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Answer 1

Note: I use this symbol "$\mathbb{R}^+$" to represent the nonnegative real numbers, because i couldnt find how to draw a small line on a Real number symbol in latex as u write in "The-2" pdf file.

a) 1)Let's say a=1, b=-1 such that a, b $\in \mathbb{R}$. So f(a)=1 and f(b)=1 since f(a)=f(b) even if a and b are not equal, so f_1 is not injective.

2) Let's say x$\in \mathbb{R}$ then $x^2 \in \mathbb{R}$ but the range of this function is (0,∞) since not all the numbers like -1 are not in the range, the function f_1 is not surjective, too.

b) 1)Let a,b $\in \mathbb{R}^+$ such that f(a)=a^2, f(b)=b^2. This implies $a^2 = b^2$ which corresponds to a=b or a=-b. Since our domain is the nonnegative real numbers, both a and b should be nonnegative which implies that a=b. so finally this means: $\forall a, b(f(a) = f(b) \rightarrow a = b)$ means f_1 is injective.

2)For example number a=-1 is in codomain of f_2, $\mathbb{R}^+$. However -1 is not the square of any real number. Therefore, there is no element of the domain that maps to the corresponds to the number -1, so f_2 is not surjective.

c) 1)Numbers a=1 and a=-1 are both in the domain of f_3 and a$\neq$b. But $f_3(a)$=$f_3(b)$=1. So f_3 is not injective.

2)Since the minimum value that x^2 can take is 0 (for the value of x=0), the range of function is $\mathbb{R}^+$ which equals to the codomain of this function. Since codomain and range are equal to each other f_3 is surjective.

d) 1)Let a,b $\in \mathbb{R}^+$ such that f(a)=a^2, f(b)=b^2. This implies $a^2 = b^2$ which corresponds to a=b or a=-b. Since our domain is the nonnegative real numbers, both a and b should be nonnegative which implies that a=b. so finally this means: $\forall a, b(f(a) = f(b) \rightarrow a = b)$ means f_1 is injective.

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Answer 2

a) Let's pick some $x_0 \in A$ and let $\epsilon > 0$. And i choose the δ=1/2. Let $x \in \mathbb{Z}$ and suppose that $|x - x_0| < \delta = 1/2$. Since there is only x_0 itself with in distance 1/2, this should satisfy that $x = x_0$.Thus $f(x) = f(x_0)$ and $|f(x) - f(x_0)| = 0$ which is certainly less than ϵ. This actullay shows that f is continous at x_0. Since I choose x_0 as arbitrary, f is continous on its domain.

b) Let's pick some x_0, $x \in A$ and let $\epsilon = 1/2$. So for this ϵ value there must be some $\delta > 0$ such that if $|x - x_0| < \delta$ then $|f(x) - f(x_0)| < \epsilon$. But for this definition to be true(which actually says f is continous function), $|f(x) - f(x_0)| < \epsilon = 1/2$ should be true. Only way for this to be true is $f(x) = f(x_0)$ because minimum distance between two distinct integer number is 1 and $1 > 1/2$. So I concluded that for a function whose domain is $\mathbb{R}$ and whose codomain is $\mathbb{Z}$, to be a continous the only way for that f is a constant function.

Answer 3

a) I will use induction for this question. In case $n = 1$ then $X_n = A_1$ which is countable. Now let's assume that $A_n(n \in k, 1 \geq n < k)$ is countable; Then $X_{n+1} = (A_1 * A_2 * ... * A_n) * A_{n+1} = X_n * X_{n+1}$ where the X_n and the X_{n+1} can be called countable. Hence the cartesian product of the countable sets is always countable. So X_{n+1} is countable.

b) Suppose S is countable, S=X*X*... . Let $(F_n : n \in \mathbb{N})$ be an enumeration of S. For each n, they correspond to $0, 1 \in X = \{0, 1\}$. Let's define another function $G(m) \in S$ as:

G(m)=

$$\begin{cases} 1 & F_m(m) = 0 \\ 0 & F_m(m) = 1 \end{cases}$$

This follows that $G \in S$ but it is different of all F_n's which is a contradiction. In concluison, I showed that infinite countable product of the set $X = \{0, 1\}$ with itself is uncountable.

Answer 4

Answer: $(n!)^2 > 5^n > 2^n > n^{51} + n^{49} > n^{50} > \sqrt{n} \log n > (\log n)^2$

a) $\lim_{x\to\infty} \dfrac{5^n}{(n!)^2} = 0$. For the proof of this i can use series. $\sum_{n=1}^{\infty} \dfrac{5^n}{(n!)^2}$ Let's use ratio test for the convergence of this series. So we should look at this limit: $\lim_{x\to\infty} \dfrac{5^{n+1} * (n!)^2}{((n+1)!)^2 * 5^n} =$ $\lim_{n\to\infty} \dfrac{5}{(n+1)^2} = 0$ Since limit goes to 0, this means our series converges to some constant k $\in \mathbb{R}$, which means $\lim_{n\to\infty}(n^{th}$ term$)=0$, in our case n^{th} term is $\dfrac{5^n}{(n!)^2}$ so $\lim_{n\to\infty} \dfrac{5^n}{(n!)^2 = 0}$ So I have showed that: $5^n = \mathcal{O}((n!)^2)$

b) $\lim_{x\to\infty} \dfrac{5^n}{2^n} = \lim_{x\to\infty} (\dfrac{5}{2})^n = \infty$. So it is shown that:

$2^n = \mathcal{O}(5^n)$

c) $\lim_{x\to\infty} \dfrac{n^{51}+n^{49}}{2^n} = 0$. For the proof this limit: I can say that since both the denominator and numerator go to ∞ I should use L'Hopital's Rule: $\lim_{n\to\infty} \dfrac{n^{51}+n^{49}}{2^n} = \lim_{x\to\infty} \dfrac{51*n^{50}+49*n^{48}}{\ln 2 * 2^n}$.
And if I keep using L Hopital's Rule 50 more times my equation will become like this:$\lim_{n\to\infty} \dfrac{51!}{(\ln 2)^{51} * 2^n}$=0 (which is obvious). So in conclusion I have showed that: $n^{51}+n^{49} = \mathcal{O}(2^n)$

d) Let's take domain as $n > 1$
$n^{51}+n^{49} = n(n^{50}+n^{48}) > n \times n^{50}$ I can divide both sides by "n" because we take domain as $n > 1$ so:
$n^{51}+n^{49} = n(n^{50}+n^{48}) > (n^{50}+n^{48}) > n^{50}$ so as we can clearly see. $n^{50} = \mathcal{O}(n^{51}+n^{49})$

e) $\lim_{x\to\infty} \dfrac{n^{50}}{\sqrt{n} * \ln n} = \infty$. To prove this, let's use L'Hopital's Rule since both the denominator and numerator goes to ∞. $\lim_{x\to\infty} \dfrac{n^{50}}{\sqrt{n} * \ln n} = \lim_{x\to\infty} \dfrac{50*n^{49}}{\dfrac{1}{2*\sqrt{n}*n}} = \dfrac{\infty}{0} = \infty$

$\sqrt{x} * \ln n = \mathcal{O}(n^5 0)$

f) Comparison between: $\sqrt{n}\log n$ and $(\log n)^2$: Lets consider our domain as $n > 4$: Let's assume that: $\sqrt{n}\log n > (\log n)^2$ (Since $\log n > 0$ for $n > 4$ I can divide both side with $\log n$)
So our duty is to show that: $\sqrt{n} > \log n$
Let $f(n) = \sqrt{n} - \log n$ So $f(4) \approx 1.4 > 0$. Also $f' = \dfrac{1}{2\sqrt{n}} - \dfrac{1}{n} > 0$ for $n > 4$. This actually means that our function keeps increasing after $n > 4$ so our proof is done. Since we have showed that $\sqrt{n} > \log n$ is true, then $\sqrt{n}\log n > (\log n)^2$ is also true. $(\log n)^2 = \mathcal{O}(\sqrt{n}\log n)$

Answer 5

a) Since 134>94;
Step 1: 134=94*(1)+40
Step 2: 94=40*(2)+14
Step 3: 40=14*(2)+12
Step 4: 14=12*(1)+2
Step 5: 12=2*(6)+0
So by Definition of Euclidean Algorithm gcd(94,134) equals to "2".

b) Let $a > 5$ be an integer. There is 2 condition to check for whether a is odd or even.
Condition 1: Let's say a is even.
Then a should be in the form of $a = 2n$ for $n \geq 3$. Also $a - 2 = 2n - 2 = 2(n-1)$ so also $a - 2$ is even.

By using definition for Golbach's Conjecture, we can write like this: $2n - 2 = p_1 + p_2$ as p_1 and p_2 are two prime numbers. Thus $2n = p_1 + p_2 + 2$ which corresponds to sum of three prime numbers, since 2 is a prime number.

Condiiton 2: Let's say a is odd, this time

Then a should be in the form of $a = 2n + 1$ for $n \geq 3$. Also $a - 3 = 2n - 2 = 2(n - 1)$ so $a - 3$ is also even.

By using definition for Goldbach's Conjecture, we can write like this: $a - 3 = p_1 + p_2$ as p_1 and p_2 are two prime numbers. Thus $a = p_1 + p_2 + 3$ which corresponds to a sum of three primes.

In addition to these, suppose every integer $a > 5$ is the sum of three primes such that $a = p_1 + p_2 + p_3$ and let $a > 2$ be an even integer. Then $a + 2 = 2n + 2$ is even and $a + 2 > 5$, thus $a + 2 = p_1 + p_2 + p_3$ is the sum of three primes p_1, p_2, p_3. Since $a + 2$ is even, this actually means that at least one of p_1, p_2, p_3 must be 2. Since my aim is to show that these two statements are equivalent. I should also show the opposite way:

Let's assume $p_3 = 2$ then $a + 2 = p_1 + p_2 + 2$ also $a = p_1 + p_2$ is a sum of two primes.

In conclusion, Goldbach's conjecture that every even integer greater than 2 is the sum of two primes is equivalent to the statement that every integer greater than 5 is the sum of three primes.