Notes

Properties of Discrete Time Systems

Consider a discrete-time system with $x[n]$ as input and $y[n]$ as output.

$$x[n] \longrightarrow \boxed{} \longrightarrow y[n]$$

The following are some of the possible properties that a system can have:

Causality

A **causal** system has the property that $y[n_0]$ only depends on $x[n]$ for $n \in (-\infty, n_0]$. An intuitive way of interpreting this condition is that the system does not "look ahead."

Linearity

A **linear system** has the properties below:

(a) **additivity**

$$x_1[n] + x_2[n] \longrightarrow \boxed{} \longrightarrow y_1[n] + y_2[n] \tag{1}$$

(b) **scaling**

$$\alpha x[n] \longrightarrow \boxed{} \longrightarrow \alpha y[n] \tag{2}$$

Here, α is some constant.

Together, these two properties are known as **superposition**:

$$\alpha_1 x_1[n] + \alpha_2 x_2[n] \longrightarrow \boxed{} \longrightarrow \alpha_1 y_1[n] + \alpha_2 y_2[n]$$

Bounded-Input, Bounded-Output (BIBO) Stability

In a BIBO stable system, if $x[n]$ is bounded, then $y[n]$ is also bounded. A signal $a[n]$ is bounded if there exists a A such that $|a[n]| \leq A < \infty \ \forall n$.

Time invariance

A system is **time-invariant** if its behavior is fixed over time:

$$x[n-n_0] \longrightarrow \boxed{} \longrightarrow y[n-n_0] \tag{3}$$

Linear Time Invariant (LTI) Systems

A system is LTI if it is both linear and time invariant. Let $h[n]$ be the **impulse response** of an LTI system. That is, $y[n] = h[n]$ if $x[n] = \delta[n]$, where $\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases}$ is the unit impulse.

An LTI system can be completely characterized by $h[n]$. The following properties hold:

- An LTI system is causal iff $h[n] = 0 \ \forall n < 0$.
- An LTI system is BIBO stable iff its impulse reponse is absolutely summable:

$$\sum_{n=-\infty}^{\infty} \left|h[n]\right| < \infty$$

Convolution Sum

Consider the following LTI system with impulse reponse $h[n]$

$$x[n] \longrightarrow \boxed{} \longrightarrow y[n]$$

Notice that we can write $x[n]$ as a sum of impulses:

$$x[n] = \ldots + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + \ldots = \sum_{m=-\infty}^{\infty} x[m]\delta[n-m]$$

In addition, we know that:

$$\delta[n] \longrightarrow \boxed{} \longrightarrow h[n]$$

By applying the LTI property of our system, we get that

$$x[n] = \sum_{m=-\infty}^{\infty} x[m]\delta[n-m] \longrightarrow \boxed{} \longrightarrow y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$

The expression $\sum_{m=-\infty}^{\infty} x[m]h[n-m]$ is known as the **convolution sum** and can be written as $x[n] * h[n]$ or $(x*h)[n]$

Questions

1. Circulant Time-Shift Systems

Imagine we have a system $S_{\to 2}$ that takes any length 5 input signal and circularly shifts it by two steps. That is, the last two entries roll over to the start and the rest are moved to the right. For example, $S_{\to 2}([3,1,4,1,5]) = [1,5,3,1,4]$.

(a) Is this system linear? That is, for any signals $\vec{x}$ and $\vec{y}$, does $S_{\to 2}$ fulfill properties (1) and (2)?

Solution: Yes. **Answer:** Yes.

(b) What does $S_{\to 2}$ look like when written as a matrix?

Solution:

$$S_{\to 2} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Answer:

$$S_{\to 2} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Determine if the following systems are linear, time-invariant, and/or causal.

2. (a) $y[t] = 2x[-2+3t] + 2x[2+3t]$

Solution: linear, not time-invariant, not causal

Let $\hat{x}[t] = x[t-t_0]$ be a delayed input signal. Then, the corresponding output $\hat{y}[t]$ is equal to $2x[-2+3t-t_0]+2x[2+3t-t_0]$

However, we can see that $\hat{y}[t] \neq y[t-t_0] = 2x[-2+3(t-t_0)] + 2x[2+3(t-t_0)]$ **Answer:** linear, not time-invariant, not causal

Let $\hat{x}[t] = x[t-t_0]$ be a delayed input signal. Then, the corresponding output $\hat{y}[t]$ is equal to $2x[-2+3t-t_0]+2x[2+3t-t_0]$

However, we can see that $\hat{y}[t] \neq y[t-t_0] = 2x[-2+3(t-t_0)] + 2x[2+3(t-t_0)]$

(b) $y[t] = 4^{x[t]}$

Solution: non-linear, time-invariant, causal

Let $\hat{x}[t] = 2x[t]$. Then $\hat{y}[t] = 16^{x[t]} \neq 2y[t]$ **Answer:** non-linear, time-invariant, causal

Let $\hat{x}[t] = 2x[t]$. Then $\hat{y}[t] = 16^{x[t]} \neq 2y[t]$

(c) $y[t] - y[t-1] + y[t-2] = x[t] - x[t-1] - x[t-2]$

Solution: linear, time-invariant, causal **Answer:** linear, time-invariant, causal

(d) $y[t] = x[t] + tx[t-1]$

Solution: linear, not time-invariant, causal **Answer:** linear, not time-invariant, causal

(e) $y[t] = 2^t cos(x[t])$

Solution: not linear, not time-invariant, causal **Answer:** not linear, not time-invariant, causal

3. Convoluted Convolution

Show that convolution is commutative. That is, show that $(x*h)[n] = (h*x)[n]$

Solution:

$$\begin{aligned}(x*h)[n] &= \sum_{m=-\infty}^{\infty} x[m]h[n-m] \\ &= \sum_{k=-\infty}^{\infty} x[n-k]h[k] && \text{Let } k = n-m \\ &= \sum_{k=-\infty}^{\infty} h[k]x[n-k] \\ &= (h*x)[n]\end{aligned}$$

Answer:

$$\begin{aligned}(x*h)[n] &= \sum_{m=-\infty}^{\infty} x[m]h[n-m] \\ &= \sum_{k=-\infty}^{\infty} x[n-k]h[k] && \text{Let } k = n-m \\ &= \sum_{k=-\infty}^{\infty} h[k]x[n-k] \\ &= (h*x)[n]\end{aligned}$$

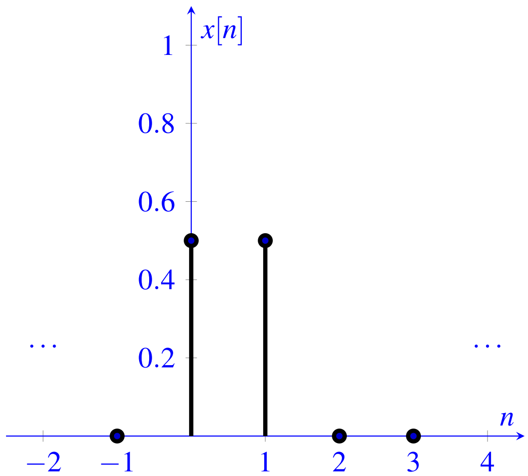
4. Mystery System

Consider an LTI system with impulse response

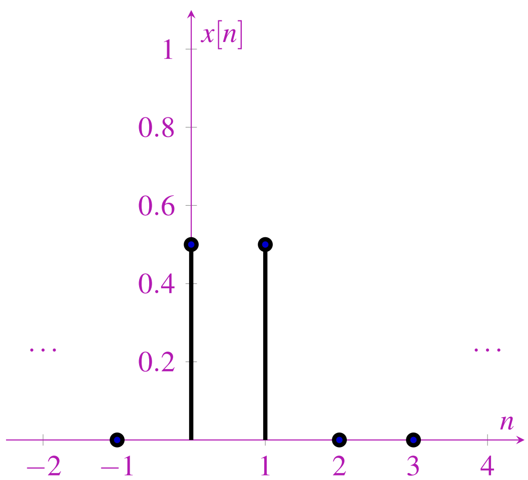
$$h[n] = \frac{1}{2}(\delta[n] + \delta[n-1])$$

(a) Create a sketch of this impulse reponse. Is this a finite or infinite impulse response system?

Solution: This is an FIR system.

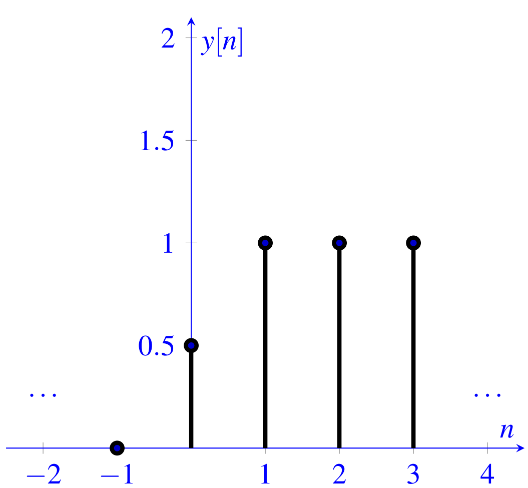


Answer: This is an FIR system.

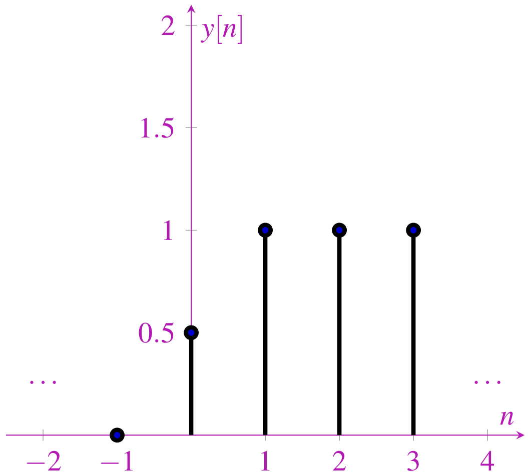


(b) What is the output of our system if the input is the unit step $U[n]$?

Solution:

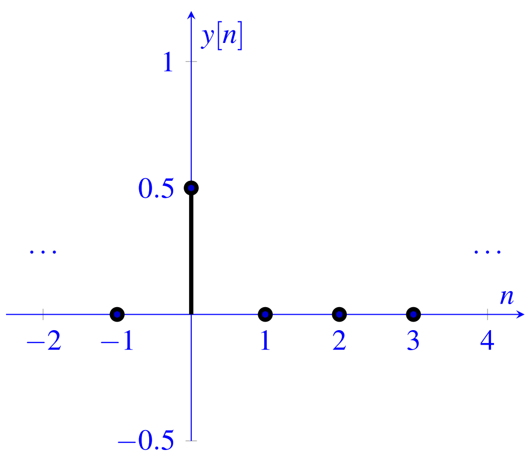


Answer:

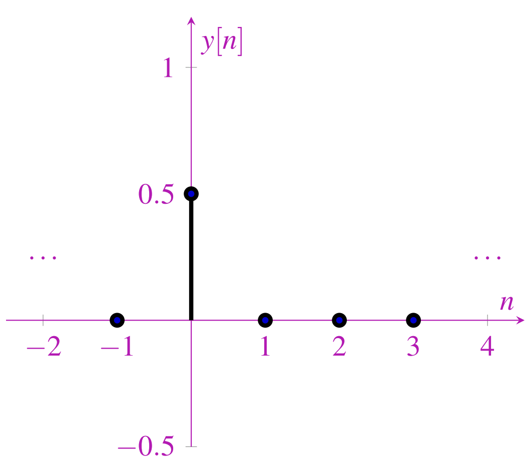


(c) What is the output of our system if our input is $x[n] = (-1)^n U[n]$?

Solution:



Answer:



(d) This system is called the two-point simple moving average (SMA) filter. Based on the previous parts, why do you thing it bears this name?

Solution: The output of the system at each timestep n is the average of $x[n]$ and $x[n-1]$. This sort of system can be used in areas like technical analysis to gain insight into stock prices and trends (usually these methods would use a longer window than just two days). There are also other variants used like the exponential moving average (EMA) filter. **Answer:** The output of the system at each timestep n is the average of $x[n]$ and $x[n-1]$. This sort of system can be used in areas like technical analysis to gain insight into stock prices and trends (usually these methods would use a longer window than just two days). There are also other variants used like the exponential moving average (EMA) filter.