Case 2: Itô's Lemma, Option Pricing, Stochastic Volatility

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1 Itô's Lemma

Geometric Brownian Motion is described by the following stochastic differential equation (SDE):

$$\begin{aligned} dS(t) &= rS(t)dt + \sigma S(t)dW(t) \\ S(0) &= S_0, \end{aligned}$$

where W is a Brownian Motion under the risk-neutral probability measure.

Geometric Brownian Motion can also be described by the following discrete-time approximation:

$$S(t_{i+1}) = S(t_i) + rS(t_i)(t_{i+1} - t_i) + \sigma S(t_i)(W_{t_{i+1}} - W_{t_i}).$$

Using Itô's Lemma the stochastic differential equation for $\ln S_t$ can be derived as:

$$d \ln S_t = \left(r - \frac{1}{2}\sigma^2\right) dt + \sigma dW(t).$$

Or in discrete form:

$$s(t_{i+1}) = \ln S(t_{i+1}) = \ln S(t_i) + \left(r - \frac{1}{2}\sigma^2\right)(t_{i+1} - t_i) + \sigma(W_{t_{i+1}} - W_{t_i}).$$

Simulate $S(t_i)$ and $s(T_i) = \ln S(t_i)$.

1. Show that the distributions of $S(t_N)$ and $\hat{S} = \exp(s(t_N))$ are the same. You can do this by means of showing histograms and reporting some characteristics (mean, standard deviation, skewness, kurtosis) of both distributions.

2. Compute log-returns $\ln X(t_N) - \ln X(t_0)$ for both processes, $S(t_i)$ and $\hat{S}(t_i)$ and show that they both have the same normal distribution. Again, you can do this by means of showing histograms and reporting some characteristics (mean, standard deviation, skewness, kurtosis) of both distributions.

You can use the following parameters for the model:

$$
\begin{aligned}
S_0 &= 1 \\
r &= 0 \\
\sigma &= 0.15 \\
t_0 &= 0 \\
t_N &= 1.
\end{aligned}
$$

You can use 252 steps (appr. the number of trading days per year) between t_0 and t_N. Please note that the parameters r and σ above are on an annualized basis. Finally, the number of paths to be used is 5,000.

You can also plot paths for $S(t_i)$ and $\exp(s(t_i))$ and observe that they are close but there is a difference. Simulation of $\ln S$ is exact in this case, while simulation of $S(t_i)$ converges to $\exp(s(t_i))$ only for small time steps.

3. Plot log-returns as a function of time for a single path. Take 1,000 time steps to obtain a dense path. Observe that the difference between $S(t_i)$ and $\exp(s(t_i))$ is smaller than in case you would take 252 steps.

2 Monte Carlo simulation

Let us again use Geometric Brownian Motion in this exercise:

$$
\begin{aligned}
dS(t) &= rS(t)dt + \sigma S(t)dW(t) \\
S(0) &= S_0,
\end{aligned}
$$

where W is a Brownian Motion under the risk-neutral probability measure.

Let the expiry time of an option be T, and let

$$
\begin{aligned}
N &= \frac{T}{\Delta t} \\
S_n &= S(n\Delta t).
\end{aligned}
$$

Then, given an initial price S_0, M realizations of the path of a risky asset are generated using the algorithm (Euler method):

$$
S_{n+1} = S_n + S_n(r\Delta t + \sigma\sqrt{\Delta t}\varphi),
$$

where φ is a normally distributed random variable with mean 0 and unit variance.

For special cases of constant coefficients, we can avoid time stepping errors for geometric Brownian Motion, since we can integrate the stochastic differential equation exactly to get:

$$
S_T = S_0 e^{(r-0.5\sigma^2)T+\sigma\sqrt{T}\varphi}.
$$

The price of an option can be calculated by computing the discounted value of the average pay-off, i.e.

$$V(S_0) = e^{-rT}\frac{\sum_{m=1}^{M} \text{payoff}^m(S_N)}{M}.$$

Write a computer program for the Monte Carlo method. As in Case 1, we want to price a European call option with maturity 3 months and strike price USD 3,000. Let the 3-month annualized interest rate be equal to 1% (with quarterly compounding). Carry out convergence studies by increasing the number of trials M (e.g. 100, 500, 1000, 5000, 10000). Use the volatility that you have also used in Case 1. How do your results compare with the results obtained in Case 1?

3 Dynamic hedging

The fundamental idea behind the Black-Scholes model is that of dynamic replication of the claim by taking positions in the underlying and the money market. In practice this means that a trader should apply a dynamic hedging strategy in order to ensure that the claim is replicated at expiry. In this part of the assignment we will apply a delta hedging strategy of a European call option. Suppose a trader sells a European call option on the S&P-500 with maturity 3 months and strike price USD 3,000 to an institutional investor. We assume the absence of dividends and a perfect Black-Scholes world. The contract size of the call option contract is 100 and the trader sells 10 contracts. The value of the S&P-500 index at trade date is USD 2,950. Use the volatility that you have also used in Case 1. The 3-month per annum interest rate is equal to 1% (with quarterly compounding).

(a) Calculate the amount of money the trader receives from the institutional investor. As we live in a perfect Black-Scholes world, you can do this by using the Black-Scholes option pricing formula.

The trader has a considerable short position in the call option on his book, i.e. if the underlying value goes up, the trader will be unhappy. So, the trader decides to setup a hedge portfolio.

(b) Setup the dynamic hedging strategy where the hedge is adjusted on a weekly basis (use 13 weeks). Generate 5,000 paths for the S&P-500 index (for the avoidance of doubt: each path has 13 weeks), evaluate the value development of both the hedging strategy and the call-option, and provide a histogram of the P&L of the trader after three months (for the total position of shorting the call-option and hedging this with a portfolio of stocks and cash). We live in a perfect Black-Scholes world so you can use the Black-Scholes delta (which is $N(d_1)$) to setup and adjust the replicating portfolio.

Note: we haven't discussed the 'Greeks' yet during our lectures. The Greeks (delta, gamma, vega, rho, theta) measure the sensitivity of an option price for changes in a variable or a parameter. Delta, for instance, measures the sensitivity of an option price for a change in the underlying value of the option. Mathematically, delta is calculated as:

$$\delta = \frac{\partial C}{\partial S}.$$

If we have an analytical option pricing formula available, as in the Black-Scholes model, then we can derive δ by calculating this partial derivative. This turns out to be $N(d_1)$.

(c) Repeat exercise (b) but now with a daily hedging frequency.

(d) Repeat exercise (c) but now with an (annual) volatility in your P&L simulation that is 5%-points higher than the volatility that you used to price the option. Comment on the results.

4 Stochastic volatility (BONUS QUESTION)

Suppose now that we believe that the S&P-500 index is better modeled by using a stochastic volatility model:

$$\begin{aligned} dS(t) &= \mu S(t)dt + \sigma_t S(t)dW^S(t) \\ d\sigma^2(t) &= -\kappa(\sigma_t^2 - \sigma)dt + \sigma_\sigma \sigma_t dW^V(t) \\ d[W^S(t), W^V(t)] &= \rho dt, \end{aligned}$$

where W^S and W^V are Brownian Motions under the real world probability measure $\mathbb{P}$.

Under the risk-neutral measure we assume the following process:

$$\begin{aligned} dS(t) &= rS(t)dt + \sigma_t S(t)d\tilde{W}^S(t) \\ d\sigma^2(t) &= -\kappa(\sigma_t^2 - \sigma)dt + \sigma_\sigma \sigma_t d\tilde{W}^V(t) \\ d[\tilde{W}^S(t)^S, \tilde{W}^V(t)] &= \rho dt. \end{aligned}$$

(a) Calculate the no-arbitrage price of the same call-option as in Exercise 2 using simulation. The starting level of volatility (annualized) is the same as you used in Exercise 2. You can assume that κ equals 3 and $\sigma_\sigma = 0.2$ (both on an annual basis). Use a time step of 1 day in your simulation. Hence, you need to generate 63 days of data to match the maturity of 0.25 years of the option.

Suppose an investor buys the call-option and hedges away the risk in the underlying by means of daily hedging program.

(b) Generate 5,000 scenarios for the return distribution of the investor's strategy. Comment on the value of the average return. What would you expect?

General information cases

The two cases of this course should be done in groups of two students. For each assignment, you have to write a short report where you must include the results, a short discussion/concluding section and the appendix with the codes. The reports should be submitted electronically, before the corresponding deadlines.

The preferred implementation tool is MatLab. During the allocated computer sessions, you will be able to get support and ask questions regarding your MatLab implementations. However, it is advised that you work on the assignments outside teaching hours and not only during the allocated sessions, as they will require significant time and effort.

In principle, you are not bound to MatLab and are free to choose the programming language/environment in which you would like to write your computer programs (so another alternative could be, e.g., R or Python). However, in this case we cannot guarantee implementation support.

The document 'Computer tutorials SPF 2019' which is published on Canvas is a good start with MatLab: it gives an introduction into this programming environment and gets you started with some simple exercises. YouTube is an excellent source of MatLab support and information.

Reporting requirements

- Report should be written in English. This means that you also should use English decimal notation in the text and in graphs.
- There is no maximum to the number of words or pages but please try to be concise but please make sure that it is clear to me what you did and what your interpretation of the results is.
- Figures and graphs should have captions such that they can be read independently from the text.
- Use your spelling checker.
- Put names and VU student numbers of all group members on the front page.
- Final reports should be uploaded on Canvas before 1pm CET on the deadline day.

Deadlines

The deadlines for the two cases are as follows:

- Case 1: Friday September 27, 2019 at 1pm CET
- Case 2: Monday October 14, 2019 at 1pm CET

Last but not least:

ENJOY AND GOOD LUCK!!