Linear Algebra: Assignment 1 Mock Midterm

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1. (4 marks) Circle either true or false for each statements.

a) True False The reduced row echelon form of a matrix is unique.

b) True False The reduced row echelon form of a matrix applies only to augmented matrices.

c) True False Whenever a system of linear equations has free variables, the solution set contains many solutions.

d) True False A 5×6 matrix has 6 rows.

2. (5 marks) Do the following lines intersect in a line, a point, or not at all? Describe the solution.

$$3x - y + 2z = 4$$
$$4x + y - 2z = 3$$
$$6x - y - z = 1$$
$$\begin{bmatrix} 3 & -1 & 2 & | & 4 \\ 4 & 1 & -2 & | & 3 \\ 6 & -1 & -1 & | & 1 \end{bmatrix}$$

$R_3 - 2R_1 = \begin{bmatrix} 3 & -1 & 2 & | & 4 \\ 4 & 1 & -2 & | & 3 \\ 0 & 1 & -5 & | & -7 \end{bmatrix}$ $R_2 + R_1 = \begin{bmatrix} 3 & -1 & 2 & | & 4 \\ 7 & 0 & 0 & | & 7 \\ 0 & 1 & -5 & | & -7 \end{bmatrix}$ $\frac{1}{7}R_2 = \begin{bmatrix} 3 & -1 & 2 & | & 4 \\ 1 & 0 & 0 & | & 1 \\ 0 & 1 & -5 & | & -7 \end{bmatrix}$

$R_1 \leftrightarrow R_2 \begin{bmatrix} 1 & 0 & 0 & | & 1 \\ 3 & -1 & 2 & | & 4 \\ 0 & 1 & -5 & | & -7 \end{bmatrix}$ $R_2 - 3R_1 = \begin{bmatrix} 1 & 0 & 0 & | & 1 \\ 0 & -1 & 2 & | & 1 \\ 0 & 1 & -5 & | & -7 \end{bmatrix}$ $R_3 + R_2 = \begin{bmatrix} 1 & 0 & 0 & | & 1 \\ 0 & -1 & 2 & | & 1 \\ 0 & 0 & -3 & | & -6 \end{bmatrix}$

$-\frac{1}{3}R_3 = \begin{bmatrix} 1 & 0 & 0 & | & 1 \\ 0 & -1 & 2 & | & 1 \\ 0 & 0 & 1 & | & 2 \end{bmatrix}$ $R_2 - 2R_3 = \begin{bmatrix} 1 & 0 & 0 & | & 1 \\ 0 & -1 & 0 & | & -3 \\ 0 & 0 & 1 & | & 2 \end{bmatrix}$ $-R_2 = \begin{bmatrix} 1 & 0 & 0 & | & 1 \\ 0 & 1 & 0 & | & 3 \\ 0 & 0 & 1 & | & 2 \end{bmatrix}$

$\therefore$ the lines intersect at $(x, y, z) = (1, 3, 2)$

3. (2×3 marks) Find all values of h that make the following matrices consistent.

a) $\begin{bmatrix} 1 & h & | & 4 \\ 3 & 6 & | & 8 \end{bmatrix}$

$R_2 - 3R_1 = \begin{bmatrix} 1 & h & | & 4 \\ 0 & -3h+6 & | & -4 \end{bmatrix}$

$\therefore$ the augmented matrix is consistent if $h \neq 2$, as that will make $R_2, 0 = -4$ which is inconsistent.

b) $\begin{bmatrix} -4 & 12 & | & h \\ 2 & -6 & | & -3 \end{bmatrix}$

$R_2 + \frac{1}{2}R_1 = \begin{bmatrix} -4 & 12 & | & h \\ 0 & 0 & | & h-3 \end{bmatrix}$

$\therefore$ the augmented matrix is consistent if and only if $h = 3$, otherwise it's inconsistent.

4. (4 marks) Circle either true or false for each statement.

a) True False In order for a matrix B to be the inverse of A, only one of the equations $AB = I$ and $BA = I$ must be true.

b) True False If A and B are $n \times n$ and invertible then $B^{-1}A^{-1}$ is the inverse of AB.

c) True False If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $ad - bc = 0$, then A is invertible.

d) True False If A is an invertible 2×2 matrix, then the equation $A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ always has no solution for $\begin{bmatrix} x \\ y \end{bmatrix}$.

5. (6 marks) Find the inverse of each matrix or explain why it doesn't exist.

a) $A = \begin{bmatrix} 3 & 4 \\ -2 & 2 \end{bmatrix}$

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $3(2) - 4(-2) = 14$

$\implies \frac{1}{14} \begin{bmatrix} 3 & 4 \\ -2 & 2 \end{bmatrix}$

$\therefore A^{-1} = \begin{bmatrix} \frac{3}{14} & \frac{2}{7} \\ -\frac{1}{7} & \frac{1}{7} \end{bmatrix}$

b) $B = \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$

If $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $1(-2) - 2(-1) = 0$

$\therefore B$ doesn't have an inverse $\because ad - bc = 0$

c) $C = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 3 \\ 1 & 1 & 2 \end{bmatrix}$

$$C^{-1} = \left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$R_1 \leftrightarrow R_2 = \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \qquad R_1 \leftrightarrow R_3 = \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \end{array}\right]$$

$$R_3 - R_1 = \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & -1 \end{array}\right] \qquad R_3 + R_2 = \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right]$$

$$R_1 - R_2 = \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & -1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right] \qquad R_1 - 2R_3 = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & -2 & 3 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right]$$

$$\therefore C^{-1} = \begin{bmatrix} -3 & -2 & 3 \\ 1 & 0 & 0 \\ 1 & 1 & -1 \end{bmatrix}$$

6. (5 marks) If $A = \begin{bmatrix} 1 & -1 & 3 \\ -2 & 0 & -5 \\ -3 & 1 & 4 \end{bmatrix}$ compute each of the following matrices or explain why it doesn't exist.

a) $A^T + I_3$

$$\implies \begin{bmatrix} 1 & -2 & -3 \\ -1 & 0 & 1 \\ 3 & -5 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -2 & -3 \\ -1 & 1 & 1 \\ 3 & -5 & 5 \end{bmatrix}$$

b) $4I_3 - 2A$

$$\implies 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2\begin{bmatrix} 1 & -1 & 3 \\ -2 & 0 & -5 \\ -3 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} - \begin{bmatrix} 2 & -2 & 6 \\ -4 & 0 & -10 \\ -6 & 2 & 8 \end{bmatrix} = \begin{bmatrix} 2 & 2 & -6 \\ 4 & 4 & 10 \\ 6 & -2 & -4 \end{bmatrix}$$

c) $(5I_3)(2A)$

$$\implies 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot 2\begin{bmatrix} 1 & -1 & 3 \\ -2 & 0 & -5 \\ -3 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} \cdot \begin{bmatrix} 2 & -2 & 6 \\ -4 & 0 & -10 \\ -6 & 2 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 5(2)+0(-4)+0(-6) & 5(-2)+0(0)+0(2) & 5(6)+0(-10)+0(8) \\ 0(2)+5(-4)+0(-6) & 0(-2)+5(0)+0(2) & 0(6)+5(-10)+0(8) \\ 0(2)+0(-4)+5(-6) & 0(-2)+0(0)+5(2) & 0(6)+0(-10)+5(8) \end{bmatrix} = \begin{bmatrix} 10 & -10 & 30 \\ -20 & 0 & -50 \\ -30 & 10 & 40 \end{bmatrix}$$

7. (2 marks) Normalize the vector $\vec{v} = \begin{bmatrix} 2 & -2 & 3 \end{bmatrix}$

$||\vec{v}|| = \sqrt{2^2 + (-2)^2 + 3^2} = \sqrt{17}$

$\frac{\vec{v}}{||\vec{v}||} = \frac{\begin{bmatrix} 2 & -2 & 3 \end{bmatrix}}{\sqrt{17}} = \begin{bmatrix} \frac{2}{\sqrt{17}} & -\frac{2}{\sqrt{17}} & \frac{3}{\sqrt{17}} \end{bmatrix}$

8. (4 marks) Find the distance between the vectors $\vec{v} = \begin{bmatrix} 2 & -4 & 4 \end{bmatrix}, \vec{w} = \begin{bmatrix} 1 & -2 & 2 \end{bmatrix}$

$\vec{v} - \vec{w} = \begin{bmatrix} 1 & -2 & 2 \end{bmatrix} = \vec{u}$

$||\vec{u}|| = \sqrt{1^2 + (-2)^2 + 2^2} = \sqrt{9} = 3$

9. (3 marks) Calculate the dot product of the following matrices $\vec{v} = \begin{bmatrix} 2 & -4 & 4 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 1 & -2 & 2 \end{bmatrix}$

$\vec{v} \cdot \vec{w} = \begin{bmatrix} 2(1) + (-4)(-2) + 4(2) \end{bmatrix} = 18$

10. (2×3 marks) Determine if the cosine of the angle between the vectors is acute, obtuse or right angle.

a) $\vec{v} = \begin{bmatrix} 2 & -4 & 4 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 1 & -2 & 2 \end{bmatrix}$

$\cos\theta = \frac{\vec{v}\cdot\vec{w}}{||\vec{v}\cdot\vec{w}||}$

$\implies \frac{\begin{bmatrix} 2(1) + (-4)(-2) + 4(2) \end{bmatrix}}{\sqrt{2^2+(-4)^2+4^2}\cdot\sqrt{1^2+(-2)^2+2^2}} = \frac{18}{\sqrt{36}\cdot\sqrt{9}} = \frac{18}{18} = 1$

$\therefore$ there is no angle between the vectors since $\cos\theta = 1$

b) $\vec{v} = \begin{bmatrix} 3 & -2 & 1 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 1 & 3 & -2 \end{bmatrix}$

$\cos\theta = \frac{\begin{bmatrix} 3(1) + (-2)(3) + 1(-2) \end{bmatrix}}{\sqrt{3^2+(-2)^2+1^2}\cdot\sqrt{1^2+3^2+(-2)^2}} = \frac{-5}{\sqrt{14}\cdot\sqrt{14}} = -\frac{5}{14}$

$\therefore$ the angle between vectors is obtuse since $-1 < \cos\theta < 0$

11. (6 marks) Consider each set of vectors in $\mathbb{R}^3$. Show why each set is NOT a vector space. Hint: Find one example that shows that at least one of the vector space properties does not hold.
a) $\{\begin{bmatrix} x & y & z \end{bmatrix} | y = 2k$, where k can be any integer $\}$

Let $c = \frac{1}{4}$ and $k = 1$ such that $c\vec{v} = \frac{1}{4}\begin{bmatrix} 4 \\ 2(1) \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ .5 \\ 1 \end{bmatrix}$

$\therefore$ not a vector space since $c\vec{v} \notin V$. If $y = 2k$ where k is any integer, then $y \in \mathbb{Z}$. But if $c \in \mathbb{R}$ such that $c = \frac{1}{4}$, and if $k = 1$ such that $y = 2(1) = 2$, then $cy \notin \mathbb{Z}$.
b) $\{\begin{bmatrix} x & y & z \end{bmatrix} | xyz = 0\}$

Where at least one of x, y or z is 0, $\vec{v} = \begin{bmatrix} 0 \\ y \\ z \end{bmatrix}$

Let $\vec{v} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$

Then $\vec{v} + \vec{u} = \begin{bmatrix} 0+1 \\ 1+0 \\ 2+3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}$

$\therefore$ not a vector space since $\vec{v} + \vec{u} \notin V$ as $\vec{v} + \vec{u}\{xyz \neq 0\}$

12. (9 marks) Determine if each set of vectors is linearly dependent or linearly independent. Show your work. Remember: Each vector should be a column NOT a row.
a) $\{\vec{v} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}, \vec{w} = \begin{bmatrix} 1 & 0 & 3 \end{bmatrix}, \vec{u} = \begin{bmatrix} 2 & 2 & 6 \end{bmatrix}, \vec{t} = \begin{bmatrix} 2 & 1 & 3 \end{bmatrix}\}$
4 vectors in $\mathbb{R}^3$ must have at least 1 free parameter $\therefore$ linearly dependent.
b) $\{\vec{v} = \begin{bmatrix} 2 & 1 & 3 \end{bmatrix}, \vec{w} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \vec{u} = \begin{bmatrix} -2 & 1 & 0 \end{bmatrix}\}$

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 0 \\ 1 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 \end{array}\right]$$

$\frac{1}{3}R_3 \leftrightarrow R_1 = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & -1 & 0 \end{array}\right]$ $R_2 \leftrightarrow R_3 = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 2 & 1 & -1 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right]$

$R_3 - R_1 = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 2 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$ $R_2 - 2R_1 = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$ $R_2 + R_3 = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$

$\therefore$ the set is linearly independent $\because$ there is no free parameters.
c) $\{\vec{v} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}, \vec{w} = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}\}$

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 2 & 0 & 0 \\ 3 & 1 & 0 \end{array}\right]$$

$R_2 - 2R_1 = \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & -2 & 0 \\ 3 & 1 & 0 \end{array}\right]$ $-\frac{1}{2}R_2 = \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 3 & 1 & 0 \end{array}\right]$ $R_1 - R_2 = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 1 & 0 \end{array}\right]$

$R_3 - 3R_1 = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{array}\right]$ $R_3 - R_2 = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\therefore$ the set is linearly independent $\because$ there is no free parameters.