Linear Algebra: Week 3 Notes and Exercises

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May 24, 2018

1 Notes

Exercises

1) Find the length of the vectors $a = \begin{bmatrix} -10 \\ 5 \end{bmatrix}$ and $b = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$

$||\vec{a}|| = \sqrt{(-10)^2 + 5^2}$
$||\vec{a}|| = \sqrt{125}$
$||\vec{b}|| = \sqrt{(3)^2 + (3)^2}$
$||\vec{b}|| = \sqrt{18}$

2) Find the magnitude of the vectors $r = \begin{bmatrix} 7 \\ -3 \end{bmatrix}$ and $q = \begin{bmatrix} -3 \\ 7 \end{bmatrix}$

$||\vec{r}|| = \sqrt{(7)^2 + (-3)^2}$
$||\vec{r}|| = \sqrt{58}$
$||\vec{q}|| = \sqrt{(-3)^2 + (7)^2}$
$||\vec{q}|| = \sqrt{58}$

3) Normalize the vectors in $\mathbb{R}^2$ $a = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$ and $b = \begin{bmatrix} 5 \\ -4 \end{bmatrix}$

$||\vec{a}|| = \sqrt{(4)^2 + (5)^2}$
$||\vec{a}|| = \sqrt{41}$
$\frac{\vec{a}}{||\vec{a}||} = \frac{\begin{bmatrix} 4 & 5 \end{bmatrix}}{\sqrt{41}} = \begin{bmatrix} \frac{4}{\sqrt{41}} & \frac{5}{\sqrt{41}} \end{bmatrix}$
$||\vec{b}|| = \sqrt{(5)^2 + (-4)^2}$
$||\vec{b}|| = \sqrt{41}$
$\frac{\vec{b}}{||\vec{b}||} = \frac{\begin{bmatrix} 5 & -4 \end{bmatrix}}{\sqrt{41}} = \begin{bmatrix} \frac{5}{\sqrt{41}} & \frac{-4}{\sqrt{41}} \end{bmatrix}$

4) Normalize the vectors in $\mathbb{R}^2$ $q = \begin{bmatrix} -3 \\ 6 \end{bmatrix}$ and $p = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$

$||\vec{q}|| = \sqrt{(-3)^2 + (6)^2}$
$||\vec{q}|| = \sqrt{41}$
$\frac{\vec{q}}{||\vec{q}||} = \begin{bmatrix} \frac{-3}{\sqrt{41}} & \frac{6}{\sqrt{41}} \end{bmatrix}$
$||\vec{p}|| = \sqrt{(-1)^2 + (-1)^2}$
$||\vec{p}|| = \sqrt{2}$
$\frac{\vec{p}}{||\vec{p}||} = \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$

5) Find the distance between the points $a = \begin{bmatrix} -10 \\ 5 \end{bmatrix}$ and $b = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$

$\vec{a} - \vec{b} = \begin{bmatrix} -13 \\ 2 \end{bmatrix}$
$||\vec{a} - \vec{b}|| = \sqrt{(-13)^2 + (2)^2}$
$||\vec{a} - \vec{b}|| = \sqrt{173}$

6) Find the distance between the points $q = \begin{bmatrix} -3 \\ 7 \end{bmatrix}$ and $r = \begin{bmatrix} 7 \\ -3 \end{bmatrix}$

$\vec{q} - \vec{r} = \begin{bmatrix} -10 \\ 10 \end{bmatrix}$
$||\vec{q} - \vec{r}|| = \sqrt{(-10)^2 + (10)^2}$
$||\vec{q} - \vec{r}|| = \sqrt{200}$

10) Find the dot product of the two vectors. Are the two vectors perpendicular?
a)

$$v_1 = \begin{bmatrix} -1 \\ -1 \end{bmatrix} v_2 = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

$v_1 \cdot v_2 = \begin{bmatrix} -1 \times 2 + (-1) \times -2 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$
$||\vec{v_1}|| = \sqrt{(-1)^2 + (-1)^2}$
$||\vec{v_1}|| = \sqrt{2}$
$||\vec{v_2}|| = \sqrt{(2)^2 + (-2)^2}$
$||\vec{v_2}|| = \sqrt{8}$
$\cos\theta = \frac{\vec{v_1} \cdot \vec{v_2}}{||\vec{v_1}|| \cdot ||\vec{v_2}||} = \frac{0}{\sqrt{2} \cdot \sqrt{8}} = \frac{0}{\sqrt{16}} = 0$
$\therefore$ it is perpendicular.

b)

$$a = \begin{bmatrix} -10 \\ 5 \end{bmatrix} b = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$a \cdot b = \begin{bmatrix} -10 \times 3 + 5 \times 3 \end{bmatrix} = \begin{bmatrix} -15 \end{bmatrix}$
$||\vec{a}|| = \sqrt{(-10)^2 + (5)^2}$
$||\vec{a}|| = \sqrt{125}$
$||\vec{b}|| = \sqrt{(3)^2 + (3)^2}$
$||\vec{b}|| = \sqrt{18}$
$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{||\vec{a}|| \cdot ||\vec{b}||} = \frac{-15}{\sqrt{125} \cdot \sqrt{18}}$
$\cos\theta = -\frac{15}{\sqrt{2250}}$
$\therefore$ it is not perpendicular.

c)

$$v = \begin{bmatrix} 3 \\ 6 \end{bmatrix} w = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$v \cdot w = \begin{bmatrix} 3 \times 1 + 6 \times 2 \end{bmatrix} = \begin{bmatrix} 15 \end{bmatrix}$
$||\vec{v}|| = \sqrt{(3)^2 + (6)^2}$
$||\vec{v}|| = \sqrt{45}$
$||\vec{w}|| = \sqrt{(1)^2 + (2)^2}$
$||\vec{w}|| = \sqrt{5}$
$\cos\theta = \frac{\vec{v} \cdot \vec{w}}{||\vec{v}|| \cdot ||\vec{w}||} = \frac{15}{\sqrt{45} \cdot \sqrt{5}}$
$\cos\theta = \frac{15}{\sqrt{225}} = 1$
$\therefore$ it is not perpendicular.

11) Compute the cosine of the angle formed by the vectors $r = \begin{bmatrix} 7 \\ -3 \end{bmatrix}$ and $m = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$

$r \cdot m = \begin{bmatrix} 7 \times 2 + (-3) \times -5 \end{bmatrix} = \begin{bmatrix} 29 \end{bmatrix}$
$||\vec{r}|| = \sqrt{(7)^2 + (-3)^2}$
$||\vec{r}|| = \sqrt{58}$
$||\vec{m}|| = \sqrt{(2)^2 + (-5)^2}$
$||\vec{m}|| = \sqrt{29}$
$\cos\theta = \frac{29}{\sqrt{58} \cdot \sqrt{29}} = \frac{29}{1682} = \frac{1}{58} > 0$
$\therefore$ the angle is going to be acute.

12) Are the following angles acute, obtuse or right angle?

$\cos\theta_1 = -0.3$, $\cos\theta_2 = 0$, $\cos\theta_3 = 0.75$
θ_1 is an obtuse angle, θ_2 is a right angle, θ_3 is an acute angle.

13) Given the target box $(min_1, min_2) = (4, 3)$, $(max_1, max_2) = (6, 5)$. Find the local point $(\frac{1}{2}, \frac{1}{2})$ in the target box.
$x_1 = (1 - u_1)min_1 + u_1 max_1$
$x_1 = (1 - \frac{1}{2})4 + \frac{1}{2}6$
$x_1 = 5$
$x_2 = (1 - u_2)min_2 + u_2 max_2$
$x_2 = (1 - \frac{1}{2})3 + \frac{1}{2}5$
$x_2 = 4$
$\therefore$ the local point $(\frac{1}{2}, \frac{1}{2})$ in the target box is $(5, 4)$.

14) Given the target box $(min_1, min_2) = (1, 3)$, $(max_1, max_2) = (6, 8)$. Find the local point $(\frac{1}{2}, \frac{1}{4})$ in the target box.
$x_1 = (1 - \frac{1}{2})1 + \frac{1}{2}6$
$x_1 = 3\frac{1}{2}$
$x_2 = (1 - \frac{1}{4})3 + \frac{1}{4}8$
$x_2 = 1\frac{1}{2} + 2$
$x_2 = 3\frac{1}{2}$
$\therefore$ the local point $(\frac{1}{2}, \frac{1}{4})$ in the target box is $(3\frac{1}{2}, 3\frac{1}{2})$.

$x_1 = min_1 + u_1\delta_1$, $x_2 = min_2 + u_2\delta_2$
Where $\delta_1 = max_1 - min_1$ and $\delta_2 = max_2 - min_2$.

15) Given the target box $(min_1, min_2) = (-2, 4)$, $(max_1, max_2) = (6, 8)$. Find the local point $(\frac{1}{2}, \frac{1}{2})$ in the target box.
$x_1 = min_1 + u_1\delta_1$
$x_1 = -2 + \frac{1}{2}(6 - (-2))$
$x_1 = 2$
$x_2 = min_2 + u_2\delta_2$
$x_2 = 4 + \frac{1}{2}(8 - 4)$
$x_2 = 6$
$\therefore$ the local point $(\frac{1}{2}, \frac{1}{2})$ in the target box is $(2, 6)$.

16) Given the target box $(min_1, min_2) = (1, 2)$, $(max_1, max_2) = (3, 7)$. Find the local point $(\frac{2}{3}, \frac{1}{3})$ in the target box.
$x_1 = 1 + \frac{2}{3}(3 - 1)$
$x_1 = 3$
$x_2 = 2 + \frac{1}{3}(7 - 2)$
$x_2 = 3\frac{1}{3}$
$\therefore$ the local point $(\frac{2}{3}, \frac{1}{3})$ in the target box is $(3, 3\frac{1}{3})$.