EECS 16B Designing Information Devices and Systems II
Fall 2018 Elad Alon and Miki Lustig **Homework 3**

This homework is due on Wednesday September 19, 2018 at 11:59PM.
Self-grades are due on Monday, September 24, 2018 at 11:59PM.

1. Mechanical 2nd Order Differential Equation

Solve $3\frac{d^2y}{dt^2} - 12\frac{dy}{dt} + 24y = 24$, where $y(0) = 1$ and $\frac{dy}{dt}(0) = 2$

Solution:

First, we'll divide both sides by 3 to get a coefficient of 1 in front of the $\frac{d^2y}{dt^2}$ term:

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 8y = 8$$

Since this is a nonhomogeneous differential equation, we will perform a change of coordinates. Choose $\tilde{y}$ such that $8\tilde{y} = 8(y-8)$, which means we need $\tilde{y} = y - 1$. Then, $\frac{d\tilde{y}}{dt} = \frac{dy}{dt}$ and $\frac{d^2\tilde{y}}{dt^2} = \frac{d^2y}{dt^2}$.

$$\begin{aligned} 8 &= \frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 8y \\ 8 &= \frac{d^2\tilde{y}}{dt^2} - 4\frac{d\tilde{y}}{dt} + 8(\tilde{y}+1) \\ 0 &= \frac{d^2\tilde{y}}{dt^2} - 4\frac{d\tilde{y}}{dt} + 8\tilde{y} \end{aligned}$$

We set up the matrix differential equation with the state vector $\vec{x}(t) = \begin{bmatrix} \tilde{y}(t) \\ \frac{d\tilde{y}}{dt}(t) \end{bmatrix}$.

$$\begin{bmatrix} \frac{d\tilde{y}}{dt}(t) \\ \frac{d^2\tilde{y}}{dt^2}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -8 & 4 \end{bmatrix} \begin{bmatrix} \tilde{y}(t) \\ \frac{d\tilde{y}}{dt}(t) \end{bmatrix}$$

Call this coefficient matrix A. We will solve for its eigenvalues, which are roots of $\det(A - \lambda I) = 0$.

$$\det\left(\begin{bmatrix} -\lambda & 1 \\ -8 & 4-\lambda \end{bmatrix}\right) = \lambda^2 - 4\lambda + 8 = 0$$

$$\lambda = \frac{4 \pm \sqrt{16-32}}{2} = 2 \pm 2j$$

The λs are non-real complex conjugates, so the general solution is given by:

$$\tilde{y}(t) = k_1 e^{(2+2j)t} + k_2 e^{(2-2j)t}$$

Because k_1, k_2 are complex conjugate pairs, we can simplify to the form shown below:

$$\tilde{y}(t) = e^{2t}(c_1 \cos(2t) + c_2 \sin(2t))$$

We reverse the change of variables $y \mapsto \tilde{y}$.

$$\begin{aligned} y(t) &= \tilde{y}(t) + 1 \\ &= e^{2t}(c_1 \cos(2t) + c_2 \sin(2t)) + 1 \\ \frac{\mathrm{d}y}{\mathrm{d}t}(t) &= 2e^{2t}(c_1 \cos(2t) + c_2 \sin(2t)) + e^{2t}(-2c_1 \sin(2t) + 2c_2 \cos(2t)) \end{aligned}$$

To find c_1 and c_2, we substitute the initial conditions for y and $\frac{\mathrm{d}y}{\mathrm{d}t}$. Recall $\sin(0) = 0$ and $\cos(0) = 1$.

$$\begin{aligned} c_1 + 1 &= 1 \\ 2c_1 + 2c_2 &= 2 \end{aligned}$$

From this we have $c_1 = 0$ and $c_2 = 1$. Returning to the solution for y,

$$y(t) = e^{2t} \sin(2t) + 1$$

2. Solution to Repeated Roots

In lecture, we claimed that the solution to a second-order differential equation with repeated eigenvalue λ_0 is $y = c_1 e^{\lambda t} + c_2 t e^{\lambda t}$. In this problem, we will show why this solution is valid.

(a) Given a differential equation $\frac{d^2y}{dt^2} + a_1 \frac{dy}{dt} + a_0 y = 0$, assume that both eigenvalues of the A matrix with the state vector defined as $\begin{bmatrix} y(t) \\ \frac{dy(t)}{dt} \end{bmatrix}$ are λ_0. Find a_0, a_1 in terms of λ_0.

Solution: Let $\vec{x} = \begin{bmatrix} y \\ \frac{dy}{dt} \end{bmatrix}$. Then our vector differential equation is

$$\begin{bmatrix} \frac{dy}{dt} \\ \frac{d^2y}{dt^2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} \begin{bmatrix} y \\ \frac{dy}{dt} \end{bmatrix}$$

If we find the characteristic polynomial of the matrix, we get $\lambda^2 + a_1 \lambda + a_0 = 0$. We need this to have two roots of λ_0. A polynomial with two roots λ_0 looks like $(\lambda - \lambda_0)(\lambda - \lambda_0) = \lambda^2 - 2\lambda_0 \lambda + \lambda_0^2$. Therefore, $a_0 = \lambda_0^2, a_1 = -2\lambda_0$, so

$$\frac{d^2y}{dt^2} - 2\lambda_0 \frac{dy}{dt} + \lambda_0^2 y = 0$$

(b) Let's assume the solution to our differential equation is $c_1 e^{\lambda_0 t} + c_2 t e^{\lambda_0 t}$. Verify that this solution satisfies the differential equation you get when using the a_0 and a_1 you found in part (a).

Solution:

$$
\begin{aligned}
y &= c_1e^{\lambda_0 t} + c_2te^{\lambda_0 t} \\
\frac{dy}{dt} &= \lambda_0 c_1 e^{\lambda_0 t} + c_2e^{\lambda_0 t} + \lambda_0 c_2 te^{\lambda_0 t} \\
\frac{d^2y}{dt^2} &= \lambda_0^2 c_1 e^{\lambda_0 t} + \lambda_0 c_2 e^{\lambda_0 t} + \lambda_0 c_2 e^{\lambda_0 t} + \lambda_0^2 c_2 te^{\lambda_0 t} \\
&= \lambda_0^2 c_1 e^{\lambda_0 t} + 2\lambda_0 c_2 e^{\lambda_0 t} + \lambda_0^2 c_2 te^{\lambda_0 t} \\
\frac{d^2y}{dt^2} - 2\lambda_0\frac{dy}{dt} + \lambda_0^2 y &= \lambda_0^2 c_1 e^{\lambda_0 t} + 2\lambda_0 c_2 e^{\lambda_0 t} + \lambda_0^2 c_2 te^{\lambda_0 t} - 2\lambda_0(\lambda_0 c_1 e^{\lambda_0 t} + c_2 e^{\lambda_0 t} + \lambda_0 c_2 te^{\lambda_0 t}) + \lambda_0^2(c_1e^{\lambda_0 t} + c_2te^{\lambda_0 t}) \\
&= c_1e^{\lambda_0 t}(\lambda_0^2 - 2\lambda_0^2 + \lambda_0^2) + c_2e^{\lambda_0 t}(2\lambda_0 - 2\lambda_0) + c_2te^{\lambda_0 t}(\lambda_0^2 - 2\lambda_0^2 + \lambda_0^2) \\
&= 0
\end{aligned}
$$

(c) Making the same assumption as part (b), show that we can always find constants c_1, c_2 such that we can satisfy initial conditions $y(0) = y_0$, $\frac{dy}{dt}(0) = y_0'$

Solution:

$$
\begin{aligned}
y(0) = y_0 &= c_1e^0 + c_2(0)e^0 \\
&= c_1 \\
\frac{dy}{dt}(0) = y_0' &= \lambda_0 c_1 e^0 + c_2e^0 + \lambda_0 c_2(0)e^0 \\
&= \lambda_0 c_1 + c_2
\end{aligned}
$$

We have two linearly independent equations and we're solving for two variables, so we should always reach a unique solution.

3. Fun with Inductors

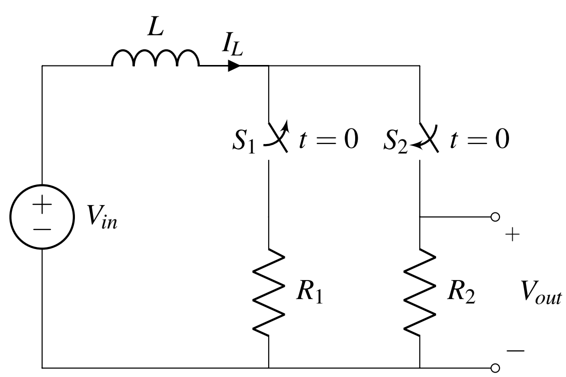


Figure 1: Circuit A

(a) Consider circuit A. Assuming that for $t < 0$, switch S_1 is on and switch S_2 is off (and both switches have been in these states indefinitely), what is $i_L(0)$?

Solution: When S_1 is on and S_2 is off for a long period of time, $\frac{di_L}{dt} = 0$ because the circuit will have reached a steady state, and the current through R_1 will be equal to i_L. We find

$$V_{in} - V_L - V_{R_1} = 0$$

$$V_{in} - L\frac{di_L}{dt}(0) - i_L(0)R_1 = 0$$

$$V_{in} - i_L(0)R_1 = 0$$

$$i_L(0) = \frac{V_{in}}{R_1}$$

(b) Now let's assume that for $t \geq 0$, S_1 is off and S_2 is on. Solve for $V_{out}(t)$ for $t \geq 0$.

Solution:

$$V_{in} - V_L - V_{out} = 0$$

$$V_{in} - L\frac{di_L}{dt} - i_L R_2 = 0$$

$$\frac{di_L}{dt} + \frac{R_2}{L}i_L = \frac{V_{in}}{L}$$

This is a non-homogenous first order differential equation in i_L. We can solve for $i_L(t)$ and then use Ohm's law to find $V_{out}(t)$ after this has been solved.

$$\frac{di_L}{dt} + \frac{R_2}{L}(i_L - \frac{V_{in}}{R_2}) = 0$$

Let $\tilde{i_L} = i_L - \frac{V_{in}}{R_2}$. We now have:

$$\frac{d\tilde{i}_L}{dt} + \frac{R_2}{L}\tilde{i}_L = 0$$

The general solution is given by:

$$\tilde{i}_L(t) = c_1 e^{-\frac{R_2}{L}t}$$

Resubstituting back i_L, we have:

$$i_L(t) = \frac{V_{in}}{R_2} + c_1 e^{-\frac{R_2}{L}t}$$

Applying initial conditions, we know:

$$i_L(0) = \frac{V_{in}}{R_2} + c_1 = \frac{V_{in}}{R_1}$$

$$c_1 = \frac{V_{in}}{R_1} - \frac{V_{in}}{R_2}$$

Our solution for $i_L(t)$ thus becomes:

$$i_L(t) = \frac{V_{in}}{R_2} + (\frac{V_{in}}{R_1} - \frac{V_{in}}{R_2})e^{-\frac{R_2}{L}t}$$

Since $V_{out}(t) = i_L(t)R_2$,

$$V_{out}(t) = V_{in}(1 + (\frac{R_2}{R_1} - 1)e^{-\frac{R_2}{L}t})$$

(c) If $V_{in} = 1V$, $L = 1nH$, $R_1 = 1k\Omega$, and $R_2 = 10k\Omega$, what is the maximum value of $V_{out}(t)$ for $t \geq 0$?

Solution: Since the coefficient in front of our time-varying component $e^{-\frac{R_2}{L}t}$, given by $\frac{R_2}{R_1} - 1 = 9$, is positive, $V_{out}(t)$ undergoes decay over time. Therefore, the maximum value is achieved at $t = 0$:

$$\max V_{out}(t) = V_{out}(0) = \frac{R_2}{R_1} V_{in} = 10V$$

(d) In general, if we want $\max V_{out}(t)$ to be greater than V_{in}, what relationship needs to be maintained between the values of R_1 and R_2?

Solution: As long as the coefficient on our exponential term, given by $\frac{R_2}{R_1} - 1$, is greater than 0 (i.e. when $\frac{R_2}{R_1} > 1$) then the maximum value of $V_{out}(t)$ will be achieved at $t = 0$ and will have a value of $\frac{R_2}{R_1} V_{in} > V_{in}$. Otherwise, if $\frac{R_2}{R_1} \leq 1$, the maximum value of $V_{out}(t)$ is reached at $t = \infty$, where $V_{out} = V_{in}$ regardless of R_2 and R_1. Therefore, our necessary condition for the maximum of V_{out} to be greater than V_{in} is:

$$R_2 > R_1$$

(e) Now assume that at time $t = t_1$, switch S_2 was turned off, and switch S_1 was turned back on. Solve for $i_L(t)$ for $t > t_1$. If $R_2 > R_1$, how does this $i_L(t)$ for $t > t_1$ compare with the initial condition $i_L(0)$ you found in part (a)?

Solution: Our new initial condition for $t > t_1$ is given by plugging in $t = t_1$ into the equation for $i_L(t)$ we found in part (b). Thus, $i_L(t_1) = \frac{V_{in}}{R_2} + (\frac{V_{in}}{R_1} - \frac{V_{in}}{R_2})e^{-\frac{R_2}{L}t_1}$.

We can write the relationship between the current through the inductor and the current through R_1:

$$i_L = i_{R_1}$$

$$i_L = \frac{V_{R_1}}{R_1}$$

$$i_L = \frac{V_{in} - V_L}{R_1}$$

$$i_L = \frac{V_{in}}{R_1} - \frac{L\frac{di_L}{dt}}{R_1}$$

$$\frac{di_L}{dt} + \frac{R_1}{L} i_L = \frac{V_{in}}{L}$$

This is a first order non-homogeneous differential equation similar to that found in part (b), except with R_1 in place of R_2. Following those steps in part (b), we find the general solution:

$$i_L(t) = \frac{V_{in}}{R_1} + c_1 e^{-\frac{R_1}{L}t}$$

To find c_1 we apply our initial condition:

$$i_L(t_1) = \frac{V_{in}}{R_1} + c_1 e^{-\frac{R_1}{L}t_1} = \frac{V_{in}}{R_2} + (\frac{V_{in}}{R_1} - \frac{V_{in}}{R_2})e^{-\frac{R_2}{L}t_1}$$

$$c_1 e^{-\frac{R_1}{L}t_1} = (\frac{V_{in}}{R_2} - \frac{V_{in}}{R_1})(1 - e^{-\frac{R_2}{L}t_1})$$

$$c_1 = \frac{(\frac{V_{in}}{R_2} - \frac{V_{in}}{R_1})(1 - e^{-\frac{R_2}{L}t_1})}{e^{-\frac{R_1}{L}t_1}}$$

$$c_1 = (\frac{V_{in}}{R_2} - \frac{V_{in}}{R_1})(e^{\frac{R_1}{L}t_1} - e^{\frac{R_1-R_2}{L}t_1})$$

Thus, we have

$$i_L(t) = \frac{V_{in}}{R_1} + (\frac{V_{in}}{R_2} - \frac{V_{in}}{R_1})(e^{\frac{R_1}{L}t_1} - e^{\frac{R_1-R_2}{L}t_1})e^{-\frac{R_1}{L}t}$$

for $t > t_1$. We also see that as $t \to \infty$, $i_L(t)$ for $t > t_1$ becomes:

$$i_L(t = \infty) = \frac{V_{in}}{R_1} + (\frac{V_{in}}{R_2} - \frac{V_{in}}{R_1})(e^{\frac{R_1}{L}t_1} - e^{\frac{R_1-R_2}{L}t_1})e^{-\frac{R_1}{L}\infty}$$

$$i_L(t = \infty) = \frac{V_{in}}{R_1} + (\frac{V_{in}}{R_2} - \frac{V_{in}}{R_1})(e^{\frac{R_1}{L}t_1} - e^{\frac{R_1-R_2}{L}t_1})(0)$$

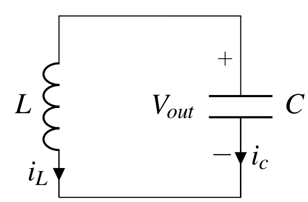
$$i_L(t = \infty) = \frac{V_{in}}{R_1} = i_L(0)$$

Thus, if we turn S_2 back off and S_1 back on as was described in this part, we will eventually revert back to the initial state from which we started! Specifically, if $R_2 > R_1$, $i_L(t)$ at $t = t_1$ will be less than our initial condition $i_L(0)$, and $i_L(t)$ will rise to $i_L(0)$ over time.

4. Oscillators

In this question, we'll be looking at an oscillator circuit. There are many types of oscillators, but this circuit is known as an LC tank. It's called an oscillator because the circuit produces a repetitive voltage waveform under the right initial conditions.

In this circuit, we have an inductor $L = 10\text{nH}$ and capacitor $C = 10\text{pF}$ in parallel:



(a) If $i_L(0) = 0\text{A}$ and $V_{out}(0) = 0\text{V}$, derive an expression for $V_{out}(t)$ for $t \geq 0$. Use V_{out} and i_L as your state variables.

Solution:

Since the inductor and capacitor are in parallel:

$$V_L = V_c = V_{out}$$

KCL gives:

$$i_L = -i_c = -C\frac{dV_{out}}{dt}$$

$$\frac{dV_{out}}{dt} = -\frac{1}{C}i_L$$

$$V_L = V_{out} = L\frac{di_L}{dt}$$
$$\frac{di_L}{dt} = \frac{1}{L}V_{out}$$

Putting it into matrix form:

$$\begin{bmatrix} \frac{dV_{out}}{dt} \\ \frac{di_L}{dt} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{C} \\ \frac{1}{L} & 0 \end{bmatrix} \begin{bmatrix} V_{out} \\ i_L \end{bmatrix}$$

Finding the eigenvalues:

$$\det\left(\begin{bmatrix} -\lambda & -\frac{1}{C} \\ \frac{1}{L} & -\lambda \end{bmatrix}\right) = \lambda^2 + \frac{1}{LC} = 0$$
$$\lambda_{1,2} = 0 \pm j\frac{1}{\sqrt{LC}}$$

The eigenvalues are purely imaginary, so the solution to the differential equation takes the form:

$$V_{out}(t) = c_1 \cos\left(\frac{1}{\sqrt{LC}}t\right) + c_2 \sin\left(\frac{1}{\sqrt{LC}}t\right)$$

We can find c_1 and c_2 with initial conditions for $V_{out}(0)$ and $\frac{dV_{out}(0)}{dt}$

$$V_{out}(0) = 0 = c_1$$
$$\frac{dV_{out}}{dt} = -\frac{c_1}{\sqrt{LC}} \sin\left(\frac{1}{\sqrt{LC}}t\right) + \frac{c_2}{\sqrt{LC}} \cos\left(\frac{1}{\sqrt{LC}}t\right)$$
$$\frac{dV_{out}(0)}{dt} = \frac{1}{C}i_c(0) = 0 = \frac{c_2}{\sqrt{LC}}$$
$$c_1 = c_2 = 0$$
$$V_{out}(t) = 0$$

(b) Now let's see how the circuit reacts with non-zero initial current. If $i_L(0) = 50\mu$A and $V_{out}(0) = 0$V, derive an expression for $V_{out}(t)$ for $t \geq 0$. How does the amplitude of V_{out} change over time?

Solution:

The only thing that changed from part (a) was the initial conditions, which means we still have:

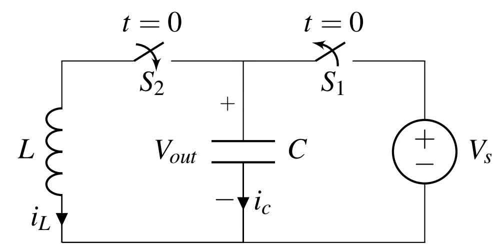
$$V_{out}(t) = c_1 \cos\left(\frac{1}{\sqrt{LC}}t\right) + c_2 \sin\left(\frac{1}{\sqrt{LC}}t\right)$$

Plugging in initial conditions:

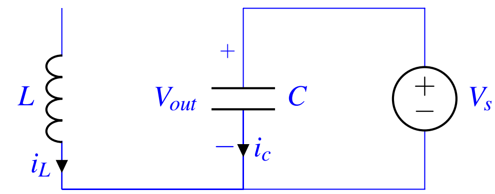
$$V_{out}(0) = 0 = c_1$$
$$i_c(0) = -i_L(0) = -50 \times 10^{-6}$$
$$\frac{dV_{out}(0)}{dt} = \frac{1}{C}i_c(0) = \frac{-50 \times 10^{-6}}{10^{-11}} = \frac{c_2}{\sqrt{10^{-8} \times 10^{-11}}}$$
$$c_1 = 0$$
$$c_2 = -\frac{5}{\sqrt{10}} \times 10^{-3} = -5\sqrt{10} \times 10^{-4}$$
$$V_{out}(t) = \left(-5\sqrt{10} \times 10^{-4}\right) \sin\left(\sqrt{10} \times 10^{9} t\right)$$

Notice that the amplitude of V_{out} is constant.

(c) In order to ensure an initial condition where we get non-zero output, some switches and a voltage source have been added to the circuit. For $t \leq 0$, switch S_1 is on while S_2 is off. Find $V_c(0)$ and $i_L(0)$. Use component values $V_s = 3\text{V}$, $L = 10\text{nH}$ and $C = 10\text{pF}$.



Solution: For $t \leq 0$, the circuit looks like:



There is no path for current to flow through the inductor, so

$$i_L(0) = 0\text{A}$$

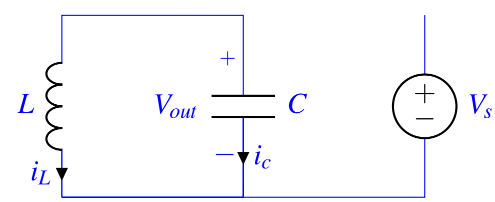
Since the capacitor is in parallel with V_s:

$$V_{out}(0) = V_s = 3\text{V}$$

(d) At $t = 0$, the switches flip state (S_1 turns off and S_2 turns on). Derive an expression for $V_{out}(t)$ for $t \geq 0$. Use the same component values as part (c).

Solution:

For $t \geq 0$, the circuit looks like:



The voltage source is in series with an open circuit, so it doesn't affect the circuit. This means the circuit is the same as in part (a), so we can say:

$$V_{out}(t) = c_1 \cos\left(\frac{1}{\sqrt{LC}}t\right) + c_2 \sin\left(\frac{1}{\sqrt{LC}}t\right)$$

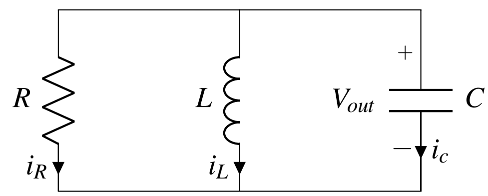
Plugging in initial conditions:

$$V_{out}(0) = 3 = c_1$$

$$\frac{dV_{out}(0)}{dt} = \frac{1}{C}i_c(0) = 0 = \frac{c_2}{\sqrt{10^{-8} \times 10^{-11}}}$$

$$c_1 = 3$$
$$c_2 = 0$$
$$V_{out}(t) = 3\cos\left(\sqrt{10}\times 10^9 t\right)$$

(e) Let's see what happens when there is a parasitic resistance R in parallel with the LC tank.



If $i_L(0) = 0\mu\text{A}$ and $V_{out}(0) = 3\text{V}$, derive an expression for $V_{out}(t)$ for $t \geq 0$. Use component values $C = 10\text{pF}$, $L = 10\text{nH}$, and $R = 100\text{k}\Omega$. How does the amplitude of V_{out} change over time?

Solution: For this problem, we'll use the same state variables as part (a), V_{out} and i_L.
Since all three components are in parallel, we can say:

$$V_R = V_L = V_c = V_{out}$$

To solve the differential equation, we need to get $\frac{dV_{out}}{dt}$ and $\frac{di_L}{dt}$ in terms of V_{out} and i_L

$$C\frac{dV_{out}}{dt} = i_c$$

Using KCL:

$$i_R + i_L + i_c = 0$$
$$\frac{V_{out}}{R} + i_L + C\frac{dV_{out}}{dt} = 0$$
$$\frac{dV_{out}}{dt} = -\frac{1}{RC}V_{out} - \frac{1}{C}i_L$$
$$V_L = V_{out} = L\frac{di_L}{dt}$$
$$\frac{di_L}{dt} = \frac{1}{L}V_{out}$$

Putting into matrix form:

$$\begin{bmatrix} \frac{dV_{out}}{dt} \\ \frac{di_L}{dt} \end{bmatrix} = \begin{bmatrix} -\frac{1}{RC} & -\frac{1}{C} \\ \frac{1}{L} & 0 \end{bmatrix} \begin{bmatrix} V_{out} \\ i_L \end{bmatrix}$$

Finding eigenvalues:

$$\det\left(\begin{bmatrix} -\frac{1}{RC} - \lambda & -\frac{1}{C} \\ \frac{1}{L} & -\lambda \end{bmatrix}\right) = \lambda^2 + \frac{1}{RC}\lambda + \frac{1}{LC} = 0$$

$$\lambda_{1,2} = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

$$\lambda_{1,2} = -\frac{1}{2 \times 10^5 \times 10^{-11}} \pm \sqrt{\frac{1}{4(10^5 \times 10^{-11})^2} - 10^{19}}$$

$$\lambda_{1,2} = -5 \times 10^5 \pm \sqrt{25 \times 10^{10} - 10^{19}}$$

Since $25 \times 10^{10} << 10^{19}$, we can estimate the square root term as $\sqrt{-10^{19}}$:

$$\lambda_{1,2} = -5 \times 10^5 \pm j\sqrt{10} \times 10^9$$

$$V_{out}(t) = e^{-5\times 10^5 t}\left(c_1 \cos(\sqrt{10} \times 10^9 t) + c_2 \sin(\sqrt{10} \times 10^9 t)\right)$$

Plugging in initial conditions:

$$V_{out}(0) = 3 = c_1$$

$$i_L(0) + i_c(0) + i_R(0) = 0$$

$$0 + C\frac{dV_{out}(0)}{dt} + \frac{V_{out}(0)}{R} = 0$$

$$\frac{dV_{out}(0)}{dt} = -\frac{3}{10^5 \times 10^{-11}} = -3 \times 10^6$$

$$\begin{aligned}\frac{dV_{out}}{dt} &= (-5 \times 10^5)e^{-5\times 10^5 t}\left(c_1 \cos(\sqrt{10} \times 10^9 t) + c_2 \sin(\sqrt{10} \times 10^9 t)\right) \\ &\quad + e^{-5\times 10^5 t}\left(-(\sqrt{10} \times 10^9)c_1 \sin(\sqrt{10} \times 10^9 t) + (\sqrt{10} \times 10^9)c_2 \cos(\sqrt{10} \times 10^9 t)\right) \qquad (1)\end{aligned}$$

$$\frac{dV_{out}(0)}{dt} = -3 \times 10^6 = (-5 \times 10^5)c_1 + (\sqrt{10} \times 10^9)c_2$$

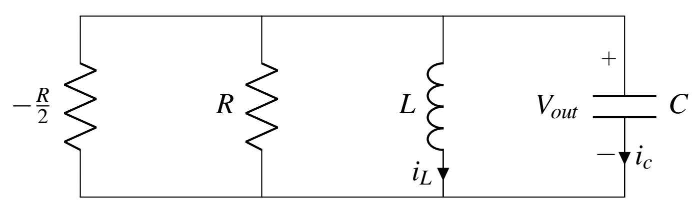
$$c_2 = \frac{-15 \times 10^{-4}}{\sqrt{10}} = -15\sqrt{10} \times 10^{-5}$$

This gives us:

$$V_{out}(t) = e^{-5\times 10^5 t}\left(3\cos(\sqrt{10} \times 10^9 t) - 15\sqrt{10} \times 10^{-5} \sin(\sqrt{10} \times 10^9 t)\right)$$

The amplitude of V_{out} decays over time. This is due to energy being dissipated across the resistor as energy transitions between the inductor and capacitor.

(f) In order to counteract the parasitic resistance, we create a negative resistance (which can be done using transistors, but will not be covered in this class) in parallel with the other components:



If $i_L(0) = 0\mu$A and $V_{out}(0) = 3$V, Derive an expression for $V_{out}(t)$ for $t \geq 0$. Use component values $C = 10$pF, $L = 10$nH, and $R = 100$kΩ. How does the amplitude of V_{out} change over time?

Solution:

Since the negative resistance and positive resistance are in parallel, we can combine the two:

$$R_{eq} = \left(R || \frac{-R}{2}\right) = \frac{R\frac{-R}{2}}{R + \frac{-R}{2}} = -R$$

Now we have the same circuit as part (e), but with R as negative. We can use the previous equations from part (e), and just plug in $-R$ for R:

$$\lambda_{1,2} = -\frac{1}{2(-R)C} \pm \sqrt{\left(\frac{1}{2(-R)C}\right)^2 - \frac{1}{LC}}$$

$$\lambda_{1,2} = 5 \times 10^5 \pm \sqrt{10} \times 10^9$$

$$V_{out}(t) = e^{5\times10^5 t}\left(c_1 \cos(\sqrt{10} \times 10^9 t) + c_2 \sin(\sqrt{10} \times 10^9 t)\right)$$

Plugging in initial conditions:

$$V_{out}(0) = 3 = c_1$$

$$i_L(0) + i_c(0) + i_R(0) = 0$$

$$0 + C\frac{dV_{out}(0)}{dt} + \frac{V_{out}(0)}{-R} = 0$$

$$\frac{dV_{out}(0)}{dt} = \frac{3}{10^5 \times 10^{-11}} = 3 \times 10^6$$

$$\begin{aligned}\frac{dV_{out}}{dt} = {} & (5 \times 10^5)e^{5\times10^5 t}\left(c_1 \cos(\sqrt{10} \times 10^9 t) + c_2 \sin(\sqrt{10} \times 10^9 t)\right) \\ & + e^{5\times10^5 t}\left(-(\sqrt{10} \times 10^9)c_1 \sin(\sqrt{10} \times 10^9 t) + (\sqrt{10} \times 10^9)c_2 \cos(\sqrt{10} \times 10^9 t)\right) \quad (2)\end{aligned}$$

$$\frac{dV_{out}(0)}{dt} = 3 \times 10^6 = (5 \times 10^5)c_1 + (\sqrt{10} \times 10^9)c_2$$

$$c_2 = \frac{15 \times 10^{-4}}{\sqrt{10}} = 15\sqrt{10} \times 10^{-5}$$

This gives us:

$$V_{out}(t) = e^{5\times10^5 t}\left(3\cos(\sqrt{10} \times 10^9 t) + 15\sqrt{10} \times 10^{-4} \sin(\sqrt{10} \times 10^9 t)\right)$$

The amplitude of V_{out} grows over time.

(g) (**BONUS**)
What value should the negative resistor have if we want to maintain a constant amplitude on $V_{out}(t)$?

Solution:

We saw in part (b) that the amplitude of V_{out} is constant when the only connection to the LC circuit is through an open circuit, i.e. when $R_{eq} = \infty$. For calculation purposes, we'll call the negative resistor R_n.

$$R_{eq} = \infty = \frac{RR_n}{R + R_n}$$

For R_{eq} to go to infinity, then we need

$$R_n = -R$$

5. Write Your Own Question And Provide a Thorough Solution.

Writing your own problems is a very important way to really learn material. The famous "Bloom's Taxonomy" that lists the levels of learning is: Remember, Understand, Apply, Analyze, Evaluate, and Create. Using what you know to create is the top level. We rarely ask you any homework questions about the lowest level of straight-up remembering, expecting you to be able to do that yourself (e.g. making flashcards). But we don't want the same to be true about the highest level. As a practical matter, having some practice at trying to create problems helps you study for exams much better than simply counting on solving existing practice problems. This is because thinking about how to create an interesting problem forces you to really look at the material from the perspective of those who are going to create the exams. Besides, this is fun. If you want to make a boring problem, go ahead. That is your prerogative. But it is more fun to really engage with the material, discover something interesting, and then come up with a problem that walks others down a journey that lets them share your discovery. You don't have to achieve this every week. But unless you try every week, it probably won't ever happen.

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