EE16B
Designing Information Devices and Systems II

Lecture 5B
Linearization
Stability of linear state models

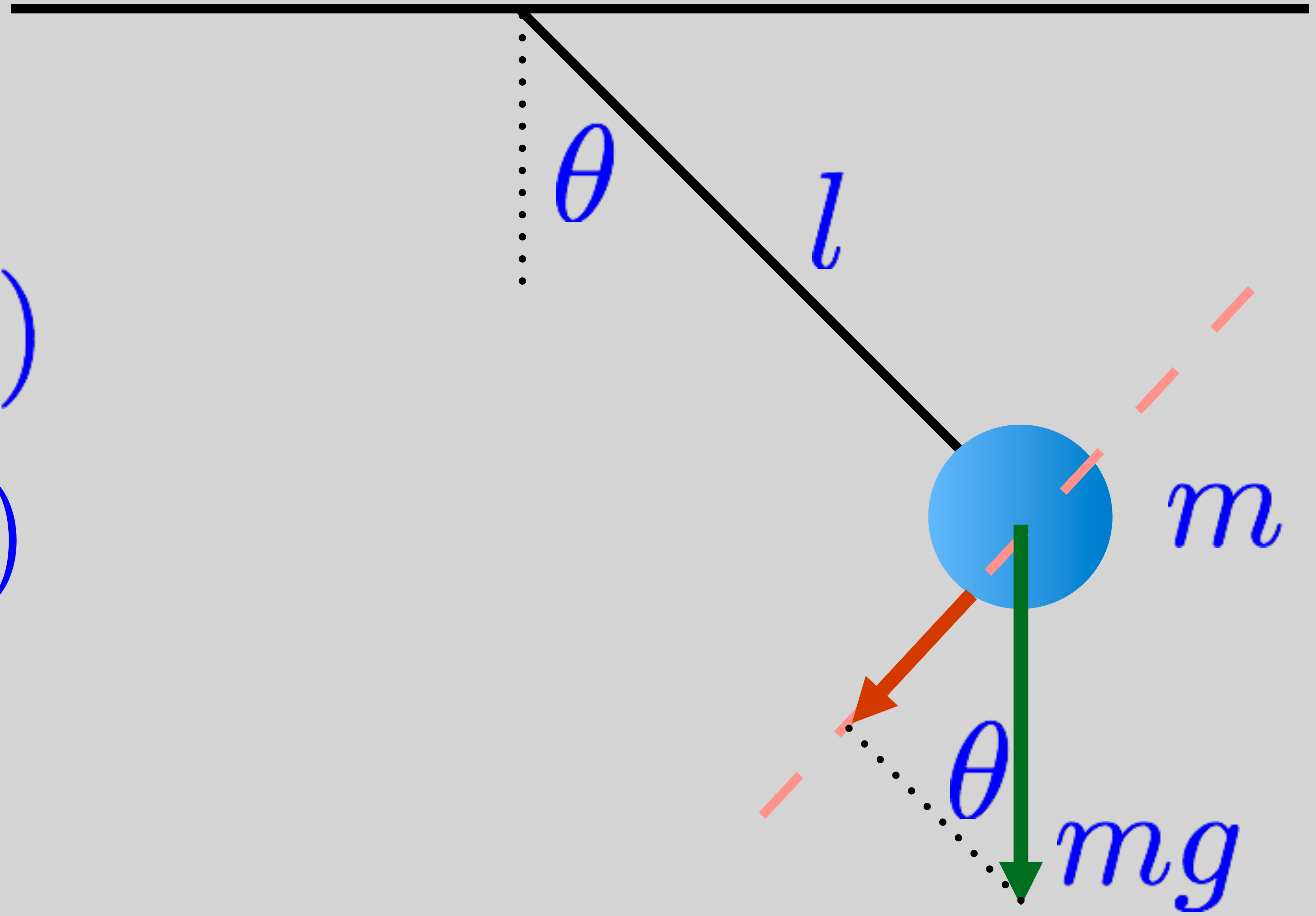
Intro

- Last time
  - Described systems with state-space model
  - Talked about linear systems
  - Change of variables

- Today
  - A bit on Linearization of non-linear systems
  - Begin Stability of linear state models
    - Scalar and discrete

https://www.youtube.com/watch?v=SPO9pVwoxVg

Linearization



State variables:

$$x_1(t) = \theta(t)$$
$$x_2(t) = \dot{\theta}(t)$$

$$\frac{dx_1(t)}{dt} = x_2(t)$$
$$\frac{dx_2(t)}{dt} = -\frac{g}{l}\underbrace{\sin(x_1(t))}_{\approx x_1} - \frac{k}{m}x_2(t)$$

Linearization:

$$\frac{dx_2(t)}{dt} = -\frac{g}{l}x_1(t) - \frac{k}{m}x_2(t)$$

Linearization

$$\frac{dx_1(t)}{dt} = x_2(t)$$

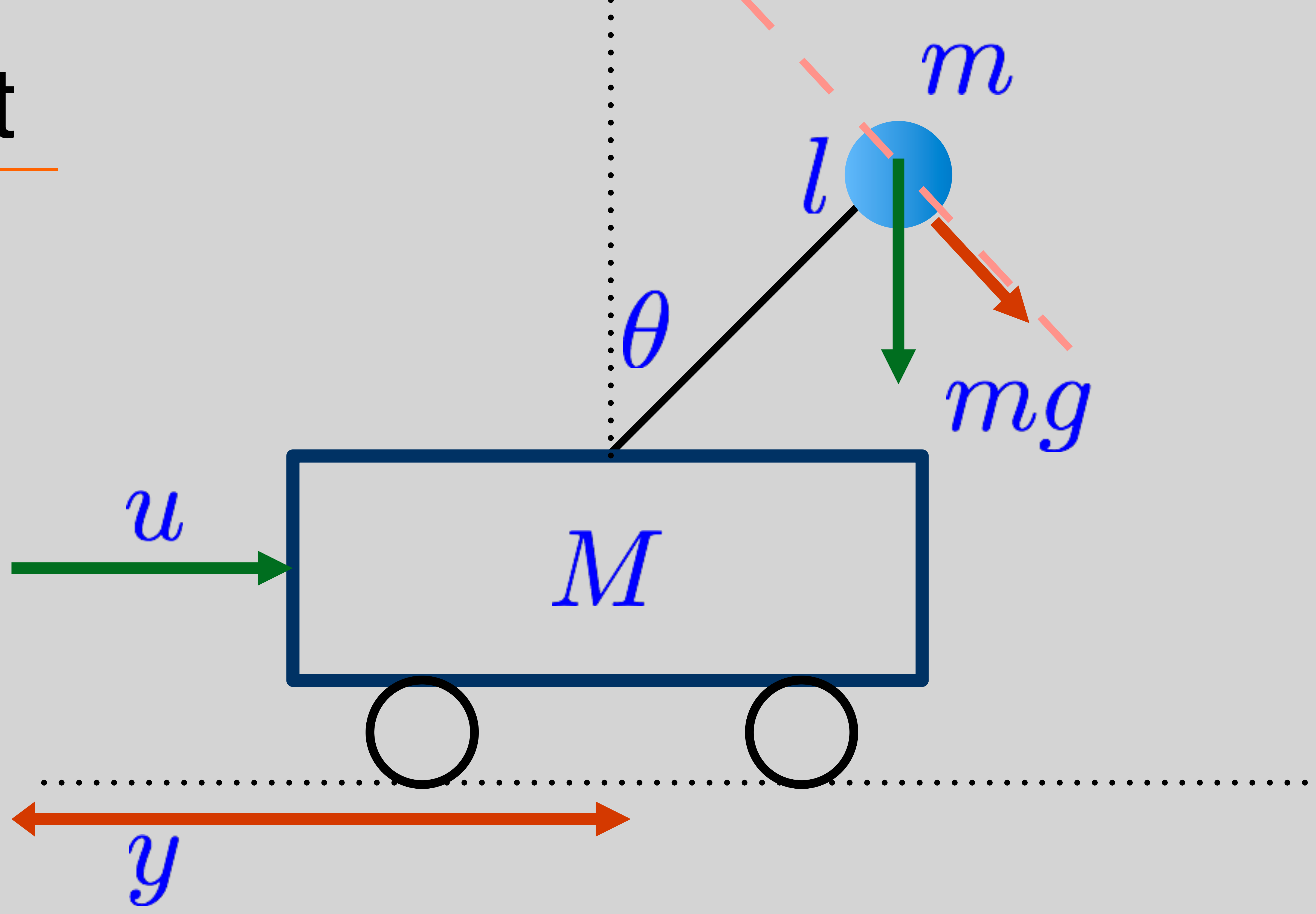
$$\frac{dx_2(t)}{dt} = -\frac{g}{l}x_1(t) - \frac{k}{m}x_2(t)$$

$$\Rightarrow \begin{bmatrix} \frac{dx_1(t)}{dt} \\ \frac{dx_2(t)}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & -\frac{k}{m} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Scary Example: Pole on a Cart

How many state variables?

How to systematically linearize?

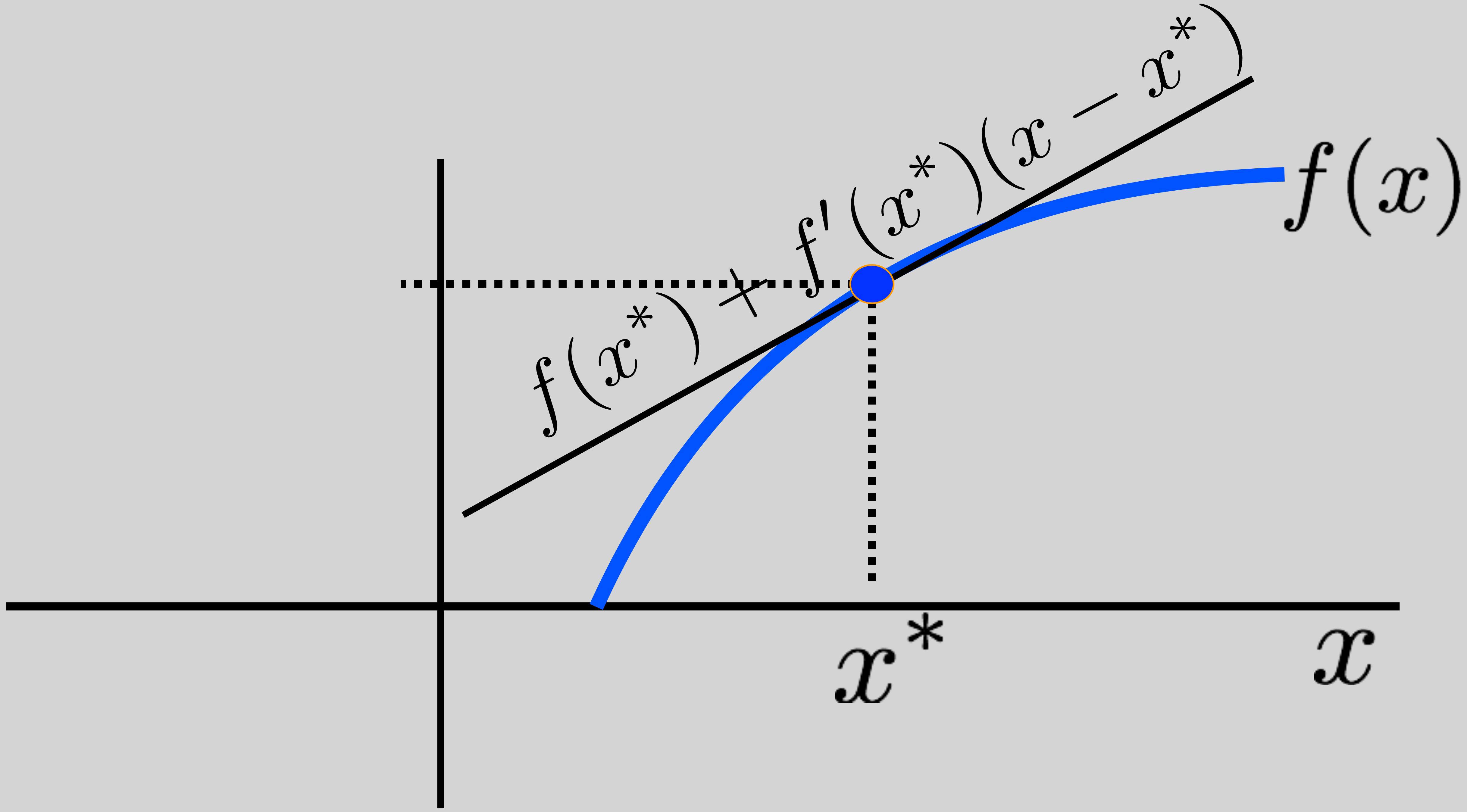


$$\ddot{y} = \frac{1}{\frac{M}{m} + \sin^2\theta}\left(\frac{u}{m} + \dot{\theta}^2 l \sin\theta - g \sin\theta\cos\theta\right)$$

$$\ddot{\theta} = \frac{1}{l(\frac{M}{m} + \sin^2\theta)}\left(-\frac{u}{m}\cos\theta - \dot{\theta}^2 l \sin\theta\cos\theta + \frac{M+m}{m} g \sin\right.$$

Taylor Approximation - scalar

$$f : \mathbb{R} \to \mathbb{R}$$



$$f(x) \approx f(x^*) + f'(x^*)(x - x^*)$$

$$\Rightarrow \sin(x) \approx \sin(x^*) + \cos(x^*)(x - x^*)$$

$$x^* = 0 \Big| \Rightarrow \sin(x) \approx \sin(0) + \cos(0)(x - 0)$$

$$\sin x \approx x$$

Taylor Approximation - scalar

$$f : \mathbb{R} \rightarrow \mathbb{R}$$

$$\Rightarrow \sin(x) \approx \sin(x^*) + \cos(x^*)(x - x^*)$$

Example:

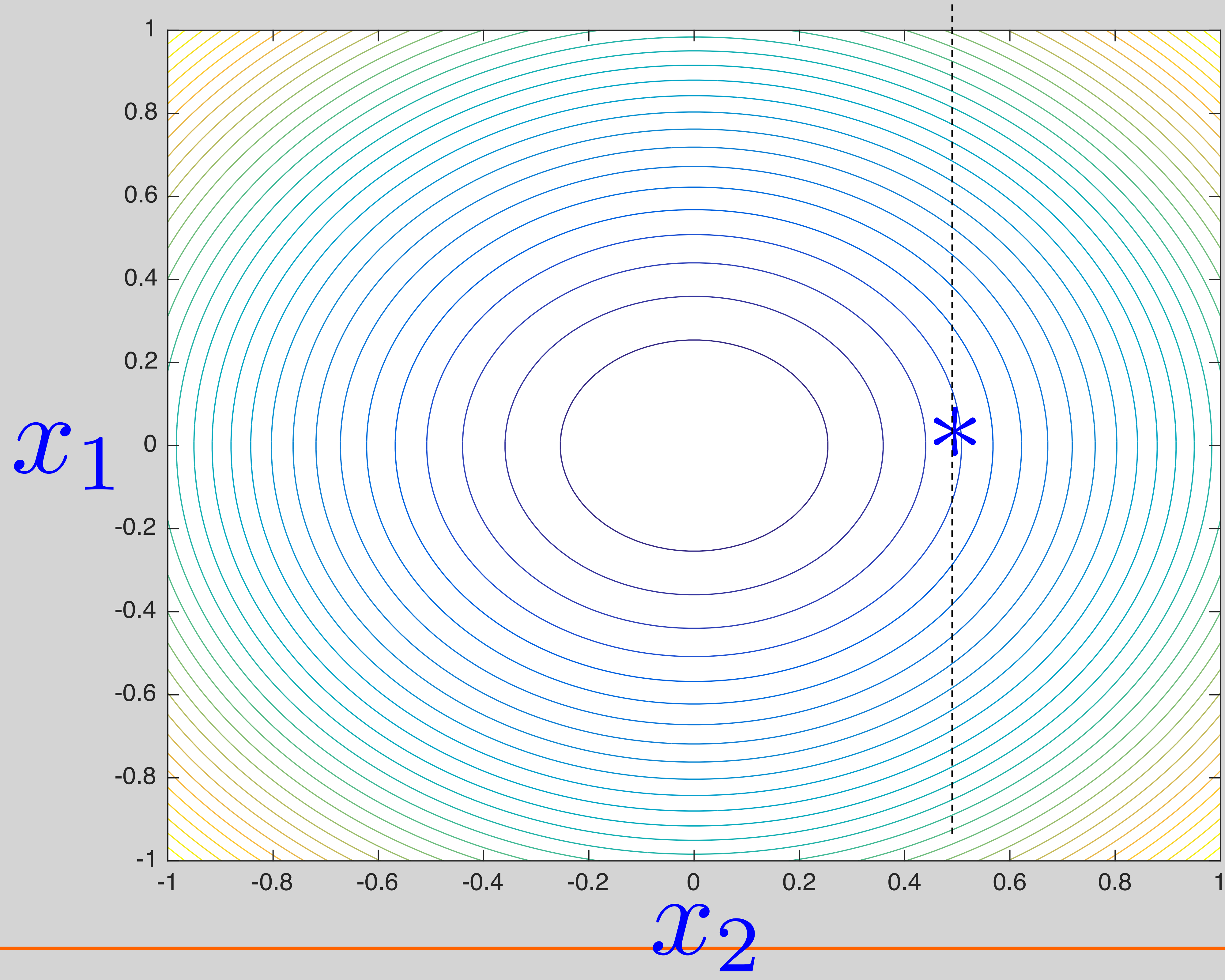
Taylor Approximation - vector

$$f : \mathbb{R}^N \to \mathbb{R}$$

Example: $f : \mathbb{R}^2 \to \mathbb{R}$ $\qquad f(\vec{x}) = ||\vec{x}||^2 = x_1^2 + x_2^2$

Let's look at

$$f(x_1, x_2 = x_2^*) = x_1^2 + {x_2^*}^2$$



Partial Derivative

Scalar "template"

$$f(x) \approx f(x^*) + f'(x^*)(x - x^*)$$

$$f(x_1, x_2 = x_2^*) = x_1^2 + {x_2^*}^2$$

$$\frac{d}{dx_1} f(x_1, x_2^*) = \frac{d}{dx_1} x_1^2 + \frac{d}{dx_1} {x_2^*}^2 = 2x_1$$

$$\frac{\partial}{\partial x_1} f(x_1, x_2) = 2x_1$$

$$\frac{\partial}{\partial x_2} f(x_1, x_2) = 2x_2$$

Taylor Approximation - vector

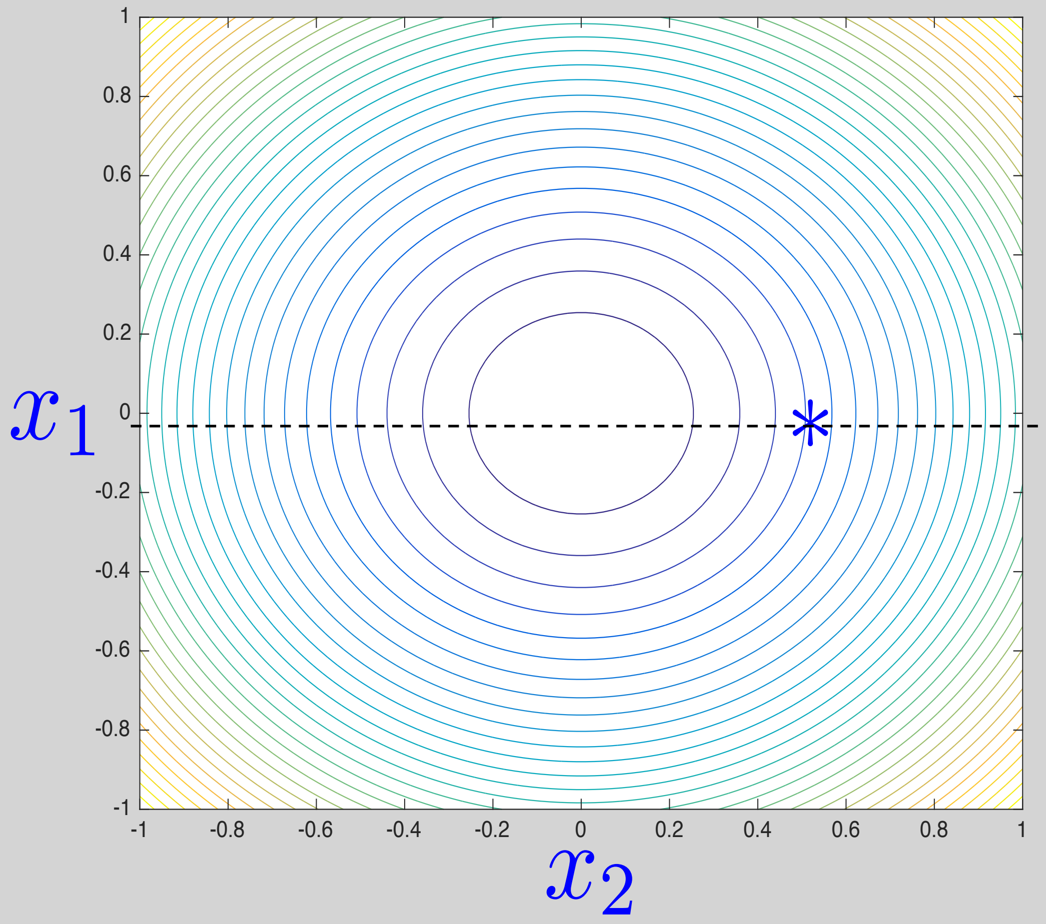
Scalar "template"

$$f(x) \approx f(x^*) + f'(x^*)(x - x^*)$$

$$f(x_1, x_2 = x_2^*) = x_1^2 + {x_2^*}^2$$

$$\frac{\partial}{\partial x_1} f(x_1, x_2)\Big|_{x_1^*, x_2^*} = 2x_1^*$$

$$f(x_1, x_2^*) \approx {x_1^*}^2 + {x_2^*}^2 \quad + 2x_1^*(x_1 - x_1^*)$$



Similarly:

$$f(x_1^*, x_2) \approx {x_1^*}^2 + {x_2^*}^2 + 2x_2^*(x_2 - x_2^*)$$

So,

$$f(x_1, x_2) \approx {x_1^*}^2 + {x_2^*}^2 + 2x_1^*(x_1 - x_1^*) + 2x_2^*(x_2 - x_2^*)$$

Taylor Approximation - vector

$$f(x_1, x_2) \approx {x_1^*}^2 + {x_2^*}^2 + 2x_1^*(x_1 - x_1^*) + 2x_2^*(x_2 - x_2^*)$$

Write in vector form:

$$f(x_1, x_2) \approx {x_1^*}^2 + {x_2^*}^2 + \begin{bmatrix} 2x_1^* & 2x_2^* \end{bmatrix}(\vec{x} - \vec{x}^*)$$

Taylor Approximation - vector

Scalar “template”

$$f(x) \approx f(x^*) + f'(x^*)(x - x^*)$$

$$f : \mathbb{R}^2 \to \mathbb{R}$$

$$f(x_1, x_2) \approx {x_1^*}^2 + {x_2^*}^2 + \begin{bmatrix} 2x_1^* & 2x_2^* \end{bmatrix}(\vec{x} - \vec{x}^*)$$

$$f(\vec{x}) \approx f(\vec{x}^*) + \begin{bmatrix} \frac{\partial}{\partial x_1} f(\vec{x}^*) & \frac{\partial}{\partial x_2} f(\vec{x}^*) \end{bmatrix} (\vec{x} - \vec{x}^*)$$

$$f(\vec{x}) \approx f(\vec{x}^*) + \nabla f(\vec{x}^*)(\vec{x} - \vec{x}^*)$$

Q: What are the dimensions of ∇f(x*) ? (gradient / Jacobian)

Taylor Approximation - vector

$$f : \mathbb{R}^N \rightarrow \mathbb{R}^N \qquad \frac{d}{dt}\vec{x} = f(\vec{x})$$

$$\underbrace{f(\vec{x})}_{\text{Nx1}} \approx \underbrace{f(\vec{x}^*)}_{\text{Nx1}} + \nabla f(\vec{x}^*)\underbrace{(\vec{x} - \vec{x}^*)}_{\text{Nx1}}$$

Q: What are the dimensions of $\nabla f(\vec{x}^*)$? (Jacobian)

A: NxN ?

Taylor Approximation - vector

$$f : \mathbb{R}^N \rightarrow \mathbb{R}^N$$

$$f(\vec{x}) = \left[\quad \right]$$

$$\nabla f(\vec{x}) = \left[\quad \right]$$

i,j th entry:

$$\frac{\partial f_i(x)}{\partial x_j}$$

Taylor Approximation - vector

$$f : \mathbb{R}^N \rightarrow \mathbb{R}^N \qquad f(\vec{x}) = \begin{bmatrix} f_1(x_1, \cdots, x_N) \\ f_2(x_1, \cdots, x_N) \\ \vdots \\ f_N(x_1, \cdots, x_N) \end{bmatrix}$$

$$\nabla f(\vec{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_N} \\ & \vdots & \\ \frac{\partial f_N}{\partial x_1} & \cdots & \frac{\partial f_N}{\partial x_N} \end{bmatrix}$$

i,j th entry:

$$\frac{\partial f_i(x)}{\partial x_j}$$

Taylor Approximation - vector

$$f : \mathbb{R}^N \to \mathbb{R}^N$$

$$f(\vec{x}) = \begin{bmatrix} f_1(x_1, \cdots, x_N) \\ f_2(x_1, \cdots, x_N) \\ \vdots \\ f_N(x_1, \cdots, x_N) \end{bmatrix}$$

$$\nabla f(\vec{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_N} \\ & \vdots & \\ \frac{\partial f_N}{\partial x_1} & \cdots & \frac{\partial f_N}{\partial x_N} \end{bmatrix}$$

i,j th entry:

$$\frac{\partial f_i(x)}{\partial x_j}$$

Linearization of State-Space

Linearize around an equilibrium, a point s.t.:

$$f(\vec{x}^*) = 0$$

Q: why?

A: no change!

$$\frac{d}{dt}\vec{x} = f(\vec{x})$$

$$\approx \underbrace{f(\vec{x}^*)}_{=0} + \nabla f(\vec{x}^*)\underbrace{(\vec{x} - \vec{x}^*)}_{\tilde{x}}$$

Which of the variables is a function of t?

write a state model for deviation!

Linearization of State-Space

$$\tilde{x} = \vec{x} - \vec{x}^*$$

$$\frac{d}{dt}\tilde{x}(t) = \frac{d}{dt}\vec{x}(t) - \underbrace{\frac{d}{dt}\vec{x}^*}_{=0}$$

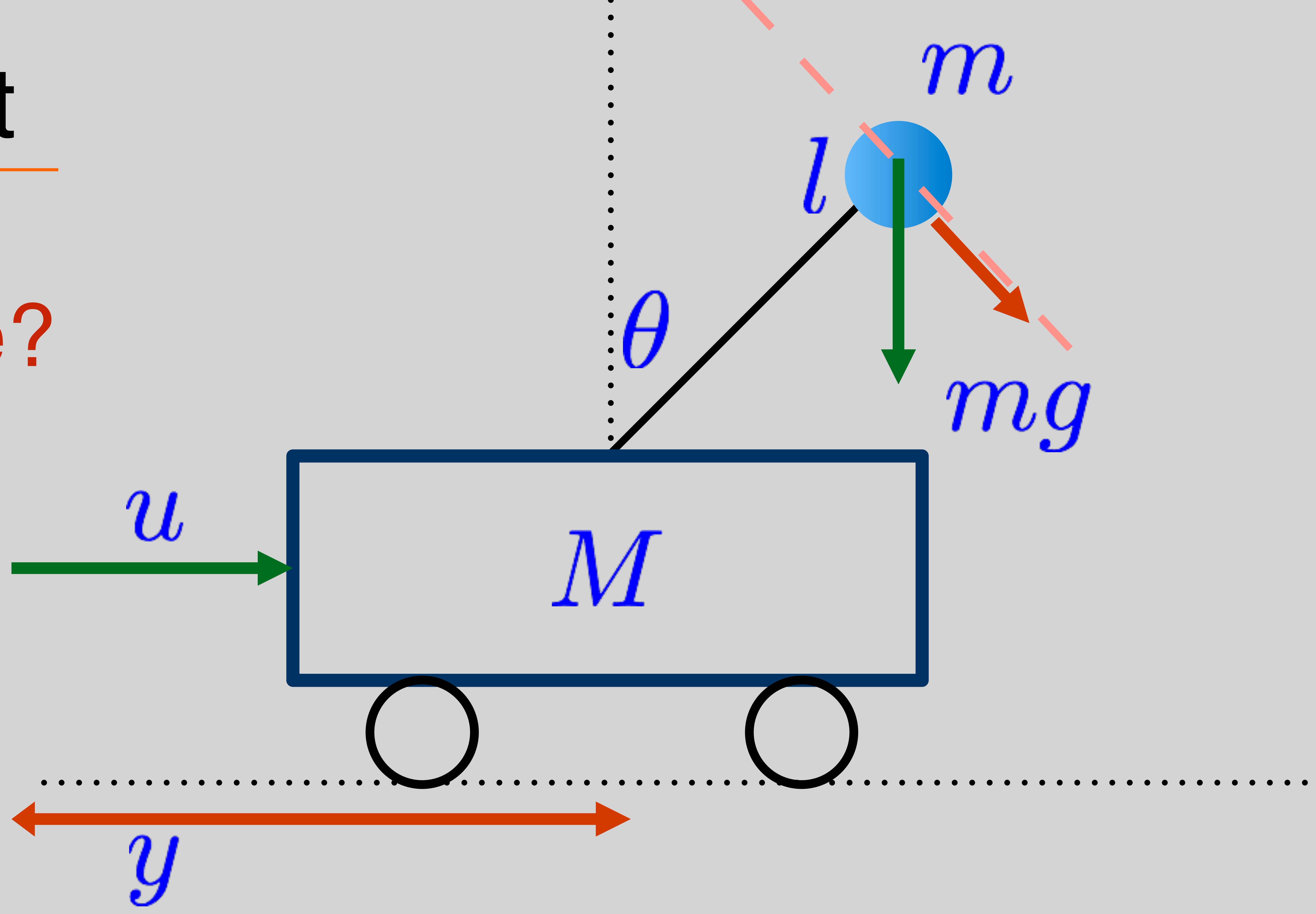
$$\Updownarrow$$

$$= f(\vec{x}(t)) \approx \underbrace{f(\vec{x}^*)}_{=0} + \nabla f(\vec{x}^*)\tilde{x}(t)$$

$$\frac{d}{dt}\tilde{x}(t) = [\nabla \underbrace{f(\vec{x}^*)}_{A}]\,\tilde{x}(t)$$

Scary Example: Pole on a Cart

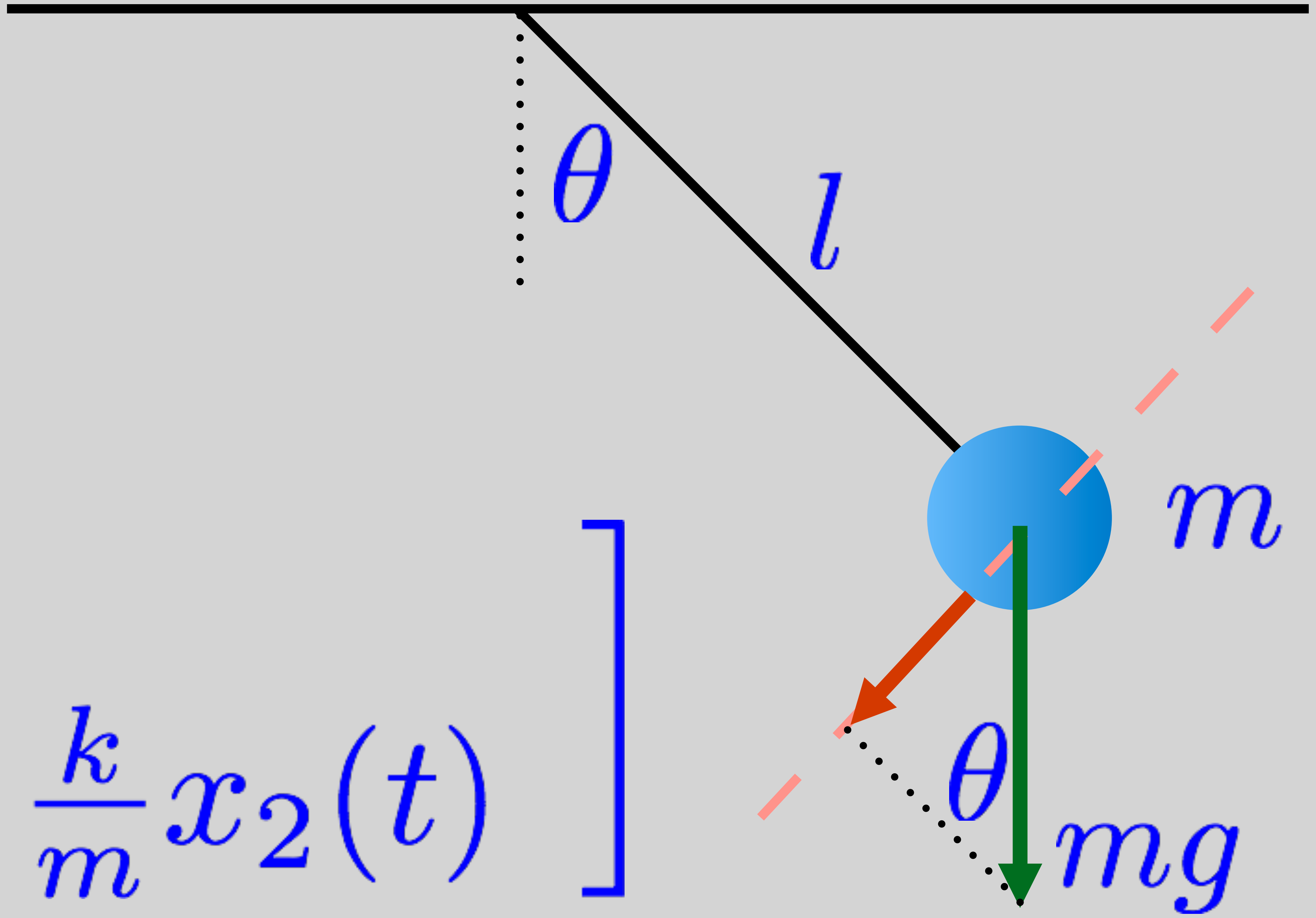
Q) Can you do it for this example?



$$\ddot{y} = \frac{1}{\frac{M}{m} + \sin^2\theta}\left(\frac{u}{m} + \dot{\theta}^2 l \sin\theta - g\sin\theta\cos\theta\right)$$

$$\ddot{\theta} = \frac{1}{l(\frac{M}{m} + \sin^2\theta)}\left(-\frac{u}{m}\cos\theta - \dot{\theta}^2 l \sin\theta\cos\theta + \frac{M+m}{m} g \sin\theta\right)$$

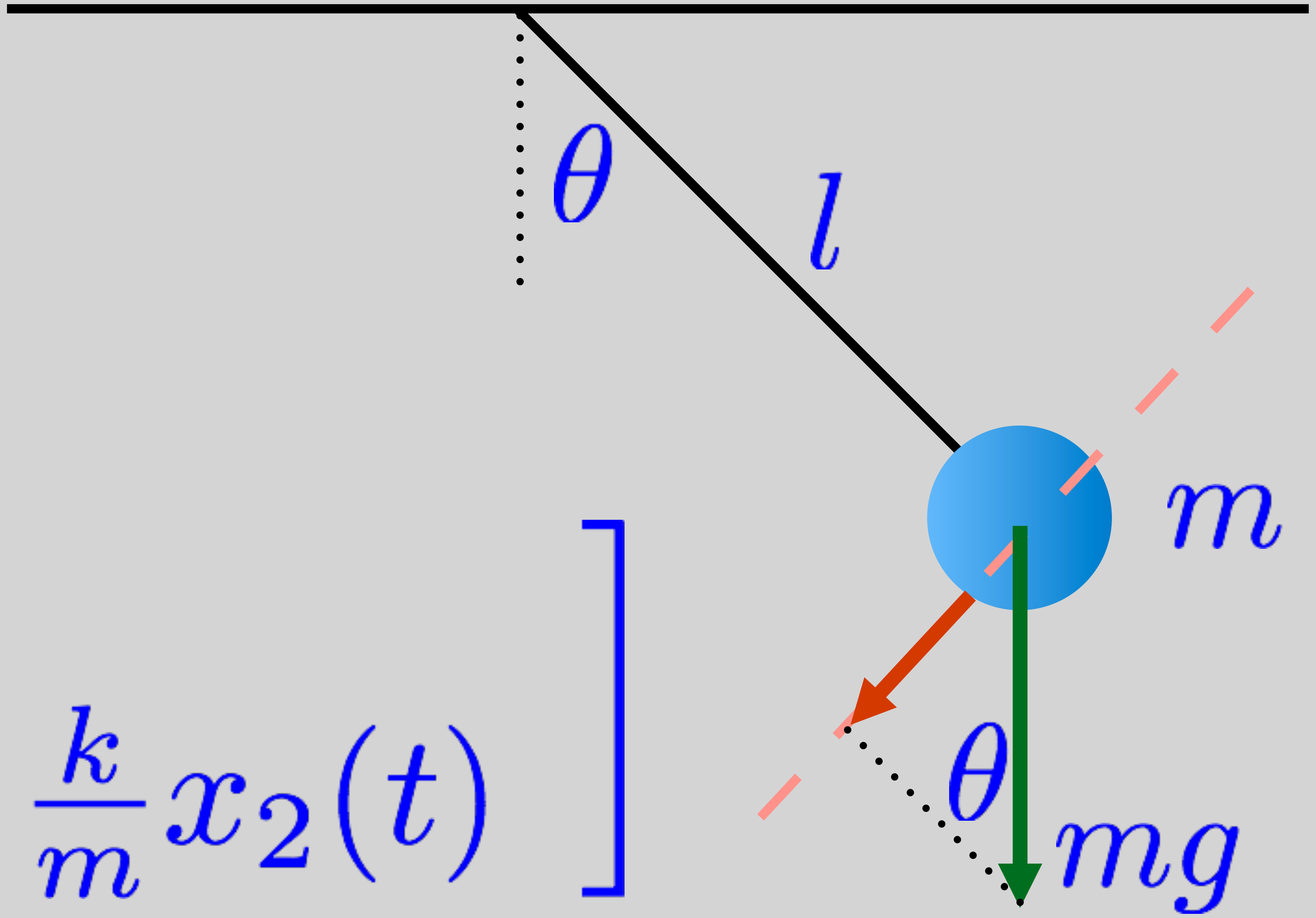
Back to the Pendulum



$$f(\vec{x}(t)) = \begin{bmatrix} x_2(t) \\ -\frac{g}{l}\sin(x_1(t)) - \frac{k}{m}x_2(t) \end{bmatrix}$$

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix}$$

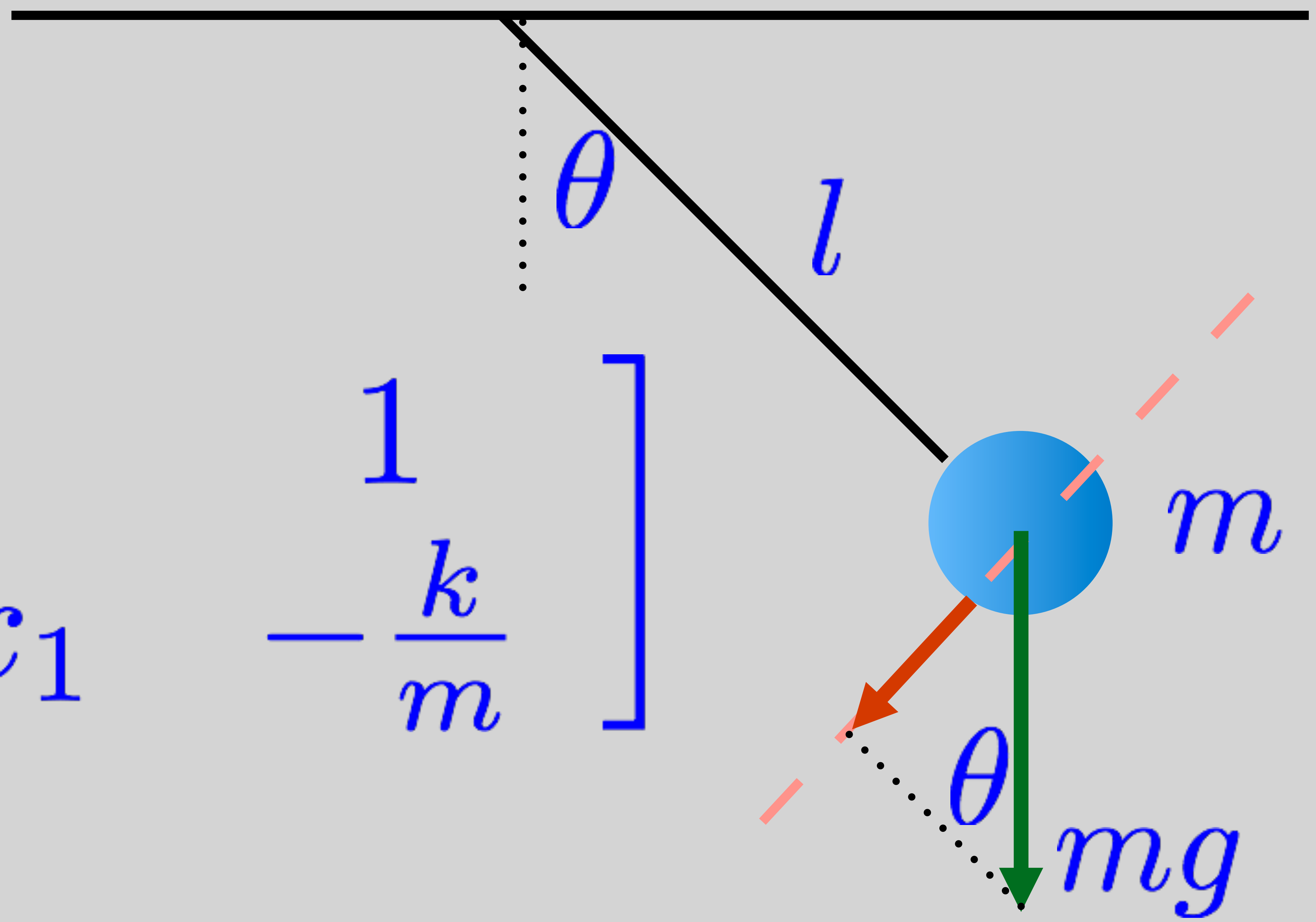
Back to the Pendulum



$$f(\vec{x}(t)) = \begin{bmatrix} x_2(t) \\ -\frac{g}{l}\sin(x_1(t)) - \frac{k}{m}x_2(t) \end{bmatrix}$$

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l}\cos x_1 & -\frac{k}{m} \end{bmatrix}$$

Pendulum at Equilibrium



$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l}\cos x_1 & -\frac{k}{m} \end{bmatrix}$$

$x_1^* = 0,\ x_2^* = 0$, Downward equilibrium

$$A_{\text{down}} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & -\frac{k}{m} \end{bmatrix}$$

This is the same as small signal analysis!

$x_1^* = \pi,\ x_2^* = 0$, Upward equilibrium

$$A_{\text{up}} = \begin{bmatrix} 0 & 1 \\ \frac{g}{l} & -\frac{k}{m} \end{bmatrix}$$

Talk about next lecture!

Discrete Time

$$\vec{x}(t+1) = f(\vec{x}(t))$$

$\vec{x} = \vec{x}*$ is an equilibrium <u>if:</u>

$$f(\vec{x}^*) = \vec{x}^*$$ (for cont. $f(\vec{x}^*) = 0$)

$$\tilde{x}(t) = \vec{x}(t) - \vec{x}^*$$

$$\tilde{x}(t+1) = \vec{x}(t+1) - \vec{x}^*$$
$$= f(\vec{x}(t)) - \vec{x}^*$$
$$\approx \cancel{f(\vec{x}^*)} + \overbrace{\nabla f(\vec{x}^*)}^{A}\tilde{x}(t) - \cancel{\vec{x}^*}$$

$$\boxed{\tilde{x}(t+1) = A\tilde{x}(t)}$$

Stability of Linear State Models

Start with scalar system 1st order system:

$$x(t+1) = ax(t) + bu(t)$$

Given initial condition x(0):

$$x(1) = ax(0) + bu(0)$$

$$\begin{aligned} x(2) &= ax(1) + bu(1) \\ &= a^2x(0) + abu(0) + bu(1) \end{aligned}$$

$$x(3) = a^3x(0) + a^2bu(0) + abu(1) + bu(2)$$

$$x(t) = a^tx(0) + a^{t-1}bu(0) + a^{t-2}bu(1) + \cdots a^0bu(t-1)$$

Stability of Linear State Models

Start with scalar system:

$$x(t+1) = ax(t) + bu(t)$$

Given initial condition x(0):

$$x(t) = a^t x(0) + \underbrace{\boxed{a^{t-1}bu(0)}}_{k=0} + a^{t-2}bu(1) + \cdots \underbrace{\boxed{a^0 bu(t-1)}}_{k=t-1}$$

$$x(t) = \underbrace{a^t x(o)}_{\text{Initial condition}} + \underbrace{\sum_{k=0}^{t-1} a^{t-k-1} bu(k)}_{\text{input}}$$

Stability - Definition

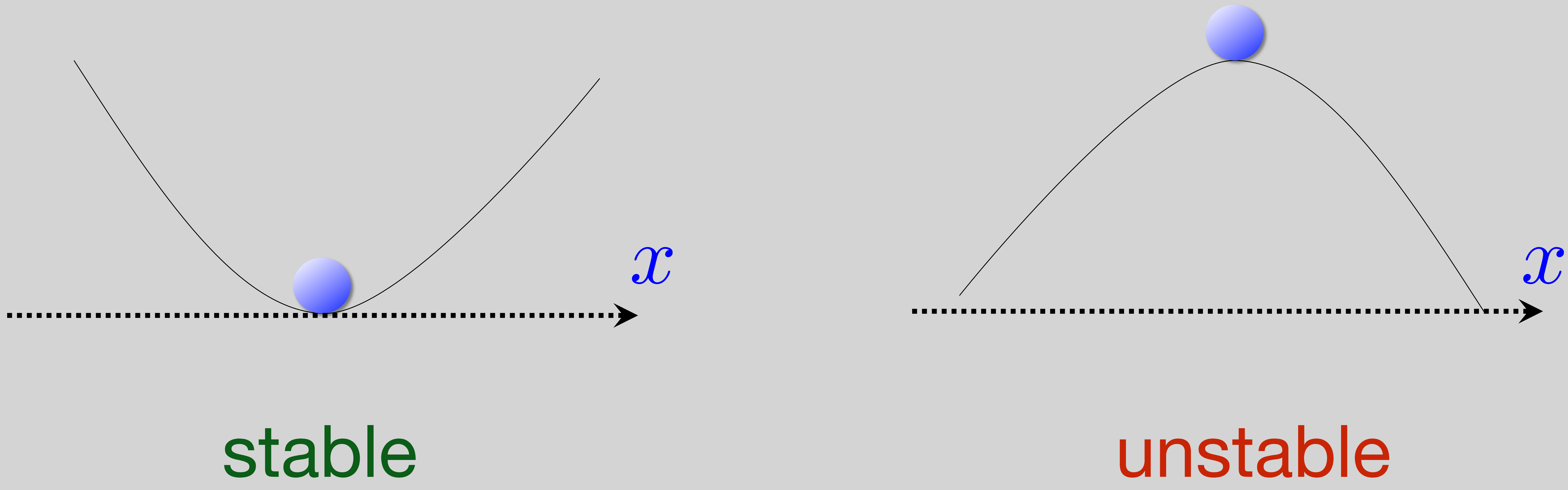
- A system is <u>stable</u> if $\vec{x}(t)$ is <u>bounded</u> for any initial condition $\vec{x}(0)$ and any bounded input sequence

$$u(0),\ u(1),\ \cdots$$

- A system is unstable if there is an $\vec{x}(0)$ or a bounded input sequence for which

$$|\vec{x}(t)| \to \infty \quad \text{as} \quad t \to \infty$$

Example



Q) Is this system stable?

$$x(t) = a^t x(o) + \sum_{k=0}^{t-1} a^{t-k-1} b u(k)$$

A) Depends on |a|

Stability Proof

$$x(t) = a^t x(o) + \sum_{k=0}^{t-1} a^{t-k-1} b u(k)$$

Claim 1: if lal < 1 then the system is stable

Proof: $a^t \rightarrow 0$ as $t \rightarrow \infty$ *because* $|a| < 1$ so,

initial condition always bounded

Sequence is bounded – there exists M s.t. $|u(t)| \leq M \; \forall t$

$$\left| \sum_{k=0}^{t-1} a^{t-k-1} b u(k) \right| \leq \sum_{k=0}^{t-1} |a^{t-k-1} b u(k)| = \sum_{k=0}^{t-1} |a|^{t-k-1} |b| \underbrace{|u(k)|}_{\leq M}$$

$$|a_1| + |a_2| \; ? \; |a_1 + a_2|$$

Stability Proof Cont.

$$\left|\sum_{k=0}^{t-1} a^{t-k-1} b u(k)\right| \le \sum_{k=0}^{t-1} |a^{t-k-1} b u(k)| = \sum_{k=0}^{t-1} |a|^{t-k-1} |b| \underbrace{|u(k)|}_{\le M}$$

Define: $s = t - k - 1$

$$\le \sum_{s=0}^{t-1} |a^s||b|M \; = |b|M \sum_{s=0}^{t-1} |a|^s \le |b|M \frac{1}{1-|a|}$$

$$\sum_{s=0}^{\infty} |a|^s = \frac{1}{1-|a|} \quad , \qquad |a| < 1$$

Stability Proof Cont.

Claim 2: unstable when |a| > 1

Proof: if $x(0) \neq 0$ (even $u(t)=0\ \forall\ t$)

$$x(t) = a^t x(0) \rightarrow \infty$$

Q: What if $|a| = 1$, i.e., $a=1$ or $a=-1$

A: Without input: $x(t) = a^t x(0)$

$$x(t) = x(0), \quad \text{or} \quad x(t) = (-1)^t x(0)$$

With input $u(t)=M$, $a=1$

$$\left|\sum_{k=0}^{t-1} a^{t-k-1} b u(k)\right| = \left|\sum_{k=0}^{t-1} bM\right| \rightarrow \infty$$

Not stable!

Quiz

With input $u(t)=M$, $a=-1$

$$\left|\sum_{k=0}^{t-1} a^{t-k-1} b u(k)\right| = \left|\sum_{k=0}^{t-1} (-1)^k bM\right| \leq bM$$

Q:what $|u(t)| \leq M$ will make it unstable?

Quiz

With input $u(t)=M,\ a=-1$

$$\left|\sum_{k=0}^{t-1} a^{t-k-1} b u(k)\right| = \left|\sum_{k=0}^{t-1} (-1)^k bM\right| \leq bM$$

Q:what $|u(t)| \leq M$ will make it unstable?

A: $u(t) = (-1)^t M$

$$\left|\sum_{k=0}^{t-1} a^{t-k-1} b u(k)\right| = \left|\sum_{k=0}^{t-1} (-1)^k b (-1)^k M\right| = \left|\sum_{k=0}^{t-1} bM\right| \to \infty$$

Summary

- Described linearization about an equilibrium point
  - Continuous time
  - Discrete time
- Conditions for stability of a linear systems
  - Covered:
    - Discrete, First order and scalar
- Next time:
  - Vector case! (which leads to Eigen-value analysis)