This homework is due on October 17, 2018 at 11:59pm
Self grades are due on October 22, 2018 at 11:59pm

1. LED Strip

I have an LED strip with 5 red LEDs whose brightnesses I want to set. These LEDs are addressed as a queue: at each time step, I can push a new brightness command between 0 and 255 to the left-most LED. Each of the following LEDs will then take on the brightness previously displayed by the LED immediately to its left.

(a) What should we use for our state vector? What does it mean that this is a state vector? What is our input?

Solution: We can use the brightnesses of each LED as our state vector. We can use these values as our state vector since together with the input, they describe everything about our system that we need to know in order to predict what our system will do in the future. Our input is the command to the left-most LED.

(b) Is our system linear? Assume that the input of the system will never exceed 255 or go below 0. If it is linear, write out the state equations in matrix form. Please choose a reasonable order for the state variables in the state vector.

Solution: The system is linear because it can be written in the form $\vec{x}(t+1) = A\vec{x}(t) + Bu$. Ordering the LED brightnesses in the state vector from left to right, we get:

$$\vec{x}(t+1) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u(t)$$

If you chose to put the left-most LED's brightness last in the state vector (so that the LEDs are ordered right to left and the state vector gets flipped upside down), the A matrix gets transposed and the B matrix is flipped upside down.

(c) Is this system controllable? Explain intuitively what this system's controllability means in terms of LED brightnesses.

Solution: Testing for controllability we see:

$$\begin{bmatrix} A^4B & A^3B & A^2B & AB & B \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

which has full rank. This means that the system is controllable. A system is called controllable if from any initial state, we can reach any final state that we desire at some time in the future.

For our LED strip, controllability means that we can display any set of brighnesses that we desire, but it may take a few time steps to get there.

(d) Is this system stable?

Solution: The eigenvalues of this discrete-time system are all 0, which is inside the unit circle. Therefore, the system is stable.

(e) Starting from the pattern of brightnesses (from left to right) [0, 127, 0, 255, 0], can we maintain this pattern for all future time steps? Can we display any fixed pattern of brightnesses for all time?

Solution: We cannot display [0, 127, 0, 255, 0] for all time. Immediately after we display this set of brightnesses, we will display [u(1), 0, 127, 0, 255].

If we want to display a fixed and unchanging set of brightnesses, every element in our state vector must be the same.

Controllability tells us only that we can *reach* any desired state (sometimes only temporarily). It does not mean we can *keep* our system at any desired state for all time.

2. Controllability in circuits

Consider the circuit in Figure 1, where V_s is an input we can control:

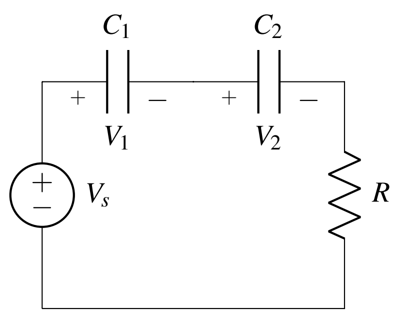


Figure 1: Controllability in circuits

(a) Write the state space model for this circuit.

Solution:

$$I = \frac{V_s - V_1 - V_2}{R} = C_1 \frac{dV_1}{dt} = C_2 \frac{dV_2}{dt}$$

$$\frac{d}{dt}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{RC_1} & -\frac{1}{RC_1} \\ -\frac{1}{RC_2} & -\frac{1}{RC_2} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{RC_1} \\ \frac{1}{RC_2} \end{bmatrix} V_s$$

(b) Show that this system is not controllable.

Solution: If we calculate AB, we find that it is a linear combination of B:

$$AB = \begin{bmatrix} -\frac{1}{RC_1}(\frac{1}{RC_1} + \frac{1}{RC_2}) \\ -\frac{1}{RC_2}(\frac{1}{RC_1} + \frac{1}{RC_2}) \end{bmatrix} = -(\frac{1}{RC_1} + \frac{1}{RC_2})B$$

This means that the controllability matrix

$$\begin{bmatrix} AB & B \end{bmatrix}$$

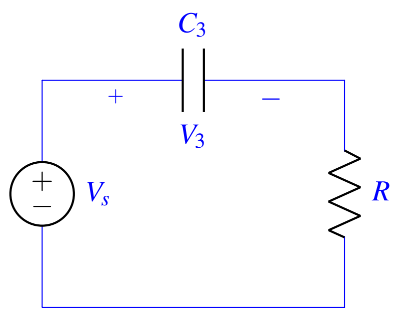
Must have rank of 1. Therefore, this system is not controllable.

(c) Explain, in terms of circuit currents and voltages, why this system isn't controllable. (Hint: think about what currents/voltages of the circuit we are controlling with V_s)

Solution: We can only control V_s, which in turn controls the amount of current flowing through the circuit. Since this current is equal through both capacitors and current directly affects the voltage across a capacitor, there is no way to individually control the voltages across the capacitors.

(d) Draw an equivalent circuit of this system that is controllable. What quantity can you control in this system?

Solution:



We can control V_3 in this circuit.

3. Controllability and discretization

In this problem, we will use the car model

$$\frac{d}{dt}p(t) = v(t)$$
$$\frac{d}{dt}v(t) = u(t)$$

that was discussed in class.

(a) Assuming that the input $u(t)$ can be varied continuously, is this system controllable?

Solution: Introducing states $x_1 = p$ and $x_2 = v$, we rewrite this system in state space form

$$\dot{x}(t) = Ax(t) + Bu(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t).$$

The controllability matrix

$$\mathscr{R}_n = \begin{bmatrix} AB & B \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

has rank 2. Therefore the continuous-time system is controllable.

(b) Now assume that we can only change our control input every T seconds. Derive a discrete-time state space model for the state updates, assuming that the input is held constant between times t and $t+T$.

Solution: By integrating both sides of the second equation from t to $t+T$ and keeping in mind that $u(t)$ is constant in this interval, that is $u(t+\tau) = u(t)$ for $\tau \in [0, T)$:

$$v(t+T) - v(t) = \int_0^T u(t+\tau)d\tau = Tu(t).$$

Now integrating the first equation and using the fact that $v(t+\tau) = v(t) + \tau u(t)$ we get

$$p(t+T) - p(t) = \int_0^T (v(t) + \tau u(t))d\tau = Tv(t) + \frac{1}{2}T^2u(t).$$

Introducing states $x_1(k) = p(kT)$ and $x_2(k) = v(kT)$ we get the state space model

$$x(k+1) = Ax(k) + Bu(k) = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} \frac{1}{2}T^2 \\ T \end{bmatrix} u(k).$$

(c) Is the discrete-time system controllable?

Solution: The controllability matrix

$$\mathscr{R}_n = \begin{bmatrix} AB & B \end{bmatrix} = \begin{bmatrix} \frac{3}{2}T^2 & \frac{1}{2}T^2 \\ T & T \end{bmatrix}$$

has rank 2. So the discrete-time system is controllable.

4. Buoyancy

An engineer would like to deploy an autonomous communications balloon (like Project Loon's balloons: `https://plus.google.com/+ProjectLoon/posts/PVitgyeYweY`) to provide internet connectivity to a particular geographical region. To provide reliable connectivity, the balloon must hold its position over the region it services. The balloon can control its altitude (a) by changing its buoyancy, but it doesn't have any engines. In order to move horizontally (horizontal position p), the balloon drifts on air currents.

Consulting meteorologists, the engineer has modeled the air currents around the desired balloon position (the point (0,0)) and found the flow field shown in Figure 2.

where the wind speed at each point is described by the equations:

$$v_p = -5p + 5a$$
$$v_a = -5p + 5a$$

Figure 2: Project Loon Balloon

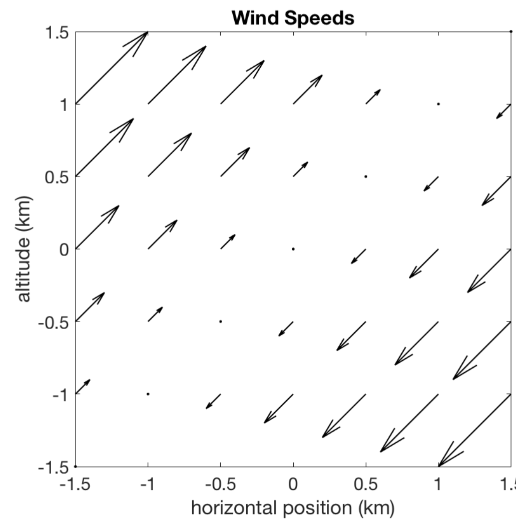


Figure 3: Wind speeds

where the velocities are in kilometers per hour and the horizontal position and altitude are in kilometers. Putting this together with the balloon's buoyancy control, the balloon's dynamics are described by:

$$\begin{bmatrix} \dot{p} \\ \dot{a} \end{bmatrix} = \begin{bmatrix} -5 & 5 \\ -5 & 5 \end{bmatrix} \begin{bmatrix} p \\ a \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u$$

(a) Is the system controllable?

Solution:

$$\mathscr{R}_n = \begin{bmatrix} AB & B \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 10 & 2 \end{bmatrix}$$

$\mathscr{R}_n$ is full rank, so the system is controllable.

(b) The engineer would like the balloon to converge to (0,0) with eigenvalues -2 and -10. What should be the state feedback gains K multiplying the original state vector $\vec{x}$ to achieve this behavior?

Solution:

$$A + BK = \begin{bmatrix} -5 & 5 \\ -5+2k_1 & 5+2k_2 \end{bmatrix}$$

$$\begin{vmatrix} -5-\lambda & 5 \\ -5+2k_1 & 5+2k_2-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - 2k_2\lambda - 10(k_1+k_2) = 0$$

We would like our characteristic polynomial to be:

$$(\lambda+2)(\lambda+10)=0$$
$$\lambda^2+12\lambda+20=0$$

Matching coefficients, we can solve for k_1 and k_2:

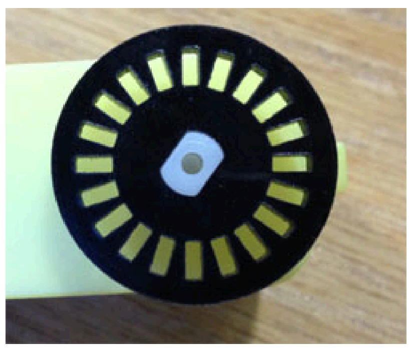
$$12=-2k_2$$

$$20=-10(k_1+k_2)$$

$$k_1=4 \quad \text{and} \quad k_2=-6$$

5. Understanding the SIXT33N Car Control Model

As we continue along the process of making the SIXT33N cars awesome, we'd like to better understand the car model that we will be using to develop a control scheme. As a wheel on the car turns, there is an encoder disc (see below) that also turns as the wheel turns. The encoder shines a light though the encoder disc, and as the wheel turns, the light is continually blocked and unblocked, allowing the encoder to detect how fast the wheel is turning by looking at the number of times that the light "ticks" between being blocked and unblocked over a specific time interval.



The following model applies separately to each wheel (and associated motor) of the car:

$$v[k]=d[k+1]-d[k]=\theta u[k]-\beta$$

Meet the variables at play in this model:

- k - The current timestep of the model. Since we model the car as a discrete system, this will advance by 1 on every new sample in the system.
- $d[k]$ - The total number of ticks advanced by a given encoder (the values may differ for the left and right motors–think about when this would be the case).
- $v[k]$ - The discrete-time velocity (in units of ticks/timestep) of the wheel, measured by finding the difference between two subsequent tick counts ($d[k+1]-d[k]$).

- $u[k]$ - The input to the system. The motors that apply force to the wheels are driven by an input voltage signal. This voltage is delivered via a technique known as pulse width modulation (PWM), where the average value of the voltage (which is what the motor is responsive to) is controlled by changing the duty cycle of the voltage waveform. The duty cycle, or percentage of the square wave's period for which the square wave is HIGH, is mapped to the range $[0,255]$. Thus, $u[k]$ takes a value in $[0,255]$ representing the duty cycle. For example, when $u[k] = 255$, the duty cycle is 100 %, and the motor controller just delivers a constant signal at the system's HIGH voltage, delivering the maximum possible power to the motor. When $u[k] = 0$, the duty cycle is 0 %, and the motor controller delivers 0 V.
- θ - Relates change in input to change in velocity: if the wheel rotates through n ticks in one timestep for a given $u[k]$ and m ticks in one timestep for an input of $u[k]+1$, then $\theta = m - n = \frac{\Delta v[k]}{\Delta u[k]} = \frac{v_{u_1[k]}[k] - v_{u_0[k]}[k]}{u_1[k] - u_0[k]}$. **Its units are ticks/(timestep · duty cycle).** Since our model is linear, we assume that θ is the same for every unit increase in $u[k]$. This is empirically measured using the car: θ depends on many physical phenomena, so for the purpose of this class, we will not attempt to create a mathematical model based on the actual physics. However, you can conceptualize θ as a "sensitivity factor", representing the idiosyncratic response of your wheel and motor to a change in power (you will have a separate θ for your left and your right wheel).
- β - Similarly to θ, β is dependent upon many physical phenomena, so we will empirically determine it using the car. β represents a constant offset in velocity, and hence **its units are ticks/timestep**. Note that you will also have a different β for your left and your right wheel.

In this problem (except parts (c) and (e)) we will assume that the wheel conforms perfectly to this model to get an intuition of how the model works.

(a) If we wanted to make the wheel move at a certain target velocity v^*, what input $u[k]$ should we provide to the motor that drives it? Your answer should be symbolic, and in terms of v^*, $u[k]$, θ, and β.

Solution:

$$\begin{aligned} v^* &= \theta u[k] - \beta \\ v^* + \beta &= \theta u[k] \\ u[k] &= \frac{v^* + \beta}{\theta} \end{aligned}$$

(b) Even if the wheel and the motor driving it conform perfectly to the model, our inputs still limit the range of velocities. Given that $0 \leq u[k] \leq 255$, determine the maximum and minimum velocities possible for the wheel. How can you slow the car down?

Solution:

The maximum is $255\theta - \beta$, and the minimum is $0 - \beta = -\beta$.

Since there are no brakes on the wheel, we slow down by reducing the input value and thereby the PWM duty cycle.

(c) Our intuition tells us that a wheel on a car should eventually stop turning if we stop applying any power to it. Find $v[k]$ assuming that $u[k] = 0$. Does the model obey your intutition? What does that tell us about our model?

Solution:

$$v[k] = -\beta$$

However, our intuition says that the car should have stopped: i.e., $v[k] = 0$. In lab, we will empirically find the value of β over a range of input duty cycles, but our fit does not work very well everywhere and our model does not match the real behavior near $u = 0$. This is a limitation of the simplified empirical model we are using, but as we will see in lab, we can still make the actual car work well over a certain range of inputs.

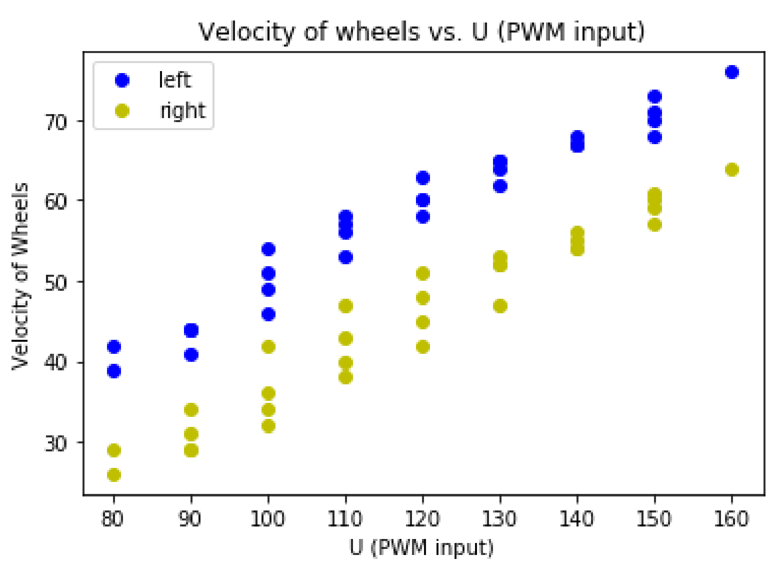
(d) In order to characterize the car, we need to find the θ and β values that model your left and right wheels and their motors: θ_l, β_l, θ_r, and β_r. How would you determine θ and β empirically? What data would you need to collect? *Hint:* keep in mind we also know the input $u[k]$ for all k.

Solution:

Given the motor model $v[k] = d[k+1] - d[k] = \theta u[k] - \beta$, we can determine θ and β by plotting velocity ($v[k]$) vs. input ($u[k]$).

By sweeping the input over a reasonable operating range and collecting multiple velocity samples (by differencing the total number of ticks, a collected quantity, between subsequent timesteps), we can collect velocity and input data. Then, we can perform least-squares linear regression on the data to determine the slope θ and y-intercept β.

Least-squares example solution:



Using the least squares linear regression method shown below, we can estimate the slope and y-intercept for each graph.

$$A^T A\hat{x} = A^T y \tag{1}$$

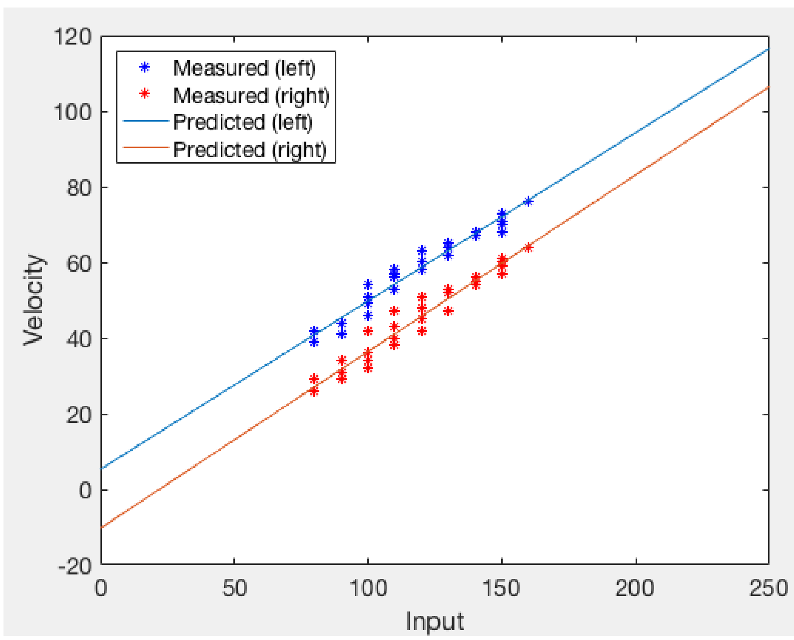
$$A = \begin{bmatrix} u_0 & 1 \\ u_1 & 1 \\ \vdots & \vdots \\ u_n & 1 \end{bmatrix} \tag{2}$$

$$y = \begin{bmatrix} v_{\text{measured},0} \\ v_{\text{measured},1} \\ \vdots & \vdots \\ v_{\text{measured},n} \end{bmatrix} \tag{3}$$

Using the θ and β values we found above, we now plot the predicted velocities of the left and right motor over the inputs u from 0 to 255:

$$v_{\text{left}} = \theta_{\text{left}} u - \beta_{\text{left}} \tag{4}$$

$$v_{\text{right}} = \theta_{\text{right}} u - \beta_{\text{right}} \tag{5}$$



(e) How can you use the data you collected in part (d) to mitigate the effect of the system's nonlinearity and/or minimize model mismatch? *Hint:* you will only wind up using a small range of the possible input values in practice. There are several reasons this is true, but one is that each motor has a characteristic attainable velocity range, and for your car to drive straight, we need the wheels to rotate with the same velocity.

Solution: Since we model the relationship between input and velocity as linear, if the actual data we collect displays nonlinearity, the car's behavior will not match the model well. To minimize model mismatch, we choose an operating point centered within an approximately linear region of the plot, *as long as both motors are capable of attaining a reasonable amount of the surrounding velocities.*

6. Write Your Own Question And Provide a Thorough Solution.

Writing your own problems is a very important way to really learn material. The famous “Bloom’s Taxonomy” that lists the levels of learning is: Remember, Understand, Apply, Analyze, Evaluate, and Create. Using what you know to create is the top level. We rarely ask you any homework questions about the lowest level of straight-up remembering, expecting you to be able to do that yourself (e.g. making flashcards). But we don’t want the same to be true about the highest level. As a practical matter, having some practice at trying to create problems helps you study for exams much better than simply counting on solving existing practice problems. This is because thinking about how to create an interesting problem forces you to really look at the material from the perspective of those who are going to create the exams. Besides, this is fun. If you want to make a boring problem, go ahead. That is your prerogative. But it is more fun to really engage with the material, discover something interesting, and then come up with a problem that walks others down a journey that lets them share your discovery. You don’t have to achieve this every week. But unless you try every week, it probably won’t ever happen.