EE16B
Designing Information Devices and Systems II

Lecture 6A
Stability of Linear State Models

Last Time

- Described linearization about an equilibrium point using Taylor approximation
  - Continuous time
  - Discrete time
- Started: Conditions for stability of linear systems

Stability of Linear State Models

Start with scalar system 1st order system:

$$x(t+1) = ax(t) + bu(t)$$

Given initial condition x(0):

$$x(1) = ax(0) + bu(0)$$

$$\begin{aligned} x(2) &= ax(1) + bu(1) \\ &= a^2x(0) + abu(0) + bu(1) \end{aligned}$$

$$x(3) = a^3x(0) + a^2bu(0) + abu(1) + bu(2)$$

$$x(t) = a^tx(0) + a^{t-1}bu(0) + a^{t-2}bu(1) + \cdots a^0bu(t-1)$$

Stability of Linear State Models

Start with scalar system:

$$x(t+1) = ax(t) + bu(t)$$

Given initial condition x(0):

$$x(t) = a^t x(0) + \underbrace{\boxed{a^{t-1}bu(0)}}_{}\!\!{}^{k=0} + a^{t-2}bu(1) + \cdots \overbrace{\boxed{a^0 bu(t-1)}}^{k=t-1}$$

$$x(t) = \underbrace{a^t x(o)}_{\text{Initial condition}} + \underbrace{\sum_{k=0}^{t-1} a^{t-k-1} bu(k)}_{\text{input}}$$

Stability - Definition

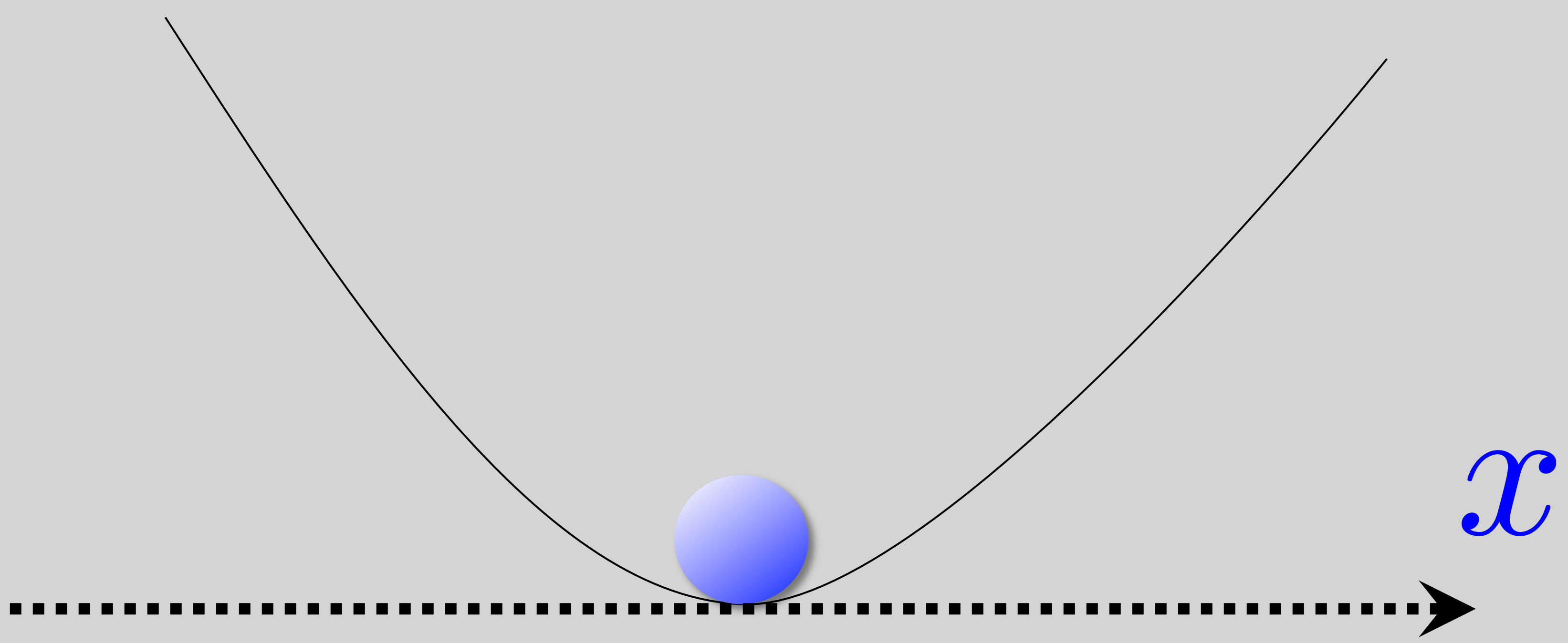
- A system is <u>stable</u> if $\vec{x}(t)$ is <u>bounded</u> for any initial condition $\vec{x}(0)$ and any bounded input sequence

$$u(0),\ u(1),\ \cdots$$

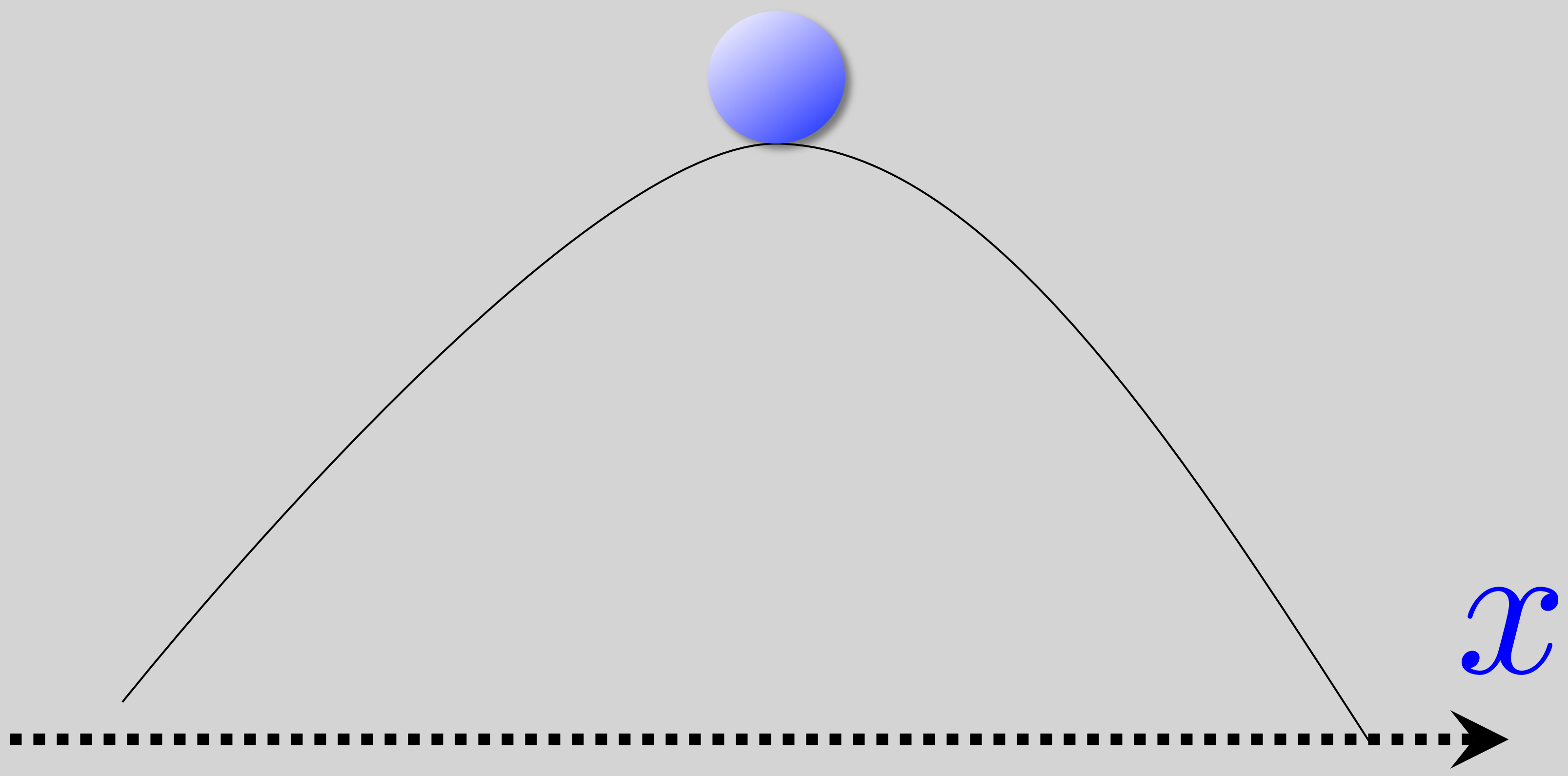
- A system is unstable if there is an $\vec{x}(0)$ or a bounded input sequence for which

$$|\vec{x}(t)| \to \infty \quad \text{as} \quad t \to \infty$$

Example



stable



unstable

Q) Is this system stable?

$$x(t) = a^t x(o) + \sum_{k=0}^{t-1} a^{t-k-1} b u(k)$$

A) Depends on $|a|$

Stability Proof

$$x(t) = a^t x(o) + \sum_{k=0}^{t-1} a^{t-k-1} bu(k)$$

Claim 1: if |a| < 1 then the system is stable

Proof: $a^t \to 0$ as $t \to \infty$ *because* $|a| < 1$ so,

initial condition always bounded

Sequence is bounded – there exists M s.t. $|u(t)| \leq M \; \forall t$

$$\left| \sum_{k=0}^{t-1} a^{t-k-1} bu(k) \right| \leq \sum_{k=0}^{t-1} |a^{t-k-1} bu(k)| = \sum_{k=0}^{t-1} |a|^{t-k-1} |b| \underbrace{|u(k)|}_{\leq M}$$

$$|a_1| + |a_2| \;\; ? \;\; |a_1 + a_2|$$

Stability Proof Cont.

$$\left|\sum_{k=0}^{t-1} a^{t-k-1} b u(k)\right| \leq \sum_{k=0}^{t-1} |a^{t-k-1} b u(k)| = \sum_{k=0}^{t-1} |a|^{t-k-1} |b| \underbrace{|u(k)|}_{\leq M}$$

Define: $s = t - k - 1$

$$\leq \sum_{s=0}^{t-1} |a^s||b|M \; = |b|M \sum_{s=0}^{t-1} |a|^s \leq |b|M \frac{1}{1-|a|}$$

$$\boxed{\sum_{s=0}^{\infty} |a|^s = \frac{1}{1-|a|} \quad , \qquad |a| < 1}$$

Stability Proof Cont.

Claim 2: unstable when |a| > 1

Proof: if $x(0) \neq 0$ (even $u(t)=0 \; \forall \; t$)

$$x(t) = a^t x(0) \rightarrow \infty$$

Q: What if $|a| = 1$, i.e., $a=1$ or $a=-1$

A: Without input: $x(t) = a^t x(0)$

$$x(t) = x(0), \quad \text{or} \quad x(t) = (-1)^t x(0)$$

With input $u(t)=M$, $a=1$

$$\left| \sum_{k=0}^{t-1} a^{t-k-1} b u(k) \right| = \left| \sum_{k=0}^{t-1} bM \right| \rightarrow \infty$$

Not stable!

Quiz

With input $u(t)=M$, $a=-1$

$$\left|\sum_{k=0}^{t-1} a^{t-k-1} b u(k)\right| = \left|\sum_{k=0}^{t-1} (-1)^k bM\right| \leq bM$$

Q:what $|u(t)| \leq M$ will make it unstable?

Quiz

With input $u(t)=M,\ a=-1$

$$\left|\sum_{k=0}^{t-1} a^{t-k-1} b u(k)\right| = \left|\sum_{k=0}^{t-1} (-1)^k bM\right| \leq bM$$

Q:what $|u(t)|\leq M$ will make it unstable?

A: $u(t) = (-1)^t M$

$$\left|\sum_{k=0}^{t-1} a^{t-k-1} b u(k)\right| = \left|\sum_{k=0}^{t-1} (-1)^k b (-1)^k M\right| = \left|\sum_{k=0}^{t-1} bM\right| \to \infty$$

Stability of Linear State Models

Previously, the scalar case:

$$x(t+1) = ax(t) + bu(t)$$

$|a| < 1 \quad \Rightarrow$ stable

$|a| \geq 1 \quad \Rightarrow$ unstable

Vector case:

$$\vec{x}(t+1) = A\vec{x}(t) + Bu(t)$$

Solve with recursion:

$$\vec{x}(1) = A\vec{x}(0) + Bu(0)$$

$$\vec{x}(2) = A^2\vec{x}(0) + ABu(0) + Bu(1)$$

$$\vec{x}(t) = A^t\vec{x}(0) + \sum_{k=0}^{t-1} A^{t-1-k}Bu(k)$$

Stability – The Vector Case

$$\vec{x}(t) = A^t \vec{x}(0) + \sum_{k=0}^{t-1} A^{t-1-k} B u(k)$$

Q: How do we determine stability?

A (partial): Not simple as in the scalar case – the state variables and inputs are coupled.

Approach: Let's change variables, to decouple them

Change of Variables - Diagonalization

$$\vec{x}(t+1) = A\vec{x}(t) + Bu(t)$$

$$\vec{z}(t) = T\vec{x}(t)$$

$$\vec{z}(t+1) = T\vec{x}(t+1)$$

$$= TA\underbrace{\vec{x}(t)}_{T^{-1}\vec{z}(t)} + TBU(t)$$

$$\underbrace{A_{\text{new}} = TAT^{-1} \qquad B_{\text{new}} = TB}_{\text{Similarity transformation}}$$

Q: What T to choose?

A: Choose T s.t. A_{new} is diagonal

Remember eigen values of new system same as original!

Diagonalization

$$A_{\text{new}} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

$$\vec{v}(t) = B_{\text{new}} u(t)$$

$$\vec{z}(t+1) = \begin{bmatrix} \boxed{\lambda_1 \quad 0 \quad \cdots \quad 0} \\ 0 \quad \lambda_2 \quad \cdots \quad 0 \\ \ddots \\ 0 \quad 0 \quad \cdots \quad \lambda_n \end{bmatrix} \vec{z}(t) + \overbrace{B_{\text{new}} u(t)}$$

$$z_1(t+1) = \lambda_1 z_1(t) + v_1(t)$$

Diagonalization

Diagonalization = decoupling!

$$
\begin{aligned}
z_1(t+1) &= \lambda_1 z_1(t) + v_1(t) \\
z_2(t+1) &= \lambda_2 z_2(t) + v_2(t) \\
&\vdots \\
z_n(t+1) &= \lambda_n z_n(t) + v_n(t)
\end{aligned}
$$

Stable if: $|\lambda_i| < 1, \qquad i = 1, 2, \cdots, n$

unstable if: $|\lambda_i| \geq 1, \qquad i = 1, 2, \cdots, n$

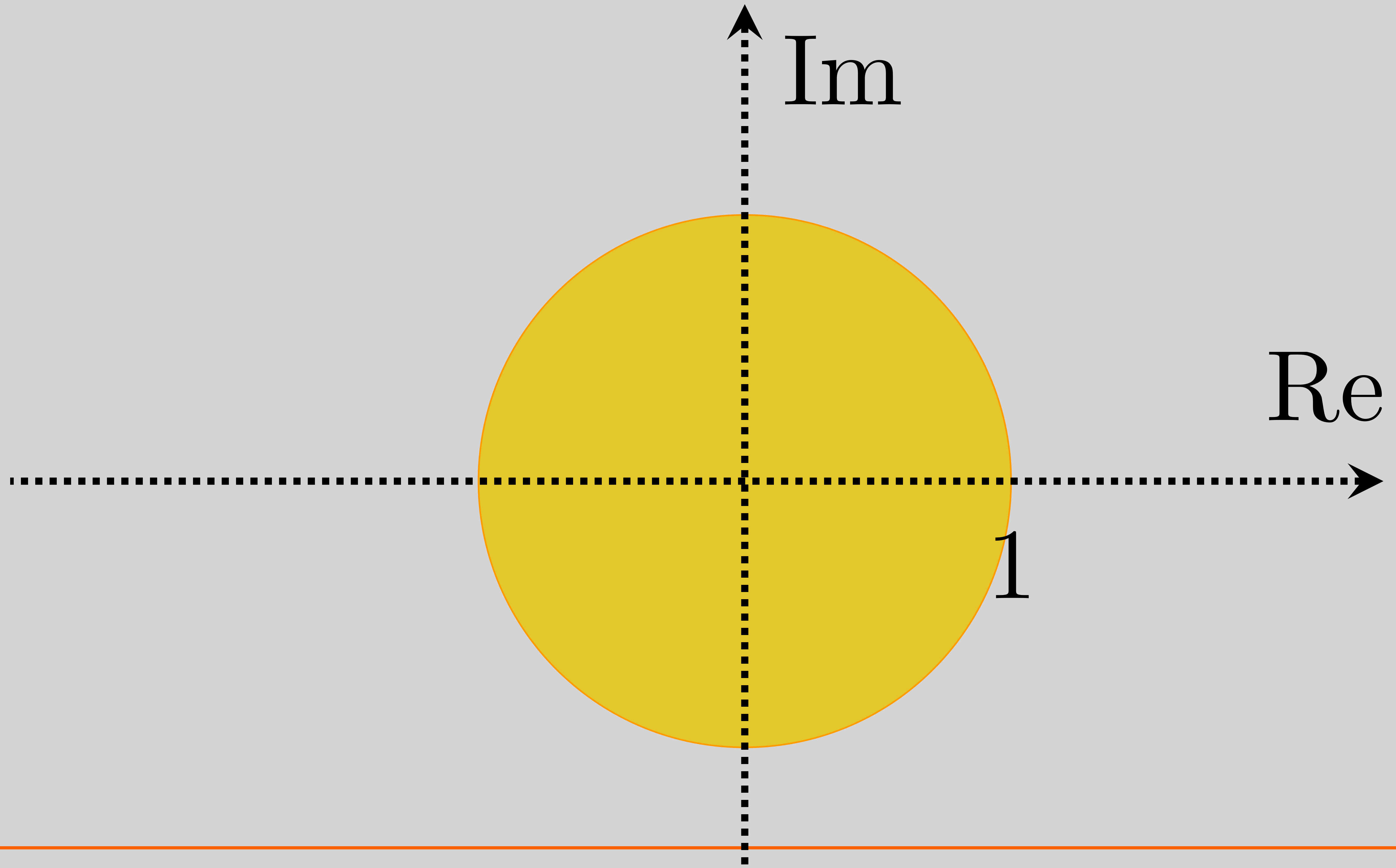
Remember eigen values of new system same as original!

Stability Cont.

What if a is complex valued?

Stable if: $|\lambda_i| < 1, \qquad i = 1, 2, \cdots, n$

unstable if: $|\lambda_i| \geq 1, \qquad i = 1, 2, \cdots, n$



Non-Diagonalizable Systems

Q: What if A is not diagonalizable?

A: Transform to upper diagonal form (always possible)

beyond 16B material

$$TAT^{-1} = \begin{bmatrix} \lambda_1 & \star & \cdots & \star \\ 0 & \lambda_2 & \cdots & \star \\ & & \ddots & \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

Stable if: $|\lambda_i| < 1, \qquad i = 1, 2, \cdots, n$

unstable if: $|\lambda_i| \geq 1, \qquad i = 1, 2, \cdots, n$

Non-Diagonalizable Proof

$$TAT^{-1} = \begin{bmatrix} \lambda_1 & \star & \cdots & \star \\ 0 & \lambda_2 & \cdots & \star \\ & & \ddots\lambda_{n-1} & \star \\ \boxed{0 & 0 & \cdots & \lambda_n} \end{bmatrix}$$

Show stability for z_n :

$$z_n(t+1) = \lambda_n z_n(t) + v_n(t) \qquad |\lambda_n| < 1$$

z_n is bounded, show stability for z_{n-1} :

$$\underbrace{z_{n-1}(t+1) = \lambda_{n-1} z_{n-1}(t)}_{\text{Bounded if } |\lambda_{n-1}| < 1} + \underbrace{\star z_n(t) + v_{n-1}(t)}_{\text{treat as bounded input}}$$

show stability for z_i recursively !

Example of non-diagonalizable:

$$\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$$

Stability of Cont.-Time Linear Systems

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + Bu(t)$$

Start with scalar x(t):

$$\frac{d}{dt}x(t) = ax(t) + bu(t)$$

$$x(t) = \underbrace{e^{at}x(0)}_{\text{Initial condition}} + \underbrace{b\int_0^t e^{a(t-s)}u(s)ds}_{\text{Due to input}}$$

Stability of Cont.-Time Linear Systems

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + Bu(t)$$

Start with scalar x(t):

$$\frac{d}{dt}x(t) = ax(t) + bu(t)$$

$$x(t) = \underbrace{e^{at}x(0)}_{\text{Initial condition}} + \underbrace{b\int_0^t e^{a(t-s)}u(s)ds}_{\text{Due to input}}$$

Q: When is the system stable?

Stability of Cont.-Time Linear Systems

$$\frac{d}{dt}x(t) = ax(t) + bu(t)$$

$$x(t) = \underbrace{e^{at}x(0)}_{\text{Initial condition}} + \underbrace{b\int_0^t e^{a(t-s)}u(s)ds}_{\text{Due to input}}$$

Q: When is the system stable?

A: For $a < 0$

Proof outline:

Show: $e^{at} \to 0, \quad t \to \infty$

$$\text{if} \quad |u(s)| \leq M \quad \forall s \Rightarrow \int \{\} < \text{Const}$$

Stability of Cont.-Time Linear Systems

$$x(t) = \underbrace{e^{at}x(0)}_{\text{Initial condition}} + \underbrace{b\int_0^t e^{a(t-s)}u(s)ds}_{\text{Due to input}}$$

Q: When is the system unstable?

A: For $a \geq 0$

Proof : choose x(0)≠0 and u(t) = M

either "due to input" or "due to initial condition" explodes

Stability of Cont.-Time Linear Systems

Summary:

$a < 0 \quad \Rightarrow$ stable

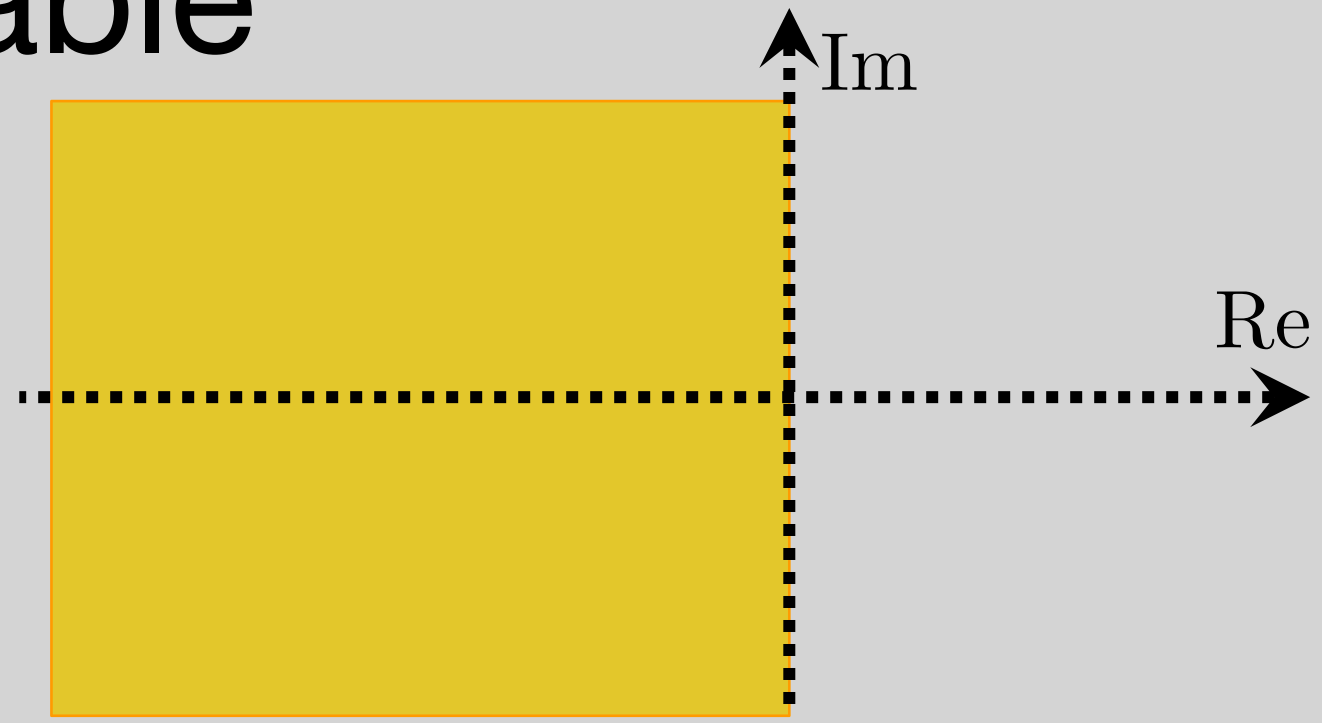
$a \geq 0 \quad \Rightarrow$ unstable

If a is complex, then:

$\text{Re}\{a\} < 0 \quad \Rightarrow$ stable

$\text{Re}\{a\} \geq 0 \quad \Rightarrow$ unstable

$$\left|e^{a_r + i a_i}\right| = \left|e^{a_r}\right| \cdot \left|e^{i a_i}\right| = \left|e^{a_r}\right|$$



Stability of Cont.-Time Linear Systems

Vector Case:

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + B\vec{u}(t)$$

Diagonalize:

$$\vec{z}(t) = T\vec{x}(t)$$

$$A_{\text{new}} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

$$\frac{d}{dt}z_i(t) = \lambda_i z_i(t) + v_i(t)$$

Stability of Cont.-Time Linear Systems

Stability test for

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + B\vec{u}(t)$$

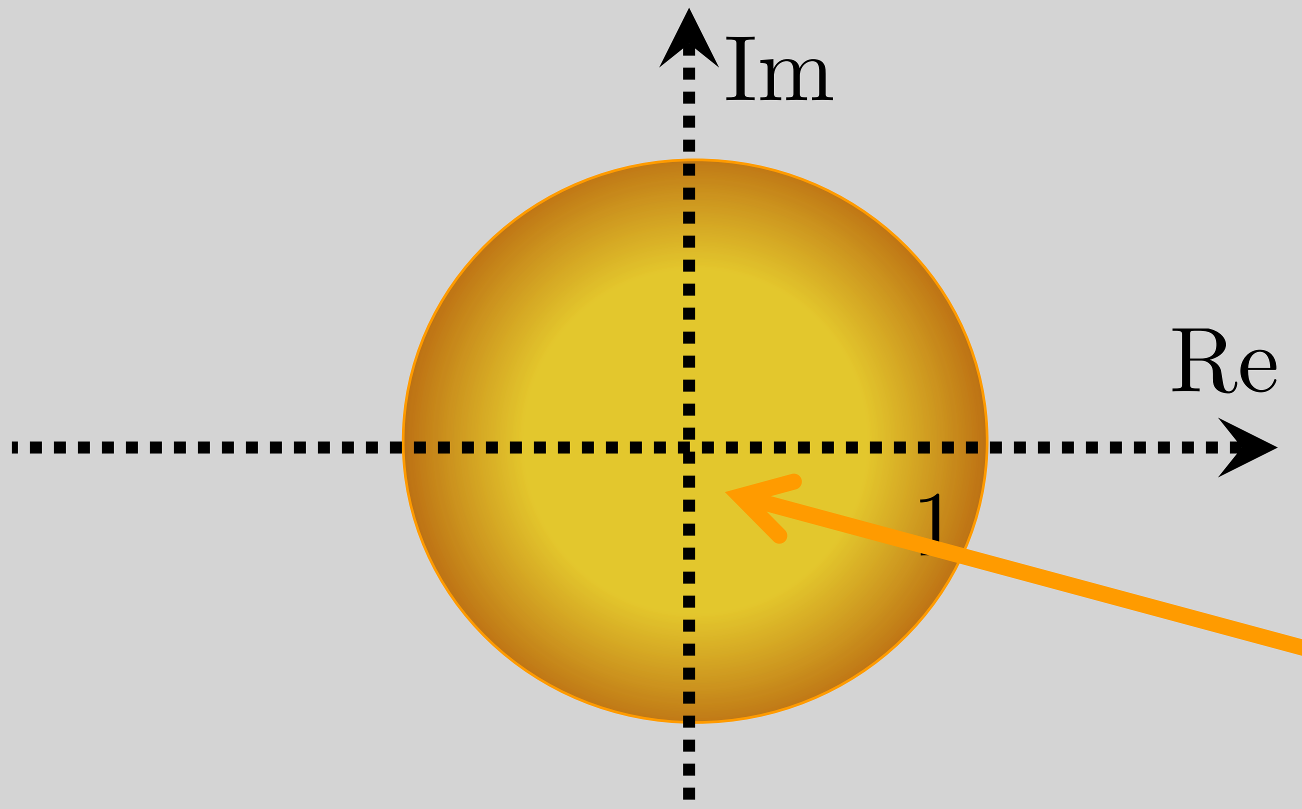
$$\mathrm{Re}\{\lambda_i(A)\} < 0 \quad \forall i| \quad i = 1, 2, \cdots, n$$ ⇒ stable

$$\mathrm{Re}\{\lambda_i(A)\} \geq 0 \quad \exists i| \quad i = 1, 2, \cdots, n$$ ⇒ unstable

Stability -- Summary

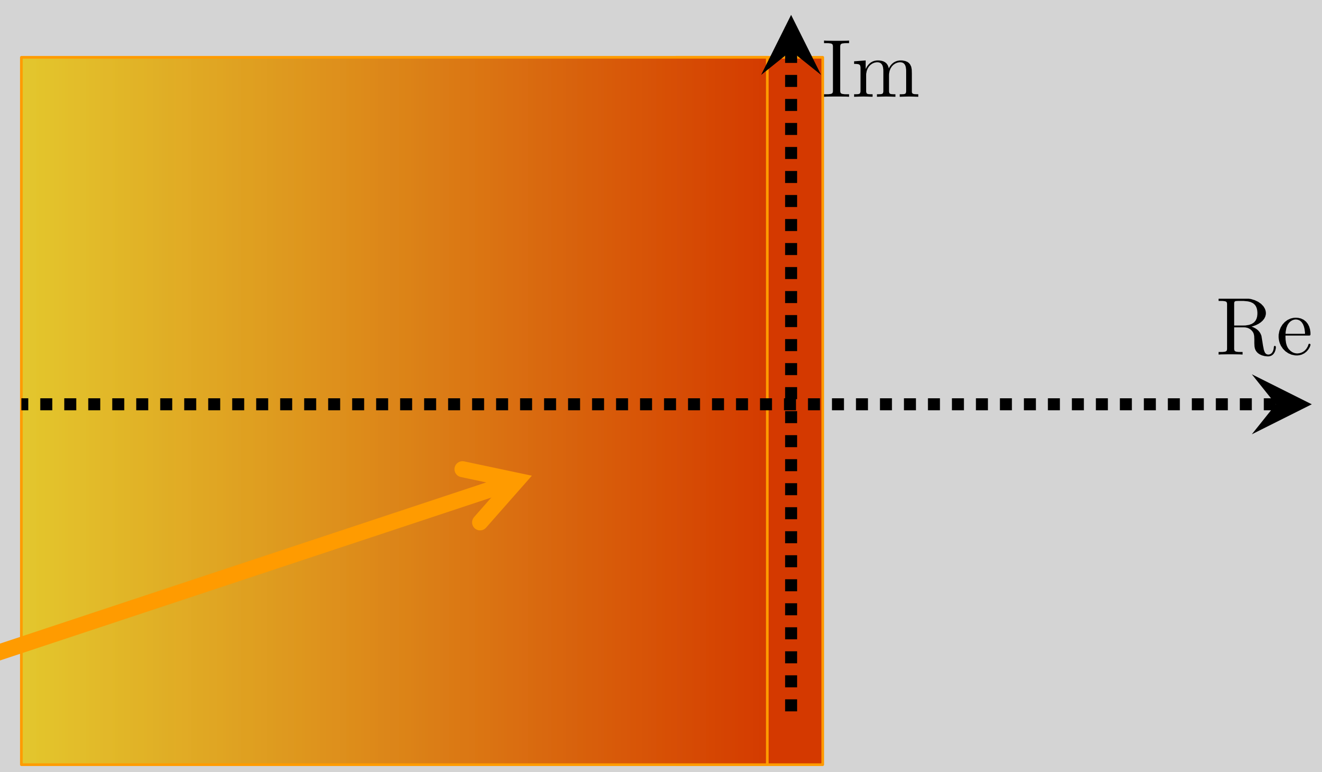
Discrete-Time

$$|\lambda_i(A)| < 1$$



Continuous-Time

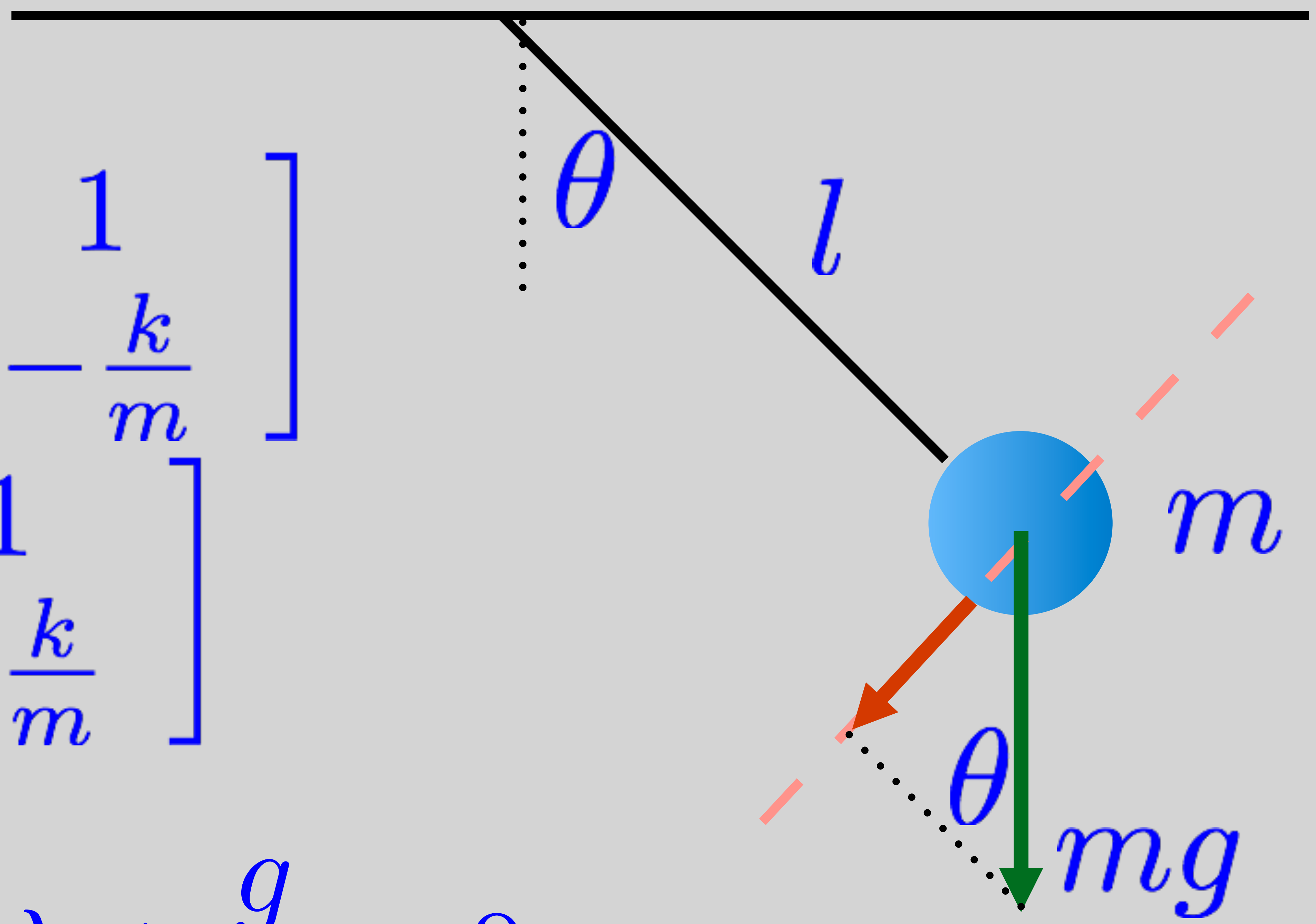
$$\mathrm{Real}\{\lambda_i(A)\} < 0$$



Stable regions

Stay away from boundaries! System uncertainty can
Move you over to unstable region

Back to the Pendulum



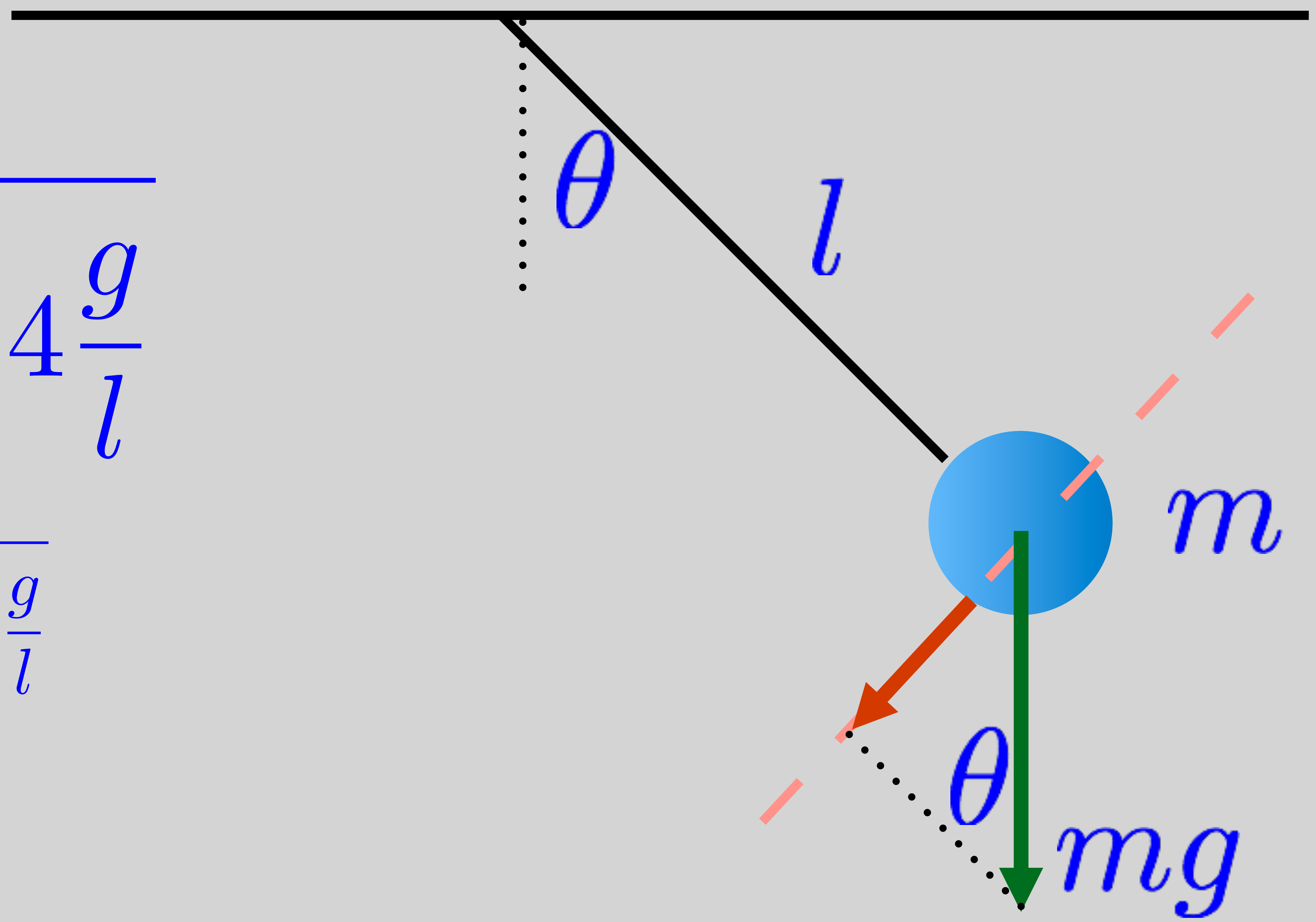
$$A_{\text{down}} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & -\frac{k}{m} \end{bmatrix}$$

$$A_{\text{up}} = \begin{bmatrix} 0 & 1 \\ \frac{g}{l} & -\frac{k}{m} \end{bmatrix}$$

$$|\lambda I - A_{\text{down}}| = \begin{bmatrix} \lambda & -1 \\ \frac{g}{l} & \lambda + \frac{k}{m} \end{bmatrix} = \lambda^2 + \frac{k}{m}\lambda + \frac{g}{l} = 0$$

$$\lambda_{1,2} = -\frac{k}{2m} \pm \frac{1}{2}\sqrt{\frac{k^2}{m^2} - 4\frac{g}{l}}$$

Back to the Pendulum



$$\lambda_{1,2} = -\frac{k}{2m} \pm \frac{1}{2}\sqrt{\frac{k^2}{m^2} - 4\frac{g}{l}}$$

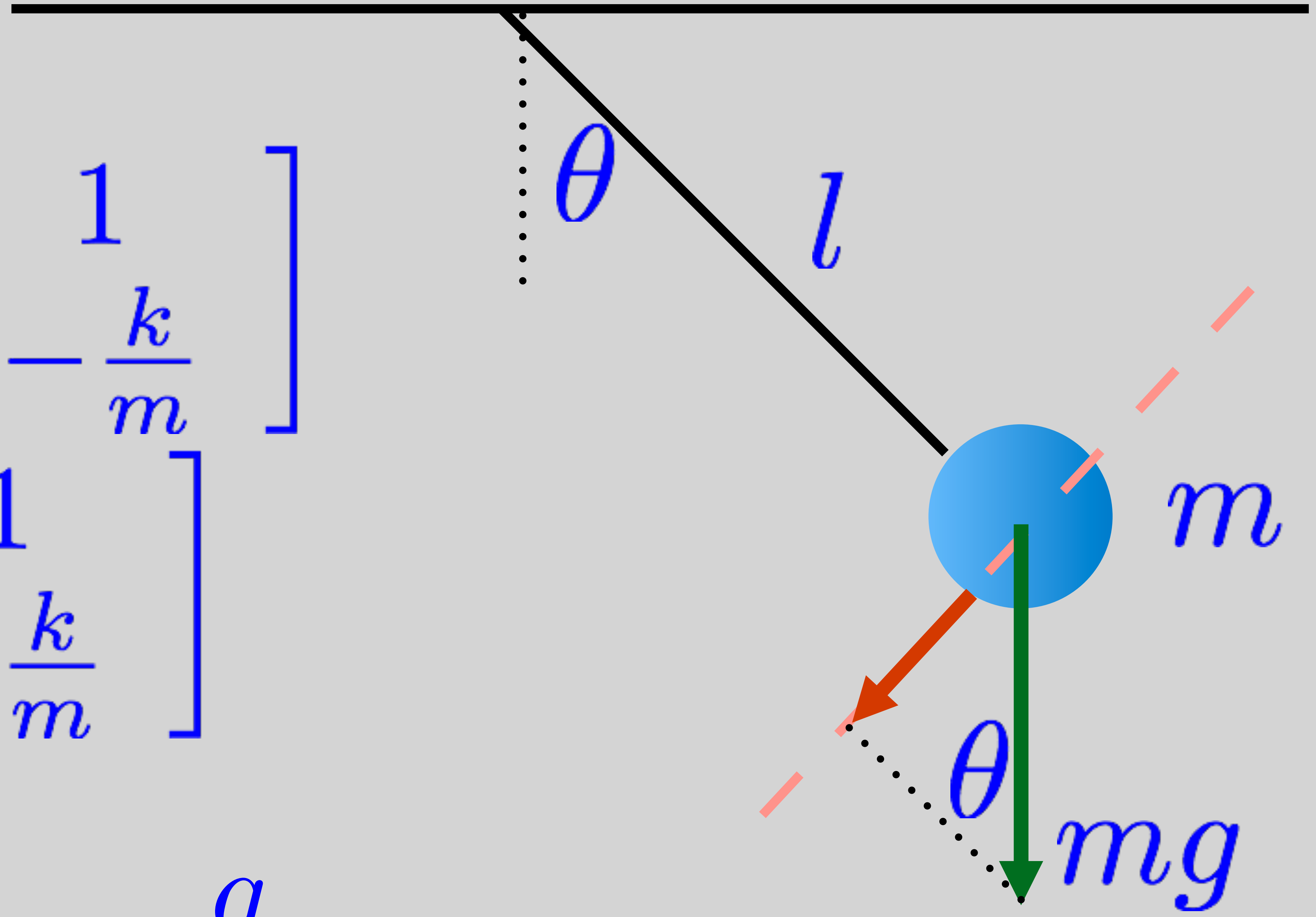
If $\frac{k^2}{m^2} \geq 4\frac{g}{l}$, i.e, sqrt is real, then $\frac{k}{2m} \geq \frac{1}{2}\sqrt{\frac{k^2}{m^2} - 4\frac{g}{l}}$

So, $\lambda_{1,2}$ always negative -- stable!

If $\frac{k^2}{m^2} < 4\frac{g}{l}$, i.e, sqrt is imaginary, then $\mathrm{Re}\{\lambda_{1,2}\} = -\frac{k}{2m}$

So, $\mathrm{Re}\{\lambda_{1,2}\}$ always negative -- stable!

Back to the Pendulum



$$A_{\text{down}} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & -\frac{k}{m} \end{bmatrix}$$

$$A_{\text{up}} = \begin{bmatrix} 0 & 1 \\ \frac{g}{l} & -\frac{k}{m} \end{bmatrix}$$

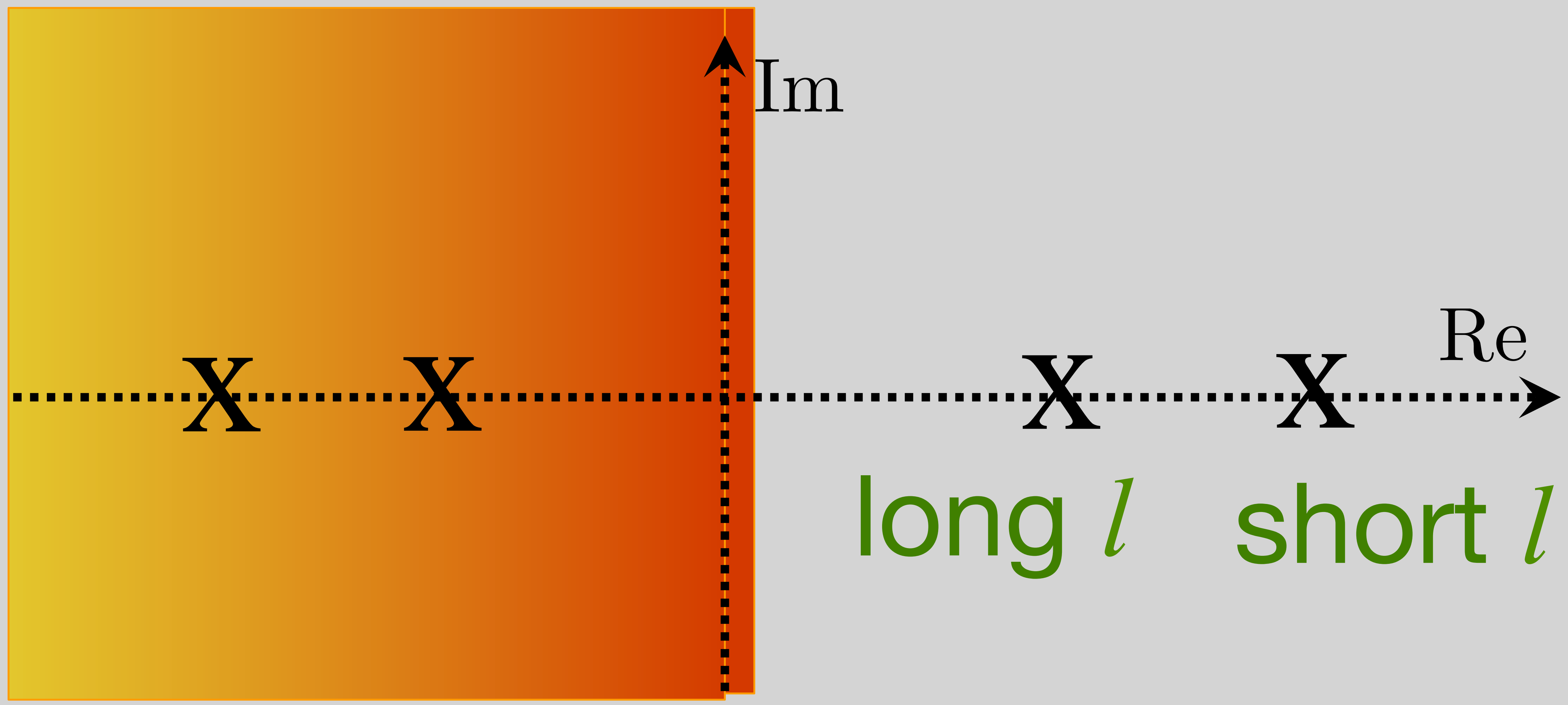
$$|\lambda I - A_{\text{up}}| = \begin{bmatrix} \lambda & -1 \\ -\frac{g}{l} & \lambda + \frac{k}{m} \end{bmatrix} = \lambda^2 + \frac{k}{m}\lambda - \frac{g}{l} = 0$$

$$\lambda_{1,2} = -\frac{k}{2m} \pm \frac{1}{2}\sqrt{\frac{k^2}{m^2} + 4\frac{g}{l}}$$

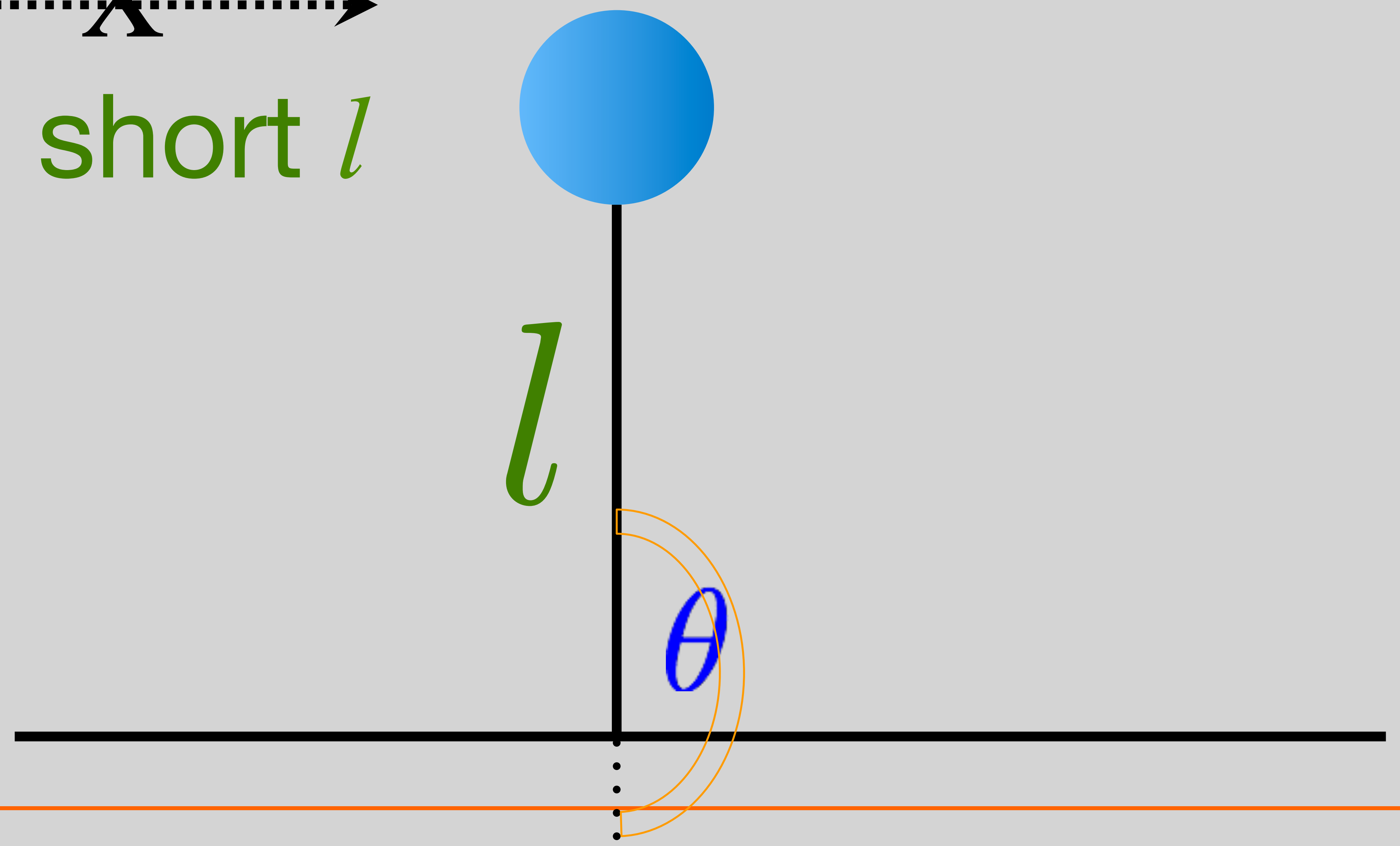
$$\lambda_1 > 0$$
$$\lambda_2 < 0$$

Back to the Pendulum

$$\lambda_{1,2} = -\frac{k}{2m} \pm \frac{1}{2}\sqrt{\frac{k^2}{m^2} + 4\frac{g}{l}}$$



$\lambda_1 > 0$
$\lambda_2 < 0$



Predicting System Behavior

$$z(t+1) = \lambda_i z(t)$$

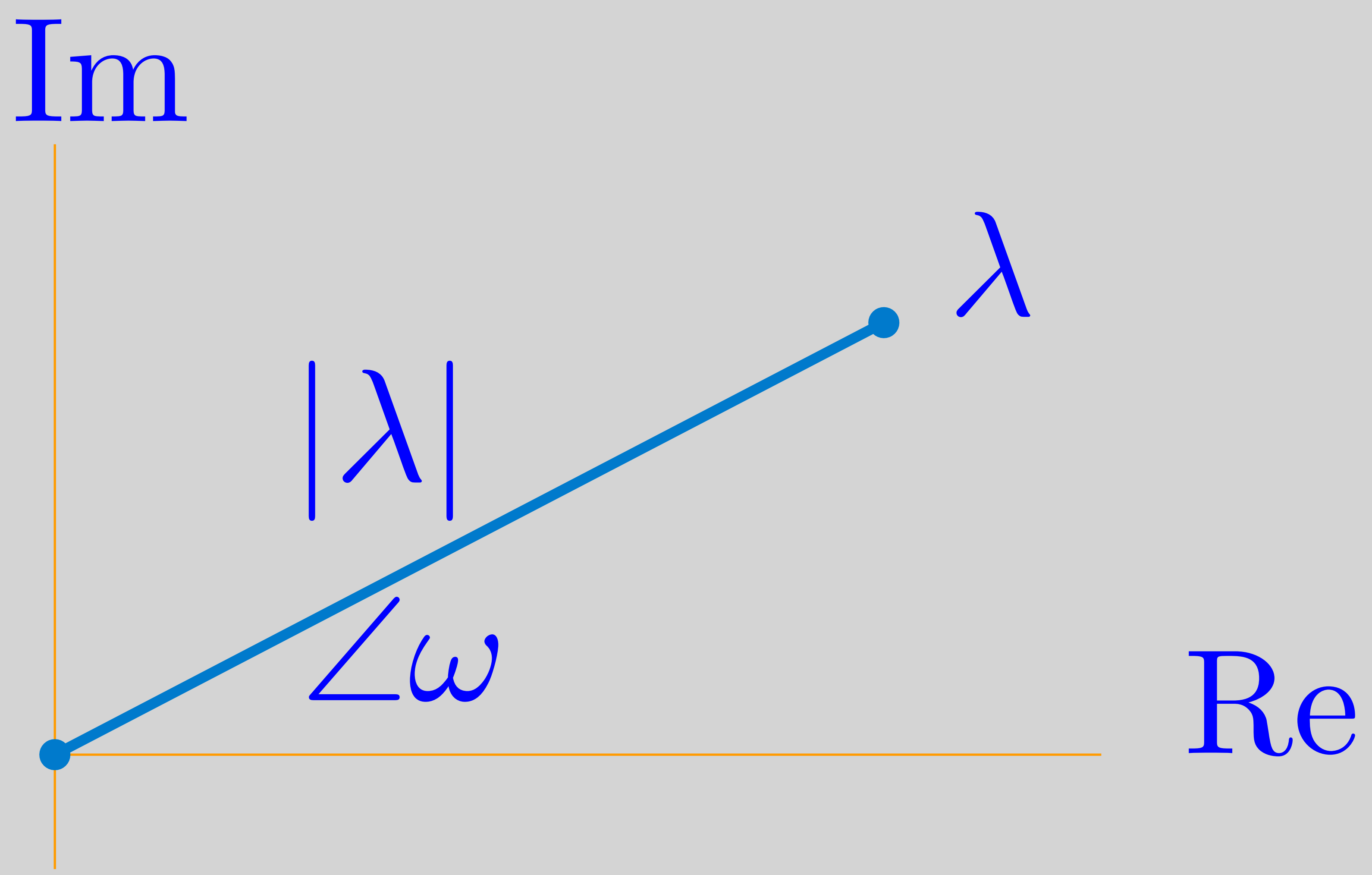
$$\text{Soln}: \quad \lambda_i^t z(0)$$

Discrete Time

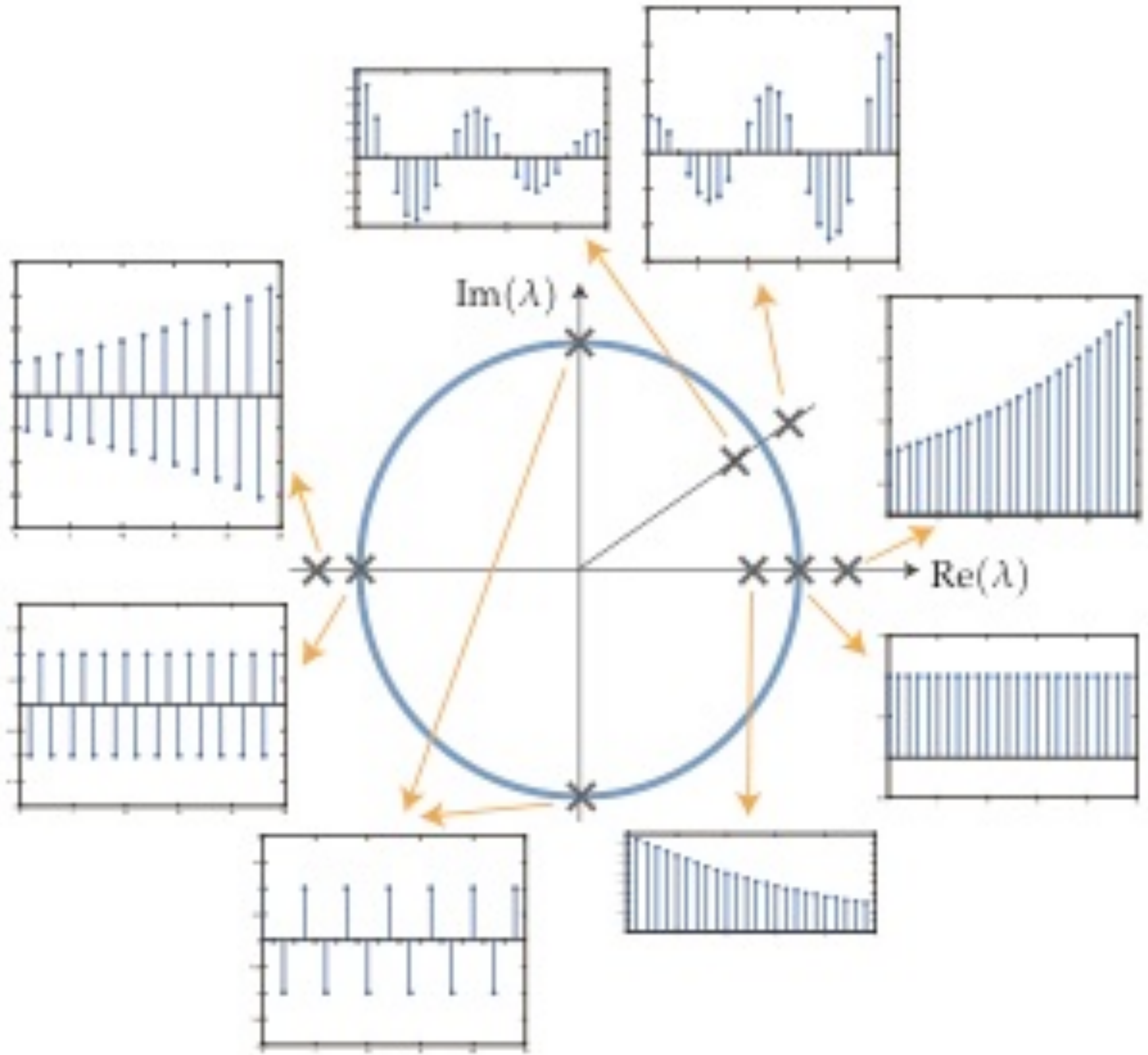
$$\lambda^t$$

- If λ is complex

$$\lambda^t = \left(|\lambda| e^{j\omega}\right)^t$$
$$= |\lambda|^t e^{j\omega t}$$



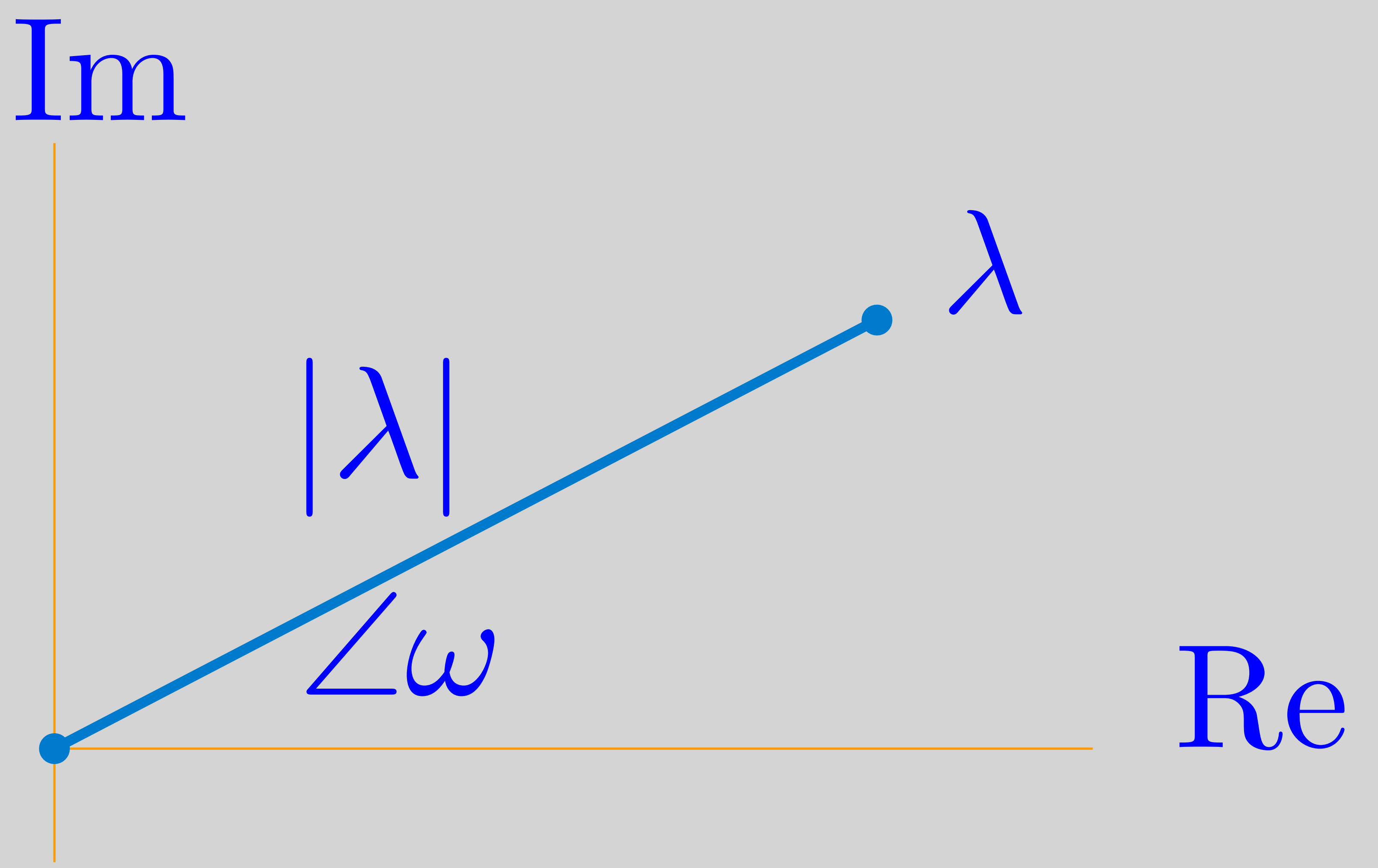
$$\lambda = |\lambda| e^{j\omega}$$ (Euler)



Im(λ)
Re(λ)

- If λ is complex

$$\lambda^t = \left(|\lambda|e^{j\omega}\right)^t$$
$$= |\lambda|^t e^{j\omega t}$$



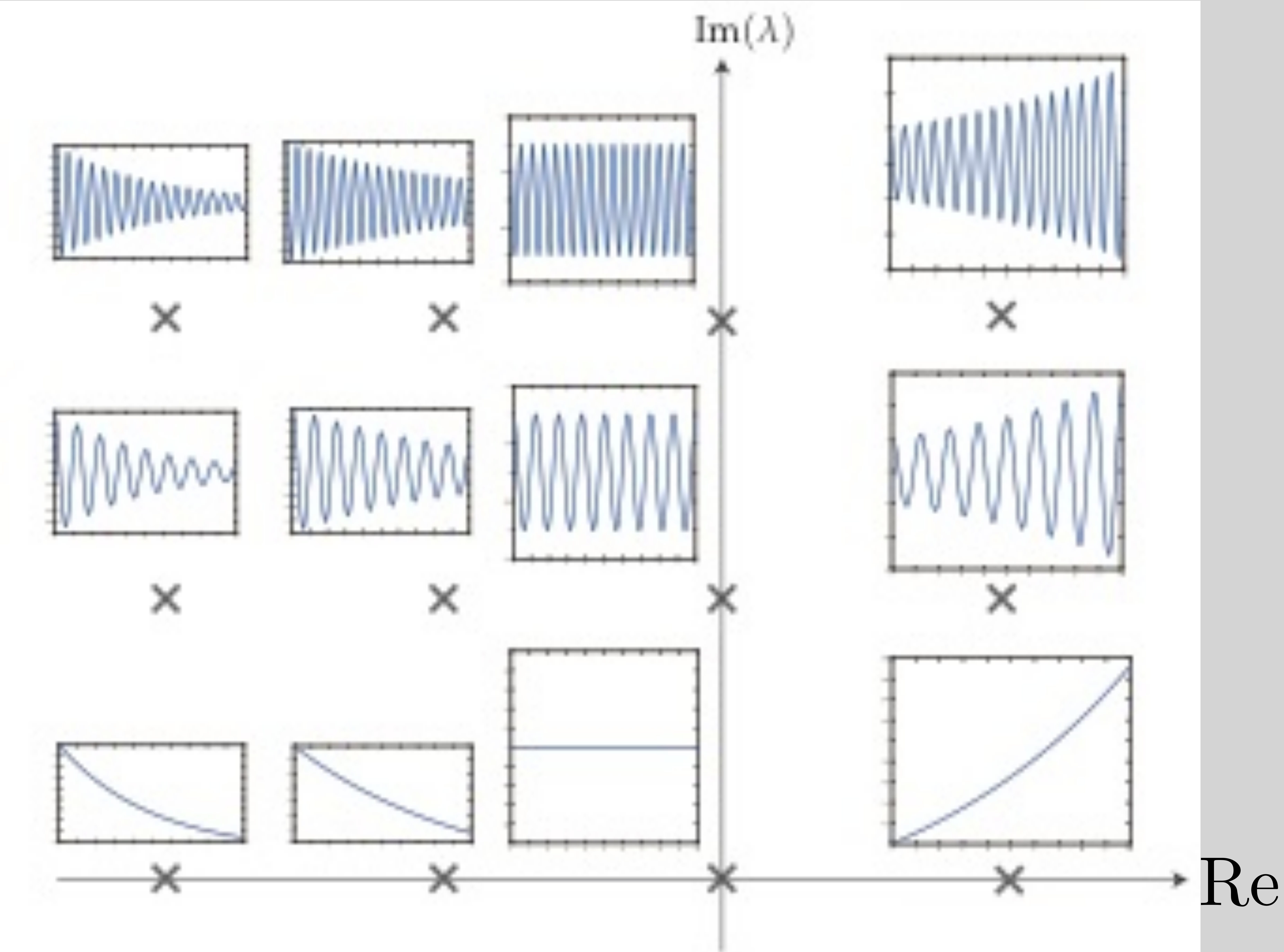
$\lambda = |\lambda|e^{j\omega}$ (Euler)

- Continuous time:

$$\frac{d}{dt}Z_i(t) = \lambda_i Z_i(t) \Rightarrow e^{\lambda_i t}Z_i(0)$$

Q) What does $e^{\lambda t}$ look like for different choices of λ?

A) $\lambda = v + j\omega \qquad \Rightarrow e^{\lambda t} = e^{vt}e^{j\omega t}$



$\mathrm{Im}(\lambda)$
Re

Summary

- Derived stability conditions for vector discrete and continuous systems
- Showed that it is easy to analyze with change of variables!
- Prediction of system behaviour for different eigen-values
  - For discrete – Phase (angle) determines frequency and magnitude determines relaxation
  - For continuous – Real part = relaxation, imaginary = frequency of oscillations
- Next time: Control design – putting the eigenvalues where we want them!