Case 2: Ito's Lemma, Option Pricing, Stochastic Volatility

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1 Introduction

This assignment will cover how Ito's Lemma is useful, how option pricing and dynamic hedging can be performed with simulation and what is the effect of stochastic volatility in option pricing.

2 Ito's Lemma

2.1

For this part, $S(t_i)$ and $s(T_i)$ are simulated using the given discrete-time approximations of Geometric Brownian Motion and exponential Geometric Brownian Motion.

Then, the distribution of terminal stock price is observed and it is proved that both processes ($S(t_N)$ and $exp(s(T_N))$) have same terminal distribution. One can observe histograms of both processes in Figure 1 and some characteristics in Table 1.

	$S(t_N)$	$exp(s(t_N))$
Mean	1.0011	1.0011
Standard Deviation	0.1513	0.1513
Skewness	0.4246	0.4246
Kurtosis	3.1318	3.1345

Table 1: Characteristics of two distributions ($S(t_N)$ and $exp(s(T_N))$)

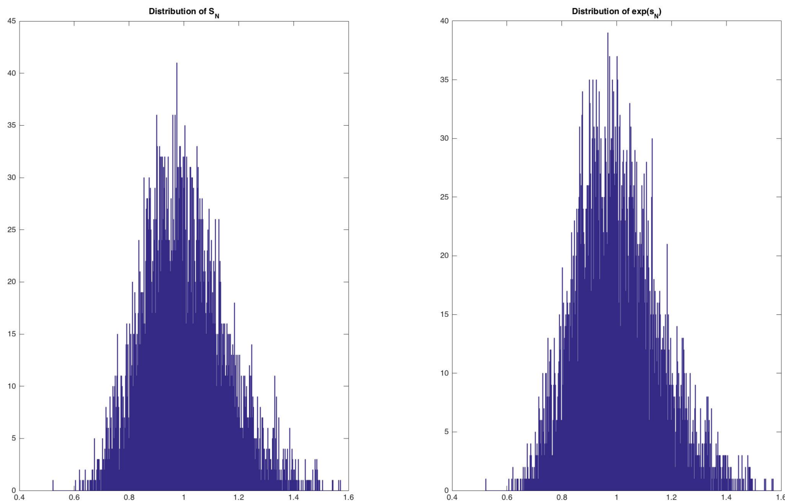

Figure 1: Distribution of $S(t_N)$ and $exp(s(T_N))$

2.2

For this part, log-returns of both processes are calculated with the following formulas:

- $ln(S(T_N)) - ln(S(T_0))$
- $ln(exp(s(T_N))) - ln(exp(s(T_0)))$

Then, it is proved that both log returns are normally distributed with same parameters. It is shown in Figure 2 and in Table 2.

	$S(t_N)$	$exp(s(t_N))$
Mean	-0.0101741	-0.0101692
Standard Deviation	0.15040	0.1539
Skewness	-0.0408	-0.0199
Kurtosis	2.9153	2.9150

Table 2: Characteristics of two distributions (log returns of $S(t_N)$ and $exp(s(T_N))$)

Table 2 shows that both distributions have negligible skewness and kurtosis very close to 3 and therefore normally distributed. It is expected that at terminal point ($t = T_N$), log returns should have a normal distribution with mean $r - 1/2 * \sigma^2$ and standard deviation of σ. Indeed, for these distributions; $r - 1/2 * \sigma^2 = 0.01125$ and $\sigma = 0.15$ which are almost identical with the values in Table 2.

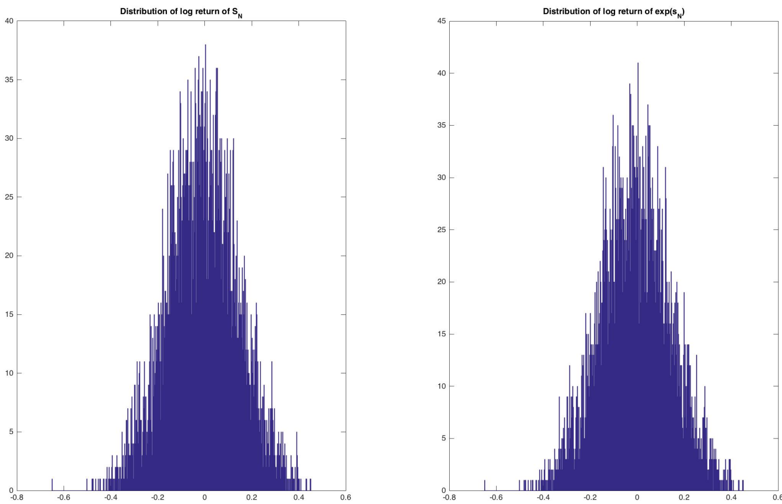

Figure 2: Distribution of log returns of $S(t_N)$ and $exp(s(T_N))$

2.3

For this part, one single path is observed for both processes with 252 and 1000 time steps for the purpose of observing the slight difference in values at large time steps.

Figure 3 is one single path with 1000 time steps for both processes. Even if they look same, there exists a difference. This difference is plotted in Figure 4. Then, the same procedure is applied to a process with 252 time steps and similar plots are observed in Figure 5 and Figure 6.

One can observe that deviation between log returns of $S(t_i)$ and $exp(s(T_N i))$ is greater when 252 time steps are used (Figure 6 has greater numbers than Figure 4).

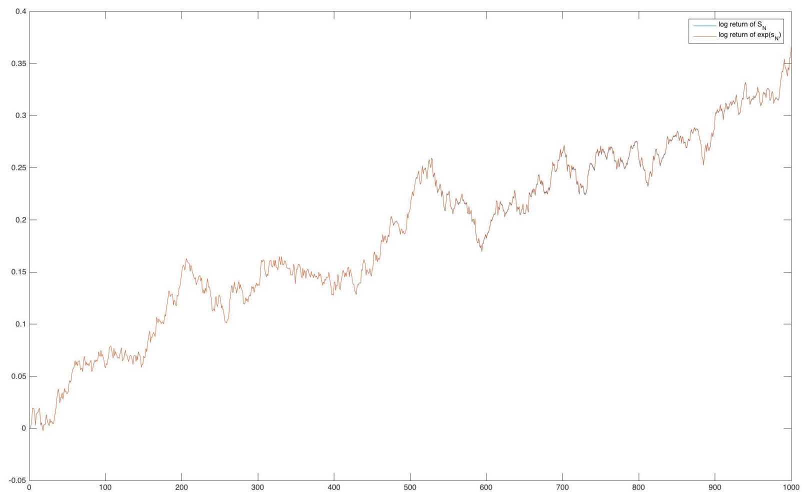

Figure 3: One single path for log returns of $S(t_i)$ and $exp(s(T_i))$ with 1000 time steps

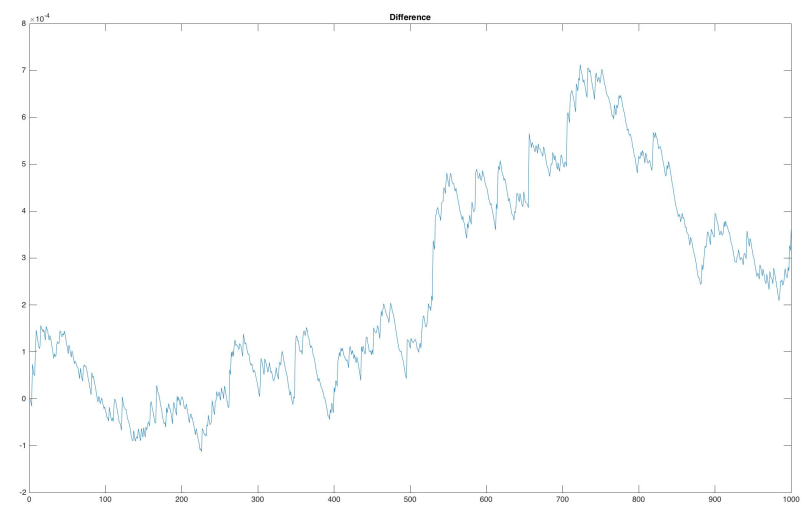

Figure 4: Difference of one single path for log returns of $S(t_i)$ and $exp(s(T_i))$ with 1000 time steps

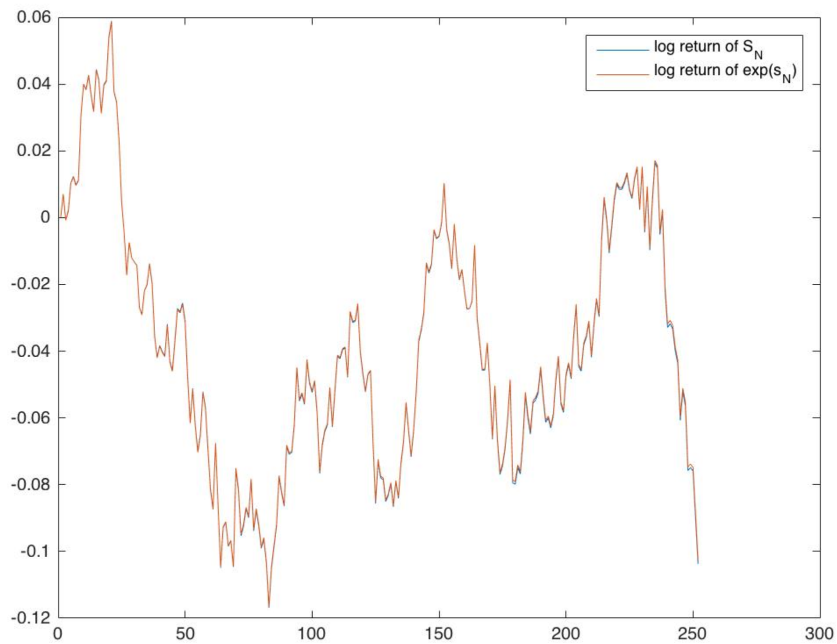

Figure 5: One single path for log returns of $S(t_i)$ and $exp(s(T_i))$ with 252 time steps

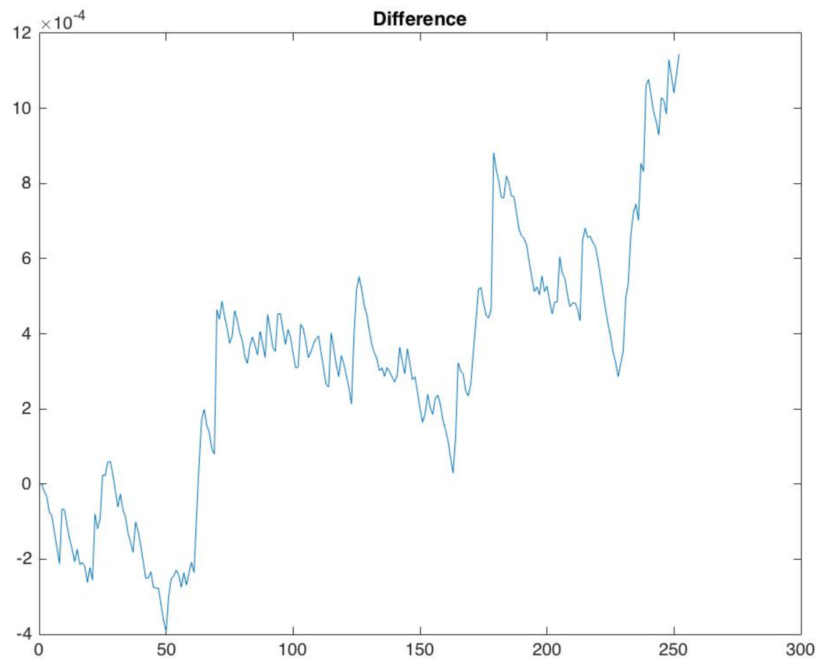

Figure 6: Difference of one single path for log returns of $S(t_i)$ and $exp(s(T_i))$ with 252 time steps

3 Monte Carlo simulation

In this section, stock price is simulated for a special case of GBM and a call option with 3000 strike, 3 months maturity, 1% quarterly compounding annualized interest rate and 14.24% volatility.

With increasing Monte Carlo trials, convergence of option price to Black-Scholes price and also price calculated in Case 1 is observed. Figure 7 shows this convergence:

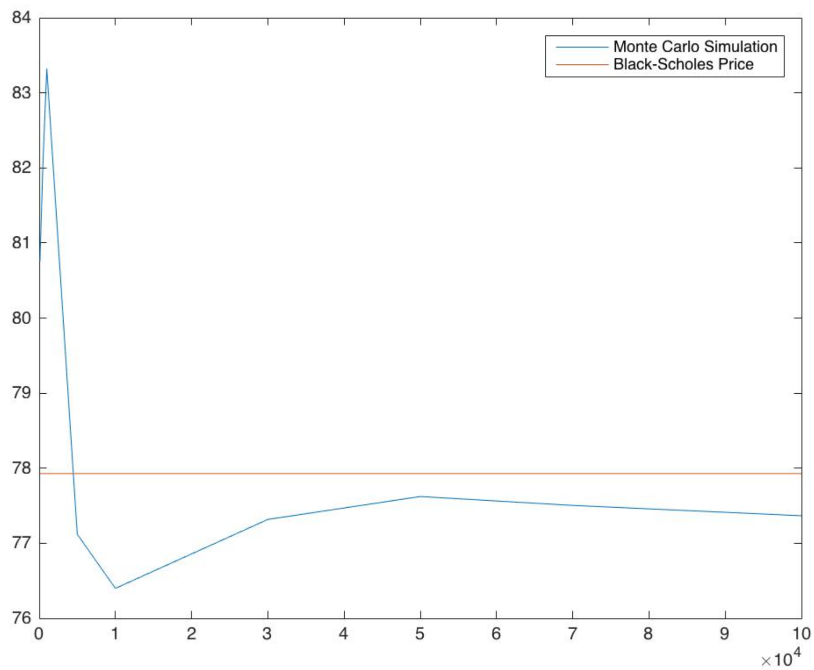

Figure 7: Call Option Price with increasing Monte Carlo trials vs. Black-Scholes Option Price

With 100,000 trials, calculated option price converged to 77.367 and BS option price is estimated to be 77.92

4 Dynamic hedging

This section investigates dynamic delta hedging of a call option on S&P 500 Index with 3 months maturity, 3000 strike, 2950 current price, 1% quarterly compounding annualized rate and 14.24% volatility for 10 contracts where each contract has size of 100.

4.1

Option price is calculated with the Black-Scholes function implemented in Appendix 6.2. With the given size, investor shorts the option and therefore receives premium equals to price of the option at $t = 0$.

The amount that investor received is calculated to be $**64822**.

4.2

In this section, a dynamic delta hedging procedure is applied. For this purpose, stock price is simulated with the same special case of GBM in Section 3 for 13 time steps (13 weeks). Then, Black-Scholes price and delta is calculated for each step for each path. For the delta hedging, a long cash position is initiated at $t = 0$ for the initial delta amount of 434.34 (43% per share). Then, cash position is adjusted (increased or decreased based on the change in delta) at each of the time steps for all paths. P%L at each step is calculated as follows:

$$P\&L_t = Delta_{t-1} * (StockPrice_t - StockPrice_{t-1})$$

Meaning that delta position from the previous time step results in a P&L based on the change in the stock price during that time step. Then, all those P&L's are summed up and carried to $t = T$ with multiplication of a factor of $exp(r * (T - t))$.

Final total P&L which consists the initial premium received, P%L from delta hedging and the terminal option value (which is a loss in case the short call option is exercised) is then calculated. Here is the histogram of the final P&L on 5000 paths (Figure 8):

As it can be observed from the Figure 8, the distribution has a mean value very close to zero. Then, the hedging frequency is increased from weekly to daily and the following histogram is observed (Figure 9):

Finally, the volatility used increased by 5% in daily hedging and the histogram in Figure 10 is observed:

All three cases are observed in Table 3:

Table 3 is very intuitive. As delta hedging is an approximation on the change of the option price, its frequency affects its success. Where mean value does not change significantly, standard deviation of the distribution decreased to its half by changing the frequency from weekly to daily. Then again very intuitively, increase in volatility causes a bigger change in stock price at each time step

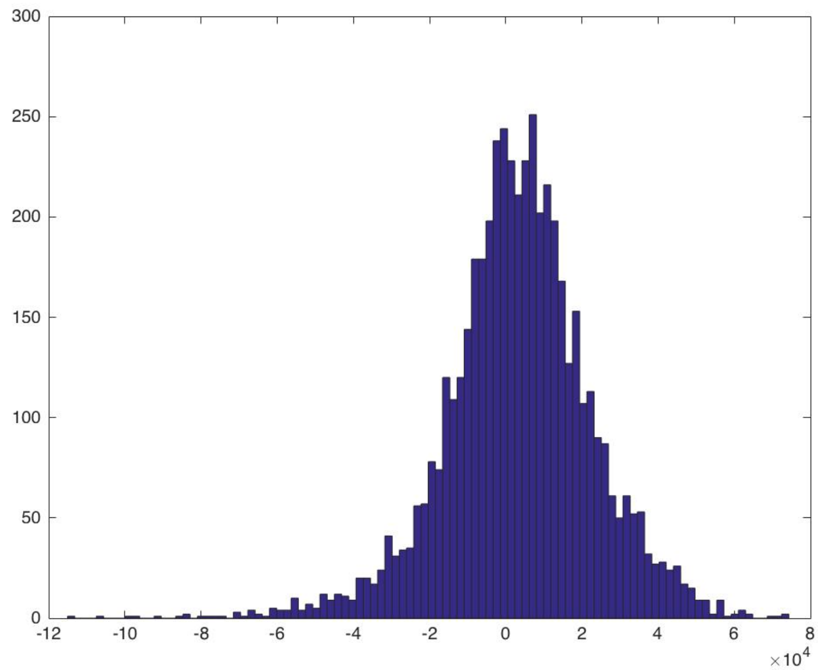

Figure 8: Histogram of Total P&L of Delta Hedged Option with weekly hedging

	Mean	**Standard Deviation**
Weekly Hedging	3140.8	19791
Daily Hedging	3301.3	9419.7
Daily Hedging with increased Volatility	3636.9	12865.7

Table 3: Comparison of Mean and Standard Deviation of different delta hedging cases

and therefore decreases success of delta hedging as delta approximation is less accurate now. Therefore, standard deviation has increased, as well.

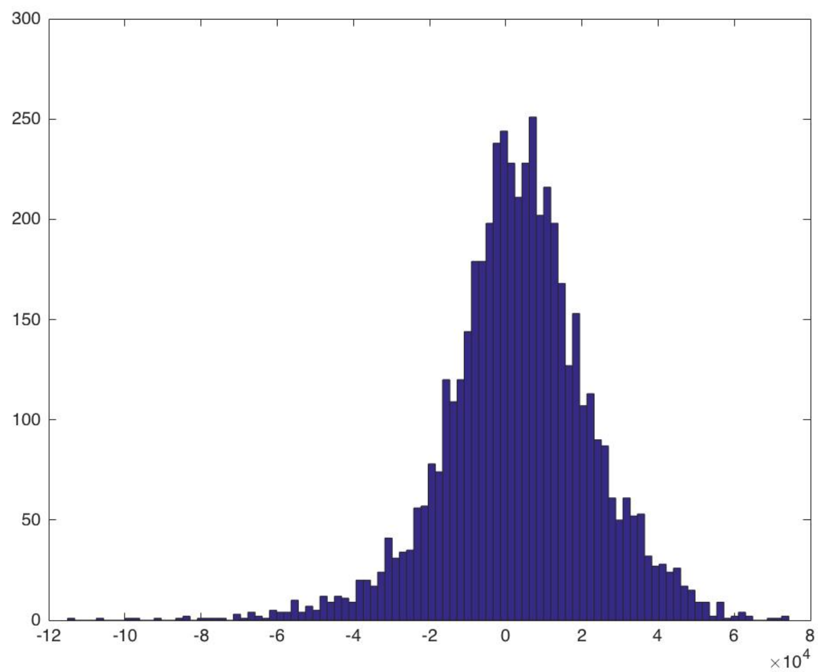

Figure 9: Histogram of Total P&L of Delta Hedged Option with daily hedging

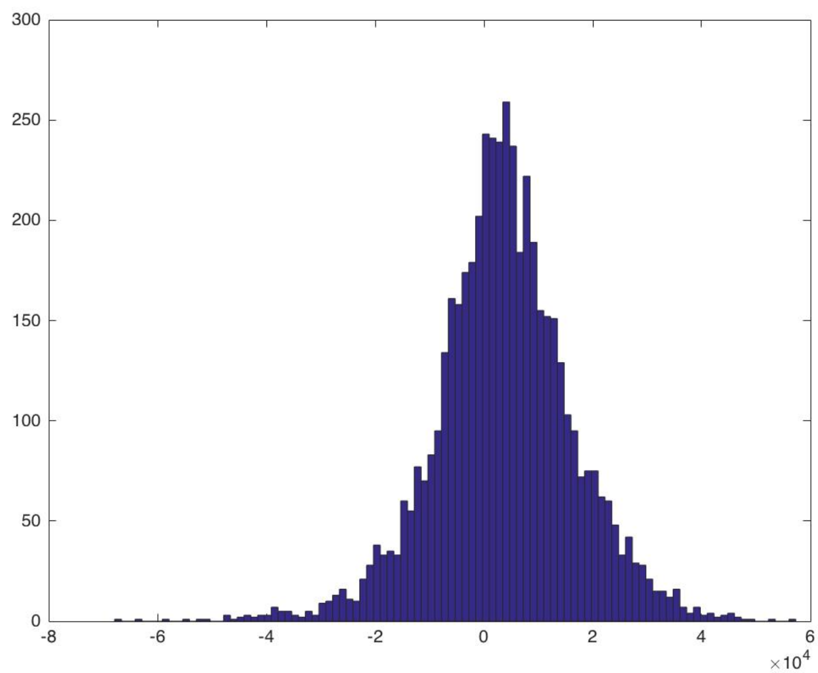

Figure 10: Histogram of Total P&L of Delta Hedged Option with daily hedging with increased volatility

5 Stochastic Volatility

In this section, stock prices are modelled with reducing the assumption of deterministic volatility. Therefore, stock price processes used in previous steps cannot be applied. After having some research, following discrete approximation to the stochastic volatility model (Heston) is applied (1):

$$S_t = S_{t-1} + \mu * S_{t-1} * dt + \sigma_{t-1}^2 * S_{t-1} * \sqrt{dt} * Z_t^S$$

$$\sigma_t^2 = \sigma_{t-1}^2 + \kappa * (\xi - \sigma_{t-1}^2) * dt + \sigma_\sigma * \sqrt{\sigma_{t-1}^2 * dt} * Z_t^V$$

$$Z_t^S = G_t^S$$

$$Z_t^V = \rho * G_t^S + \sqrt{1 - \rho^2} * G_t^V$$

where G_t^S and G_t^V are independent identically distributed standard normal random variables. Moreover, ρ is the correlation of two Brownian motions; σ_σ is the volatility of volatility; ξ is the long-term variance of of the Heston model and Z_t^S and Z_t^V are standard normal random variables.

With the model in hand, stock prices and volatilities are simulated with 63 time steps, κ of 3, σ_σ of 0.2, ρ of -0.3 and ξ of 0.04. The volatility simulation is initiated with σ of 0.1424 and stock price simulation is initiated with 2978.4

Then, the following plot is obtained (Figure 11). The plot shows that option price is between the initial volatility valuation with Black-Scholes and long-term volatility valuation with Black-Scholes.

Then, in the last step; a daily delta hedging is applied for a call option with the simulation of stock prices and volatility.

Here is the histogram of the returns in Figure 12:

The average return on the hedging strategy is observed to be around -25%.

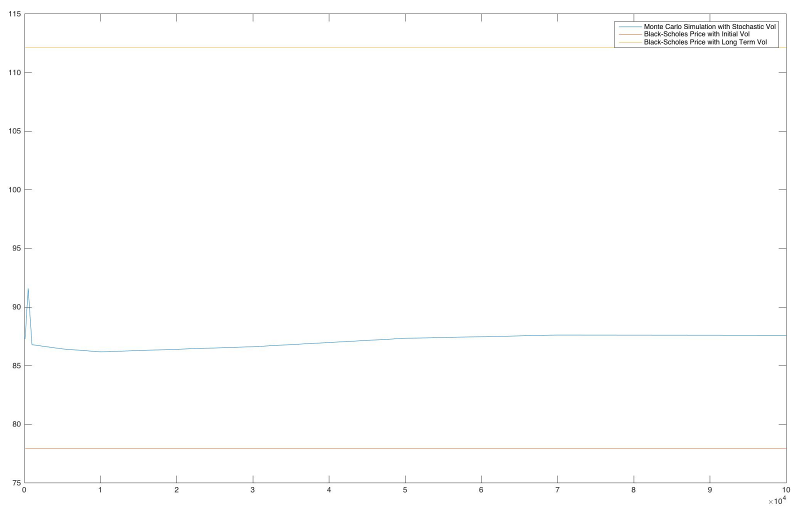

Figure 11: Option Price calculated with MC Simulation with Stochastic Volatility vs. Black-Scholes Price with initial volatility and long-term volatility

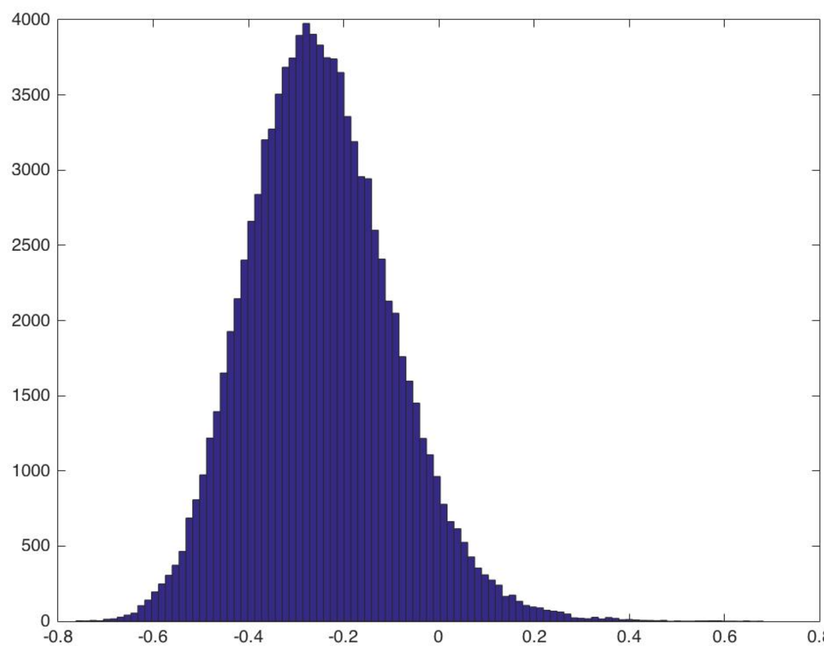

Figure 12: Histogram of Returns of daily delta hedging with Stochastic Volatility Model

6 Annex

6.1 Main Code

```matlab
%%Ito's lemma

S0=1;
r=0;
sigma = 0.15;
t0=0;
t_N = 1;
NumSteps = 252;
%NumSteps = 1000;

NumPaths = 5000;
S = zeros(NumPaths,NumSteps);
s = zeros(NumPaths,NumSteps);
S(:,1)=1;
s(:,1)=log(1);
time_step = (t_N-t0)/NumSteps;

for i=2:NumSteps
    brownian = randn(NumPaths,1);
    S(:,i) = S(:,i-1).*(1 + r*time_step + sigma*brownian*sqrt(time_step));
    s(:,i) = s(:,i-1) + (r-power(sigma,2)/2)*time_step + sigma*brownian*sqrt(time_step);
end

S_hat = exp(s);

S_N = S(:,NumSteps);
S_hat_N = S_hat(:,NumSteps);

S_0 = S(:,1);
S_hat_0 = S_hat(:,1);

%1
subplot(1,2,1)
hist(S_N,500);
title('Distribution of S_N')
subplot(1,2,2)
hist(S_hat_N,500);
title('Distribution of exp(s_N)')

mean(S_N)
```

```
mean(S_hat_N)

std(S_N)
std(S_hat_N)

skewness(S_N)
skewness(S_hat_N)

kurtosis(S_N)
kurtosis(S_hat_N)

%test if same distribution
kruskalwallis([S_N,S_hat_N])

%2
log_S_N_0 = log(S_N) - log(S_0);
log_S_hat_N_0 = log(S_hat_N) - log(S_hat_0);

subplot(1,2,1)
hist(log_S_N_0,500);
title('Distribution of log return of S_N')
subplot(1,2,2)
hist(log_S_hat_N_0,500);
title('Distribution of log return of exp(s_N)')

mean(log_S_N_0)
mean(log_S_hat_N_0)
%mean should be (r-1/2 sigma^2)
r - 0.5*sigma*sigma

std(log_S_N_0)
std(log_S_hat_N_0)
%std should be sigma
sigma

skewness(log_S_N_0)
skewness(log_S_hat_N_0)

kurtosis(log_S_N_0)
kurtosis(log_S_hat_N_0)

%3
%log returns
logS = log(S);
logShat = log(S_hat);
```

```
logreturnS = zeros(NumPaths,NumSteps);
logreturnShat = zeros(NumPaths,NumSteps);



for i=1:NumSteps
    logreturnS(:,i) = logS(:,i) - log(S_0);
    logreturnShat(:,i) = logShat(:,i) - log(S_hat_0);
end


plot(logreturnS(2,:)')
hold on

plot(logreturnShat(2,:)')
legend({'log return of S_N','log return of exp(s_N)'})

plot(logreturnShat(2,:)' - logreturnS(2,:)')
title('Difference')

%%%monte carlo
int_rate=0.01;
last_price = 2978.4;
compound_freq = 0.25;
option_maturity = 0.25;
annual_simple_int_rate = power((1+int_rate*compound_freq)
    ,1/compound_freq)-1;
sigma = 0.1424;
strike = 3000;
M = 100000;

discount  = exp(-annual_simple_int_rate*option_maturity);
MC = last_price*exp((annual_simple_int_rate - 0.5*power(
    sigma,2))*option_maturity+sigma*sqrt(option_maturity)*
    randn(M,1));
payoffs =  discount * max(MC-strike,0);
Ms= [100,500,1000,5000,10000,30000,50000,70000,100000];
payoff_list = [mean(payoffs(1:100,:)),mean(payoffs
    (1:500,:)),mean(payoffs(1:1000,:)),mean(payoffs
    (1:5000,:)),mean(payoffs(1:10000,:)),mean(payoffs
    (1:30000,:)),mean(payoffs(1:50000,:)),mean(payoffs
    (1:70000,:)),mean(payoffs(1:100000,:))];
[call_price,call_delta] = bs_call(last_price,strike,
    annual_simple_int_rate,option_maturity,sigma,1);

plot(Ms,payoff_list,Ms,call_price*ones(1,length(Ms)))
```

```
legend({'Monte Carlo Simulation','Black-Scholes Price'})


%Dynamic Hedge
strike  = 3000;
spot = 2950;
T = 0.25;
sigma = 0.1424;
%sigma = 0.1924;

int_rate= 0.01;
size = -100*10;
NumScenarios = 5000;
NumPeriods = 13;
%NumPeriods = 252/4;

[call_price,call_delta] = bs_call(spot,strike,int_rate,T,
    sigma,size);


SpotPrices  = zeros(NumScenarios,NumPeriods+1);
SpotPrices(:,1) = spot;

BSCallPrices = zeros(NumScenarios,NumPeriods+1);
BSCallPrices(:,1) = call_price;

Deltas = zeros(NumScenarios,NumPeriods+1);
Deltas(:,1) = call_delta;

PnL = zeros(NumScenarios,NumPeriods+1);

for i=2:(NumPeriods+1)
    time_to_mat = T*((NumPeriods+1-i)/NumPeriods);
    passed_time = T - time_to_mat;
    SpotPrices(:,i) = SpotPrices(:,i-1) .* exp((int_rate
        - 0.5*power(sigma,2))*T/NumPeriods+sigma*sqrt(T/
        NumPeriods)*randn(NumScenarios,1));
    [BSCallPrices(:,i),Deltas(:,i)] = bs_call(SpotPrices
        (:,i),strike,int_rate,time_to_mat,sigma,size);
    PnL(:,i) = PnL(:,i-1) + exp(int_rate*time_to_mat) *
        Deltas(:,i-1) .* (SpotPrices(:,i) - SpotPrices(:,i
        -1));
end

Final_PnL = -exp(int_rate*T) * BSCallPrices(:,1) + PnL(:,
    NumPeriods+1) + BSCallPrices(:,NumPeriods+1);
```

```
plot(BSCallPrices(1,:))
hold on
plot(PnL(1,:))
legend({'Short Call P&L','Delta Hedge P&L'})

hist(Final_PnL,100)
mean(Final_PnL)
std(Final_PnL)



%heston
int_rate=0.01;
last_price = 2978.4;
compound_freq = 0.25;
option_maturity = 0.25;
rate = power((1+int_rate*compound_freq),1/compound_freq)
    -1;
M = 100000;
delta_t = 1/252;
num_periods = option_maturity / delta_t;


sigma_initial = 0.1424;
strike = 3000;
k = 3;
vol_vol = 0.2;
p=-0.3;
long_term_sigma = 0.2;
long_term_var = power(long_term_sigma,2);

stock_prices = zeros(M,num_periods+1);
stock_prices(:,1) = last_price;

variances = zeros(M,num_periods+1);
variances(:,1) = power(sigma_initial,2);

for i=2:(num_periods+1)

    g_s = randn(M,1);
    g_v = randn(M,1);
    rand1 = g_s;
    rand2 = p*g_s + sqrt(1-power(p,2))*g_v;

    stock_prices(:,i) = stock_prices(:,i-1).*(1 + rate*
```

```
delta_t + sqrt(variances(:,i-1)*delta_t).*rand1);
    variances(:,i) = variances(:,i-1) + k*(long_term_var - variances(:,i-1))*delta_t + vol_vol*sqrt(variances(:,i-1)*delta_t).*rand2;
end

discount  = exp(-rate*option_maturity);
payoffs =  discount * max((stock_prices(:,num_periods+1)-strike),0);
Ms= [100,500,1000,5000,10000,30000,50000,70000,100000];
payoff_list = [mean(payoffs(1:100,:)),mean(payoffs(1:500,:)),mean(payoffs(1:1000,:)),mean(payoffs(1:5000,:)),mean(payoffs(1:10000,:)),mean(payoffs(1:30000,:)),mean(payoffs(1:50000,:)),mean(payoffs(1:70000,:)),mean(payoffs(1:100000,:))];

[call_price1,call_delta1] = bs_call(last_price,strike,rate,option_maturity,sigma_initial,1);
[call_price2,call_delta2] = bs_call(last_price,strike,rate,option_maturity,long_term_sigma,1);

plot(Ms,payoff_list,Ms,call_price1*ones(1,length(Ms)),Ms,call_price2*ones(1,length(Ms)))
legend({'Monte Carlo Simulation with Stochastic Vol','Black-Scholes Price with Initial Vol','Black-Scholes Price with Long Term Vol'})

BSCallPrices = zeros(M,num_periods+1);
BSCallPrices(:,1) = call_price;

Deltas = zeros(M,num_periods+1);
Deltas(:,1) = call_delta;

PnL = zeros(M,num_periods+1);

for i=2:(num_periods+1)
    time_to_mat = option_maturity*((num_periods+1-i)/num_periods);
    passed_time = option_maturity - time_to_mat;
    [BSCallPrices(:,i),Deltas(:,i)] = bs_call(stock_prices(:,i),strike,rate,time_to_mat,sqrt(variances(:,i)),1);
    PnL(:,i) = PnL(:,i-1) + exp(rate*time_to_mat) * Deltas(:,i-1) .* (stock_prices(:,i) - stock_prices(:,i-1));
end
```

```
Final_PnL = -exp(int_rate*option_maturity) * BSCallPrices(:,1) + PnL(:,num_periods+1) + BSCallPrices(:,num_periods+1);
returns = Final_PnL./BSCallPrices(:,1);

mean(returns)
std(returns)
hist(returns,100)
```

6.2 BS-Call

```
function [f,delta] = bs_call(price,strike,int_rate,expiry,vol,size)
    lso = (log(price/strike)+(int_rate+(power(vol,2))/2)*expiry);
    d1 = lso./(vol*sqrt(expiry));
    d2 = d1 - vol*sqrt(expiry);
    f = size*(price.*normcdf(d1)-strike*exp(-int_rate*expiry)*normcdf(d2));
    delta = -size*normcdf(d1);
end
```

References

[1] Patrik Karlsson. *The Heston Model,Stochastic Volatility and Approximation.* Department of Economics, Lund University.