EE16B
Designing Information Devices and Systems II

Lecture 12A
LTI Systems, Convolution sum
Finite sequences

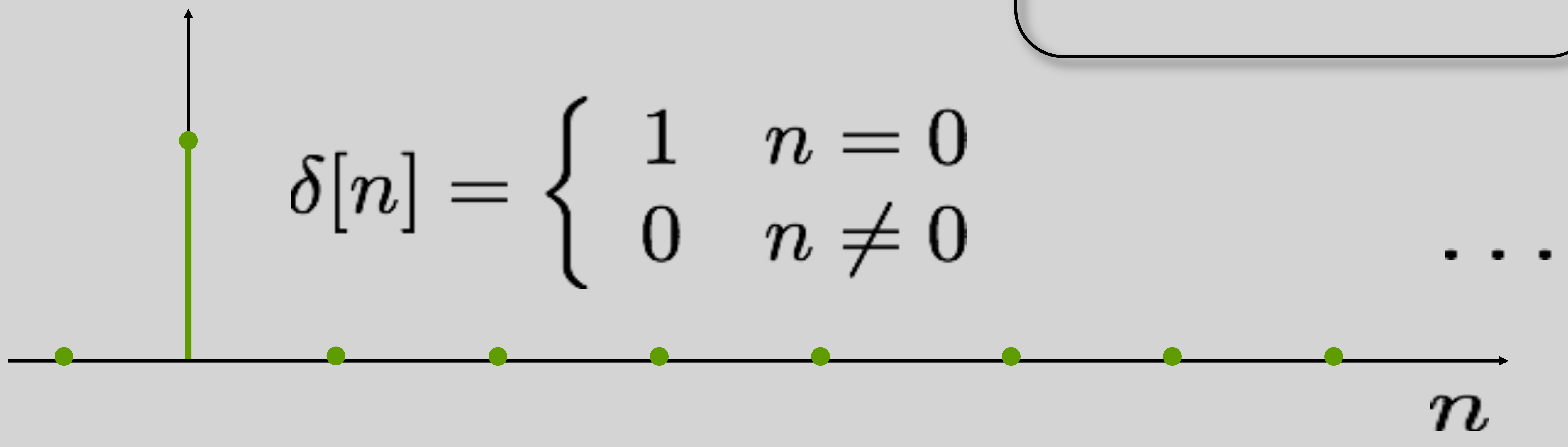
Intro

- Last time:
  - Discrete Systems
  - LTI Systems

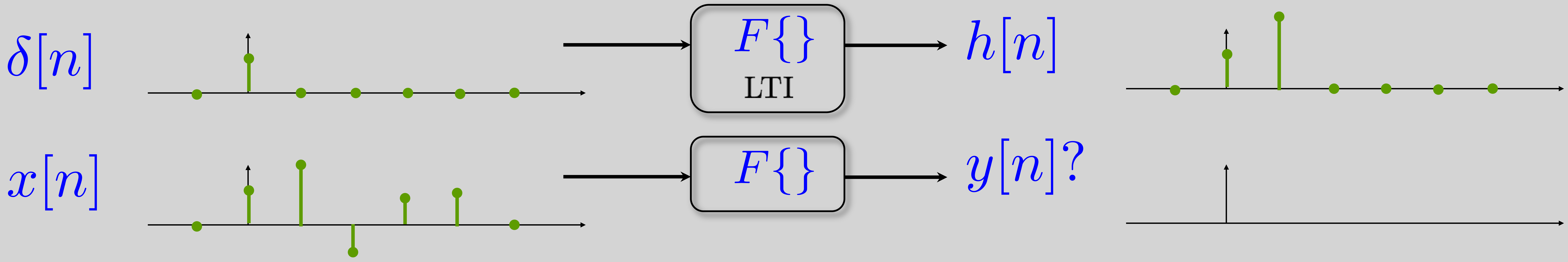
- Today
  - LTI Systems
  - Convolution sum
  - Finite sequences as vectors

Linear Time Invariant Systems

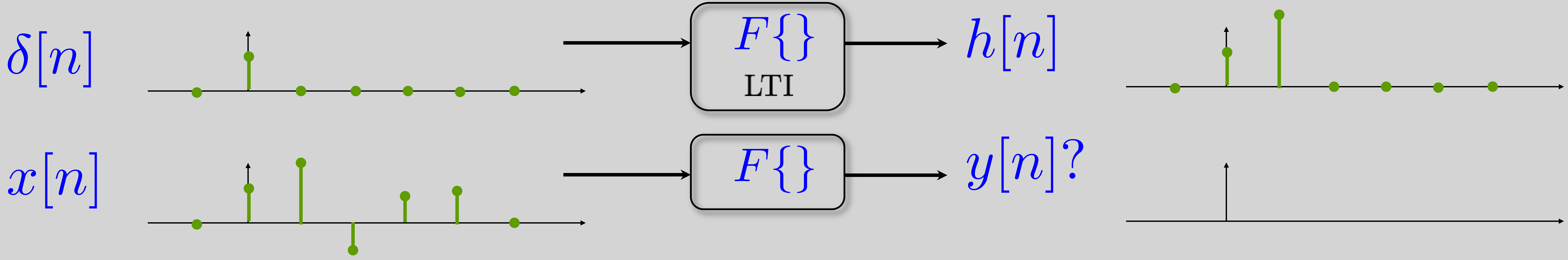
- Linear Time/Shift Invariant (LTI/LSI) systems are completely characterized by their impulse response h[n]



$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$



$\delta[n]$
$F\{\}$
LTI
$h[n]$
$x[n]$
$F\{\}$
$y[n]?$



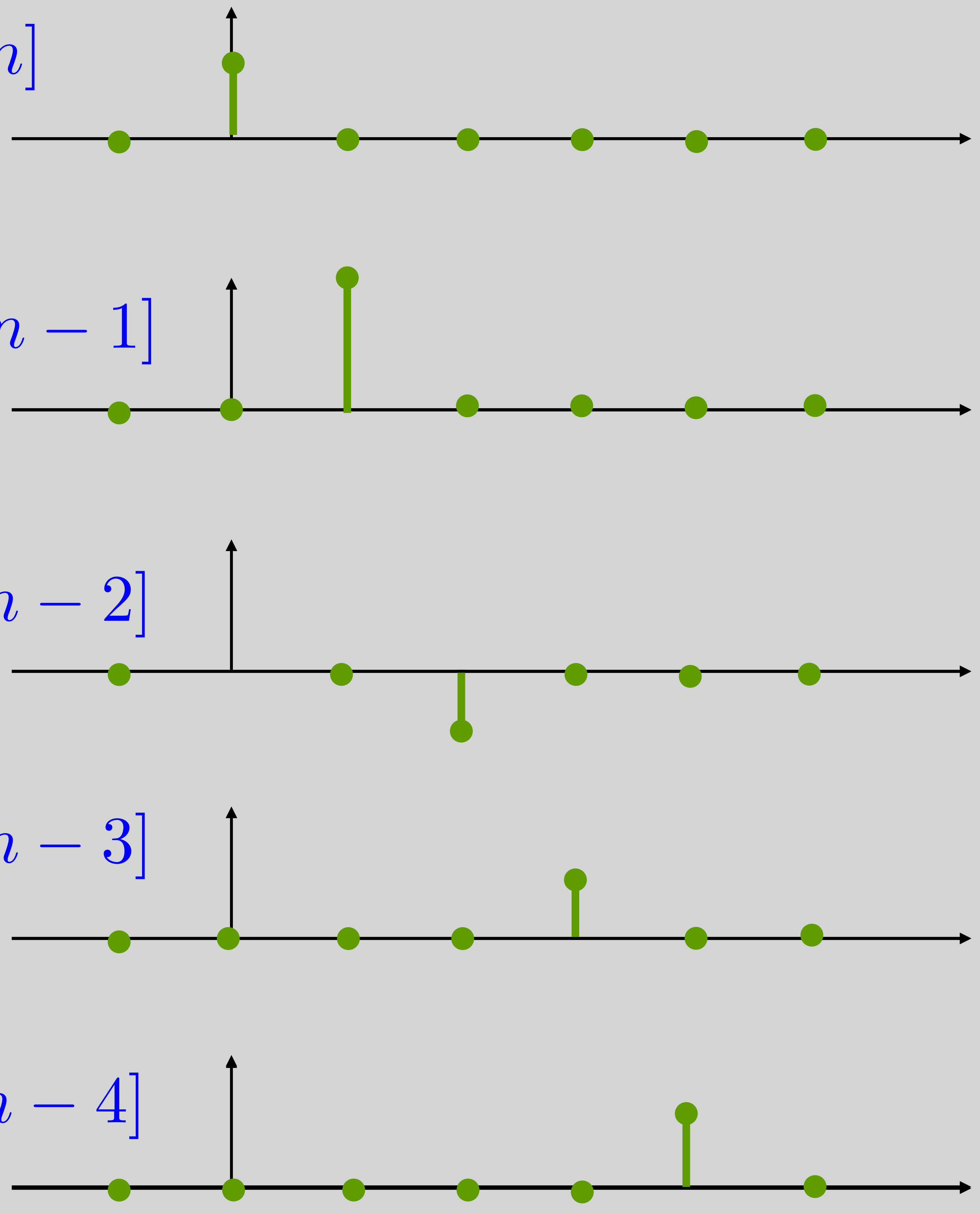
$x_0[n] = x[0]\delta[n]$

$x_1[n] = x[1]\delta[n-1]$

$x_2[n] = x[2]\delta[n-2]$

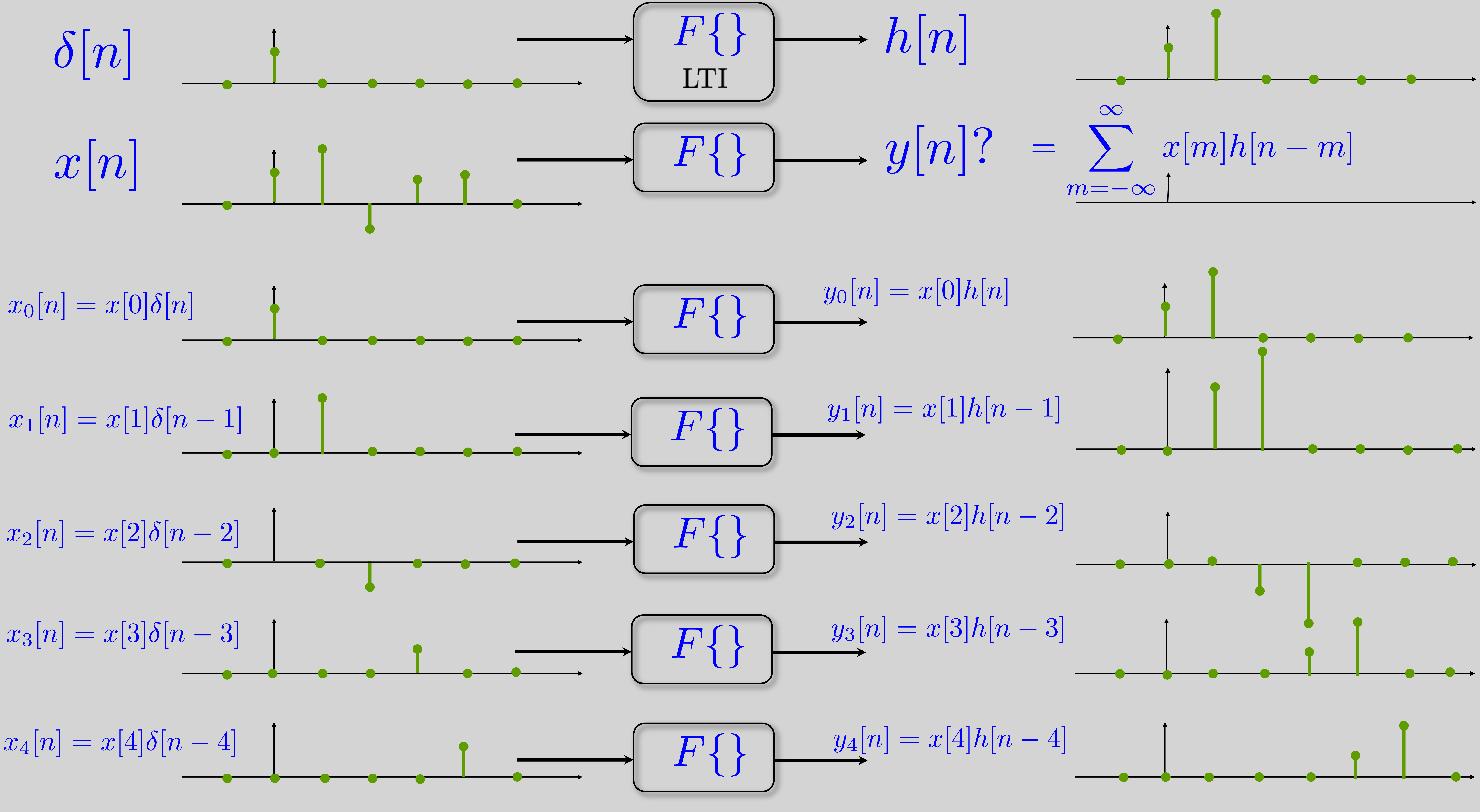
$x_3[n] = x[3]\delta[n-3]$

$x_4[n] = x[4]\delta[n-4]$



$$x[n] = \sum_{m=-\infty}^{\infty} x_m[n]$$

$$= \sum_{m=-\infty}^{\infty} x[m]\delta[n-m]$$


$\delta[n]$
$F\{\}$
LTI
$h[n]$
$x[n]$
$F\{\}$
$y[n]? = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$
$x_0[n] = x[0]\delta[n]$
$F\{\}$
$y_0[n] = x[0]h[n]$
$x_1[n] = x[1]\delta[n-1]$
$F\{\}$
$y_1[n] = x[1]h[n-1]$
$x_2[n] = x[2]\delta[n-2]$
$F\{\}$
$y_2[n] = x[2]h[n-2]$
$x_3[n] = x[3]\delta[n-3]$
$F\{\}$
$y_3[n] = x[3]h[n-3]$
$x_4[n] = x[4]\delta[n-4]$
$F\{\}$
$y_4[n] = x[4]h[n-4]$

Linear Time Invariant Systems

$$\delta[n] \longrightarrow \boxed{\text{LTI}} \longrightarrow h[n]$$

- Decompose x[n]:

$$x[n] = \sum_{m=-\infty}^{\infty} x[m]\delta[n-m]$$

$$= \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases}$$

- Compute output:

Convolution sum

$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m] \quad = x[n] * h[n]$$

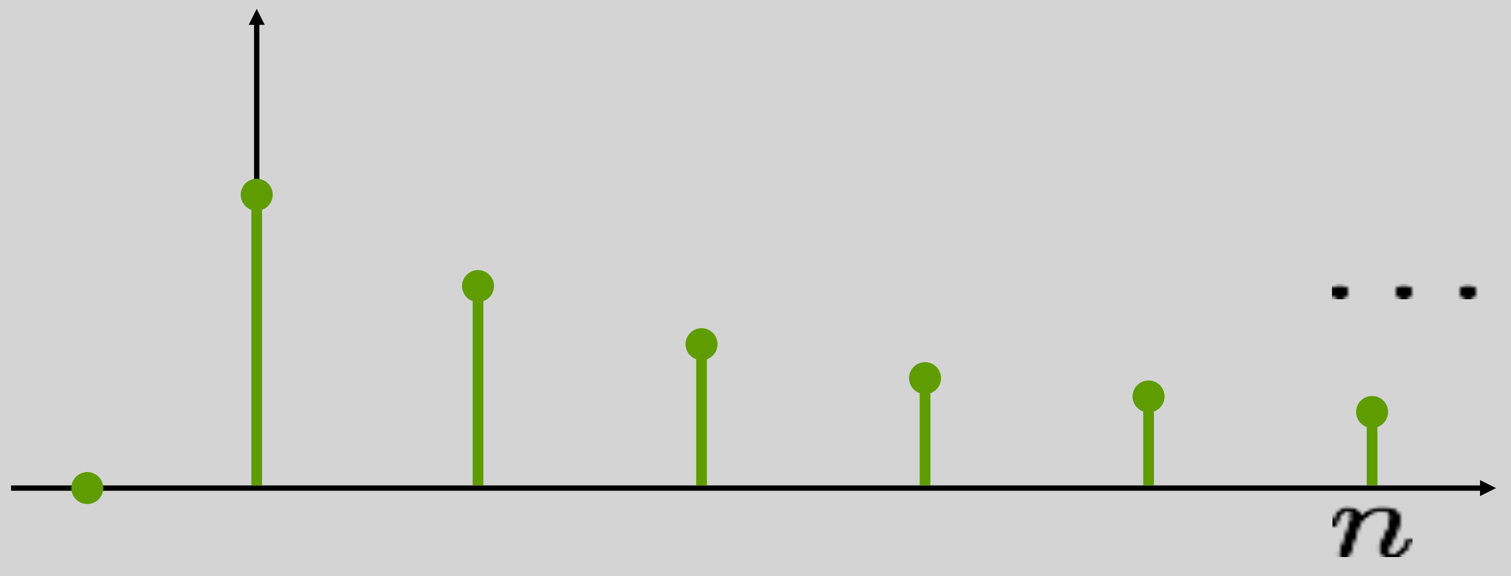
Sum of weighted, delayed impulse responses!

Example:

$$y[n] = ay[n-1] + x[n]$$

$$h[n] = \begin{cases} a^n & n \geq 0 \\ 0 & n < 0 \end{cases}$$

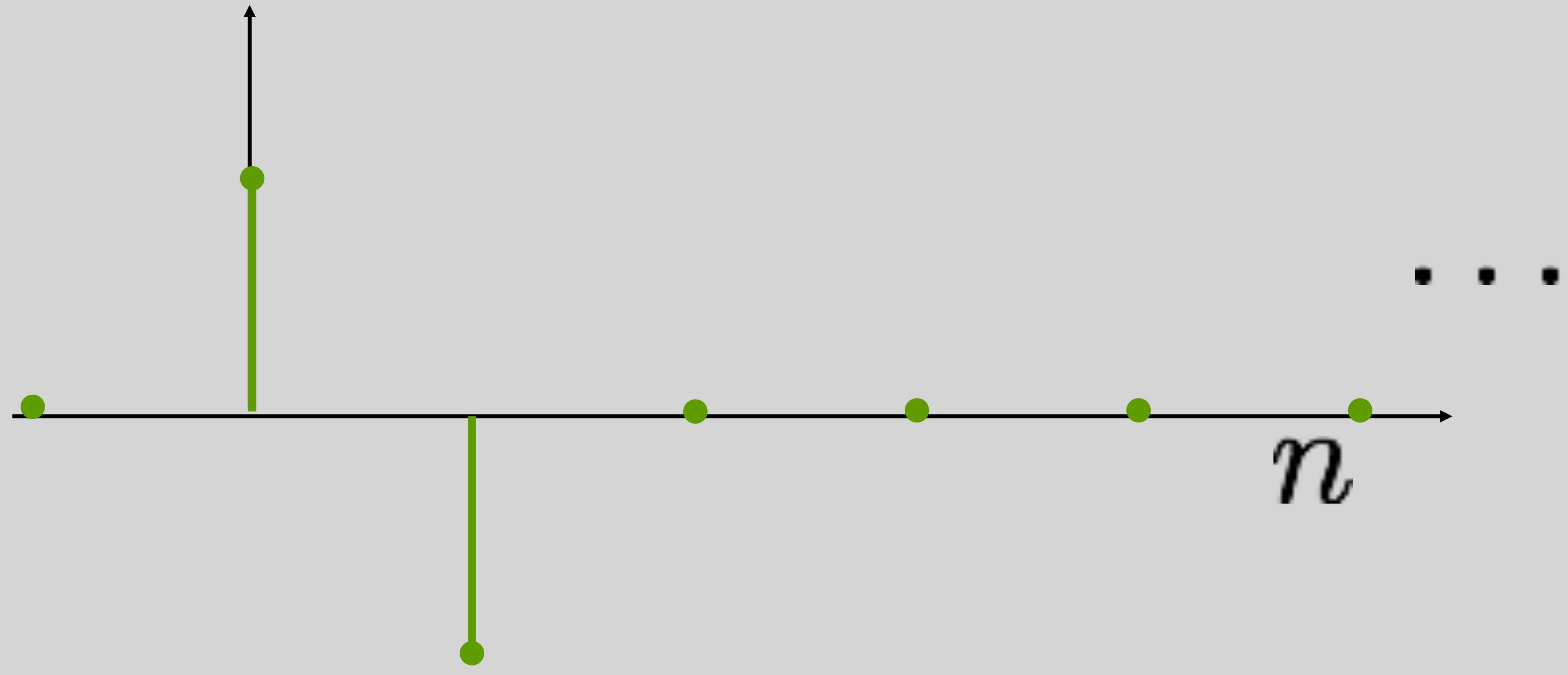
Infinite impulse response (IIR)



$$y[n] = x[n] - x[n-1]$$

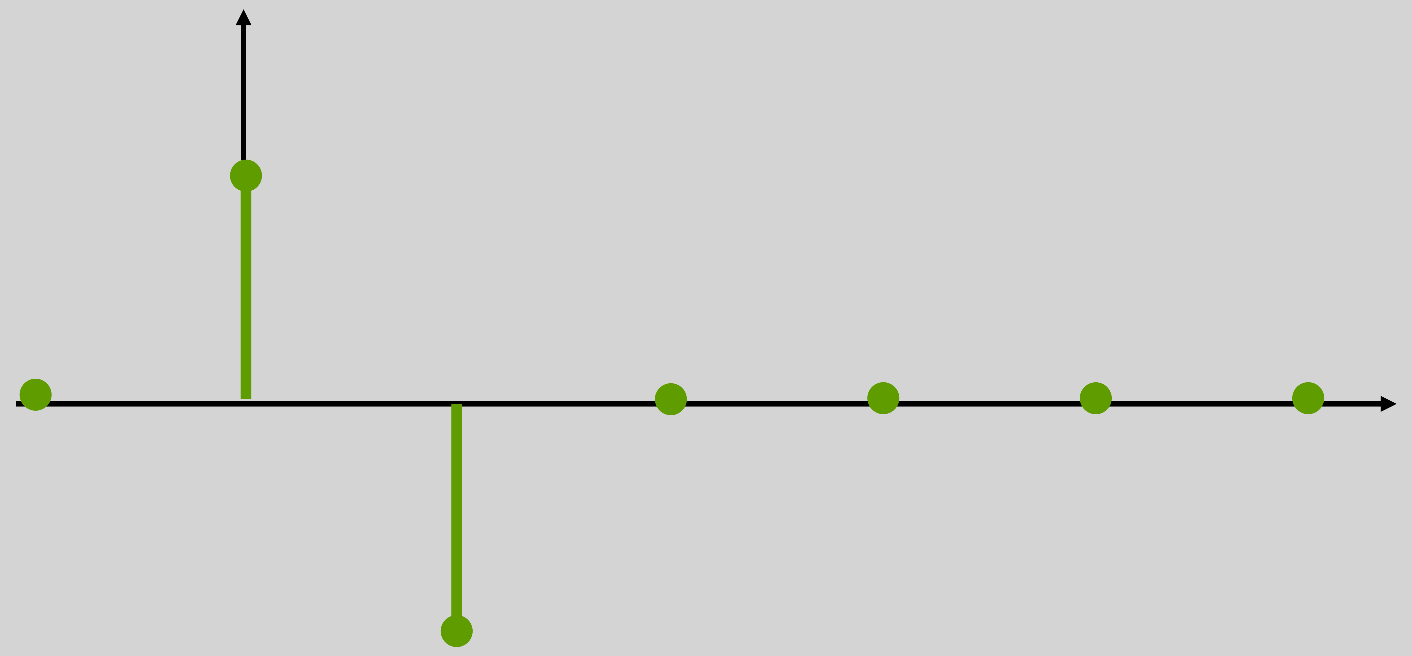
$$h[n] = \delta[n] - \delta[n-1]$$

finite impulse response (FIR)



Convolution Sum

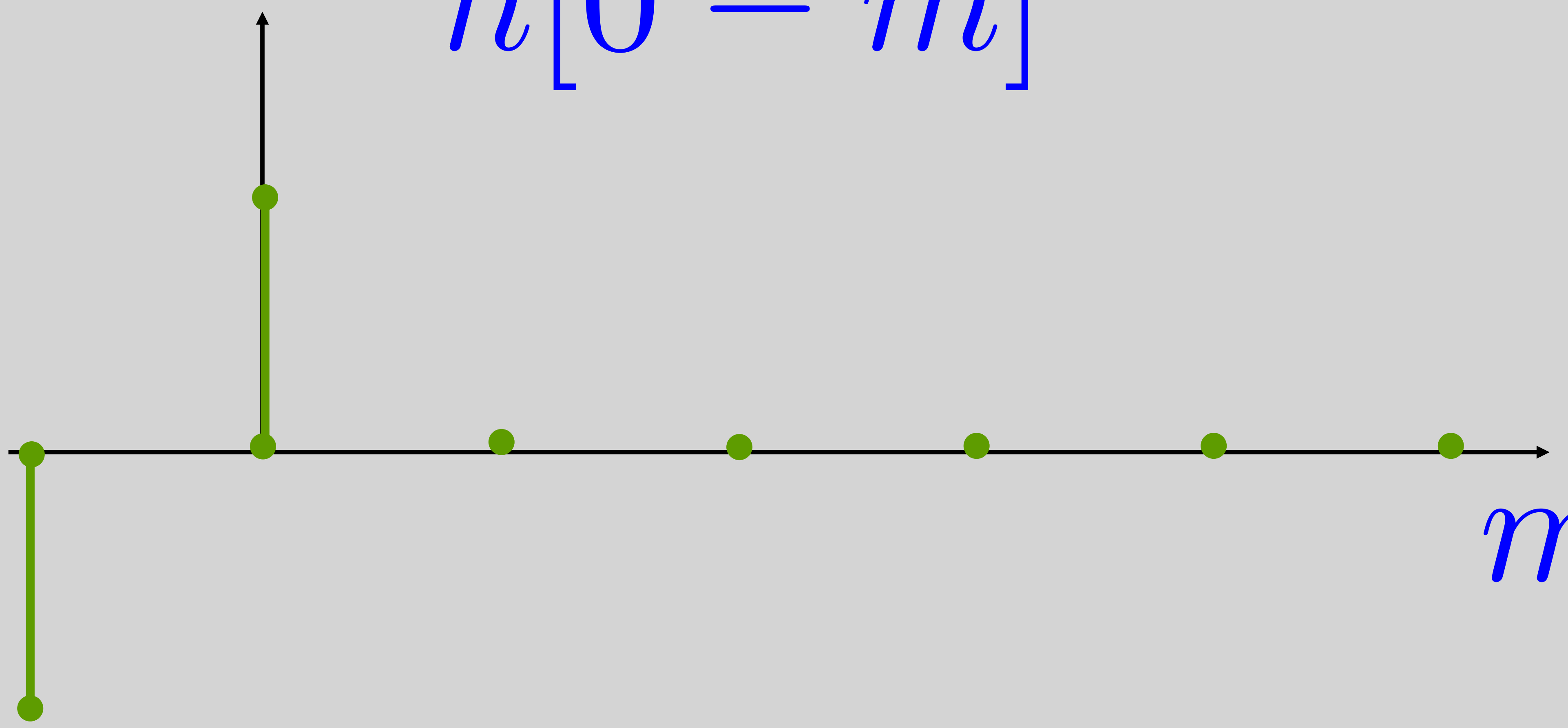
$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m] = x[n] * h[n]$$



$$h[n] = \delta[n] - \delta[n-1]$$

- What is h[n-m] for different n's?

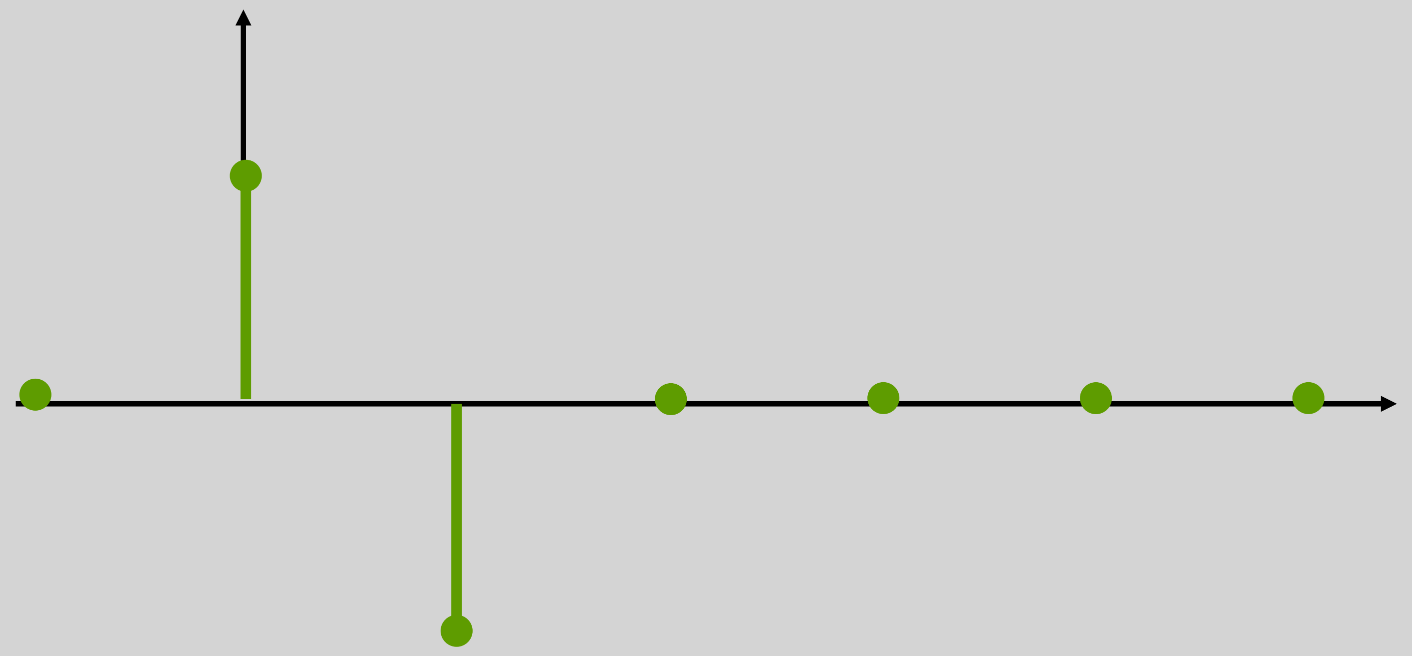
$$h[0-m]$$



m

Convolution Sum

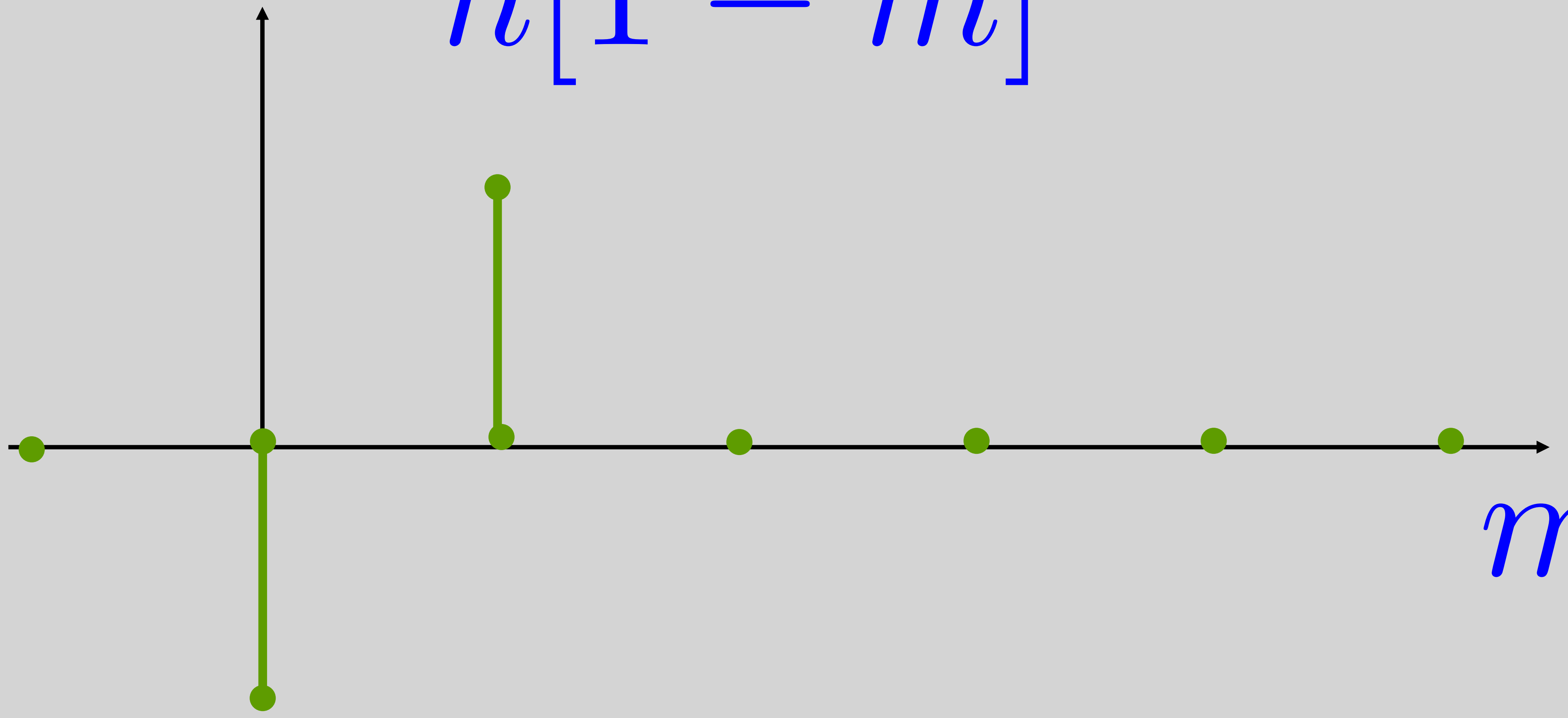
$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m] = x[n] * h[n]$$



$$h[n] = \delta[n] - \delta[n-1]$$

- What is h[n-m] for different n's?

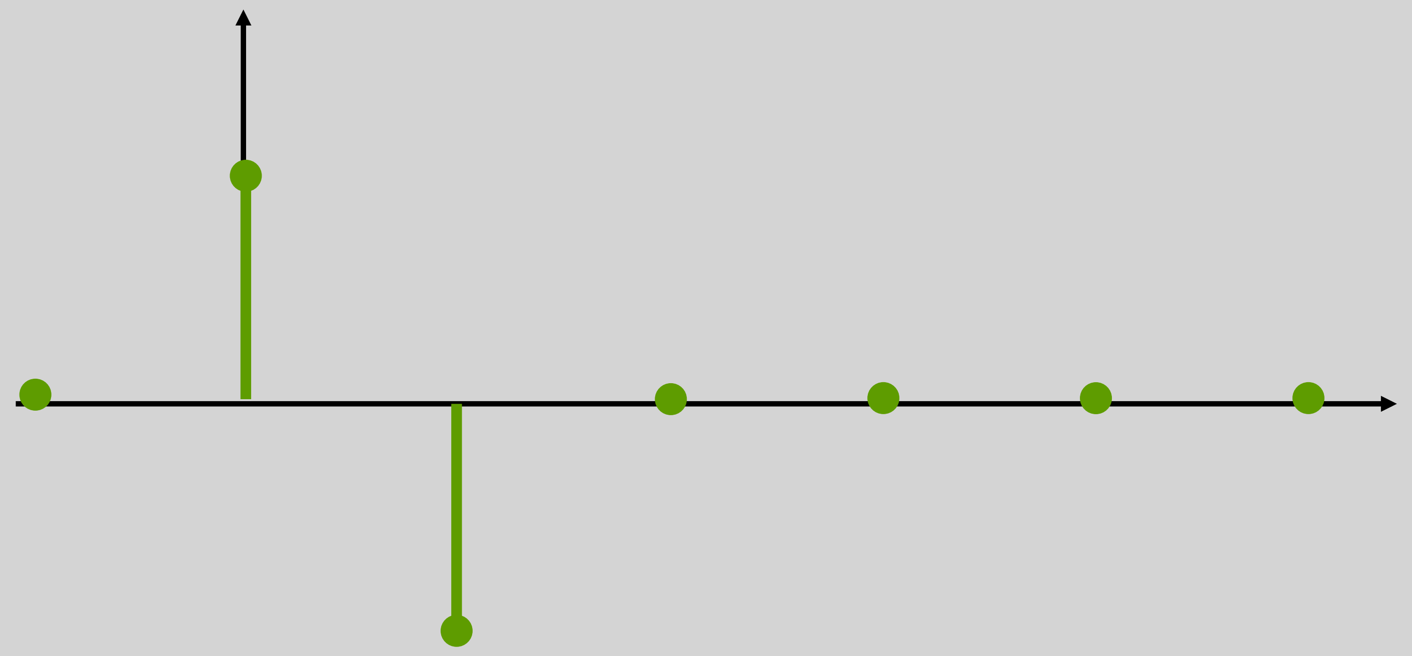
$$h[1-m]$$



m

Convolution Sum

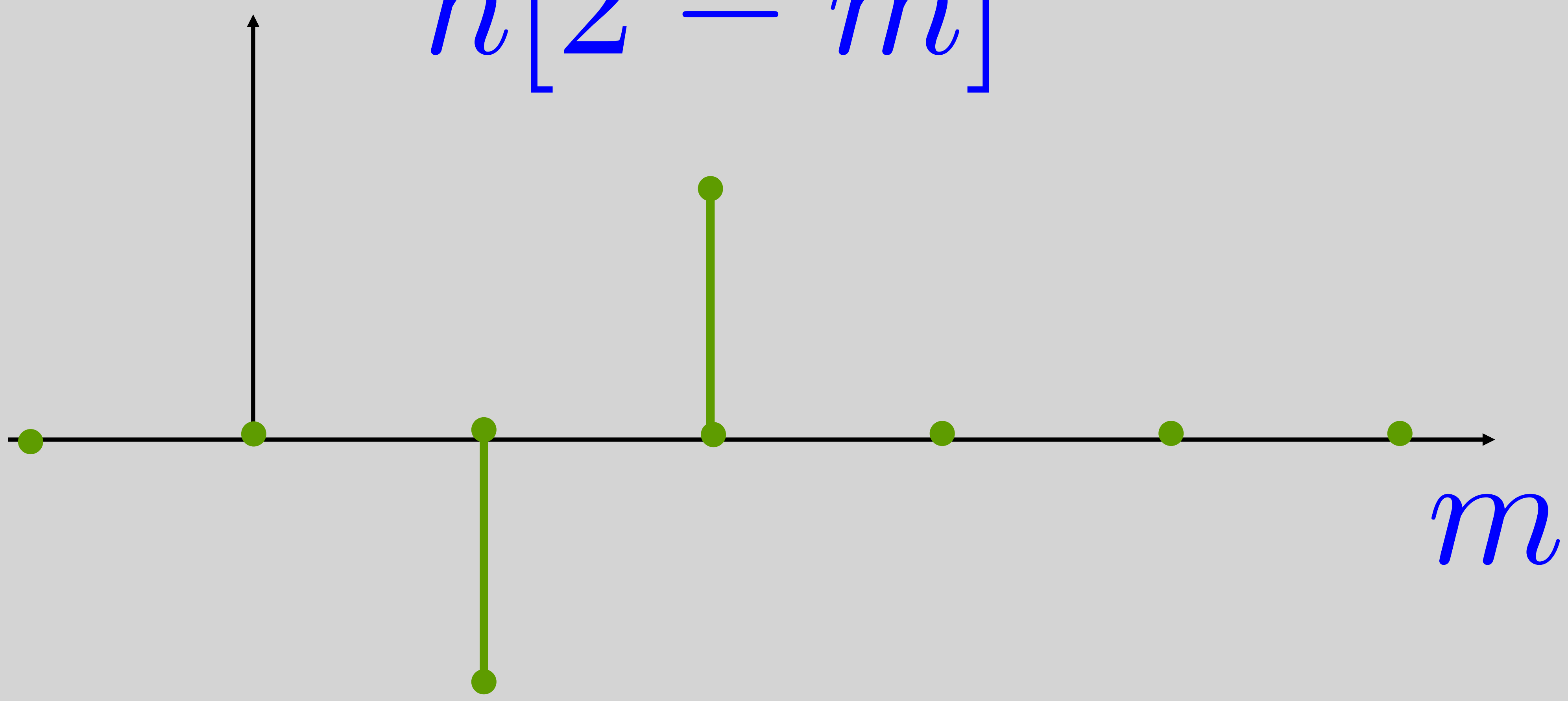
$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m] = x[n] * h[n]$$



$$h[n] = \delta[n] - \delta[n-1]$$

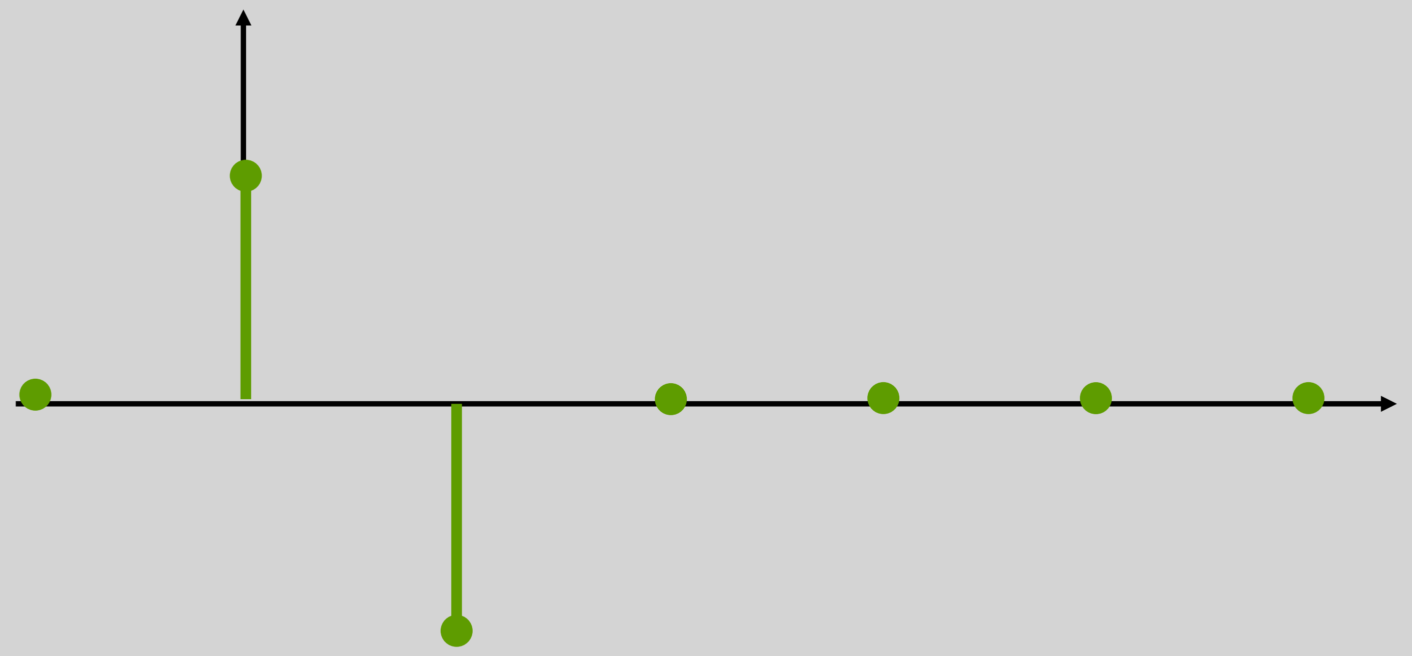
- What is h[n-m] for different n's?

$$h[2-m]$$



Convolution Sum

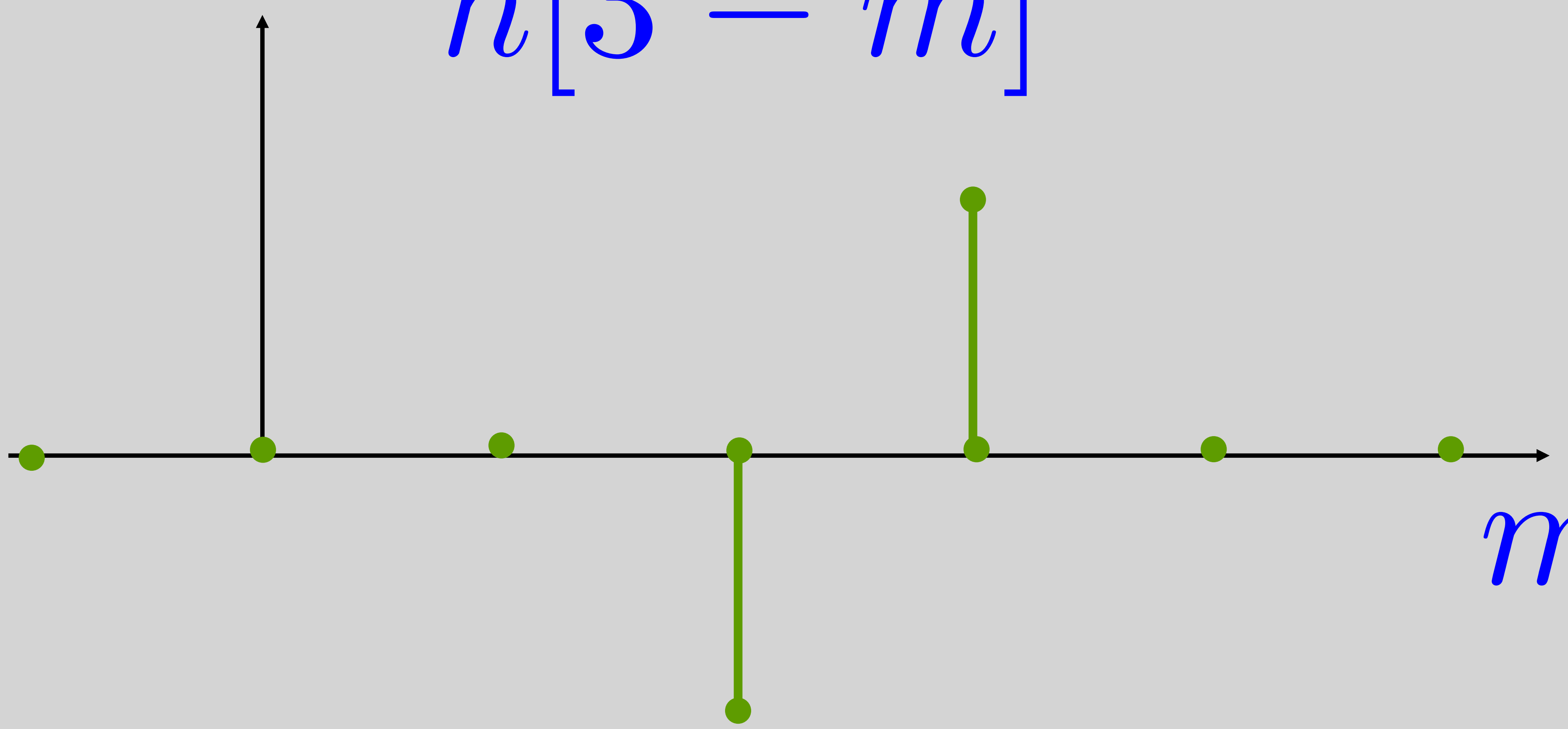
$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m] = x[n] * h[n]$$



- What is h[n-m] for different n's?

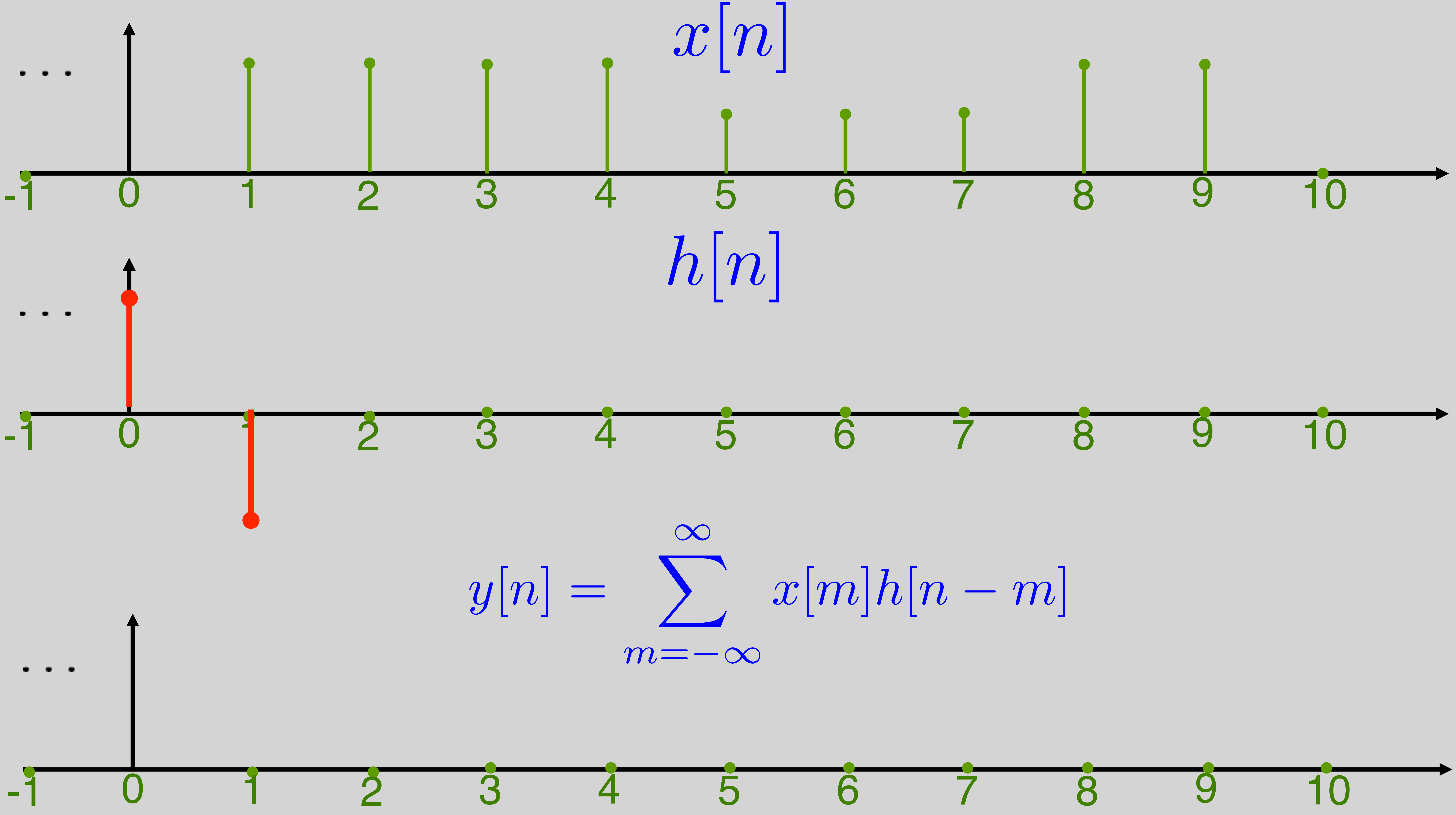
$$h[n] = \delta[n] - \delta[n-1]$$

$$h[3-m]$$



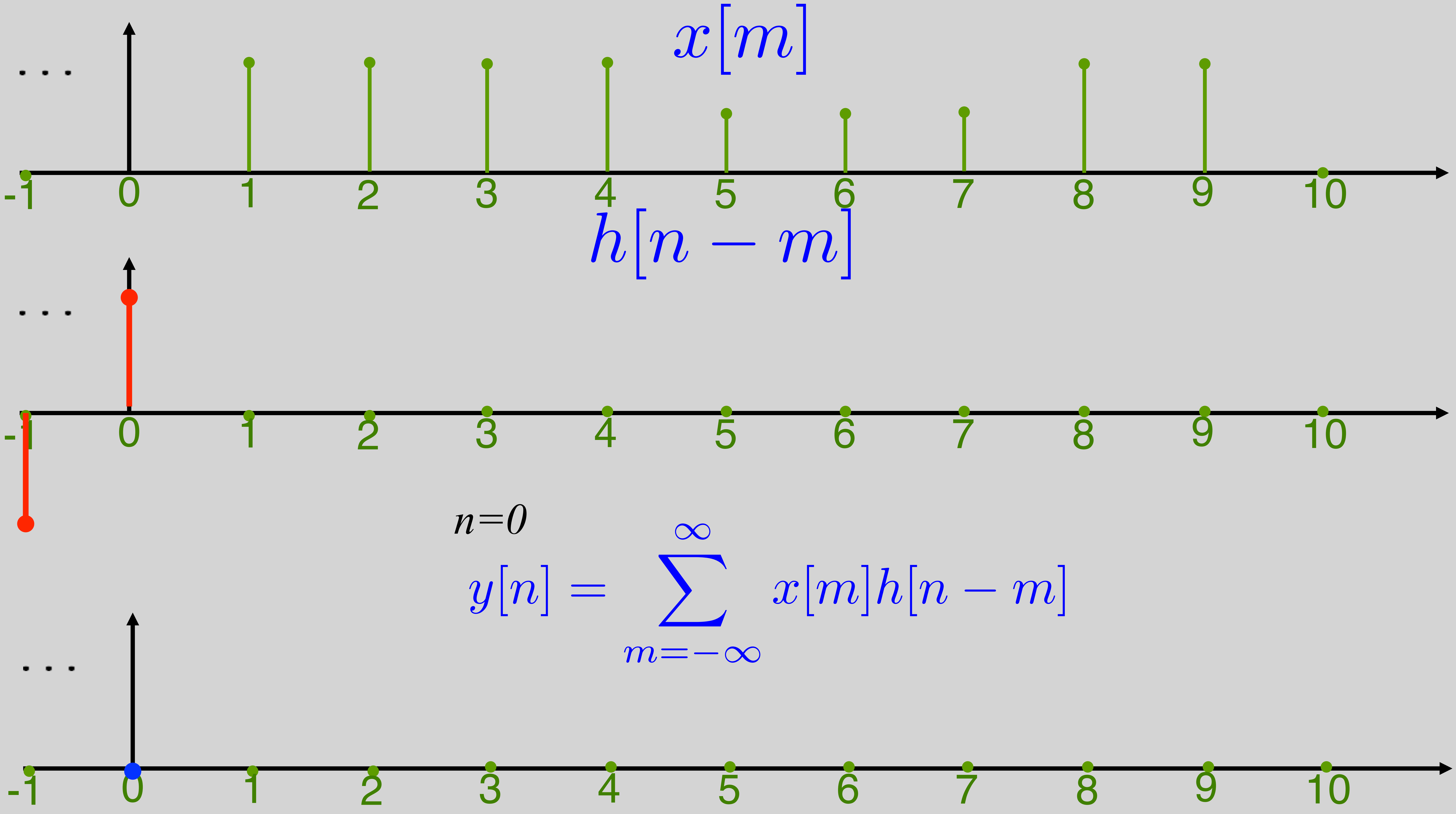
m

Graphical Example of Convolution



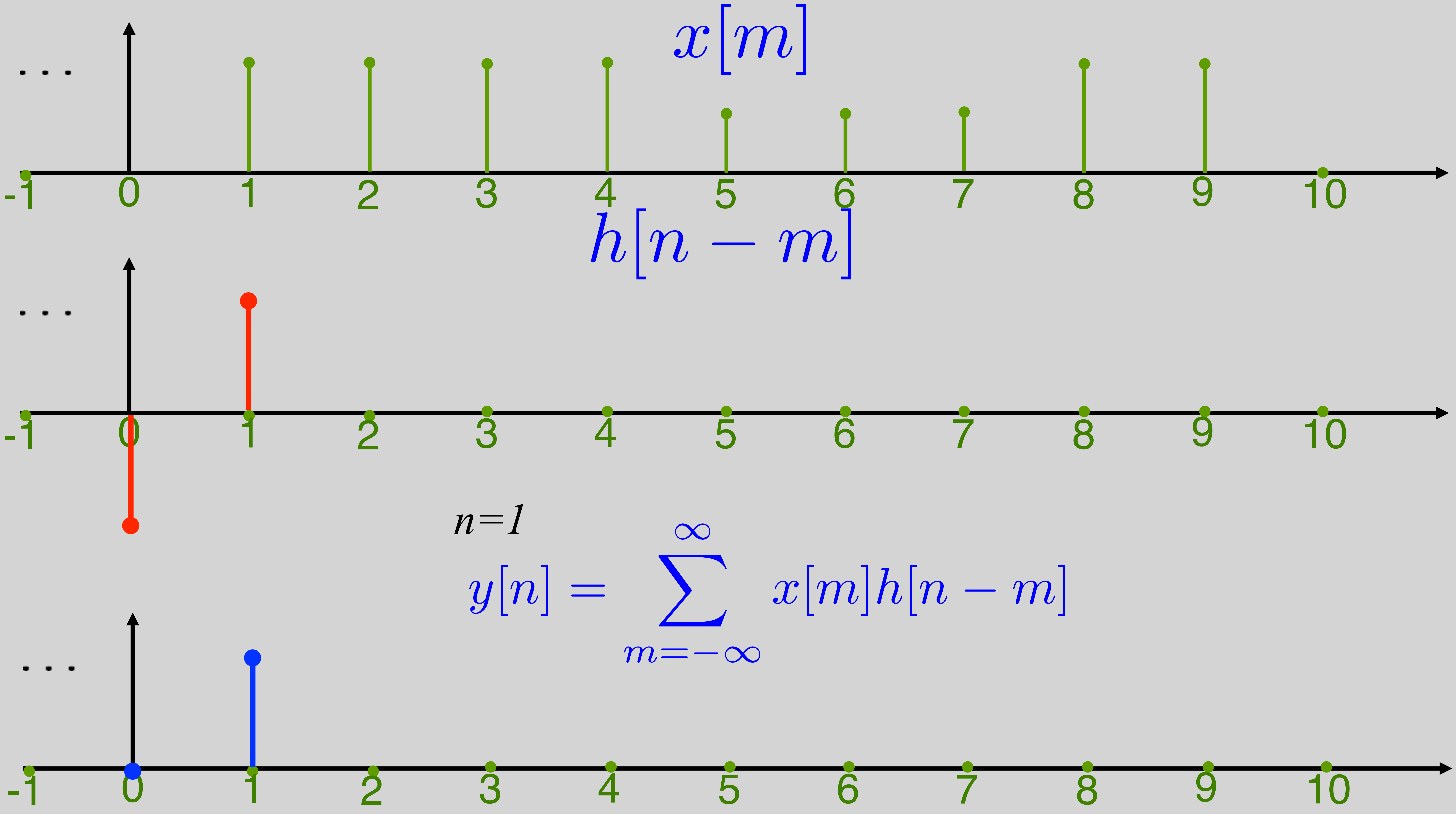
$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$

Graphical Example of Convolution



$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$

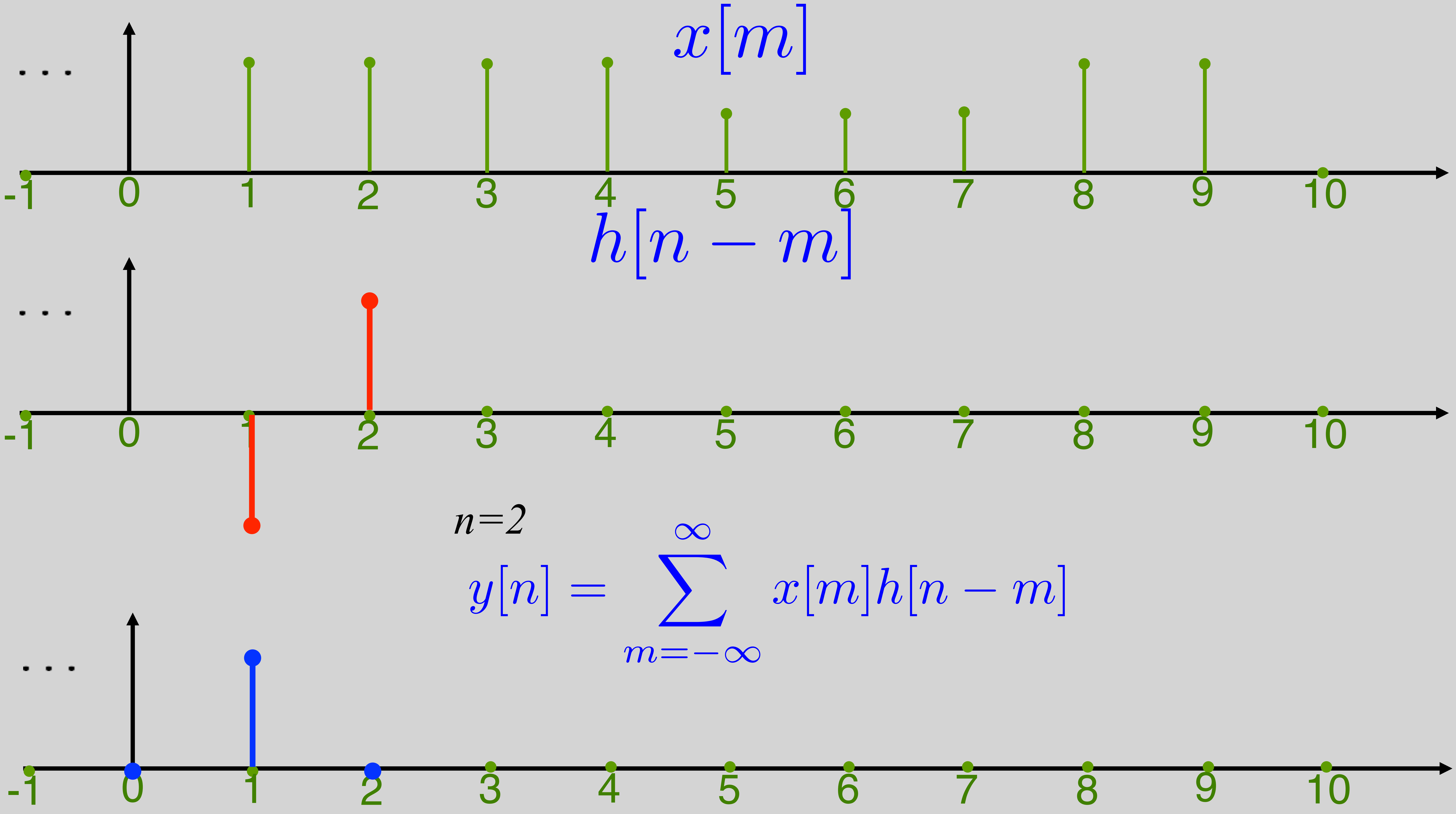
Graphical Example of Convolution



n=1

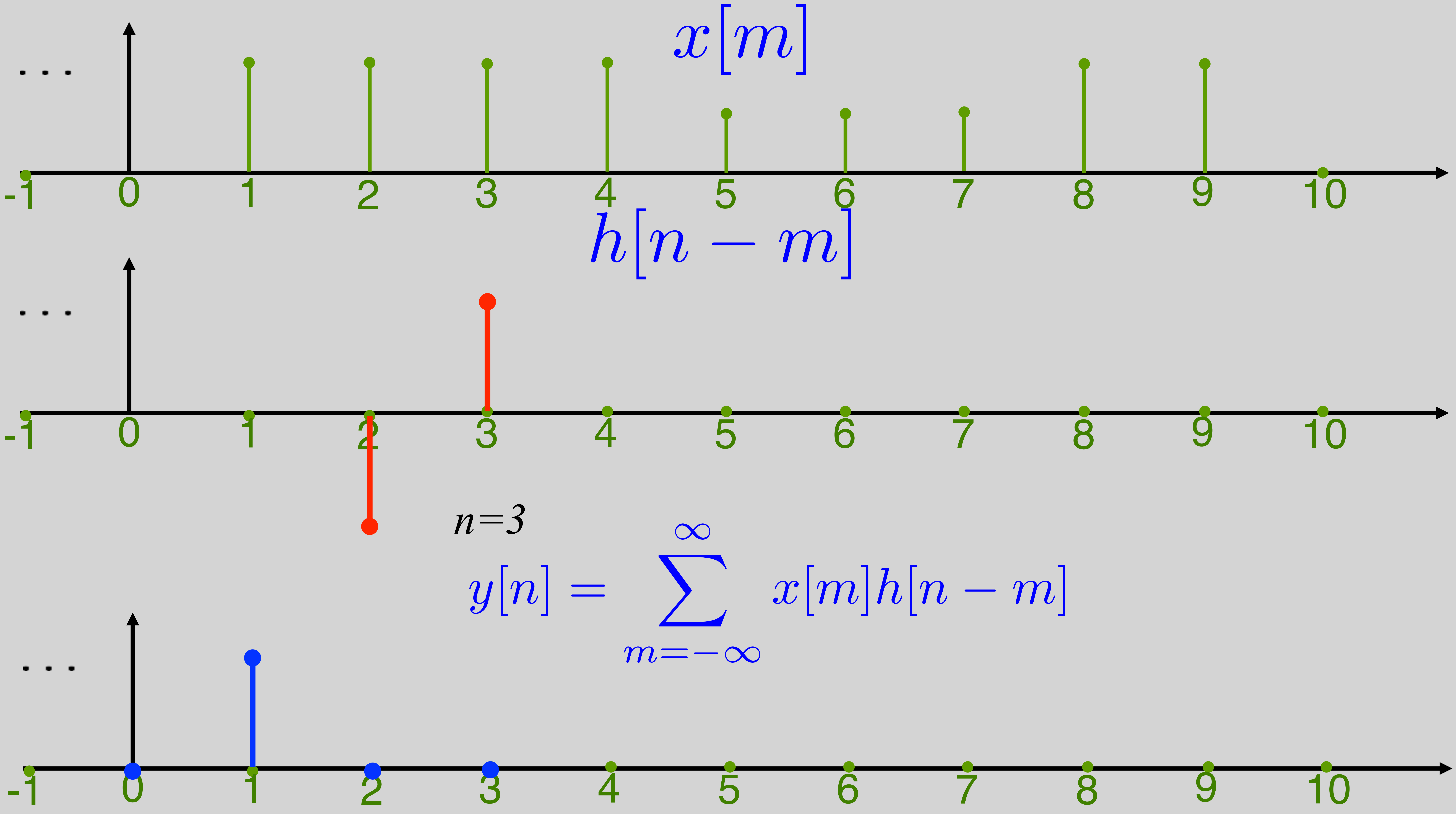
$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$

Graphical Example of Convolution



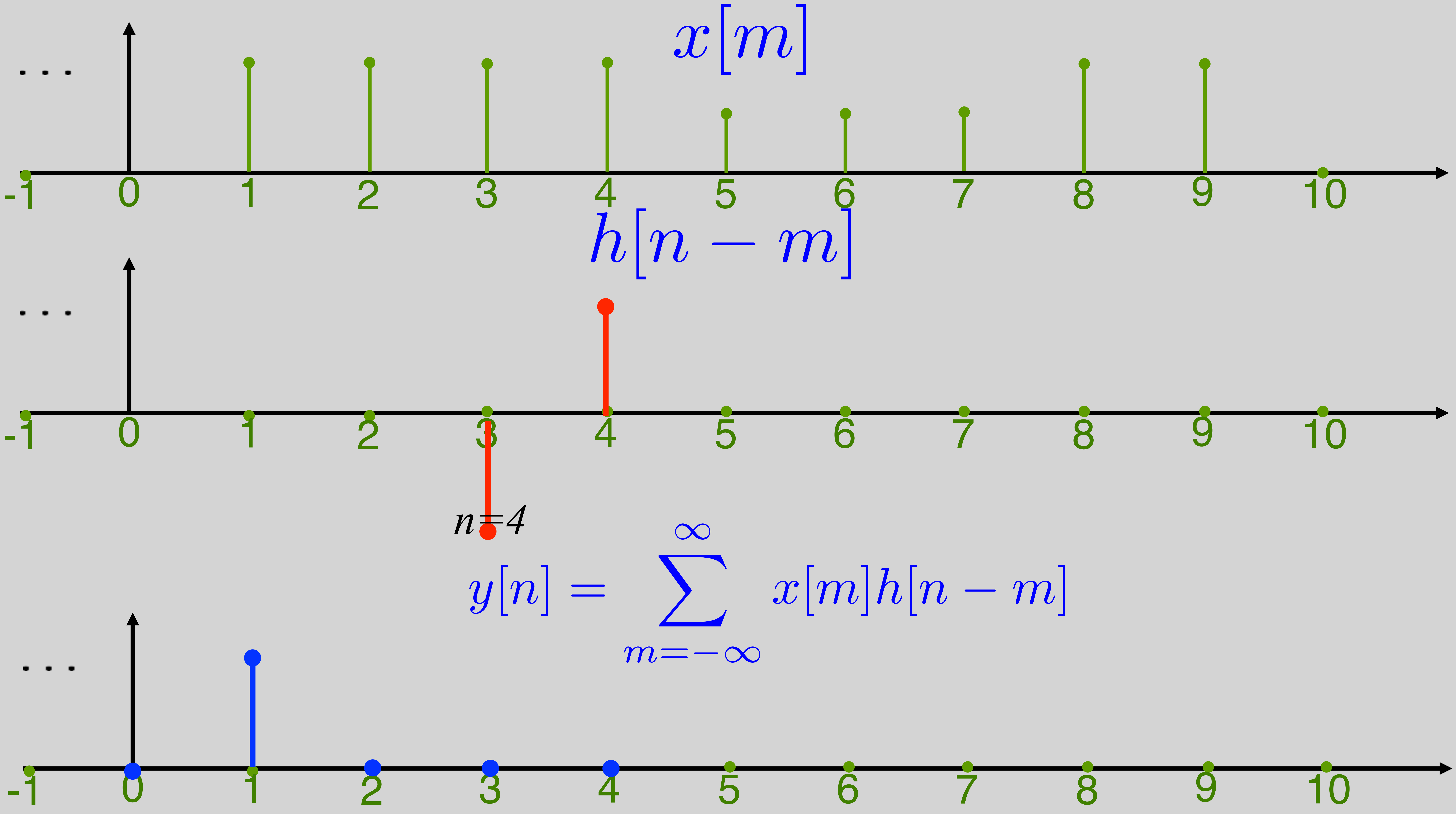
$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$

Graphical Example of Convolution



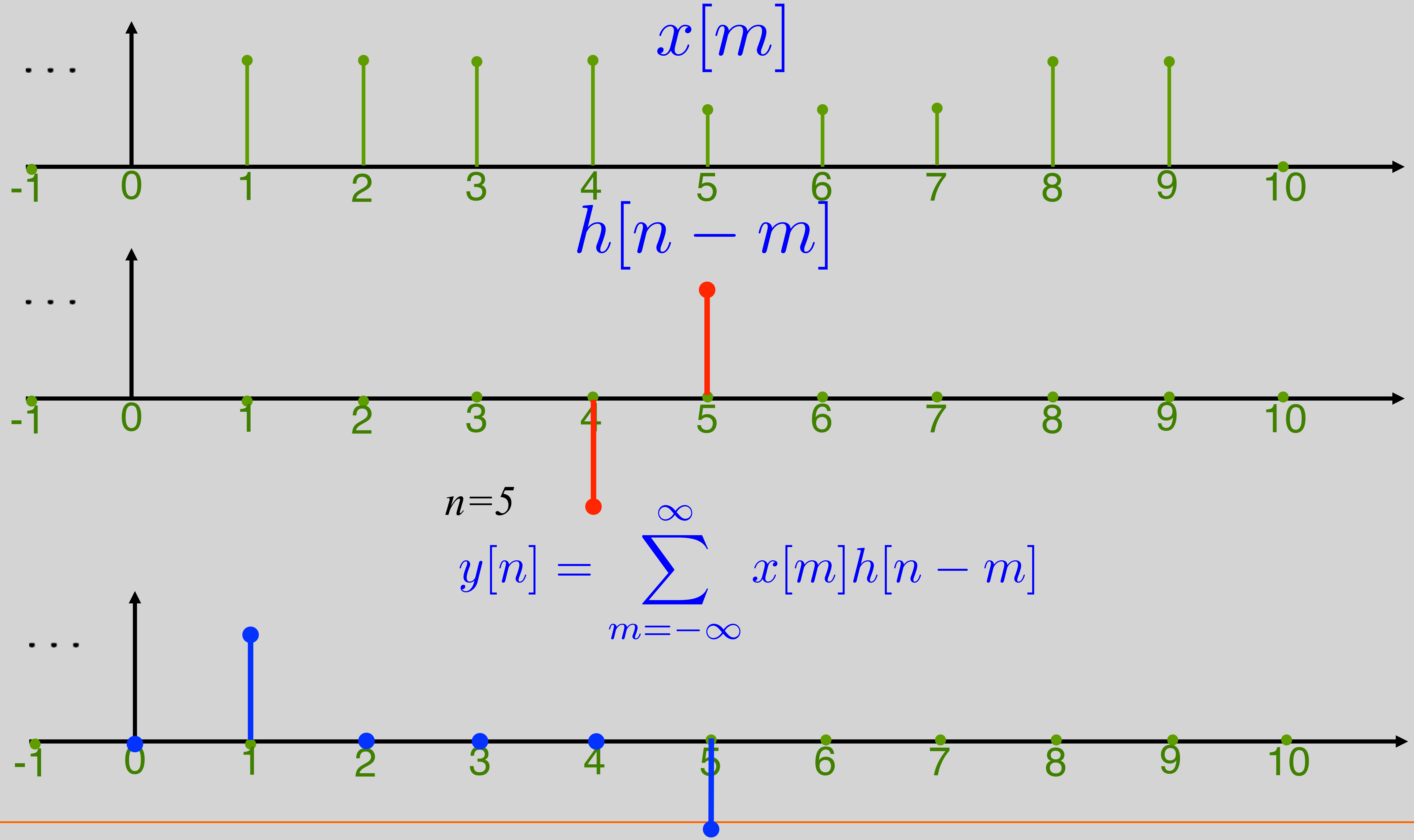
$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$

Graphical Example of Convolution



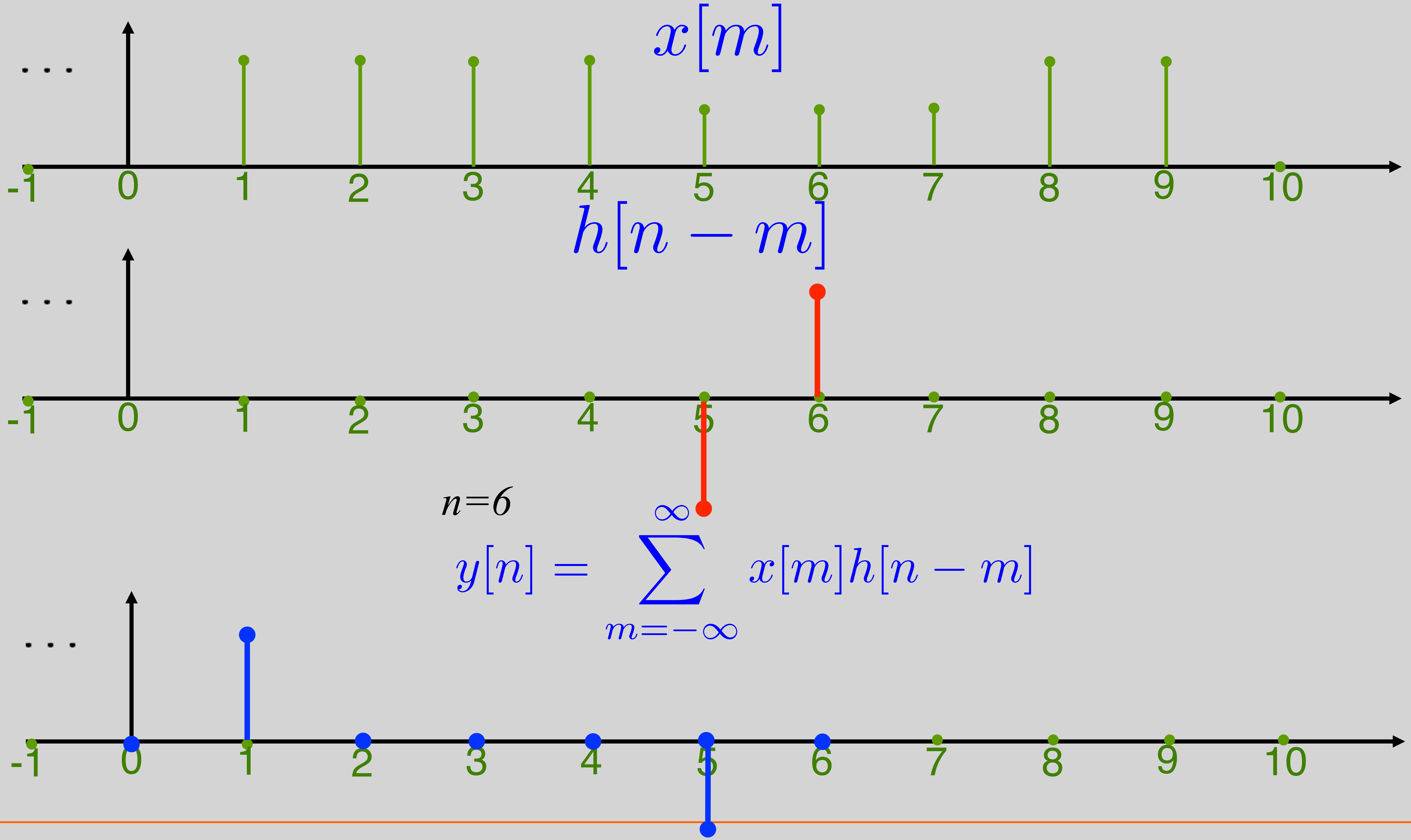
$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$

Graphical Example of Convolution



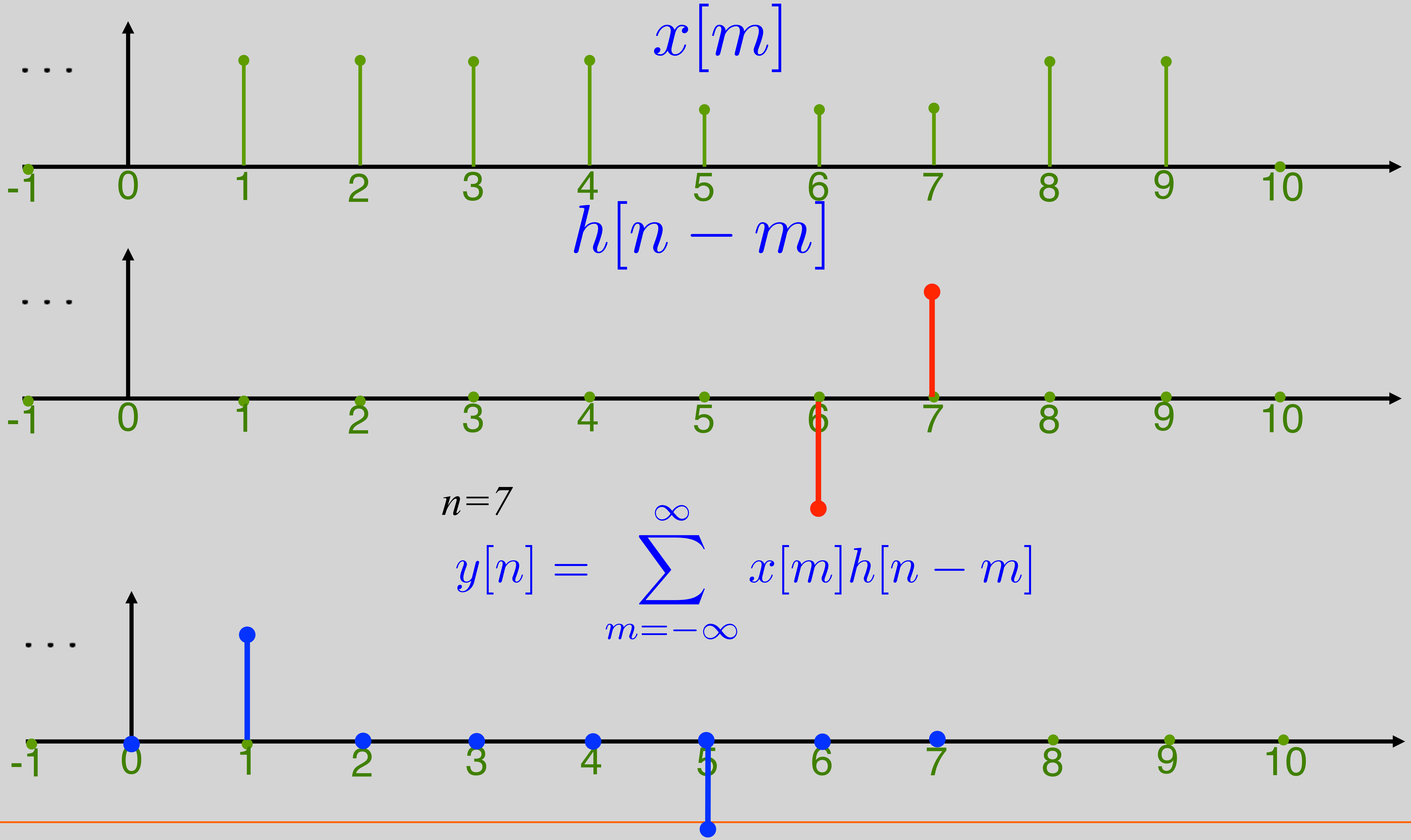
$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$

Graphical Example of Convolution



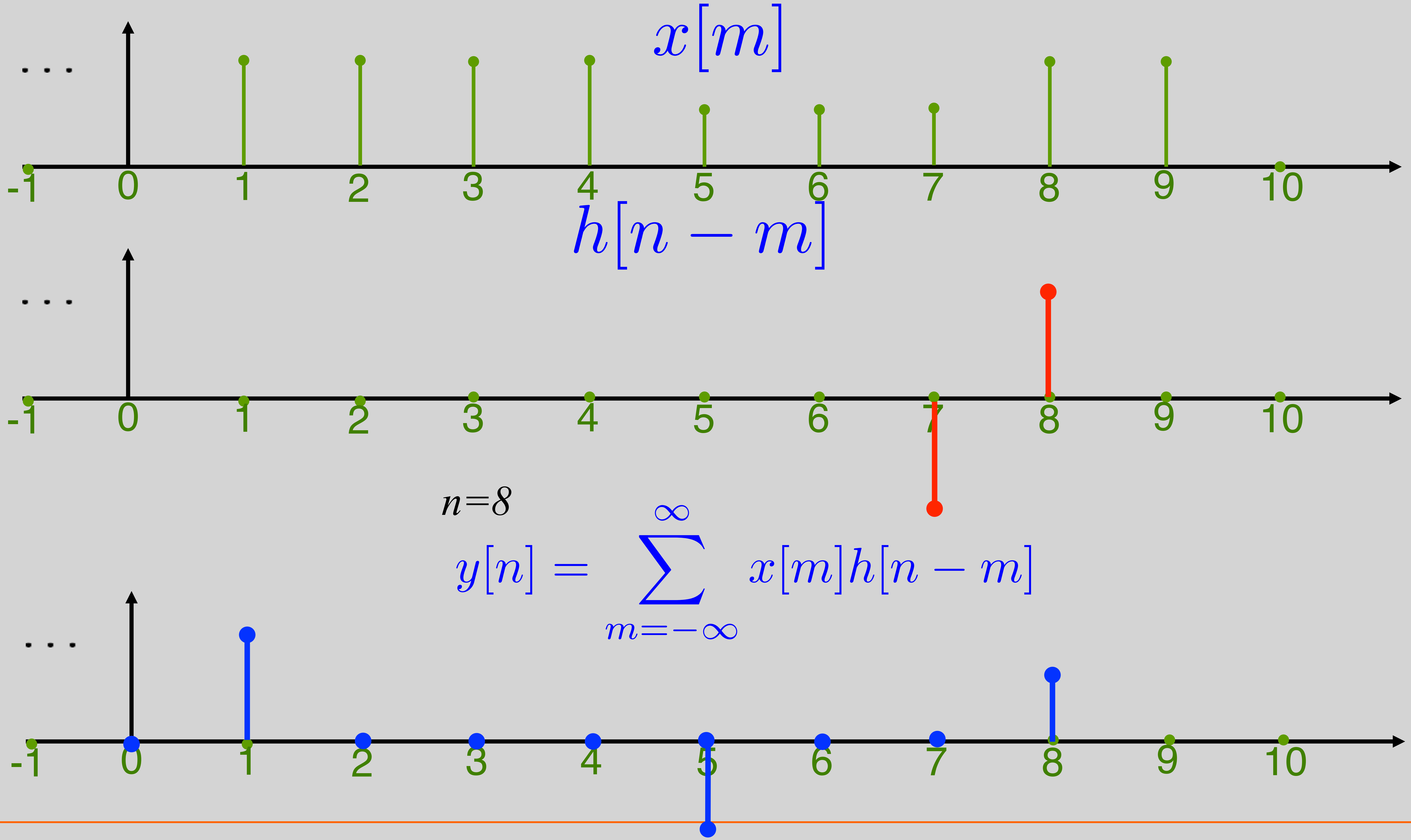
$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$

Graphical Example of Convolution



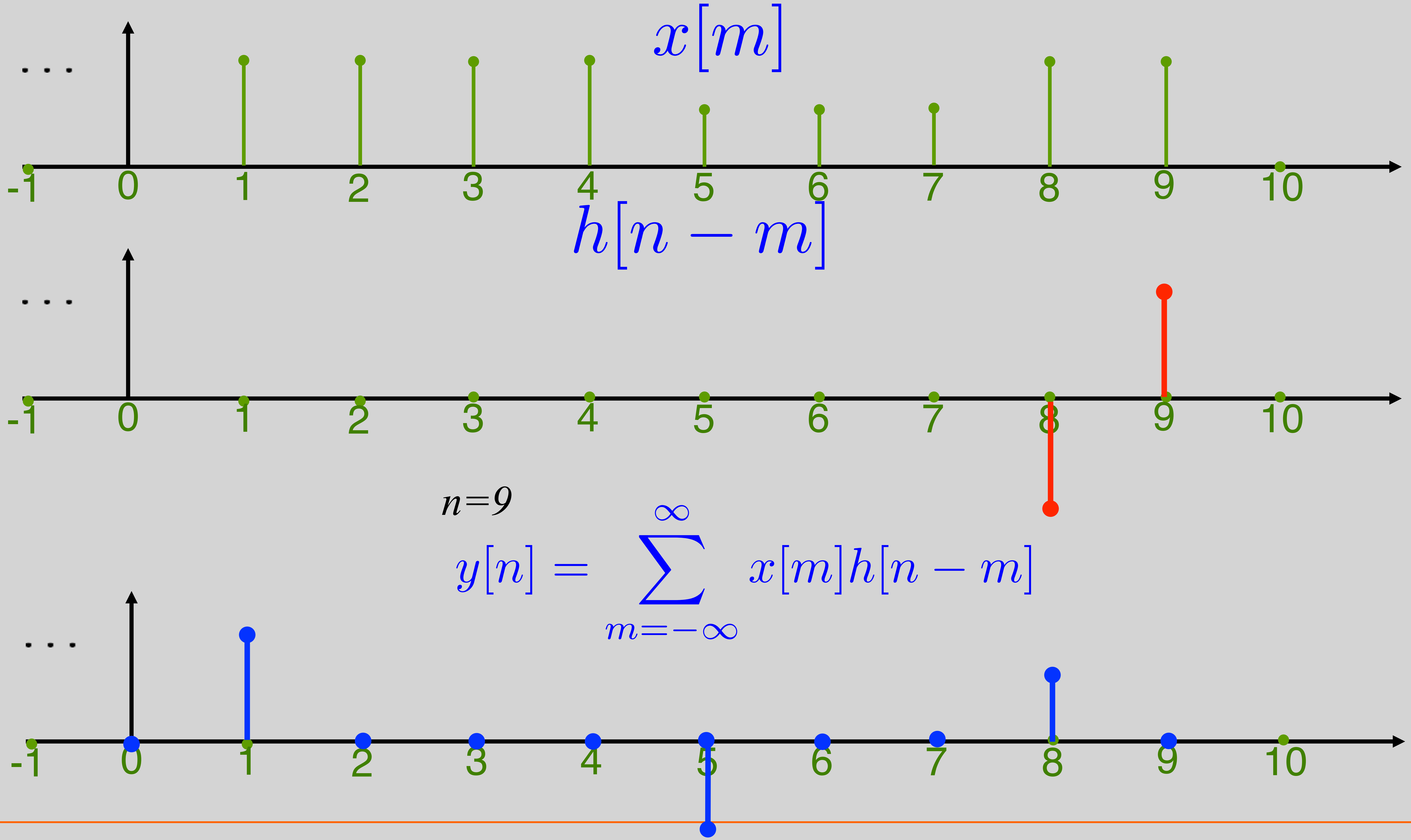
$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$

Graphical Example of Convolution



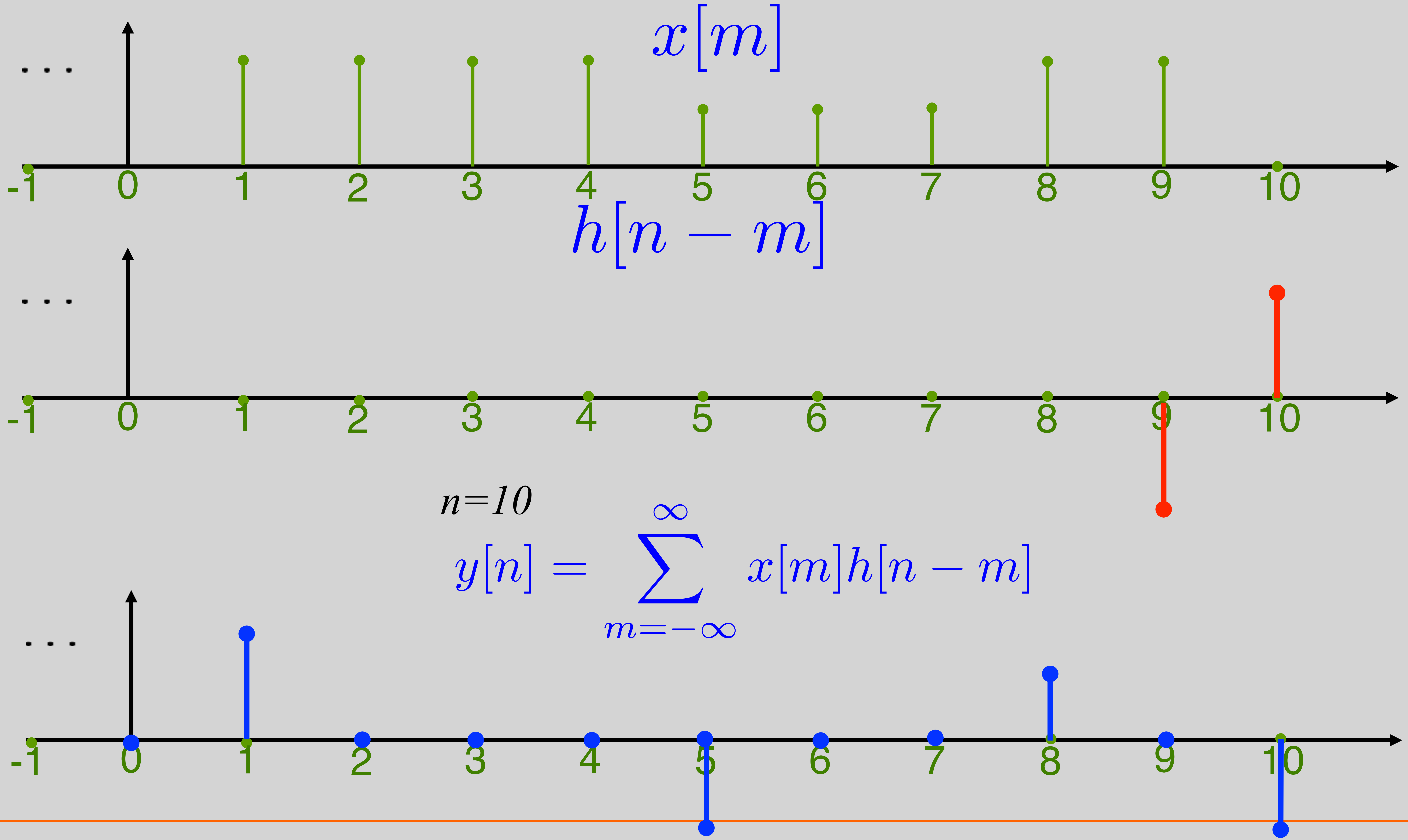
$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$

Graphical Example of Convolution



$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$

Graphical Example of Convolution



$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$

Example

BIBO Stability of LTI systems

- LTI system is BIBO stable if, and only if h[n] is absolutely summable.

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

Proof (if):

$$|y[n]| = \left| \sum_{m=-\infty}^{\infty} x[m]h[n-m] \right| \leq \sum_{m=-\infty}^{\infty} |x[m]| \cdot |h[n-m]|$$

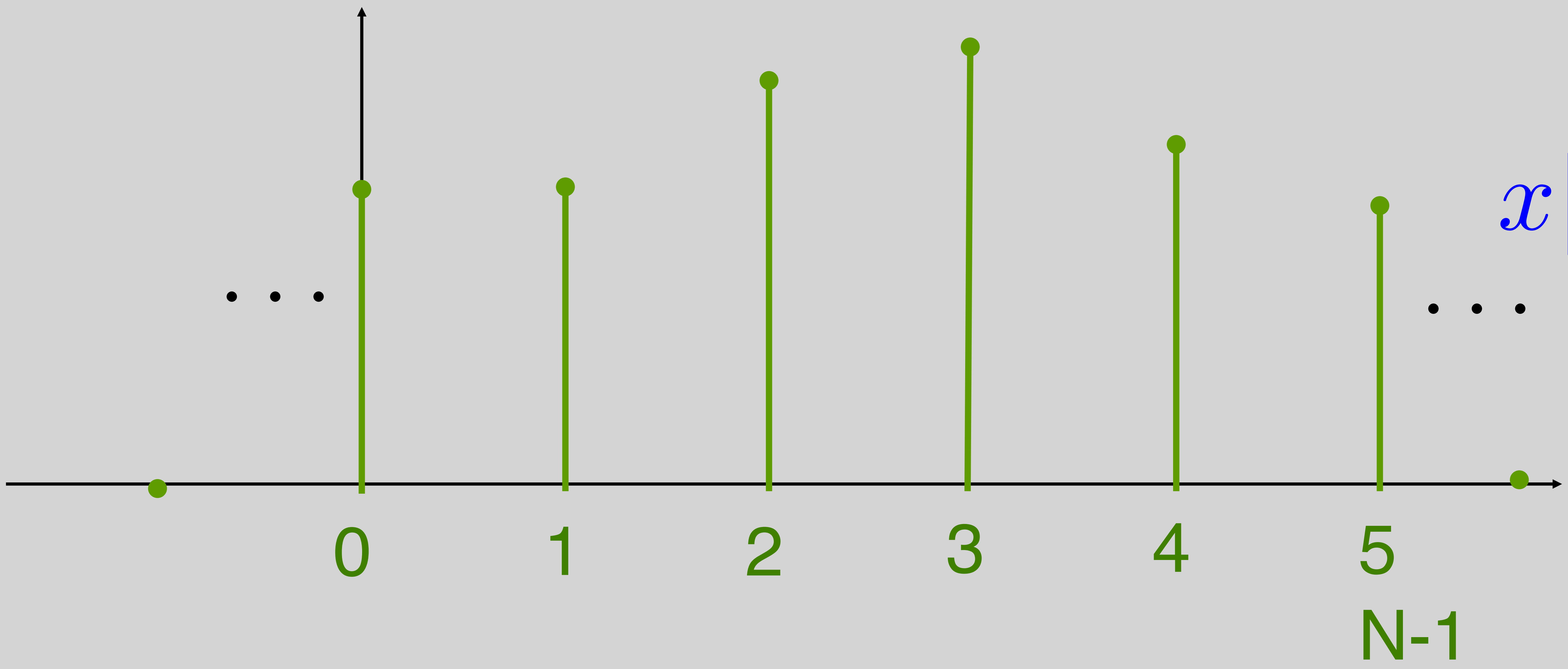
$$|x[m]| < M$$

$$\leq M \sum_{m=-\infty}^{\infty} |h[n-m]| = M \sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

Only if in EE123

Finite Sequences

- Consider a finite sequence of length N



$$x[n] = \begin{cases} \text{something} & 0 \leq n < N \\ 0 & \text{otherwise} \end{cases}$$

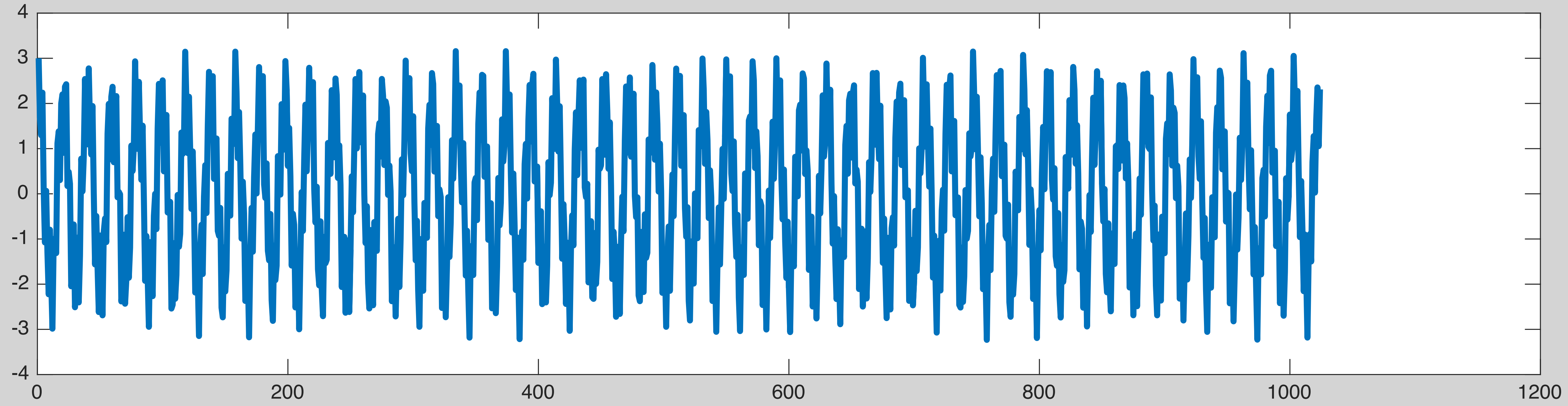
- Can also be written as a vector

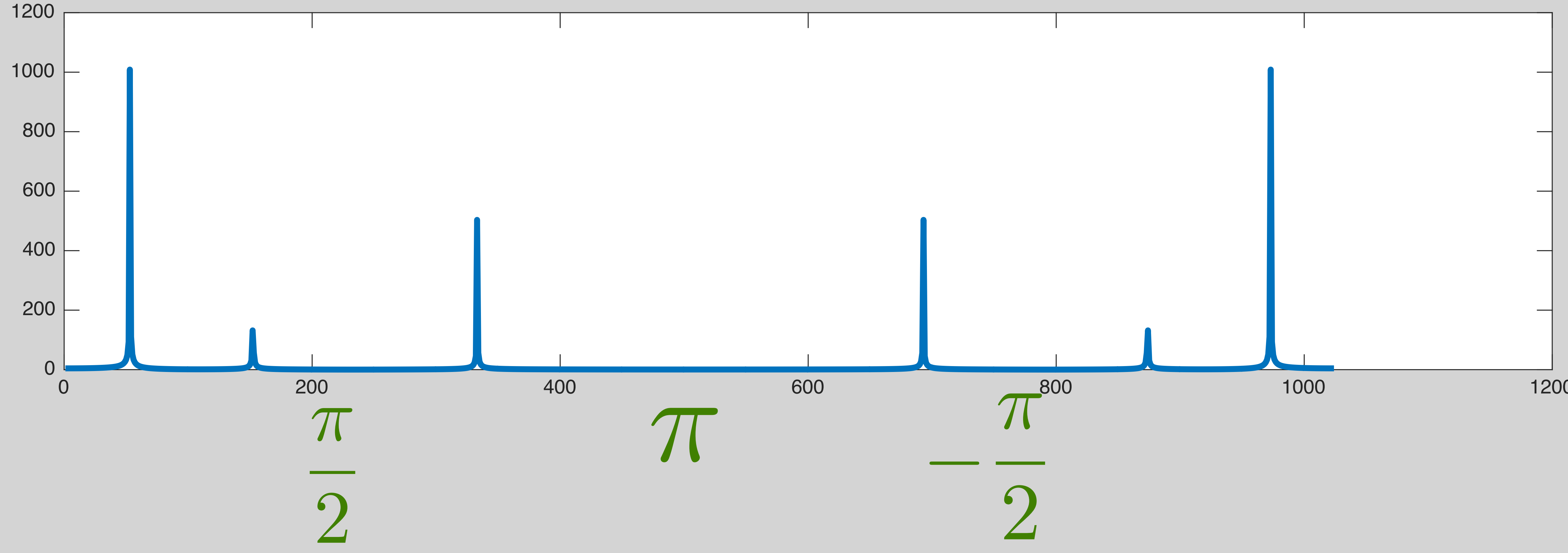
$$\vec{x} = \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}$$

Why?

- To compute this:

$$x[n] \Rightarrow X[k]$$





Finite Sequences as Vectors

- Define an inner-product (for R^N):

$$< \vec{x}, \vec{y} >= \vec{x} \cdot \vec{y} = \sum_{n=0}^{N-1} x[n]y[n] =$$

$$= \vec{x}^T \vec{y}$$

So,

$$< \vec{x}, \vec{x} >= \sum_{n=0}^{N-1} x[n]x[n] = \sum_{n=0}^{N-1} x^2[n] = ||\vec{x}||^2$$

$$\Rightarrow \vec{x}^T \vec{x} = ||\vec{x}||^2$$

Finite Sequences as Vectors

- What about complex?

$$x \cdot x = x^2 = (x_r + jx_i)(x_r + jx_i) = x_r^2 - x_i^2 + 2jx_r x_i \neq ||x||^2$$

but,

$$x^* \cdot x = (x_r - jx_i)(x_r + jx_i) = x_r^2 + x_i^2 = ||x||^2$$

- Transpose vs Transpost conjugate

$$\vec{x} = \begin{bmatrix} 1 \\ j \\ 1+j \end{bmatrix} \qquad \vec{x}^T = \begin{bmatrix} 1 & j & 1+j \end{bmatrix}$$

$$\vec{x}^* = \begin{bmatrix} 1 & -j & 1-j \end{bmatrix}$$

Finite Sequences as Vectors

- Define Complex inner product

$$< \vec{x}, \vec{y} >= \overline{\vec{x}} \cdot \vec{y} = \sum_{n=0}^{N-1} x[n]^* y[n] = \vec{x}^* \vec{y} = \vec{x}^H \vec{y}$$

$$\vec{x} = \begin{bmatrix} 1 \\ j \end{bmatrix} \Rightarrow \vec{x}^* x = \begin{bmatrix} 1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ j \end{bmatrix} = 2$$

Projections

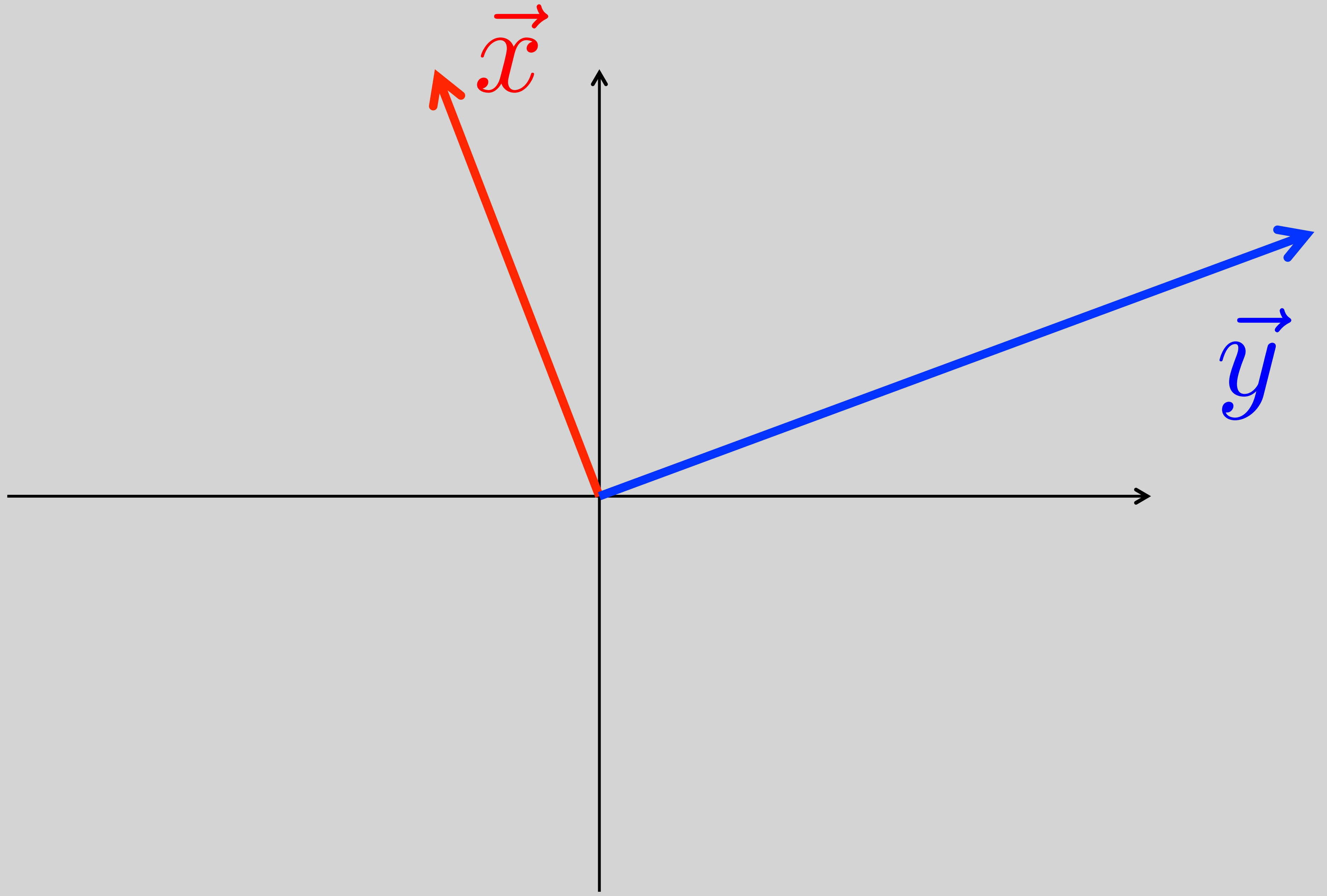
- Orthogonality:

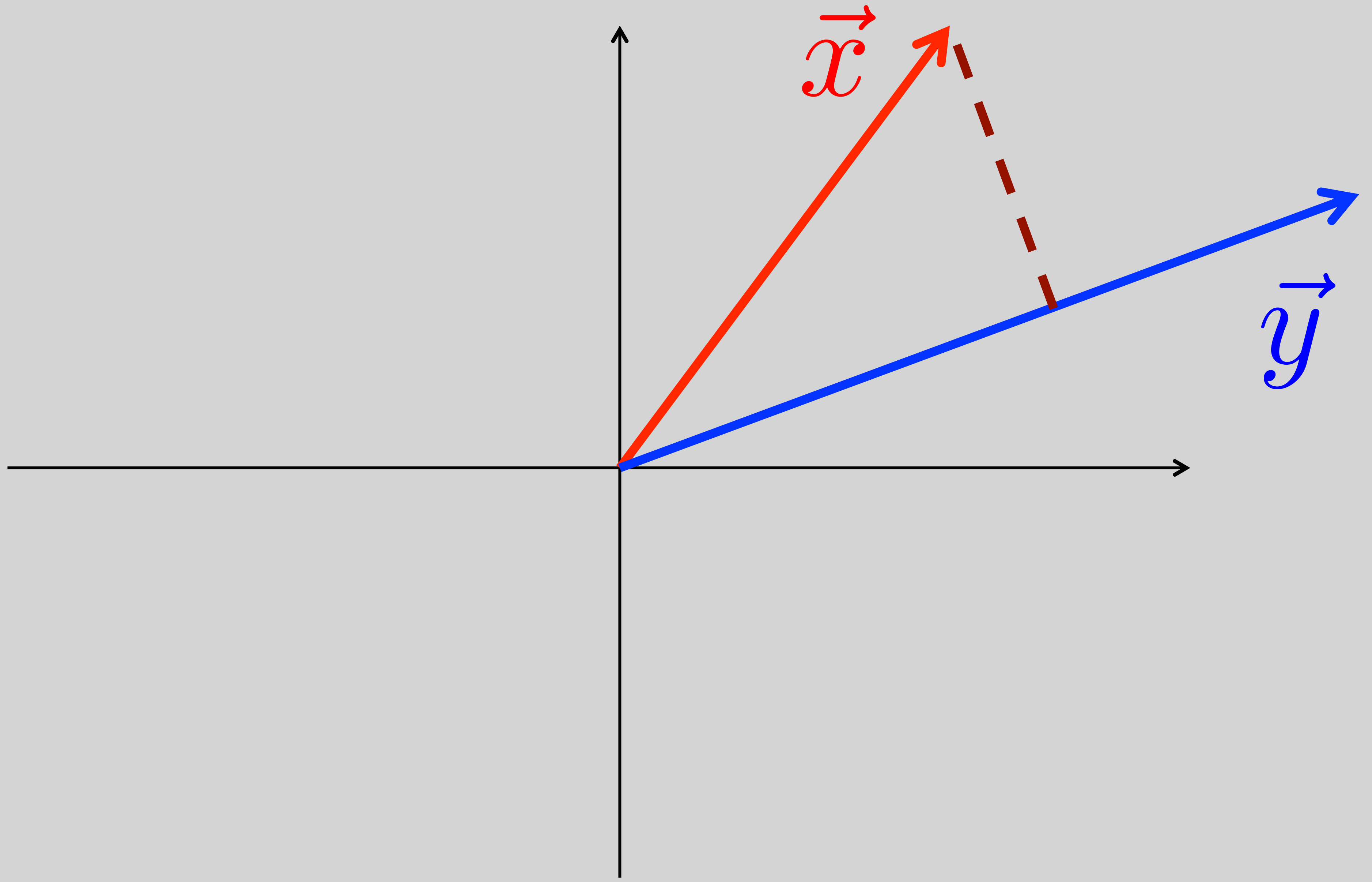
$$\vec{x}^*\vec{y} = \sum_{n=0}^{N-1} x[n]^* y[n] = 0$$

- Unit vector: $||\hat{x}|| = 1$

$$\hat{x} = \frac{\vec{x}}{||\vec{x}||}$$

- Define projection as: $\dfrac{\vec{y}^* x}{||\vec{y}||}$

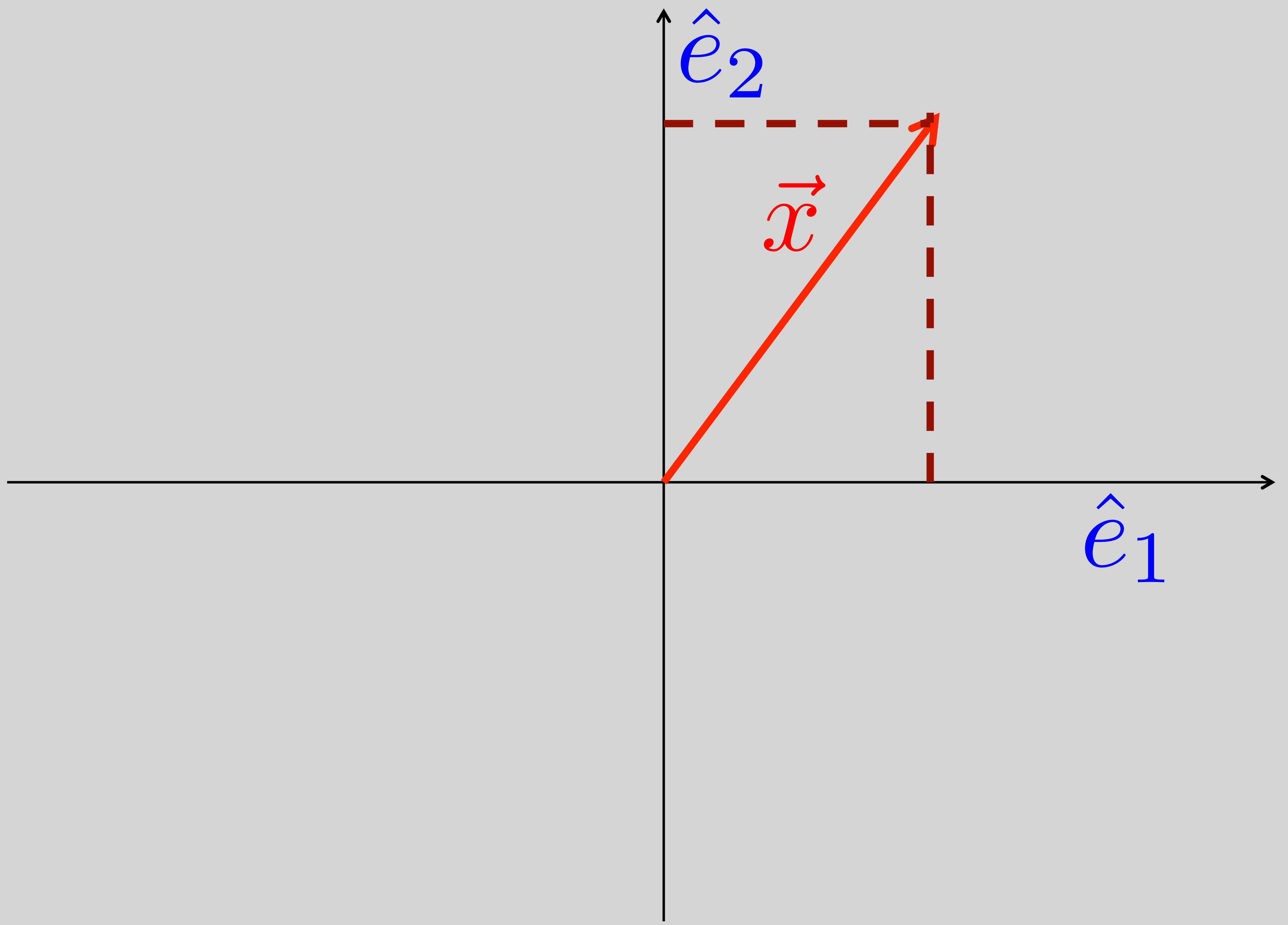




Change of Coordinates (Basis)

- We can compute new coordinates by projections onto orthonormal basis vectors

$$\hat{e}_1^* \vec{x} = [1 \quad 0]\vec{x} = x_1$$
$$\hat{e}_2^* \vec{x} = [0 \quad 1]\vec{x} = x_2$$



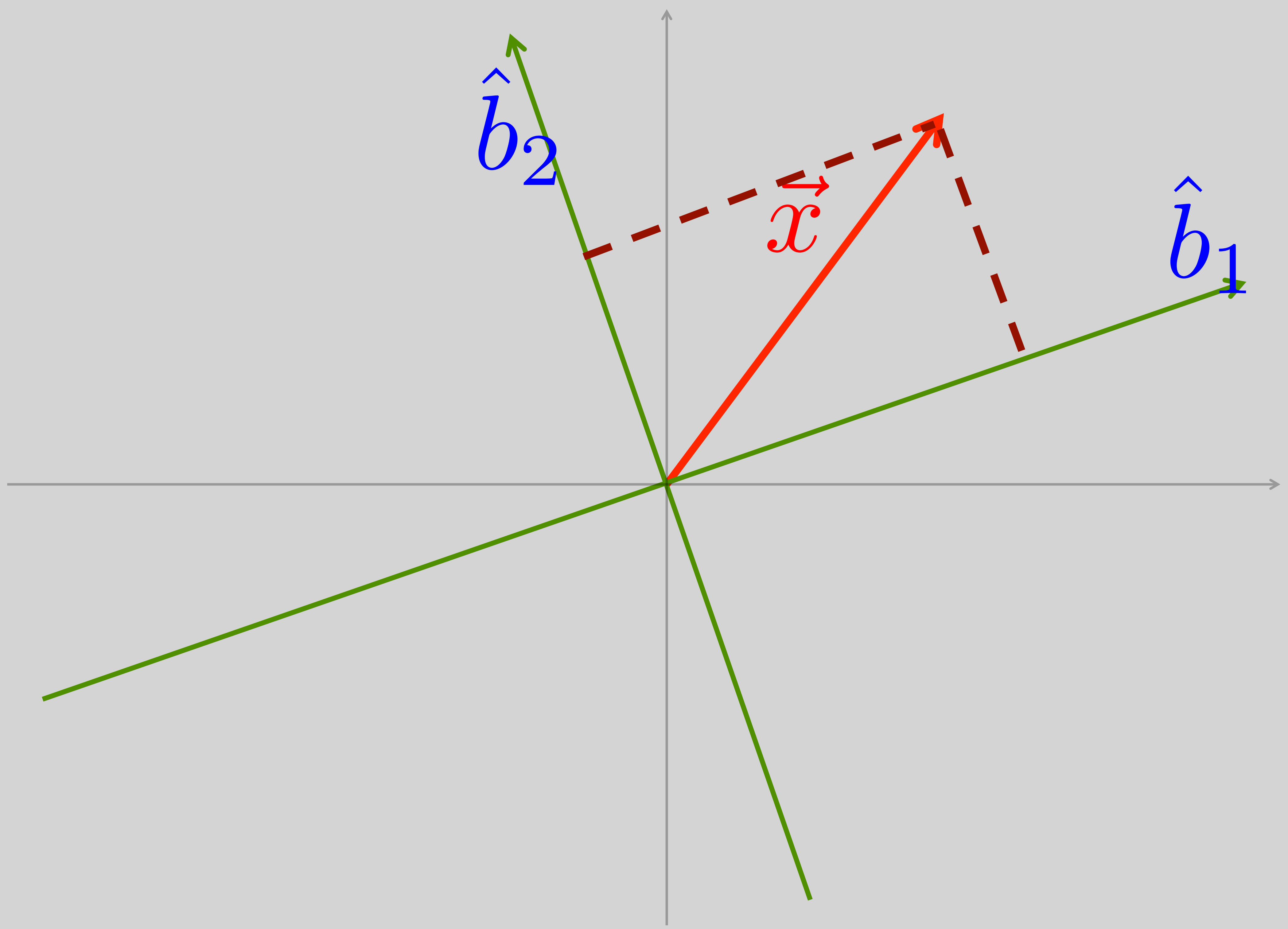
Change of Coordinates (Basis)

- We can compute new coordinates by projections onto orthonormal basis vectors

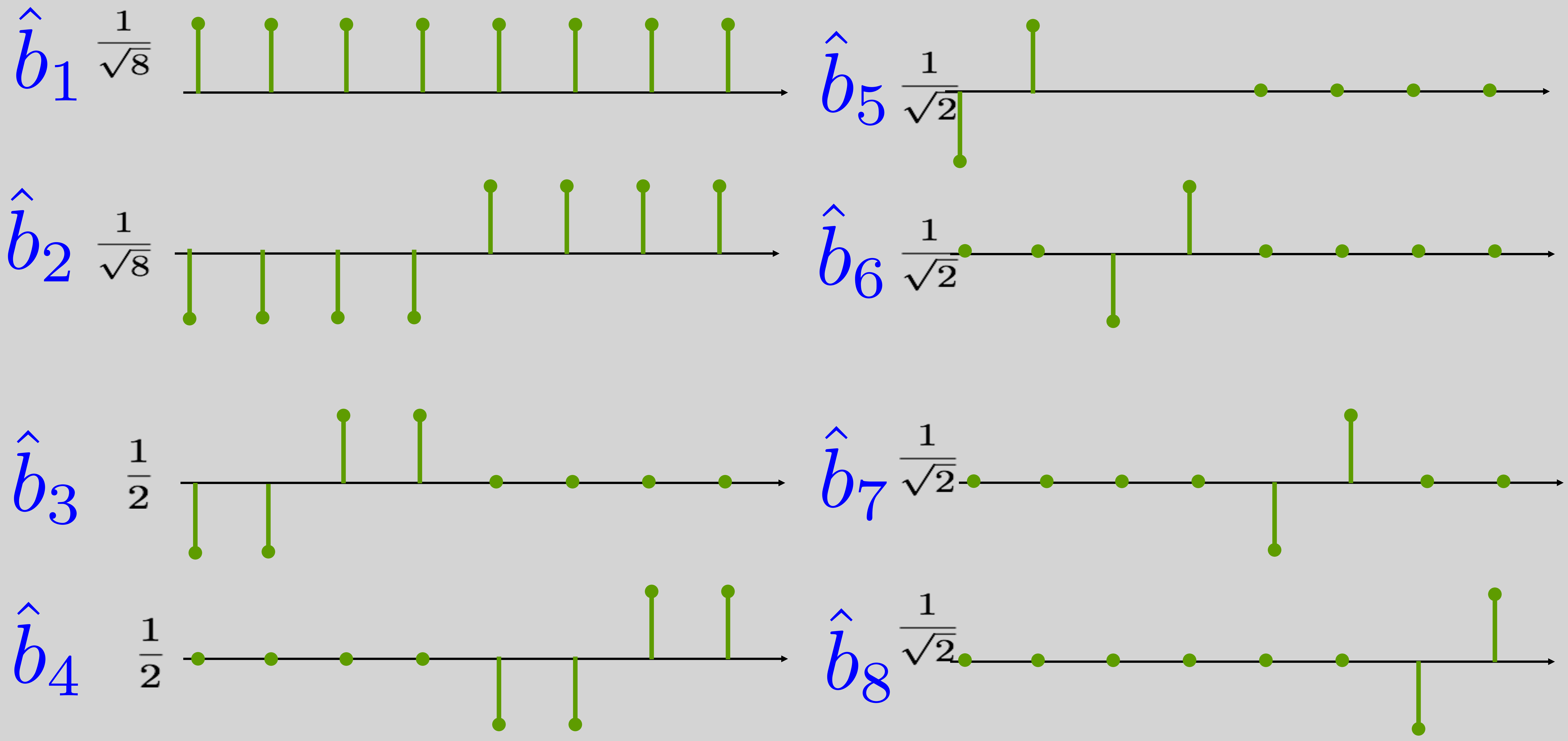
New coordinates:

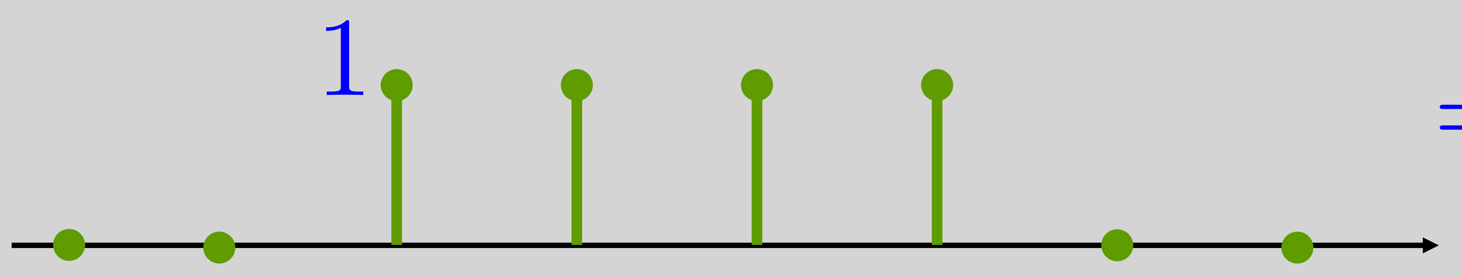
$$\begin{bmatrix} \hat{b}_1^* \vec{x} \\ \hat{b}_2^* \vec{x} \end{bmatrix} = \begin{bmatrix} \hat{b}_1 & \hat{b2} \end{bmatrix}^* \vec{x}$$

$$\Rightarrow \vec{x} = (\hat{b}_1^* \vec{x})\hat{b}_1 + (\hat{b}_2^* \vec{x})\hat{b}_2$$



Change of basis





$$= \quad \hat{b}_1 + \quad \hat{b}_2 + \quad \hat{b}_3 + \quad \hat{b}_4 + \quad \hat{b}_5 + \quad \hat{b}_6 + \quad \hat{b}_7 + \quad \hat{b}_8$$