EECS 16A   Designing Information Devices and Systems I
Fall 2018
                                                                           Homework 5

This homework is due September 28, 2018, at 23:59.

Self-grades are due October 2, 2018, at 23:59.

Submission Format

Your homework submission should consist of **two** files.

- `hw5.pdf`: A single PDF file that contains all of your answers (any handwritten answers should be scanned) as well as your IPython notebook saved as a PDF.

  If you do not attach a PDF of your IPython notebook, you will not receive credit for problems that involve coding. Make sure that your results and your plots are visible.

- `hw5.ipynb`: A single IPython notebook with all of your code in it.

  In order to receive credit for your IPython notebook, you must submit both a "printout" and the code itself.

Submit each file to its respective assignment on Gradescope.

1. **Mechanical Eigenvalues and Eigenvectors**

   Find the eignvalues and their eigenspaces — give a basis for the eigenspace when it is more than 1 dimensional

   (a) $\begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix}$

   (b) $\begin{bmatrix} 22 & 6 \\ 6 & 13 \end{bmatrix}$

   (c) **(PRACTICE)** $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

   (d) $\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$ (What special matrix is this?)

   (e) $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

2. **Counting The Paths of a Random Surfer**

   In homework and discussion, we have discussed the behavior of water flowing in reservoirs and the people flowing in social networks. We now consider the behavior of a random web-surfer who jumps from webpage to webpage. We would like to know how many possible paths there are for a random surfer to get from one webpage to another webpage. To do this, we represent the webpages as a graph.

   If webpage 1 has a link to webpage 2, we have a directed edge from webpage 1 to webpage 2. This graph can further be represented by what is known as an "adjacency matrix", $\mathbf{A}$, with elements a_{ij}. We define

$a_{ji} = 1$ if there is link from page i to page j. Note the ordering of the indices! Matrix operations on the adjacency matrix make it very easy to compute the number of paths to get from a particular webpage i to webpage j.

Consider the following graphs.
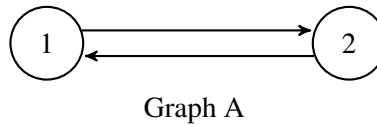
Graph A

(a) Based on this definition, the "adjacency matrix" for graph A, will be,

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{1}$$

The element a_{ji} of $\mathbf{A}$ gives the number of one-hop paths from from website i to website j. Similarly, the elements of $\mathbf{A}^2$ give the number of two-hop paths from website i to website j. How many one-hop paths are there from webpage 1 to webpage 2? How many two-hop paths are there from webpage 1 to webpage 2? How about three-hop paths?

(b) This path counting aspect is very related to the steady-state frequency for the fraction of people for each webpage. The steady-state frequency for a graph of websites is related to the eigenspace associated with eigenvalue 1 for the "transition matrix" of the graph.

The "transition matrix", $\mathbf{T}$, is slightly different from the "adjacency matrix". Its values, t_{ji}, are the *proportion* of the people who are at website i that click the link for website j. We assume people divide equally among the links on the website (e.g. if there are three links on a website, $\frac{1}{3}$ of the people will click each link).

Once computed, an eigenvector with eigenvalue 1 will have values which correspond to the steady-state frequency for the fraction of people for each webpage. When this eigenvector's values are made to sum to one (to conserve people), the ith element of the eigenvector corresponds to the fraction of people on the ith website.

For graph A, what are the steady-state frequencies for the two webpages?
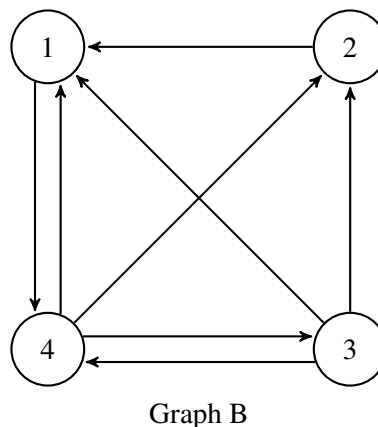
Graph B

(c) Write out the adjacency matrix for graph B.

(d) For graph B, how many two-hop paths are there from webpage 1 to webpage 3? How many three-hop paths are there from webpage 1 to webpage 2? You may use your IPython notebook for this.

(e) For graph B, what are the steady-state frequencies for the webpages? You may use your IPython notebook and the Numpy command numpy.linalg.eig for this.
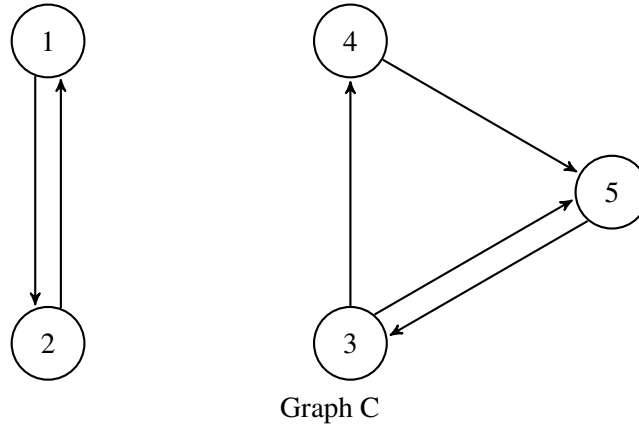
Graph C

(f) Write out the adjacency matrix for graph C.

(g) For graph C, how many paths are there from webpage 1 to webpage 3?

(h) **(PRACTICE)** Find the eigenspace that corresponds to the steady-state for graph C. How many independent systems (disjoint sets of webpages) are there in graph C versus in graph B? What is the dimension of the eigenspace corresponding to the steady-state for graph C?

3. Noisy Images

In lab, we used a single pixel camera to capture many measurements of an image $\vec{x}$. A single measurement y_i is captured using a mask $\vec{a}_i$ such that $y_i = \vec{a}_i^T \vec{x}$. Many measurements can be expressed as a matrix-vector multiplication of the masks with the image, where the masks lie along the rows of the matrix.

$$\begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix} = \begin{bmatrix} \vec{a}_1 \\ \vdots \\ \vec{a}_N \end{bmatrix} \vec{x} \tag{2}$$

$$\vec{y} = \mathbf{A}\vec{x} \tag{3}$$

In the real world, noise, $\vec{n}$, creeps into our measurements, so instead,

$$\vec{y} = \mathbf{A}\vec{x} + \vec{n} \tag{4}$$

(a) Express $\vec{x}$ in terms of $\mathbf{A}$ (or its inverse), $\vec{y}$, and $\vec{n}$. (*Hint:* Think about what you did in the imaging lab.)

(b) Now, because there is noise in our measurements, there will be noise in our recovered image, however, the noise is scaled. The noise in the recovered image, $\hat{\vec{w}}$, is related to $\vec{w}$, but it is transformed by $\mathbf{A}^{-1}$. Specifically,

$$\hat{\vec{w}} = \mathbf{A}^{-1}\vec{w} \tag{5}$$

To analyze how this transformation alters $\vec{w}$, consider representing $\vec{w}$ as a linear combination of the eigenvectors of $\mathbf{A}^{-1}$,

$$\vec{w} = \alpha_1 \vec{b}_1 + \ldots + \alpha_N \vec{b}_N, \tag{6}$$

where, $\vec{b}_i$ is $\mathbf{A}^{-1}$'s eigenvector with eigenvalue λ_i. Now we can express the recovered image's noise as,

$$\hat{\vec{w}} = \mathbf{A}^{-1}\vec{w} = \mathbf{A}^{-1}(\alpha_1\vec{b}_1 + \ldots + \alpha_N\vec{b}_N) \tag{7}$$

$$= \alpha_1\mathbf{A}^{-1}\vec{b}_1 + \ldots + \alpha_N\mathbf{A}^{-1}\vec{b}_N \tag{8}$$

$$= \alpha_1\lambda_1\vec{b}_1 + \ldots + \alpha_N\lambda_N\vec{b}_N \tag{9}$$

Depending on the size of the eigenvalues, noise in the recovered image will be amplified or attenuated. For eigenvectors with large eigenvalues, will the noise signal along those eigenvectors be amplified of attenuated? For eigenvectors with small eigenvalues, will the noise signal along those eigenvectors be amplified of attenuated?

(c) We are going to try different $\mathbf{A}$ matrices in this problem and compare how they deal with noise. Run the associated cells in the attached IPython notebook. What special matrix is $\mathbf{A}_1$? Are there any differences between the matrices $\mathbf{A}_2$ and $\mathbf{A}_3$?

(d) Run the associated cells in the attached IPython notebook. Notice that each plot returns the result of trying to image a noisy image as well as the minimum absolute value of the eigenvalue of each matrix. Comment on the effect of small eigenvalues on the noise in the image.

(e) Depending on how large or small the eigenvalues of $\mathbf{A}^{-1}$ are, we will amplify or attenuate our measurement's noise. These eigenvalues are actually related to the eigenvalues of $\mathbf{A}$! Inverting the matrix $\mathbf{A}$ turns these small eigenvalues into large eigenvalues. Show that if λ is an eigenvalue of a matrix $\mathbf{A}$, then $\frac{1}{\lambda}$ is an eigenvalue of the matrix $\mathbf{A}^{-1}$.

Hint: Start with an eigenvalue λ and one corresponding eigenvector $\vec{v}$, such that they satisfy $\mathbf{A}\vec{v} = \lambda\vec{v}$.

4. The Dynamics of Romeo and Juliet's Love Affair

In this problem, we will study a discrete-time model of the dynamics of Romeo and Juliet's love affair—adapted from Steven H. Strogatz's original paper, *Love Affairs and Differential Equations*, Mathematics Magazine, 61(1), p.35, 1988, which describes a continuous-time model.

Let $R[n]$ denote Romeo's feelings about Juliet on day n, and let $J[n]$ denote Juliet's feelings about Romeo on day n. The sign of $R[n]$ (or $J[n]$) indicates like or dislike. For example, if $R[n] > 0$, it means Romeo likes Juliet. On the other hand, $R[n] < 0$ indicates that Romeo dislikes Juliet. $R[n] = 0$ indicates that Romeo has a neutral stance towards Juliet.

The magnitude (i.e. absolute value) of $R[n]$ (or $J[n]$) represents the intensity of that feeling. For example, a larger $|R[n]|$ means that Romeo has a stronger emotion towards Juliet (love if $R[n] > 0$ or hatred if $R[n] < 0$). Similar interpretations hold for $J[n]$.

We model the dynamics of Romeo and Juliet's relationship using the following linear system:

$$R[n+1] = aR[n] + bJ[n], \quad n = 0, 1, 2, \ldots$$

and

$$J[n+1] = cR[n] + dJ[n], \quad n = 0, 1, 2, \ldots,$$

which we can rewrite as

$$\vec{s}[n+1] = \mathbf{A}\,\vec{s}[n],$$

where $\vec{s}[n] = \begin{bmatrix} R[n] \\ J[n] \end{bmatrix}$ denotes the state vector and $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ the state transition matrix for our dynamic system model.

The selection of the parameters a, b, c, d results in different dynamic scenarios. The fate of Romeo and Juliet's relationship depends on these model parameters (i.e. a, b, c, d) in the state transition matrix and the initial state ($\vec{s}[0]$). In this problem, we'll explore some of these possibilities.

(a) Consider the case where $a + b = c + d$ in the state-transition matrix

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Show that

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

is an eigenvector of $\mathbf{A}$, and determine its corresponding eigenvalue λ_1. Show that

$$\vec{v}_2 = \begin{bmatrix} b \\ -c \end{bmatrix}$$

is an eigenvector of $\mathbf{A}$, and determine its corresponding eigenvalue λ_2. Now, express the first and second eigenvalues and their eigenspaces in terms of the parameters $a, b, c,$ and d.

Hint: You could use the characteristic polynomial approach to find the eigenvalues and eigenvectors. You may find it easier to use the following approach instead:

- First find λ_1 by showing $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of $\mathbf{A}$.

- Then find λ_2 by showing $\vec{v}_2 = \begin{bmatrix} b \\ -c \end{bmatrix}$ is an eigenvector of $\mathbf{A}$.

For parts (b) - (d), consider the following state-transition matrix:

$$\mathbf{A} = \begin{bmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{bmatrix}$$

(b) Determine the eigenpairs (i.e. $(\lambda_1, \vec{v}_1)$ and $(\lambda_2, \vec{v}_2)$) for this system. Note that this matrix is a special case of the matrix explored in part (a), so you can use results from that part to help you.

(c) Determine all of the *steady states* of the system. That is, find the set of points such that if Romeo and Juliet start at, or enter, any of those points, their states will stay in place forever: $\{\vec{s}_* \mid \mathbf{A}\vec{s}_* = \vec{s}_*\}$.

(d) Suppose Romeo and Juliet start from an initial state $\vec{s}[0] \in \text{span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$. What happens to their relationship over time? Specifically, what is $\vec{s}[n]$ as $n \to \infty$?

Now suppose we have the following state-transition matrix:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Use this state-transition matrix for parts (e) - (g).

(e) Determine the eigenpairs (i.e. $(\lambda_1, \vec{v}_1)$ and $(\lambda_2, \vec{v}_2)$) for this system. Note that this matrix is a special case of the matrix explored in part (a), so you can use results from that part to help you.

(f) Suppose Romeo and Juliet start from an initial state $\vec{s}[0] \in \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$. What happens to their relationship over time? Specifically, what is $\vec{s}[n]$ as $n \to \infty$?

(g) Now suppose that Romeo and Juliet start from an initial state $\vec{s}[0] \in \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$. What happens to their relationship over time? Specifically, what is $\vec{s}[n]$ as $n \to \infty$?

Finally, we consider the case where we have the following state-transition matrix:

$$\mathbf{A} = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$$

Use this state-transition matrix for parts (h) - (j).

(h) Determine the eigenpairs (i.e. $(\lambda_1, \vec{v}_1)$ and $(\lambda_2, \vec{v}_2)$) for this system. Note that this matrix is a special case of the matrix explored in part (a), so you can use results from that part to help you.

(i) Suppose Romeo and Juliet start from an initial state $\vec{s}[0] \in \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$. What happens to their relationship over time if $R[0] > 0$ and $J[0] < 0$? What about if $R[0] < 0$ and $J[0] > 0$? Specifically, what is $\vec{s}[n]$ as $n \to \infty$?

(j) Now suppose that Romeo and Juliet start from an initial state $\vec{s}[0] \in \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$. What happens to their relationship over time? Specifically, what is $\vec{s}[n]$ as $n \to \infty$?

5. (CHALLENGE PRACTICE) Is There A Steady State?

So far, we've seen that for a conservative state transition matrix $\mathbf{A}$, we can find the eigenvector, $\vec{v}$, corresponding to the eigenvalue $\lambda = 1$. This vector is the steady state since $\mathbf{A}\vec{v} = \vec{v}$. However, we've so far taken for granted that the state transition matrix even has the eigenvalue $\lambda = 1$. Let's try to prove this fact.

(a) Show that if λ is an eigenvalue of a matrix $\mathbf{A}$, then it is also an eigenvalue of the matrix $\mathbf{A}^T$.

Note: The transpose of the matrix is a new matrix where the columns of $\mathbf{A}$ are now along the rows of $\mathbf{A}^T$

Hint: The determinants of $\mathbf{A}$ and $\mathbf{A}^T$ are the same. This is because the volumes which these matrices represent are the same.

(b) Let a square matrix $\mathbf{A}$ have rows that sum to one. Show that $\vec{1} = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}^T$ is an eigenvector of $\mathbf{A}$. What is the corresponding eigenvalue?

(c) Let's put it together now. From the previous two parts, show that any conservative state transition matrix will have the eigenvalue $\lambda = 1$. Recall that conservative state transition matrices are those that have columns that sum to 1.

6. Homework Process and Study Group

Who else did you work with on this homework? List names and student ID's. (In case of homework party, you can also just describe the group.) How did you work on this homework?