

Week 1 Notes and Exercises

Network Flow Diagrams

$$400 + x_2 = x_1$$

$$400 = x_1 - x_2$$

$$x_1 + x_3 - x_4 = 600$$

$$x_4 + x_5 = 100$$

$$x_2 + x_3 + x_5 = 300$$

Solutions to Systems of Linear Equations

A unique solution(Consistent)

$$x - 2y = -1$$

$$x = 2y - 1$$

$$-2y + 1 + 3y = 3$$

$$y = 2$$

$$x - 4 = -1$$

$$x = 3$$

$$(x, y) = (3, 2)$$

Infinitely many solutions(Consistent)

$$x - 2y = -1$$

$$-x + 2y = 3$$

$$x = 2y - 1$$

$$-2y + 1 + 2y = 3$$

$$0y = 2$$

No solutions(Inconsistent)

$$x - 2y = -1$$

$$-x + 2y = 1$$

$$x = 2y - 1$$

$$-2y + 1 + 2y = 1$$

$$0y = 0$$

Linear Equations

Method 1

$$1) x_1 - 2x_2 = -1$$

$$2) -x_1 + 3x_2 = 3$$

Rewrite (1) as:

$$3) x_1 = -1 + 2x_2$$

Sub (3) into (2):

$$-(-1 + 2x_2) + 3x_2 = 3$$

$$1 - 2x_2 + 3x_2 + 2 = 3$$

$$1 + x_2 = 3$$

$$x_2 = 3 - 1$$

$$4) x_2 = 2$$

Sub (4) into (1):

$$x_1 - 2(2) = -1$$

$$x_1 - 4 = -1$$

$$x_1 = -1 + 4$$

$$x_1 = 3$$

Method 2

$$1) x_1 - 2x_2 + x_3 = 0$$

$$2) 2x_2 - 8x_3 = 8$$

$$3) -4x_2 + 5x_3 = -9$$

Multiply (1) by 4

$$4x_1 - 8x_2 + 4x_3 = 0$$

$$-4x_1 + 5x_2 + x_3 = -9$$

Add both

$$-3x_2 + 5x_3 = -9$$

$$2x_2 - 8x_3 = 8$$

Make the coefficient the same (By multiplication)

$$-6x_2 + 10x_3 = -18$$

$$6x_2 - 24x_3 = 24$$

Addition

$$-14x_3 = 6$$

$$x_3 = \frac{-6}{14}$$

Exercise

Problem 1

$$1) x_1 - 3x_3 = 8$$

$$2) 2x_1 + 2x_2 + 9x_3 = 7$$

$$3) x_2 + 5x_3 = -2$$

$$x_1 = 8 + 3x_3$$

$$x_2 = -2 - 5x_3$$

Sub

$$2(8 + 3x_3) + 2(-2 - 5x_3) + 9x_3 = 7$$

$$16 + 6x_3 - 4 - 10x_3 + 9x_3 = 7$$

$$6x_3 - 10x_3 + 9x_3 = 7 - 16 + 4$$

$$5x_3 = -5$$

$$x_3 = -1$$

$$x_1 - 3(-1) = 8$$

$$x_1 = 8 - 3$$

$$x_1 = 5$$

$$2x_2 + 18(-1) = -9$$

$$2x_2 = 6$$

$$x_2 = 3$$

$$(x_1, x_2, x_3) = (5, 3, -1)$$

Problem 2

$$1) \ x_2 + 4x_3 = -5$$

$$2) \ x_1 + 3x_2 + 5x_3 = -2$$

$$3) \ 3x_1 + 7x_2 + 7x_3 = 4$$

$$x_2 = -5 - 4x_3$$

$$x_1 = -3(-5 - 4x_3) - 5x_3$$

$$x_1 = 15 + 4x_3 - 5x_3$$

$$x_1 = -x_3 + 15$$

$$3(-x_3 + 15) + 7(-5 - 4x_3) + 7x_3 = 4$$

$$-3x_3 + 45 - 35 - 28x_3 + 7x_3 = 4$$

$$-24x_3 = -6$$

$$x_3 = \frac{1}{4}$$

$$x_2 + 4(\frac{1}{4}) = -5$$

$$x_2 = -6$$

$$x_1 + 3(-6) + 5(\frac{1}{4}) = -2$$

$$x_1 = -2 + 18 - \frac{5}{4}$$

$$x_1 = 15\frac{1}{4}$$

$$(x_1, x_2, x_3) = (15\frac{1}{4}, -6, \frac{1}{4})$$

Matrices Example

Reduced Row Echelon Form

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array} \right]$$

$$4R_1 + R_3$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array} \right]$$

$$\frac{1}{2}R_2$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{array} \right]$$

$$3R_2 + R_3$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array} \right]$$

$$R_2 + 4R_3$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array} \right]$$

$$R_1 - R_3$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array} \right]$$

$$x_1 = 29, x_2 = 16, x_3 = 3$$

More examples

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 32 \\ 0 & 1 & 2 & 16 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

It is a reduced row echelon.

Let $x_3 = S$

$$x_1 = 32 - S$$

$$x_2 = 16 - 2S$$

$$x_3 = 0 + S$$

Exercises

Solve each system in Exercise 1-4 by using elementary row operations on the

equations or on the augmented matrix. Follow the systematic elimination procedure described in this section.

$$1) \ x_1 + 5x_2 = 7, \ -2x_1 - 7x_2 = -5$$

$$x_1 = -5x_2 + 7$$

$$-2(-5x_2 + 7) - 7x_2 = -5$$

$$10x_2 - 14 - 7x_2 = -5$$

$$3x_2 = 9$$

$$x_2 = 3$$

$$x_1 = -5(3) + 7$$

$$x_1 = -15 + 7$$

$$x_1 = -8$$

$$\left[\begin{array}{cc|c} 1 & 5 & 7 \\ -2 & -7 & -5 \end{array} \right]$$

$$R_2 + 2R_1$$

$$\left[\begin{array}{cc|c} 1 & 5 & 7 \\ -2 + 2(1) & -7 + 2(5) & -5 + 2(7) \end{array} \right]$$

$$\left[\begin{array}{cc|c} 1 & 5 & 7 \\ 0 & 3 & 9 \end{array} \right]$$

$$\frac{1}{3}R_2$$

$$\left[\begin{array}{cc|c} 1 & 5 & 7 \\ 0 * \frac{1}{3} & 3 * \frac{1}{3} & 9 * \frac{1}{3} \end{array} \right]$$

$$\left[\begin{array}{cc|c} 1 & 5 & 7 \\ 0 & 1 & 3 \end{array} \right]$$

$$R_1 + -5R_2$$

$$\left[\begin{array}{cc|c} 1 + -5(0) & 5 + -5(1) & 7 + -5(3) \\ 0 & 1 & 3 \end{array} \right]$$

$$\left[\begin{array}{cc|c} 1 & 0 & -8 \\ 0 & 1 & 3 \end{array} \right]$$

$$(x_1, x_2) = (-8, 3)$$

$$2) \ 3x_1 + 6x_2 = -3, \ 5x_1 + 7x_2 = 10$$

$$3x_1 = -6x_2 - 3$$

$$x_1 = -2x_2 - 1$$

$$5(-2x_2 - 1) + 7x_2 = 10$$

$$-10x_2 - 5 + 7x_2 = 10$$

$$-3x_2 = 15$$

$$x_2 = -5$$

$$x_1 = -2(-5) - 1$$

$$x_1 = 9$$

$$\left[\begin{array}{cc|c} 3 & 6 & -3 \\ 5 & 7 & 10 \end{array} \right]$$

3) Find the point (x_1, x_2) that lies on the line $x_1 + 2x_2 = 4$ and on the line

$$x_1 - x_2 = 1.$$

$$x_1 + 2x_2 = 4$$

$$x_1 - x_2 = 1$$

$$\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 1 & -1 & 1 \end{array} \right]$$

$$R_2 - R_1$$

$$\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 1-1 & -1-2 & 1-4 \end{array} \right]$$

$$\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & -3 & -3 \end{array} \right]$$

$$-\frac{1}{3}R_2$$

$$\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 * -\frac{1}{3} & -3 * \frac{1}{3} & -3 * -\frac{1}{3} \end{array} \right]$$

$$\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 1 & 1 \end{array} \right]$$

$$R_1 - 2R_2$$

$$\left[\begin{array}{cc|c} 1 - 2(0) & 2 - 2(1) & 4 - 2(1) \\ 0 & 1 & 1 \end{array} \right]$$

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array} \right]$$

$$(x_1, x_2) = (2, 1)$$

4) Find the point of intersection of the lines $x_1 + 2x_2 = -13$ and $3x_1 - 2x_2 = 1$.

$$x_1 + 2x_2 = -13$$

$$3x_1 - 2x_2 = 1$$

$$\left[\begin{array}{cc|c} 1 & 2 & -13 \\ 3 & -2 & 1 \end{array} \right]$$

$$R_2 - 3R_1$$

$$\left[\begin{array}{cc|c} 1 & 2 & -13 \\ 3 - 3(1) & -2 - 3(2) & 1 - 3(-13) \end{array} \right]$$

$$\left[\begin{array}{cc|c} 1 & 2 & -13 \\ 0 & -8 & -38 \end{array} \right]$$

$$R_1 - \frac{1}{4}(R_2)$$

$$\left[\begin{array}{cc|c} 1 - \frac{1}{4}(0) & 2 - \frac{1}{4}(-8) & -13 - \frac{1}{4}(-38) \\ 0 & -8 & -38 \end{array} \right]$$

$$\left[\begin{array}{cc|c} 1 & 0 & 22\frac{1}{2} \\ 0 & -8 & -38 \end{array} \right]$$

$$-\frac{1}{8}(R_2)$$

$$\left[\begin{array}{cc|c} 1 & 0 & 22\frac{1}{2} \\ 0 * -\frac{1}{8} & -8 * -\frac{1}{8} & -38 * -\frac{1}{8} \end{array} \right]$$

$$\left[\begin{array}{cc|c} 1 & 0 & 22\frac{1}{2} \\ 0 & 1 & 4\frac{3}{4} \end{array} \right]$$

$$(x_1, x_2) = (22\frac{1}{2}, 4\frac{3}{4})$$

17) Do the three lines $2x_1 + 3x_2 = -1$, $6x_1 + 5x_2 = 0$, and $2x_1 - 5x_2 = 7$ have a common point of intersection? Explain.

$$\left[\begin{array}{cc|c} 2 & 3 & -1 \\ 6 & 5 & 0 \\ 2 & -5 & 7 \end{array} \right]$$

18) Do the three planes $2x_1 + 4x_2 + 4x_3 = 4$, $x_2 - 2x_3 = -2$, and $2x_1 + 3x_2 = 0$ have at least one common point of intersection? Explain.

$$\left[\begin{array}{ccc|c} 2 & 4 & 4 & 4 \\ 0 & 1 & -2 & -2 \\ 2 & 3 & 0 & 0 \end{array} \right]$$

$$R_3 - \frac{1}{2}(R_2)$$

$$\left[\begin{array}{ccc|c} 2 & 4 & 4 & 4 \\ 0 & 1 & -2 & -2 \\ 0 - \frac{1}{2}(0) & 3 - \frac{1}{2}(-2) & 0 - \frac{1}{2}(-2) & 0 - \frac{1}{2}(-2) \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 2 & 4 & 4 & 4 \\ 0 & 1 & -2 & -2 \\ 0 & 4 & 1 & 1 \end{array} \right]$$