Homework 3

This homework is due on Wednesday September 19, 2018 at 11:59PM.
Self-grades are due on Monday, September 24, 2018 at 11:59PM.

1. **Mechanical 2nd Order Differential Equation**

   Solve $3\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - 12\frac{\mathrm{d}y}{\mathrm{d}t} + 24y = 24$, where $y(0) = 1$ and $\frac{\mathrm{d}y}{\mathrm{d}t}(0) = 2$

2. **Solution to Repeated Roots**

   In lecture, we claimed that the solution to a second-order differential equation with repeated eigenvalue λ_0 is $y = c_1e^{\lambda t} + c_2te^{\lambda t}$. In this problem, we will show why this solution is valid.

   (a) Given a differential equation $\frac{d^2y}{dt^2} + a_1\frac{dy}{dt} + a_0y = 0$, assume that both eigenvalues of the A matrix with the state vector defined as $\begin{bmatrix} y(t) \\ \frac{dy(t)}{dt} \end{bmatrix}$ are λ_0. Find a_0, a_1 in terms of λ_0.

   (b) Let's assume the solution to our differential equation is $c_1e^{\lambda_0 t} + c_2te^{\lambda_0 t}$. Verify that this solution satisfies the differential equation you get when using the a_0 and a_1 you found in part (a).

   (c) Making the same assumption as part (b), show that we can always find constants c_1, c_2 such that we can satisfy initial conditions $y(0) = y_0$, $\frac{dy}{dt}(0) = y_0'$

3. **Fun with Inductors**

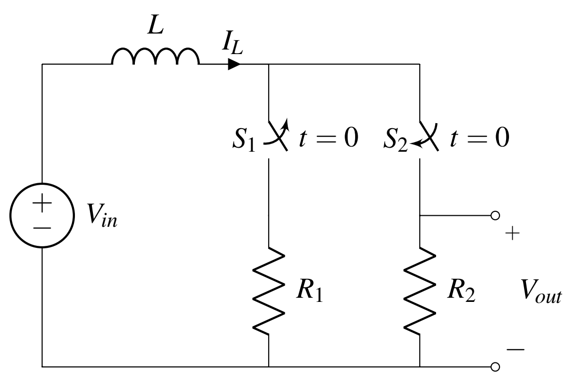


Figure 1: Circuit A

   (a) Consider circuit A. Assuming that for $t < 0$, switch S_1 is on and switch S_2 is off (and both switches have been in these states indefinitely), what is $i_L(0)$?

   (b) Now let's assume that for $t \geq 0$, S_1 is off and S_2 is on. Solve for $V_{out}(t)$ for $t \geq 0$.

   (c) If $V_{in} = 1V$, $L = 1nH$, $R_1 = 1k\Omega$, and $R_2 = 10k\Omega$, what is the maximum value of $V_{out}(t)$ for $t \geq 0$?

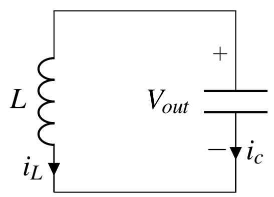
(d) In general, if we want $\max V_{out}(t)$ to be greater than V_{in}, what relationship needs to be maintained between the values of R_1 and R_2?

(e) Now assume that at time $t = t_1$, switch S_2 was turned off, and switch S_1 was turned back on. Solve for $i_L(t)$ for $t > t_1$. If $R_2 > R_1$, how does this $i_L(t)$ for $t > t_1$ compare with the initial condition $i_L(0)$ you found in part (a)?

4. Oscillators

In this question, we'll be looking at an oscillator circuit. There are many types of oscillators, but this circuit is known as an LC tank. It's called an oscillator because the circuit produces a repetitive voltage waveform under the right initial conditions.

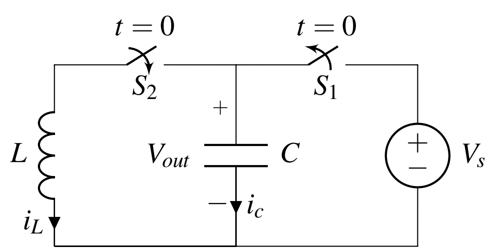
In this circuit, we have an inductor $L = 10\text{nH}$ and capacitor $C = 10\text{pF}$ in parallel:



(a) If $i_L(0) = 0\text{A}$ and $V_{out}(0) = 0\text{V}$, derive an expression for $V_{out}(t)$ for $t \geq 0$. Use V_{out} and i_L as your state variables.

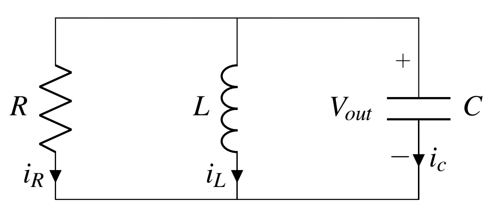
(b) Now let's see how the circuit reacts with non-zero initial current. If $i_L(0) = 50\mu\text{A}$ and $V_{out}(0) = 0\text{V}$, derive an expression for $V_{out}(t)$ for $t \geq 0$. How does the amplitude of V_{out} change over time?

(c) In order to ensure an initial condition where we get non-zero output, some switches and a voltage source have been added to the circuit. For $t \leq 0$, switch S_1 is on while S_2 is off. Find $V_c(0)$ and $i_L(0)$. Use component values $V_s = 3\text{V}$, $L = 10\text{nH}$ and $C = 10\text{pF}$.



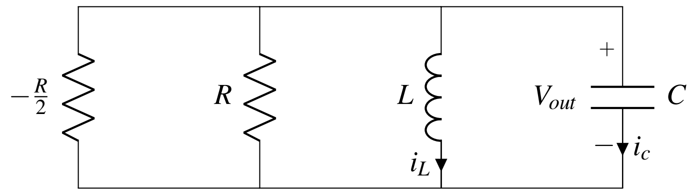
(d) At $t = 0$, the switches flip state (S_1 turns off and S_2 turns on). Derive an expression for $V_{out}(t)$ for $t \geq 0$. Use the same component values as part (c).

(e) Let's see what happens when there is a parasitic resistance R in parallel with the LC tank.



If $i_L(0) = 0\mu\text{A}$ and $V_{out}(0) = 3\text{V}$, derive an expression for $V_{out}(t)$ for $t \geq 0$. Use component values $C = 10\text{pF}$, $L = 10\text{nH}$, and $R = 100\text{k}\Omega$. How does the amplitude of V_{out} change over time?

(f) In order to counteract the parasitic resistance, we create a negative resistance (which can be done using transistors, but will not be covered in this class) in parallel with the other components:



If $i_L(0) = 0\mu$A and $V_{out}(0) = 3$V, Derive an expression for $V_{out}(t)$ for $t \geq 0$. Use component values $C = 10$pF, $L = 10$nH, and $R = 100$kΩ. How does the amplitude of V_{out} change over time?

(g) **(BONUS)**
What value should the negative resistor have if we want to maintain a constant amplitude on $V_{out}(t)$?

5. Write Your Own Question And Provide a Thorough Solution.

Writing your own problems is a very important way to really learn material. The famous "Bloom's Taxonomy" that lists the levels of learning is: Remember, Understand, Apply, Analyze, Evaluate, and Create. Using what you know to create is the top level. We rarely ask you any homework questions about the lowest level of straight-up remembering, expecting you to be able to do that yourself (e.g. making flashcards). But we don't want the same to be true about the highest level. As a practical matter, having some practice at trying to create problems helps you study for exams much better than simply counting on solving existing practice problems. This is because thinking about how to create an interesting problem forces you to really look at the material from the perspective of those who are going to create the exams. Besides, this is fun. If you want to make a boring problem, go ahead. That is your prerogative. But it is more fun to really engage with the material, discover something interesting, and then come up with a problem that walks others down a journey that lets them share your discovery. You don't have to achieve this every week. But unless you try every week, it probably won't ever happen.

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