1 Polynomial Interpolation

Given n distinct points, we can find a unique degree $n-1$ polynomial that passes through these points. Let the polynomial p be

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1}.$$

Let the n points be

$$p(x_1) = y_1, p(x_2) = y_2, \cdots, p(x_n) = y_n,$$

where $x_1 \neq x_2 \neq \cdots \neq x_n$.

We can construct a matrix-vector equation as follows to recover the polynomial p.

$$\underbrace{\begin{bmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \dots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^{n-1} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{bmatrix}}_{\vec{a}} = \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}}_{\vec{y}}$$

We can solve for the a values by setting:

$$\vec{a} = A^{-1}\vec{y}$$

Note that the matrix A is known as a Vandermonde matrix whose determinant is given by

$$\det(A) = \prod_{1 \leq i < j \leq n} (x_j - x_i)$$

Since $x_1 \neq x_2 \neq \cdots \neq x_n$, the determinant is non-zero and A is always invertible.

2 Polynomial Regression

Sometimes we may want to fit our data to a polynomial with an order less than $n-1$. If we fit the data to a polynomial of order $m < n$ we get:

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{m-1} x^{m-1}$$

Now when we construct the matrix-vector equation to recover polynomial p, we get:

$$\underbrace{\begin{bmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{m-1} \\ 1 & x_2 & x_2^2 & \dots & x_2^{m-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_m & x_m^2 & \dots & x_m^{m-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^{m-1} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{m-1} \end{bmatrix}}_{\vec{a}} = \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \\ \vdots \\ y_n \end{bmatrix}}_{\vec{y}}$$

With this matrix equation, we have n equations with m unknowns, which means our system is over-defined (since $m < n$). One way to find the best fitting a values for this polynomial is to use least-squares, where you set:

$$\vec{a} = (A^T A)^{-1} A^T \vec{y}$$

1. **Interpolation Example**

   Use polynomial interpolation to find the polynomial that passes through the points $(1,5)$, $(2,15)$ and $(3,33)$

   Solution:

   Setting up the matrix-vector equation:

$$\begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 15 \\ 33 \end{bmatrix}$$

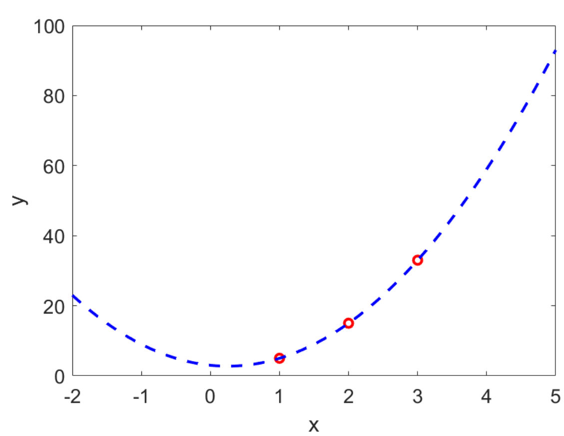
$$\vec{a} = A^{-1} \vec{y}$$

   using row reduction:

$$A^{-1} = \begin{bmatrix} 3 & -3 & 1 \\ -\frac{5}{2} & 4 & -\frac{3}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{bmatrix}$$

$$\vec{a} = \begin{bmatrix} 3 & -3 & 1 \\ -\frac{5}{2} & 4 & -\frac{3}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 5 \\ 15 \\ 33 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}$$

$$p(x) = 3 - 2x + 4x^2$$



2. Regression Example

Using least-squares, find the best-fit quadratic equation for the data set: $(-2,28)$, $(-1,-14)$, $(0,0)$, $(1,-42)$, and $(2,56)$.

Solution:

$$\begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \\ 1 & x_4 & x_4^2 \\ 1 & x_5 & x_5^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 4 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 28 \\ -14 \\ 0 \\ -42 \\ 56 \end{bmatrix}$$

For least-squares, we set:

$$\vec{a} = (A^T A)^{-1} A^T \vec{y}$$

$$(A^T A) = \begin{bmatrix} 5 & 0 & 10 \\ 0 & 10 & 0 \\ 10 & 0 & 34 \end{bmatrix}$$

Using row reduction:

$$(A^T A)^{-1} = \begin{bmatrix} \frac{17}{35} & 0 & -\frac{1}{7} \\ 0 & \frac{1}{10} & 0 \\ -\frac{1}{7} & 0 & \frac{1}{14} \end{bmatrix} = \frac{1}{14} \begin{bmatrix} 6.8 & 0 & -2 \\ 0 & 1.4 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

$$(A^T A)^{-1} A^T = \frac{1}{14} \begin{bmatrix} 6.8 & 0 & -2 \\ 0 & 1.4 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \\ 4 & 1 & 0 & 1 & 4 \end{bmatrix} = \frac{1}{14} \begin{bmatrix} -1.2 & 4.8 & 6.8 & 4.8 & -1.2 \\ -2.8 & -1.4 & 0 & 1.4 & 2.8 \\ 2 & -1 & -2 & -1 & 2 \end{bmatrix}$$

$$(A^T A)^{-1} A^T \vec{y} = \frac{1}{14} \begin{bmatrix} -1.2 & 4.8 & 6.8 & 4.8 & -1.2 \\ -2.8 & -1.4 & 0 & 1.4 & 2.8 \\ 2 & -1 & -2 & -1 & 2 \end{bmatrix} \begin{bmatrix} 28 \\ -14 \\ 0 \\ -42 \\ 56 \end{bmatrix}$$

We can factor out a 14 from $\vec{y}$ to cancel out with the $\frac{1}{14}$:

$$(V^TV)^{-1}V^T\vec{y} = \begin{bmatrix} -1.2 & 4.8 & 6.8 & 4.8 & -1.2 \\ -2.8 & -1.4 & 0 & 1.4 & 2.8 \\ 2 & -1 & -2 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 0 \\ -3 \\ 4 \end{bmatrix}$$

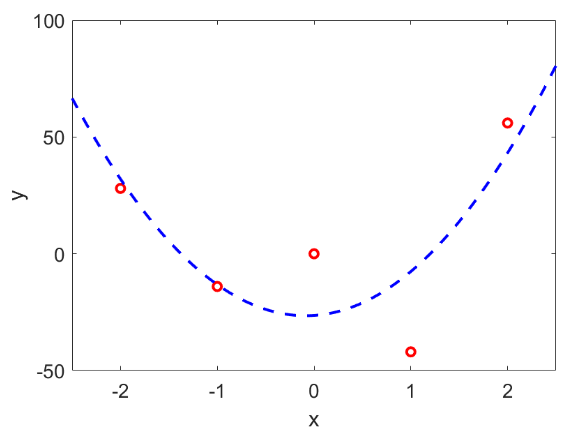
$$(V^TV)^{-1}V^T\vec{y} = \begin{bmatrix} -26.4 \\ 2.8 \\ 16 \end{bmatrix}$$

$$\vec{a} = \begin{bmatrix} -26.4 \\ 2.8 \\ 16 \end{bmatrix}$$

So the best fit quadratic equation is:

$$p(x) = -26.4 + 2.8x + 16x^2$$

Here's a graph showing the best-fit curve with the data points:



3. Minimum Norm Polynomial Interpolation

We have two data points: (0,0) and (1,1).

(a) Find a linear fit curve for the two data points.

Solution:

$$p(x) = x$$

(b) Find the second order polynomial for which the coefficients have the smallest norm. Compare the norm of the result to the first order polynomial found in part (a). Note that a first order polynomial is also a (degenerate) second order polynomial.

Solution:

To fit to a 2nd order polynomial, we would have the matrix-vector equation:

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

To solve for $\vec{a}$, we can use the pseudo inverse. Recall from last discussion that using the pseudo inverse gives the minimum norm solution.

$$\vec{a} = A^{\dagger}\vec{y} = V_1 S^{-1} U_1^T \vec{y}$$

Where V_1, S and U_1 come from the SVD of A.

If you take the SVD of A, you'll get:

$$U_1 = \begin{bmatrix} 0.3827 & 0.9239 \\ 0.9239 & -0.3827 \end{bmatrix}$$

$$S = \begin{bmatrix} 1.8478 & 0 \\ 0 & 0.7654 \end{bmatrix}$$

$$V_1 = \begin{bmatrix} 0.7071 & 0.7071 \\ 0.5 & 0.5 \\ 0.5 & -0.5 \end{bmatrix}$$

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0.7071 & 0.7071 \\ 0.5 & 0.5 \\ 0.5 & -0.5 \end{bmatrix} \begin{bmatrix} 0.5412 & 0 \\ 0 & 1.3066 \end{bmatrix} \begin{bmatrix} 0.3827 & 0.9239 \\ 0.9239 & -0.3827 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.5 \\ 0.5 \end{bmatrix}$$

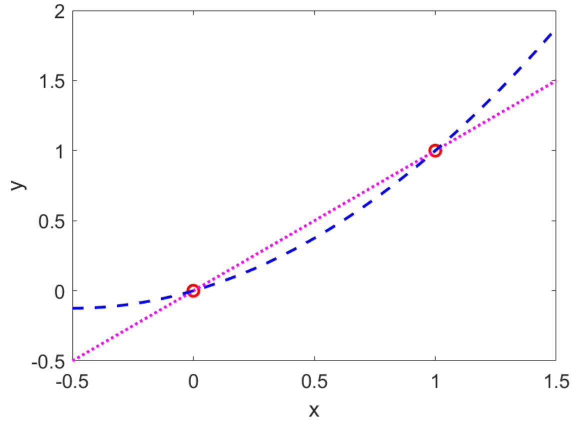
Which means the corresponding polynomial is:

$$p(x) = \frac{1}{2}x + \frac{1}{2}x^2$$

The norm of the coefficients is:

$$||\vec{a}|| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{1}{\sqrt{2}}$$

The norm of the linear approximation is 1. Surprisingly, the polynomial fit with the smallest norm is not the linear fit, but a quadratic fit.



2
1.5
1
0.5
0
-0.5
y
-0.5
0
0.5
1
1.5
x