EECS 16B Designing Information Devices and Systems II
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Discussion 14B

1 Notes

1.1 Discrete Fourier Transform

Assume we are working with an N length discrete signal and we would like to find its discrete frequencies. This is done through the Discrete Fourier Transform (DFT), which is simply a change of basis to what is called the DFT basis.

First, let us vectorize our signal. If $x[n]$ is our input signal, we model it as a vector by letting the n^{th} coordinate be $x[n]$. In other words,

$$\vec{x} = \left[x[0], x[1], x[2], \ldots, x[N-1]\right]^T$$

In order to decompose $\vec{x}$ into its constituent frequencies, we must find the vector representation of these frequencies.

Given that we have an N length signal, we have N different discrete frequencies of the following form.

$$u_k[n] = \frac{1}{\sqrt{N}} e^{j\frac{2\pi}{N}kn} \text{ for } k = 0, 1, \ldots N-1$$

To simplify we let

$$W_N = e^{j\frac{2\pi}{N}}$$

and we rewrite

$$u_k[n] = \frac{1}{\sqrt{N}} W_N^{kn} \text{ for } k = 0, 1, \ldots N-1$$

In building up a frequency basis, we vectorize the above frequencies in a manner similar to how we vectorized $\vec{x}$. Define $\vec{u}_k$ as follows.

$$\vec{u}_k = \frac{1}{\sqrt{N}} \left[1, W_N^k, W_N^{k(2)}, \ldots, W_N^{k(N-1)}\right]^T$$

$\{\vec{u}_k\}_{k=0}^{N-1}$ is an orthonormal set of vectors. Recall that an orthonormal set of vectors satisfies the following:

$$\langle \vec{u}_p, \vec{u}_q \rangle = \sum_{n=0}^{N-1} \overline{\vec{u}_p} \vec{u}_q = \begin{cases} 0, p \neq q \\ 1, p = q \end{cases}$$

To see why this set is orthonormal, first consider arbitrary $\vec{u}_p$ and $\vec{u}_q$ such that $p \neq q$.

$$\begin{aligned}\langle \vec{u}_p, \vec{u}_q \rangle &= \frac{1}{N}\sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}pn} e^{j\frac{2\pi}{N}qn} \\ &= \frac{1}{N}\sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(q-p)n}\end{aligned}$$

Before we continue, let us remind ourselves about the sum of a finite geometric series. Let S be the sum of the series. Then,

$$S = 1 + a + a^2 + \cdots a^{N-1}$$

Then,

$$aS = a + a^2 + a^3 + \cdots a^N$$

Subtracting the two, we get,

$$(1-a)S = 1 - a^N \implies S = \frac{1-a^N}{1-a}$$

Applying this, we get,

$$\begin{aligned}\langle \vec{u}_p, \vec{u}_q \rangle &= \frac{1}{N}\sum_{n=0}^{N-1} \Bigg(\underbrace{e^{j\frac{2\pi}{N}(q-p)}}_{a}\Bigg)^n \\ &= \frac{1}{N}\left(\frac{1-a^N}{1-a}\right) \\ &= \frac{1}{N}\left(\frac{1-e^{j\frac{2\pi}{N}(q-p)N}}{1-e^{j\frac{2\pi}{N}(q-p)}}\right)\end{aligned}$$

Note that $q-p$ is an non-zero integer. This means that,

$$e^{j\frac{2\pi}{N}(q-p)N} = e^{j2\pi(q-p)} = 1$$

Applying this, we get,

$$\langle \vec{u}_p, \vec{u}_q \rangle = 0$$

Finally, we also observe that, for a particular DFT basis vector,

$$\langle \vec{u}_p, \vec{u}_p \rangle = \frac{1}{N}\sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}pn} e^{j\frac{2\pi}{N}pn} = \frac{1}{N}\sum_{n=0}^{N-1} 1 = 1$$

Thus, $\{\vec{u}_k\}_{k=0}^{N-1}$ is an orthonormal set of vectors and is a valid basis. The coefficients of $\vec{x}$ within this basis are called the frequency components of $\vec{x}$ and are often denoted by $\vec{X}$.

$$\vec{X} = [\langle \vec{u}_0, \vec{x} \rangle, \langle \vec{u}_1, \vec{x} \rangle, \cdots, \langle \vec{u}_{N-1}, \vec{x} \rangle]^T$$

The k^{th} frequency component is the k^{th} coordinate of $\vec{X}$ and is denoted as $X[k]$. If we want to get the component in the same space as $\vec{x}$, we compute the projection.

$$\text{proj}_{\vec{u}_k}\vec{x} = X[k]\vec{u}_k$$

It is worthwhile to note that there is a conjugate property we often exploit.

$$e^{j\frac{2\pi}{N}pn} = e^{-j\frac{2\pi}{N}(N-p)n}$$

This means that,

$$\vec{u}_k = \overline{\vec{u}_{N-k}}$$

In fact, since we are using complex exponentials, there is a periodicity that can be succinctly expressed with the remainder operation (also called mod). Let p be any arbitrary integer.

$$\vec{u}_p = \vec{u}_{p \bmod N}$$

2 Questions

1. DFT of pure sinusoids

(a) Consider the continuous-time signal $x(t) = \cos\left(\frac{2\pi}{3}t\right)$. Suppose that we sampled it every 1 second to get (for $n = 3$ time steps):

$$\vec{x} = \begin{bmatrix} \cos\left(\frac{2\pi}{3}(0)\right) & \cos\left(\frac{2\pi}{3}(1)\right) & \cos\left(\frac{2\pi}{3}(2)\right) \end{bmatrix}^T.$$

Compute $\vec{X}$ and the basis vectors $\vec{u}_k$ for this signal.

Answer:

$$\vec{X} = \frac{\sqrt{3}}{2}\begin{bmatrix} 0 & 1 & 1 \end{bmatrix}^T$$

Solution: Directly apply $F^*\vec{x}$ to derive $\vec{X}$.

$$\vec{X} = \frac{\sqrt{3}}{2}\begin{bmatrix} 0 & 1 & 1 \end{bmatrix}^T$$

(b) Now for the same signal as before, suppose that we took $n = 6$ samples. In this case we would have:

$$\vec{x} = \begin{bmatrix} \cos\left(\frac{2\pi}{3}(0)\right) & \cos\left(\frac{2\pi}{3}(1)\right) & \cos\left(\frac{2\pi}{3}(2)\right) & \cos\left(\frac{2\pi}{3}(3)\right) & \cos\left(\frac{2\pi}{3}(4)\right) & \cos\left(\frac{2\pi}{3}(5)\right) \end{bmatrix}^T.$$

Repeat what you did above. What are $\vec{X}$ and the basis vectors $\vec{u}_k$ for this signal.

Answer:

$$\vec{X} = \frac{\sqrt{6}}{2}\begin{bmatrix}0 & 0 & 1 & 0 & 1 & 0\end{bmatrix}^T$$

Solution: Directly apply $F^*\vec{x}$ to derive $\vec{X}$.

$$\vec{X} = \frac{\sqrt{6}}{2}\begin{bmatrix}0 & 0 & 1 & 0 & 1 & 0\end{bmatrix}^T$$

(c) Let's do this more generally. For the signal $x(t) = \cos\left(\frac{2\pi k}{N}t\right)$, compute $\vec{X}$ of its vector form in discrete time, $\vec{x}$, of length $n = N$:

$$\vec{x} = \begin{bmatrix}\cos\left(\frac{2\pi k}{N}(0)\right) & \cos\left(\frac{2\pi k}{N}(1)\right) & \cdots & \cos\left(\frac{2\pi k}{N}(N-1)\right)\end{bmatrix}^T.$$

Answer:

$$X[k] = X[N-k] = \frac{\sqrt{N}}{2}$$
$$X[m] = 0 \text{ for } m \neq k, N-k.$$

Solution:

i. Show that $\vec{u}_k + \vec{u}_{N-k} = \frac{2}{\sqrt{N}}\vec{x}$.

ii. Then we have

$$X[k] = X[N-k] = \frac{\sqrt{N}}{2}$$
$$X[m] = 0 \text{ for } m \neq k, N-k.$$