EECS 16B Fall 2018 | Designing Information Devices and Systems II
Elad Alon and Miki Lustig | Discussion 4A

Phasors

We consider sinusoidal voltages and currents of a specific form.

Voltage	$v(t) = V_0 \cos(\omega t + \phi_v)$
Current	$i(t) = I_0 \cos(\omega t + \phi_i)$

Where,

(a) V_0 is the voltage amplitude and is the highest value of voltage $v(t)$ will attain at any time. Similarly, I_0 is the current amplitude.

(b) ω is the frequency of oscillation.

(c) ϕ_v and ϕ_i are the phase terms of the voltage and current respectively. These capture a delay, or a shift in time.

Recall the trigonometric properties:

$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$

$$\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$$

By using these properties, we can take any sinusoid and represent it with a linear combination of $\cos()$ and $-\sin()$:

$$v(t) = a\cos(\omega t) - b\sin(\omega t)$$

If we put it into this form, then all we need to do when analyzing a circuit is keep track of the coefficients a and b with a single complex number. This complex number is called the phasor of $v(t)$. So how do we cast a and b into a single complex number? Recall that:

$$\cos(\omega t) = \frac{1}{2}\left(e^{j\omega t} + e^{-j\omega t}\right)$$

As well as:

$$-\sin(\omega t) = \frac{1}{2}\left(je^{j\omega t} - je^{-j\omega t}\right)$$

Notice that for both $\cos(\omega t)$ and $\sin(\omega t)$ the coefficients of the $e^{j\omega t}$ and $e^{-j\omega t}$ terms are complex conjugates.

- For $\cos(\omega t)$, the coefficients are $1 \pm j0$
- For $\sin(\omega t)$, the coefficients are $0 \pm j1$

In other words, the coefficients of the cosine part of a sinusoid is purely real while the coefficients of the sine part is purely imaginary. This implies that we can just add the coefficients together to keep track of everything we need to know about the sinusoid (amplitude and phase).
To show that we can just add the coefficients together, recall that we can represent any sinusoid as:

$$v(t) = a\cos(\omega t) - b\sin(\omega t)$$

converting to complex exponentials:

$$v(t) = \frac{1}{2}\left(ae^{j\omega t} + ae^{-j\omega t} + jbe^{j\omega t} - jbe^{-j\omega t}\right)$$

$$v(t) = \frac{1}{2}\left((a+jb)e^{j\omega t} + (a-jb)e^{-j\omega t}\right)$$

This means we can represent the phasor as:

$$\tilde{v} = a + jb$$

We can use this single complex term since we know that after all the complex math, the resulting coefficients is equivalent to specifying a sinusoid

$$R\cos(\omega t + \phi)$$

in the form

$$a\cos(\omega t) - b\sin(\omega t)$$

In general, if we have a sinusoidal signal in the form:

$$v_1(t) = R\cos(\omega t + \phi)$$

The phasor will end up being:

$$\tilde{v_1} = R\cos(\phi) + jR\sin(\phi)$$

If instead we have a sin rather than cosine:

$$v_2(t) = R\sin(\omega t + \phi)$$

we can use the same coefficients as if it was a cosine, but multiply them by $-j$:

$$\tilde{v_2} = (-j)\left(R\cos(\phi) + jR\sin(\phi)\right)$$

$$\tilde{v_2} = R\sin(\phi) - jR\cos(\phi)$$

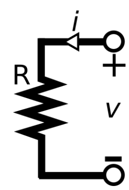


Figure 1: A simple resistor circuit

Phasor Relationship for Resistors

Consider a simple resistor circuit as in Figure 1, with current being,

$$i(t) = I_0 \cos(\omega t + \phi)$$

By Ohm's law,

$$\begin{aligned} v(t) &= i(t)R \\ &= I_0 R \cos(\omega t + \phi) \end{aligned}$$

In phasor domain,

$$\tilde{V} = R\tilde{I}$$

Phasor Relationship for Capacitors

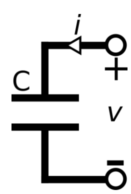


Figure 2: A simple capacitor circuit

Consider a capacitor circuit as in Figure 2, with voltage being,

$$v(t) = V_0 \cos(\omega t + \phi)$$

By the capacitor equation,

$$\begin{aligned} i(t) &= C\frac{\mathrm{d}v}{\mathrm{d}t}(t) \\ &= -CV_0 \omega \sin(\omega t + \phi) \\ &= -(\omega C)V_0 \sin(\omega t + \phi) \\ &= (\omega C)\big(-V_0 \sin(\phi)\cos(\omega t) - V_0 \cos(\phi)\sin(\omega t)\big) \end{aligned}$$

In phasor domain,

$$\tilde{I} = (\omega C)(-V_0 \sin(\phi) + jV_0 \cos(\phi))$$
$$\tilde{I} = (\omega C)\big(j(V_0 \cos(\phi) + jV_0 \sin(\phi))\big)$$
$$\tilde{I} = (\omega C)(j\tilde{V})$$

The impedence of a capacitor is an abstraction to model the capacitor as a resistor in the phasor domain. This is denoted $\tilde{Z}_C$.

$$\tilde{Z}_C = \frac{\tilde{V}}{\tilde{I}} = \frac{1}{j\omega C}$$

Questions

1. Proof of Induction

Given the voltage-current relationship of an inductor $V = L\frac{di}{dt}$, show that its complex impedance is $Z_L = j\omega L$.

Answer:

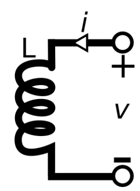


Figure 3: A simple inductor circuit

Consider a simple resistor circuit as in Figure 3, with current being,

$$i(t) = I_0 \cos(\omega t + \phi)$$

By the inductor equation,

$$\begin{aligned} v(t) &= L\frac{\mathrm{d}i}{\mathrm{d}t}(t) \\ &= -LI_0\omega \sin(\omega t + \phi) \\ &= (\omega L)\left(-I_0 \sin(\phi)\cos(\omega t) - I_0 \cos(\phi)\sin(\omega t)\right) \end{aligned}$$

In phasor domain,

$$\tilde{V} = (\omega C)(-I_0 \sin(\phi) + jI_0 \cos(\phi))$$
$$\tilde{V} = (\omega C)\left(j(I_0 \cos(\phi) + jI_0 \sin(\phi))\right)$$
$$\tilde{V} = (\omega C)(j\tilde{I})$$

The impedence of an inductor is an abstraction to model the inductor as a resistor in the phasor domain. This is denoted $\tilde{Z}_L$.

$$\tilde{Z}_L = \frac{\tilde{V}}{\tilde{I}} = j\omega L$$

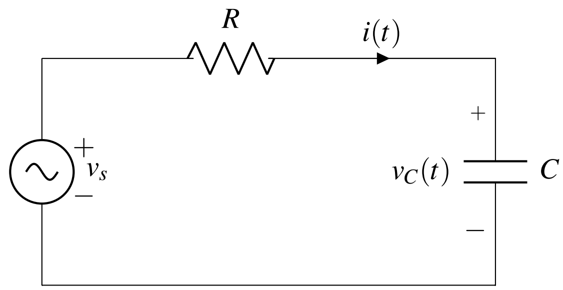
2. Phasor analysis

Any sinusoidal time-varying function $x(t)$, representing a voltage or a current, can be expressed in the form

$$x(t) = \frac{1}{2}\left(\tilde{X}e^{j\omega t} + \tilde{X}^* e^{-j\omega t}\right) \tag{1}$$

where $\tilde{X}$ is the phasor of $x(t)$. Thus, $x(t)$ is defined in the time domain, while $\tilde{X}$ is defined in the phasor domain.

The phasor analysis method consists of four steps. Consider the RC circuit below.



The voltage source is given by

$$v_s = 12\sin\left(\omega t - \frac{\pi}{4}\right), \tag{2}$$

with $\omega = 10^3$ rad/s, $R = 1$ kΩ, and $C = 1\ \mu F$.

Our goal is to obtain a solution for $i(t)$ with the sinusoidal voltage source v_s.

(a) **Step 1: Convert sources to phasor format**

Find the phasor $\tilde{v}$ of $v_s(t)$

Answer:

$$v_s(t) = 12\sin\left(\omega t - \frac{\pi}{4}\right)$$

Since the voltage is a sine:

$$\tilde{v_s} = R\sin(\phi) - jR\cos(\phi)$$
$$\tilde{v_s} = 12\sin\left(\frac{-\pi}{4}\right) - j12\cos\left(\frac{-\pi}{4}\right)$$
$$\tilde{v_s} = -\frac{12}{\sqrt{2}} - j\frac{12}{\sqrt{2}}$$
$$\tilde{v_s} = -6\sqrt{2}(1+j)$$

(b) **Step 2: Calculate the impedance of each component**

Calculate the impedances of the resistor and capacitor.

Answer:

$$Z_R = R = 10^3\Omega \tag{3}$$
$$Z_C = \frac{1}{j\omega C} = -j10^3\Omega \tag{4}$$

(c) **Step 3: Solve for unknown variables**

Derive equations for the $\tilde{I}$ and $\tilde{v_c}$, where $\tilde{I}$ and $\tilde{v_c}$ are the phasor representations of current and capacitor voltage, respectively.

Answer:

In the phasor domain, impedances act like resistors. To find $\tilde{v_c}$, we can use the equation for a voltage divider:

$$\tilde{v_c} = \frac{Z_C}{Z_C + Z_R}\tilde{v_s}$$

$$\tilde{v_c} = \frac{-j10^3}{10^3 - j10^3}(-6\sqrt{2}(1+j)) = -6\sqrt{2}\frac{1-j}{1-j} = -6\sqrt{2}$$

$$\tilde{i} = \frac{\tilde{v_c}}{Z_C} = \frac{-6\sqrt{2}}{-j10^3} = -j6\sqrt{2} \times 10^{-3}$$

(d) **Step 4: Transform solutions back to time domain**

To return to time domain, we apply the fundamental relation between a sinusoidal function and its phasor counterpart. What is $i(t)$ and $v_C(t)$?

Answer: If the phasor is in the form of:

$$\tilde{v} = a + jb$$

Then the time domain signal is in the form:

$$v(t) = a\cos(\omega t) - b\sin(\omega t)$$

So:

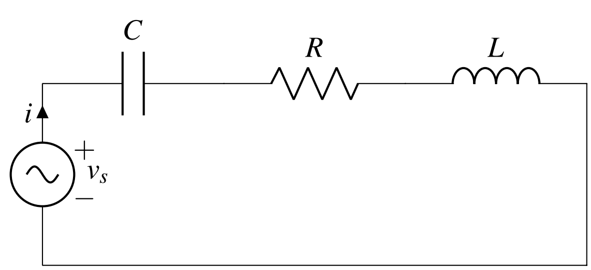
$$v_c(t) = -6\sqrt{2}\cos(\omega t)$$

$$i_c(t) = 6\sqrt{2} \times 10^{-3}\sin(\omega_t)$$

3. RLC circuit in AC

We study a simple RLC circuit with an AC voltage source given by

$$v_s = B\cos(\omega t - \phi)$$



(a) Write out the phasor representation of v_s and calculate the impedances of R, C, and L.

Answer:

$$\tilde{v_s} = B\cos(-\phi) + jB\sin(-\phi)$$
$$Z_R = R$$
$$Z_C = \frac{1}{j\omega C}$$
$$Z_L = j\omega L$$

(b) Derive an equation for $\tilde{i}$

Answer:

$$\tilde{i} = \frac{\tilde{v_s}}{Z_C + Z_R + Z_L} = \frac{B\cos(-\phi) + jB\sin(-\phi)}{R + j\omega L + \frac{1}{j\omega C}}$$

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