# Linearization of systems

**One dimensional linear approximation**

Consider a differentiable function $f$ of one variable.

$$f : \mathbb{R} \to \mathbb{R}$$

Say we are interested in $f$ in a small, open neighborhood about a particular point $t^*$. $t^*$ is often called the fixed point of the system. Let's call this open neighborhood $U$. In this case, we can construct a linear approximation of $f$ about the neighborhood $U$. Recall Taylor Approximation,

$$\frac{\mathrm{d}f}{\mathrm{d}t}(t^*) \approx \frac{f(t) - f(t^*)}{t - t^*} \text{ for } t \in U$$

We can use the above to construct a linear approximation of $f$. Let $f_l$ denote the linear approximation of $f$ about $U$.

$$f_l(t) = \frac{\mathrm{d}f}{\mathrm{d}t}(t^*)(t - t^*) + f(t^*) \tag{1}$$

Strictly speaking, $f_l$ is an affine approximation unless $t^* = 0$, but the process of obtaining $f_l$ is colloquially called the linearization of $f$.

For example, consider $f(t) = t^2$. We will set the fixed point of the system to be $t^* = 1$. Then,

$$\begin{aligned} f_l(t) &= \tfrac{\mathrm{d}f}{\mathrm{d}t}(t^*)(t - t^*) + f(t^*) \\ &= 2(t - 1) + 1 \\ &= 2t - 1 \end{aligned}$$

Let $\varepsilon = 10^{-2}$. Consider the open neighborhood $U = (1 - \varepsilon, 1 + \varepsilon)$. Let's plot $f$ and $f_l$ when their respective domains are restricted to $U$. This is seen in Figure 1 on the next page.

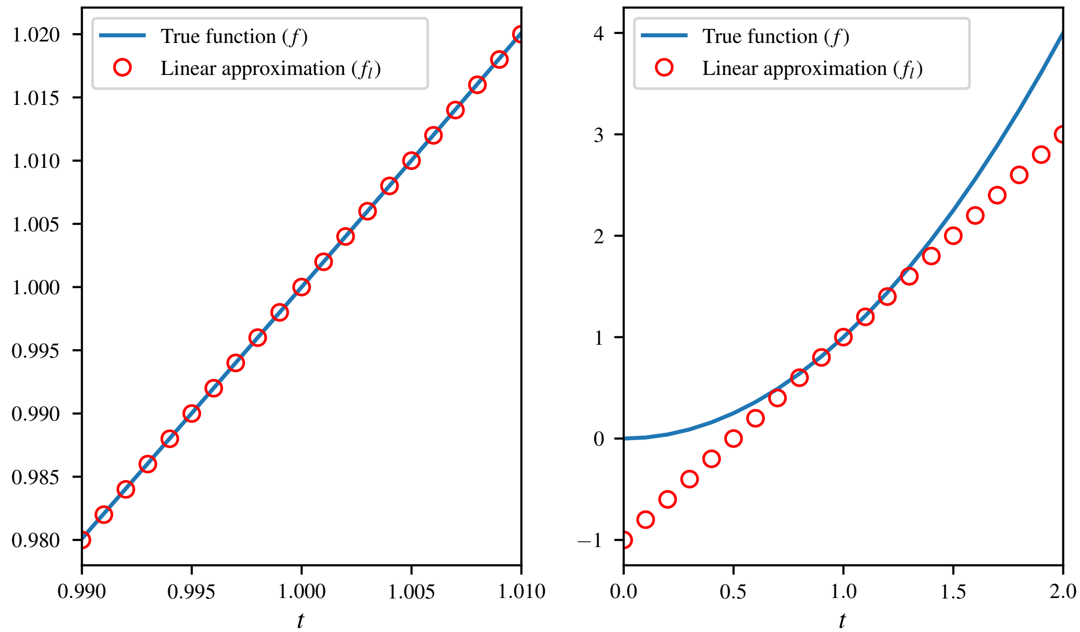



Figure 1: *Left*: The linear approximation of the function $f(t) = t^2$ about $t = 1$ is fairly accurate in the range $U = [0.99, 1.01]$. *Right*: Outside of this range, the approximation does not track the function very well.

### Linearization of a system

Consider a continuous, non-linear system with state $\vec{x}(t)$ (which is $n$ dimensional) and input $\vec{u}(t)$ (which is $m$ dimensional) of the form,

$$\frac{\mathrm{d}\vec{x}}{\mathrm{d}t}(t) = f(\vec{x}(t), \vec{u}(t))$$

To clarify and establish notation, $f$ is a function that takes in $\vec{x}(t)$ and $\vec{u}(t)$ and outputs an $n$ dimensional vector. $f_k(\vec{x}(t), \vec{u}(t))$ refers to the function that is the $k^{th}$ coordinate of the output of $f(\vec{x}(t), \vec{u}(t))$.

Let $\vec{x}^*(t)$ be the desired state trajectory and $\vec{u}^*(t)$ be the desired input. Analogous to (1), we will construct an affine estimate about a neighborhood around the fixed points.

$$f(\vec{x}(t), \vec{u}(t)) \approx f(\vec{x}^*(t), \vec{u}^*(t)) + \nabla_{\vec{x}} f(\vec{x}^*(t), \vec{u}^*(t))(\vec{x}(t) - \vec{x}^*(t)) + \nabla_{\vec{u}} f(\vec{x}^*(t), \vec{u}^*(t))(\vec{u}(t) - \vec{u}^*(t)) \tag{2}$$

Let's define $\tilde{x} = \vec{x} - \vec{x}^*, \tilde{u} = \vec{u} - \vec{u}^*$. $\nabla f$ is the multidimensional generalization of the derivative, which is constructed as follows.

$$\nabla_{\mathbf{x}} f(\mathbf{x}, \mathbf{u}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{x}, \mathbf{u}) & \frac{\partial f_1}{\partial x_2}(\mathbf{x}, \mathbf{u}) & \dots & \frac{\partial f_1}{\partial x_n}(\mathbf{x}, \mathbf{u}) \\ \frac{\partial f_2}{\partial x_1}(\mathbf{x}, \mathbf{u}) & \frac{\partial f_2}{\partial x_2}(\mathbf{x}, \mathbf{u}) & \dots & \frac{\partial f_2}{\partial x_n}(\mathbf{x}, \mathbf{u}) \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_n}{\partial x_1}(\mathbf{x}, \mathbf{u}) & \frac{\partial f_n}{\partial x_2}(\mathbf{x}, \mathbf{u}) & \dots & \frac{\partial f_n}{\partial x_n}(\mathbf{x}, \mathbf{u}) \end{bmatrix} \text{ and } \nabla_{\mathbf{u}} f(\mathbf{x}, \mathbf{u}) = \begin{bmatrix} \frac{\partial f_1}{\partial u_1}(\mathbf{x}, \mathbf{u}) & \frac{\partial f_1}{\partial u_2}(\mathbf{x}, \mathbf{u}) & \dots & \frac{\partial f_1}{\partial u_m}(\mathbf{x}, \mathbf{u}) \\ \frac{\partial f_2}{\partial u_1}(\mathbf{x}, \mathbf{u}) & \frac{\partial f_2}{\partial u_2}(\mathbf{x}, \mathbf{u}) & \dots & \frac{\partial f_2}{\partial u_m}(\mathbf{x}, \mathbf{u}) \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_n}{\partial u_1}(\mathbf{x}, \mathbf{u}) & \frac{\partial f_n}{\partial u_2}(\mathbf{x}, \mathbf{u}) & \dots & \frac{\partial f_n}{\partial u_m}(\mathbf{x}, \mathbf{u}) \end{bmatrix}$$

Note the dimensions of the matrices. It must be notated that, when calculating $\nabla_x f$, $\vec{u}$ is considered constant. Similarly, when calculating $\nabla_u f$, $\vec{x}$ is considered constant.

To continue from (2), note that,

$$\frac{\mathrm{d}x}{\mathrm{d}t}(t) = f(x(t), u(t)), \frac{\mathrm{d}x^*}{\mathrm{d}t}(t) = f(x^*(t), u^*(t)) \text{ and } \frac{\mathrm{d}x}{\mathrm{d}t}(t) = \frac{\mathrm{d}x^*(t)}{\mathrm{d}t} + \frac{\mathrm{d}\tilde{x}}{\mathrm{d}t}(t)$$

Plugging this back into (2), we get,

$$\cancel{\tfrac{\mathrm{d}x^*}{\mathrm{d}t}(t)} + \frac{\mathrm{d}\tilde{x}}{\mathrm{d}t}(t) \approx \cancel{\tfrac{\mathrm{d}x^*}{\mathrm{d}t}(t)} + \nabla_{\vec{x}} f(\vec{x}^*(t), \vec{u}^*(t))\tilde{x}(t) + \nabla_{\vec{u}} f(\vec{x}^*(t), \vec{u}^*(t))\tilde{u}(t)$$

Thus, we get linearized version of our system.

$$\frac{\mathrm{d}\tilde{x}}{\mathrm{d}t}(t) \approx \nabla_{\vec{x}} f(\vec{x}^*(t), \vec{u}^*(t))\tilde{x}(t) + \nabla_{\vec{u}} f(\vec{x}^*(t), \vec{u}^*(t))\tilde{u}(t) \tag{3}$$

Note that, unlike the one dimensional linearization example, we are **linearizing with respect to $\vec{x}$ and $\vec{u}$**. Also, observe that our state variables are now the pertubations $\tilde{x}(t)$ and $\tilde{u}(t)$.

# Questions

1. **Jacobian Warm-Up**

Consider the following function $f : \mathbb{R}^2 \mapsto \mathbb{R}^3$

$$f(x_1, x_2) = \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \\ f_3(x_1, x_2) \end{pmatrix} = \begin{pmatrix} x_1^2 - x_2^2 \\ x_1^2 + x_1 x_2^2 \\ x_1 \end{pmatrix}$$

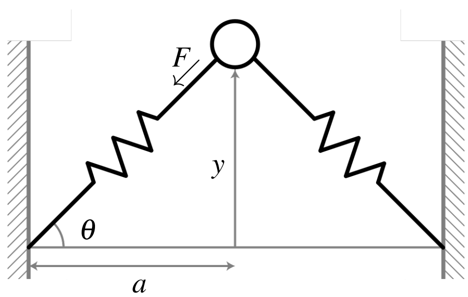
Calculate its Jacobian.

**Answer:**

$$\frac{\mathrm{d}f}{\mathrm{d}\vec{x}} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} \end{pmatrix}$$
$$= \begin{pmatrix} 2x_1 & -2x_2 \\ 2x_1 + x_2^2 & 2x_1 x_2 \\ 1 & 0 \end{pmatrix}$$

2. **Linearization**

Consider a mass attached to two springs:



We assume that each spring is linear with spring constant $k$ and resting length $X_0$. We want to build a state space model that describes how the displacement $y$ of the mass from the spring base evolves. The differential equation modeling this system is $\frac{d^2y}{dt} = -\frac{2k}{m}(y - X_0\frac{y}{\sqrt{y^2+a^2}})$.

(a) Write this model in state space form $\dot{x} = f(x)$.

**Answer:** We introduce states $x_1 = y$ and $x_2 = \dot{y}$. Writing the model in state space form gives

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{-2k}{m}\left(x_1 - X_0\frac{x_1}{\sqrt{x_1^2+a^2}}\right) \end{bmatrix}.$$

(b) Find the equilibrium of the state-space model. You can assume $X_0 < a$.

**Answer:** We find the equilibrium by solving $0 = \dot{x} = f(x)$:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{-2k}{m}\left(x_1 - X_0\frac{x_1}{\sqrt{x_1^2+a^2}}\right) \end{bmatrix}.$$

The unique solution is the equilibrium at $(x_1, x_2) = (0, 0)$.

(c) Linearize your model about the equilibrium.

**Answer:**

$$\left.\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}\right|_{x=(0,0)} = \left.\begin{bmatrix} 0 & 1 \\ -\frac{2k}{m}\left(1 - X_0\frac{a^2}{(x_1^2+a^2)^{3/2}}\right) & 0 \end{bmatrix}\right|_{x=(0,0)} = \begin{bmatrix} 0 & 1 \\ -\frac{2k}{m}\left(1 - \frac{X_0}{a}\right) & 0 \end{bmatrix}$$

So the linearized system is

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -\frac{2k}{m}\left(1 - \frac{X_0}{a}\right) & 0 \end{bmatrix} x.$$