EECS 16B Designing Information Devices and Systems II
Fall 2018 Elad Alon and Miki Lustig Discussion 11A

See also attached ipython notebook.

Notes

Interpolation with Basis Functions

Assume there exists a set of functions $\phi_i(x)$ such that

$$\phi_i(x_i) = 1 \quad \text{and} \quad \phi_i(x_j) = 0 \text{ when } j \neq i$$

We can interpolate between the data points (x_i, y_i) with the function

$$f(x) = \sum_{k=1}^{n} y_k \phi_k(x) \quad \text{because } f(x_i) = \sum_{k=1}^{n} y_k \phi_k(x_i) = y_i$$

We call this set of functions "basis functions".

Sampling theorem

Let f be a signal bandlimited by frequency ω_{max}, and we sample with a period of Δ then we can write the sinc-interpolated signal $\hat{f}$

$$\hat{f} = \sum_{n=-\infty}^{\infty} y[n]\Phi(x - n\Delta)$$

Where $\Phi(x) = \text{sinc}\left(\frac{x}{\Delta}\right)$

Then we can recover the signal, i.e. $f = \hat{f}$, if $\omega_{max} < \frac{\pi}{\Delta}$

Questions

1. **Interpolation**

   Samples from the sinusoid $f(x) = \sin(0.2\pi x)$ are shown in Figure 1. Draw the results of interpolation using each of the following three methods:

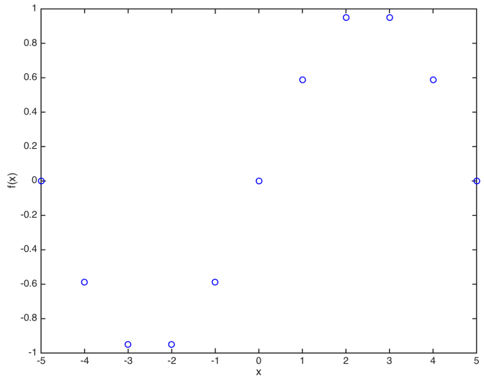

Figure 1: Samples of $f(x)$.

.

(a) Zero order hold interpolation.

Answer: Figure 2.

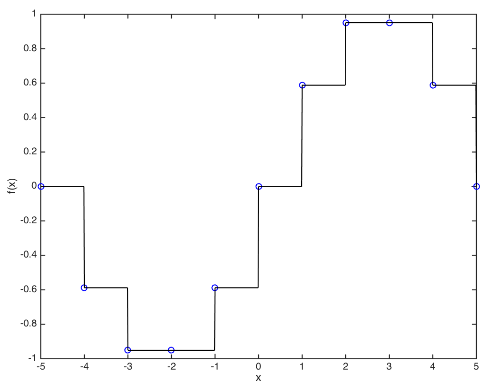

Figure 2: Zero order hold.

.

(b) Linear interpolation.

Answer: Figure 3.

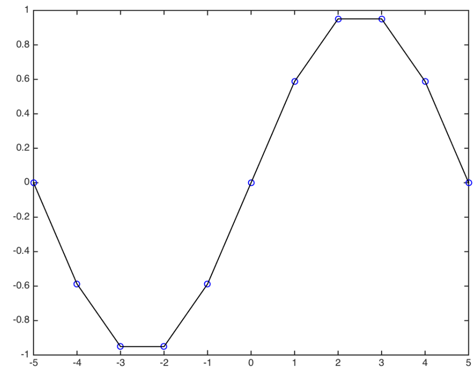

Figure 3: Linear interpolation.

.

(c) Sinc interpolation assuming the Nyquist limit has been satisfied.

Answer: Figure 4.

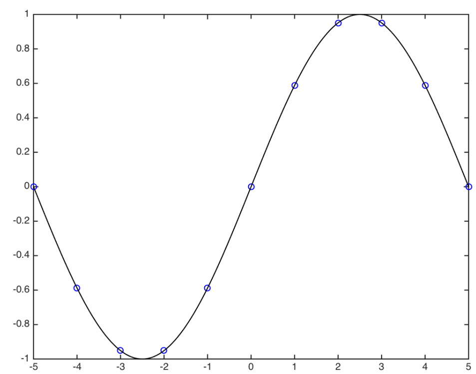

Figure 4: Sinc interpolation.

.

2. Sampling Theorem basics

Consider the following signal, $f(x)$ defined as,

$$f(x) = \cos(2\pi x)$$

(a) Find the maximum frequency, ω_{max}, in radians per second? In Hertz? (From now on, frequencies will refer to radians per second.)

Answer: $\omega_{\text{max}} = 2\pi$ in radians per second, which is 1 Hertz.

(b) What is the smallest sampling Δ that would result in an imperfect reconstruction?

Answer: From the sampling theorem, we know that Δ has an upperbound of $\frac{\pi}{\omega_{max}}$ for perfect reconstruction. Hence the smallest Δ for which we cannot reconstruct our signal is,

$$\Delta = \frac{\pi}{2\pi} = \frac{1}{2}$$

.

(c) If I sample every Δ_s seconds, what is the sampling frequency?

Answer: $\omega_s = \frac{2\pi}{\Delta s}$.

3. More Sampling

Let's sample the signal from the previous question f with sampling period $\Delta_m = \frac{1}{4}$s and $\Delta_n = 1$s and perform sinc interpolation on the resulting samples. Let the reconstructed functions be g_m and g_n.

(a) Have we satisfied the Nyquist limit (i.e. the sampling theorem) in any case?

Answer: To satisfy the Nyquist limit, we need the sampling period $\Delta < \frac{1}{2}$. Hence, Δ_m satisfies Nyquist, but Δ_n does not.

(b) What is the highest frequency we can reconstruct with the sampling rate Δ_n?

Answer: The sinc functions used to reconstruct g_n are,

$$\left\{ \text{sinc}\left(\frac{t-k}{1}\right) \right\}_{k\in\mathbb{Z}}.$$

These functions can represent a maximum frequency of π.

(c) Based on this answer, can you think of any periodic function that has a frequencies less than or equal to π that samples the same as g_n?

Answer: Since the frequencies vary from 0 to π, the smallest period that can be represented is 2. That is to say, functions of period < 2 cannot be captured with the sinc function derived from Δ_n. Since the period must be greater than 2, no sine or cosin function can give the same samples as g_n. This means suggests looking into a fairly trivial kind of periodic function: a constant. In particular, the answer to this problem is the constant function that is 1 everywhere.

4. Aliasing

Consider the signal $f(x) = sin(0.2\pi x)$.

(a) At what period T should we sample so that sinc interpolation recovers a function that is identically zero?

Answer: We want to sample such that our resultant discrete time signal is all zeros. To do this, we can sample at $x = 5k$, for integral values of k. Hence, T = 5.

(b) At what period T should we sample so that sinc interpolation recovers the function $g(x) = -\sin\left(\frac{\pi}{15}x\right)$?

Answer: T = 7.5

$$\begin{aligned}
f[n] &= \sin(0.2\pi nT) && \text{sampling f(x)}\\
&= \cos\left(0.2\pi nT - \frac{\pi}{2}\right) && \sin(x) = \cos\left(x - \frac{\pi}{2}\right)\\
&= \cos\left(2\pi n - (0.2\pi nT - \frac{\pi}{2})\right) && \cos(x) = \cos(2\pi - x)\\
&= \cos\left((2\pi - 0.2\pi T)n + \frac{\pi}{2}\right) && \\
&= -\sin\big((2\pi - 0.2\pi T)n\big) && -\sin(x) = \cos\left(x + \frac{\pi}{2}\right)\\
&= -\sin\left(\frac{\pi}{15}nT\right) && \text{equivalent to sampling g(x)}\\
2\pi - 0.2\pi T &= \frac{\pi}{15}T &&\\
T &= 7.5 &&
\end{aligned}$$