EECS 16A   Designing Information Devices and Systems I
Spring 2018 Lecture Notes   Note 16

16.1   Touchscreen Revisited

We've seen how a resistive touchscreen works by using the concept of voltage dividers. Essentially, for a resistive touchscreen, we map the value of analog voltag to the position touched. Another way to design a touchscreen is to break the screen down into a bunch of pixels. At each one of the pixels, we can detect whether the finger is touching it or not. This allows the touchscreen to sense multiple touch points at a time. In order to do this efficiently, we need a new element: **capacitors**. One of the nice parts about the capacitor is that you don't have to bend anything like we did for the resistive touchscreen — the presence or absence of the finger can modify the capacitance directly!

16.2   Capacitor

Capacitance represents how much charge can be stored for a given amount of voltage. We usually denote capacitance with C. Let's represent voltage across the capacitor as V and the charge stored on the capacitor as Q. This relation is often written as:

$$Q = CV. \tag{1}$$

When we draw circuits, the symbol for a capacitor is:
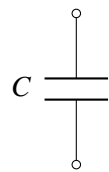
The unit of capacitance is **Farad (F)**. A capacitor with capacitance of $1F$ stores $1C$ of charge across is when it has $1V$ of potential energy across its electrodes.

Let's examine the defining relationship of a capacitor:

$$Q = CV_C. \tag{2}$$

Differentiating both sides with respect to t and assuming a constant capacitance, we have

$$\frac{dQ}{dt} = \frac{dCV_C}{dt} = C\frac{dV_C}{dt}. \tag{3}$$

However, we know that current $I = \frac{dQ}{dt}$. Hence,

$$I = C\frac{dV_C}{dt}. \tag{4}$$

Thus, the current through a capacitor is just the product of the capacitance and the rate of change of the voltage across the capacitor. This yields the following graph, with slope equal to C:
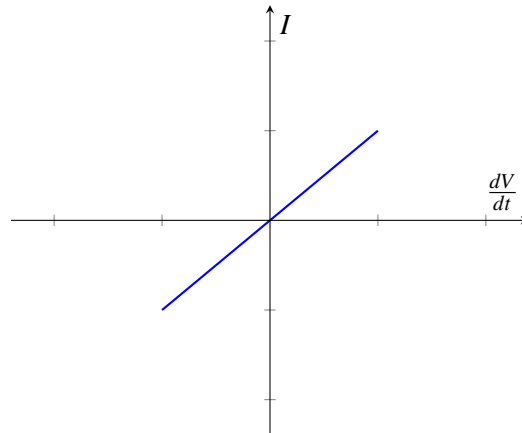
An important implication of this is **in order for current to flow through the capacitor, the voltage across that capacitor must be changing with time.** If the voltage is no longer increasing, then the current through the capacitor will equal 0.

What about solving for the voltage across the capacitor; how do we find $V_{elem}(t)$? There are many ways to solve this equation, but we will use separation of variables with our equation for current, and then integrating both sides. Note that it's okay if you haven't seen this technique before, it is not required for this class.
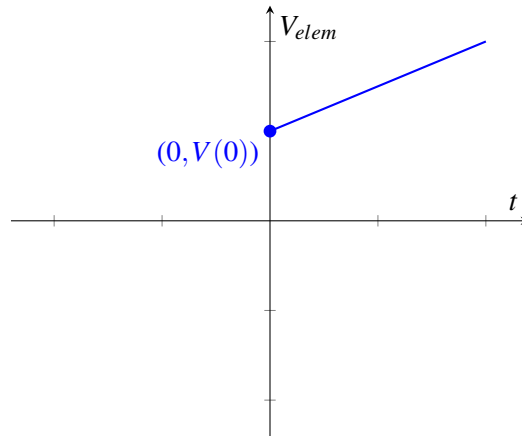
$$I = C\frac{dV_C}{dt}. \tag{5}$$

$$Idt = CdV_C \tag{6}$$
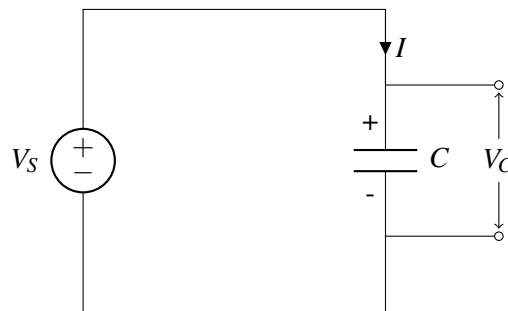
$$\int_0^t Idt = \int_0^{V_C} CdV_C \tag{7}$$

$$It = C(V_C(t) - V_C(0)) \tag{8}$$

Don't forget about the initial voltage across the capacitor! Our limits of integration in the derivation set the point of our initial condition- in this case, at $t = 0$. This yields the following graph, with slope equal to $\frac{I}{C}$.

$$V_C(t) = \frac{I}{C}t + V_C(0). \tag{9}$$

Let's see what happens when we connect a capacitor with a voltage source.

We know that initially, charges are going to build up on the two plates — positive charges on the top plate and negative charges on the bottom plate. Once enough charge has been stored, the voltage across the capacitor becomes $V_C = V_S$; because the voltage can no longer increase past this point, the current I flowing through the circuit becomes zero.

16.3 Capacitor Equivalence

Note that when we initially talked about equivalence it was in terms of $I - V$. But what about for capacitors? Now we will specify equivalence in terms of $I - \frac{dV}{dt}$. That means we need to apply I_{test} and measure $\frac{dV_{test}}{dt}$ or apply $\frac{dV_{test}}{dt}$ and measure I_{test}.

Just like resistors, we can connect capacitors in series and in parallel. We'll first focus on capacitor-only circuits, which means that we don't need to find Thevenin/Norton equivalent circuits; it will just be a capacitor.

Remember that a capacitor has an $I - \frac{dV}{dt}$ curve, not an IV curve. To be clear, this means that when solving for equivalence we will use a test $\frac{dV}{dt}$ or measure $\frac{dV}{dt}$. For example, we may apply a test current I_{test} and then measure a $\frac{dV}{dt}_{test}$.

16.3.1 Capacitors in parallel

Given the following circuit:

$$C_{eq} = \frac{I_{total}}{\frac{dV_t}{dt}}$$
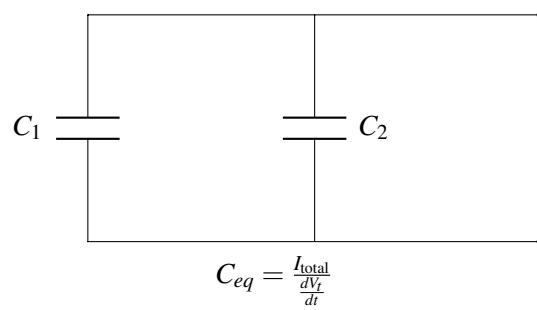
What is the equivalent capacitance? By analogy with our resistor case, we will try and apply a test voltage to this circuit:
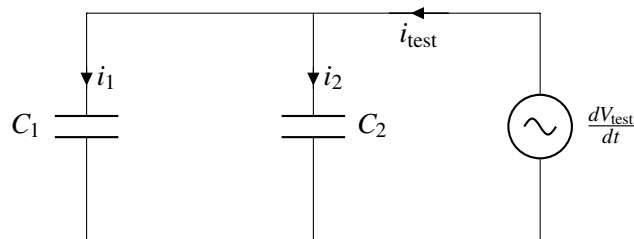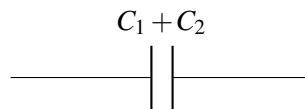

$$i_1 = C_1 \frac{dV_T}{dt}$$

$$i_2 = C_2 \frac{dV_T}{dt}$$

$$i_T = i_1 + i_2 = (C_1 + C_2) \frac{dV_T}{dt}$$

$$\therefore C_{eq} = C_1 + C_2$$
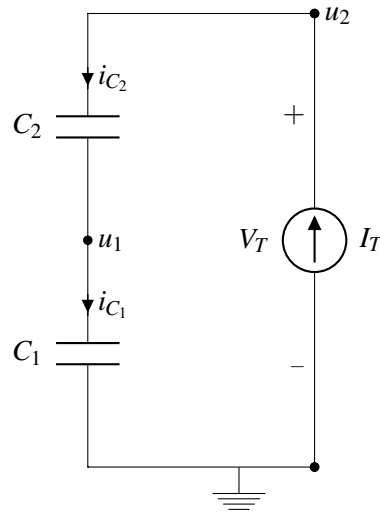
We can thus reduce our original circuit to:

$$C_1 + C_2$$

We find that in general, the equivalent capacitance of capacitors in parallel is the sum of their capacitance

16.3.2   Capacitors in series

Given the circuit:

$$C_1 \qquad C_2$$

By symmetry with the resistor case, let's apply a test current and measure the resulting voltage:

$$i_{C_1} = i_{C_2} = I_T$$

$$i_{C_1} = I_{\text{test}} = C_1 \frac{du_1}{dt}$$

$$i_{C_2} = I_{\text{test}} = C_2 \frac{d(u_2 - u_1)}{dt} = C_2 \left(\frac{du_2}{dt} - \frac{du_1}{dt} \right)$$

$$V_{\text{test}} = u_2$$

$$\frac{du_1}{dt} = \frac{I_T}{C_1} \Rightarrow I_{\text{test}} = C_2 \frac{du_2}{dt} - \frac{C_2}{C_1} I_T$$
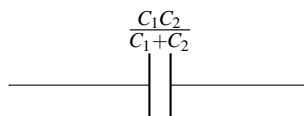
$$I_T = \frac{C_2}{1 + \frac{C_2}{C_1}} \frac{du_2}{dt}$$

$$I_T = \frac{C_2}{1 + \frac{C_2}{C_1}} \frac{dV_{\text{test}}}{dt}$$

$$\Rightarrow C_{eq} = \frac{C_2}{1 + \frac{C_2}{C_1}} = \frac{C_1 C_2}{C_1 + C_2}$$

Note that this is in fact the same as saying $C_1 \parallel C_2$. Remember that the $\parallel$ operator is mathematical notation; in this case, the capacitors are actually in series, but *mathematically* their equivalent circuit is found via the "parallel resistor" operation.

We can thus reduce our original circuit to:

16.4   A Little Bit of Physics

Remember that this is 16A physics - designed not to be exact or fully detailed, but close enough to get us the understanding we need to create circuit models from physical structures.

Any two pieces of conductive material, typically metal, that are separated by some other material that is not in general conductive (aka an insulator), forms a capacitor.
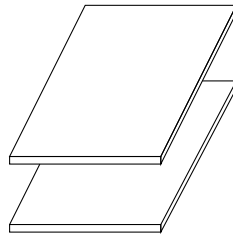
Note: What does this mean about **you** in the case of a capacitive touch screen? That's right, you're a conductor!

If there is voltage across the two pieces of conductors, charges will build up on the surface of the capacitor. As depicted below, when we apply voltage V across the two plates, positive charges build up on the (bottom) surface of the plate connected to the positive terminal and negative charges build up on the (top) surface of the plate connected to the negative terminal.
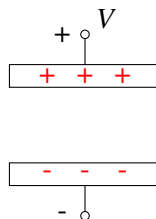

Let's try to get some intuition on capacitors by drawing an analogy between capacitors and water buckets — simply put, we can think of a capacitor as a bucket. The more water (charge) you put in the bucket, the higher the water level (voltage) would be. The water level in the bucket is determined by its dimensions. Similarly, the capacitance is set by the dimensions of the conductors and some properties of the material separating them. Please note that the water analogy is imperfect for many cases in electrical engineering- be careful not to try and extend or use it arbitrarily.

In particular, the capacitance of a capacitor is:

$$C = \varepsilon \frac{A}{d} \tag{10}$$

where A is the area of the surface of the plates facing each other, d is the separation between the two plates (illustrated below), and ε is the "permitivity" of the material between the plates. We can deduce that ε has unit F/m. For example, if the material between the capacitors is air (vacuum), then $\varepsilon = 8.85 \times 10^{-12} F/m$.
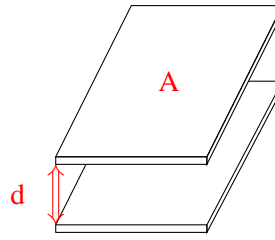
When a capacitor is charged, there is voltage across it. What does this mean? This means that there is energy stored in the capacitor. Why is there energy stored? Recall that like charges repel each other. So for example if we take a positive charge and move it closer to another positive charge, we need to exert force and thus supply energy to do so. Let's now ask the question: when the capacitor is fully charged, how much energy is stored in it? We know from previous lectures that the energy required to store an additional dq amount of charge when the voltage across the capacitor is V_C is

$$dE = V_C dq. \tag{11}$$

We also know that $dq = CdV_C$. With this in mind, we have

$$dE = V_C (CdV_C) = CV_C dV_C. \tag{12}$$

Integrating both sides, we have

$$\int_0^E dE = \int_0^{V_S} CV_C dV_C = C\int_0^{V_S} V_C dV_C \tag{13}$$

$$E = \frac{1}{2}CV_S^2. \tag{14}$$

Hence, the energy stored in the capacitor after it's fully charged is $\frac{1}{2}CV_S^2$. The charges across the capacitor cannot just disappear for no reason, and therefore the energy continues to be stored - that means we can use this element as a tool to design interesting circuit which measure this stored charge/energy.

Let's return to our capacitor equivalence models and try to gain some more physical intuition:

Suppose we combine two capacitors in parallel as follows
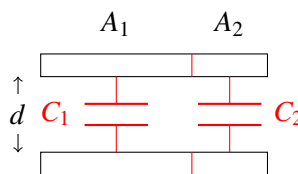
If we look at the combined capacitor above, its capacitance is equal to

$$C_{eq} = \varepsilon \frac{A_1 + A_2}{d}, \tag{15}$$

which is equal to the sum of the capacitance of the two capacitors

$$C_{eq} = \varepsilon \frac{A_1 + A_2}{d} = \varepsilon \frac{A_1}{d} + \varepsilon \frac{A_2}{d} = C_1 + C_2, \tag{16}$$

If we take the following capacitor and draw a horizontal line in between, then essentially we can view it as two capacitors connected in series
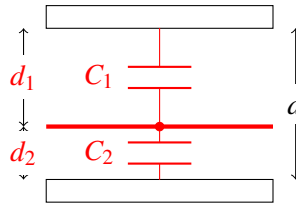
Assume that the surface area is A. Intuitively, we know that the equivalent capacitance should be smaller than both C_1 and C_2 since the plates are farther from each other. We know that $C_1 = \varepsilon \frac{A}{d_1}$ and $C_2 = \varepsilon \frac{A}{d_2}$. Let the capacitance of the combined capacitor be C_{eq}. It satisfies

$$\frac{1}{C_{eq}} = \frac{1}{\varepsilon} \frac{d_1 + d_2}{A} = \frac{1}{\varepsilon} \frac{d_1}{A} + \frac{1}{\varepsilon} \frac{d_2}{A} = \frac{1}{C_1} + \frac{1}{C_2}. \tag{17}$$

Hence, the equivalent capacitance is equal to

$$C_{eq} = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2}} = \frac{C_1 C_2}{C_1 + C_2}. \tag{18}$$

The capacitor on the left has capacitance $C_1 = \varepsilon \frac{A_1}{d}$ and the capacitor on the right has capacitance $C_2 = \varepsilon \frac{A_2}{d}$. Intuitively, the equivalent capacitance for capacitors in parallel would just be the sum of the capacitance of each of the capacitors since connecting them in parallel is analogous to summing the surface area of the two plates.