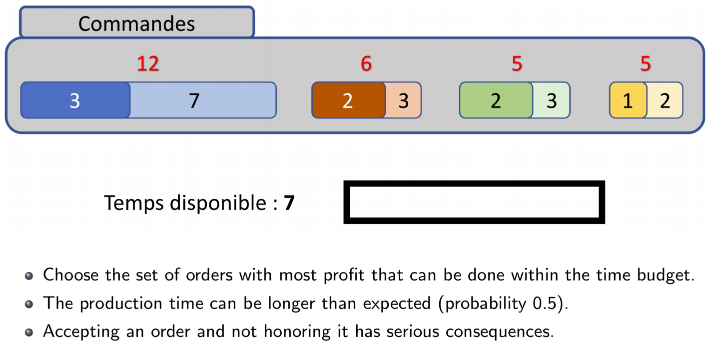

FIGURE 1 – An instance of robust knapsack problem with uncertain weights.

Exercice 11. Knapsack problem with uncertain weights

We are interested in the uncertain variant of the 0-1 knapsack problem where the weights of items are uncertain. We assume that they lie in a known interval, and consider the budgeted uncertainty set.

1. Give a formal definition of the uncertainty set $\mathcal{U}^\Gamma$ for the instance in Figure 1, where the dark colour indicates the minimal value $(\bar{a}_i)$ and the light colour indicates the total value $(\bar{a}_i + \hat{a}_i)$.
2. Write a (potentially very large or even infinite size, potentially non-linear) formulation for the robust problem, which consists in maximizing the total profit of items such that the capacity constraint is satisfied under all possible weight realizations in $\mathcal{U}^\Gamma$.
3. Solve the instance of Figure 1 with $\Gamma = 2$.
4. $\star$ Write a mixed integer linear programming formulation of the robust problem, whose size is polynomial in the number of items.

Exercice 12. Robust location-transportation problem (adapted from Zeng and Zhao, 2013) Consider the previous location problem, assuming now that each facility has a maximum capacity C_j and d_i denotes the quantity of product demanded by the customer i. We assume that $\sum C_j \geq \sum d_i$ so that the problem always has a feasible solution.

1. Adapt formulation (WF) for this problem, assuming that demand maybe be split among the facilities. Here, x_{ij} could play two distinct roles, leading to two different formulations : (i) x_{ij} can represent the total amount shipped from j to i, and (ii) x_{ij} can represent the fraction of total demand shipped from j to i. Write both formulations.

In reality, the demand of customers is unknown. To capture this uncertainty, assume that the demands d_i are replaced by the estimations $\bar{d}_i$ and deviations $\hat{d}_i$ in such a way that the uncertain demand $\tilde{d}_i$ is assumed to be in the interval $[\bar{d}_i, \bar{d}_i + \hat{d}_i]$. In order to avoid being overly conservative, the company assumes that at most Γ customers will have their demand deviate from its estimated value, giving rise to the following uncertainty set :

$$D^\Gamma = \left\{ \tilde{d} \in \mathbb{R}^n \;\middle|\; \sum_{j \in [n]} \frac{\tilde{d}_j - \bar{d}_j}{\hat{d}_j} \leq \Gamma, \tilde{d}_j \in [\bar{d}_i, \bar{d}_i + \hat{d}_i] \text{ for } j \in [n] \right\}.$$

We will then assume that $\sum_{i \in [m]} C_i \geq \max_{d \in D^\Gamma} \sum_{j \in [n]} d_j$ so that the problem always has a feasible solution.

Remark 2. Recall that the robust counterparts of equivalent deterministic formulations can be non-equivalent to each other.

2. Write the robust counterpart of the two deterministic formulations that you proposed for Question 1.

3. $\star$ Write a deterministic mixed integer linear programming model with a number of variables and constraints polynomial in m and n starting from the formulation that you proposed for Question 2.

Exercice 13. Robust uncapacitated lot-sizing problem In the uncapacitated lot-sizing (ULS) problem, one must plan the production of a single product over a planning horizon which is discretized into a set of time periods $\mathcal{T} = \{1, \ldots, T\}$. For each time period $t \in \mathcal{T}$, the customer demand d_t, is assumed to be known. The cost incurred for producing x_t units of product during period t is equal to 0 if $x_t = 0$, and to $f_t + c_t x_t$ otherwise (f_t is the fixed *setup cost*, while c_t is the *unit production cost*). Products can be stored for later delivery. The inventory cost at period t is equal to $h_t s_t$, where h_t is the *unit inventory cost* and s_t is the amount of stock at the end of period t. We assume that the level s_0 of stock at the beginning of period $t = 1$ is known, and that all costs are non-negative.

The problem consists in determining, for each period $t \in \mathcal{T}$, which quantity x_t to produce and which quantity s_t to have in stock, in order to satisfy all the demands at minimum cost. Using auxiliary binary variables $y_t = 1$ if and only if there is a positive production at time period t, we can write the following model for (ULS) :

$$
\begin{aligned}
(\mathrm{ULS}_1) : \min \ & \sum_{t \in \mathcal{T}} (f_t y_t + c_t x_t + h_t s_t) && && (1)\\
\text{s.t. } & s_{t-1} + x_t - s_t = d_t && \forall t \in \mathcal{T} && (2)\\
& x_t \leq M y_t && \forall t \in \mathcal{T} && (3)\\
& x_t \geq 0, s_t \geq 0 && \forall t \in T && (4)\\
& y_t \in \{0, 1\} && \forall t \in T && (5)
\end{aligned}
$$

We now define the uncertain version of (ULS) by assuming that the demands are not known in advance. More precisely, the production levels over the whole horizon must be decided assuming that the demand at each period t can vary by %1 around d_t, and that the sum of the demands over the horizon cannot vary by more than %10 around $\sum_{t \in \mathcal{T}} d_t$.

1. Write the robust counterpart (R-ULS$_1$) of formulation (ULS$_1$). Comment on the set of feasible solutions of (R-ULS$_1$).

2. Based on the non-negativity of the costs, we can write another formulation for (ULS), by changing the equality constraint (2) to an inequality. Let us denote this model by (ULS$_2$). Write the robust counterpart (R-ULS$_2$) of formulation (ULS$_2$). Comment on the set of feasible solutions of (R-ULS$_2$).

3. Propose another formulation for (ULS), whose robust counterpart is, in general, less conservative than (R-ULS$_2$).