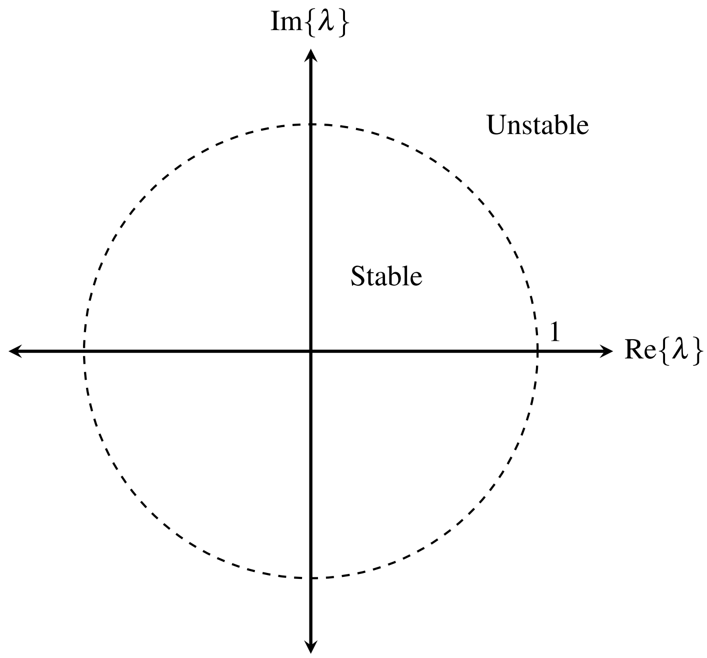
1 Stability

1.1 Discrete time systems

A discrete time system is of the form:

$$\vec{x}[t+1] = A\vec{x}[t] + B\vec{u}[t]$$

Let λ be any particular eigenvalue of A. This system is stable if $|\lambda| < 1$ for all λ. If we plot all λ for A on the real-imaginary axis, if all λ lie within (not on) the unit circle, then the system is stable.

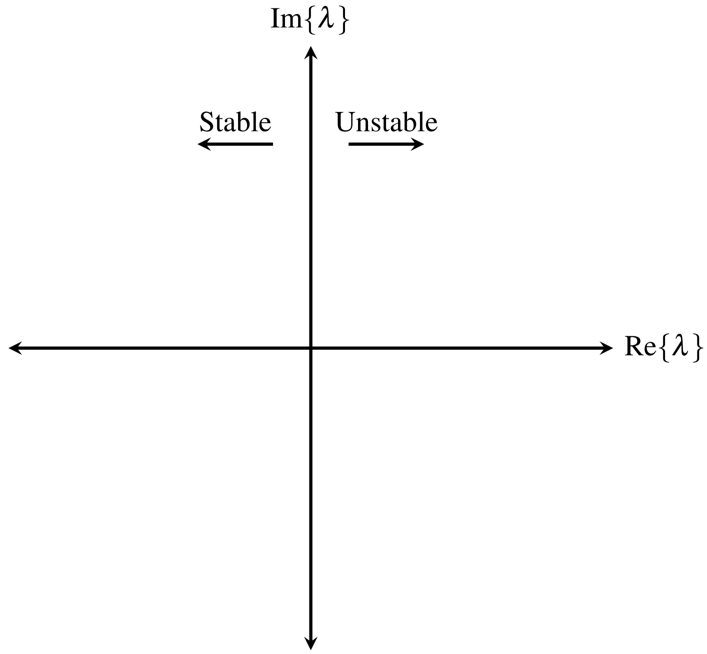


1.2 Continuous time systems

A continuous time system is of the form:

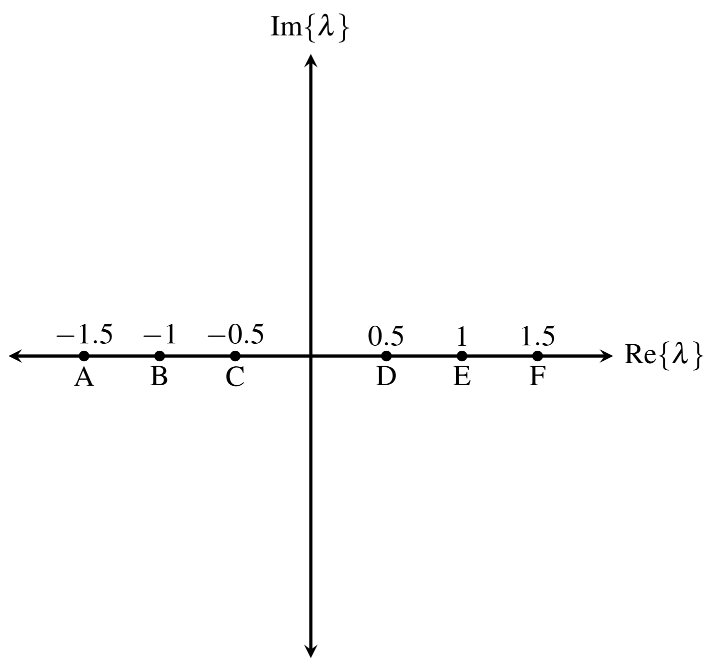
$$\frac{\mathrm{d}\vec{x}}{\mathrm{d}t}(t) = A\vec{x}(t) + B\vec{u}(t)$$

Let λ be any particular eigenvalue of A. This system is stable if $\mathrm{Re}\{\lambda\} < 0$ for all λ. If we plot all λ for A on the real-imaginary axis, if all λ lie to the left of $\mathrm{Re}\lambda = 0$, then the system is stable.

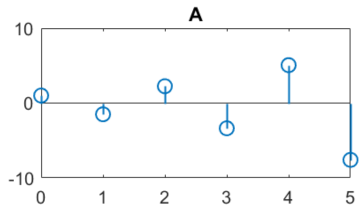


1. Discrete time system responses

We have a system $x[k+1] = \lambda x[k]$. For each λ value plotted on the real-imaginary axis, sketch $x[k]$ with an initial condition of $x[0] = 1$. Determine if each system is stable.



Answer:



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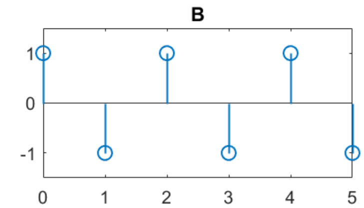

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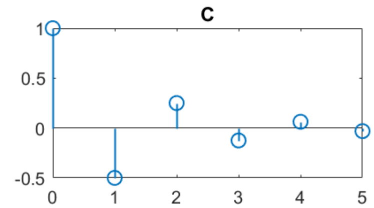

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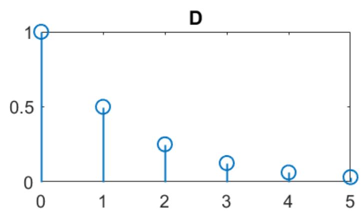

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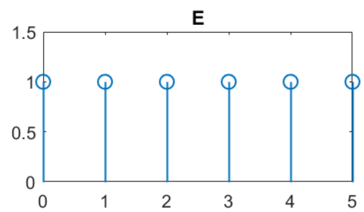

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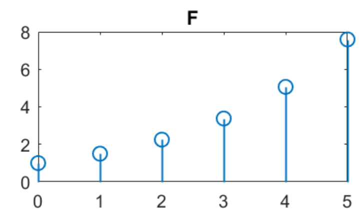
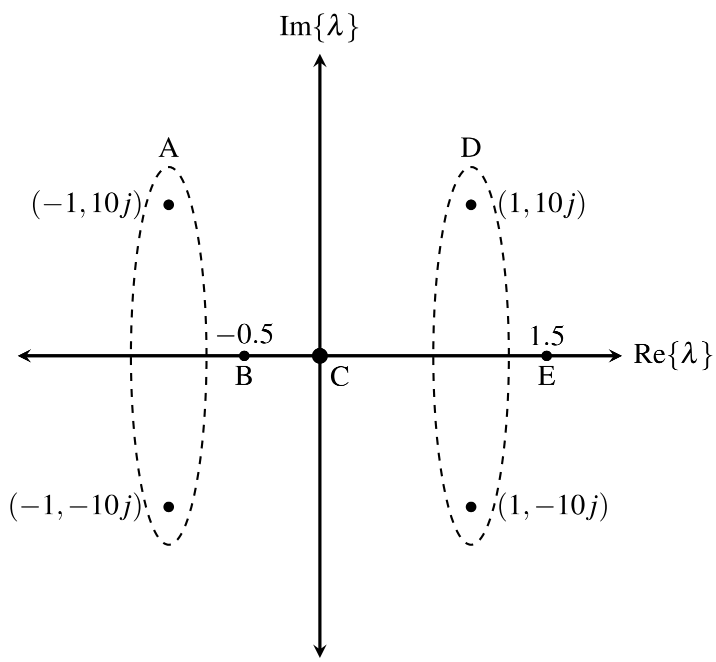


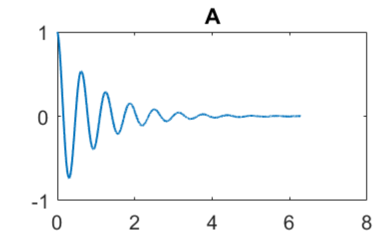
Unstable

2. Continuous time system responses

We have a system $\frac{d\vec{x}}{dt} = A\vec{x}$ with eigenvalues λ. For each set of λ values plotted on the real-imaginary axis, sketch $\vec{x}(t)$ with an initial condition of $x(0) = 1$. Determine if each system is stable.



Answer:



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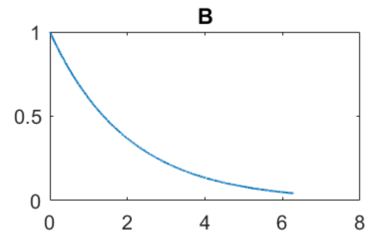

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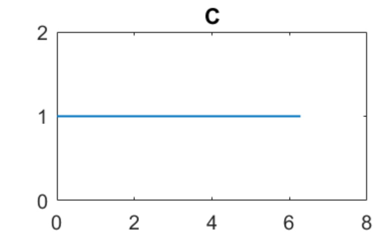

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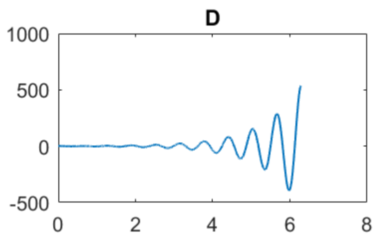

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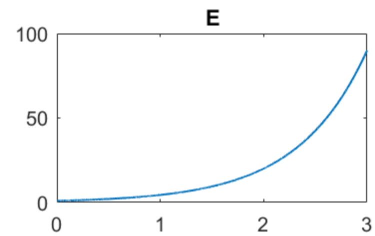

Unstable

3. Discrete-Time Stability

Determine which values of α and β will make the following discrete-time state space models stable:

(a)

$$x[t+1] = \alpha x[t]$$

Answer:

$$|\alpha| < 1$$

(b)

$$\vec{x}[t+1] = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \vec{x}[t]$$

Answer:
The eigenvalues of this system are:

$$\lambda = \alpha \pm j\beta$$
$$|\lambda| = \sqrt{\alpha^2 + \beta^2}$$

For this system to be stable, $|\lambda| < 1$, so

$$\alpha^2 + \beta^2 < 1$$

(c)

$$\vec{x}[t+1] = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix} \vec{x}[t]$$

Answer:

The eigenvalues of this system are

$$\lambda = 1, 1$$

This means that regardless of α, this system is always unstable.

4. Linearization and Stability

We have a system:

$$\frac{dx_1(t)}{dt} = x_1(t)x_2(t) - 3$$
$$\frac{dx_2(t)}{dt} = u(t)x_2(t) + 8x_1(t) - x_2(t)x_1(t) - 5$$

(a) Find the equilibrium point of this system when $u(t) = 0$.

Answer:

To find the equilibrium point, we set both $\frac{dx_1(t)}{dt}$ and $\frac{dx_2(t)}{dt}$ equal to 0.

$$0 = x_1x_2 - 3 \tag{1}$$
$$0 = 8x_1 - x_2x_1 - 5 \tag{2}$$

From Equation 1:

$$x_1x_2 = 3$$

Plugging into Equation 2:

$$0 = 8x_1 - 3 - 5$$

$$x_1 = 1$$
$$x_2 = 3$$

Therefore, our equilibrium point is $x_1(t) = 1$, $x_2(t) = 3$, and $u(t) = 0$.

(b) Linearize the system around its equilibrium point.

Answer:

To linearize the system, we take the Jacobian and plug in the equilibrium point values. Let $\vec{\tilde{x}}(t) = \vec{x}(t) - [1,3]^T$ and $\vec{\tilde{u}}(t) = \vec{u}(t) - 0 = \vec{u}(t)$

$$\frac{d\vec{\tilde{x}}(t)}{dt} = \begin{bmatrix} \frac{\partial}{\partial x_1}(\frac{dx_1}{dt}) & \frac{\partial}{\partial x_2}(\frac{dx_1}{dt}) \\ \frac{\partial}{\partial x_1}(\frac{dx_2}{dt}) & \frac{\partial}{\partial x_2}(\frac{dx_2}{dt}) \end{bmatrix}\Bigg|_{x_1=1,x_2=3,u=0} \vec{\tilde{x}}(t) + \begin{bmatrix} \frac{\partial}{\partial u}(\frac{dx_1}{dt}) \\ \frac{\partial}{\partial u}(\frac{dx_2}{dt}) \end{bmatrix}\Bigg|_{x_1=1,x_2=3,u=0} \tilde{u}(t)$$

$$\frac{d\vec{\tilde{x}}(t)}{dt} = \begin{bmatrix} x_2 & x_1 \\ 8 - x_2 & u - x_1 \end{bmatrix}\Bigg|_{x_1=1,x_2=3,u=0} \vec{\tilde{x}}(t) + \begin{bmatrix} 0 \\ x_2 \end{bmatrix}\Bigg|_{x_1=1,x_2=3,u=0} \tilde{u}(t)$$

$$\frac{d\vec{\tilde{x}}(t)}{dt} = \begin{bmatrix} 3 & 1 \\ 5 & -1 \end{bmatrix} \vec{\tilde{x}}(t) + \begin{bmatrix} 0 \\ 3 \end{bmatrix} \tilde{u}(t)$$

(c) Is the linearized system stable?

Answer:

The system is stable if the real parts of both eigenvalues are negative.

$$\begin{aligned}
\det(\lambda I - A) &= (\lambda - 3)(\lambda + 1) - 5 = 0 \\
\lambda^2 - 2\lambda - 8 &= (\lambda - 4)(\lambda + 2) = 0 \\
\lambda_1 &= 4 \\
\lambda_2 &= -2
\end{aligned}$$

$\lambda_1 > 0$, so the system is unstable.