Linear Algebra: Week 4 Notes and Exercises

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1 Notes

Notes

Vector Space
1) $\vec{0} \in V$
2) $\vec{u}, \vec{v} \in V$ then must have $\vec{u} + \vec{v} \in V$
3) $c \in \mathbb{R}, \vec{u} \in V$ then must have $c\vec{u} \in V$
Exercise

2) a) True because scaling $\vec{u}$ with either a positive or negative value will still be in W.
$\vec{u} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, x_1 \geq 0, y_1 \geq 0$
$c \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} cx_1 \\ cy_1 \end{bmatrix}$
$c \geq 0 \Rightarrow cx_1 \geq 0 \Rightarrow cy_1 \geq 0$
$c \leq 0 \Rightarrow cx_1 \leq 0 \Rightarrow cy_1 \leq 0$
b)Let $\vec{u} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} -1 \\ -8 \end{bmatrix}$ $\therefore \vec{u} + \vec{v} = \begin{bmatrix} 3 \\ -6 \end{bmatrix}$ which is not in any of the quadrants associated with $W \therefore W$ is not a vector space .

6) $p(t) = a + t^2$
$V = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \in \mathbb{R}^3 | x_1 \in \mathbb{R}, x_2 = 0, x_3 = 1$
$\begin{bmatrix} a & 0 & 1 \end{bmatrix}$ Its not a vector space because scaling or adding vectors will change the x_3.

2 Exercises

1. Let V be the first quadrant in the xy-plane; that is let $V = \{ \begin{bmatrix} x \\ y \end{bmatrix} : x \geq 0, y \geq 0\}$

a) If u and v are in V, is $u + v$ in V? Why?
If $\vec{u}, \vec{v} \in V$ is $\vec{u} + \vec{v} \in V$
$\vec{u} \in V \Rightarrow \vec{u} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, x_1 \geq 0, y_1 \geq 0$
$\vec{v} \in V \Rightarrow \vec{v} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}, x_2 \geq 0, y_2 \geq 0$
$\vec{u} + \vec{v} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix} = x_1 + x_2 \geq 0, y_1 + y_2 \geq 0$
$\Rightarrow \vec{u} + \vec{v} \in V$
b) Find a specific vector u in V and a specific scalar c such that cu is not in V. (This is enough to show that V is not a vector space.)
Let $\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ then $\vec{u} \in V$,
Let $c = -1$ then $c\vec{u} = -1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \notin V$

2. Let W be the union of the first and third quadrants in the xy-plane. That is, let $W = \{ \begin{bmatrix} x \\ y \end{bmatrix} : xy \geq 0\}$

a) If u is in W and c is any scalar, is cu in W? Why?
If $\vec{u} \in W, c \in \mathbb{R}$, is $c\vec{u} \in W$
Let $\vec{u} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, x_1 y_1 \geq 0$, Let $c \in \mathbb{R}$
$cu = c \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} cx_1 \\ cy_1 \end{bmatrix}$
$(cx_1)(cy_1) = c^2 x_1 y_1$
But $c^2 \geq 0, \forall c \in \mathbb{R}$
$c^2 x_1 y_1 \geq 0 \Rightarrow c\vec{u} \in W$
b) Find specific vectors u and v in W such that $u + v$ is not in W. This is enough to show that W is not a vector space.
Let $\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, let $\vec{v} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$
Then $\vec{u} + \vec{v} = \begin{bmatrix} 1-2 \\ 2-1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \notin W, since (-1)(1) = -1$

In Exercise 5-8, determine if the given set is a subspace of $\mathbb{P}_n$ for an appropriate value of n. Justify your answers.
All polynomials $p(t) = at^2$, $a \in \mathbb{R}$
5. All polynomials of the form $p(t) = at^2$, where a is in $\mathbb{R}$
If $a = 0, p(t) = 0t^2 = \vec{0} \therefore \vec{0} \in \{p(t)\}$
If $u(t) = a_1 t^2$, $v(t) = a_2 t^2$, then $\vec{u} + \vec{v} = a_1 t^2 + a_2 t^2 = (a_1 + a_2)t^2 \in \{p(t)\}$
If $u(t) = a_1 t^2$ and $c \in \mathbb{R}$, $c\vec{u} = ca_1 t^2 = (ca_1)t^2 \in \{p(t)\}$
$\therefore \{p(t)\}$ is a subspace
6. All polynomials of the form $p(t) = a + t^2$, where a is in $\mathbb{R}$
$\vec{0} = 0 + 0t + t^2$ is not in $\{p(t)\}$
If $n(t) = a_1 + t^2$, $c = 0$
$c\vec{u} = 0(a_1 + t^2) = 0 \notin \{p(t)\}$
$\therefore$ not in subspace
7. All polynomials of degree at most 3, with integers of coefficients.
If the scalars are any number in $\mathbb{R}$ then $c = \frac{1}{10}$, $p(t) = 1 + x + x^2 \Rightarrow cp(t) = \frac{1}{10} + \frac{1}{10}x + \frac{1}{10}x^2 \notin$ the set.
But if we restrict scalars to the set of integers then it will be a subspace.
8. All polynomials in $\mathbb{P}_n$ such that $p(0) = 0$.
$\vec{0} = 0 + 0t + 0t^2 \cdots, \vec{0} = 0$
For $u(t) = a_0 + a_1 t + \cdots, u(0) = 0$
$v(t) = b_0 + b_1 t + \cdots, v(0) = 0$
$u(t) + v(t) = a_0 + b_0 + (a_1 + b_1)t + \cdots, (u + v) = 0$
if $u(0) = 0$, $c \in \mathbb{R}$ then $cu(0) = c(0) = 0$
$\therefore$ is a subspace.
9. Let H be the set of all vectors of the form $\begin{bmatrix} -2t \\ 5t \\ 3t \end{bmatrix}$. Find a vector v in $\mathbb{R}^3$ such that $H = Span\{v\}$.
Why does this show that H is a subspace of $\mathbb{R}^3$?
$\vec{v} = t \begin{bmatrix} -2 \\ 5 \\ 3 \end{bmatrix}$
$t = 0 \Rightarrow \begin{bmatrix} -2(0) \\ 5(0) \\ 3(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in H$, so $\vec{0} \in H$
$\vec{u} = \begin{bmatrix} -2t_1 \\ 5t_1 \\ 3t_1 \end{bmatrix}, \vec{v} = \begin{bmatrix} -2t_2 \\ 5t_2 \\ 3t_2 \end{bmatrix} \Rightarrow \vec{u} + \vec{v} = \begin{bmatrix} -2(t_1 + t_2) \\ 5(t_1 + t_2 \\ 3(t_1 + t_2) \end{bmatrix} \in H$
$\forall c \in \mathbb{R}\ c\vec{u} = \begin{bmatrix} -2(ct) \\ 5(ct) \\ 3(ct) \end{bmatrix} \in H$
$\therefore H$ is a subspace.

Which of the following sets are linearly independent?

Which form a basis for $\mathbb{R}^3$

$\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}2\\3\\0\end{bmatrix} \begin{bmatrix}1 & 2 & \mid & 0\\0 & 3 & \mid & 0\\0 & 0 & \mid & 0\end{bmatrix} \frac{1}{2}R_2 = \begin{bmatrix}1 & 2 & \mid & 0\\0 & 1 & \mid & 0\\0 & 0 & \mid & 0\end{bmatrix} R_1 - R_2 = \begin{bmatrix}1 & 0 & \mid & 0\\0 & 1 & \mid & 0\\0 & 0 & \mid & 0\end{bmatrix}$

t has no free parameter.

$\therefore$ are linearly independent.

$\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}2\\3\\0\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}$

$\begin{bmatrix}1 & 2 & 4 & \mid & 0\\0 & 3 & 5 & \mid & 0\\0 & 0 & 6 & \mid & 0\end{bmatrix} \frac{1}{6}R_3 = \begin{bmatrix}1 & 2 & 4 & \mid & 0\\0 & 3 & 5 & \mid & 0\\0 & 0 & 1 & \mid & 0\end{bmatrix} R_1 - 4R_3 = \begin{bmatrix}1 & 2 & 0 & \mid & 0\\0 & 3 & 5 & \mid & 0\\0 & 0 & 1 & \mid & 0\end{bmatrix} R_2 - 5R_3 = \begin{bmatrix}1 & 2 & 0 & \mid & 0\\0 & 3 & 0 & \mid & 0\\0 & 0 & 1 & \mid & 0\end{bmatrix}$

$\frac{1}{3}R_2 = \begin{bmatrix}1 & 2 & 0 & \mid & 0\\0 & 1 & 0 & \mid & 0\\0 & 0 & 1 & \mid & 0\end{bmatrix} R_1 - 2R_2 = \begin{bmatrix}1 & 0 & 0 & \mid & 0\\0 & 1 & 0 & \mid & 0\\0 & 0 & 1 & \mid & 0\end{bmatrix}$

t has no free parameter, $\therefore$ are linearly independent.

$\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}2\\3\\0\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}, \begin{bmatrix}7\\8\\9\end{bmatrix}$

4 Vectors in $\mathbb{R}^3$ must have at least 1 free parameter. $\therefore$ linearly dependent.