

1 RC Circuit Theory

The RC circuit is a fundamental component of any real world circuit. Many electronic systems' specifications, like clock speed and bandwidth, are direct results of RC circuits. We will use differential equation methods to find the time domain behavior of RC systems. We first set up our problem by defining two functions of time: $I_C(t)$ is the current into the capacitor at time t , and $V_C(t)$ is the voltage across the capacitor at time t .

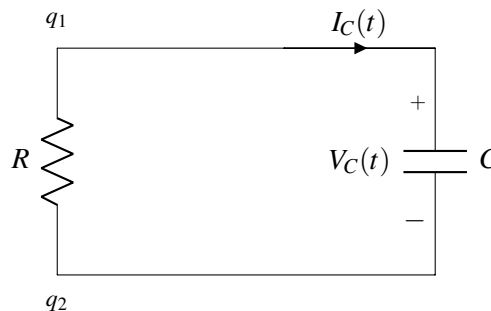


Figure 1: RC Circuit

Let's consider the RC circuit above in Figure 1. Assume that the capacitor is initially fully charged to V_{DD} . Now, we essentially have a pull-down network bringing the capacitor voltage to 0; current starts to flow out of the capacitor through the resistor. As the current flows out, the charge stored in the capacitor decreases. This causes the voltage across the capacitor to decrease. How can we describe this behavior mathematically?

Real life is continuous, so we need to use differential equations. Let's first define the node voltages as q_1 and q_2 . The current across the resistor, I_R , is

$$I_R(t) = \frac{q_2 - q_1}{R}$$

From KCL, $I_R(t) = I_C(t)$, and we also know that

$$I_C(t) = C \frac{dV_C}{dt}$$

Plugging these relations above, with $q_1 = V_C$ and $q_2 = 0$, we get

$$C \frac{dV_C(t)}{dt} = -\frac{V_C(t)}{R}$$

We end up with $-\frac{V_C(t)}{R}$ because the current through the resistor is flowing against the direction we defined for $I_C(t)$.

Rearranging the term, we get

$$\begin{aligned}\frac{dV_C(t)}{dt} &= -\frac{1}{RC}V_C(t) \\ \frac{d}{dt}V_C(t) &= -\frac{1}{RC}V_C(t)\end{aligned}$$

The differentiation operator is a linear operator, so we can view this equation as a linear system $\mathbf{A}V_C(t) = \lambda V_C(t)$, where λ is the eigenvalue of the system and \mathbf{A} is the differentiation operator. Essentially, we need to find a function $V_C(t)$ that, when operated on by \mathbf{A} , results in that same function multiplied by a scaling factor. In linear systems, this function is called the eigenfunction of an operator. The eigenfunction of $\frac{d}{dt}$ is $Ke^{\lambda t}$. This means that $V_C(t)$ is of the form $Ke^{\lambda t}$.

When we operate on it with $\frac{d}{dt}$, we get

$$\frac{d}{dt}Ke^{\lambda t} = K\lambda e^{\lambda t} = \lambda Ke^{\lambda t}$$

and λ is the corresponding eigenvalue.

In our RC system, the constant K is defined by the initial conditions and $\lambda = -\frac{1}{RC}$. Plugging our eigenvalue into our eigenfunction gives us

$$V_C(t) = Ke^{-\frac{t}{RC}}$$

To solve for K , we need to take into account the initial conditions of our problem. At $t = 0$, $V_C(t) = V_{DD}$, so

$$V_C(0) = Ke^{-\frac{0}{RC}} = V_{DD}$$

$$K = V_{DD}$$

Finally, we have

$$V_C(t) = V_{DD}e^{-\frac{t}{RC}}$$

Now, we can evaluate how long it takes to discharge half the voltage.

$$t_{\text{half life}} = \ln(2)RC \approx 0.693RC$$

The equation is derived by setting $V_C(t) = \frac{1}{2}V_{DD}$ and solving for t . We see that bigger the values of R and C , the longer it takes for the voltage to drop. RC is also called the time constant τ . It's useful to have a general idea of how many τ it takes for a capacitor to reach its final steady state value. After one τ , the capacitor voltage is within 36.8% of its final steady state value. After 5τ , it is within 1% of its final steady state value.

1. RC Circuits

In this problem, we will be using differential equations to find the voltage across a capacitor V_C over time in an RC circuit. We set up our problem by first defining three functions over time: $I(t)$ is the current at time t , $V(t)$ is the voltage across the circuit at time t , and $V_C(t)$ is the voltage across the capacitor at time t .

Recall from 16A that the voltage across a resistor is defined as $V_R = RI_R$ where I_R is the current across the resistor. Also, recall that the voltage across a capacitor is defined as $V_C = \frac{Q}{C}$ where Q is the charge across the capacitor.

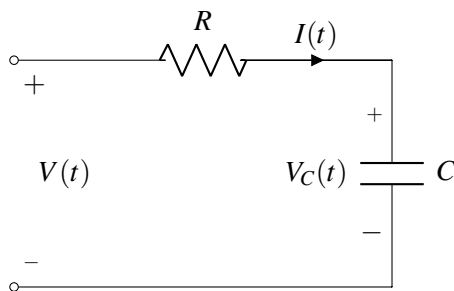


Figure 2: Example Circuit

- (a) First, find an equation that relates the current across the capacitor $I(t)$ with the voltage across the capacitor $V_C(t)$.

Answer:

Differentiating $V_C(t) = \frac{Q(t)}{C}$ in terms of t , we get

$$\frac{dV_C(t)}{dt} = \frac{dQ(t)}{dt} \frac{1}{C}$$

By definition, the change in charge is the current across the capacitor, so

$$\frac{dV_C(t)}{dt} = I(t) \frac{1}{C}$$

- (b) Using Kirchhoff's law, write an equation that relates the functions $I(t)$, $V_C(t)$, and $V(t)$.

Answer:

Kirchhoff's law states that the voltage across a closed loop is 0.

$$RI(t) + V_C(t) - V(t) = 0$$

$$RI(t) + V_C(t) = V(t) \tag{1}$$

- (c) So far, we have three unknown functions and only one equation, but we can remove $I(t)$ from the equation using what we found in part (a). Rewrite the previous equation in part (b) in the form of a differential equation.

Answer:

From part (a), we have

$$I(t) = \frac{dV_C(t)}{dt} C$$

Substituting this into Equation 1 gives us

$$RC \frac{dV_C(t)}{dt} + V_C(t) = V(t)$$

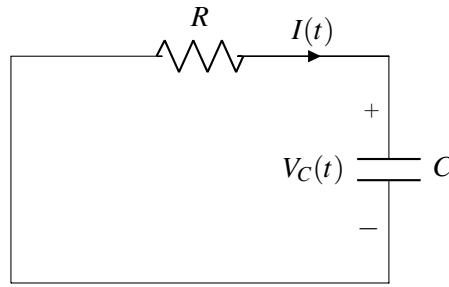


Figure 3: Circuit for part (d)

- (d) Let's suppose that at $t = 0$, the capacitor is charged to a voltage V_{DD} ($V_C(0) = V_{DD}$). Let's also assume that $V(t) = 0 \forall t \geq 0$, i.e. shorted to ground. Use the initial condition of $V_C(t)$ to solve the differential equation for $V_C(t)$ for $t \geq 0$.

Answer:

Because $V(t) = 0$, our differential equation simplifies to

$$RC \frac{dV_C(t)}{dt} + V_C(t) = 0$$

Doing some algebraic manipulations gives us

$$\frac{dV_C(t)}{dt} = -\frac{1}{RC} V_C(t)$$

This equation tells us that we are looking for some function $V_C(t)$ such that when we take its derivative, we get the same function $V_C(t)$ multiplied by a scalar $-\frac{1}{RC}$. Because the derivative is equal to a scalar times itself, we think that the solution $V_C(t)$ will probably be of the form Ae^{bt} , where A and b are both constants. In this case we see that $b = -\frac{1}{RC}$, and we find that

$$V_C(t) = Ae^{-\frac{1}{RC}t}$$

We still need to solve for the constant A in front of the exponential, and we use $V_C(0) = K$ to help us find A . Setting $t = 0$ in the equation gives us

$$\begin{aligned} V_C(0) &= Ae^{-\frac{1}{RC}0} \\ &= Ae^0 \\ &= A \\ &= V_{DD} \end{aligned}$$

Thus, we see that $A = V_{DD}$, and our solution is

$$V_C(t) = V_{DD}e^{-\frac{1}{RC}t}$$

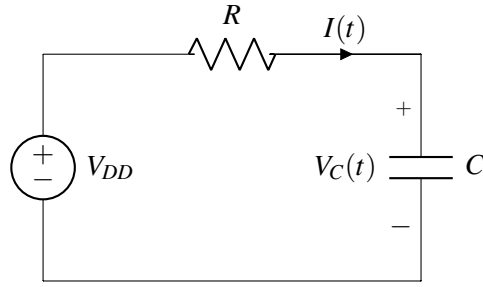


Figure 4: Circuit for part (e)

- (e) Now, let's suppose that we start with an uncharged capacitor $V_C(0) = 0$. We apply some constant voltage $V(t) = V_{DD}$ across the circuit. Solve the differential equation for $V_C(t)$ for $t \geq 0$.

Answer:

Plugging in $V(t) = V_{DD}$ into our solution from part (c):

$$RC \frac{dV_C(t)}{dt} + V_C(t) = V_{DD}$$

Since this is a non-homogeneous differential equation, let's define a new equation to model the difference $\tilde{V}_C(t) = V_C(t) - V_{DD}$ over time. Note that $\frac{d\tilde{V}_C(t)}{dt} = \frac{dV_C(t)}{dt}$. We can substitute these into our differential equation and obtain

$$RC \frac{dV_C(t)}{dt} + V_C(t) - V_{DD} = 0$$

$$RC \frac{d\tilde{V}_C(t)}{dt} + \tilde{V}_C(t) = 0$$

In this equation, we have now removed V_{DD} from the left hand because of how we defined $\tilde{V}_C(t)$. We can now solve the differential equation using the same method as in the previous part to get

$$\tilde{V}_C(t) = Ae^{-\frac{t}{RC}}$$

Substituting $V_C(t) = V_{DD} + \tilde{V}_C(t)$ back into this equation gives us

$$V_C(t) = V_{DD} + Ae^{-\frac{t}{RC}}$$

Plugging in the initial condition $V_C(0) = 0$, we get:

$$0 = V_{DD} + Ae^{-\frac{0}{RC}} = V_{DD} + A \implies A = -V_{DD}$$

Therefore,

$$\begin{aligned} V_C(t) &= V_{DD} - V_{DD}e^{-\frac{t}{RC}} \\ &= V_{DD}(1 - e^{-\frac{t}{RC}}) \end{aligned}$$

2. Two-capacitor RC Circuit

Let's now consider a slightly more complicated RC circuit.

In this problem, we will explore what happens when we change the voltage in between the capacitors.

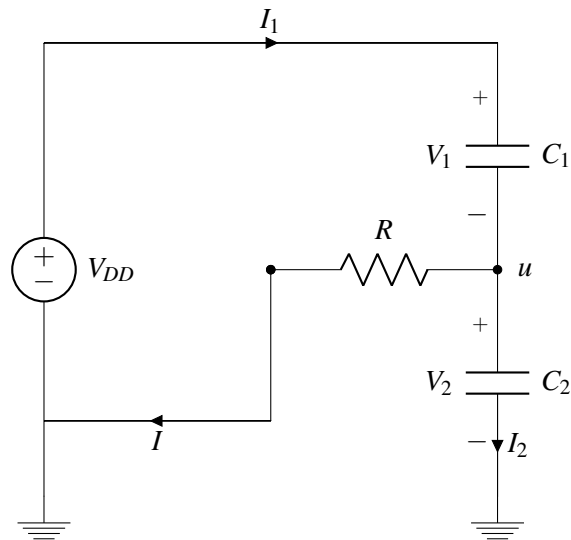


Figure 5: Inverter Input

- (a) Suppose that $V_1 = 0$ and $V_2 = V_{DD}$ at $t = 0$. Express the voltages $V_1(t)$ and $V_2(t)$ as a function of time.

Answer: Using KCL, we get that $I - I_1 + I_2 = 0$

$$I = \frac{u}{R}$$

$$I_1 = C_1 \frac{d(V_{DD} - u)}{dt} = -C_1 \frac{du}{dt}$$

$$I_2 = C_2 \frac{du}{dt}$$

Now, we can replace the currents in the KCL equation as

$$\frac{u}{R} + C_1 \frac{du}{dt} + C_2 \frac{du}{dt} = 0$$

$$\frac{du}{dt} = -\frac{u}{R(C_1 + C_2)}$$

This is the same equation we saw in the previous part where we wanted to discharge a capacitor. With this, we get a time constant of $\tau = R(C_1 + C_2)$. As such, the solution is

$$u = V_{DD} e^{\frac{-t}{R(C_1 + C_2)}}$$

$$V_2(t) = V_{DD} e^{\frac{-t}{R(C_1 + C_2)}}$$

$$V_1(t) = V_{DD} - V_{DD} e^{\frac{-t}{R(C_1 + C_2)}}$$

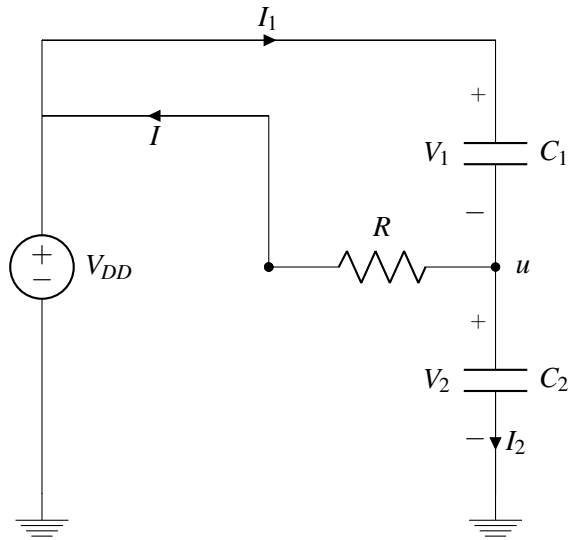


Figure 6: Configuration 1

- (b) Suppose that $V_1 = V_{DD}$ and $V_2 = 0$ at $t = 0$. Express the voltages $V_1(t)$ and $V_2(t)$ as a function of time.

Answer: We can follow the same steps as in the previous part to arrive at a similar solution. The only difference this time is that

$$I = \frac{u - V_{DD}}{R}$$

We specifically write the current this way because we defined the current as leaving the node V_{out} .

As such, we can replace the function in the original KCL equation as in the previous part and get a differential equation for V_{out} .

$$R(C_1 + C_2) \frac{du}{dt} + u = V_{DD}$$

This again looks very similar to what we did when we wanted to charge a capacitor. As such, we would get a similar solution. With this, we get a time constant of $\tau = R(C_1 + C_2)$. As such, the solution is

$$\begin{aligned} u &= V_{DD} - V_{DD} e^{\frac{-t}{R(C_1 + C_2)}} \\ V_2(t) &= V_{DD} - V_{DD} e^{\frac{-t}{R(C_1 + C_2)}} \\ V_1(t) &= V_{DD} e^{\frac{-t}{R(C_1 + C_2)}} \end{aligned}$$

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