# EECS 16A Designing Information Devices and Systems I
# Fall 2018 Homework 4

**This homework is due September 21, 2018, at 23:59.**
**Self-grades are due September 25, 2018, at 23:59.**

**Submission Format**

Your homework submission should consist of **two** files.

- `hw4.pdf`: A single PDF file that contains all of your answers (any handwritten answers should be scanned) as well as your IPython notebook saved as a PDF.

  If you do not attach a PDF of your IPython notebook, you will not receive credit for problems that involve coding. Make sure that your results and your plots are visible.

- `hw4.ipynb`: A single IPython notebook with all of your code in it.

  In order to receive credit for your IPython notebook, you must submit both a “printout” and the code itself.

Submit the file to the appropriate assignment on Gradescope.

**1. Finding Null Spaces**

(a) Consider the column vectors of any $3 \times 5$ matrix. What is the maximum possible number of linearly independent column vectors?

**Solution:**

Since each column vector is in $\mathbb{R}^3$, then there are at most 3 linearly independent column vectors. Hence we can say that the column space of this matrix has at most dimension 3.

For example,

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

The first three columns span $\mathbb{R}^3$, therefore any fourth and fifth columns can be written as a linear combination of the first three columns. This means the columns of $\mathbf{A}$ will always be linearly dependent, no matter what the fourth and fifth columns are.

(b) Someone performed Gaussian elimination and got the following upper triangular matrix:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & -2 & 3 \\ 0 & 0 & 2 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Find a set of vectors which span the column space of $\mathbf{A}$. How many unique vectors are required to span the column space of $\mathbf{A}$? (This is the dimension of the column space of $\mathbf{A}$)

**Solution:**

For any vector $\vec{x}$, $\mathbf{A}\vec{x}$ is a linear combination of the columns of $\mathbf{A}$. Thus the column space of $\mathbf{A}$ is the space spanned by its columns. We can see that there are only two linearly independent columns because the third component for each column vector is 0. Therefore a set of linearly independent vectors spanning the range of $\mathbf{A}$ is:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \right\}$$

This set has two vectors in it, thus the column space has dimension 2.
Another valid set of vectors which span the columns could be,

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$$

Again, this set as two vectors in it, thus the column space has dimension 2.

(c) Recall that for every vector $\vec{x}$ in the null space of $\mathbf{A}$, we have $\mathbf{A}\vec{x} = \vec{0}$. The dimension of a the null space is the minimum number of vectors needed to span it. Find vectors that span the null space of $\mathbf{A}$ (the matrix in the previous part). What is the dimension of the null space of $\mathbf{A}$? Use the same $\mathbf{A}$ from part b.

**Solution:**

Finding the null space of $\mathbf{A}$ is the same as solving the following system of linear equations:

$$\begin{bmatrix} 1 & 1 & 0 & -2 & 3 \\ 0 & 0 & 2 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} x_1 + x_2 - 2x_4 + 3x_5 &= 0 \\ x_3 - x_4 + x_5 &= 0 \end{aligned}$$

We have 5 unknowns but only 2 linearly independent equations. Therefore, there are 3 degrees of freedom in the null space. Hence, the dimension of the null space is 3. Note that because of the way Gaussian elimination is performed, $x_1$ and $x_3$ only appear once in each equation at the head of their respective rows/equations. Thus, we let $x_2$, $x_4$ and $x_5$ be free variables $a$, $b$ and $c$. Now we rewrite the equations as:

$$\begin{aligned} x_1 &= -a + 2b - 3c \\ x_2 &= a \\ x_3 &= b - c \\ x_4 &= b \\ x_5 &= c \end{aligned}$$

We can then write this in vector form:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = a \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} -3 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

Therefore, the null space of **A** is spanned by the vectors:

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(d) (**Practice**) Now consider the new matrix, B, which is related to A,

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \\ -2 & -2 & 0 \\ 3 & 2 & 0 \end{bmatrix}$$

Find a set of vectors which span the column space of **B**. How many unique vectors are required to span the column space of **B**?

**Solution:**

We see that only the first two column vectors of B span the column space of B. The third column does not contribute to the column space of B. There are only two unique vectors in the set and therefore the column space has dimension 2.

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \\ -2 \\ 2 \end{bmatrix} \right\}$$

An additional interesting observeration is that the dimension of the column space of B, $C(B)$, and the dimension of the nullspace of A, $N(A)$, always sum to the number of columns in A (or the number of rows in B).

$$\dim(N(A)) + \dim(C(B)) = 5 \tag{1}$$

(e) Find vector(s) that span the null space of the following matrix:

$$\mathbf{C} = \begin{bmatrix} 2 & -4 & 4 & 8 \\ 1 & -2 & 3 & 6 \\ 2 & -4 & 5 & 10 \\ 3 & -6 & 7 & 14 \end{bmatrix}$$

**Solution:**

Using Gaussian elimination, we can row-reduce the matrix:

$$\begin{bmatrix} 2 & -4 & 4 & 8 \\ 1 & -2 & 3 & 6 \\ 2 & -4 & 5 & 10 \\ 3 & -6 & 7 & 14 \end{bmatrix} \overbrace{\Rightarrow}^{\frac{1}{2}R_1 \to R_1} \begin{bmatrix} 1 & -2 & 2 & 4 \\ 1 & -2 & 3 & 6 \\ 2 & -4 & 5 & 10 \\ 3 & -6 & 7 & 14 \end{bmatrix} \overbrace{\Rightarrow}^{\substack{R_1 - R_2 \to R_2 \\ 2R_1 - R_3 \to R_3 \\ 3R_1 - R_4 \to R_4}} \begin{bmatrix} 1 & -2 & 2 & 4 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & -1 & -2 \end{bmatrix}$$

$$\overset{\substack{2R_2+R_1\rightarrow R_1\\ R_2-R_3\rightarrow R_3\\ R_2-R_4\rightarrow R_4}}{\overbrace{\Rightarrow}} \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \overset{-R_2\rightarrow R_2}{\overbrace{\Rightarrow}} \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Vectors in the null space satisfy the following equations:

$$\begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

$$x_1 - 2x_2 = 0$$

$$x_3 + 2x_4 = 0$$

We then set $x_2$ and $x_4$ to be free variables and substitute in:

$$x_1 = 2a$$
$$x_2 = a$$
$$x_3 = -2b$$
$$x_4 = b$$

We then write these equations in vector form:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = a \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

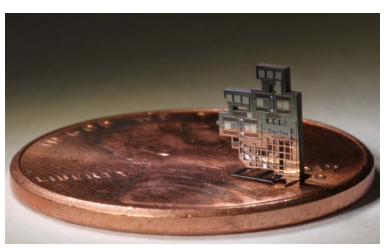
Therefore, the null space of the matrix is spanned by the vectors:

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$$

Notice the same pattern as before in terms of the relationship of the null space basis found to the fully row-reduced matrix.

## 2. Technical Issues

After a long and arduous search involving dozens of e-mails and way too many copies of your résumé, you've been offered a position with a research lab working on microelectromechanical systems (MEMS)!



Source: Contreras et al., *Transducers* (2017)

Your group has just starting working with another university, and your collaborators have sent you the design file of what they think is the next big break in microrobotics. The design file contains the layout information of the devices you're hoping to make; that is, it contains the shapes and polygons that will be fabricated in your real devices. **The design is a polygon defined by the coordinates of its corners (A,B,C,D)**. Unfortunately, the designs they sent don't make sense when you open them! When they see the rhombus in Figure 1

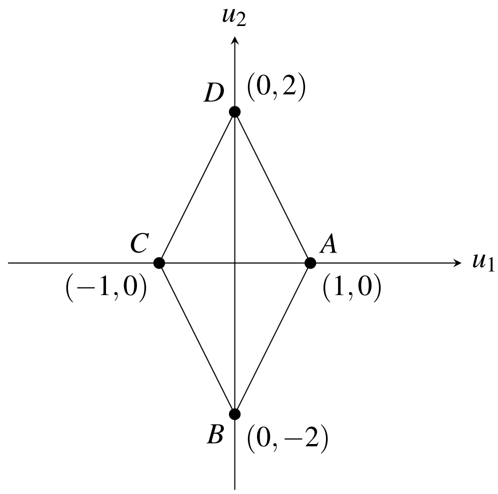



Figure 1: Collaborators' view (Basis: $\{\vec{u}_1, \vec{u}_2\}$)

you see the parallelogram in Figure 2 instead!

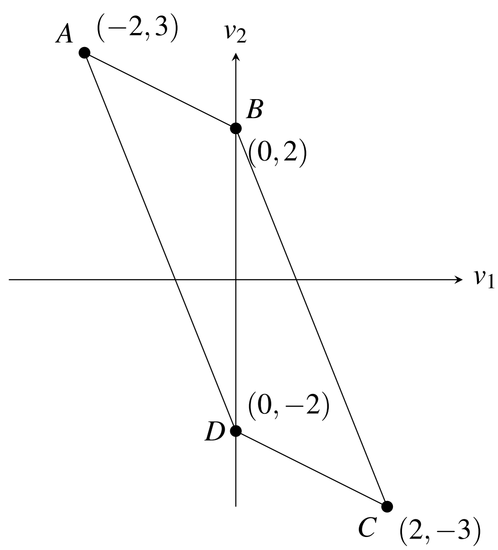



Figure 2: Your view (Basis: $\{\vec{v}_1, \vec{v}_2\}$)

(a) After some thought, you conclude that there must be a linear transformation between your view and your collaborators', i.e. you're viewing things in different coordinate systems. More specifically, your collaborators are drawing their designs using the basis $\{\vec{u}_1, \vec{u}_2\}$, and you're viewing in the basis

$\{\vec{v}_1, \vec{v}_2\}$.

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Using point $A$ as an example, $(1,0)$ to your collaborators is the equivalent of saying $\vec{a} = 1\vec{u}_1 + 0\vec{u}_2$. Similarly, $(-2,3)$ in your basis is the equivalent of saying $\vec{a} = -2\vec{v}_1 + 3\vec{v}_2$. You want to modify the design files you receive so you can view the rest of the shapes as their senders intended. Find the basis vectors $\vec{u}_1$ and $\vec{u}_2$ your collaborators used.

**Solution:**

There are a number of ways this can be solved, but the basic idea is that we want to write the vectors of the points in Figure 1 in terms of our basis vectors, i.e. as a linear combination of $\vec{v}_1$ and $\vec{v}_2$. Using the points from the figures, we can pull a few equations out

$$\begin{array}{rlll} \vec{u}_1 & = -2\vec{v}_1 & +3\vec{v}_2 & \longleftarrow \text{point } A \\ -2\vec{u}_2 & = & +2\vec{v}_2 & \longleftarrow \text{point } B \\ -\vec{u}_1 & = +2\vec{v}_1 & -3\vec{v}_2 & \longleftarrow \text{point } C \\ 2\vec{u}_2 & = & -2\vec{v}_2 & \longleftarrow \text{point } D \end{array}$$

We can see that the equations for $A$ and $C$ provide the same information, as do $B$ and $D$, so shaving the redundant equations, we're left with

$$\begin{array}{rll} \vec{u}_1 & = -2\vec{v}_1 & +3\vec{v}_2 \\ -2\vec{u}_2 & = & +2\vec{v}_2 \end{array}$$

Plugging in $\vec{v}_1$ and $\vec{v}_2$, we get

$$\vec{u}_1 = \begin{bmatrix} -2 \\ 3 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

(b) After making some modifications to your collaborators' designs, you now want to send them back for review. Unfortunately, their linear algebra is extremely rusty, so the task falls to you to convert your file so that when your collaborators view it, they see what you intend. In other words, find $\mathbf{A}_{V\to U}$ such that

$$\mathbf{A}_{V\to U} \begin{bmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ \vec{u}_1 & \vec{u}_2 \\ | & | \end{bmatrix}$$

**Solution:**

Plugging in $\vec{v}_1$ and $\vec{v}_2$, the second matrix is just the identity. From this,

$$\mathbf{A}_{V\to U} = \begin{bmatrix} | & | \\ \vec{u}_1 & \vec{u}_2 \\ | & | \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 3 & -1 \end{bmatrix}$$

Alternatively, we could use the definition of matrix multiplication to arrive at the same conclusion:

$$\mathbf{A}_{V\to U}\vec{v}_1 = \vec{u}_1$$
$$\mathbf{A}_{V\to U}\vec{v}_2 = \vec{u}_2$$

$$\mathbf{A}_{V\to U} = \begin{bmatrix} -2 & 0 \\ 3 & -1 \end{bmatrix}$$

(c) Just as you fixed this issue, the company providing the software for design viewing introduced a feature (not a bug!) to only your version of the software. You know you're supposed to be seeing the left-leaning parallelogram in Figure 2, but now you see a right-leaning parallelogram in Figure 3! Once again your superior linear algebra skills come to the rescue, and you realize that your basis vectors have been changed from $\{\vec{v}_1, \vec{v}_2\}$ to a new set $\{\vec{w}_1, \vec{w}_2\}$. Solve for the new basis vectors $\{\vec{w}_1, \vec{w}_2\}$ your software now uses.

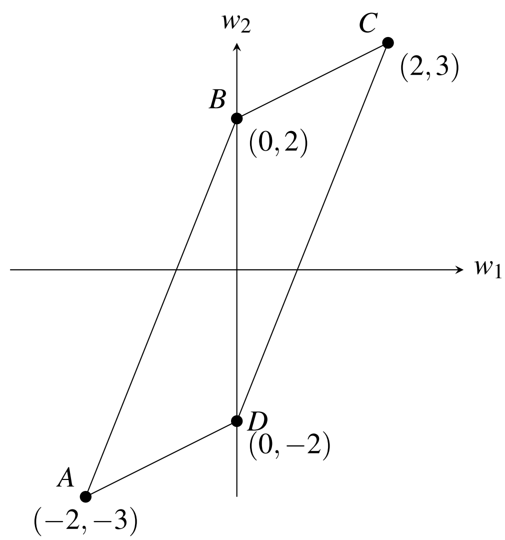



Figure 3: Your new view (Basis: $\{\vec{w}_1, \vec{w}_2\}$)

**Solution:**

Using the same process as part (a) and the information from Figures 2 and 3, we get the equations

$$\begin{array}{rrcrrl} -2\vec{w}_1 & -3\vec{w}_2 & = & -2\vec{v}_1 & +3\vec{v}_2 & \longleftarrow \text{point } A \\ & +2\vec{w}_2 & = & & +2\vec{v}_2 & \longleftarrow \text{point } B \\ +2\vec{w}_1 & +3\vec{w}_2 & = & +2\vec{v}_1 & -3\vec{v}_2 & \longleftarrow \text{point } C \\ & -2\vec{w}_2 & = & & -2\vec{v}_2 & \longleftarrow \text{point } D \end{array}$$

and once again can remove two redundant equations to get

$$\begin{array}{rrcrr} -2\vec{w}_1 & -3\vec{w}_2 & = & -2\vec{v}_1 & +3\vec{v}_2 \\ & +2\vec{w}_2 & = & & +2\vec{v}_2 \end{array}$$

We see from these that $\vec{w}_2 = \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Plugging this into the other equation

$$-2\vec{w}_1 - \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$
$$-2\vec{w}_1 = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$$

$$\vec{w}_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \vec{w}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

(d) Repeat part (b) for your new vectors $\{\vec{w}_1, \vec{w}_2\}$. That is, find $\mathbf{A}_{W\to U}$ such that

$$\mathbf{A}_{W\to U}\begin{bmatrix} | & | \\ \vec{w}_1 & \vec{w}_2 \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ \vec{u}_1 & \vec{u}_2 \\ | & | \end{bmatrix}$$

**Solution:**
We'll say the matrix of $w$ vectors and the matrix of $u$ vectors are $\mathbf{W}$ and $\mathbf{U}$, respectively. Rewriting the equation from the problem statement, we can isolate $\mathbf{A}_{W\to U}$ by multiplying on the right by $\mathbf{W}^{-1}$

$$\mathbf{A}_{W\to U}\mathbf{W} = \mathbf{U}$$
$$\mathbf{A}_{W\to U} = \mathbf{U}\mathbf{W}^{-1}$$

From previous parts, we know that

$$\mathbf{W} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$$
$$\mathbf{U} = \begin{bmatrix} -2 & 0 \\ 3 & -1 \end{bmatrix}$$

Solving for $\mathbf{W}^{-1}$, we can use Gaussian elimination to find the inverse

$$\left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ -3 & 1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 1 \end{array}\right]$$

or we can use the more graphical argument that for the $y$-component of a vector, the $\mathbf{W}$ matrix subtracts 3 of the $x$-component, so adding 3 of the original $x$-component will reverse the operation. Either method warrants full credit.

$$\mathbf{W}^{-1} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$$

Plugging in values for the matrices,

$$\begin{aligned} \mathbf{A}_{W\to U} &= \mathbf{U}\mathbf{W}^{-1} \\ &= \begin{bmatrix} -2 & 0 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \end{aligned}$$

(e) **(PRACTICE)** Using your answers from the previous parts, show that

$$\mathbf{A}_{W\to U} = \begin{bmatrix} | & | \\ \vec{w}_1 & \vec{w}_2 \\ | & | \end{bmatrix}^{-1}\begin{bmatrix} | & | \\ \vec{u}_1 & \vec{u}_2 \\ | & | \end{bmatrix}\begin{bmatrix} | & | \\ \vec{w}_1 & \vec{w}_2 \\ | & | \end{bmatrix}$$

**Solution:**

$$\begin{aligned} \mathbf{W}^{-1}\mathbf{U}\mathbf{W} &= \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} -2 & 0 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} -2 & 0 \\ 6 & -1 \end{bmatrix} \\ &= \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \end{aligned}$$

**Note:** This may not seem like an intuitive answer, but surprisingly, it's true! You may have noticed that the $w$ vectors are eigenvectors of the $\mathbf{U}$ matrix—in this case, the answer from (d) happens to have the eigenvalues on the diagonal in the order corresponding to $\mathbf{W}$'s columns. Had the columns of $\mathbf{W}$ been scaled differently, the relationship in part (d) would be off by some constant value.

**3. Traffic Flows**

Your goal is to measure the flow rates of vehicles along roads in a town. However, it is prohibitively (too) expensive to place a traffic sensor along every road. You realize, however, that the number of cars flowing into an intersection must equal the number of cars flowing out. You can use this "flow conservation" to determine the traffic along all roads in a network by only measuring flow along only some roads. In this problem, we will explore this concept.

(a) Let's begin with a network with three intersections, $A$, $B$ and $C$. Define the flow $t_1$ as the rate of cars (cars/hour) on the road between $B$ and $A$, flow $t_2$ as the rate on the road between $C$ and $B$, and flow $t_3$ as the rate on the road between $C$ and $A$.

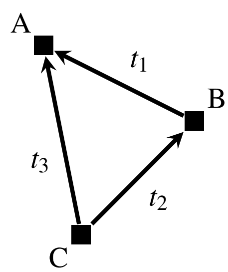



Figure 4: A simple road network.

(Note: The directions of the arrows in the figure are the way that we define positive flow by convention. For example, if there were 100 cars per hour traveling from $A$ to $C$, then $t_3 = -100$.)

We assume the "flow conservation" constraints: the net number of cars per hour flowing into each intersection is zero. For example at intersection $B$, we have the constraint $t_2 - t_1 = 0$. The full set of constraints (one per intersection) is:

$$\begin{cases} t_1 + t_3 = 0 \\ t_2 - t_1 = 0 \\ -t_3 - t_2 = 0 \end{cases}$$

As mentioned earlier, we can place sensors on a road to measure the flow through it, but we have a limited budget, and we would like to determine all of the flows with the smallest possible number of sensors.

Suppose for the network above we have one sensor reading, $t_1 = 10$. Can we figure out the flows along the other roads? (That is, the values of $t_2$ and $t_3$). If we can, find the values of $t_2$ and $t_3$.

**Solution:**

Yes, since we know that $t_1 = t_2 = -t_3$, so we must have $t_2 = 10$ and $t_3 = -10$.

(b) Now suppose we have a larger network, as shown in Figure 5.

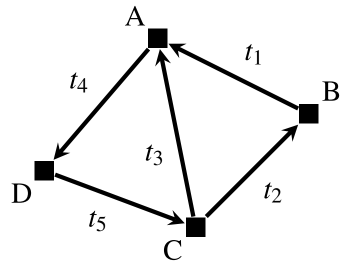



Figure 5: A larger road network.

We would again like to determine the traffic flows on all roads, using measurements from some sensors. A Berkeley student claims that we need two sensors placed on the roads AD (measuring $t_4$) and BA (measuring $t_1$). A Stanford student claims that we need two sensors placed on the roads CB (measuring $t_2$) and BA (measuring $t_1$). Write out the system of linear equations that represents this flow graph. Is it possible to determine all traffic flows, $\left[t_1, t_2, t_3, t_4, t_5\right]^T$, with the Berkeley student's suggestion? How about the Stanford student's suggestion?

**Solution:** Since we have 4 intersections, we can write 4 linear equations describing the flows into and out of each intersection. We know that the flows into and out of an intersection must sum to 0. The set of linear equations that represents this flow graph is:

$$\begin{cases} t_1 + t_3 - t_4 = 0 \\ t_2 - t_1 = 0 \\ t_5 - t_2 - t_3 = 0 \\ t_4 - t_5 = 0 \end{cases}$$

The Stanford student is wrong (obviously). Observing $t_1$ and $t_2$ is not sufficient, as $t_3$, $t_4$ and $t_5$ can still not be uniquely determined. Specifically, for any $\alpha \in \mathbb{R}$, the following flow satisfies the constraints and the measurements:

$$t_4 = \alpha$$
$$t_5 = \alpha$$
$$t_3 = \alpha - t_1$$

On the other hand, if we're given $t_1$ and $t_4$, we can uniquely solve for all the traffic densities as follows since we know the flow conservation constraints. We know that $t_2$ is the same as $t_1$ and that $t_4$ is the same as $t_5$ since the flow going into $B$ and $D$ must equal the flow going out. The flow into $A$, $t_1 + t_3$, must equal the flow going out, $t_4$, so:

$$t_2 = t_1$$
$$t_5 = t_4$$
$$t_3 = t_4 - t_1$$

This is related to the fact that $t_1$ and $t_4$ are parts of different cycles in the graph, whereas $t_1$ and $t_2$ are in the same cycle, so measuring both of them would not give additional information.

(c) We would like a more general way of determining the possible traffic flows in a network. Suppose we write the traffic flow on all roads as a vector $\vec{t} = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{bmatrix}$. As a first step, let us try to write all the flow conservation constraints (one per intersection) as a matrix equation.

Construct a $4 \times 5$ matrix $\mathbf{B}$ such that the equation $\mathbf{B}\vec{t} = \vec{0}$

$$\begin{bmatrix} & & \\ & \mathbf{B} & \\ & & \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

represents the flow conservation constraints for the network in Figure 5.

*Hint:* Each row is the constraint of an intersection. You can construct $\mathbf{B}$ using only $0, 1$, and $-1$ entries. This matrix is called the **incidence matrix.** What constraint does each column of $\mathbf{B}$ represent?

**Solution:**

$$\mathbf{B} = \begin{array}{c} \begin{bmatrix} 1 & 0 & 1 & -1 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{matrix} A \\ B \\ C \\ D \end{matrix} \\ \begin{matrix} t_1 & \; t_2 & \; t_3 & \; t_4 & \; t_5 \end{matrix} \end{array}$$

(The rows of this matrix can be in any order and your solution can differ by a factor of -1.) Each row represents an intersection, and each column represents a road between two intersections. Each 1 on a row represents a road flowing into an intersection, and each $-1$ represents a road flowing out of an intersection. Each $-1$ in a column represents the source intersection of a road (where the arrow starts), and each 1 in a column represents the destination intersection of a road (where the arrow ends).

Each column of $\mathbf{B}$ must sum to 0. The columns represent the flows into and out of an intersection. Since we know that each intersection's flows in equal flows out, we know that each column must sum to 0

(d) Again, suppose we write the traffic flow on all roads as a vector $\vec{t} = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{bmatrix}$. Then, determine the subspace of all valid traffic flows for the network of Figure 5. Specifically, express this space as the span of two linearly independent vectors.

*Hint:* Use the claim of the Berkeley student in part (b). Justify why you can use their claim. Then write all valid flows as a vector in terms of $t_1 = \alpha$ and $t_4 = \beta$.

**Solution:** To determine the subspace of traffic flows for the above network, use the solution in the previous part to see what $\vec{t}$ looks like in terms of $t_1 = \alpha$ and $t_4 = \beta$:

$$\vec{t} = \begin{bmatrix} \alpha \\ \alpha \\ \beta - \alpha \\ \beta \\ \beta \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \alpha \vec{u}_1 + \beta \vec{u}_2$$

Clearly, $\vec{u}_1$ and $\vec{u}_2$ are linearly independent, and the space of all possible traffic flows is the span of them.

**Note to grader:** If you used Gaussian Elimination in this step and found another basis for the nullspace, that is also fine for this part. The student did not need to use the hint.

**Note:** We show here, for your reference, that the space of all possible traffic flows is a subspace. You don't not need to include this proof in your solution. Suppose we have a set of valid flows $\vec{t}$. Then, for any intersection, the total flow into it is the same as the total flow out of it. If we scale $\vec{t}$ by a constant $a$, each $t_i$ will also get scaled by $a$. The total flows into and out of the intersection would be scaled by the same amount and remain equal to each other. Thus any scaling of a valid flow is still a valid flow. Suppose now we add valid flows $\vec{t}_1$ and $\vec{t}_2$ to get $\vec{t} = \vec{t}_1 + \vec{t}_2$. For any intersection $I$,

$$\text{total flow into } I = \text{total flow into } I \text{ from } \vec{t}_1 + \text{total flow into } I \text{ from } \vec{t}_2$$
$$\text{total flow out of } I = \text{total flow out of } I \text{ from } \vec{t}_1 + \text{total flow out of } I \text{ from } \vec{t}_2$$

Since the total flow into $I$ from $\vec{t}_1$ is the same as the total flow out of $I$ from $\vec{t}_1$ and similarly for $\vec{t}_2$, the total flow into $I$ is the same as the total flow out of $I$. Therefore, the sum of any two valid flows is still a valid flow. Also, $\vec{t} = \vec{0}$ is a valid flow. Therefore the set of valid flows forms a subspace.

(e) Notice that the set of all vectors $\vec{t}$ that satisfy $\mathbf{B}\vec{t} = \vec{0}$ is exactly the null space of the matrix $\mathbf{B}$. That is, we can find all valid traffic flows by computing the null space of $\mathbf{B}$. Use Gaussian elimination to determine the dimension of the null space of $\mathbf{B}$ and compute a basis for the null space. Does this match your answer to part (d)?

**Challenge (optional):** Can you interpret the dimension of the null space of $\mathbf{B}$ for the road networks of Figure 4 and Figure 5?

**Solution:**

After row-reducing, we get the following matrix:

$$\begin{bmatrix} +1 & 0 & +1 & 0 & -1 \\ 0 & +1 & +1 & 0 & -1 \\ 0 & 0 & 0 & +1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We have 5 unknowns but from the matrix above, we can see that we have only 3 non-zero rows. Therefore we have 2 free parameters. So, the dimension of the null space is 2. We can find the following basis for the null space by designating $t_3 = a$ and $t_5 = b$ $(a, b \in \mathbb{R})$ as the free variables and solving for the parametric solution:

$$a \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

This does not match the answer in the earlier part because these are two different bases, but the null space they span is the same.

**Note to grader**: It is fine to give yourself full credit as long as you found a valid basis for the null space using Gaussian Elimination. It doesn't have to be this particular one. Also, if you have the same answer for part (d) and part (e), that is fine (assuming the answer is a valid basis for the null space).

**Challenge Solution:** By itself, the first vector weighted by $a$ is a vector corresponding to the small cycle in the graph. The second vector weighted by $b$ corresponds to the bigger cycle. These two cycles are still independent of each other, which is why the dimension of the null space can be interpreted as the number of "independent cycles" in the graph.

(f) Now let us analyze more general road networks. Say there is a road network graph $G$, with incidence matrix $\mathbf{B}_G$. If $\mathbf{B}_G$ has a $k$-dimensional null space, does this mean measuring the flows along ***any** $k$ **roads*** is always sufficient to recover the exact flows? Prove or give a counterexample.

*Hint:* Consider the Stanford student from part (b).

**Solution:**

No, consider the network of Figure 5. The corresponding incidence matrix has a $k = 2$ dimensional null space, as you showed in part (e). However, measuring $t_1$ and $t_2$ (as the Stanford student suggested) is not sufficient, as you showed in part (b).

(g) Let $G$ be a network of $n$ roads with the incidence matrix $\mathbf{B}_G$, which has a $k$-dimensional null space. We would like to characterize exactly when it is sufficent to measure a set of $k$ roads to recover the exact flow along all roads.

To do this, it will help to generalize the problem and consider measuring *linear combinations* of flows. Let $t_i$ be the flow on one road. We measure some linear combination of $t_i$'s or $m_0 \cdot t_0 + m_1 \cdot t_1 + \cdots + m_n \cdot t_n$. Now we measure many of these linear combinations, which we will represent using matrix vector multiplication. Then, making $k$ measurements is equivalent to observing the vector $\mathbf{M}\vec{t}$ for some $k \times n$ "measurement matrix" $\mathbf{M}$.

For example, for the network of Figure 5, the measurement matrix corresponding to measuring $t_1$ and $t_4$ (as the Berkeley student suggests) is:

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Similarly, the measurement matrix corresponding to measuring $t_1$ and $t_2$ (as the Stanford student suggests) is:

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

For general networks $G$ and measurements $\mathbf{M}$, give a condition for when the exact traffic flows can be recovered in terms of the null space of $\mathbf{M}$ and the null space of $\mathbf{B}_G$.

*Hint:* Recovery will fail iff (if and only if) there are two valid flows with the same measurements, that is, there exist distinct $\vec{t}_1$ and $\vec{t}_2$ satisfying the flow conservation constraints, such that $\mathbf{M}\vec{t}_1 = \mathbf{M}\vec{t}_2$. Can you express this in terms of the null spaces of $\mathbf{M}$ and $\mathbf{B}_G$?

**Solution:**

As stated in the hint, we cannot uniquely determine the flow iff (if and only if) there are two valid flows that yield the same set of measurements. That is, there should not be two *distinct* valid flows $\vec{t}_1$ and $\vec{t}_2$, such that $\mathbf{M}\vec{t}_1 = \mathbf{M}\vec{t}_2$, or equivalently, such that $\mathbf{M}(\vec{t}_1 - \vec{t}_2) = 0$.

The set of valid flows is the null space of $\mathbf{B}_G$, denoted $\text{Null}(\mathbf{B}_G)$. So recovery fails if $\mathbf{M}(\vec{t}_1 - \vec{t}_2) = \vec{0}$ for some $\vec{t}_1, \vec{t}_2 \in \text{Null}(\mathbf{B}_G)$, with $\vec{t}_1 \neq \vec{t}_2$. The set of valid flows is a subspace, so we can equivalently state this as: Recovery fails iff $\mathbf{M}\vec{t} = \vec{0}$ for some $\vec{t} \neq \vec{0}$, $\vec{t} \in \text{Null}(\mathbf{B}_G)$.

In other words, *there should be no vector $\vec{t} \neq 0$ that is both in the null space of $\mathbf{B}_G$* and *in the null space of $\mathbf{M}$*.

This can also be stated as: *We can recover the exact traffic flows iff (if and only if) the null space of $\mathbf{B}_G$ does not non-trivially intersect the null space of $\mathbf{M}$.*

Full credit for stating any condition that is equivalent to this, using the null spaces of $\mathbf{M}$ and $\mathbf{B}_G$.

(h) **Challenge (optional):** If the incidence matrix $\mathbf{B}_G$ has a $k$-dimensional null space, does this mean we can **always pick a set of** $k$ **roads** such that measuring the flows along these roads is sufficient to recover the exact flows? Prove or give a counterexample.

**Solution:**

Yes.

Let $\mathbf{U}$ be a matrix whose columns form a basis of the null space of $\mathbf{B}_G$, as above. The $k$ columns of $\mathbf{U}$ are linearly independent since they form a basis. Since there are $k$ linearly independent columns, when we run Gaussian elimination on $\mathbf{U}$, we must get $k$ pivots. (Recall that "pivot" is the technical term for being able to row-reduce and turn a column into something that has exactly one 1 in it. The pivot is the entry that we found and turned into that 1.)

Therefore, the row space of $\mathbf{U}$ is $k$ dimensional since there are some $k$ linearly independent rows in $\mathbf{U}$ — namely the ones where we found pivots. Choose to measure the roads corresponding to these rows.

This will work because:

For a given valid flow $\vec{t} = \mathbf{U}\vec{x}$, the results of measuring this flow vector are $\mathbf{U}^{(k)}\vec{x}$, where the matrix $\mathbf{U}^{(k)}$ is some $k$ linearly independent rows of $\mathbf{U}$. By construction, the $k \times k$ matrix $\mathbf{U}^{(k)}$ has all linearly independent rows, so we can invert $\mathbf{U}^{(k)}$ to find $\vec{x}$ from $\mathbf{U}^{(k)}\vec{x}$ and then recover the flows along all the edges as $\mathbf{U}\vec{x}$.

This isn't the only set of $k$ roads that will work. But it does provide a set of $k$ roads that are guaranteed to work.

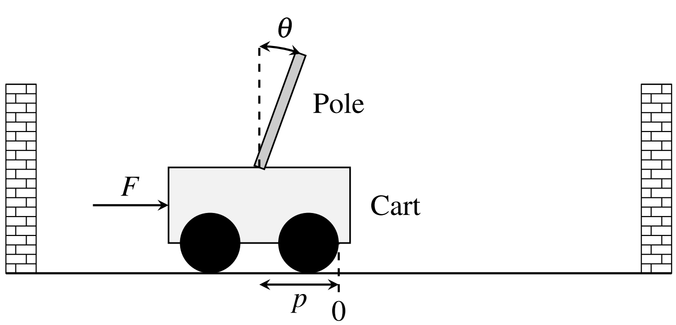
## 4. Segway Tours

Your friend has decided to start a new SF tour business, and you suggest he use segways.

He becomes intrigued by your idea and asks you how it works.

You let him know that a force (through the spinning wheel) is applied to the base of the segway, and this in turn controls both the position of the segway and the angle of the stand. As you push on the stand, the segway tries to bring itself back to the upright position, and it can only do this by moving the base.

Your friend is impressed, to say the least, but he is a little concerned that only one input (force) is used to control two outputs (position and angle). He finally asks if it's possible for the segway to be brought upright and to a stop from any initial configuration. He calls up a friend who's majoring in mechanical engineering, who tells him that a segway can be modeled as a cart-pole system:




A cart-pole system can be fully described by its position $p$, velocity $\dot{p}$, angle $\theta$, and angular velocity $\dot{\theta}$. We write this as a "state vector":

$$\vec{x} = \begin{bmatrix} p \\ \dot{p} \\ \theta \\ \dot{\theta} \end{bmatrix}$$

The input to this system $u$ will just be the force applied to the cart (or base of the segway).[1]

At time step $n$, we can apply scalar input $u[n]$. The cart-pole system can be represented by a linear model:

$$\vec{x}[n+1] = \mathbf{A}\vec{x}[n] + \vec{b}u[n], \tag{2}$$

where $\mathbf{A} \in \mathbb{R}^{4\times 4}$ and $\vec{b} \in \mathbb{R}^{4\times 1}$. The model tells us how the the state vector will evolve over (discrete) time as a function of the current state vector and control inputs.

To answer your friend's question, you look at this general linear system and try to answer the following question: Starting from some initial state $\vec{x}_0$, can we reach a final desired state, $\vec{x}_f$, in $N$ steps?

**The challenge seems to be that the state is 4-dimensional and keeps evolving and that we can only apply a one dimensional control at each time. Is it possible to control something 4-dimensional with only one degree of freedom that we can control?**

You will solve this problem by walking through several steps.

(a) Express $\vec{x}[1]$ in terms of $\vec{x}[0]$ and the input $u[0]$. (*Hint:* This is easy.)

**Solution:**

From Equation (2), we get (by substituting $n = 0$):

$$\vec{x}[1] = \mathbf{A}\vec{x}[0] + \vec{b}u[0] \tag{3}$$

(b) Express $\vec{x}[2]$ in terms of *only* $\vec{x}[0]$ and the inputs, $u[0]$ and $u[1]$. Then express $\vec{x}[3]$ in terms of *only* $\vec{x}[0]$ and the inputs, $u[0]$, $u[1]$, and $u[2]$, and express $\vec{x}[4]$ in terms of *only* $\vec{x}[0]$ and the inputs, $u[0]$, $u[1]$, $u[2]$, and $u[3]$.

**Solution:**

From Equation (2), we get (by substituting $n = 1$):

$$\vec{x}[2] = \mathbf{A}\vec{x}[1] + \vec{b}u[1]$$

By substituting $\vec{x}[1]$ from Equation (3), we get

$$\begin{aligned} \vec{x}[2] &= \mathbf{A}\vec{x}[1] + \vec{b}u[1] \\ &= \mathbf{A}\left(\mathbf{A}\vec{x}[0] + \vec{b}u[0]\right) + \vec{b}u[1] \\ &= \mathbf{A}^2\vec{x}[0] + \mathbf{A}\vec{b}u[0] + \vec{b}u[1] \end{aligned} \tag{4}$$

From Equation (2), we get (by substituting $n = 2$):

$$\vec{x}[3] = \mathbf{A}\vec{x}[2] + \vec{b}u[2]$$

[1] You might note that velocity and angular velocity are derivatives of position and angle respectively. Differential equations are used to describe continuous time systems, which you will learn more about in EE 16B. But even without these techniques, we can still approximate the solution to be a continuous time system by modeling it as a discrete time system where we take very small steps in time. We think about applying a force constantly for a given finite duration and we see how the system responds after that finite duration.

By substituting $\vec{x}[2]$ from Equation (4), we get

$$\begin{aligned}\vec{x}[3] &= \mathbf{A}\vec{x}[2] + \vec{b}u[2] \\ &= \mathbf{A}\left(\mathbf{A}^2\vec{x}[0] + \mathbf{A}\vec{b}u[0] + \vec{b}u[1]\right) + \vec{b}u[1] \\ &= \mathbf{A}^3\vec{x}[0] + \mathbf{A}^2\vec{b}u[0] + \mathbf{A}\vec{b}u[1] + \vec{b}u[2]\end{aligned} \tag{5}$$

From Equation (2), we get (by substituting $n = 3$):

$$\vec{x}[4] = \mathbf{A}\vec{x}[3] + \vec{b}u[3]$$

By substituting $\vec{x}[3]$ from Equation (5), we get

$$\begin{aligned}\vec{x}[4] &= \mathbf{A}\vec{x}[3] + \vec{b}u[3] \\ &= \mathbf{A}\left(\mathbf{A}^3\vec{x}[0] + \mathbf{A}^2\vec{b}u[0] + \mathbf{A}\vec{b}u[1] + \vec{b}u[2]\right) + \vec{b}u[3] \\ &= \mathbf{A}^4\vec{x}[0] + \mathbf{A}^3\vec{b}u[0] + \mathbf{A}^2\vec{b}u[1] + \mathbf{A}\vec{b}u[2] + \vec{b}u[3]\end{aligned} \tag{6}$$

(c) Now, derive an expression for $\vec{x}[N]$ in terms of $\vec{x}[0]$ and the inputs from $u[0], \ldots, u[N-1]$. (*Note: To obtain a compact expression, you can use a summation from* $0$ *to* $N-1$.)

**Solution:**

Use the same procedure as above for $N$ steps. You will obtain the following expression:

$$\vec{x}[N] = \mathbf{A}^N\vec{x}[0] + \mathbf{A}^{N-1}\vec{b}u[0] + \ldots + \mathbf{A}\vec{b}u[N-2] + \vec{b}u[N-1] \tag{7}$$

You might also use the compact expression:

$$\vec{x}[N] = \mathbf{A}^N\vec{x}[0] + \sum_{i=0}^{N-1} \mathbf{A}^i\vec{b}u[N-i-1] \tag{8}$$

Note that $\mathbf{A}^0$ is the identity matrix.

As a sanity check, plug the values $N = 1, 2, 3,$ and $4$ to obtain Equations (3), (4), (5), and (6), respectively.

For the next four parts of the problem, you are given the matrix $\mathbf{A}$ and the vector $\vec{b}$:

$$\mathbf{A} = \begin{bmatrix} 1 & 0.05 & -0.01 & 0 \\ 0 & 0.22 & -0.17 & -0.01 \\ 0 & 0.10 & 1.14 & 0.10 \\ 0 & 1.66 & 2.85 & 1.14 \end{bmatrix}$$

$$\vec{b} = \begin{bmatrix} 0.01 \\ 0.21 \\ -0.03 \\ -0.44 \end{bmatrix}$$

Assume the cart-pole starts in an initial state $\vec{x}[0] = \begin{bmatrix} -0.3853493 \\ 6.1032227 \\ 0.8120005 \\ -14 \end{bmatrix}$, and you want to reach the desired state $\vec{x}_f = \vec{0}$ using the control inputs $u[0], u[1], \ldots$. The state vector $\vec{x}_f = \vec{0}$ corresponds to the cart-pole (or segway) being upright and stopped at the origin. (Reaching $\vec{x}_f = \vec{0}$ in $N$ steps means that, given that we start at $\vec{x}[0]$, we can find control inputs, such that we get $\vec{x}[N]$, the state vector at the $N$th time step, equal to $\vec{x}_f$.)

*Note:* You may use IPython to solve the next three parts of the problem. You may use the function we provided (`gauss_elim(matrix)`) to help you find the upper triangular form of matrices. An example of Gaussian Elimination using this code is provide in the iPython notebook. You may also use the function (`np.linalg.solve`) to solve the equations.

(d) Can you reach $\vec{x}_f$ in *two* time steps? (*Hint: Express* $\vec{x}[2] - \mathbf{A}^2\vec{x}[0]$ *in terms of the inputs* $u[0]$ *and* $u[1]$. *Then determine if the system of equations can be solved to obtain* $u[0]$ *and* $u[1]$. *If we obtain valid solutions for* $u[0]$ *and* $u[1]$, *then we can say we will reach* $\vec{x}_f$ *in two time steps.*)

**Solution:**

No.

From Equation (4), we know that $\mathbf{A}^2\vec{x}[0] + \mathbf{A}\vec{b}u[0] + \vec{b}u[1] = \vec{x}[2]$ which is equivalent to $\mathbf{A}\vec{b}u[0] + \vec{b}u[1] = \vec{x}[2] - \mathbf{A}^2\vec{x}[0]$.

This means that in order to reach any state $\vec{x}_f$ in two time steps (that is, $\vec{x}[2] = \vec{x}_f$), we have to solve the following system of linear equations:

$$\mathbf{A}\vec{b}u[0] + \vec{b}u[1] = \vec{x}_f - \mathbf{A}^2\vec{x}[0],$$

where $u[0]$ and $u[1]$ are the unknowns.

Since in our case we want to reach $\vec{x}_f = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, the system of linear equations simplifies to

$$\mathbf{A}\vec{b}u[0] + \vec{b}u[1] = -\mathbf{A}^2\vec{x}[0].$$

In matrix form, this system of linear equations is

$$\begin{bmatrix} | & | \\ \mathbf{A}\vec{b} & \vec{b} \\ | & | \end{bmatrix} \begin{bmatrix} u[0] \\ u[1] \end{bmatrix} = -\mathbf{A}^2\vec{x}[0],$$

which yields the following augmented matrix:

$$\left[\begin{array}{cc|c} | & | & | \\ \mathbf{A}\vec{b} & \vec{b} & -\mathbf{A}^2\vec{x}[0] \\ | & | & | \end{array}\right].$$

By plugging in the values of $\mathbf{A}$, $\vec{b}$, and $\vec{x}[0]$, we get the following augmented matrix:

$$\left[\begin{array}{cc|c} 0.0208 & 0.01 & 0.02243475295 \\ 0.0557 & 0.21 & -0.30785116611 \\ -0.0572 & -0.03 & 0.0619347608 \\ -0.2385 & -0.44 & 1.38671325508 \end{array}\right].$$

Applying Gaussian elimination, we get the upper triangular form to be

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right],$$

which means that the system is inconsistent (due to the third row) and that there are no solutions for $u[0]$ and $u[1]$. It is fine if you did not row reduce all the way to the upper triangular form as long as you showed that the system of equations is inconsistent.

(e) Can you reach $\vec{x}_f$ in *three* time steps?

**Solution:**

No.

Similar to the previous part, from Equation (5), we know that $\mathbf{A}^3\vec{x}[0]+\mathbf{A}^2\vec{b}u[0]+\mathbf{A}\vec{b}u[1]+\vec{b}u[2]=\vec{x}[3]$, which is equivalent to $\mathbf{A}^2\vec{b}u[0]+\mathbf{A}\vec{b}u[1]+\vec{b}u[2]=\vec{x}[3]-\mathbf{A}^3\vec{x}[0]$.

This means that in order to reach any state $\vec{x}_f$ in three time steps (that is, $\vec{x}[3]=\vec{x}_f$), we have to solve the following system of linear equations:

$$\mathbf{A}^2\vec{b}u[0]+\mathbf{A}\vec{b}u[1]+\vec{b}u[2]=\vec{x}_f-\mathbf{A}^3\vec{x}[0],$$

where $u[0],u[1]$, and $u[2]$ are the unknowns.

Since in our case we want to reach $\vec{x}_f=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$, the system of linear equations simplifies to

$$\mathbf{A}^2\vec{b}u[0]+\mathbf{A}\vec{b}u[1]+\vec{b}u[2]=-\mathbf{A}^3\vec{x}[0].$$

In matrix form, this system of linear equations is

$$\begin{bmatrix}| & | & | \\ \mathbf{A}^2\vec{b} & \mathbf{A}\vec{b} & \vec{b} \\ | & | & |\end{bmatrix}\begin{bmatrix}u[0]\\u[1]\\u[2]\end{bmatrix}=-\mathbf{A}^3\vec{x}[0],$$

which yields the following augmented matrix:

$$\left[\begin{array}{ccc|c} | & | & | & | \\ \mathbf{A}^2\vec{b} & \mathbf{A}\vec{b} & \vec{b} & -\mathbf{A}^3\vec{x}[0] \\ | & | & | & | \end{array}\right].$$

By plugging in the values of $\mathbf{A}$, $\vec{b}$, and $\vec{x}[0]$, we get the following augmented matrix:

$$\left[\begin{array}{ccc|c} 0.024157 & 0.0208 & 0.01 & 0.0064228470365 \\ 0.024363 & 0.0557 & 0.21 & -0.092123298431 \\ -0.083488 & -0.0572 & -0.03 & 0.178491836209001 \\ -0.342448 & -0.2385 & -0.44 & 1.246334243328597 \end{array}\right].$$

Applying Gaussian elimination, we get the upper triangular form to be

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right],$$

which means that the system is inconsistent (due to the fourth row) and that there are no solutions for $u[0],u[1]$, and $u[2]$. It is fine if you did not row reduce all the way to the upper triangular form as long as you showed that the system of equations is inconsistent.

(f) Can you reach $\vec{x}_f$ in *four* time steps?

**Solution:**

Yes.

Similar to the previous part, from Equation (6), we know that $\mathbf{A}^4\vec{x}[0] + \mathbf{A}^3\vec{b}u[0] + \mathbf{A}^2\vec{b}u[1] + \mathbf{A}\vec{b}u[2] + \vec{b}u[3] = \vec{x}[4]$ which is equivalent to $\mathbf{A}^3\vec{b}u[0] + \mathbf{A}^2\vec{b}u[1] + \mathbf{A}\vec{b}u[2] + \vec{b}u[3] = \vec{x}[4] - \mathbf{A}^4\vec{x}[0]$.

This means that in order to reach any state $\vec{x}_f$ in four time steps (that is, $\vec{x}[4] = \vec{x}_f$), we have to solve the following system of linear equations:

$$\mathbf{A}^3\vec{b}u[0] + \mathbf{A}^2\vec{b}u[1] + \mathbf{A}\vec{b}u[2] + \vec{b}u[3] = \vec{x}_f - \mathbf{A}^4\vec{x}[0],$$

where $u[0], u[1], u[2]$, and $u[3]$ are the unknowns.

Since in our case we want to reach $\vec{x}_f = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, the system of linear equations simplifies to

$$\mathbf{A}^3\vec{b}u[0] + \mathbf{A}^2\vec{b}u[1] + \mathbf{A}\vec{b}u[2] + \vec{b}u[3] = -\mathbf{A}^4\vec{x}[0].$$

In matrix form, this system of linear equations is

$$\begin{bmatrix} | & | & | & | \\ \mathbf{A}^3\vec{b} & \mathbf{A}^2\vec{b} & \mathbf{A}\vec{b} & \vec{b} \\ | & | & | & | \end{bmatrix} \begin{bmatrix} u[0] \\ u[1] \\ u[2] \\ u[3] \end{bmatrix} = -\mathbf{A}^4\vec{x}[0].$$

By defining $\mathbf{Q} = \begin{bmatrix} | & | & | & | \\ \mathbf{A}^3\vec{b} & \mathbf{A}^2\vec{b} & \mathbf{A}\vec{b} & \vec{b} \\ | & | & | & | \end{bmatrix}$ and $\vec{u}_4 = \begin{bmatrix} u[0] \\ u[1] \\ u[2] \\ u[3] \end{bmatrix}$, we can now rewrite our system of linear equations as

$$\mathbf{Q}\vec{u}_4 = -\mathbf{A}^4\vec{x}[0].$$

Refer to the code in the solution IPython notebook for a solution of the system above. The solution of the system is

$$\vec{u}_4 = \begin{bmatrix} -13.24875075 \\ 23.73325125 \\ -11.57181872 \\ 1.46515973 \end{bmatrix},$$

that is, the control input sequence is: $u[0] = -13.24875075$, $u[1] = 23.73325125$, $u[2] = -11.57181872$, and $u[3] = 1.46515973$.

(g) If you have found that you can get to the final state in 4 time steps, find the required correct control inputs using IPython and verify the answer by entering these control inputs into the *Plug in your controller* section of the code in the IPython notebook. The code has been written to simulate this system, and you should see the system come to a halt in four time steps! *Suggestion: See what happens if you enter all four control inputs equal to 0. This gives you an idea of how the system naturally evolves!*

**Solution:**

See the solution to the previous part.

(h) Let's return to a general matrix $\mathbf{A}$ and a general vector $\vec{b}$. What condition do we need on

$$\text{span}\left\{\vec{b}, \mathbf{A}\vec{b}, \mathbf{A}^2\vec{b}, \ldots, \mathbf{A}^{N-1}\vec{b}\right\}$$

for $\vec{x}_f = \vec{0}$ to be "reachable" from $\vec{x}_0$ in $N$ steps?

**Solution:**

Similar to the previous parts, the key step here is to rewrite the equation you derived in part ((c)) (Equation (8)) as

$$\sum_{i=0}^{N-1} \mathbf{A}^i \vec{b} u[N-i-1] = \vec{x}[N] - \mathbf{A}^N x[0].$$

We want $\vec{x}[N] = \vec{x}_f = \vec{0}$. Therefore, the system of linear equations simplifies to

$$\sum_{i=0}^{N-1} \mathbf{A}^i \vec{b} u[N-i-1] = -\mathbf{A}^N x[0].$$

If we extend this sum, we get

$$\mathbf{A}^{N-1}\vec{b}u[0] + \mathbf{A}^{N-2}\vec{b}u[1] + \cdots + \mathbf{A}\vec{b}u[N-2] + \vec{b}u[N-1] = -\mathbf{A}^N \vec{x}[0].$$

This system of linear equations can be rewritten as

$$\begin{bmatrix} | & | & \cdots & | & | \\ \mathbf{A}^{N-1}\vec{b} & \mathbf{A}^{N-2}\vec{b} & \cdots & \mathbf{A}\vec{b} & \vec{b} \\ | & | & \cdots & | & | \end{bmatrix} \begin{bmatrix} u[0] \\ u[1] \\ \vdots \\ u[N-1] \end{bmatrix} = -\mathbf{A}^N \vec{x}[0].$$

We need to find $\{u[0], u[1], \ldots, u[N-1]\}$ that satisfy this system of linear equations. For this system to be solvable, we need $-\mathbf{A}^N x[0] \in \text{span}\left\{\vec{b}, A\vec{b}, A^2\vec{b}, \ldots, A^{N-1}\vec{b}\right\}$. That is, we need $-\mathbf{A}^N x[0]$ to be in the range (column space) of the matrix $\begin{bmatrix} | & | & \cdots & | & | \\ \mathbf{A}^{N-1}\vec{b} & \mathbf{A}^{N-2}\vec{b} & \cdots & \mathbf{A}\vec{b} & \vec{b} \\ | & | & \cdots & | & | \end{bmatrix}$.

(i) What condition would we need on span $\left\{\vec{b}, \mathbf{A}\vec{b}, \mathbf{A}^2\vec{b}, \ldots, \mathbf{A}^{N-1}\vec{b}\right\}$ for *any* valid state vector to be reachable from $\vec{x}_0$ in $N$ steps?

Wouldn't this be cool?

**Solution:**

Similar to the previous parts, the key step here is to rewrite the equation you derived in part ((c)) (Equation (8)) as

$$\sum_{i=0}^{N-1} \mathbf{A}^i \vec{b} u[N-i-1] = \vec{x}[N] - \mathbf{A}^N x[0].$$

The difference is that $\vec{x}[N] = \vec{x}_f$ can be anything in $\mathbb{R}^4$. Therefore, the system of linear equations can be written as

$$\sum_{i=0}^{N-1} \mathbf{A}^i \vec{b} u[N-i-1] = \vec{x}_f - \mathbf{A}^N x[0].$$

If we extend this sum, we get

$$\mathbf{A}^{N-1}\vec{b}u[0]+\mathbf{A}^{N-2}\vec{b}u[1]+\cdots+\mathbf{A}\vec{b}u[N-2]+\vec{b}u[N-1]=\vec{x}_f-\mathbf{A}^N x[0].$$

This system of linear equations can be further rewritten as

$$\begin{bmatrix} | & | & \cdots & | & | \\ \mathbf{A}^{N-1}\vec{b} & \mathbf{A}^{N-2}\vec{b} & \cdots & \mathbf{A}\vec{b} & \vec{b} \\ | & | & \cdots & | & | \end{bmatrix}\begin{bmatrix} u[0] \\ u[1] \\ \vdots \\ u[N-1] \end{bmatrix}=\vec{x}_f-\mathbf{A}^N x[0].$$

For this system to be solvable, we need $\vec{x}_f-\mathbf{A}^N x[0]\in\operatorname{span}\left\{\vec{b},\mathbf{A}\vec{b},\mathbf{A}^2\vec{b},\ldots,\mathbf{A}^{N-1}\vec{b}\right\}$. Since $\vec{x}_f$ can be any vector in $\mathbb{R}^4$, it also means that $\vec{x}_f-\mathbf{A}^N x[0]$ can be any vector in $\mathbb{R}^4$. This means that in order to be able to reach any state $\vec{x}_f\in\mathbb{R}^4$, the range (column space) of the matrix $\begin{bmatrix} | & | & \cdots & | & | \\ \mathbf{A}^{N-1}\vec{b} & \mathbf{A}^{N-2}\vec{b} & \cdots & \mathbf{A}\vec{b} & \vec{b} \\ | & | & \cdots & | & | \end{bmatrix}$ has to be all of $\mathbb{R}^4$.

P.S.: Congratulations! You have just derived the condition for "controllability" for systems with linear dynamics. When dealing with a system that evolves over time, we can sometimes influence the behavior of the system through various control inputs (for example, the steering wheel and gas pedal of a car or the rudder of an airplane). It is of great importance to know what states (think positions and velocities of a car or configurations of an aircraft) that our system can be controlled to. Controllability is the ability to control the system to any possible state or configuration.

**5. (PRACTICE) Codes Revisited**

Alice and Bob are back and they've successfully figured out how to avoid dropping symbols when sending messages. In this problem, Alice is using a similar encoding scheme as last time where she uses vectors $\{\vec{v}_1,\vec{v}_2,\vec{v}_3\}$ to encode her message $\begin{bmatrix} a & b & c \end{bmatrix}^T$. (Assume Bob knows the vectors, $\{\vec{v}_1,\vec{v}_2,\vec{v}_3\}$, that Alice is using.) Namely, she tries to send $\vec{k}$:

$$\vec{k}=\begin{bmatrix} - & \vec{v}_1{}^T & - \\ - & \vec{v}_2{}^T & - \\ - & \vec{v}_3{}^T & - \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} \tag{9}$$

Unfortunately, their arch-nemesis, Eve, is trying to interfere with Alice's messages to Bob and has found a way to add noise to the transmission! But, Eve's interference must pass through a linear transformation, $\mathbf{U}$, before the interference hits Alice's message. Now instead of seeing $\vec{k}$, Bob is receiving $\vec{y}$:

$$\vec{y}=\begin{bmatrix} - & \vec{v}_1{}^T & - \\ - & \vec{v}_2{}^T & - \\ - & \vec{v}_3{}^T & - \end{bmatrix}\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}+\begin{bmatrix} - & \vec{u}_1{}^T & - \\ - & \vec{u}_2{}^T & - \\ - & \vec{u}_3{}^T & - \end{bmatrix}\begin{bmatrix} p \\ q \\ r \end{bmatrix}\right) \tag{10}$$

where Eve inserts $\begin{bmatrix} p & q & r \end{bmatrix}^T$ and it undergoes a transformation by the matrix $\mathbf{U}$ and then undergoes the same transformation as Alice's original 3-symbol vector. There are two ways Eve's meddling can mess up the transmission:

- If Bob receives $\vec{0}$, he doesn't even realize he's getting a message
- Bob receives a nonzero transmission but can't determine the original $\begin{bmatrix} a & b & c \end{bmatrix}^T$

(a) **(PRACTICE)** Alice is using the following vectors for her encoding scheme

$$\vec{v_1}^T = \begin{bmatrix}1 & 2 & 0\end{bmatrix}, \vec{v_2}^T = \begin{bmatrix}0 & 0 & 1\end{bmatrix}, \vec{v_3} = \begin{bmatrix}1 & 2 & 1\end{bmatrix}$$

If Eve is not interfering ($p = q = r = 0$), will Bob be able to uniquely determine what $a$, $b$, and $c$ are?

**Solution:**

Nope! We can either use Gaussian elimination to see that there's no unique solution:

$$\begin{bmatrix}1 & 2 & 0\\0 & 0 & 1\\1 & 2 & 1\end{bmatrix} \to \begin{bmatrix}1 & 2 & 0\\0 & 0 & 1\\1 & 2 & 0\end{bmatrix} \longleftarrow R_3 - R_2 \mapsto R_3 \to \begin{bmatrix}1 & 2 & 0\\0 & 0 & 1\\0 & 0 & 0\end{bmatrix} \longleftarrow R_1 - R_3 \mapsto R_3 \tag{11}$$

or we can see that $\vec{v_3} = \vec{v_1} + \vec{v_2}$. Therefore, the rows are linearly dependent and we are not getting 3 unique measurements. Since we have 3 unknowns, we need at least 3 measurements to find a unique solutions. Alice chose a bit of an unfortunate encoding scheme; even if Eve isn't interfering, Bob won't be able to determine what she's sending!

(b) **(PRACTICE)** Eve decides to change her strategy—now she's sending her interference through its own transformation before adding it to Alice's message. Bob receives $\vec{y}$ according to Equation (10).

Eve's interference is transformed by the vectors:

$$\vec{u_1}^T = \begin{bmatrix}1 & 2 & 0\end{bmatrix}, \vec{u_2}^T = \begin{bmatrix}0 & 0 & 1\end{bmatrix}, \vec{u_3} = \begin{bmatrix}1 & 2 & 1\end{bmatrix} \tag{12}$$

Find the null space of $\mathbf{U} = \begin{bmatrix}- & \vec{u_1}^T & -\\- & \vec{u_2}^T & -\\- & \vec{u_3}^T & -\end{bmatrix}$

**Solution:**

Calculating the null space of $\mathbf{U}$:

$$\begin{bmatrix}1 & 2 & 0\\0 & 0 & 1\\1 & 2 & 1\end{bmatrix} \to \begin{bmatrix}1 & 2 & 0\\0 & 0 & 1\\0 & 0 & 0\end{bmatrix} \longleftarrow R_3 - R_2 - R_1 \mapsto R_3$$

$$p + 2q = 0$$

$$r = 0$$

$$\text{null}(\mathbf{U}) = \text{span}\left\{\begin{bmatrix}-2\\1\\0\end{bmatrix}\right\}$$

(c) **(PRACTICE)** Given $\mathbf{U}$ from Equation (12), find $p$, $q$, and $r$ such that Bob is guaranteed to never realize he's receiving a message, regardless of Alice's encoding scheme, i.e. $\vec{y} = \vec{0}$ (this corresponds to the fact that the null space of any matrix always includes the zero vector).

State any constraints on $a$, $b$, and $c$ which are necessary for Eve's cancellation to work. In other words, find $\begin{bmatrix}p & q & r\end{bmatrix}^T$ and define any restrictions on $a$, $b$, and $c$ such that

$$\begin{bmatrix}a\\b\\c\end{bmatrix} + \begin{bmatrix}- & \vec{u_1}^T & -\\- & \vec{u_2}^T & -\\- & \vec{u_3}^T & -\end{bmatrix}\begin{bmatrix}p\\q\\r\end{bmatrix} = \vec{0}$$

Do not take into account the vectors Alice is using for her encoding.

**Solution:**

Rearranging the equation a bit,

$$\begin{bmatrix} - & \vec{u}_1^{\,T} & - \\ - & \vec{u}_2^{\,T} & - \\ - & \vec{u}_3^{\,T} & - \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} -a \\ -b \\ -c \end{bmatrix}$$

we can use Gaussian elimination to determine what $p$, $q$, and $r$ are with respect to $a$, $b$, and $c$.

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & -a \\ 0 & 0 & 1 & -b \\ 1 & 2 & 1 & -c \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 0 & -a \\ 0 & 0 & 1 & -b \\ 0 & 0 & 0 & a+b-c \end{array}\right]$$

In other words,

$$\begin{aligned} p+2q &= -a \\ r &= -b \\ 0 &= a+b-c \end{aligned}$$

For this to be a consistent set of equations and for Eve to even have a chance of canceling out Alice's original message, $a+b-c=0$. To have Eve's interference cancel out Alice's symbols, $p+2q=-a$ and $r=-b$.

(d) **(PRACTICE)** Realizing her scheme from (a) was flawed, Alice has also chosen to change things up and is using new vectors for her encoding:

$$\vec{v}_1^{\,T} = \begin{bmatrix} 1 & 2 & 0 \end{bmatrix}, \vec{v}_2^{\,T} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1 & 3 & 1 \end{bmatrix}$$

For each of the following cases, determine whether Bob will receive a message at all and—if he does—whether he'll be able to correctly and uniquely determine what $a$, $b$, and $c$ are. Justify your answer. **Do not use IPython to solve this problem**. *Hint: Use your answers from (b) and (c)*

i. $\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$

ii. $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

iii. $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$

**Solution:**

To save on some writing, we'll define the following:

$$\vec{x}_{\text{Alice}} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \vec{x}_{\text{Eve}} = \begin{bmatrix} p \\ q \\ r \end{bmatrix},$$

From Equation (10), we can use the distributive property of matrix multiplication

$$\mathbf{V}\left(\vec{x}_{\text{Alice}} + \mathbf{U}\vec{x}_{\text{Eve}}\right) = \mathbf{V}\vec{x}_{\text{Alice}} + \mathbf{V}\mathbf{U}\vec{x}_{\text{Eve}}$$

For Bob to receive Alice's uncorrupted transmission, he should only receive $\mathbf{V}\vec{x}_{\text{Alice}}$. In other words, if $\mathbf{V}\mathbf{U}\vec{x}_{\text{Eve}} = \vec{0}$, Bob will receive exactly what Alice intended. Because the columns of $\mathbf{V}$ are linearly independent, we know from lecture that means that $\mathbf{V}\mathbf{U}\vec{x}_{\text{Eve}} = \vec{0}$ only when when $\mathbf{U}\vec{x}_{\text{Eve}} = \vec{0}$. Written another way, Eve's interference will have no effect if $\vec{x}_{\text{Eve}}$ falls in the null space of $\mathbf{U}$ calculated in part (b).

For Bob to not realize that he's receiving a message, i.e. for him to receive $\vec{0}$, $\vec{x}_{\text{Alice}} + \mathbf{U}\vec{x}_{\text{Eve}}$ must fall in the null space of $\mathbf{V}$. Again, the null space of $\mathbf{V}$ is just the zero vector, so we can use our answer from part (c) to determine if Eve's interference is canceling out Alice's message.

i. Bob will receive a message and will be able to determine what $a$, $b$, and $c$ are. There are two parts to this: (1) Eve's interference vector falls in null($\mathbf{U}$), she's adding zero interference to Alice's original transmission.

$$\begin{aligned}\mathbf{V}\left(\vec{v}_{\text{Alice}} + \mathbf{U}\vec{v}_{\text{Eve}}\right) &= \mathbf{V}\left(\vec{v}_{\text{Alice}} + \vec{0}\right)\\ &= \mathbf{V}\vec{v}_{\text{Alice}}\end{aligned}$$

(2) The columns of $\mathbf{V}$ are linearly independent with 3 equations and 3 unknowns, meaning Bob will be able to uniquely find the elements of $\vec{x}_{\text{Alice}}$.

ii. Bob will receive a message and be unable to determine what $a$, $b$, and $c$ are. We know that $\vec{v}_{\text{Eve}}$ does not fall in the null space of $\mathbf{U}$, so now we should check if Bob will receive a message by using our answer from part (c). First, we see that $a+b-c = 1+1-2 = 0$, so it's not impossible for Eve to cancel Alice's message. $p+2q = 1+3 = 4 \neq -a$, so Bob receives a message, but it's been garbled by Eve's interference.

iii. Bob won't realize he's being sent a message. Using our answer from part (c), we can see that Bob will receive $\vec{0}$, meaning he won't even be aware that Alice is attempting to transmit.

**6. Homework Process and Study Group**

Who else did you work with on this homework? List names and student ID's. (In case of homework party, you can also just describe the group.) How did you work on this homework?

**Solution:**

I worked on this homework with...

I first worked by myself for 2 hours, but got stuck on problem 5, so I went to office hours on...

Then I went to homework party for a few hours, where I finished the homework.