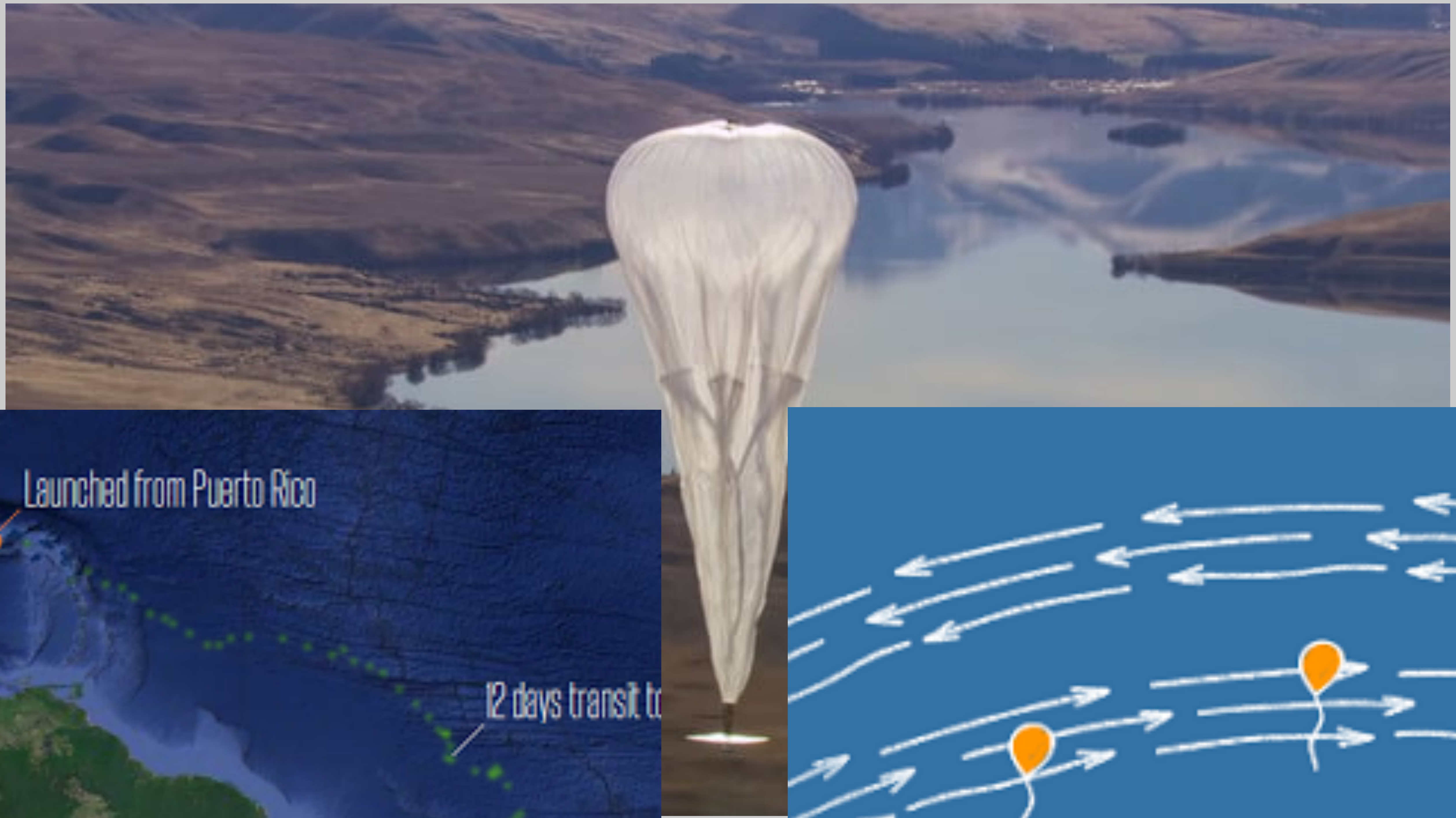
EE16B
Designing Information Devices and Systems II

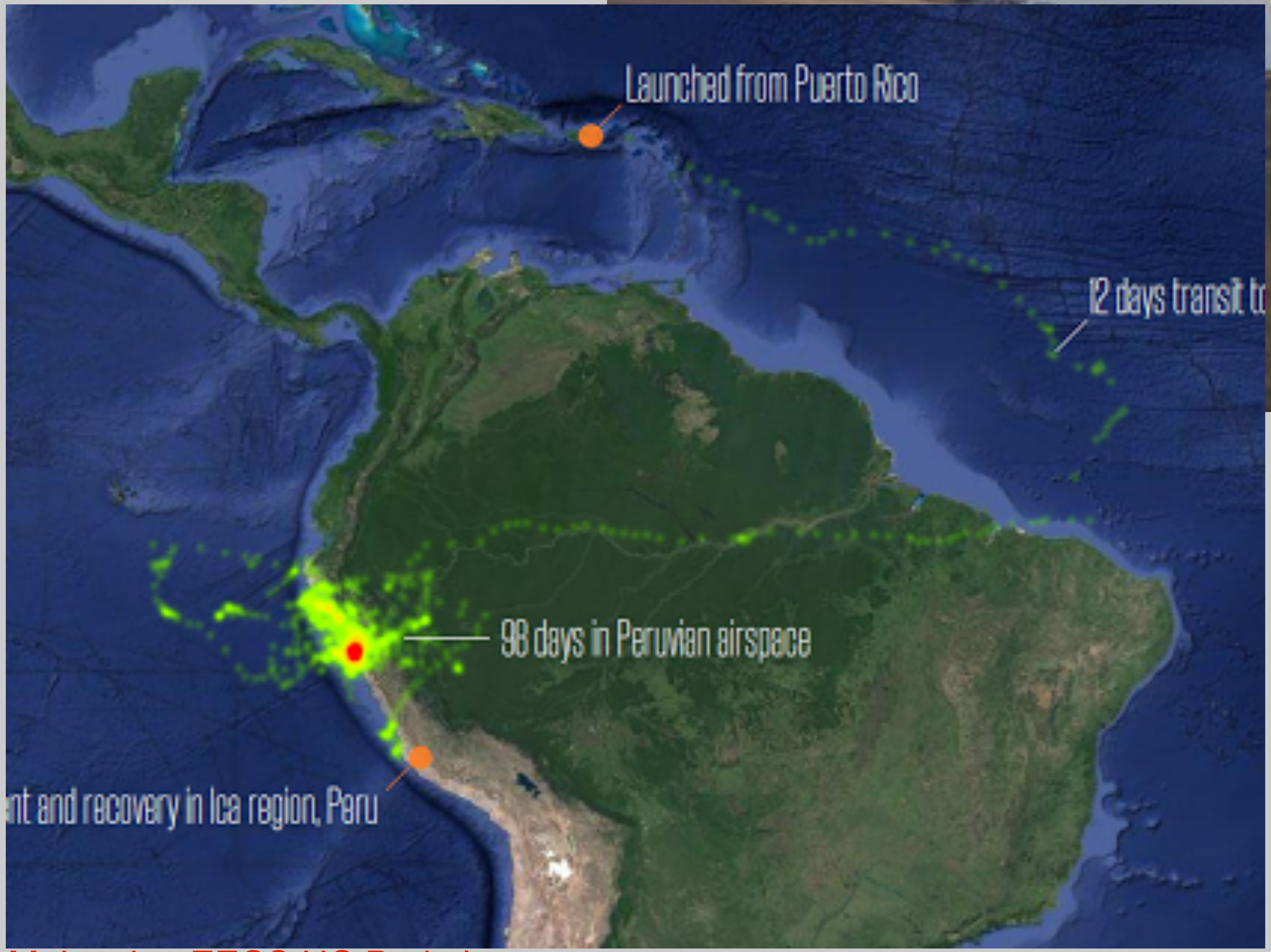
Lecture 7B
Outputs and Observability

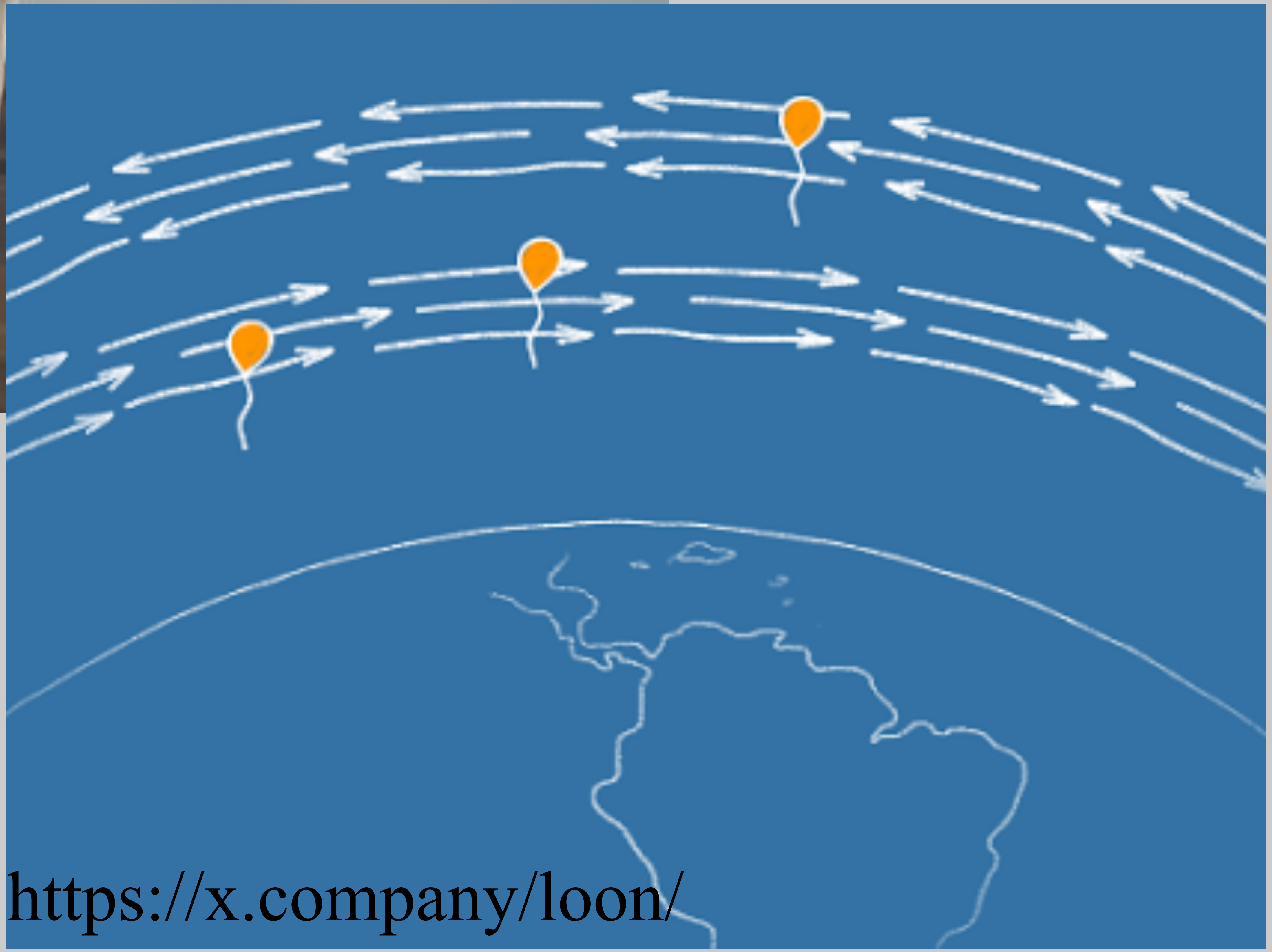
Intro

- Last time
  - State feedback control
  - Eigen value assignment
- Today:
  - Example of cooperative, adaptive cruise control
  - Output and observability
  - (MRI as a dynamical system – maybe)

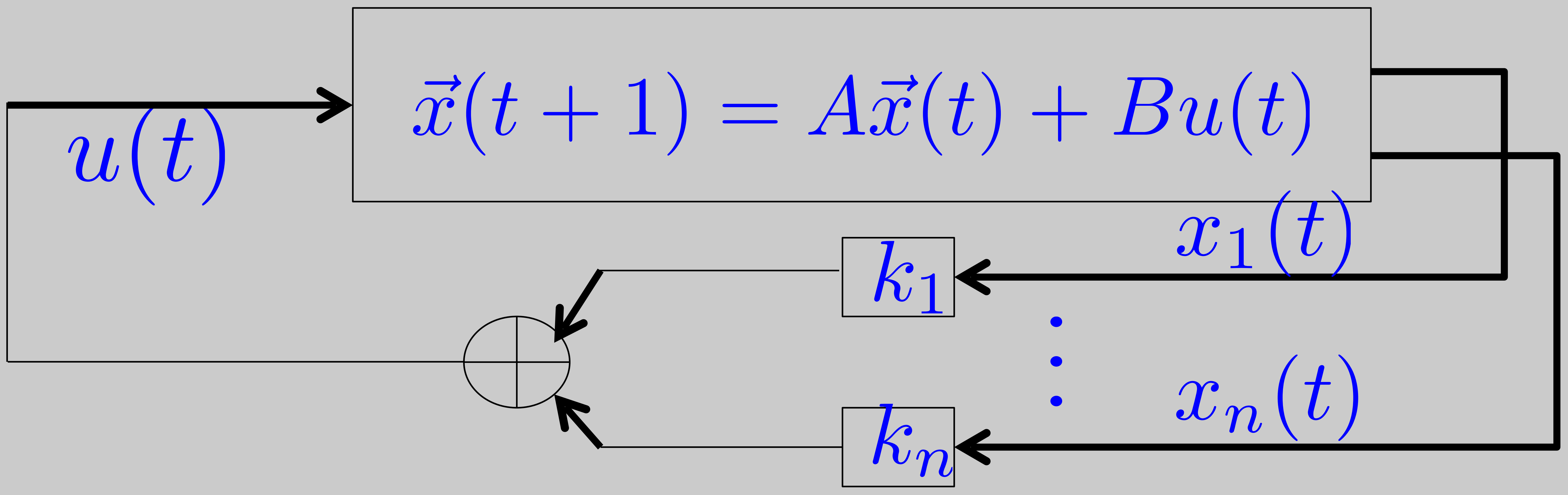
GOOGLE PROJECT LOON BALLOONS







https://x.company/loon/



$$\Rightarrow \quad \vec{x}(t+1) = (A+BK)\vec{x}(t)$$

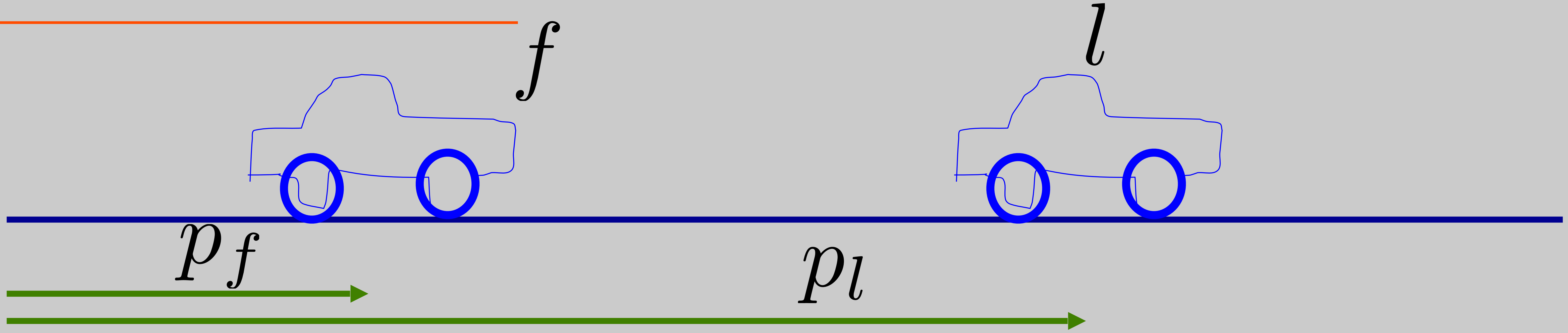
$$A + BK = \begin{bmatrix} 0 & 1 & 0 \\ 11 - k_1 & -k_2 & -k_3 \\ -1 + k_1 & k_2 & k_3 \end{bmatrix}$$

$$\lambda^3 + (k_2 - k_3)\lambda^2 + (k_1 - 11)\lambda + 10k_3 = 0$$

$$(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) \quad \Leftrightarrow \quad \lambda^3 - (\lambda_1 + \lambda_2 + \lambda_3)\lambda^2 - (\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3)\lambda - \lambda_1\lambda_2\lambda_3 = 0$$

Cooperative Adaptive Cruise control

Example:



$$\frac{d}{dt} p_l(t) = v_l(t) \qquad \frac{d}{dt} p_f(t) = v_f(t)$$

$$\frac{d}{dt} v_l(t) = u_l(t) \qquad \frac{d}{dt} v_f(t) = u_f(t)$$

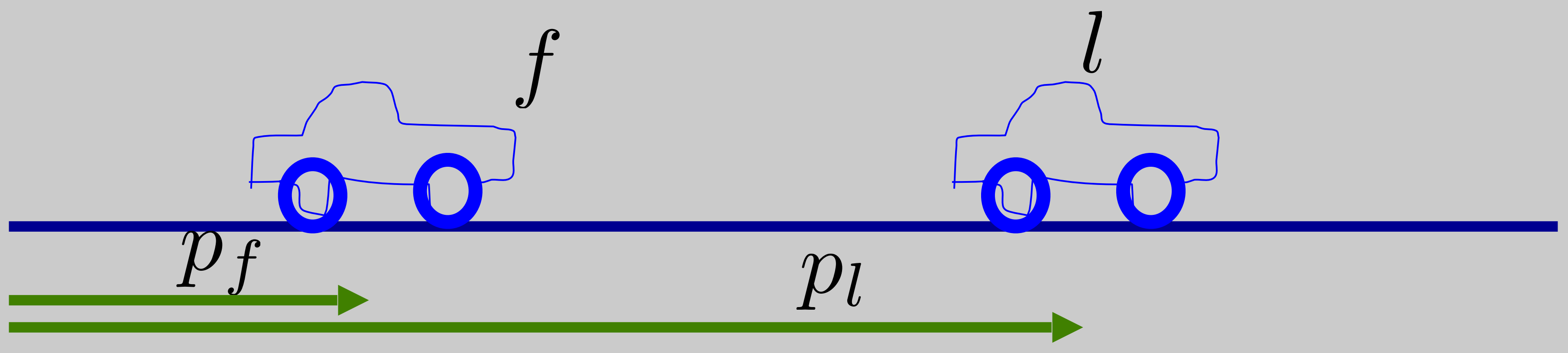
$$x_1(t) = p_l(t) - p_f(t) - \delta$$

$$x_2(t) = v_l(t) - v_f(t)$$

$$\frac{d}{dt} x_1(t) = v_l(t) - v_f(t)$$

$$\frac{d}{dt} x_2(t) = u_l(t) - u_f(t) \triangleq u(t)$$

Cooperative Adaptive Cruise control



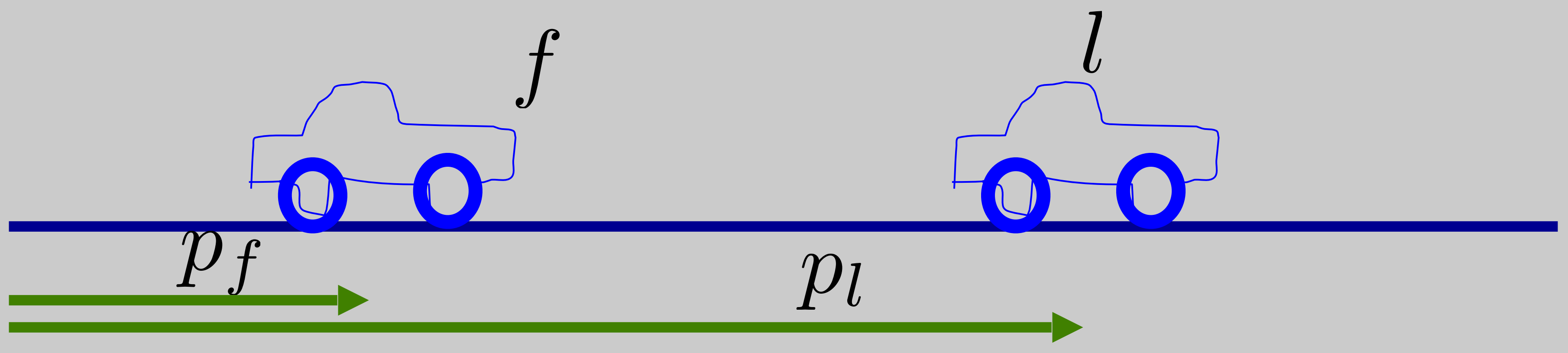
$$\frac{d}{dt}x_1(t) = v_l(t) - v_f(t)$$

$$\frac{d}{dt}x_2(t) = u_l(t) - u_f(t) \triangleq u(t)$$

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \Rightarrow A + BK = \begin{bmatrix} 0 & 1 \\ k_1 & k_2 \end{bmatrix}$$

Q) What eigen-values will you want here?

Let's look at input more closely…



$$u(t) = k_1 x_1(t) + k_2 x_2(t)$$

$$\Rightarrow u(t) = k_1(p_l(t) - p_f(t) - \delta) + k_2(v_l(t) - v_f(t))$$

But leader chooses his own acceleration u_l(t)

$$u(t) = u_l(t) - u_f(t)$$

$$\begin{aligned} u_f(t) &= u_l(t) - u(t) \\ &= u_l(t) - k_1(p_l(t) - p_f(t) - \delta) - k_2(v_l(t) - v_f(t)) \end{aligned}$$

Q) What does the follower need to know to implement?

A) Cooperative (vehicle2vehicle comm.)
range sensor (for distance and velocity)

Outputs

$$\vec{x}(t+1) = A\vec{x}(t) + Bu(t)$$

Can’t always measure state directly or all states…

Define output:

$$\vec{y}(t) = C\vec{x}(t)$$

p x n matrix for p outputs

Outputs

$$\vec{x}(t+1) = A\vec{x}(t) + Bu(t)$$

Can't always measure state directly or all states...

Define output:

$$\vec{y}(t) = C\vec{x}(t)$$

p x n matrix for p outputs

$$y = x_1 \qquad \Rightarrow C = [1\ 0\ 0\ \cdots\ 0]$$

$$y = x_1 + x_2 \qquad \Rightarrow C = [1\ 1\ 0\ \cdots\ 0]$$

$$\vec{y} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \Rightarrow C = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Observability

A system is "observable" if, by watching

y(0),y(1),y(2),... we can determine the full state

Two stage approach:

1) Determine initial state *x(0)* from *y(0),y(1),....*

2) $\vec{x}(t) = A^t\vec{x}(0) + Bu(t)$

Ignore input: $u(t) = 0$

$$\left.\begin{aligned} y(0) &= C\vec{x}(0) \\ y(1) &= C\vec{x}(1) = CA\vec{x}(0) \\ &\vdots \\ y(t) &= CA^t\vec{x}(0) \end{aligned}\right\} \vec{y} = \begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(t) \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^t \end{bmatrix} \vec{x}(0)$$

Observability

Q: What conditions on O_t, to determine x(0) uniquely?

$$\vec{y} = \begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(t) \end{bmatrix} = \underbrace{\begin{bmatrix} C \\ CA \\ \vdots \\ CA^t \end{bmatrix}}_{\triangleq O_t} \vec{x}(0)$$

A: O_t must have n independent <u>rows</u>

strictly O_{n-1} has full rank

null-space is {0}

$$\text{Observability} \quad \Leftrightarrow \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \text{ has rank = n}$$

Observability

With input:

$$\begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(n-1) \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \vec{x}(0) + \begin{bmatrix} Bu(0) \\ Bu(1) \\ \vdots \\ Bu(n-1) \end{bmatrix}$$

$$\begin{bmatrix} y(0) - Bu(0) \\ y(1) - Bu(1) \\ \vdots \\ y(n-1) - Bu(n-1) \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \vec{x}(0)$$

Example

$$\vec{x}(t+1) = \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{A} \vec{x}(t)$$

rotation matrix

$$\underbrace{y(t) = x_1(t)}_{C = [1 \;\; 0]}$$

Example

$$\vec{x}(t+1) = \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{A} \vec{x}(t)$$

rotation matrix

$$\underbrace{y(t) = x_1(t)}_{C = [1 \;\; 0]}$$

$$\begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} \end{bmatrix} \Rightarrow$$

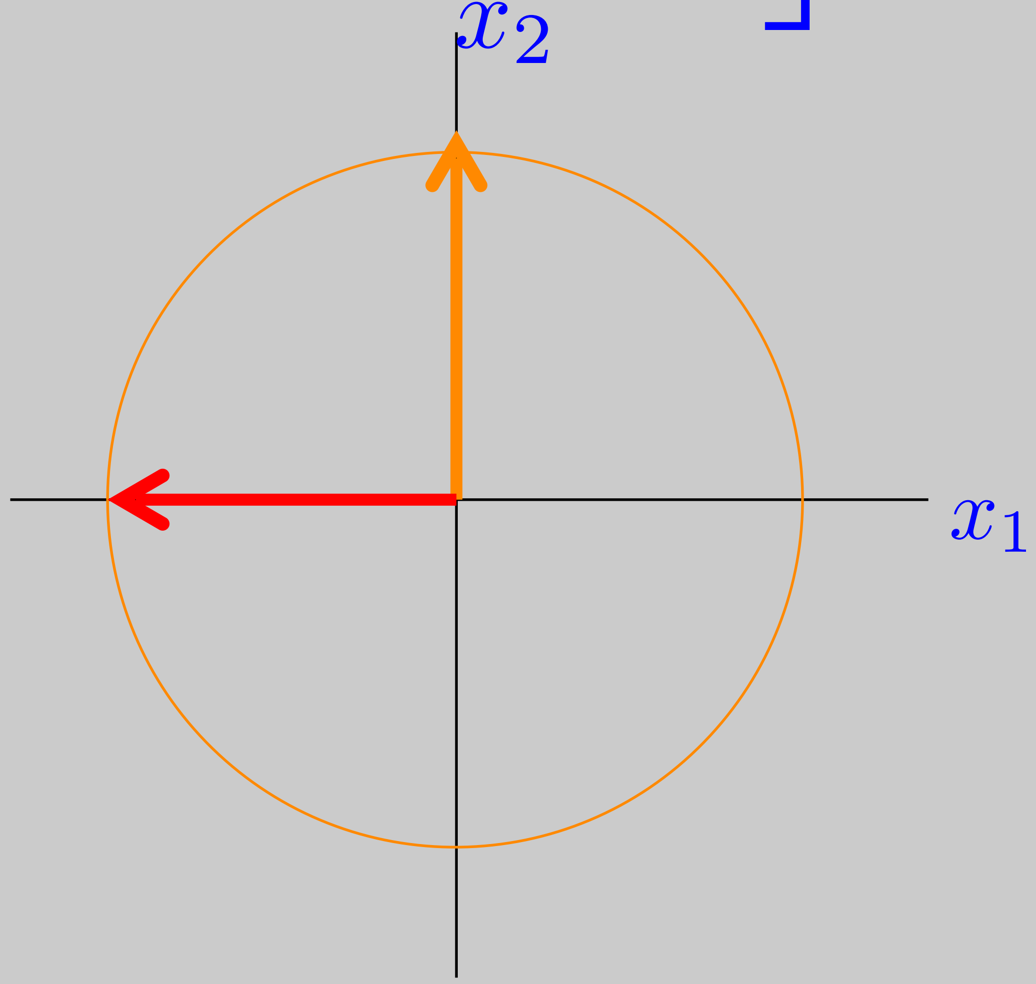
Example

$$\vec{x}(t+1) = \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{A} \vec{x}(t) \qquad \underbrace{y(t) = x_1(t)}_{C = [1 \;\; 0]}$$

rotation matrix

$$\begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \cos\theta & -\sin\theta \end{bmatrix} \Rightarrow \text{rank} = 2 \quad \text{if} \quad \theta \neq k\pi$$

$$\theta = \frac{\pi}{2}$$



$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} C \\ CA \end{bmatrix}^{-1} \begin{bmatrix} y(0) \\ y(1) \end{bmatrix}$$

Example

$$\vec{x}(t+1) = \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{A} \vec{x}(t)$$

A rotation matrix

$$\underbrace{y(t) = x_1(t)}_{C = [1 \;\; 0]}$$

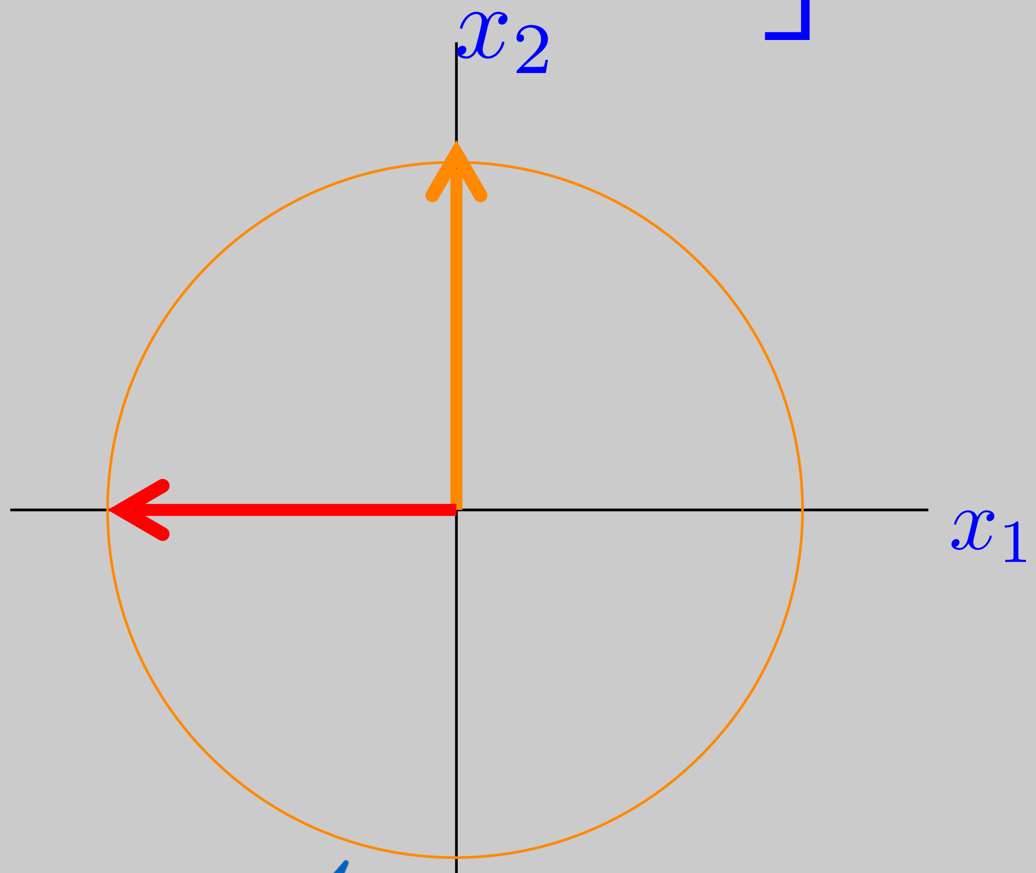
$$\begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \cos\theta & -\sin\theta \end{bmatrix} \Rightarrow \text{rank} = 2 \quad \text{if} \quad \theta \neq k\pi$$

$$\theta = \frac{\pi}{2}$$

$$x_1(0) = 0$$

$$x_1(1) = -1$$

$$x_2(1) = 0$$



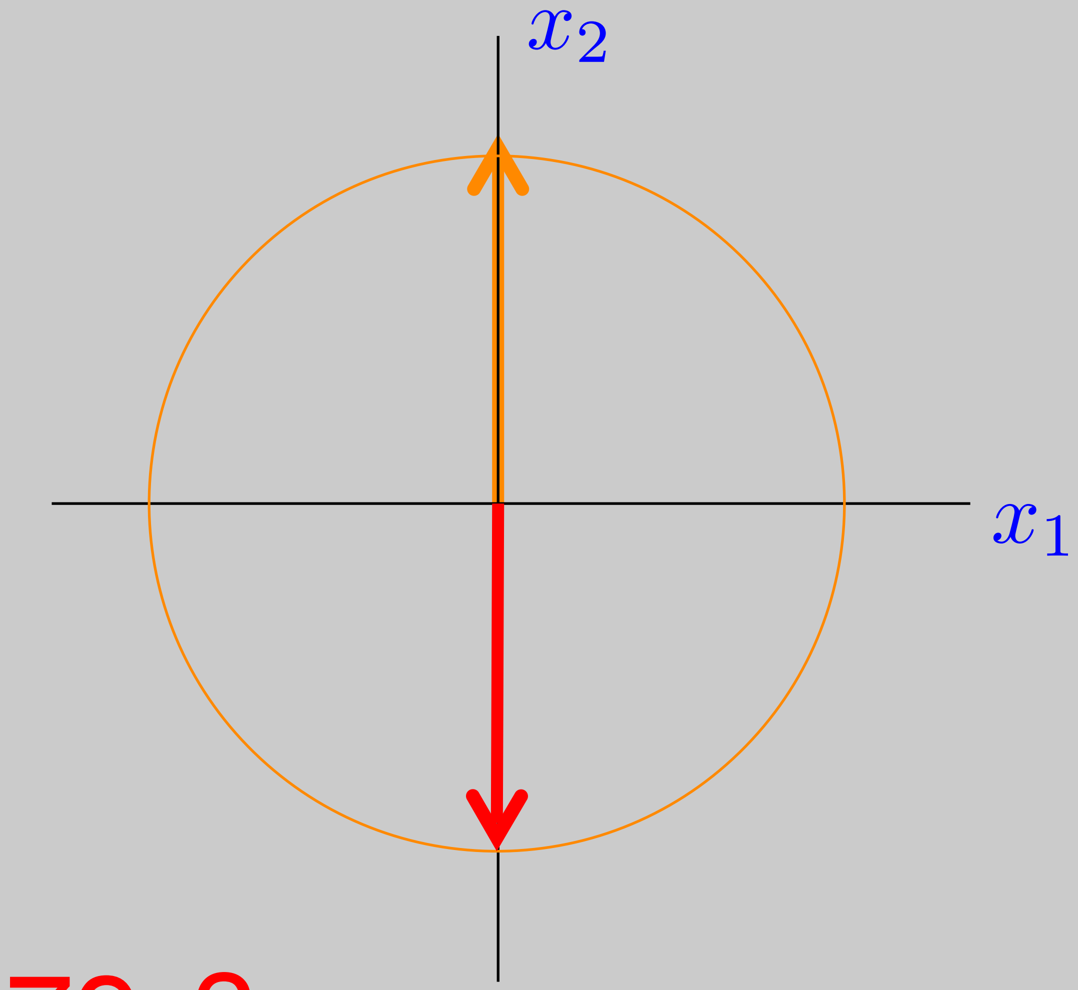
$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^{-1} \cdot \begin{bmatrix} -0 \\ -1 \end{bmatrix}$$

$$\theta = \pi$$

$$x_1(0) = 0$$

$$x_1(1) = 0$$

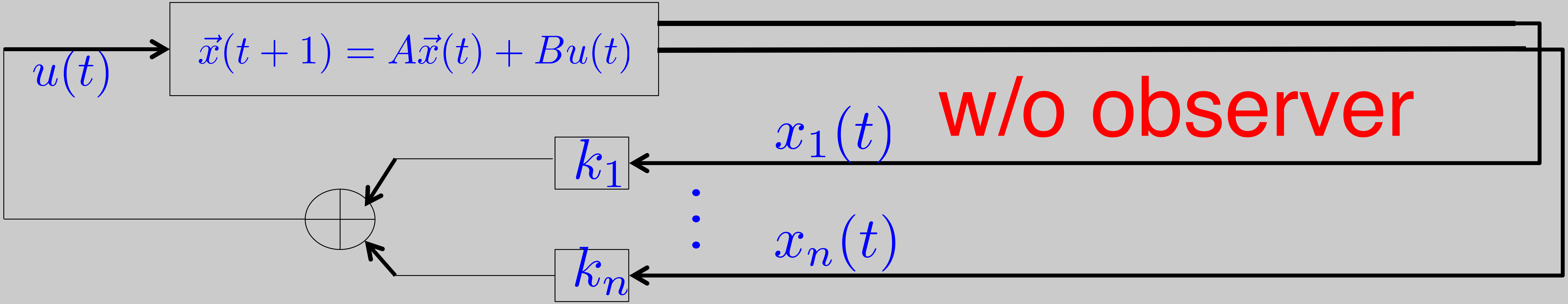
$$x_2(1) = ?$$



Q: What is ϴ = 179°?

State Feedback Control

$$\vec{x}(t+1) = A\vec{x}(t) + Bu(t) \qquad \vec{y}(t) = C\vec{x}(t)$$

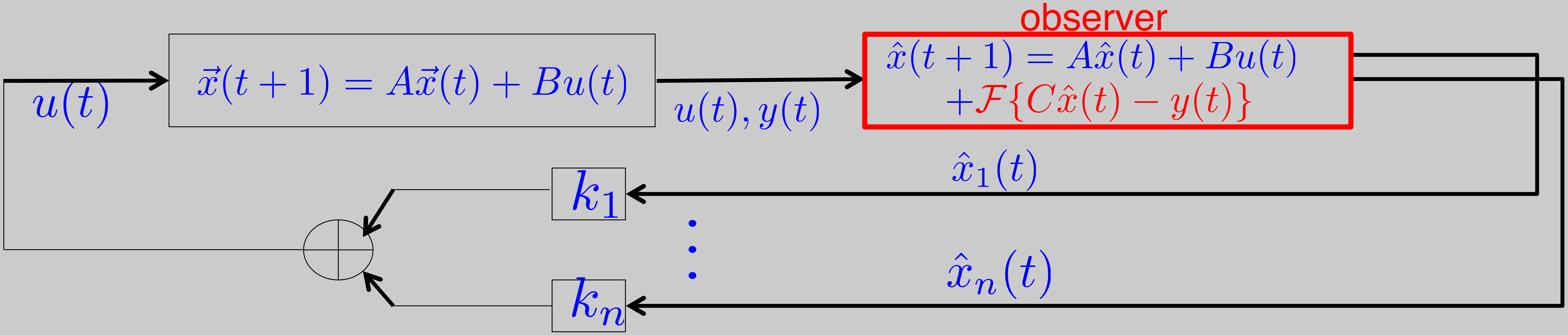


A Common Observer Algorithm

Start with initial guess $\hat{x}(0)$

Update estimate each time using:

$$\hat{x}(t+1) = \underbrace{A\hat{x}(t) + Bu(t)}_{\text{Copy of system model}} + \underbrace{\mathcal{F}\{C\hat{x}(t) - y(t)\}}_{\text{correction}}$$



Kalman Filter

- Accounts for noise and errors in our system model and inputs

$$\hat{x}(t+1) = \underbrace{A\hat{x}(t) + Bu(t)}_{\text{Copy of system model}} + \mathcal{F}\{\underbrace{C\hat{x}(t) - y(t)}_{\text{correction}}\}$$

A more elaborate form of the observer where the matrix L is also updated at each time, is known as the Kalman Filter and is the industry standard in navigation. The Kalman Filter takes into account the statistical properties of the noise that corrupts measurements and minimizes the mean square error between x(t) and xˆ(t)

Figure 3: **Rudolf Kalman (1930-2016)** introduced the Kalman Filter as well as many of the state space concepts we studied, such as controllability and observability. He was awarded the National Medal of Science in 2009.

Control Recap

- Controllability:

$$\vec{x}(n) - A^n\vec{x}(0) = \begin{bmatrix} A^{n-1}B & A^{n-2}B & \cdots & AB & B \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(n-1) \end{bmatrix}$$

If Rn is full rank then we can move to any target value

Same rank test for continuous time

- Open loop control:

Can use the above equation to design an input sequence – and apply it blindly. Accuracy of result will depend on accuracy of model.

Control Recap – State Feedback

$$u(t) = K\vec{x}(t)$$

Closed-loop system: $\Rightarrow \quad \vec{x}(t+1) = (A + BK)\vec{x}(t)$

Must choose K s.t. A+BK has eigenvalues inside the unit circle (or left half-plane for continuous time)

If controllable, can assign eigenvalues for A+BK arbitrarily

If not, some eigenvalues of A can not be changed!

(could be OK, if stable, bad news if not)

Control Recap - Observers

Not all state variable are measured, but we get “outputs”

$$\vec{y}(t) = C\vec{x}(t)$$

O_{n-1} must have n independent rows (full rank) to determine x(0) uniquely from output

$$\begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(n-1) \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \vec{x}(0)$$

How Does MRI Work? (some today – more later!)

- Magnetic Polarization
  -- Very strong uniform magnet
- Excitation
  -- Very powerful RF transmitter
- Acquisition
  -- Location is encoded by gradient magnetic fields
  -- Very powerful audio amps