EE16B
Designing Information Devices and Systems II

~~12B~~

Lecture 14A
Change of Basis
The Discrete Fourier Transform

Intro

- Last time:
  - LTI Systems
  - Convolution sum
  - Finite sequences as vectors

- Today
  - Change of basis
  - Discrete Fourier Transform

–Announcements:
  - Midterm II
  - Course evaluations — Fill in to get extra 2 points for Midterm 2 (see Piazza)
  - Good news
  - Bad news
  - Worse News

Good News

- Competition!

TI Contest
Hi everyone,

We're excited to announce a competition between SIXT33N cars sponsored by Texas Instruments. The competition will consist of an obstacle course in which contestants can score points by completing various objectives as quickly as possible.

The competition will be Friday, December 7th from 10am-12pm. We will provide food and monetary prizes.

To sign up, fill out this form: https://goo.gl/forms/wfEw1tcybEg8CXBP2

We encourage everyone to participate, no matter how straight your car may drive!

See you there!

Bad news

- How do you feel about an extra lecture in RRR week?

Worse news – Big Game

98	1995	Stanford	29–24
99	1996	Stanford	42–21
100	1997	Stanford	21–20
101	1998	Stanford	10–3
102	1999	Stanford	31–13
103	2000	Stanford	36–30
104	2001	Stanford	35–28
105	2002	California	30–7
106	2003	California	28–16
107	2004	California	41–6
108	2005	California	27–3
109	2006	California	26–17
110	2007	Stanford	20–13
111	2008	California	37–16
112	2009	California	34–28
113	2010	Stanford	48–14
114	2011	Stanford	31–28
115	2012	Stanford	21–3
116	2013	Stanford	63–13
117	2014	Stanford	38–17
118	2015	Stanford	35–22
119	2016	Stanford	45–31
120	2017	Stanford	17–14
121	2018	?	

Finite Sequences as Vectors

- Define Complex inner product

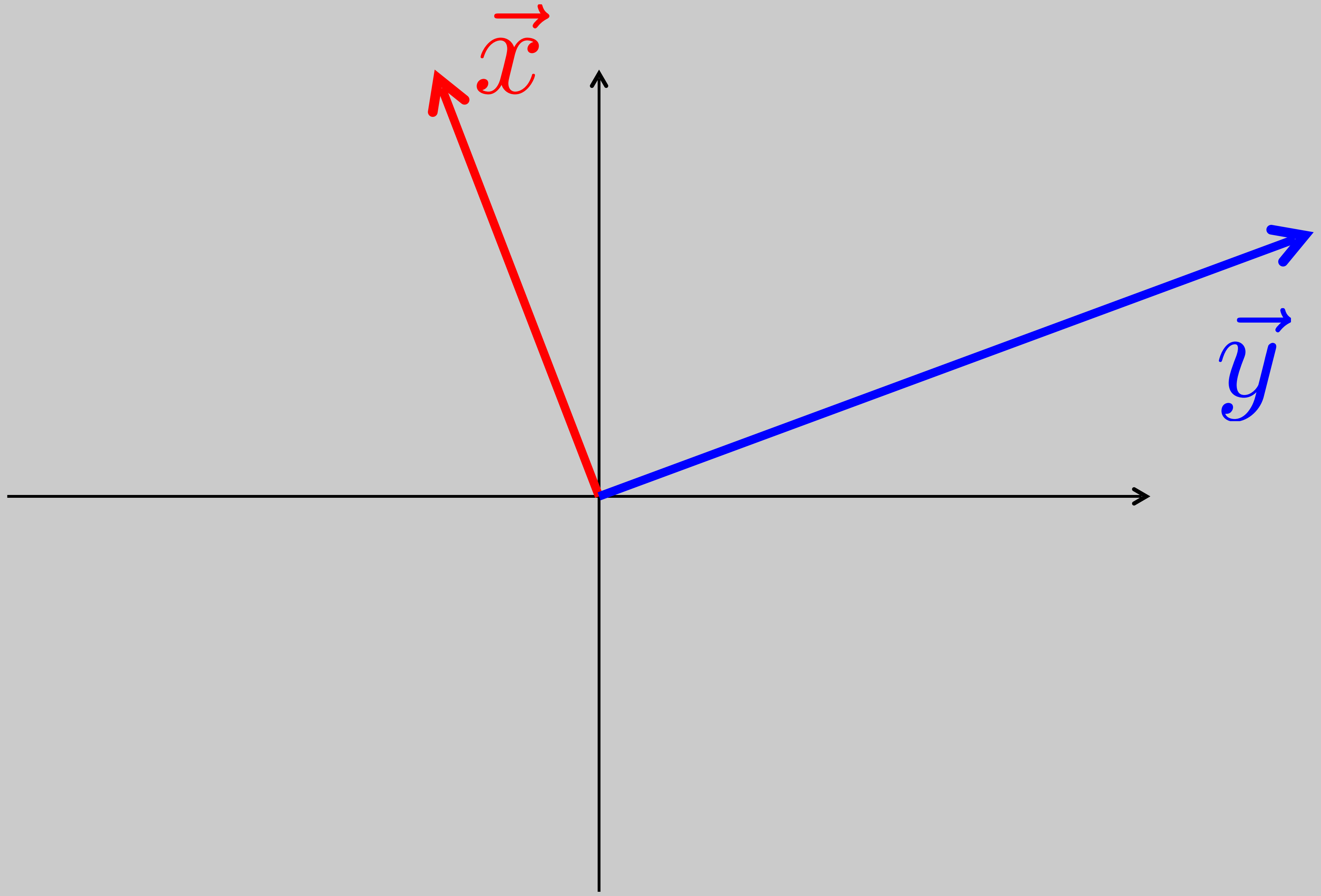
$$< \vec{x}, \vec{y} >= \overline{\ \vec{x}\ } \cdot \vec{y} = \sum_{n=0}^{N-1} x[n]^* y[n] = \vec{x}^* \vec{y} = \vec{x}^H \vec{y}$$

$$\vec{x} = \begin{bmatrix} 1 \\ j \end{bmatrix} \Rightarrow \vec{x}^* x = \begin{bmatrix} 1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ j \end{bmatrix} = 2$$

Projections

- Orthogonality:

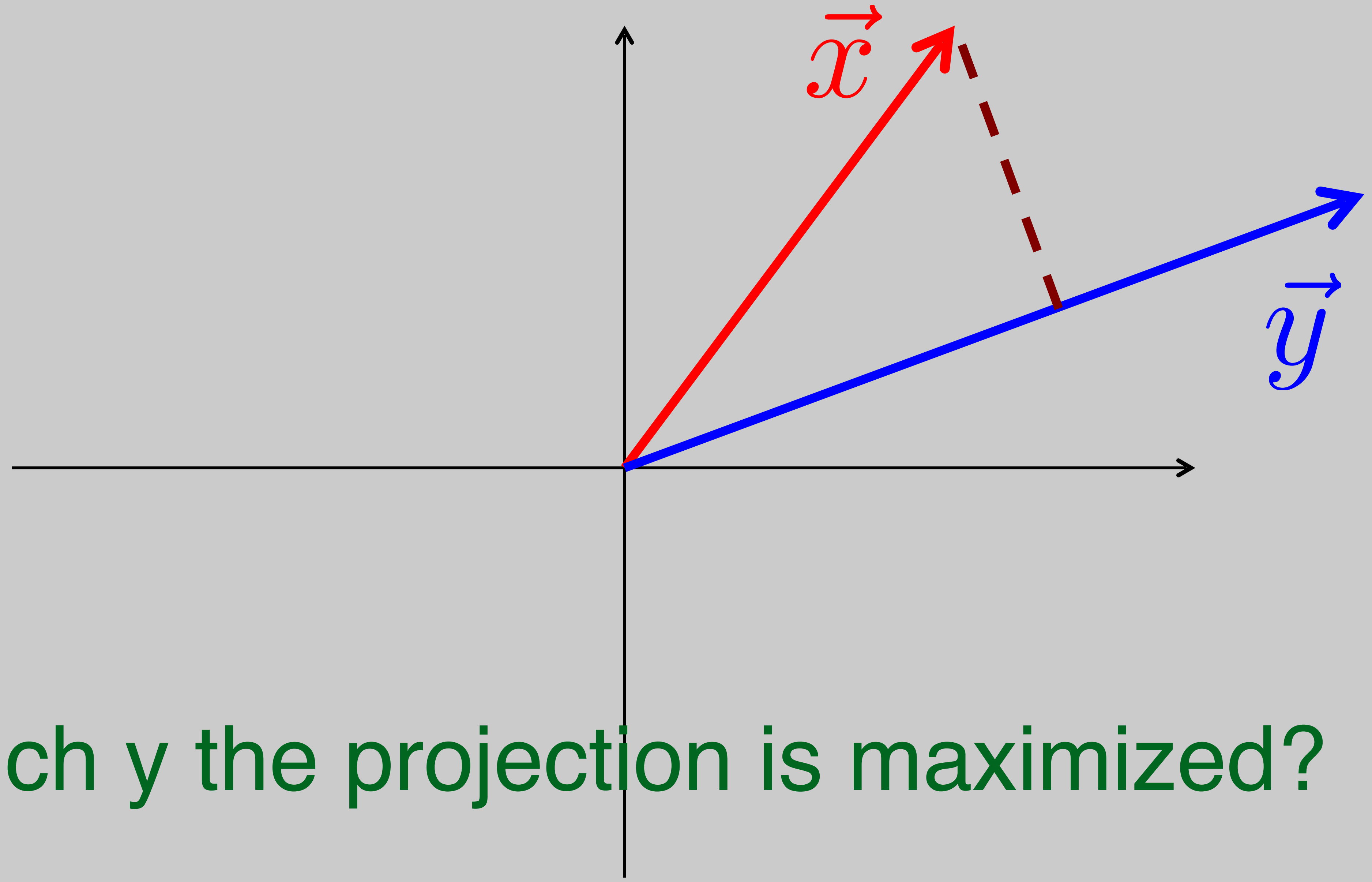
$$\vec{x}^*\vec{y} = \sum_{n=0}^{N-1} x[n]^* y[n] = 0$$



- Unit vector: $||\hat{x}|| = 1$

$$\hat{x} = \frac{\vec{x}}{||\vec{x}||}$$

- Define projection as: $\dfrac{\vec{y}^* x}{||\vec{y}||}$

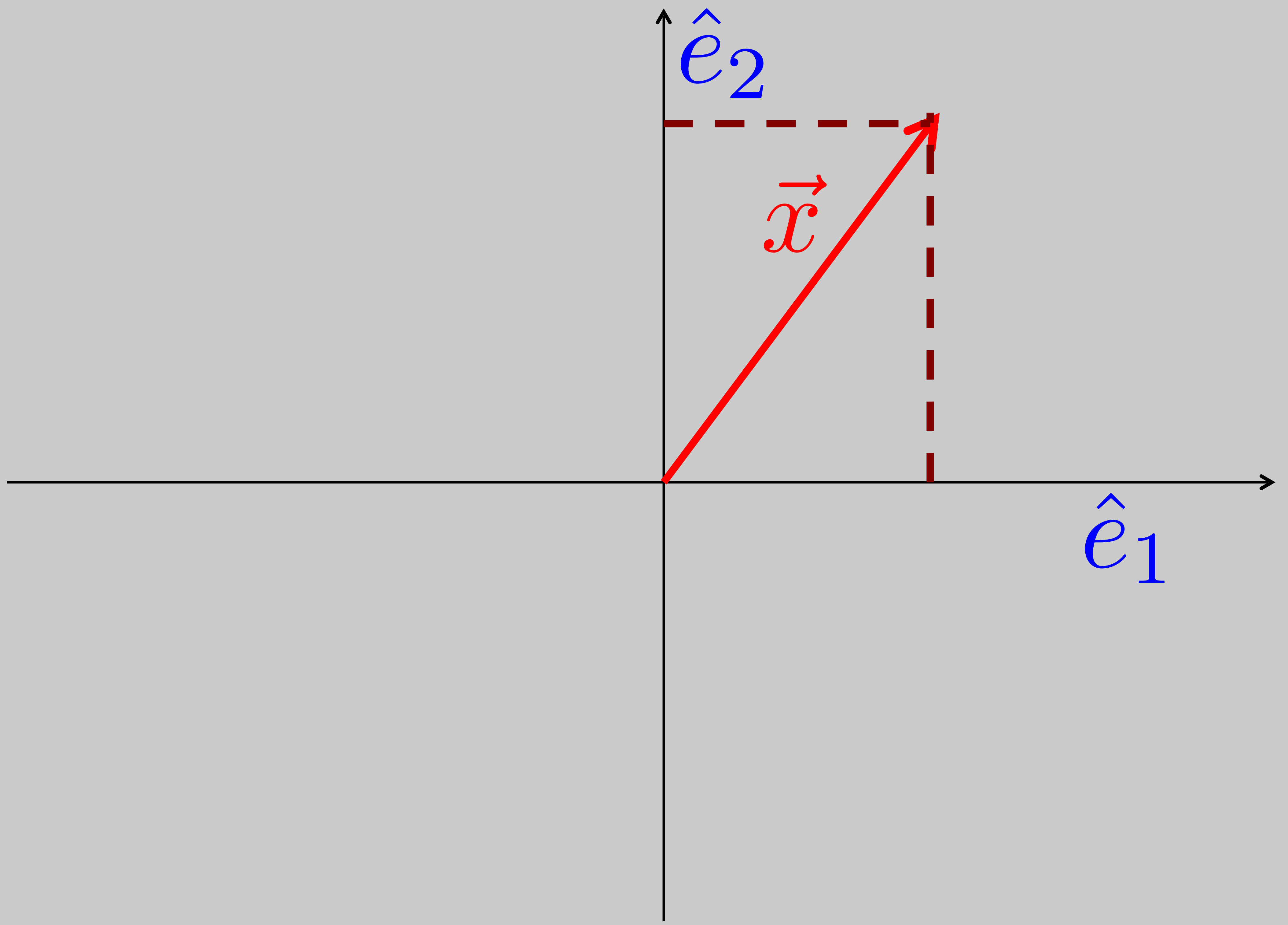


For which y the projection is maximized?

Change of Coordinates (Basis)

- We can compute new coordinates by projections onto orthonormal basis vectors

$$\hat{e}_1^* \vec{x} = [1 \quad 0]\vec{x} = x_1$$
$$\hat{e}_2^* \vec{x} = [0 \quad 1]\vec{x} = x_2$$



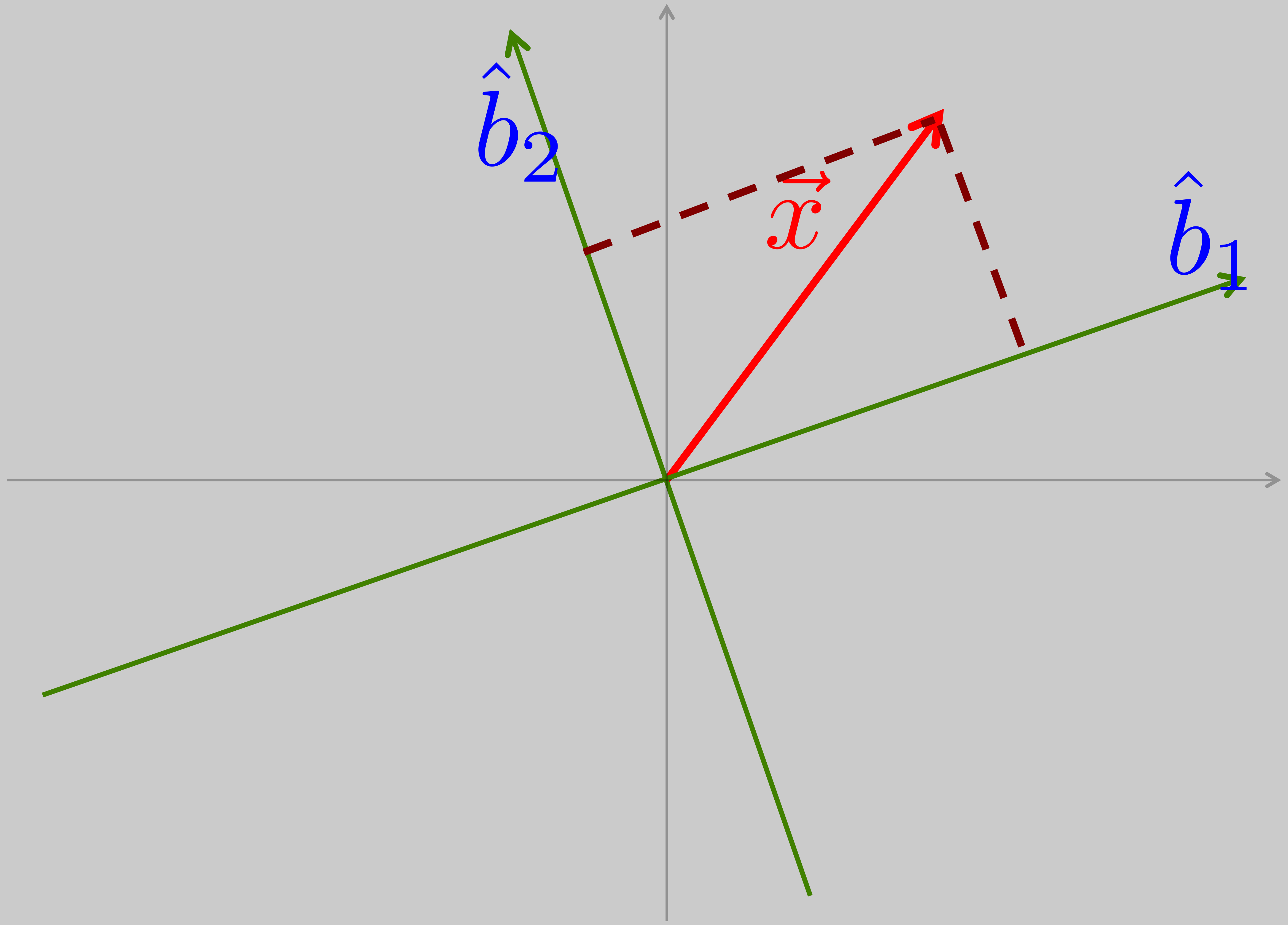
Change of Coordinates (Basis)

- We can compute new coordinates by projections onto orthonormal basis vectors

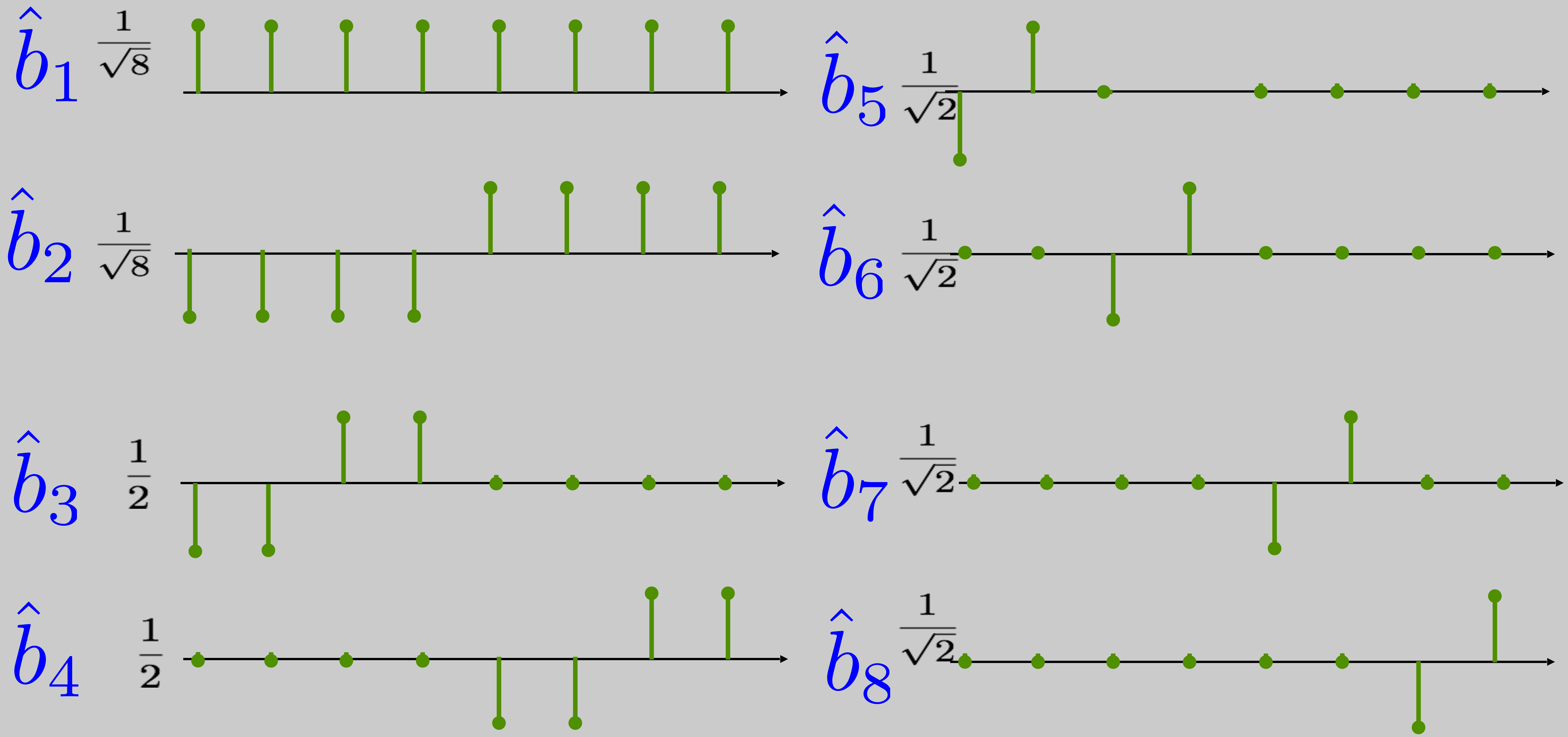
New coordinates:

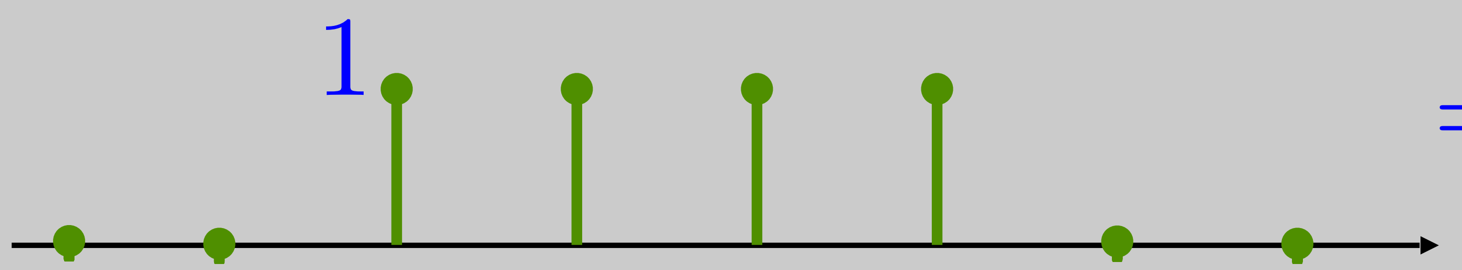
$$\begin{bmatrix} \hat{b}_1^* \vec{x} \\ \hat{b}_2^* \vec{x} \end{bmatrix} = \begin{bmatrix} \hat{b}_1 & \hat{b2} \end{bmatrix}^* \vec{x}$$

$$\Rightarrow \vec{x} = (\hat{b}_1^* \vec{x})\hat{b}_1 + (\hat{b}_2^* \vec{x})\hat{b}_2$$



Change of basis

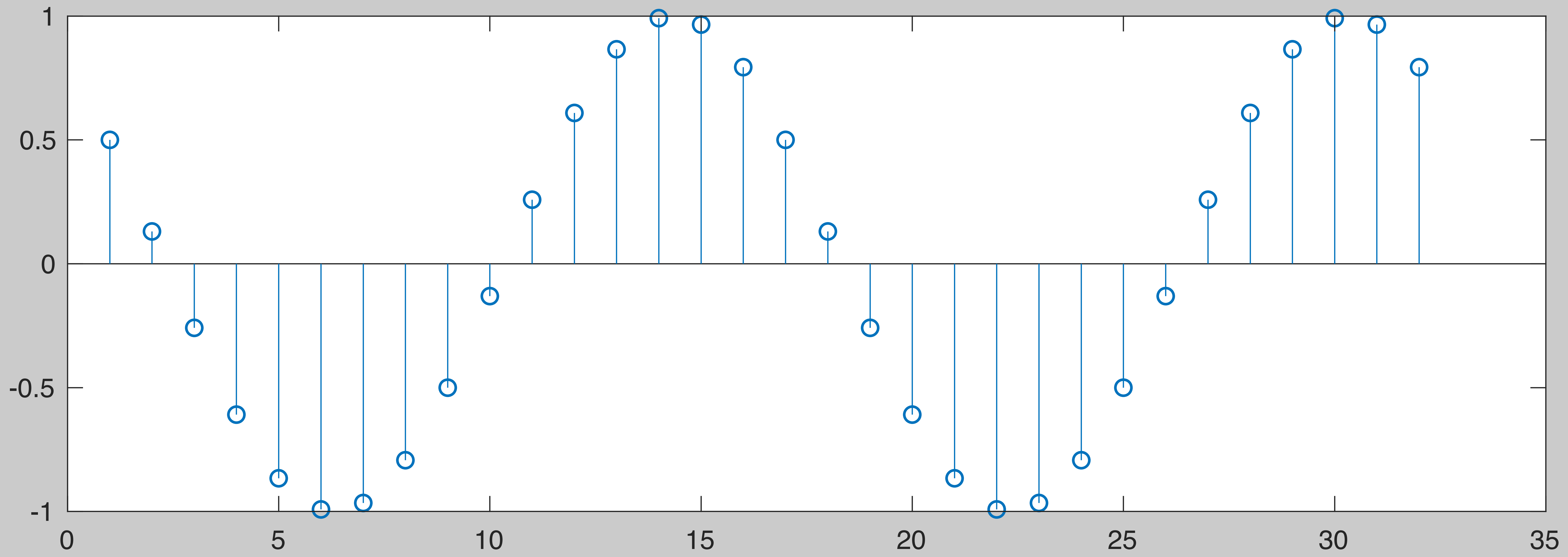




$$= \frac{4}{\sqrt{8}} \cdot \hat{b}_1 + 0 \cdot \hat{b}_2 + 1 \cdot \hat{b}_3 + (-1) \cdot \hat{b}_4 + 0 \cdot \hat{b}_5 + 0 \cdot \hat{b}_6 + 0 \cdot \hat{b}_7 + 0 \cdot \hat{b}_8$$

Frequency Analysis

- How can we find the frequency of this N=32 length signal?



Project on unit sinusoidal vectors?

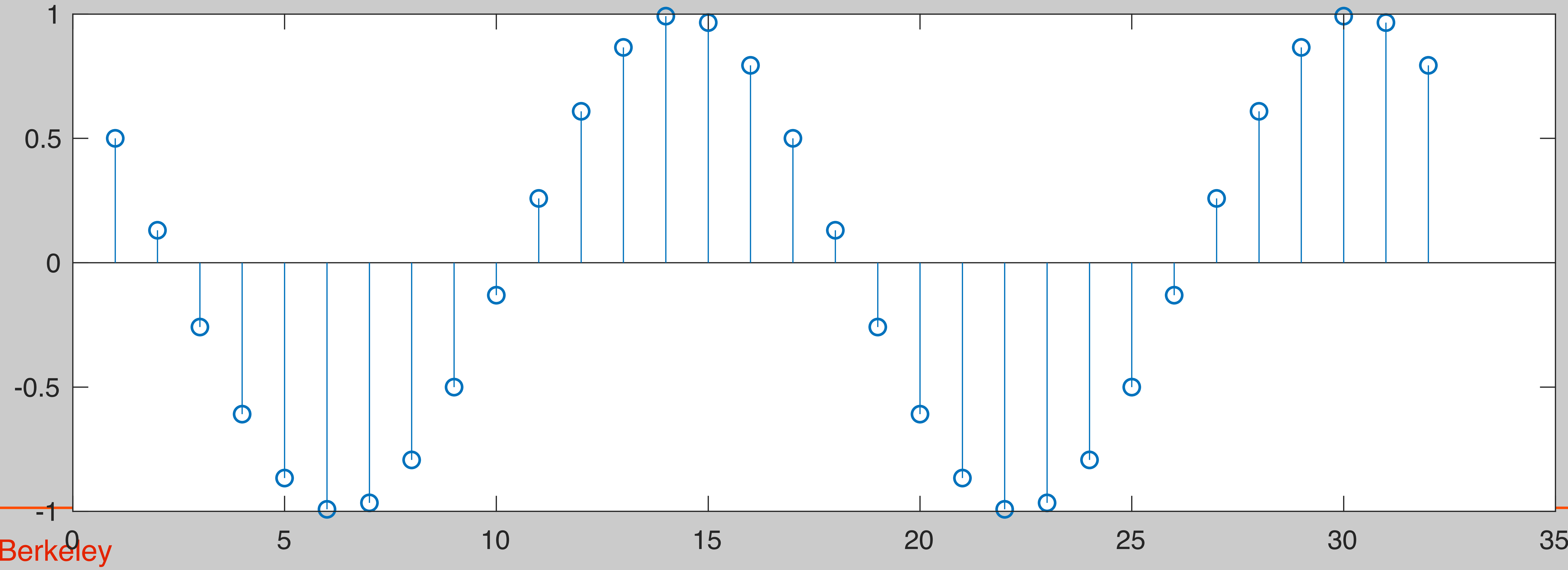
Complex Exponential Basis

- Phase is a problem! (inside a cosine)

$$\vec{x} \quad \Big| \quad x[n] = \cos(\omega_0 n + \phi_0)$$

- Solution: Phase is a coefficient for complex exponentials!

$$\vec{x} \quad \Big| \quad x[n] = \frac{1}{2}e^{j\omega n} \cdot e^{j\phi} \; + \frac{1}{2}e^{-j\omega n} \cdot e^{-j\phi}$$



Frequency Analysis Through Projections

- N-length normalized discrete frequency:

$$u_\omega[n] = \frac{1}{\sqrt{N}} e^{j\omega n} \qquad 0 \le n < N \qquad 0 \le \omega < 2\pi$$

$$\vec{u}_\omega = \frac{1}{\sqrt{N}} \begin{bmatrix} e^{j\omega 0} \\ e^{j\omega 1} \\ \vdots \\ e^{j\omega (N-1)} \end{bmatrix} \Rightarrow X(\omega) = \vec{u}_w^* \vec{x}$$

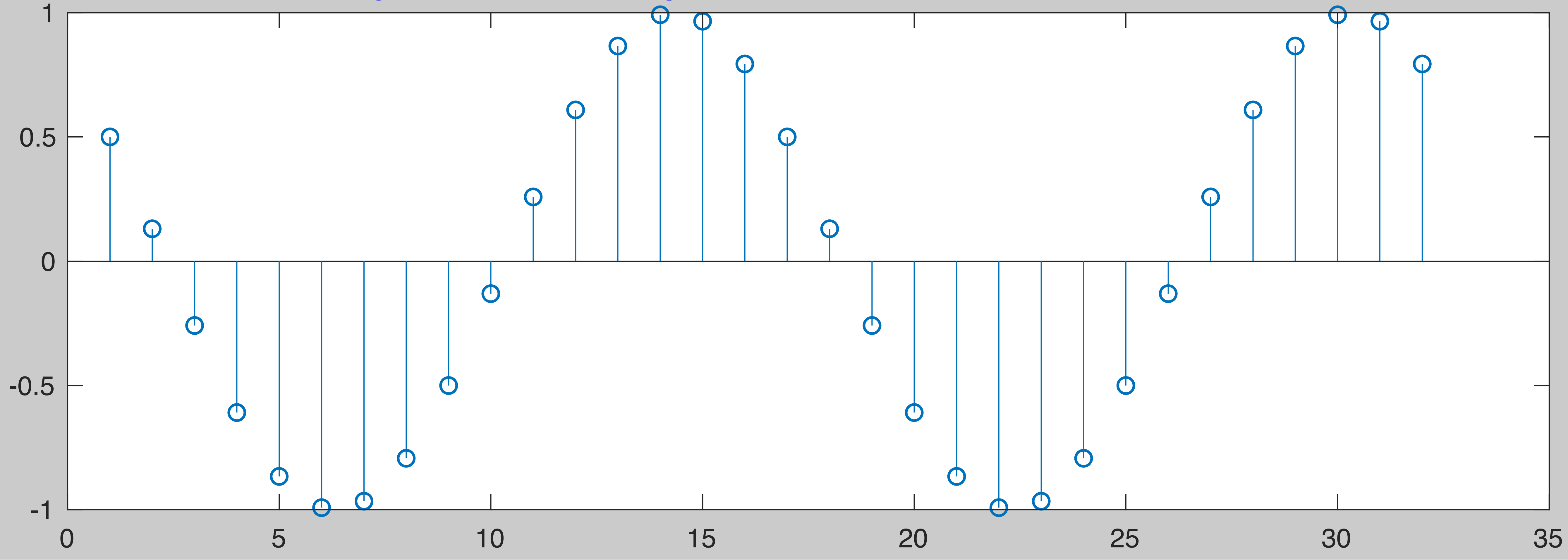
$$= \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x[n] e^{-j\omega n}$$

Also the DTFT of the finite sequence x

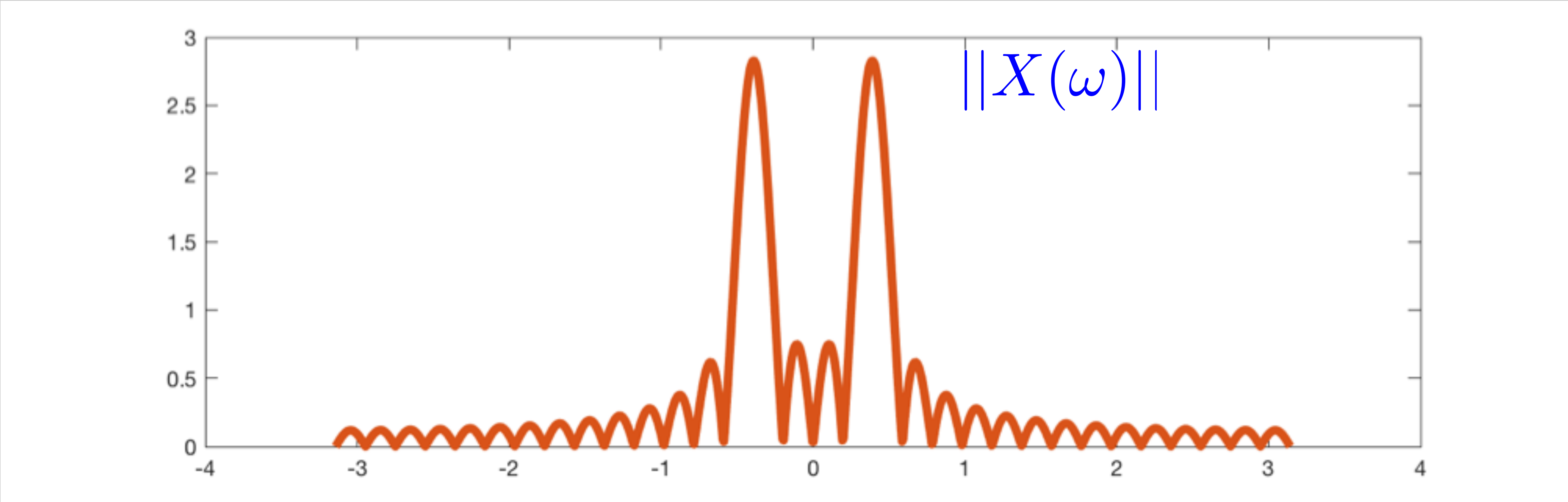
Frequency Analysis Through Projections

- Example: $x[n] = \cos\left(\frac{\pi}{8}n + \frac{\pi}{3}n\right)$

N = 32



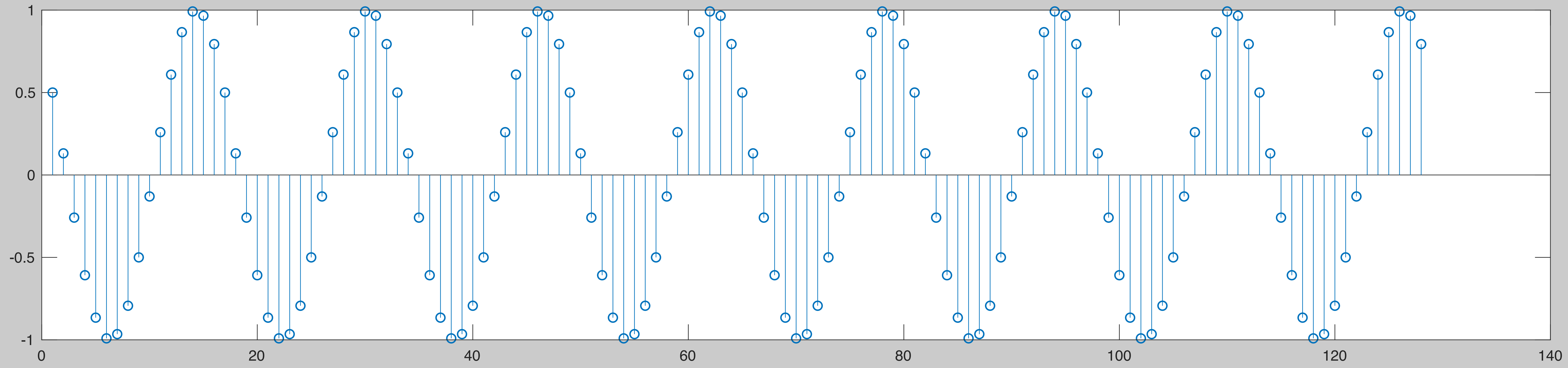
$$\Rightarrow X(\omega) = \vec{u}_w^* \vec{x}$$



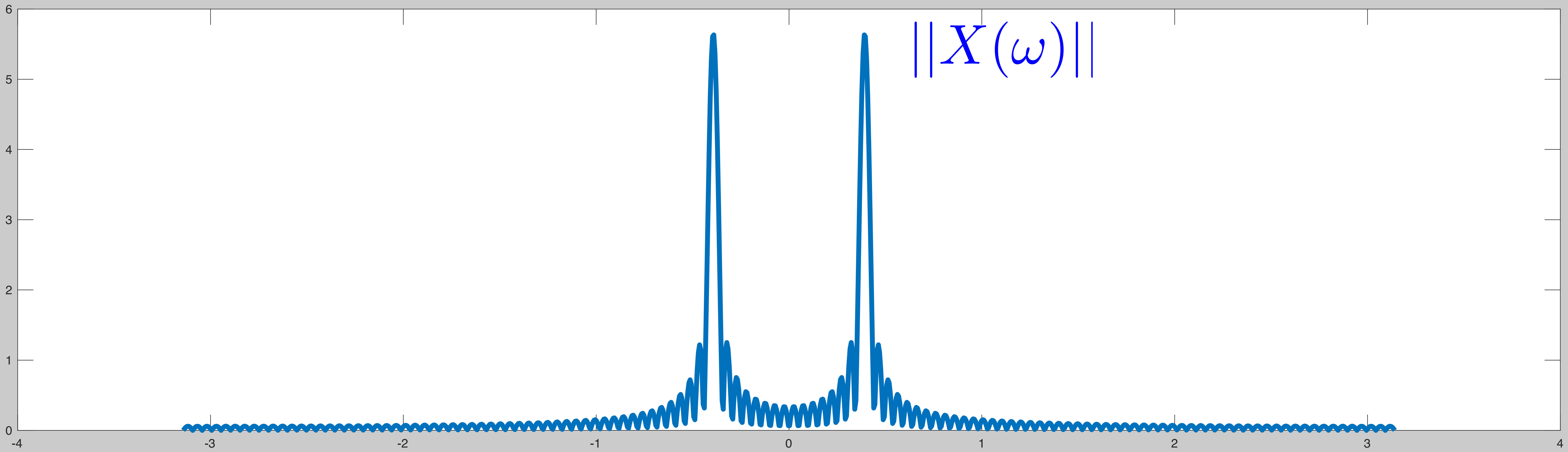
Frequency Analysis Through Projections

- Example: $x[n] = \cos\left(\frac{\pi}{8}n + \frac{\pi}{3}n\right)$

N = 128

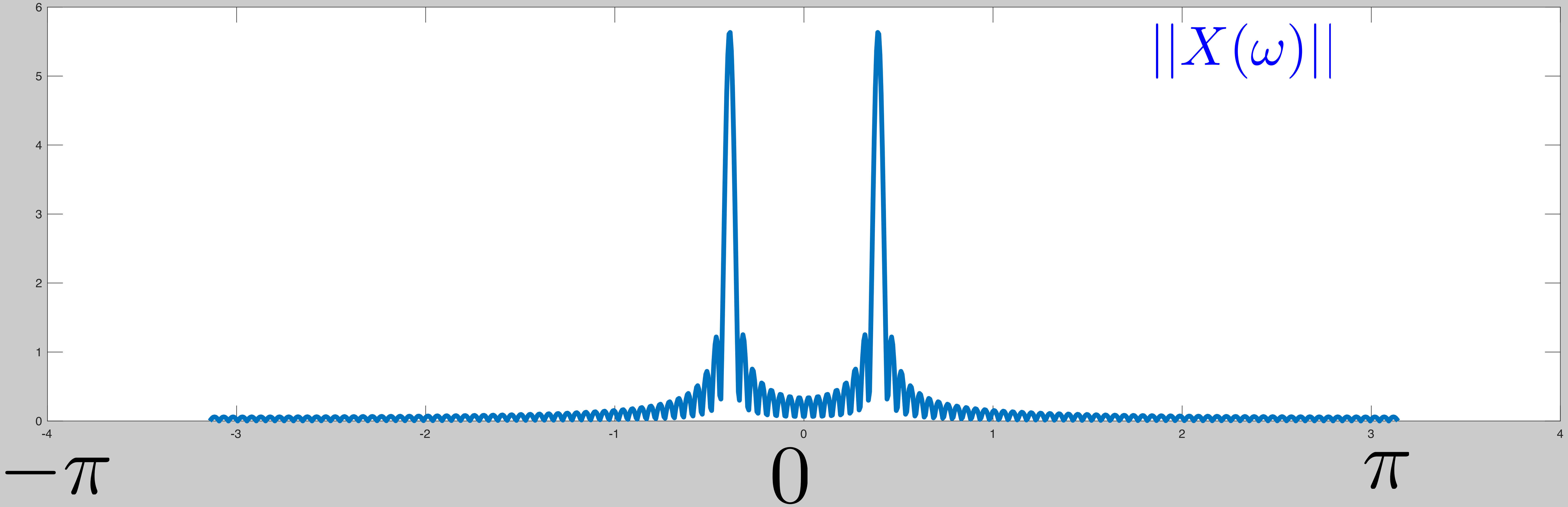


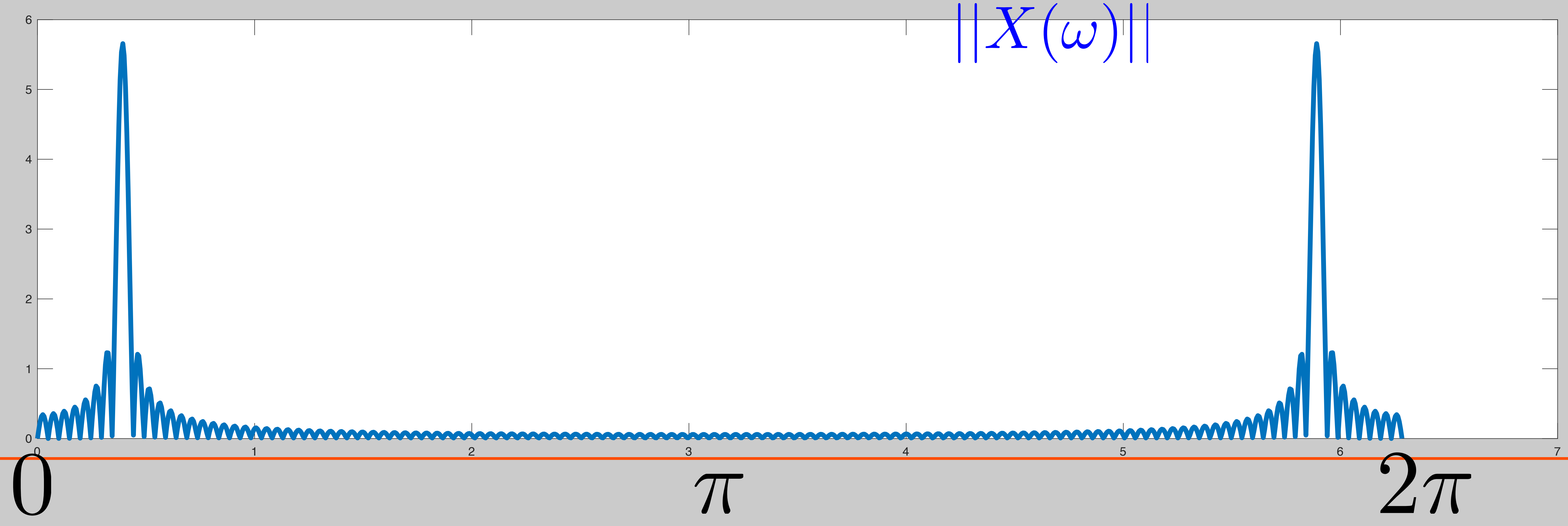
$$\Rightarrow X(\omega) = \vec{u}_w^{*}\vec{x}$$



Frequency Analysis Through Projections

- Example: $x[n] = \cos\left(\frac{\pi}{8}n + \frac{\pi}{3}n\right)$

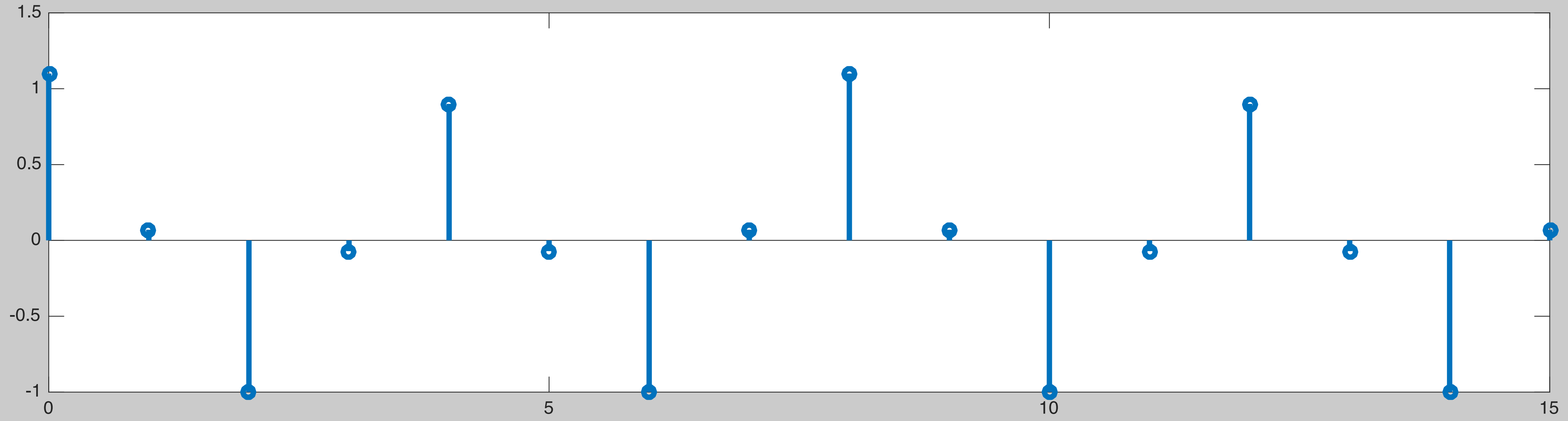




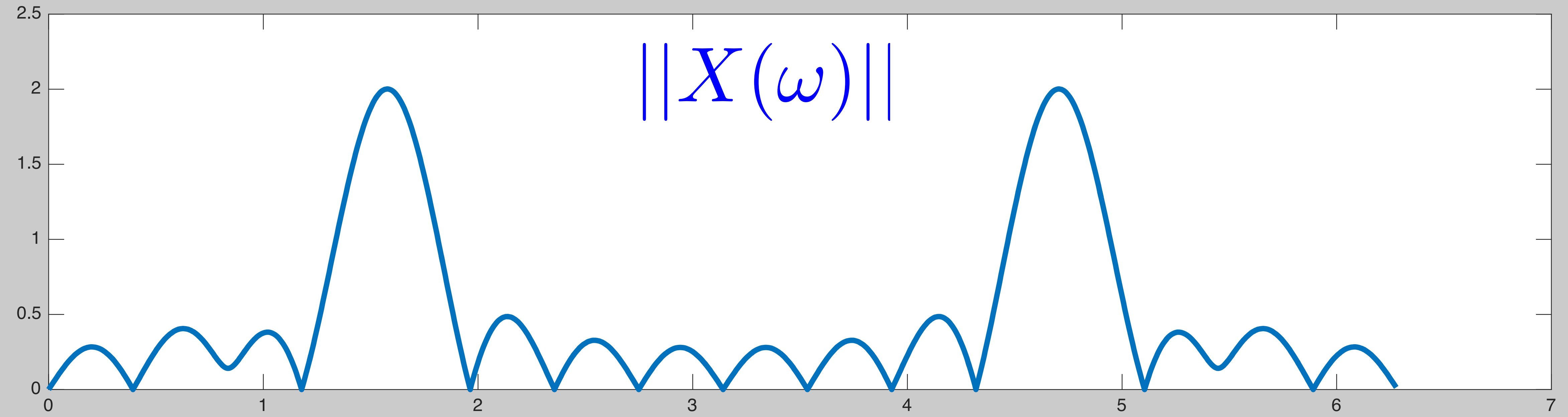
Frequency Analysis Through Projections

- Example: $x[n] = \cos(\frac{\pi}{2}n) + 0.1\cos(\frac{\pi}{4}n)$

N = 16



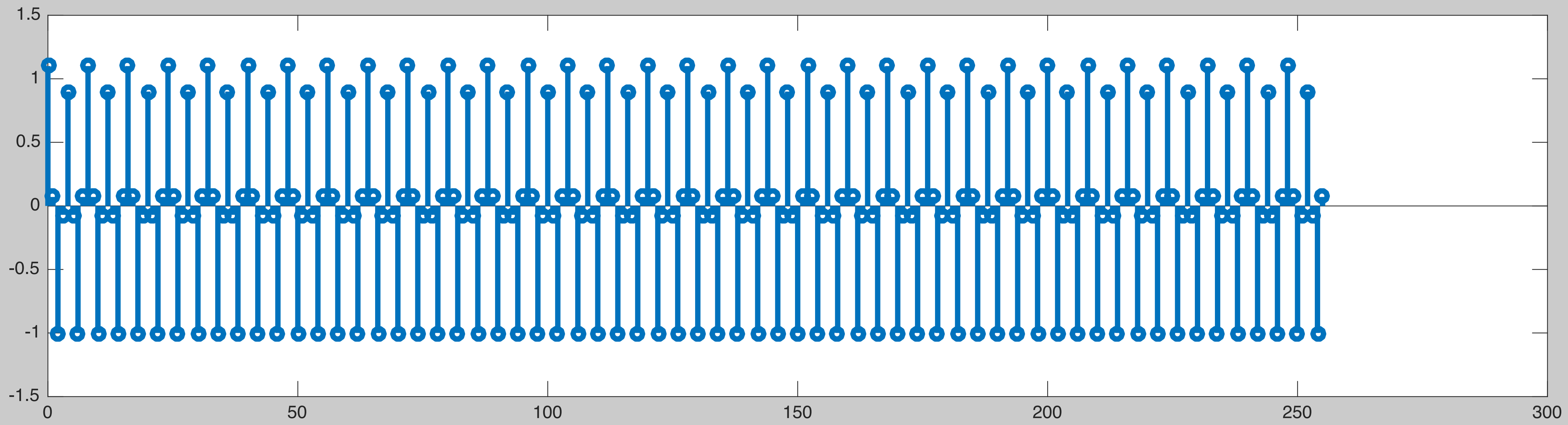
$$\Rightarrow X(\omega) = \vec{u}_w^{\,*}\vec{x}$$

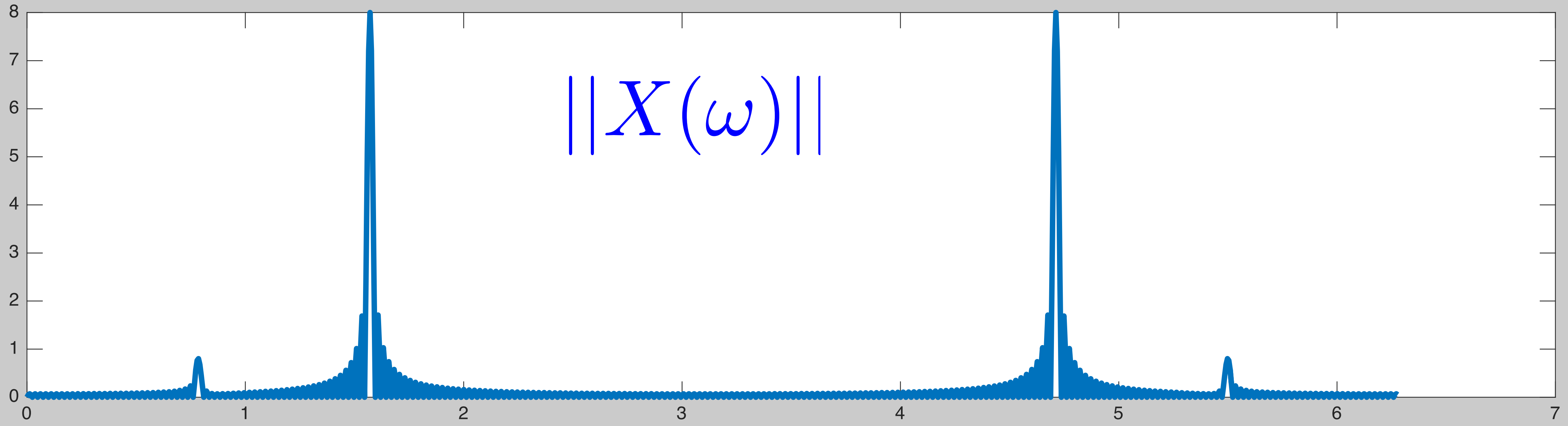


Frequency Analysis Through Projections

- Example: $x[n] = \cos(\frac{\pi}{2}n) + 0.1\cos(\frac{\pi}{4}n)$

N = 256





$$\Rightarrow X(\omega) = \vec{u}_w^* \vec{x}$$

Discrete-Time-Fourier-Transform

- DTFT (not DFT)

$$\vec{u}_\omega = \frac{1}{\sqrt{N}} \begin{bmatrix} e^{j\omega 0} \\ e^{j\omega 1} \\ \vdots \\ e^{j\omega (N-1)} \end{bmatrix}$$

$$X(\omega) = \vec{u}_w^* \vec{x}$$

$$X(\omega) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x[n] e^{-j\omega n}$$

Discrete Fourier Transform (DFT)

- For $u_\omega[n] = \frac{1}{\sqrt{N}} e^{j\omega n}$, pick a set of N frequencies, which will result in an orthogonal basis

- Choose: $\omega_k = \frac{2\pi k}{N} \Rightarrow \frac{1}{\sqrt{N}} e^{j\frac{2\pi k}{N} n}$

$$k \in [0, N-1]$$
$$n \in [0, N-1]$$

$$W_N \triangleq e^{j2\pi/N} \quad \Rightarrow \frac{1}{\sqrt{N}} W_N^{kn}$$

DFT Basis

$$\vec{u}_k = \frac{1}{\sqrt{N}} \begin{bmatrix} e^{j\frac{2\pi k \cdot 0}{N}} \\ e^{j\frac{2\pi k \cdot 1}{N}} \\ \vdots \\ e^{j\frac{2\pi k \cdot (N-1)}{N}} \end{bmatrix} = \frac{1}{\sqrt{N}} \begin{bmatrix} W_N^{k \cdot 0} \\ W_N^{k \cdot 1} \\ \vdots \\ W_N^{k \cdot (N-1)} \end{bmatrix} \quad k \in [0, N-1]$$

$$\Rightarrow X[k] = \vec{u}_k^* \vec{x}$$

DFT vs DTFT

$$\vec{u}_k = \frac{1}{\sqrt{N}} \begin{bmatrix} e^{j\frac{2\pi k \cdot 0}{N}} \\ e^{j\frac{2\pi k \cdot 1}{N}} \\ \vdots \\ e^{j\frac{2\pi k \cdot (N-1)}{N}} \end{bmatrix}$$

$$\vec{u}_\omega = \frac{1}{\sqrt{N}} \begin{bmatrix} e^{j\omega 0} \\ e^{j\omega 1} \\ \vdots \\ e^{j\omega (N-1)} \end{bmatrix}$$

$$X[k] = \vec{u}_k^* \vec{x}$$

$$X(\omega) = \vec{u}_w^* \vec{x}$$

$$X[k] = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi k n}{N}}$$

$$X(\omega) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x[n] e^{-j\omega n}$$

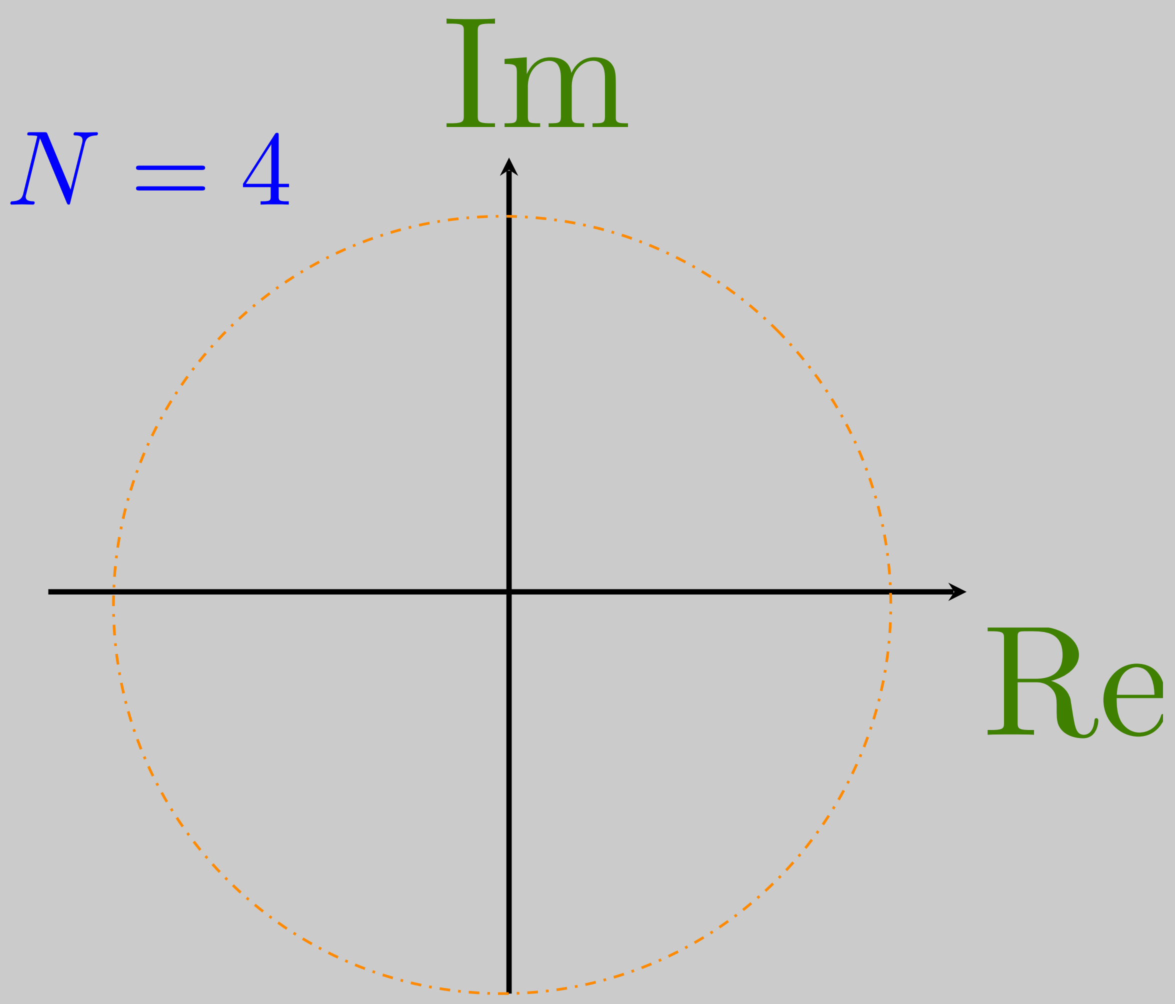
DFT Basis

$$\vec{u}_k = \frac{1}{\sqrt{N}} \begin{bmatrix} e^{j\frac{2\pi k\cdot 0}{N}} \\ e^{j\frac{2\pi k\cdot 1}{N}} \\ \vdots \\ e^{j\frac{2\pi k\cdot (N-1)}{N}} \end{bmatrix} = \frac{1}{\sqrt{N}} \begin{bmatrix} W_N^{k\cdot 0} \\ W_N^{k\cdot 1} \\ \vdots \\ W_N^{k\cdot (N-1)} \end{bmatrix} k \in [0, N-1]$$

$$\Rightarrow X[k] = \vec{u}_k^* \vec{x}$$

$k = 1, N = 4$

$$\Rightarrow \frac{1}{\sqrt{N}} W_N^{kn}$$



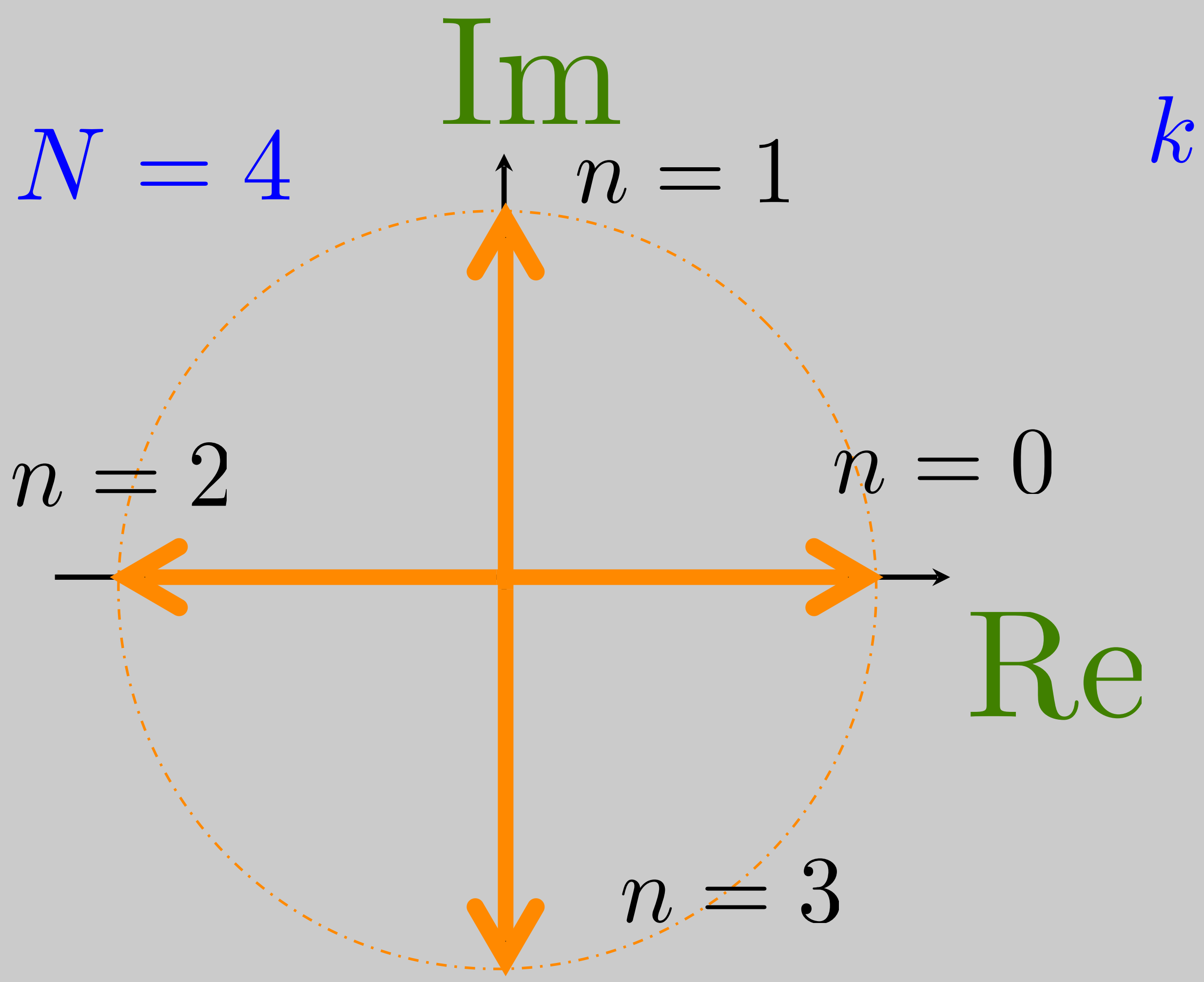
DFT Basis

$$\vec{u}_k = \frac{1}{\sqrt{N}} \begin{bmatrix} e^{j\frac{2\pi k\cdot 0}{N}} \\ e^{j\frac{2\pi k\cdot 1}{N}} \\ \vdots \\ e^{j\frac{2\pi k\cdot (N-1)}{N}} \end{bmatrix} = \frac{1}{\sqrt{N}} \begin{bmatrix} W_N^{k\cdot 0} \\ W_N^{k\cdot 1} \\ \vdots \\ W_N^{k\cdot (N-1)} \end{bmatrix} k \in [0, N-1]$$

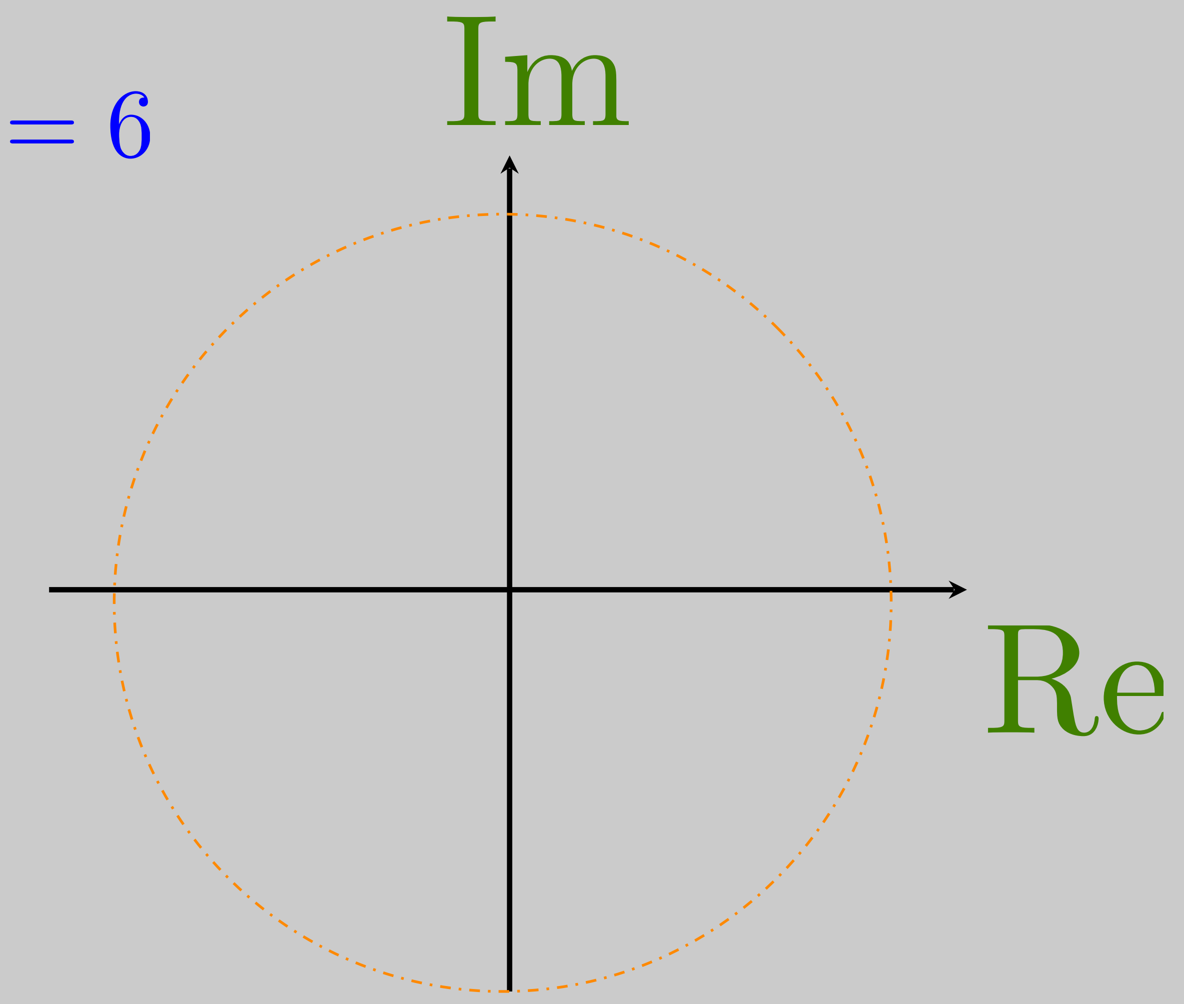
$$\Rightarrow X[k] = \vec{u}_k^* \vec{x}$$

$$\Rightarrow \frac{1}{\sqrt{N}} W_N^{kn}$$

$k = 1, N = 4$



$k = 2, N = 6$



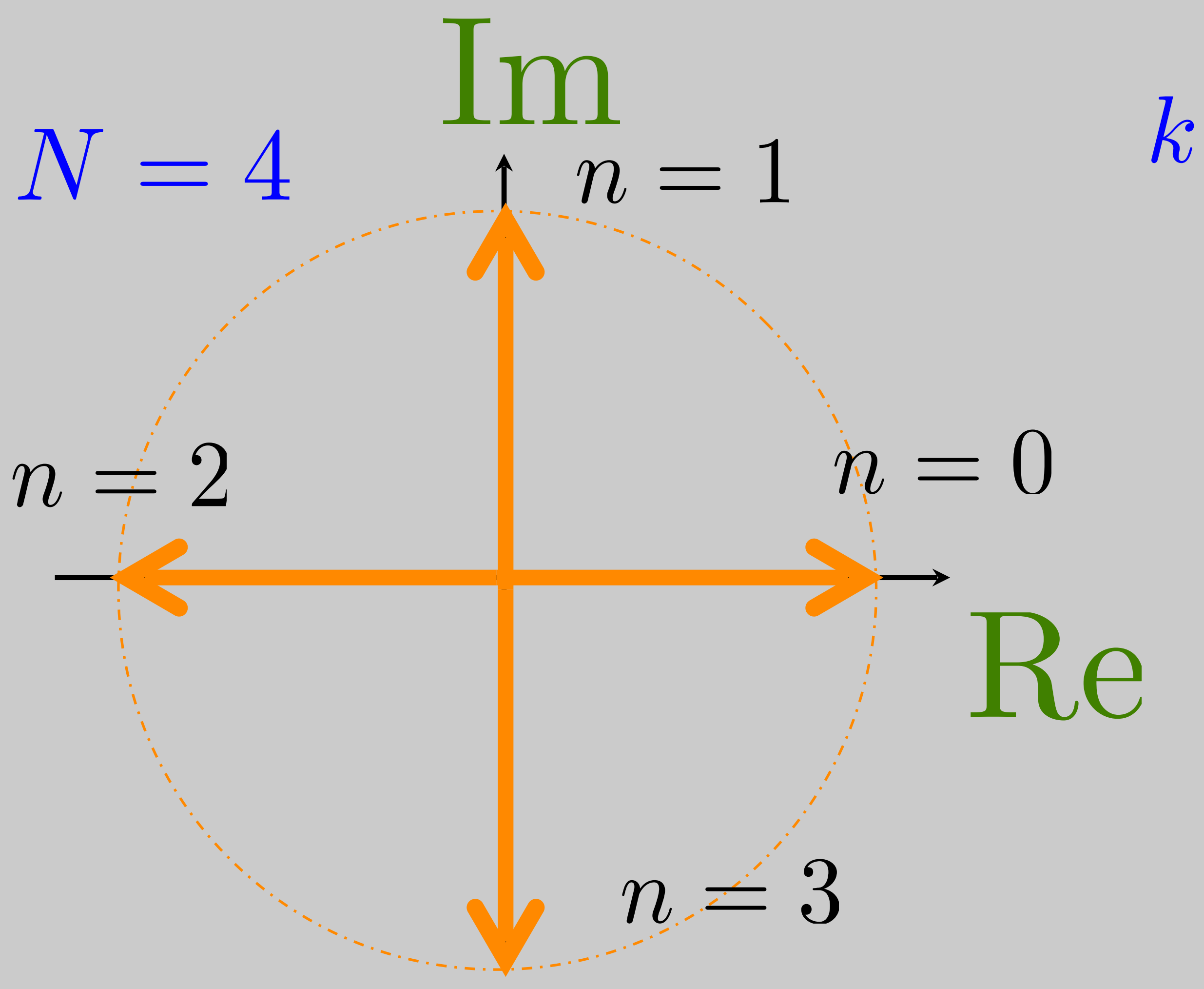
DFT Basis

$$\vec{u}_k = \frac{1}{\sqrt{N}} \begin{bmatrix} e^{j\frac{2\pi k\cdot 0}{N}} \\ e^{j\frac{2\pi k\cdot 1}{N}} \\ \vdots \\ e^{j\frac{2\pi k\cdot (N-1)}{N}} \end{bmatrix} = \frac{1}{\sqrt{N}} \begin{bmatrix} W_N^{k\cdot 0} \\ W_N^{k\cdot 1} \\ \vdots \\ W_N^{k\cdot (N-1)} \end{bmatrix} \quad k \in [0, N-1]$$

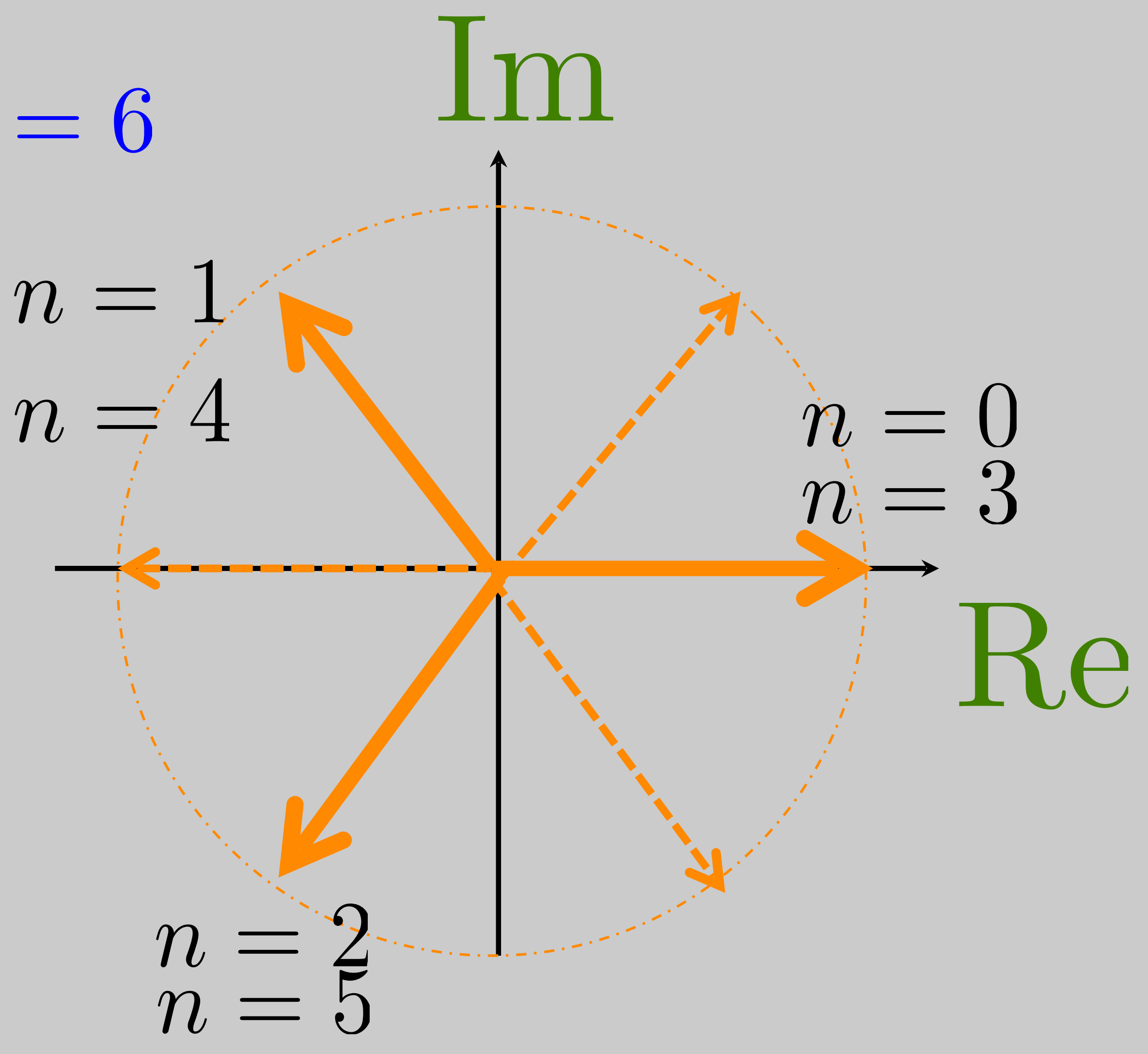
$$\Rightarrow X[k] = \vec{u}_k^* \vec{x}$$

$$\Rightarrow \frac{1}{\sqrt{N}} W_N^{kn}$$

$k = 1, N = 4$



$k = 2, N = 6$



$$\sum_{n=0}^{N-1} W_N^{nk} =?= \begin{cases} N & k = 0 \\ 0 & k \neq 0 \end{cases}$$

Orthonormality of DFT Basis

- DFT basis vectors are orthonormal. Proof:

$$\sum_{n=0}^{N-1} W_N^{nk} = \begin{cases} N & k = 0 \\ 0 & k \neq 0 \end{cases}$$

$$\vec{u}_k^* \vec{u}_m = \frac{1}{N} \sum_{n=0}^{N-1} W_N^{-nk} W_N^{nm} = \frac{1}{N} \sum_{n=0}^{N-1} W_N^{n(m-k)} = \begin{cases} 1 & k = m \\ 0 & k \neq m \end{cases}$$

Example

$$N = 16 \qquad \vec{u}_k = \frac{1}{\sqrt{16}} \begin{bmatrix} e^{j\frac{2\pi k \cdot 0}{16}} \\ e^{j\frac{2\pi k \cdot 1}{16}} \\ \vdots \\ e^{j\frac{2\pi k \cdot (15)}{16}} \end{bmatrix} = \frac{1}{\sqrt{N}} \begin{bmatrix} W_{16}^{k\cdot 0} \\ W_{16}^{k\cdot 1} \\ \vdots \\ W_{16}^{k\cdot 15} \end{bmatrix}$$

$$x[n] = \cos(\frac{\pi}{2}n) + 0.1\cos(\frac{\pi}{4}n) \quad = 0.5(e^{j\frac{\pi n}{2}} + e^{-j\frac{\pi n}{2}} + 0.1e^{j\frac{\pi n}{4}} + 0.1e^{-j\frac{\pi n}{2}})$$

$$= 0.5(e^{j\frac{2\pi 4n}{16}} + e^{-j\frac{2\pi 4n}{16}} + 0.1e^{j\frac{2\pi 2n}{16}} + 0.1e^{-j\frac{2\pi 2n}{16}})$$

$$= 0.5(e^{j\frac{2\pi 4n}{16}} + e^{j\frac{2\pi 12n}{16}} + 0.1e^{j\frac{2\pi 2n}{16}} + 0.1e^{j\frac{2\pi 14n}{16}})$$

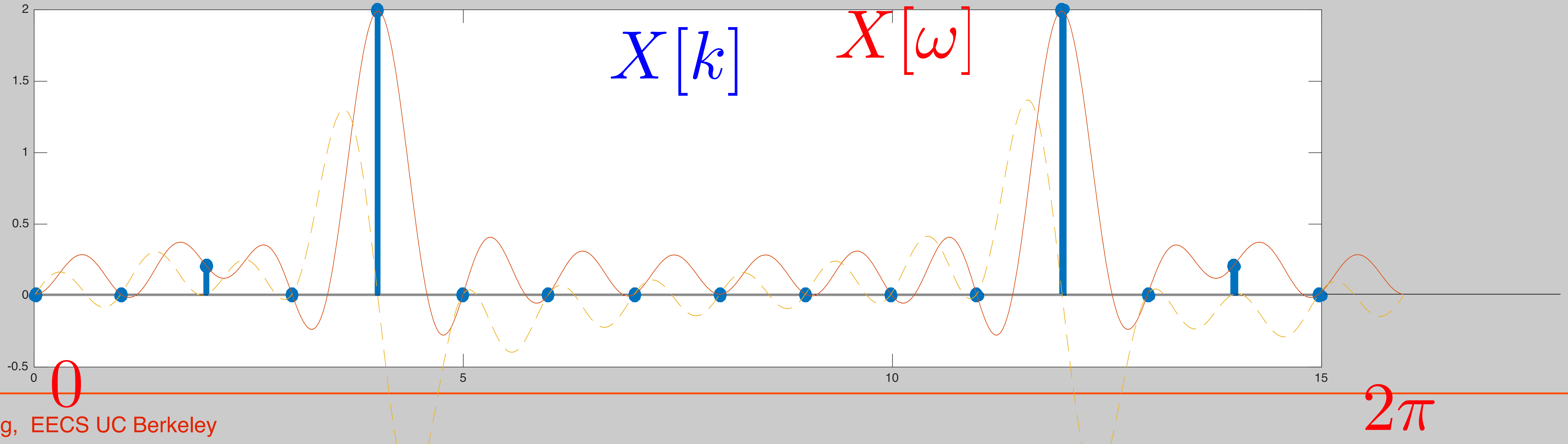
$$= \frac{2}{\sqrt{16}} W_{16}^{4n} + \frac{2}{\sqrt{16}} W_{16}^{12n} + \frac{0.2}{\sqrt{16}} W_{16}^{2n} + \frac{0.2}{\sqrt{16}} W_{16}^{14n}$$

Example

$$x[n] = \cos(\frac{\pi}{2}n) + 0.1\cos(\frac{\pi}{4}n)$$

$$= \frac{2}{\sqrt{16}}W_{16}^{4n} + \frac{2}{\sqrt{16}}W_{16}^{12n} + \frac{0.2}{\sqrt{16}}W_{16}^{2n} + \frac{0.2}{\sqrt{16}}W_{16}^{14n}$$

$$= 0.2\vec{u}_2 + 2\vec{u}_4 + 2\vec{u}_{12} + 0.2\vec{u}_{14}$$

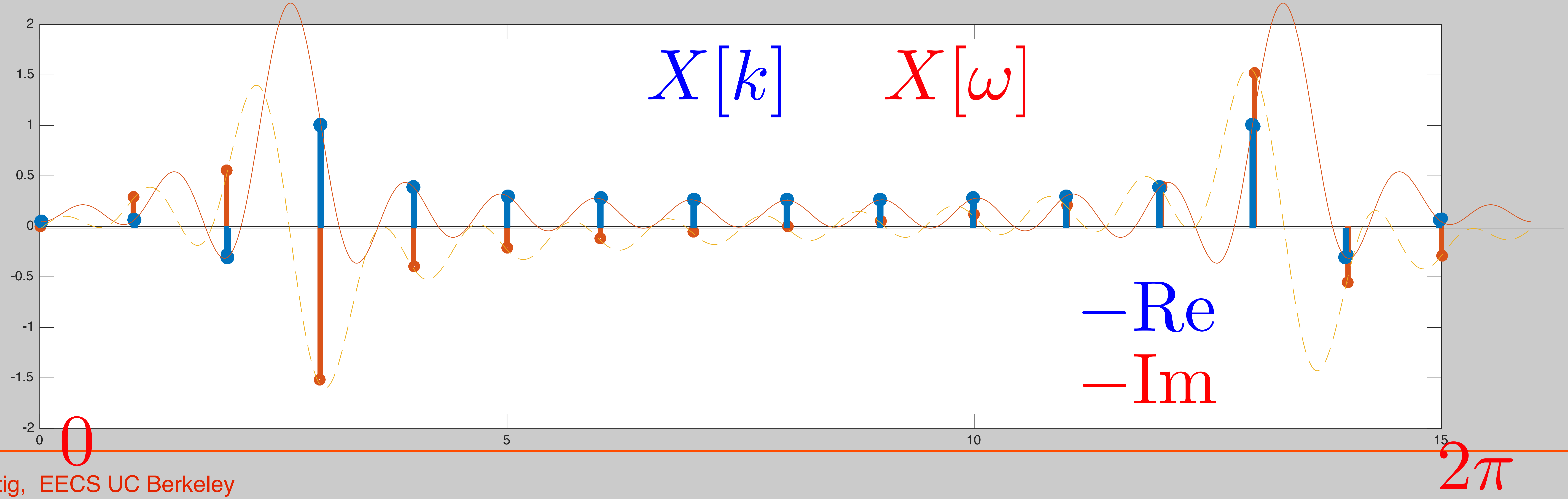


Example 2

What if there is no integer k to fit the frequency

$$\omega_k = \frac{2\pi k}{N}$$

$$x[n] = \cos(\frac{\pi}{3}n) + 0.1\cos(\frac{\pi}{6}n)$$



DFT

$$\vec{u}_k = \frac{1}{\sqrt{N}} \begin{bmatrix} e^{j\frac{2\pi k\cdot 0}{N}} \\ e^{j\frac{2\pi k\cdot 1}{N}} \\ \vdots \\ e^{j\frac{2\pi k\cdot (N-1)}{N}} \end{bmatrix} = \frac{1}{\sqrt{N}} \begin{bmatrix} W_N^{k\cdot 0} \\ W_N^{k\cdot 1} \\ \vdots \\ W_N^{k\cdot (N-1)} \end{bmatrix} \qquad k \in [0, N-1]$$

$$\Rightarrow X[k] = \vec{u}_k^* \vec{x}$$

$$\vec{X} = \underbrace{\frac{1}{\sqrt{N}} \begin{bmatrix} | & | & & | \\ \vec{u}_0 & \vec{u}_1 & \cdots & \vec{u}_{N-1} \\ | & | & & | \end{bmatrix}^*}_{\triangleq F^*} \vec{x}$$

DFT

- DFT Analysis

$$F = \frac{1}{\sqrt{N}} \begin{bmatrix} | & | & & | \\ \vec{u}_0 & \vec{u}_1 & \cdots & \vec{u}_{N-1} \\ | & | & & | \end{bmatrix}$$

$$\begin{bmatrix} X[0] \\ \vdots \\ X[N-1] \end{bmatrix} = \frac{1}{\sqrt{N}} \begin{bmatrix} - & \vec{u}_0^* & - \\ - & \vec{u}_1^* & - \\ & \vdots & \\ - & \vec{u}_{N-1}^* & - \end{bmatrix} \begin{bmatrix} x[0] \\ \vdots \\ x[N-1] \end{bmatrix}$$

$$\vec{X} = F^* \vec{x}$$

$$X[k] = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x[n] W_N^{-nk}$$

DFT

- DFT Synthesis

$$F = \frac{1}{\sqrt{N}} \begin{bmatrix} | & | & & | \\ \vec{u}_0 & \vec{u}_1 & \cdots & \vec{u}_{N-1} \\ | & | & & | \end{bmatrix}$$

$$\begin{bmatrix} x[0] \\ \vdots \\ x[N-1] \end{bmatrix} = \frac{1}{\sqrt{N}} \begin{bmatrix} | & | & & | \\ \vec{u}_0 & \vec{u}_1 & \cdots & \vec{u}_{N-1} \\ | & | & & | \end{bmatrix} \begin{bmatrix} X[0] \\ \vdots \\ X[N-1] \end{bmatrix}$$

$$\vec{x} = F\vec{X} \qquad = F(F^*\vec{x})$$

$$x[n] = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} X[k] W_N^{+nk}$$

Quiz

$$\vec{u}_k = \frac{1}{\sqrt{N}} \begin{bmatrix} e^{j\frac{2\pi k \cdot 0}{N}} \\ e^{j\frac{2\pi k \cdot 1}{N}} \\ \vdots \\ e^{j\frac{2\pi k \cdot (N-1)}{N}} \end{bmatrix} = \frac{1}{\sqrt{N}} \begin{bmatrix} W_N^{k \cdot 0} \\ W_N^{k \cdot 1} \\ \vdots \\ W_N^{k \cdot (N-1)} \end{bmatrix}$$

Compute a 2 point DFT of: $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\vec{u}_1 =$ $\quad\quad$ $\vec{u}_2 =$

$\vec{u}_1^* \vec{x} =$ $\quad\quad$ $\vec{u}_2^* \vec{x} =$

$\vec{X} =$

Example cont

- DFT$_2$ matrix:

$$F = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\vec{X} = F^* \vec{x} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix}$$