Week 1 Notes and Exercises

Network Flow Diagrams

$400 + x_2 = x_1$
$400 = x_1 - x_2$
$x_1 + x_3 - x_4 = 600$
$x_4 + x_5 = 100$
$x_2 + x_3 + x_5 = 300$

Solutions to Systems of Linear Equations

A unique solution(Consistent)
$x - 2y = -1$
$x = 2y - 1$
$-2y + 1 + 3y = 3$
$y = 2$
$x - 4 = -1$
$x = 3$
$(x, y) = (3, 2)$
Infinitely many solutions(Consistent)
$x - 2y = -1$
$-x + 2y = 3$
$x = 2y - 1$
$-2y + 1 + 2y = 3$
$0y = 2$
No solutions(Inconsistent)
$x - 2y = -1$
$-x + 2y = 1$
$x = 2y - 1$
$-2y + 1 + 2y = 1$
$0y = 0$

Linear Equations

Method 1
1) $x_1 - 2x_2 = -1$
2) $-x_1 + 3x_2 = 3$
Rewrite (1) as:
3) $x_1 = -1 + 2x_2$
Sub (3) into (2):
$-(-1 + 2x_2) + 3x_2 = 3$
$1 - 2x_2 + 3x_2 + 2 = 3$

$1 + x_2 = 3$
$x_2 = 3 - 1$
4) $x_2 = 2$
Sub (4) into (1):
$x_1 - 2(2) = -1$
$x_1 - 4 = -1$
$x_1 = -1 + 4$
$x_1 = 3$
Method 2
1) $x_1 - 2x_2 + x_3 = 0$
2) $2x_2 - 8x_3 = 8$
3) $-4x_2 + 5x_2 + x_3 = -9$
Multiply (1) by 4
$4x_1 - 8x_2 + 4x_3 = 0$
$-4x_1 + 5x_2 + x_3 = -9$
Add both
$-3x_2 + 5x_3 = -9$
$2x_2 - 8x_3 = 8$
Make the coefficient the same(By multiplication)
$-6x_2 + 10x_3 = -18$
$6x_2 - 24x_3 = 24$
Addition
$-14x_3 = 6$
$x_3 = \frac{-6}{14}$

Exercise

Problem 1
1) $x_1 - 3x_3 = 8$
2) $2x_1 + 2x_2 + 9x_3 = 7$
3) $x_2 + 5x_3 = -2$
$x_1 = 8 + 3x_3$
$x_2 = -2 - 5x_3$
Sub
$2(8 + 3x_3) + 2(-2 - 5x_3) + 9x_3 = 7$
$16 + 6x_3 - 4 - 10x_3 + 9x_3 = 7$
$6x_3 - 10x_3 + 9x_3 = 7 - 16 + 4$
$5x_3 = -5$
$x_3 = -1$
$x_1 - 3(-1) = 8$
$x_1 = 8 - 3$

$x_1 = 5$
$2x_2 + 18(-1) = -9$
$2x_2 = 6$
$x_2 = 3$
$(x_1, x_2, x_3) = (5, 3, -1)$
Problem 2
1) $x_2 + 4x_3 = -5$
2) $x_1 + 3x_2 + 5x_3 = -2$
3) $3x_1 + 7x_2 + 7x_3 = 4$
$x_2 = -5 - 4x_3$
$x_1 = -3(-5 - 4x_3) - 5x_3$
$x_1 = 15 + 4x_3 - 5x_3$
$x_1 = -x_3 + 15$
$3(-x_3 + 15) + 7(-5 - 4x_3) + 7x_3 = 4$
$-3x_3 + 45 - 35 - 28x_3 + 7x_3 = 4$
$-24x_3 = -6$
$x_3 = \frac{1}{4}$
$x_2 + 4(\frac{1}{4}) = -5$
$x_2 = -6$
$x_1 + 3(-6) + 5(\frac{1}{4}) = -2$
$x_1 = -2 + 18 - \frac{5}{4}$
$x_1 = 15\frac{1}{4}$
$(x_1, x_2, x_3) = (15\frac{1}{4}, -6, \frac{1}{4})$

Matrices Example

Reduced Row Echelon Form

$$\begin{bmatrix} 1 & -2 & 1 & | & 0 \\ 0 & 2 & -8 & | & 8 \\ -4 & 5 & 9 & | & -9 \end{bmatrix}$$

$4R_1 + R_3$

$$\begin{bmatrix} 1 & -2 & 1 & | & 0 \\ 0 & 2 & -8 & | & 8 \\ 0 & -3 & 13 & | & -9 \end{bmatrix}$$

$\frac{1}{2}R_2$

$$\begin{bmatrix} 1 & -2 & 1 & | & 0 \\ 0 & 1 & -4 & | & 4 \\ 0 & -3 & 13 & | & -9 \end{bmatrix}$$

$3R_2 + R_3$

$$\begin{bmatrix} 1 & -2 & 1 & | & 0 \\ 0 & 1 & -4 & | & 4 \\ 0 & 0 & 1 & | & 3 \end{bmatrix}$$

$R_2 + 4R_3$

$$\begin{bmatrix} 1 & -2 & 1 & | & 0 \\ 0 & 1 & 0 & | & 16 \\ 0 & 0 & 1 & | & 3 \end{bmatrix}$$

$R_1 - R_3$

$$\begin{bmatrix} 1 & -2 & 0 & | & -3 \\ 0 & 1 & 0 & | & 16 \\ 0 & 0 & 1 & | & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & | & 29 \\ 0 & 1 & 0 & | & 16 \\ 0 & 0 & 1 & | & 3 \end{bmatrix}$$

$x_1 = 29, x_2 = 16, x_3 = 3$
More examples

$$\begin{bmatrix} 1 & 0 & 1 & | & 32 \\ 0 & 1 & 2 & | & 16 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

It is a reduced row echelon.
Let $x_3 = S$
$x_1 = 32 - S$
$x_2 = 16 - 2S$
$x_3 = 0 + S$

Exercises

Solve each system in Exercise 1-4 by using elementary row operations on the equations or on the augmented matrix. Follow the systematic elimination procedure described in this section.

1) $x_1 + 5x_2 = 7,\ -2x_1 - 7x_2 = -5$
$x_1 = -5x_2 + 7$
$-2(-5x_2 + 7) - 7x_2 = -5$
$10x_2 - 14 - 7x_2 = -5$
$3x_2 = 9$
$x_2 = 3$
$x_1 = -5(3) + 7$
$x_1 = -15 + 7$
$x_1 = -8$

$$\begin{bmatrix} 1 & 5 & | & 7 \\ -2 & -7 & | & -5 \end{bmatrix}$$

$R_2 + 2R_1$

$$\begin{bmatrix} 1 & 5 & | & 7 \\ -2+2(1) & -7+2(5) & | & -5+2(7) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5 & | & 7 \\ 0 & 3 & | & 9 \end{bmatrix}$$

$\frac{1}{3}R_2$

$$\begin{bmatrix} 1 & 5 & | & 7 \\ \frac{1}{3}(0) & \frac{1}{3(3)} & | & \frac{1}{3}(9) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5 & | & 7 \\ 0 & 1 & | & 3 \end{bmatrix}$$

$R_1 + -5R_2$

$$\begin{bmatrix} 1+-5(0) & 5+-5(1) & | & 7+-5(3) \\ 0 & 1 & | & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & | & -8 \\ 0 & 1 & | & 3 \end{bmatrix}$$

$$(x_1, x_2) = (-8, 3)$$

2) $3x_1 + 6x_2 = -3$, $5x_1 + 7x_2 = 10$
$3x_1 = -6x_2 - 3$
$x_1 = -2x_2 - 1$
$5(-2x_2 - 1) + 7x_2 = 10$
$-10x_2 - 5 + 7x_2 = 10$
$-3x_2 = 15$
$x_2 = -5$
$x_1 = -2(-5) - 1$
$x_1 = 9$

$$\begin{bmatrix} 3 & 6 & | & -3 \\ 5 & 7 & | & 10 \end{bmatrix}$$

$\frac{1}{3}R_1$

$$\begin{bmatrix} \frac{1}{3}(1) & \frac{1}{3}(2) & | & \frac{1}{3}(-1) \\ 5 & 7 & | & 10 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & | & -1 \\ 5 & 7 & | & 10 \end{bmatrix}$$

$R_2 - 5R_1$

$$\begin{bmatrix} 1 & 2 & | & -1 \\ 5 - 5(1) & 7 - 5(2) & | & 10 - 5(-1) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & | & -1 \\ 0 & -3 & | & 15 \end{bmatrix}$$

$-\frac{1}{3}R_2$

$$\begin{bmatrix} 1 & 2 & | & -1 \\ -\frac{1}{3}(0) & -\frac{1}{3}(-3) & | & -\frac{1}{3}(15) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & | & -1 \\ 0 & 1 & | & -5 \end{bmatrix}$$

$R_1 - 2R_2$

$$\begin{bmatrix} 1 - 2(0) & 2 - 2(1) & | & -1 - 2(-5) \\ 0 & 1 & | & -5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & | & 9 \\ 0 & 1 & | & -5 \end{bmatrix}$$

$$(x_1, x_2) = (9, -5)$$

3) Find the point (x_1, x_2) that lies on the line $x_1 + 2x_2 = 4$ and on the line $x_1 - x_2 = 1$.
$x_1 + 2x_2 = 4$
$x_1 - x_2 = 1$

$$\begin{bmatrix} 1 & 2 & | & 4 \\ 1 & -1 & | & 1 \end{bmatrix}$$

$R_2 - R_1$

$$\begin{bmatrix} 1 & 2 & | & 4 \\ 1-1 & -1-2 & | & 1-4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & | & 4 \\ 0 & -3 & | & -3 \end{bmatrix}$$

$-\frac{1}{3}R_2$

$$\begin{bmatrix} 1 & 2 & | & 4 \\ -\frac{1}{3}(0) & -\frac{1}{3}(-3) & | & -\frac{1}{3}(-3) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & | & 4 \\ 0 & 1 & | & 1 \end{bmatrix}$$

$R_1 - 2R_2$

$$\begin{bmatrix} 1-2(0) & 2-2(1) & | & 4-2(1) \\ 0 & 1 & | & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & | & 2 \\ 0 & 1 & | & 1 \end{bmatrix}$$

$$(x_1, x_2) = (2, 1)$$

4) Find the point of intersection of the lines $x_1 + 2x_2 = -13$ and $3x_1 - 2x_2 = 1$.
$x_1 + 2x_2 = -13$
$3x_1 - 2x_2 = 1$

$$\begin{bmatrix} 1 & 2 & | & -13 \\ 3 & -2 & | & 1 \end{bmatrix}$$

$R_2 - 3R_1$

$$\begin{bmatrix} 1 & 2 & | & -13 \\ 3-3(1) & -2-3(2) & | & 1-3(-13) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & | & -13 \\ 0 & -8 & | & 40 \end{bmatrix}$$

$-\frac{1}{8}R_2$

$$\begin{bmatrix} 1 & 2 & | & -13 \\ -\frac{1}{8}(0) & -\frac{1}{8}(-8) & | & -\frac{1}{8}(40) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & | & -13 \\ 0 & 1 & | & -5 \end{bmatrix}$$

$R_1 - 2R_2$

$$\begin{bmatrix} 1-2(0) & 2-2(1) & | & -13-2(-5) \\ 0 & 1 & | & -5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & | & -3 \\ 0 & 1 & | & -5 \end{bmatrix}$$

$$(x_1, x_2) = (-3, -5)$$

17) Do the three lines $2x_1 + 3x_2 = -1$, $6x_1 + 5x_2 = 0$, and $2x_1 - 5x_2 = 7$ have a common point of intersection? Explain.

$$\begin{bmatrix} 2 & 3 & | & -1 \\ 6 & 5 & | & 0 \\ 2 & -5 & | & 7 \end{bmatrix}$$

$R_2 - 3R_1$ and $R_3 - R_1$

$$\begin{bmatrix} 2 & 3 & | & -1 \\ 6-3(2) & 5-3(3) & | & 0-3(-1) \\ 2-2 & -5-3 & | & 7-(-1) \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 & | & -3 \\ 0 & -4 & | & 3 \\ 0 & -8 & | & 8 \end{bmatrix}$$

$R_3 - 2R_2$

$$\begin{bmatrix} 2 & 3 & | & -3 \\ 0 & -4 & | & 3 \\ 0-2(0) & -8-2(-4) & | & 8-2(3) \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 & | & -3 \\ 0 & -4 & | & 3 \\ 0 & 0 & | & 2 \end{bmatrix}$$

Therefore, the three lines doesn't have a common point of intersection because the third equation shows the system is inconsistent with $0 = 2$

18) Do the three planes $2x_1 + 4x_2 + 4x_3 = 4$, $x_2 - 2x_3 = -2$, and $2x_1 + 3x_2 = 0$ have at least one common point of intersection? Explain.

$$\begin{bmatrix} 2 & 4 & 4 & | & 4 \\ 0 & 1 & -2 & | & -2 \\ 2 & 3 & 0 & | & 0 \end{bmatrix}$$

$R_3 - R_1$

$$\begin{bmatrix} 2 & 4 & 4 & | & 4 \\ 0 & 1 & -2 & | & -2 \\ 2-2 & 3-4 & 0-4 & | & 0-4 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 4 & 4 & | & 4 \\ 0 & 1 & -2 & | & -2 \\ 0 & -1 & -4 & | & -4 \end{bmatrix}$$

$R_3 + R_2$

$$\begin{bmatrix} 2 & 4 & 4 & | & 4 \\ 0 & 1 & -2 & | & -2 \\ 0+0 & -1+1 & -4+(-2) & | & -4+(-2) \end{bmatrix}$$

$$\begin{bmatrix} 2 & 4 & 4 & | & 4 \\ 0 & 1 & -2 & | & -2 \\ 0 & 0 & -6 & | & -6 \end{bmatrix}$$

$\frac{1}{2}R_1$ and $-\frac{1}{6}R_3$

$$\begin{bmatrix} \frac{1}{2}(2) & \frac{1}{2}(4) & \frac{1}{2}(4) & | & \frac{1}{2}(4) \\ 0 & 1 & -2 & | & -2 \\ -\frac{1}{6}(0) & -\frac{1}{6}(0) & -\frac{1}{6}(-6) & | & -\frac{1}{6}(-6) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 2 & | & 2 \\ 0 & 1 & -2 & | & -2 \\ 0 & 0 & 1 & | & 1 \end{bmatrix}$$

Therefore, since the Reduced Row Echelon form of a matrix has a leading 1 in each row, then the corresponding system is consistent and has at least one solution.

In Exercise 19-22, determine the value(s) of h such that the matrix is the augmented matrix of a consistent linear system.

19)

$$\begin{bmatrix} 1 & h & | & 4 \\ 3 & 6 & | & 8 \end{bmatrix}$$

$R_2 - 3R_1$

$$\begin{bmatrix} 1 & h & | & 4 \\ 3-3(1) & 6-3(h) & | & 8-3(4) \end{bmatrix}$$

$$\begin{bmatrix} 1 & h & 4 & \\ 0 & 3h-6 & | & -4 \end{bmatrix}$$

If $h = 2$, then the system has no solution, because $3(2) - 6 = 0$ cannot equal -4. Otherwise, if $h \neq 2$, the system has a solution.

20)

$$\begin{bmatrix} 1 & h & | & -5 \\ 2 & -8 & | & 6 \end{bmatrix}$$

$R_2 - 2R_1$

$$\begin{bmatrix} 1 & h & | & -5 \\ 2-2(1) & -8-2(h) & | & 6-2(-5) \end{bmatrix}$$

$$\begin{bmatrix} 1 & h & | & -5 \\ 0 & -2h-8 & | & 16 \end{bmatrix}$$

If $h = -4$, then the system has no solution, because $-2(-4) - 8 = 0$ cannot equal 16. Otherwise, if $h \neq -4$, the system has a solution.

21)

$$\begin{bmatrix} 1 & 4 & | & -2 \\ 3 & h & | & -6 \end{bmatrix}$$

$R_2 - 3R_1$

$$\begin{bmatrix} 1 & 4 & | & -2 \\ 3-3(1) & h-3(4) & | & -6-3(-2) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & | & -2 \\ 0 & h-12 & | & 0 \end{bmatrix}$$

The system has infinite solutions because $R_2 = 0$.

22)

$$\begin{bmatrix} -4 & 12 & | & h \\ 2 & -6 & | & -3 \end{bmatrix}$$

$R_2 + \frac{1}{2}R_1$

$$\begin{bmatrix} -4 & 12 & | & h \\ 2+\frac{1}{2}(-4) & -6+\frac{1}{2}(12) & | & -3+\frac{h}{2} \end{bmatrix}$$

$$\begin{bmatrix} -4 & 12 & | & h \\ 0 & 0 & | & -3+\frac{h}{2} \end{bmatrix}$$

The system is only consist if and only if $h = 6$.

23)
a) Every elementary row operation is reversible.

True, scaling and adding or subtracting rows are reversible.

b) A 5×6 matrix has six rows.
False, a 5×6 matrix has five rows and 6 columns.

c) The solution set of a linear system involving variables $x_1, \cdots, x_n$ is a list of numbers $(s_1, \cdots, s_n)$ that makes each equation in the system a true statement when the values $s_1, \cdots, s_n$ are substituted for $x_1, \cdots, x_n$ respectively.
False, the solution set consists of all possible solutions as a statement is only true if it is always true.

d) Two fundamental questions about a linear system involve existence and uniqueness.
True, uniqueness implies existence and existence implies uniqueness. Then A is invertible.