Week 2 Notes and Exercises

1 Notes

Matrix operations

Calculate $A+B$ and $A-B$ or state why you can't.

a)

$$A = \begin{bmatrix} 1 & 0 & 4 \\ -1 & 2 & 2 \\ 0 & -2 & -3 \end{bmatrix}, B = \begin{bmatrix} -1 & 3 & 5 \\ 2 & 2 & -3 \\ 2 & -3 & 0 \end{bmatrix}$$

$$A+B = \begin{bmatrix} 0 & 3 & 9 \\ 1 & 4 & -1 \\ 2 & -5 & -3 \end{bmatrix}, A-B = \begin{bmatrix} 2 & -3 & -1 \\ -3 & 0 & 5 \\ -2 & 1 & -3 \end{bmatrix}$$

b)

$$A = \begin{bmatrix} -1 & 0 & 5 & 6 \\ -4 & -3 & 5 & -2 \end{bmatrix}, B = \begin{bmatrix} -3 & 9 & -3 & 4 \\ 0 & -2 & -1 & 2 \end{bmatrix}$$

$$A+B = \begin{bmatrix} -4 & 9 & 2 & 10 \\ -4 & -5 & 4 & 0 \end{bmatrix}$$

Exercises find AB or state why it doesn't exist.

a)

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & -1 \\ -1 & 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 & -1 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

b)

$$A = \begin{bmatrix} 4 & -3 \\ 3 & -1 \\ 0 & -2 \\ -1 & 5 \end{bmatrix}, B = \begin{bmatrix} -1 & 3 & 2 & 2 \\ 0 & -1 & 4 & 3 \end{bmatrix}$$

2 Exercise 2.1

Questions 1,2,3,7,8,9,10,12

In Exercise 1 and 2, compute each matrix sum or product if it is defined. If an expression is undefined, explain why. Let $A = \begin{bmatrix} 2 & 0 & -1 \\ 4 & -5 & 2 \end{bmatrix}, B = \begin{bmatrix} 7 & -5 & 1 \\ 1 & -4 & -3 \end{bmatrix}, C = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}, D = \begin{bmatrix} 3 & 5 \\ -1 & 4 \end{bmatrix}, E = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$.

Sum

$$A+B = \begin{bmatrix} 2+7 & 0+(-5) & -1+1 \\ 4+1 & -5+(-4) & 2+(-3) \end{bmatrix} = \begin{bmatrix} 9 & -5 & 0 \\ 5 & -9 & -1 \end{bmatrix}$$

$$C+D = \begin{bmatrix} 1+3 & 2+5 \\ -2+(-1) & 1+4 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ -3 & 5 \end{bmatrix}$$

Rest of matrices can't be added together since they don't have the same dimensions.

Product

$$C_{2\times2}\cdot A_{2\times3}=\begin{bmatrix}1\cdot2+2\cdot4 & 1\cdot0+2\cdot-5 & 1\cdot-1+2\cdot2\\ -2\cdot2+1\cdot4 & -2\cdot0+1\cdot-5 & -2\cdot-1+1\cdot2\end{bmatrix}=\begin{bmatrix}10 & -10 & 3\\ 0 & -5 & 4\end{bmatrix}$$

$$C_{2\times2}\cdot B_{2\times3}=\begin{bmatrix}1\cdot7+2\cdot1 & 1\cdot-5+2\cdot-4 & 1\cdot1+2\cdot-3\\ -2\cdot7+1\cdot1 & -2\cdot-5+1\cdot-4 & -2\cdot1+1\cdot-3\end{bmatrix}=\begin{bmatrix}9 & -13 & -5\\ -13 & 6 & -5\end{bmatrix}$$

$$C_{2\times2}\cdot D_{2\times2}=\begin{bmatrix}1\cdot3+2\cdot5 & 1\cdot5+2\cdot4\\ -2\cdot3+1\cdot-1 & -2\cdot5+1\cdot4\end{bmatrix}=\begin{bmatrix}13 & 13\\ -7 & -5\end{bmatrix}$$

$$C_{2\times2}\cdot E_{2\times1}=\begin{bmatrix}1\cdot-5+2\cdot3\\ -2\cdot-5+1\cdot3\end{bmatrix}=\begin{bmatrix}1\\ 13\end{bmatrix}$$

$$D_{2\times2}\cdot A_{2\times3}=\begin{bmatrix}3\cdot2+5\cdot4 & 3\cdot0+5\cdot-5 & 3\cdot-1+5\cdot2\\ -1\cdot2+4\cdot4 & -1\cdot0+4\cdot-5 & -1\cdot-1+4\cdot2\end{bmatrix}=\begin{bmatrix}26 & -25 & 7\\ 6 & -20 & 9\end{bmatrix}$$

$$D_{2\times2}\cdot B_{2\times3}=\begin{bmatrix}3\cdot7+5\cdot1 & 3\cdot-5+5\cdot-4 & 3\cdot1+5\cdot-3\\ -1\cdot7+4\cdot1 & -1\cdot-5+4\cdot-4 & -1\cdot1+4\cdot-3\end{bmatrix}=\begin{bmatrix}26 & -35 & -12\\ -3 & -11 & -13\end{bmatrix}$$

$$D_{2\times2}\cdot E_{2\times1}=\begin{bmatrix}3\cdot-5+5\cdot3\\ -1\cdot-5+4\cdot3\end{bmatrix}=\begin{bmatrix}0\\ 17\end{bmatrix}$$

Rest of matrices can't have a dot product since they don't have the required dimensions.

1)

$$-2A=2\cdot\begin{bmatrix}2 & 0 & -1\\ 4 & -5 & 2\end{bmatrix}=\begin{bmatrix}2\cdot2 & 2\cdot0 & 2\cdot-1\\ 2\cdot4 & 2\cdot-5 & 2\cdot2\end{bmatrix}=\begin{bmatrix}4 & 0 & -2\\ 8 & -10 & 2\end{bmatrix}$$

$$B-2A=\begin{bmatrix}7 & -5 & 1\\ 1 & -4 & -3\end{bmatrix}+\begin{bmatrix}4 & 0 & -2\\ 8 & -10 & 2\end{bmatrix}=\begin{bmatrix}11 & -5 & -1\\ 9 & -14 & -1\end{bmatrix}$$

$A_{2\times3}\cdot C_{2\times2}$ can't have a dot product because they don't have the proper dimensions.

$$CD=\begin{bmatrix}1\cdot3+2\cdot5 & 1\cdot5+2\cdot4\\ -2\cdot3+1\cdot-1 & -2\cdot5+1\cdot4\end{bmatrix}=\begin{bmatrix}13 & 13\\ -7 & -5\end{bmatrix}$$

2)

$$A+2B=\begin{bmatrix}2 & 0 & -1\\ 4 & -5 & 2\end{bmatrix}+2\begin{bmatrix}7 & -5 & 1\\ 1 & -4 & -3\end{bmatrix}=\begin{bmatrix}2 & 0 & -1\\ 4 & -5 & 2\end{bmatrix}+\begin{bmatrix}14 & -10 & 2\\ 2 & -8 & -6\end{bmatrix}=\begin{bmatrix}16 & -10 & 1\\ 6 & -13 & -4\end{bmatrix}$$

$3C-E$ can't have a sum because they don't have the same dimensions.

$$C\cdot B=\begin{bmatrix}1\cdot7+2\cdot1 & 1\cdot-5+2\cdot-4 & 1\cdot1+2\cdot-3\\ -2\cdot7+1\cdot1 & -2\cdot-5+1\cdot-4 & -2\cdot1+1\cdot-3\end{bmatrix}=\begin{bmatrix}9 & -13 & -5\\ -13 & 6 & -5\end{bmatrix}$$

EB can't have a dot product because they don't have the proper dimensions.

3) Let $A=\begin{bmatrix}4 & -1\\ 5 & -2\end{bmatrix}$. Compute $3I_2-A$ and $(3I_2)A$.

$$3I_2-A=3\cdot\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}-\begin{bmatrix}4 & -1\\ 5 & -2\end{bmatrix}=\begin{bmatrix}3 & 0\\ 0 & 3\end{bmatrix}-\begin{bmatrix}4 & -1\\ 5 & -2\end{bmatrix}=\begin{bmatrix}-1 & 0\\ -5 & 5\end{bmatrix}$$

$$(3I_2)A=\begin{bmatrix}3 & 0\\ 0 & 3\end{bmatrix}\cdot\begin{bmatrix}4 & -1\\ 5 & -2\end{bmatrix}=\begin{bmatrix}3\cdot4+0\cdot5 & 3\cdot-1+0\cdot-2\\ 0\cdot4+3\cdot5 & 0\cdot-1+3\cdot-2\end{bmatrix}=\begin{bmatrix}12 & -3\\ 15 & -6\end{bmatrix}$$

7) If a matrix A is 5×3 and the product AB is 5×7, what is the size of B?
$A_{5\times 3} \cdot B_{x\times y} = AB_{5\times 7}$. To be able to multiply AB, the column number of A has to be the same as the row number of B. Since AB has a size of 5×3, the row size came from the row size of A and the column size came from the column size B. Therefore, $B_{3\times 7}$.

8) How many rows does B have if BC is a 3×4 matrix?
$B = 3$ rows.

9) Let $A = \begin{bmatrix} 2 & 5 \\ -3 & 1 \end{bmatrix}, B = \begin{bmatrix} 4 & -5 \\ 3 & k \end{bmatrix}$. What value(s) of k, if any, will make $AB = BA$?

$$A \cdot B = \begin{bmatrix} 2\cdot 4 + 5 \cdot 3 & 2 \cdot -5 + 5 \cdot k \\ -3 \cdot 4 + 1 \cdot 3 & -3 \cdot -5 + 1 \cdot k \end{bmatrix} = \begin{bmatrix} 23 & 5k - 10 \\ -9 & k + 15 \end{bmatrix}$$
$$B \cdot A = \begin{bmatrix} 4 \cdot 2 + (-5) \cdot -3 & 4 \cdot 5 + (-5) \cdot 1 \\ 3 \cdot 2 + k \cdot -3 & 3 \cdot 5 + k \cdot 1 \end{bmatrix} = \begin{bmatrix} 23 & 15 \\ -3k + 6 & k + 15 \end{bmatrix}$$
$$if \begin{bmatrix} 23 & 5k - 10 \\ -9 & k + 15 \end{bmatrix} = \begin{bmatrix} 23 & 15 \\ -3k + 6 & k + 15 \end{bmatrix}, then \begin{bmatrix} 23 & 5k - 10 \\ -9 & k + 15 \end{bmatrix} - \begin{bmatrix} 23 & 15 \\ -3k + 6 & k + 15 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$
$$AB - BA = \begin{bmatrix} 23 - (23) & 5k - 10 - (15) \\ -9 - (-3k + 6) & k + 15 - (k + 15) \end{bmatrix} = \begin{bmatrix} 0 & 5k - 25 \\ 3k - 15 & 0 \end{bmatrix}$$
$$5k - 25 = 0, 5k = 25, k = 5$$
$$3k - 15 = 0, 3k = 15, k = 5$$

10) Let $A = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}, B = \begin{bmatrix} 8 & 4 \\ 5 & 5 \end{bmatrix}, C = \begin{bmatrix} 5 & -2 \\ 3 & 1 \end{bmatrix}$. Verify that $AB = AC$ and yet $B \neq C$.

The cancellation laws do not hold for matrix multiplication. Such that

$$AB = AC$$
$$\frac{A}{A}B = C$$
$$I_A B = C$$
$$B = C$$

$$A \cdot B = \begin{bmatrix} 2 \cdot 8 + (-3) \cdot 5 & 2 \cdot 4 + (-3) \cdot 5 \\ -4 \cdot 8 + 6 \cdot 5 & -4 \cdot 4 + 6 \cdot 5 \end{bmatrix} = \begin{bmatrix} 1 & -7 \\ -2 & 14 \end{bmatrix}$$
$$A \cdot C = \begin{bmatrix} 2 \cdot 5 + (-3) \cdot 3 & 2 \cdot -2 + (-3) \cdot 1 \\ -4 \cdot 5 + 6 \cdot 3 & -4 \cdot -2 + 6 \cdot 1 \end{bmatrix} = \begin{bmatrix} 1 & -7 \\ -2 & 14 \end{bmatrix}$$
$$\begin{bmatrix} 1 & -7 \\ -2 & 14 \end{bmatrix} = \begin{bmatrix} 1 & -7 \\ -2 & -2 \end{bmatrix}$$

12) Let $A = \begin{bmatrix} 3 & -6 \\ -1 & 2 \end{bmatrix}$. Construct a 2×2 matrix B such that AB is the zero matrix. Use two different non-zero columns for B.

Consider $B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$, then $Ab_1 = 0$ and $Ab_2 = 0$ or $Ab_3 = 0$ and $Ab_4 = 0$.

$$B = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$$
$$A \cdot B = \begin{bmatrix} 3 \cdot 2 + (-6) \cdot 1 & 3 \cdot 2 + (-6) \cdot 1 \\ -1 \cdot 2 + 2 \cdot 1 & -1 \cdot 2 + 2 \cdot 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

3 Exercise 2.2

Questions 1,2,3,4,5,6,7a,8,29-32

Find the inverse of the matrices in Exercise 1-4.

1) $\begin{bmatrix} 8 & 6 \\ 5 & 4 \end{bmatrix}$

$$A^{-1} = \begin{bmatrix} 8 & 6 & | & 1 & 0 \\ 5 & 4 & | & 0 & 1 \end{bmatrix} = \frac{1}{8}R_1, -\frac{1}{5}R_2 \begin{bmatrix} \frac{1}{8}(8) & \frac{1}{8}(6) & | & \frac{1}{8}(1) & \frac{1}{8}(0) \\ -\frac{1}{5}(5) & -\frac{1}{5}(4) & | & -\frac{1}{5}(0) & -\frac{1}{5}(1) \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{3}{4} & | & \frac{1}{8} & 0 \\ -1 & -\frac{4}{5} & | & 0 & -\frac{1}{5} \end{bmatrix} = R_2 + R_1 \begin{bmatrix} 1 & \frac{3}{4} & | & \frac{1}{8} & 0 \\ -1+1 & -\frac{4}{5}+\frac{3}{4} & | & 0+\frac{1}{8} & -\frac{1}{5}+0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{3}{4} & | & \frac{1}{8} & 0 \\ 0 & -\frac{1}{20} & | & \frac{1}{8} & -\frac{1}{5} \end{bmatrix} = R_2 \times 20 \begin{bmatrix} 1 & \frac{3}{4} & | & \frac{1}{8} & 0 \\ 0 \times -20 & -\frac{1}{20} \times -20 & | & \frac{1}{8} \times -20 & -\frac{1}{5} \times -20 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{3}{4} & | & \frac{1}{8} & 0 \\ 0 & 1 & | & -2\frac{1}{2} & 4 \end{bmatrix} = R_1 - \frac{3}{4}R_2 \begin{bmatrix} 1-\frac{3}{4}(0) & \frac{3}{4}-\frac{3}{4}(1) & | & \frac{1}{8}-\frac{3}{4}(-\frac{5}{2}) & 0-\frac{3}{4}(4) \\ 0 & 1 & | & -2\frac{1}{2} & 4 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1 & 0 & | & 2 & 3 \\ 0 & 1 & | & -2\frac{1}{2} & 4 \end{bmatrix}$$

2) $\begin{bmatrix} 3 & 2 \\ 7 & 4 \end{bmatrix}$

$$A^{-1} = \begin{bmatrix} 3 & 2 & | & 1 & 0 \\ 7 & 4 & | & 0 & 1 \end{bmatrix} = R_1 - \frac{1}{2}R_2 \begin{bmatrix} 3-\frac{1}{2}(7) & 2-\frac{1}{2}(4) & | & 1-\frac{1}{2}(0) & 0-\frac{1}{2}(1) \\ 7 & 4 & | & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -\frac{1}{2} & 0 & | & 1 & -\frac{1}{2} \\ 7 & 4 & | & 0 & 1 \end{bmatrix} = R_1 \times -2 \begin{bmatrix} -\frac{1}{2} \times -2 & 0 \times -2 & | & 1 \times -2 & -\frac{1}{2} \times -2 \\ 7 & 4 & | & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & | & -2 & 1 \\ 7 & 4 & | & 0 & 1 \end{bmatrix} = R_2 - 7(R_1) \begin{bmatrix} 1 & 0 & | & -2 & 1 \\ 7-7(1) & 4-7(0) & | & 0-7(-2) & 1-7(1) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & | & -2 & 1 \\ 0 & 4 & | & 14 & -6 \end{bmatrix} = \frac{1}{4}R_2 \begin{bmatrix} 1 & 0 & | & -2 & 1 \\ \frac{1}{4}(0) & \frac{1}{4}(4) & | & \frac{1}{4}(14) & \frac{1}{4}(-6) \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1 & 0 & | & -2 & 1 \\ 0 & 1 & | & 3\frac{1}{2} & -1\frac{1}{2} \end{bmatrix}$$

3) $\begin{bmatrix} 8 & 5 \\ -7 & -5 \end{bmatrix}$

$$A^{-1} = \frac{1}{-40-(-35)} \begin{bmatrix} 8 & 5 \\ -7 & -5 \end{bmatrix} = -\frac{1}{5} \begin{bmatrix} 8 & 5 \\ -7 & -5 \end{bmatrix} = \begin{bmatrix} -\frac{8}{5} & -1 \\ \frac{7}{5} & 1 \end{bmatrix}$$

4) $\begin{bmatrix} 3 & -4 \\ 7 & -8 \end{bmatrix}$

$$A^{-1} = \frac{1}{-24-(28)} \begin{bmatrix} 3 & -4 \\ 7 & -8 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 3 & -4 \\ 7 & -8 \end{bmatrix} = \begin{bmatrix} \frac{3}{4} & -1 \\ \frac{7}{4} & -2 \end{bmatrix}$$