Homework 11

This homework is due Wednesday, November 21, 2018, at 11:59 pm. Self grades are due Monday, November 26, 2018, at 11:59 pm.

1. **Complex Transpose (Mechanical)**

   Find the complex transpose A^* of the following matrices:

   (a) $A = \begin{bmatrix} 1-3j & 7+j \\ j & 1+j \\ 12 & 9-6j \end{bmatrix}$

   (b) $A = \begin{bmatrix} 13 & 4+j \\ 4-j & 2 \end{bmatrix}$

2. **Complex Inner Product (Mechanical)**

   Calculate the complex inner product $\langle \vec{u}, \vec{v} \rangle$ for the following set of vectors:

   (a) $\vec{u} = \begin{bmatrix} 1 \\ 1-j \\ -2j \\ 1+1j \end{bmatrix}, \vec{v} = \begin{bmatrix} 1-3j \\ 2j \\ -2+j \\ 1+4j \end{bmatrix}$

   (b) $\vec{u} = \begin{bmatrix} 1-3j \\ 2j \\ -2+j \\ 1+4j \end{bmatrix}, \vec{v} = \begin{bmatrix} 1 \\ 1-j \\ -2j \\ 1+1j \end{bmatrix}$

   (c) $\vec{u} = \begin{bmatrix} 1 \\ 1-j \\ -2j \\ 1+1j \end{bmatrix}, \vec{v} = \begin{bmatrix} 1 \\ 1-j \\ -2j \\ 1+1j \end{bmatrix}$

3. **Change of Basis (Mechanical)**

   We have a vector $\vec{x} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$. Perform a change of basis on $\vec{x}$ using the orthonormal basis $\hat{b}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ and $\hat{b}_2 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$.

4. **LTI Response Length (Mechanical)**

(a) You have a discrete-time LTI system with input $u[n]$ and output $y[n]$. The system has a finite impulse response $h[n]$ of length 4. If we input a signal that has a length of 10, then what will the length of the output be?

(b) Find the convolution between $x[n]$ and $h[n]$ where:

$$x[n] = \delta[n] + 4\delta[n-1] - 2\delta[n-3]$$
$$h[n] = \delta[n] - \frac{1}{2}\delta[n-1]$$

5. Roots of Unity

The DFT is a coordinate transformation to a basis made up of roots of unity. In this problem we explore some properties of the roots of unity. An Nth root of unity is a complex number z satisfying the equation $z^N = 1$ (or equivalently $z^N - 1 = 0$).

(a) Show that $z^N - 1$ factors as

$$z^N - 1 = (z-1)\left(\sum_{k=0}^{N-1} z^k\right).$$

(b) Show that any complex number of the form $W_N^k = e^{j\frac{2\pi}{N}k}$ for $k \in \mathbb{Z}$ is an N-th root of unity.

(c) Draw the fifth roots of unity in the complex plane. How many unique fifth roots of unity are there?

(d) Let $W_5 = e^{j\frac{2\pi}{5}}$. What is another expression for W_5^{42}?

(e) What is the complex conjugate of W_5? What is the complex conjugate of W_5^{42}? What is the complex conjugate of W_5^4?

6. LTI Low Pass Filters

Given a sequence of discrete samples with high frequency noise, we can de-noise our signal with a discrete low-pass filter. Two examples are given below:

$$y[n] = 0.5y[n-1] + x[n] \tag{1}$$
$$y[n] = 0.25x[n] + 0.25x[n-1] + 0.25x[n-2] + 0.25x[n-3] \tag{2}$$

(a) Write the impulse responses $h[n]$ for (1) and (2). Are they IIR or FIR?

(b) Are either of these filters causal? Are either of these filters stable?

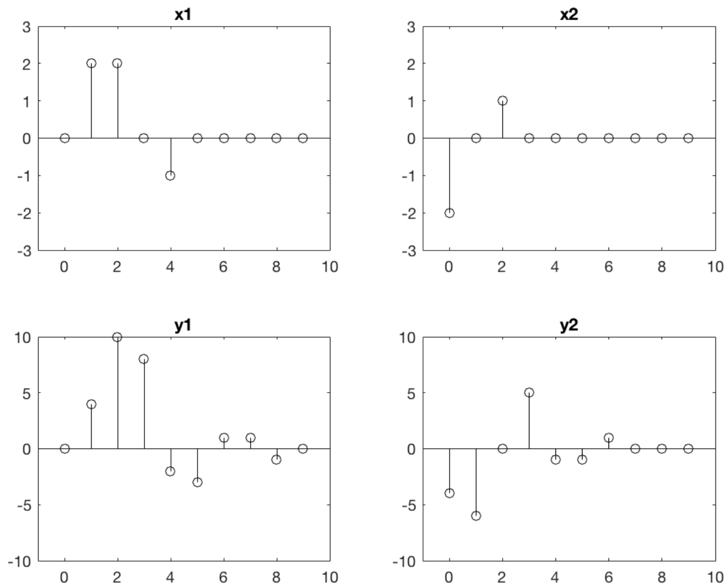
(c) Give the output y for each filter given the input sequence $x[n] = 2cos(\pi n) + n$ from $n = 0$ to $n = 7$. Assume that $y[n] = 0$ for $n < 0$.

7. LTI Inputs

We have an LTI system whose exact characteristics we do not know. However, we know that it has a finite impulse response that isn't longer than 5 samples. We also observed two sequences, x_1 and x_2, pass through the system and observed the system's responses y_1 and y_2.

x_1	0	2	2	0	-1	0	0	0	0	0
y_1	0	4	10	8	-2	-3	1	1	-1	0

x_1	-2	0	1	0	0	0	0	0	0	0
y_1	-4	-6	0	5	-1	-1	1	0	0	0



(a) Given the above sequences, what would be the output for the input:

x_3	0	0	-2	0	1	0	0	0	0	0

(b) What is the output of the system for the input $x_1 - x_2$?

(c) Given the above information, how could you find the impulse response of this system? What is the impulse response?

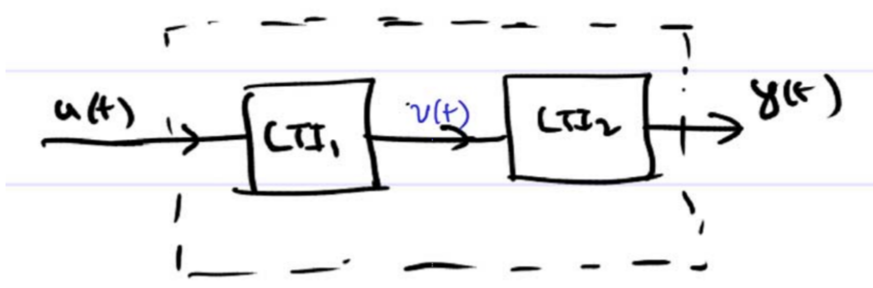
(d) What is the output of this system for the following input:

x_4	1	1	1	0	0	0	0	0	0	0

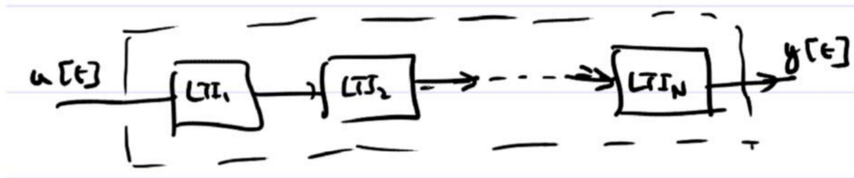
8. LTI Exam Question

This question is from spring 2017's final exam.

(a) Prove that the composition of two LTI systems is LTI. In other words, that if each block in the figure below is LTI, then $u(t) \rightarrow y(t)$ is LTI.



(b) Using reasoning similar to part (a), it is easy to show that the composition of N LTI systems, as depicted below, is LTI.



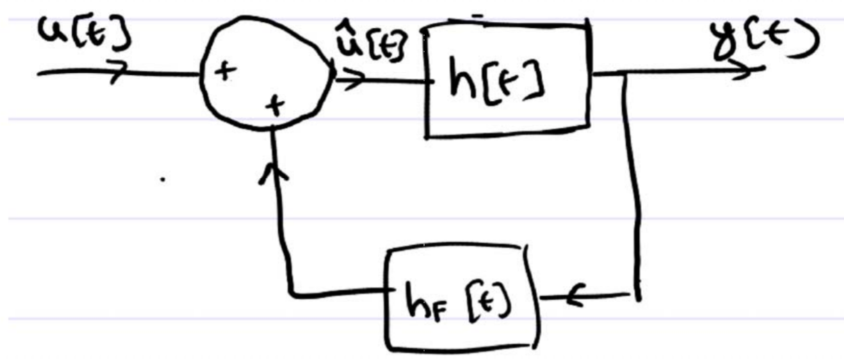
Suppose all the internal LTI blocks LTI_1 to LTI_N are identical, with impulse response

$$h[t] = \begin{cases} 1, & t = 1 \\ 0, & \text{otherwise} \end{cases}$$

Find the impulse response $h_c[t]$ of the composed system, i.e. from $u[t] \to y[t]$.

(c) You are a discrete-time, causal, LTI system with impulse response $h[t]$.
You are not told what $h[t]$ is.
However, you **are** told that if the input $u[t]$ to the system is chosen to be $h[t]$, then the first five samples of the output $y[t]$ are: $y[0] = 1$, $y[1] = 0$, $y[2] = -2$, $y[3] = 0$, $y[4] = 3$.
Find **two** different possible solutions for $h[t]$ (only for $t = 0, \cdots, 4$) that satisfy the above condition. Are other solutions for $t = 0, \cdots, 4$ also possible?

(d) Given two discrete-time LTI systems in a feedback loop:



with $h[t] = \begin{cases} 1, & t = 0 \\ -1, & t = 1 \\ 0, & \text{otherwise} \end{cases}$, and $h_F[t] = \begin{cases} 1, & t = 1 \\ 0, & \text{otherwise} \end{cases}$

Please read the expressions above for $h[t]$ and $h_F[t]$ **very carefully** and make sure you understand them right. Note also that the feedback **adds** to the input.

Assuming that $y[t] = 0$ for all $t < 0$, find the impulse response $h_c[t]$ of the closed-loop system (i.e. from $u[t] \to y[t]$).

Hint: write out $y[t]$ for $t = 0, \cdots, 15$ at least (not required, but highly recommended), examine the values, and use it to devise a general formula for $y[t]$. Be very careful to avoid mistakes in your calculations.

(e) Is the system in part (d) BIBO stable or unstable? If stable, explain why. If unstable, write an input $u[t]$ that would make $y[\infty] \to \infty$

9. Write Your Own Question And Provide a Thorough Solution.

Writing your own problems is a very important way to really learn material. The famous "Bloom's Taxonomy" that lists the levels of learning is: Remember, Understand, Apply, Analyze, Evaluate, and Create. Using what you know to create is the top level. We rarely ask you any homework questions about the lowest level of straight-up remembering, expecting you to be able to do that yourself (e.g. making flashcards). But we don't want the same to be true about the highest level. As a practical matter, having some practice at trying to create problems helps you study for exams much better than simply counting on solving existing practice problems. This is because thinking about how to create an interesting problem forces you to really look at the material from the perspective of those who are going to create the exams. Besides, this is fun. If you want to make a boring problem, go ahead. That is your prerogative. But it is more fun to really engage with the material, discover something interesting, and then come up with a problem that walks others down a journey that lets them share your discovery. You don't have to achieve this every week. But unless you try every week, it probably won't ever happen.