EECS 16A   Designing Information Devices and Systems I
Fall 2018   Lecture Notes   Note 3

3.1   Linear Dependence

Recall the simple tomography example from Note 1, in which we tried to determine the composition of a box of bottles by shining light at different angles and measuring light absorption. The Gaussian elimination algorithm implied that we needed to take at least 9 measurements to properly identify the 9 bottles in a box so that we had at least one equation per variable. However, will taking *any* 9 measurements guarantee that we can find a solution? Answering this question requires an understanding of linear dependence. In this note, we will define linear dependence (and independence), and take a look at what it implies for systems of linear equations.

3.1.1   What is Linear Dependence?

Linear dependence is a very useful concept that is often used to characterize the "redundancy" of information in real world applications. We will give two equivalent definitions of linear dependence.

Definition 3.1 (Linear Dependence (I)): A set of vectors $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is linearly dependent if there exist scalars $\alpha_1, \ldots, \alpha_n$ such that $\alpha_1 \vec{v}_1 + \cdots + \alpha_n \vec{v}_n = \vec{0}$ and not all α_i's are equal to zero.

Definition 3.2 (Linear Dependence (II)): A set of vectors $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is linearly dependent if there exist scalars $\alpha_1, \ldots, \alpha_n$ and an index i such that $\vec{v}_i = \sum_{j \neq i} \alpha_j \vec{v}_j$.[1] In words, a set of vectors is linearly dependent if one of the vectors could be written as a linear combination of the rest of the vectors.

Why did we introduce two equivalent definitions? They could be useful in different settings. For example, it is often easier mathematically to show linear dependence with definition (I). Can you see why? If we would like to prove linear dependence with definition (II), we need to first choose a vector $\vec{v}_i$ and show that it is a linear combination of the other vectors. However, with definition (I), we don't need to try to "single out" a vector to get started with the proof. We can blindly write down the equation $\alpha_1 \vec{v}_1 + \cdots + \alpha_n \vec{v}_n = \vec{0}$ and begin our proof from there. On the other hand, definition (II) gives us a more intuitive way to talk about redundancy. If a vector can be constructed from the rest of the vectors, then this vector does not contribute any information that is not already captured by the other vectors.

Now we will show that the two definitions are equivalent. This is the first formal proof in the course! We will walk you through it.[2] First, we ask the question, "What does it mean when we say two definitions are equivalent?" It means that when the condition in definition (I) holds, the condition in definition (II) must

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[1] In case you are unfamiliar with this notation, the $\sum$ symbol is simply shorthand for addition. For instance, $\alpha_1 \vec{v}_1 + \cdots + \alpha_n \vec{v}_n$ can be written as $\sum_{i=1}^{n} \alpha_i \vec{x}_i$ or $\sum_i \alpha_i \vec{x}_i$, which is a sum over all possible i values. In this instance, $\sum_{j \neq i} \alpha_j \vec{v}_j$ is the sum over all $\alpha_j \vec{v}_j$ excluding the $\alpha_i \vec{v}_i$ term, which can also be calculated as $\alpha_1 \vec{v}_1 + \cdots + \alpha_n \vec{v}_n - \alpha_i \vec{v}_i$.

[2] Note 4 provides a more in depth treatment on how to approach proofs.

hold as well. And when the condition in definition (II) holds, the condition in definition (I) must also hold. So there are two directions that we have to show:

(i) To see how definition (II) implies definition (I), we start from the condition in definition (II) — suppose there exist scalars $\alpha_1, \ldots, \alpha_n$ and an index i such that $\vec{v}_i = \sum_{j \neq i} \alpha_j \vec{v}_j$. We want to somehow transform this equation into the form that appears in definition (I). How can we achieve that? We can move $\vec{v}_i$ to the right:

$$\vec{0} = -1 \times \vec{v}_i + \sum_{j \neq i} \alpha_j \vec{v}_j. \tag{1}$$

Now setting $\alpha_i = -1$, we have

$$\vec{0} = \alpha_i \times \vec{v}_i + \sum_{j \neq i} \alpha_j \vec{v}_j = \sum_j \alpha_j \vec{v}_j. \tag{2}$$

Since $\alpha_i = -1$, at least one of the α_j terms is not zero, and the condition in definition (I) is satisfied.

(ii) Now let's show the reverse — that definition (I) implies definition (II). Suppose the condition in definition (I) is true. Then there exist scalars $\alpha_1, \ldots, \alpha_n$ such that $\alpha_1 \vec{v}_1 + \cdots + \alpha_n \vec{v}_n = 0$ and not all α_i's are equal to zero. Since at least one of the α_i's is nonzero, let's assume that α_1 is nonzero since we can always reorder terms in the summation because addition is commutative. Now how do we get the equation into the form identical to that in definition (II)? Observe that if we move $\alpha_1 \vec{v}_1$ to the opposite side of equation and divide both sides by α_1, we have

$$\vec{v}_1 = \sum_{j \neq 1} \left(\frac{\alpha_j}{\alpha_1} \right) \vec{v}_j. \tag{3}$$

We see that this is identical to the second definition. (In our proof, we made the assumption that $\alpha_1 \neq 0$. However, notice that we could as well have supposed that $\alpha_2 \neq 0$, $\alpha_3 \neq 0$, or any index i so that $\alpha_i \neq 0$. The convention is to set the first index, in this case 1, to be nonzero. In mathematical texts, we typically write *"Without loss of generality (W.L.O.G.), we let $\alpha_1 \neq 0$."*)

Now that we have introduced the notion of linear dependence, what does it mean to be linearly *independent*?

3.1.2 Linear Independence

Definition 3.3 (Linear Independence): A set of vectors is linearly independent if it is not linearly dependent. More specifically, from the first definition of linear dependence we can deduce that a set of vectors $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is linearly independent if $\alpha_1 \vec{v}_1 + \cdots + \alpha_n \vec{v}_n = \vec{0}$ implies $\alpha_1 = \cdots = \alpha_n = 0$.

Let's see some simple examples of linear dependence and linear independence.

Example 3.1 (Linear dependence of 2 vectors): Consider vectors $\vec{a} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$. These vectors are linearly dependent because we can write $\vec{b}$ as a scaled version of $\vec{a}$:

$$\vec{b} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 2 \times \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2 \times \vec{a}.$$

Example 3.2 (Linear independence of 2 vectors): Consider vectors $\vec{a} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$. We will show that the two vectors are linearly independent. Consider scalars α_1, α_2 such that $\alpha_1 \vec{a} + \alpha_2 \vec{b} = \vec{0}$. We can write this vector equation as a system of linear equations:

$$\alpha_1 \vec{a} + \alpha_2 \vec{b} = \vec{0}$$
$$\Rightarrow \alpha_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\Rightarrow \begin{bmatrix} 2\alpha_1 \\ \alpha_1 \end{bmatrix} + \begin{bmatrix} \alpha_2 \\ 5\alpha_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\Rightarrow \begin{bmatrix} 2\alpha_1 + \alpha_2 \\ \alpha_1 + 5\alpha_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\Rightarrow \begin{cases} 2\alpha_1 + \alpha_2 = 0 \\ \alpha_1 + 5\alpha_2 = 0 \end{cases}$$

Solving this system of linear equations with Gaussian elimination yields a unique solution, $\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

By definition, $\vec{a}$ and $\vec{b}$ are linearly independent.

> **Additional Resources** For more on the definition of linear dependence, read *Strang* pages 164-167 or read *Schuam's* pages 121-124. For additional practice with these ideas, try *Schuam's* Problems 4.17 to 4.22, and 4.89 to 4.96.

3.1.3 Linear Dependence and Systems of Linear Equations

Previously, we saw that a system of linear equations can have zero solutions, a unique solution, or infinitely many solutions. Is there a way to tell what kind of solution a system of linear equations has without running Gaussian elimination or explicitly solving for the solution? Yes! Recall that a system of linear equations can be written in matrix-vector form as $A\vec{x} = \vec{b}$, where A is a matrix of variable coefficients, $\vec{x}$ is a vector of variables, and $\vec{b}$ is a vector of values that these weighted sums must equal. We will show that just looking at the columns or rows of the matrix A can help tell us about the solutions to $A\vec{x} = \vec{b}$.

Theorem 3.1: **If the system of linear equations $A\vec{x} = \vec{b}$ has infinite number of solutions, then the columns of A are linearly dependent.**

Let's see why this is the case: If the system has infinite number of solutions, it must have at least two distinct solutions. Let's call them $\vec{x}_1$ and $\vec{x}_2$. (Note that $\vec{x}_1, \vec{x}_2$ are full vectors, not vector elements.) Then $\vec{x}_1$ and $\vec{x}_2$ must satisfy

$$A\vec{x}_1 = \vec{b} \tag{4}$$
$$A\vec{x}_2 = \vec{b}. \tag{5}$$

Subtracting the first equation from the second equation, we have $A(\vec{x}_2 - \vec{x}_1) = \vec{0}$. Let $\vec{\alpha} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = \vec{x}_2 - \vec{x}_1$.

Because $\vec{x}_1$ and $\vec{x}_2$ are distinct, not all α_i's are equal to zero. Let the columns of A be $\vec{a}_1, \ldots, \vec{a}_n$. Then,

$A\vec{\alpha} = \sum_{i=1}^{n} \alpha_i \vec{a}_i = \vec{0}$. By definition, the columns of A are linearly dependent.

Theorem 3.2: **If the columns of A in the system of linear equations $A\vec{x} = b$ are linearly dependent, then the system does not have a unique solution.**

Let's walk through this proof step by step: we'll start by assuming we have a matrix A with linearly dependent columns, and then we will show that this means that the system does not have a unique solution.

Since we are interested in the columns of A, let's start by explicitly defining the columns of A:

$$A = \begin{bmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \dots & \vec{a}_n \\ | & | & & | \end{bmatrix},$$

We've defined A to have linearly dependent columns, so by the definition of linear dependence, there exist scalars $\alpha_1, \dots, \alpha_n$ such that $\alpha_1 \vec{a}_1 + \dots + \alpha_n \vec{a}_n = \vec{0}$ where not all of the α_i's are zero. We can put these α_i's in a vector

$$\vec{\alpha} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$$

and by the definition of matrix-vector multiplication, we can compactly write the expression above:

$$A\vec{\alpha} = \vec{0}$$

where $\vec{\alpha} \neq \vec{0}$.

Recall that we are trying to show that the system of equations $A\vec{x} = \vec{b}$ does not have a unique solution. We know that systems of equations can have either zero, one, or infinite solutions. If our system of equations has zero solutions, then it cannot have a unique solution, so we don't need to consider this case. Now let's consider the case where we have at least one solution, $\vec{x}$:

$$A\vec{x} = \vec{b}$$
$$A\vec{x} + \vec{0} = \vec{b}$$
$$A\vec{x} + A\vec{\alpha} = \vec{b}$$
$$A(\vec{x} + \vec{\alpha}) = b$$

Therefore, $\vec{x} + \vec{\alpha}$ is also a solution to the system of equations! Since both $\vec{x}$ and $\vec{x} + \vec{\alpha}$ are solutions, and $\vec{\alpha} \neq \vec{0}$, the system has more than one solution. We've now proven the theorem.

Note that we can add any multiple of $\vec{\alpha}$ to $\vec{x}$ and it will still be a solution – therefore, if there is at least one solution to the system and the columns of A are linearly dependent, then there are infinite solutions. Let's think about why this makes sense intuitively. In an experiment, each column in matrix A represents the influence of each variable x_i on the measurements. If the columns are linearly dependent, this means that some of the variables influence the measurement in the same way, and therefore cannot be disambiguated. The next example illustrates this idea.

Example 3.3 (Intuition): Suppose we have a black and white image with two pixels. We cannot directly see the shade of each pixel, but we can measure how much light the two pixels absorb in total. Can we

figure out the shade of each pixel? Let's model this as a system of linear equations. Suppose pixel 1 absorbs x_1 units of light and pixel 2 absorbs x_2 units of light. Our measurement indicates that total amount of light absorbed by the image are 10 units of light. Then we could write down the equation,

$$x_1 + x_2 = 10. \tag{6}$$

Written in matrix form, we have

$$\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 10 \end{bmatrix}. \tag{7}$$

We see that the columns are $\begin{bmatrix} 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \end{bmatrix}$. The total amount of light absorbed is influenced by 1 unit of x_1 and 1 unit of x_2. However, we cannot pin down the exact influence by x_1 and x_2 because if pixel 1 absorbs c units less, we can just have pixel 2 absorb c units more. This is connected with the fact that the two columns are linearly dependent — if one pixel absorbs less, it is possible to find a way such that the other pixel absorbs more to make up for the loss (the column of that pixel can be written as a linear combination of the columns of the other pixels).

This result has important implications to the design of engineering experiments. Often times, we can't directly measure the values of the variables we're interested in. However, we can measure the total weighted contribution of each variable. The hope is that we can fully recover each variable by taking several of such measurements. Now we can ask: "What is the minimum number of measurements we need to fully recover the solution?" and "How do we design our experiment so that we can fully recover our solution with the minimum number of measurements?" Consider the tomography example. We are confident that we can figure out the configuration of the stack when the columns of the lighting pattern matrix A in $A\vec{x} = \vec{b}$ are linearly independent. On the other hand, if the columns of the lighting pattern matrix are linearly dependent, we know that we don't yet have enough information to figure out the configuration. Checking whether the columns are linearly independent essentially gives us a way to validate whether we've effectively designed our experiment.

Now let's consider how the rows of A in the system of linear equations $A\vec{x} = \vec{b}$ relate to its solution. Remember that each row corresponds to an equation in a system of linear equations.

Theorem 3.3: **If the system of linear equations $A\vec{x} = \vec{b}$ has an infinite number of solutions and the number of rows in A is greater than or equal to the number of columns (A is a square or a tall matrix), then the rows of A are linearly dependent.**

Let's see why this is true. Since $A\vec{x} = \vec{b}$ has an infinite number of solutions and A is a square or tall matrix, we must end up with at least one row of zeros after running Gaussian elimination. What does it mean if we end up with a row of zeros when we run Gaussian elimination? At least one of the rows can be written as a linear combination of the rest of the rows, because row operations correspond exactly to computing linear combinations of the rows. Thus, the rows of A must be linearly dependent.

Now let's think about this intuitively. In an experiment, each row of A represents a measurement. If the number of measurements taken is at least the number of variables and we still cannot completely determine the variables, then at least one of our measurements must be redundant (it doesn't give us any new information). If we think back to the tomography example, this means that we need to ensure each measurement is unique — for instance, it wouldn't work to shine a large light so we can measure absorption across two rows in addition to measuring the light absorption of each row, because we could obtain the same information by simply adding the measurements from individual rows.

In summary, looking from the perspective of columns, linear dependency gives us a way to characterize ambiguity in the influence of each variable to our collection of measurements; looking from the perspective of rows, linear dependency gives us a way to characterize redundancy in the measurements.

3.2 Span

Let's introduce **span**, a closely related concept to linear dependence that will be used throughout this course.

Definition 3.4 (Span): The span of a set of vectors $\{v_1, \ldots, v_n\}$ is the set of *all* linear combinations of $\{v_1, \ldots, v_n\}$. We can write this mathematically as

$$span(v_1, \ldots, v_n) = \left\{ \sum_{i=1}^{n} \alpha_i \vec{v}_i \mid \alpha_i \in \mathbb{R} \right\}$$

We can now rephrase our second definition of linear dependence: A set of vectors is linearly dependent if any one of the vectors is in the *span* of the remaining vectors.

Additional Resources For more on linear span, read *Schaum's* pages 119-121. For additional practice with these ideas, try Problems 4.13 to 4.16, 4.66, 4.69, and 4.83 to 4.88.

3.3 Practice Problems

These practice problems are also available in an interactive form on the course website.

1. Are the vectors $\vec{a} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ linearly independent?

2. Are the vectors $\vec{a} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \vec{b} = \begin{bmatrix} 1 \\ 6 \\ 2 \end{bmatrix}, \vec{c} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ linearly independent?

3. Suppose for some matrix $\mathbf{A}$, $\mathbf{A}\vec{x}_1 = \vec{b}$ and $\mathbf{A}\vec{x}_2 = \vec{b}$, where $\vec{x}_1 \neq \vec{x}_2$. Are the columns of $\mathbf{A}$ linearly independent?

4. Is $\vec{v} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ in the *span* $\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix} \right)$?