EE16B
Designing Information Devices and Systems II

Lecture 10B
Sampling and Interpolation

Polynomial Interpolation

- Given n distinct points, then there's exist a unique (n-1) order polynomial passing through them

$$y = a_0 + a_1 x + a_2 x^2 + \cdots a_{n-1} x^{n-1}$$

$$\rightarrow \begin{bmatrix} \phantom{x_1^{n-1} \quad x_1^{n-1} \quad x_1^{n-1}} \\ \\ \\ \\ \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

Polynomial Interpolation

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$$y = a_0 + a_1 x + a_2 x^2 + \cdots a_{n-1} x^{n-1}$$

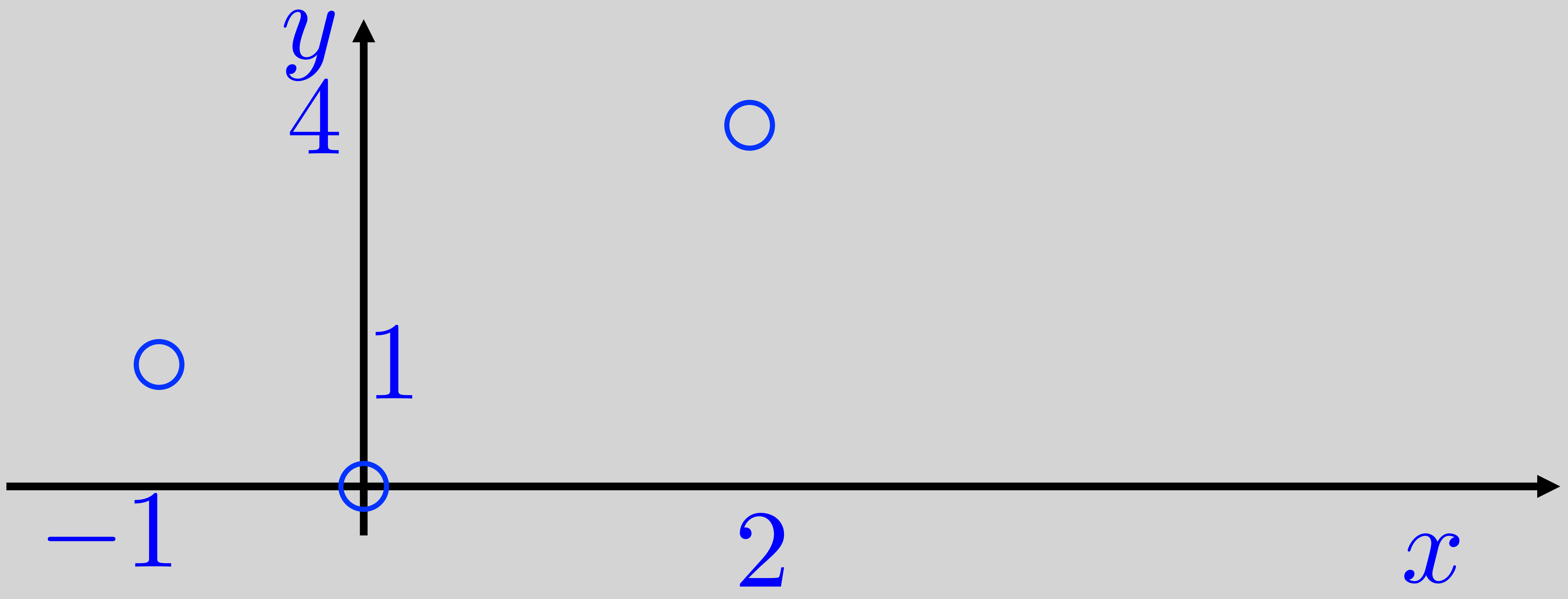
$$\rightarrow \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ & \vdots & & & \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

"Vandermonde" Matrix $\det(v) = \Pi_{1 \le i < j \le n}(x_j - x_i)$

Quiz

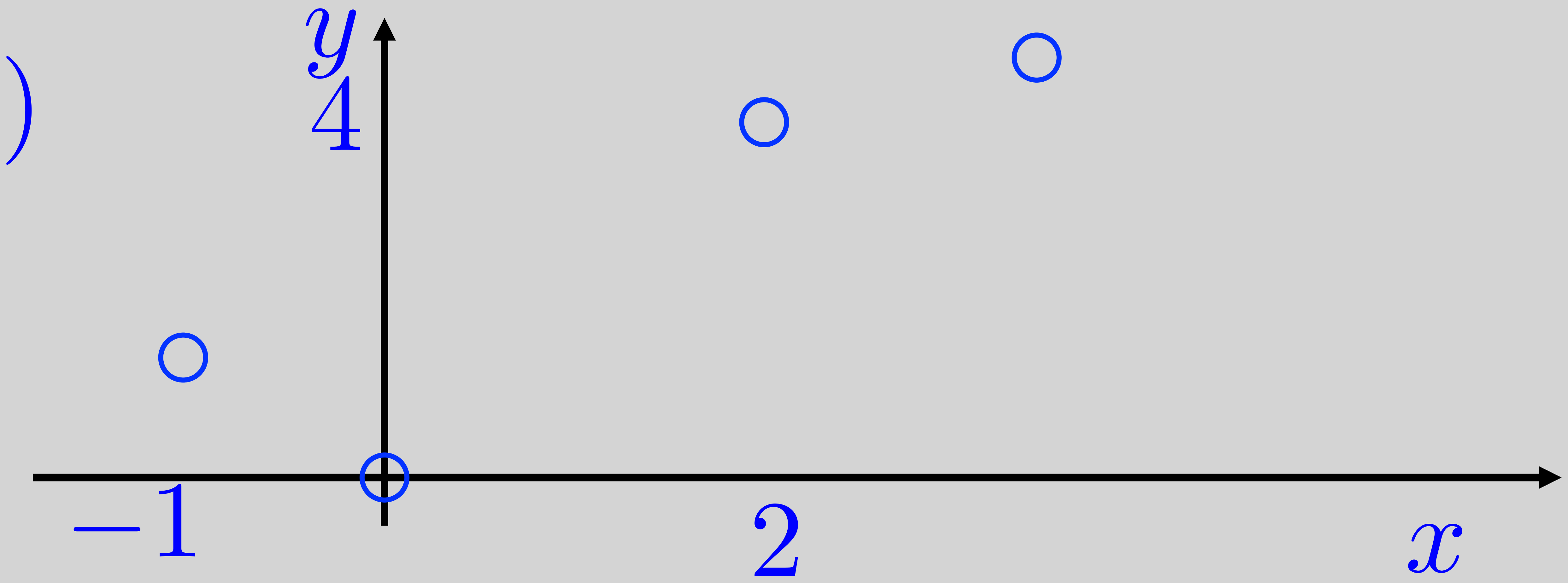
- What's the polynomial that passes through these points:

$$(-1,1),(0,0),(2,4)$$



Polynomial Regression

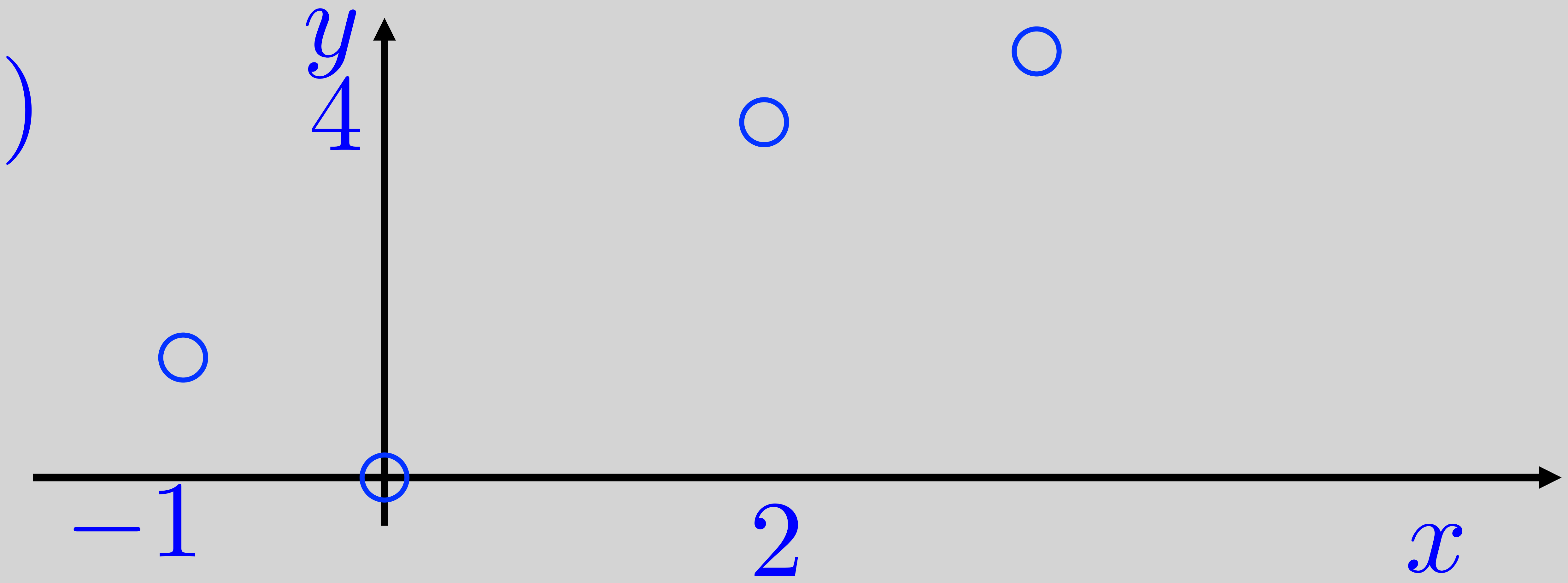
$(-1,1),(0,0),(2,4),(3,5)$



- What is the “best” quadratic polynomial that passes through the points?

Polynomial Regression

$$(-1,1),(0,0),(2,4),(3,5)$$

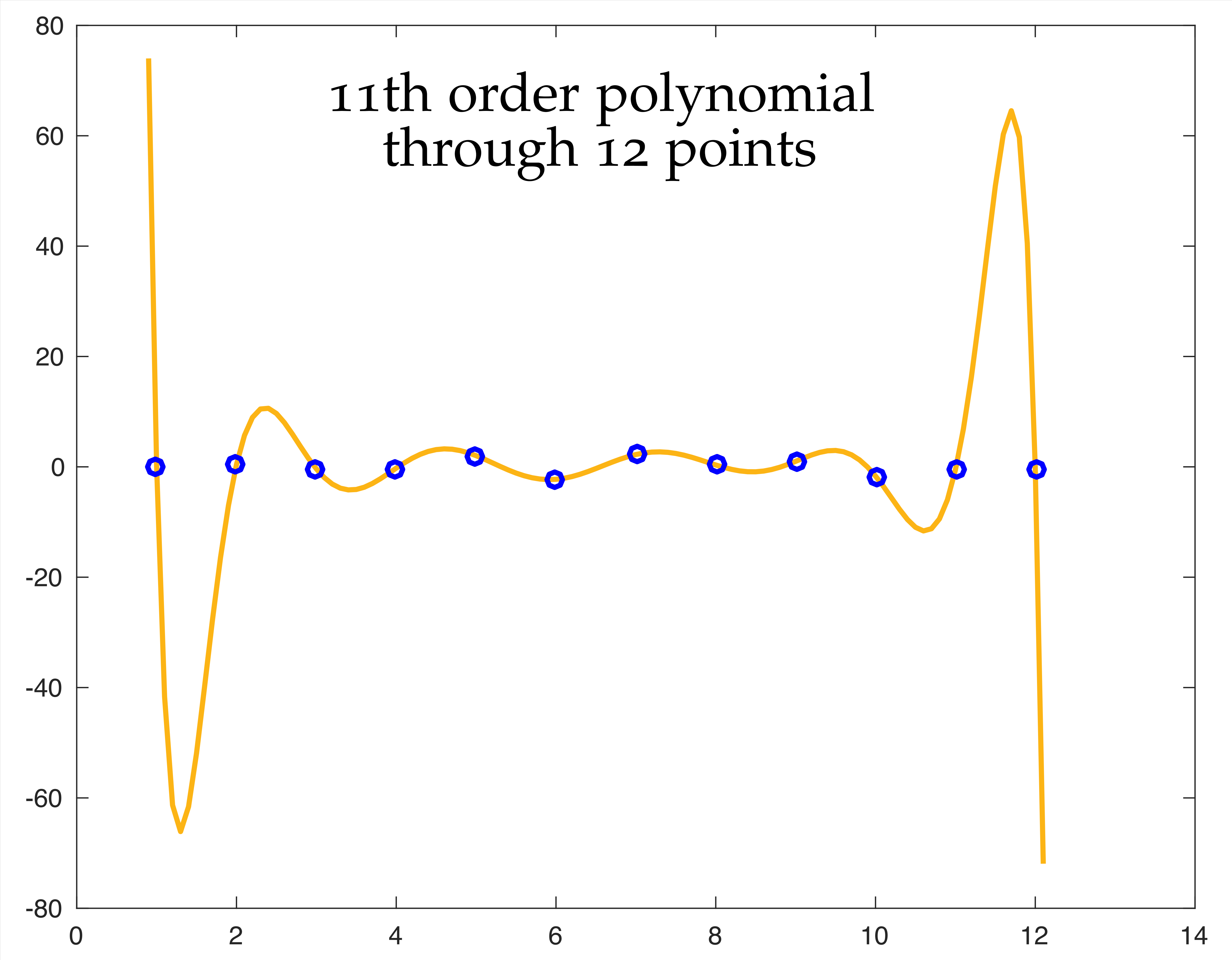


What is the “best” quadratic polynomial that passes through the points? $y = a_0 + a_1 x + a_2^2 + \cdots a_{n-1} x^{n-1}$

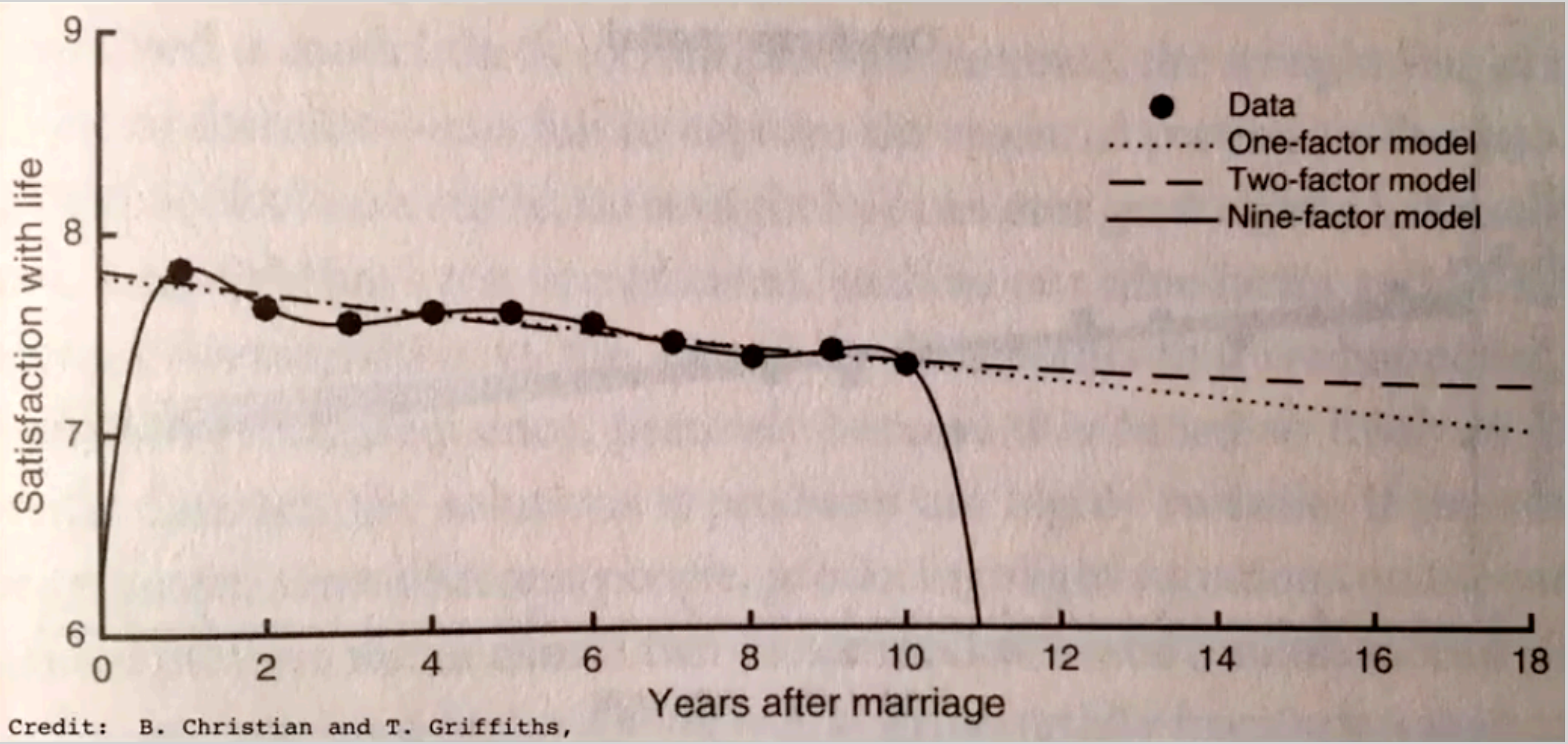
$$\begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \\ 1 & x_4 & x_4^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

Issue with Polynomial Interpolation

- Tend to be oscilatory in high order interp/regression
- Not numerically stable! x^{n-1}



Example

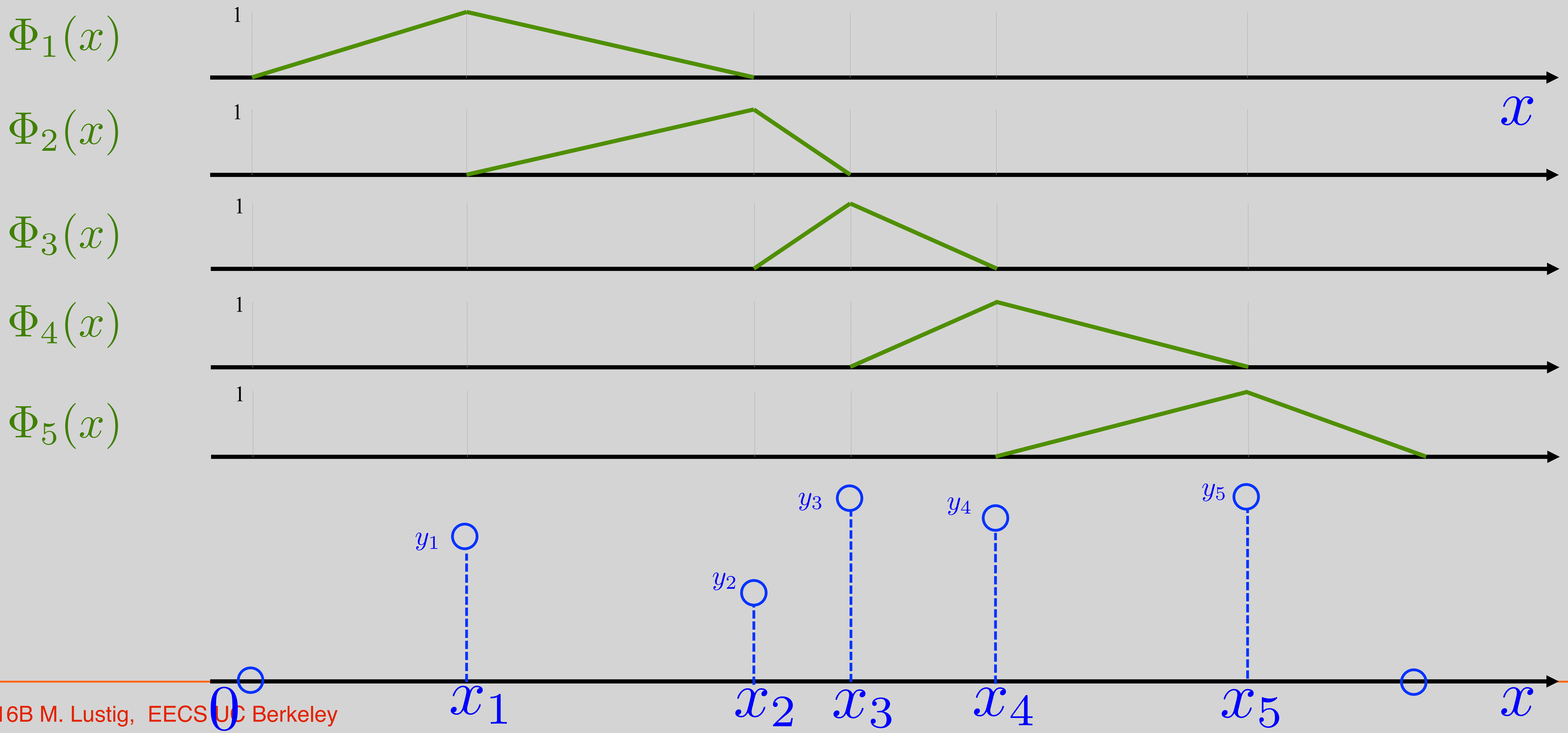

Satisfaction with life
9
8
7
6
0
2
4
6
8
10
12
14
16
18
Years after marriage
Data
One-factor model
Two-factor model
Nine-factor model
Credit: B. Christian and T. Griffiths,

Interpolation with Basis Functions

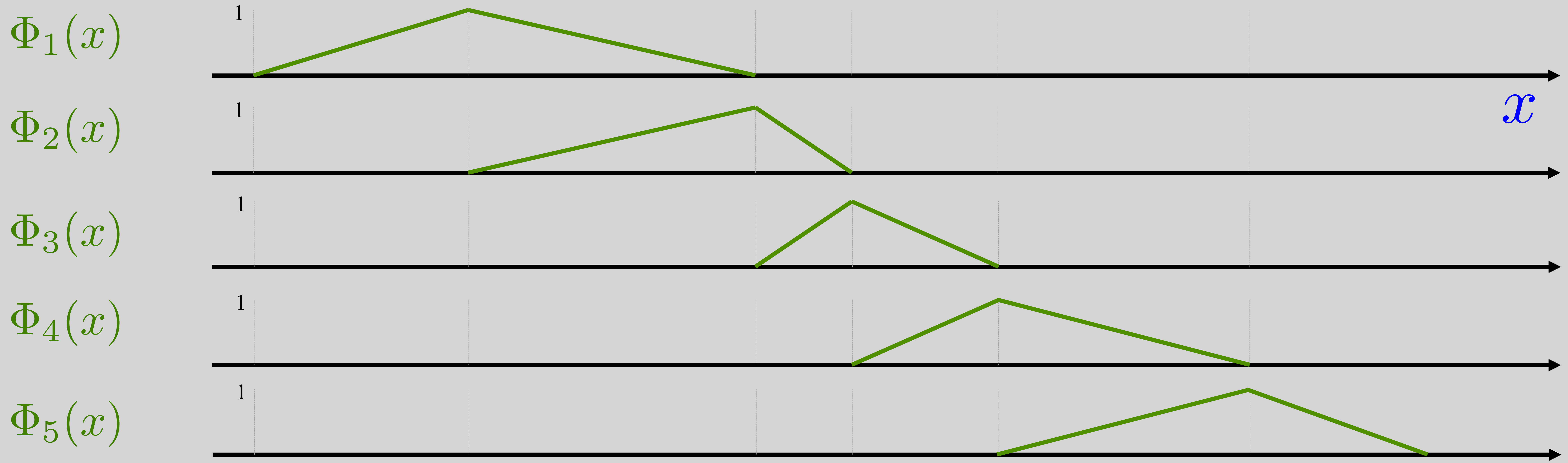
Given, (x_i, y_i) $i=1,\ldots,n$

Define a set of Cont. functions $\Phi_i(x)$

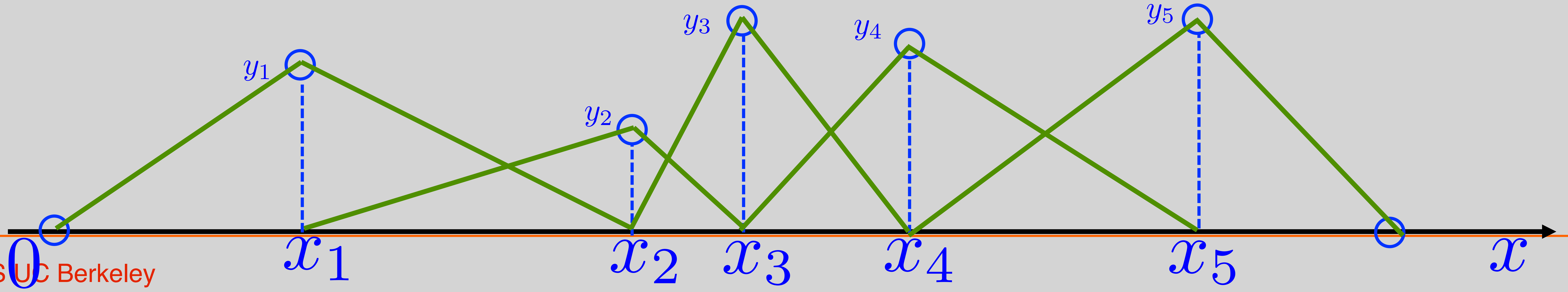
$$\Phi_i(x_i) = 1$$
$$\Phi_i(x_j) = 0$$



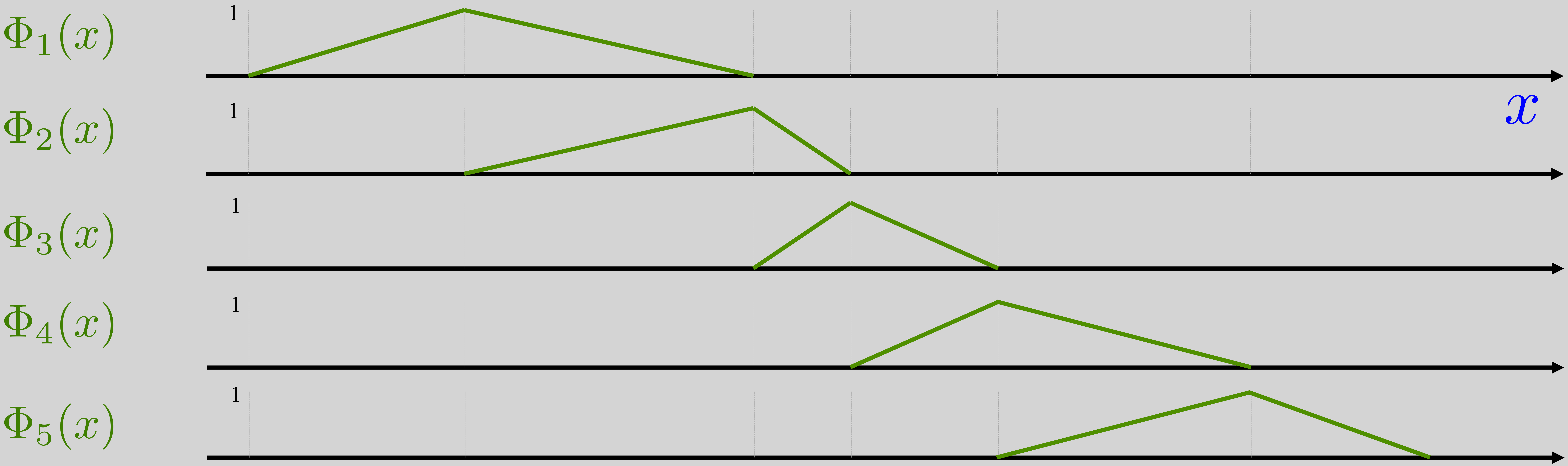
Interpolation with Basis Function



$$y(x) = y_1\Phi_1(x) + y_2\Phi_2(x) + y_3\Phi_3(x) + y_4\Phi_4(x) + y_5\Phi_5(x)$$



Interpolation with Basis Function



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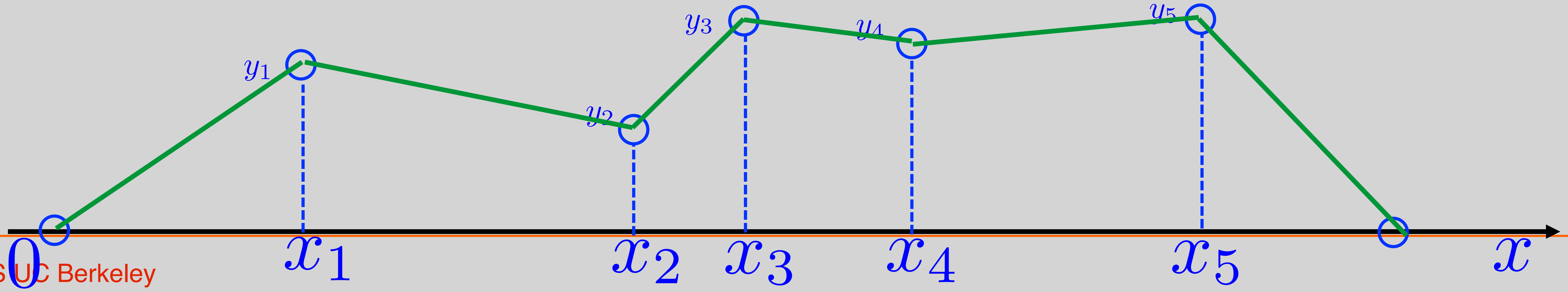
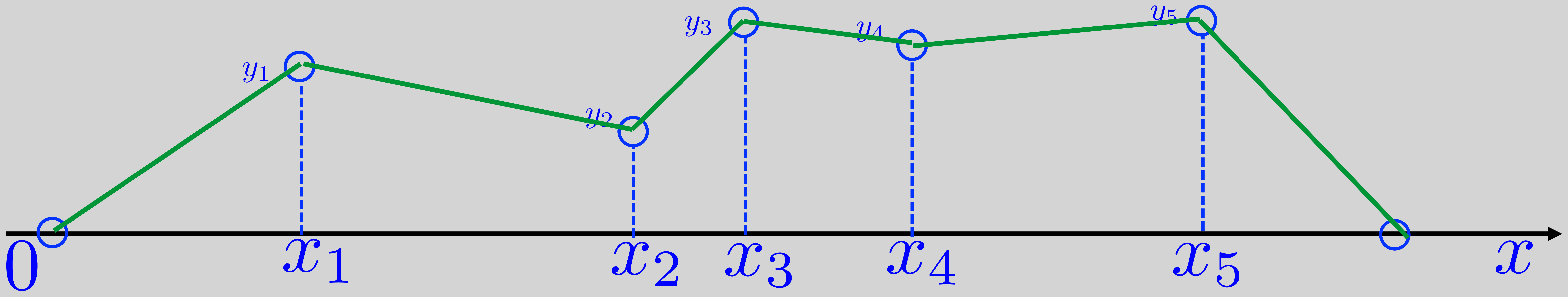


Interpolation with Basis Functions

- Interpolation:

$$y(x) = \sum_{i=1}^{n} y_i \Phi_i(x) \quad \Rightarrow y(x_i) = \sum_{i=1}^{n} y_i \Phi_i(x_i) = y_i$$



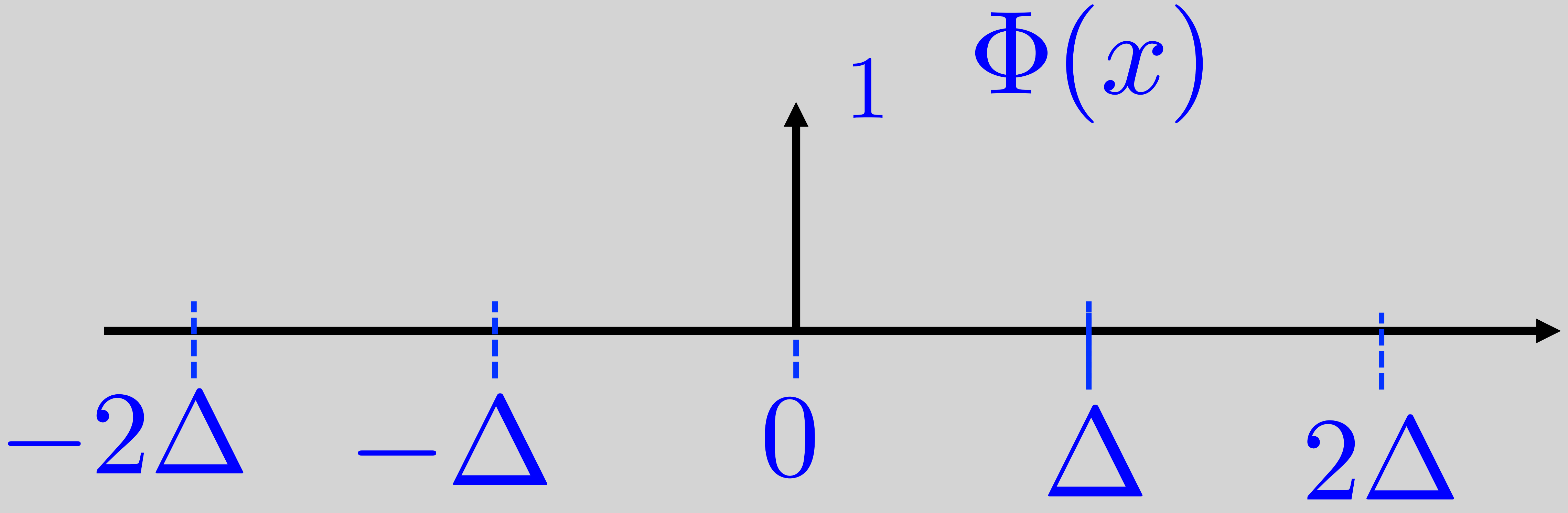
Interpolation with Basis Functions

- Guarenteed value of known points
- Control of continuous function behaviour between known points

- Often used for equispaced points
  – Can use a single “kernel” and shift it to get a basis

Kernel Interpolation, equispaced sampling

$$(x_i, y_i) \qquad i = 1, 2, 3, \cdots, N \qquad x_{i+1} - x_i = \Delta \quad \forall i$$



Define: $\Phi(x)$

$$\Phi(0) = 1$$

$$\Phi(k\Delta) = 0 \qquad k = \text{integer} \neq 0$$

$$y(x) = \sum_{k=1}^{N} y_k \Phi(x - k\Delta)$$

Kernel Interpolation, equispaced sampling

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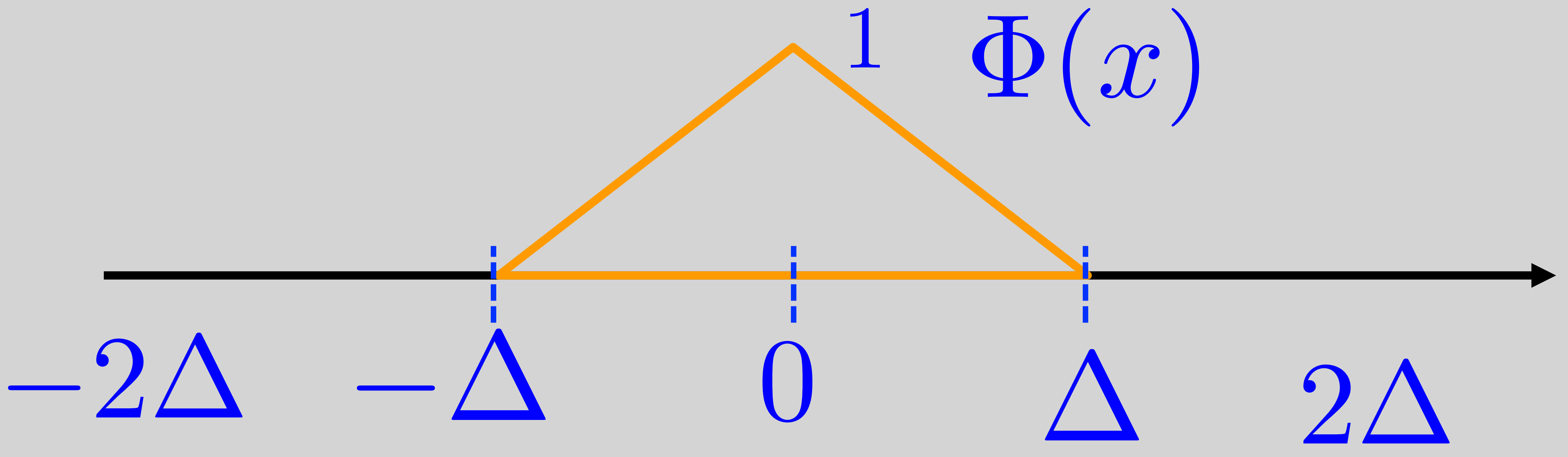

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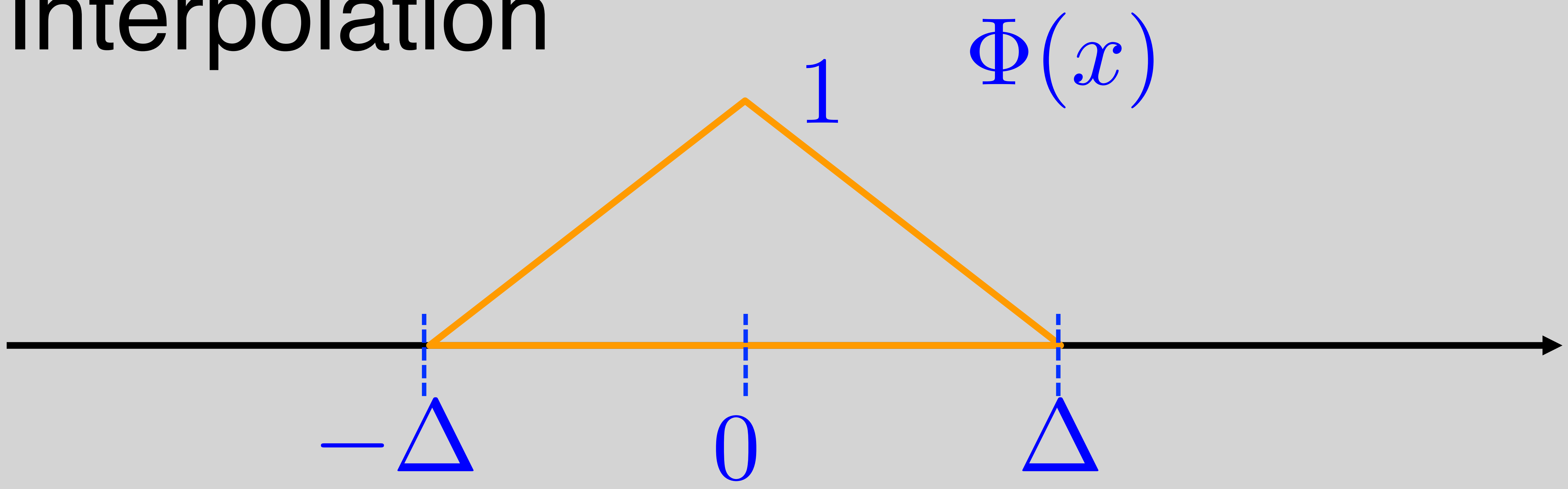
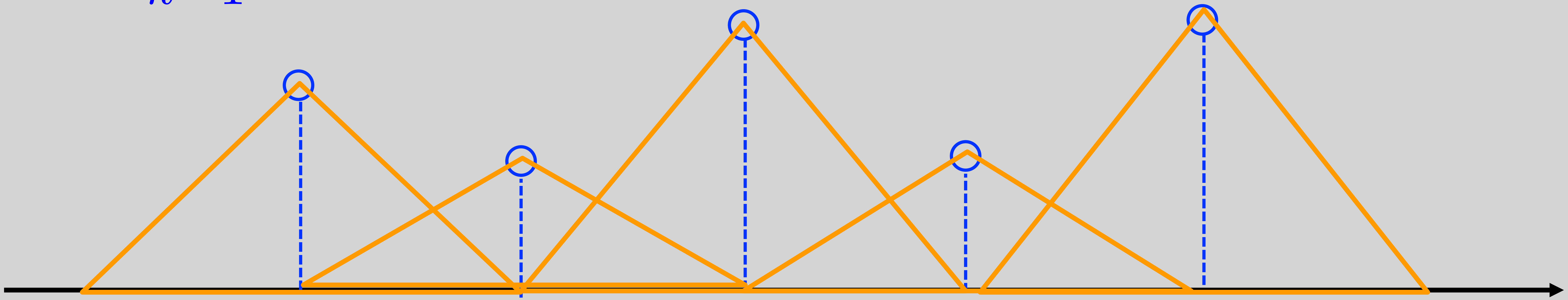
Example:

- Linear Interpolation

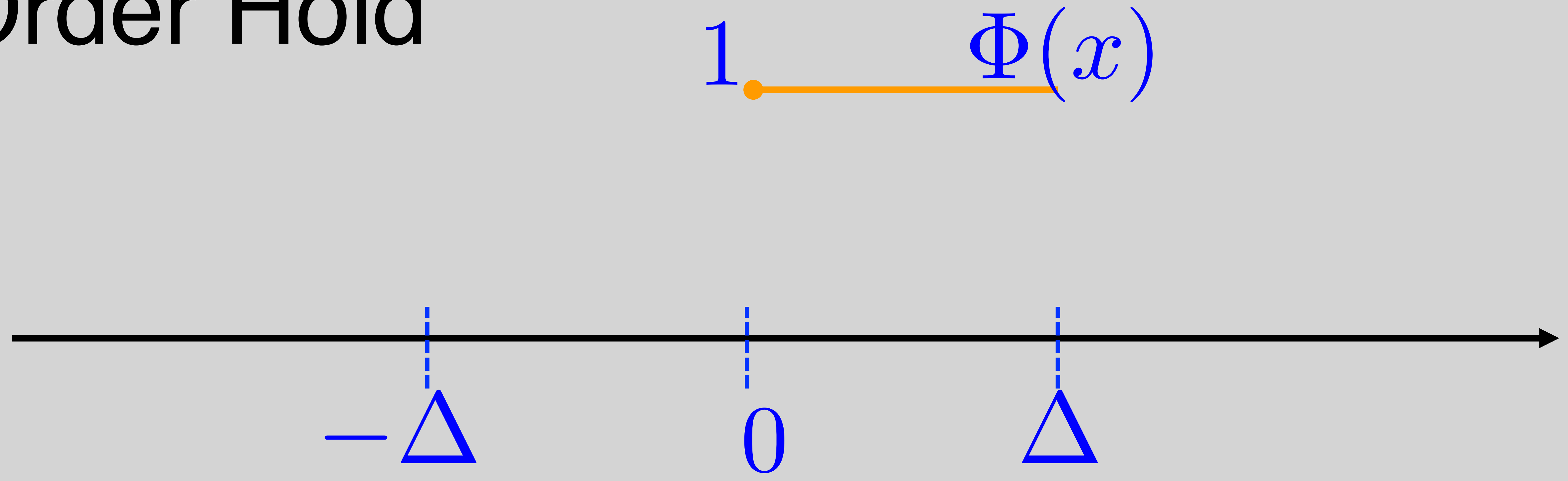


$$y(x) = \sum_{k=1}^{N} y_k \Phi(x - k\Delta)$$

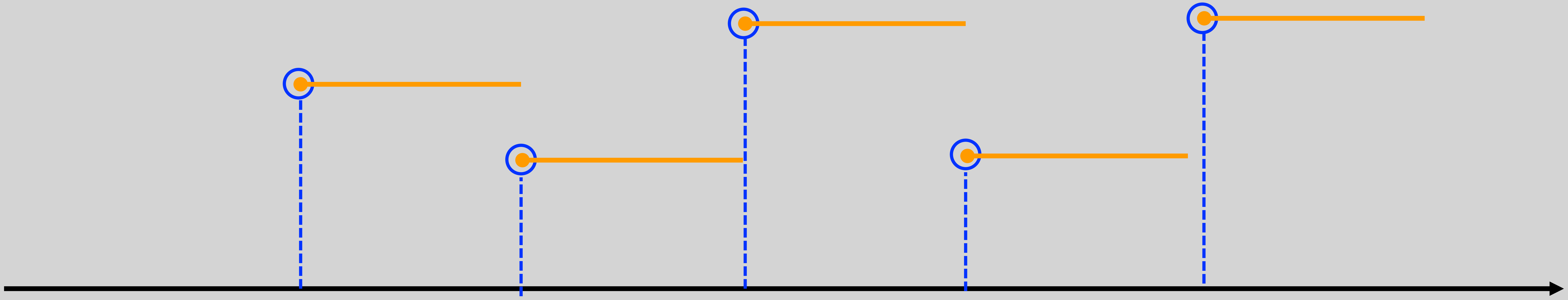


Example:

- Zero-Order Hold



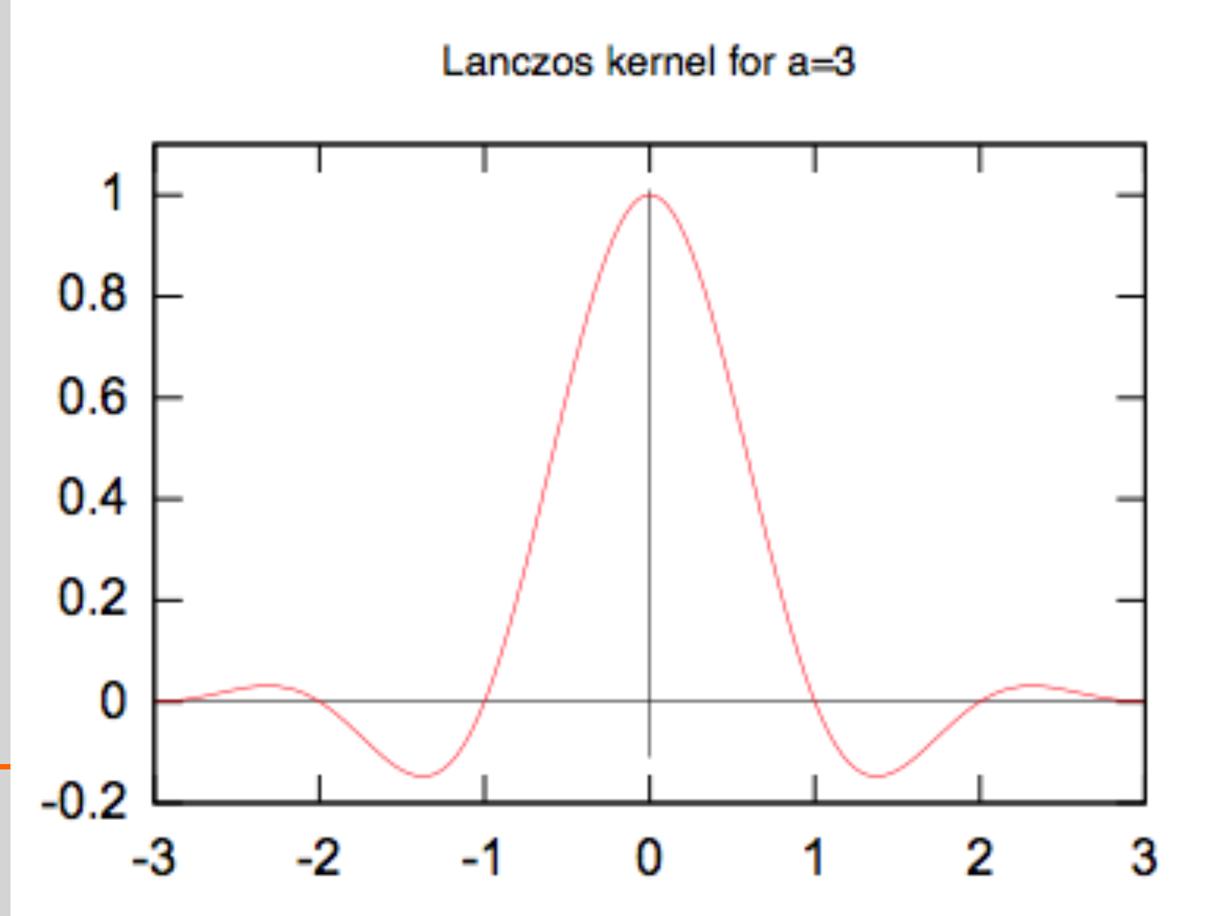
$$y(x) = \sum_{k=1}^{N} y_k \Phi(x - k\Delta)$$



Interpolating Elad

Zero-order-hold

linear



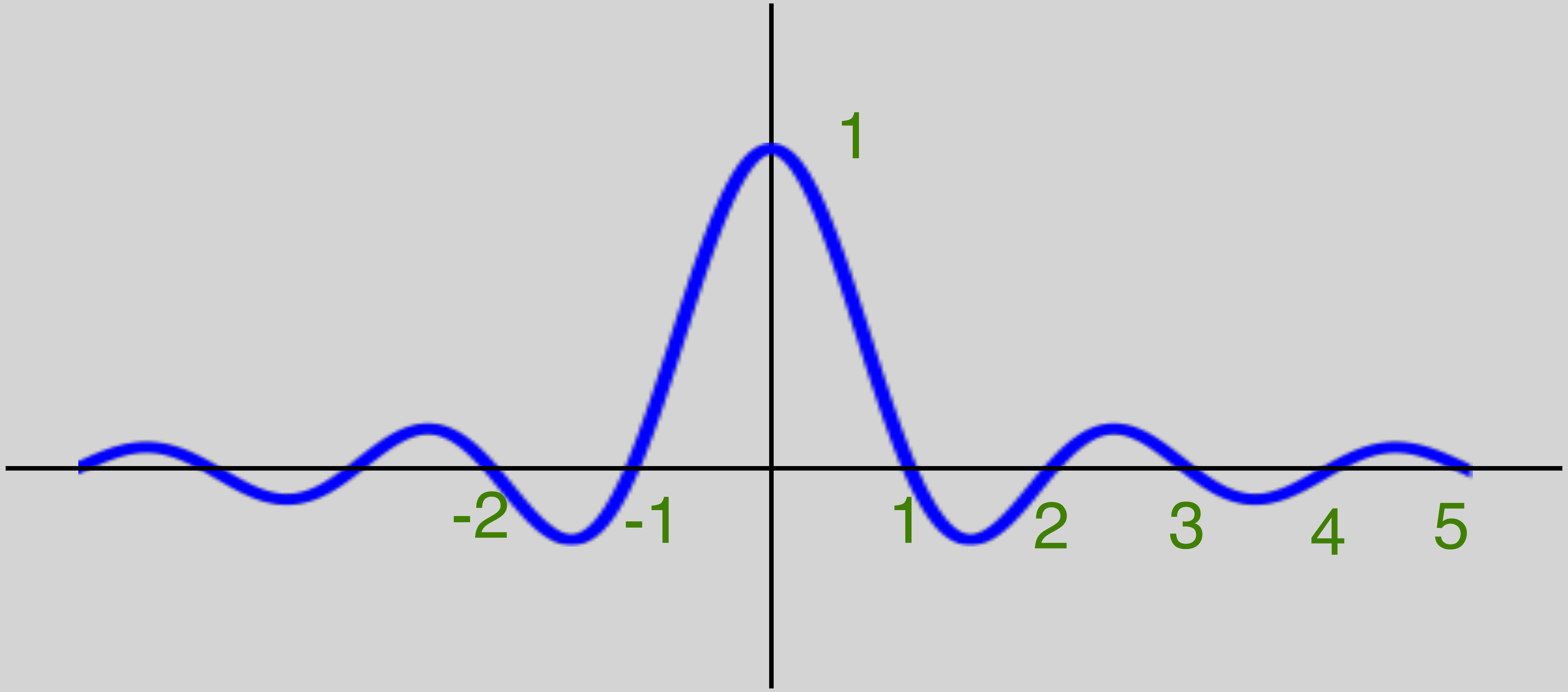
What is common to all these logos?

IEEE
Signal Processing Society ®

The sinc function

- Sinc:

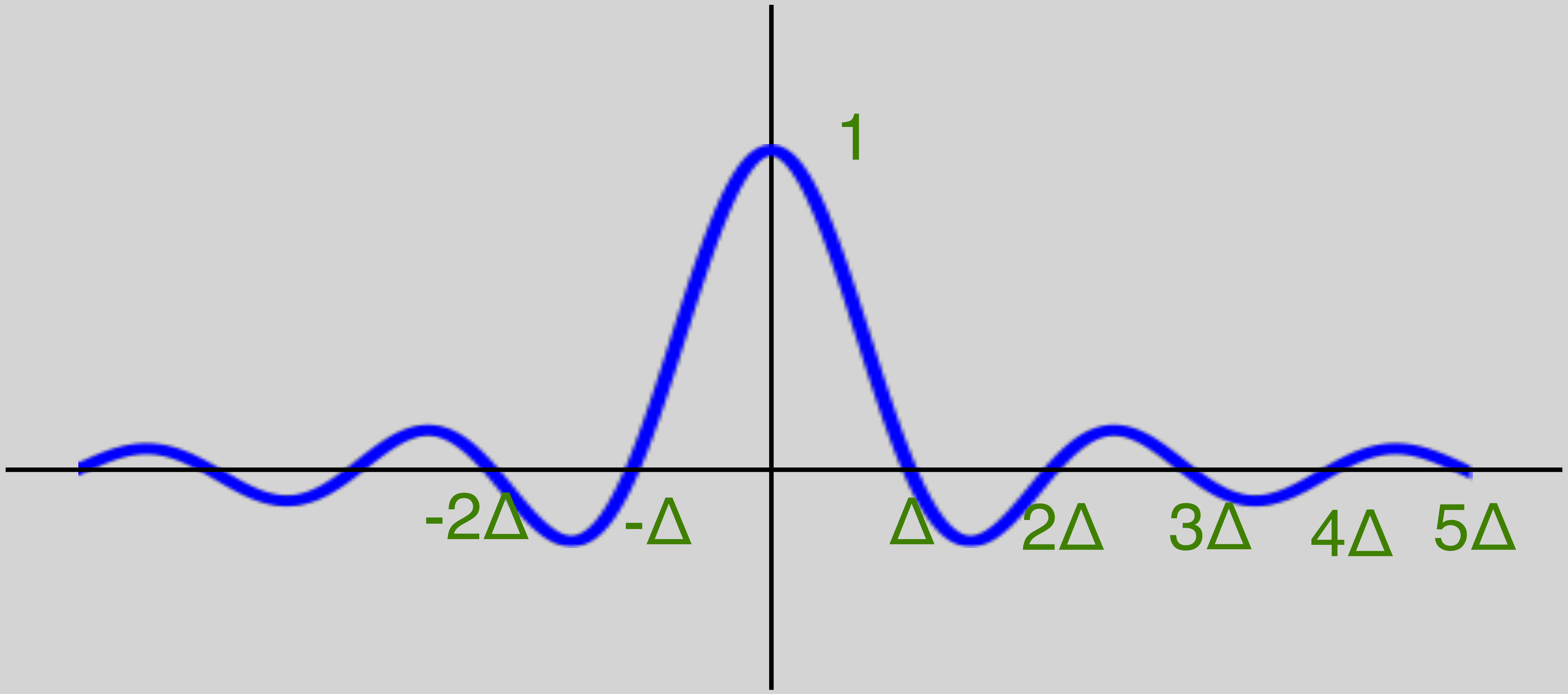
$$\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x} \quad \Rightarrow \begin{cases} \frac{\sin(\pi x)}{\pi x} & x \neq 0 \\ 1 & x = 0 \end{cases}$$



The sinc function

- Let $$\Phi(x) = \text{sinc}\left(\frac{x}{\Delta}\right)$$

then $$\Phi(k\Delta) = \text{sinc}(k) = ?$$



The sinc function

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then $$\Phi(k\Delta) = \mathrm{sinc}(k) = \begin{cases} 0 & k \neq 0 \\ 1 & k = 0 \end{cases}$$

Interpolation with sinc: $$y(x) = \sum_{k=-\infty}^{\infty} y_k \Phi(x - k\Delta)$$

The sinc function

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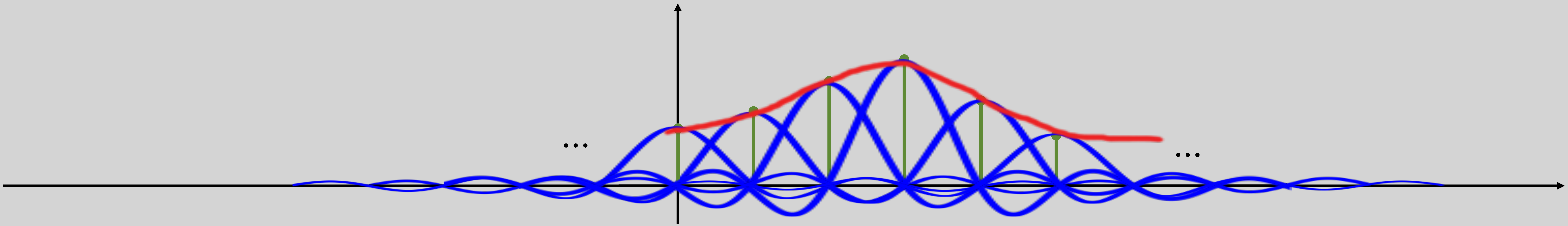


Bandlimitedness

- The sinc function does not contain frequencies beyond a certain bandwidth

$$\text{sinc}(x) = \frac{1}{\pi}\int_0^{\pi} \cos(\omega x)d\omega$$

$$\left.\frac{\sin(\omega x)}{\pi x}\right|_0^{\pi} = \frac{\sin \pi x}{\pi x} \quad x \neq 0$$

Sinc is an infinite sum of cosine functions with frequencies in the range $\omega \in [0, \pi]$

More in EE120, EE123!

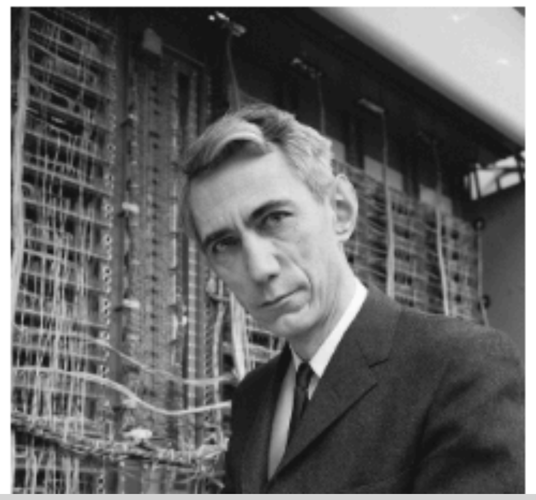
Sampling and Recovey

- Due to Shanon – Nyquist

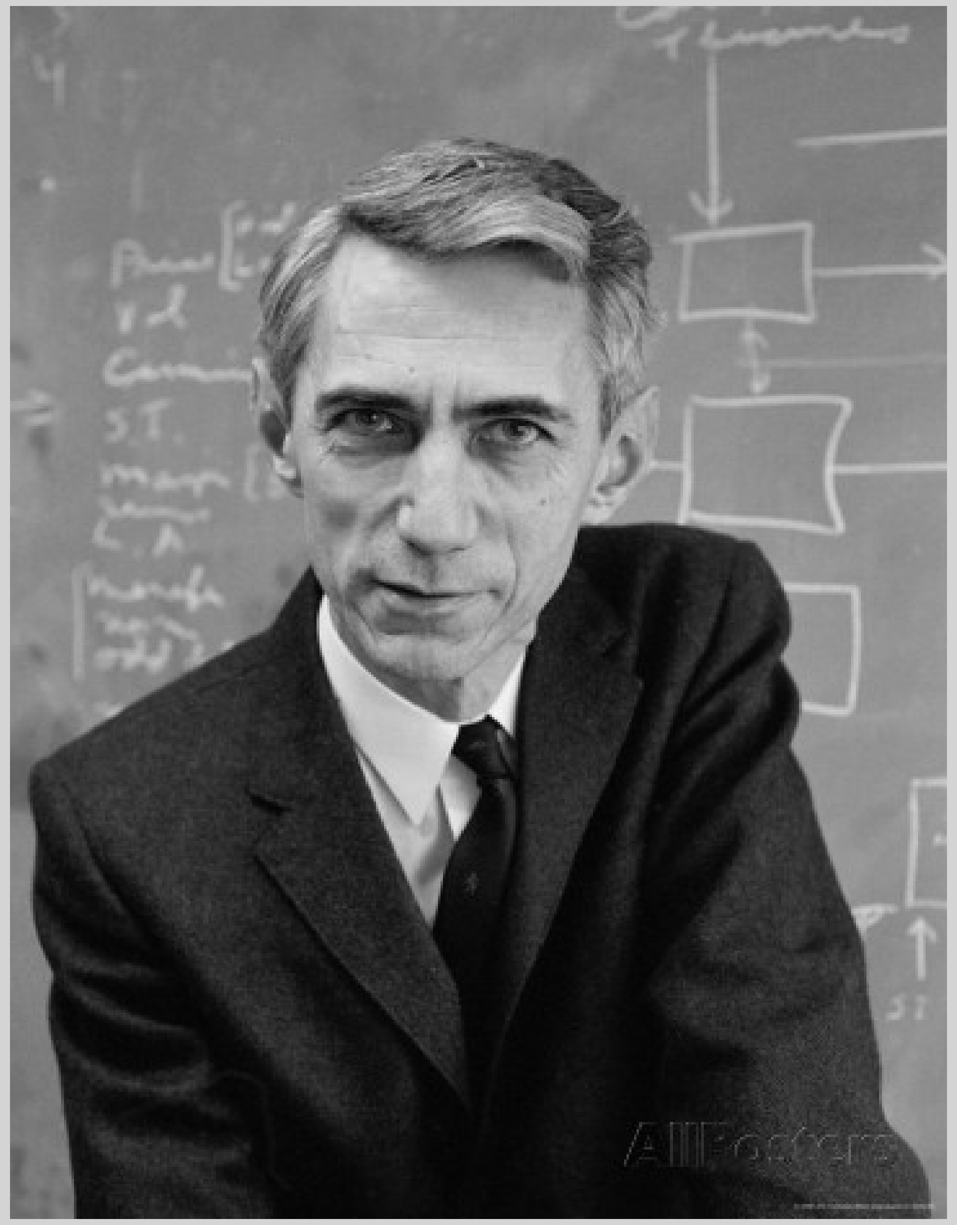
CLAUDE SHANNON, THE FATHER OF THE INFORMATION AGE, TURNS 1100100

By Siobhan Roberts April 30, 2016

Twelve years ago, Robert McEliece, a mathematician and engineer at Caltech, won the Claude E. Shannon Award, the highest honor in the field of information theory. During his acceptance lecture, at an international symposium in Chicago, he discussed the prize's namesake, who died in 2001. Someday, McEliece imagined, many



rarely. Yet he still tinkered, in the time he might have spent cultivating the big reputation that scientists of his stature tend to seek. In 1973, the Institute of Electrical and Electronics Engineers christened the Shannon Award by bestowing it on the man himself, at the International Symposium on Information Theory in Ashkelon, Israel. Shannon had a bad case of nerves, but he pulled himself together and delivered a fine lecture on feedback, then dropped off the scene again. In 1985, at the International Symposium in



Claude Shannon 1916-2001

Harry Nyquist 1889-1976

https://www.newyorker.com/tech/elements/claude-shannon-the-father-of-the-information-age-turns-1100100

Sampling and Recovery

- Can we perfectly recover an analog signal from its samples?

Analog signal:

$$y(x) = f(x)$$

Sample:

$$y[n] = f(n\Delta)$$

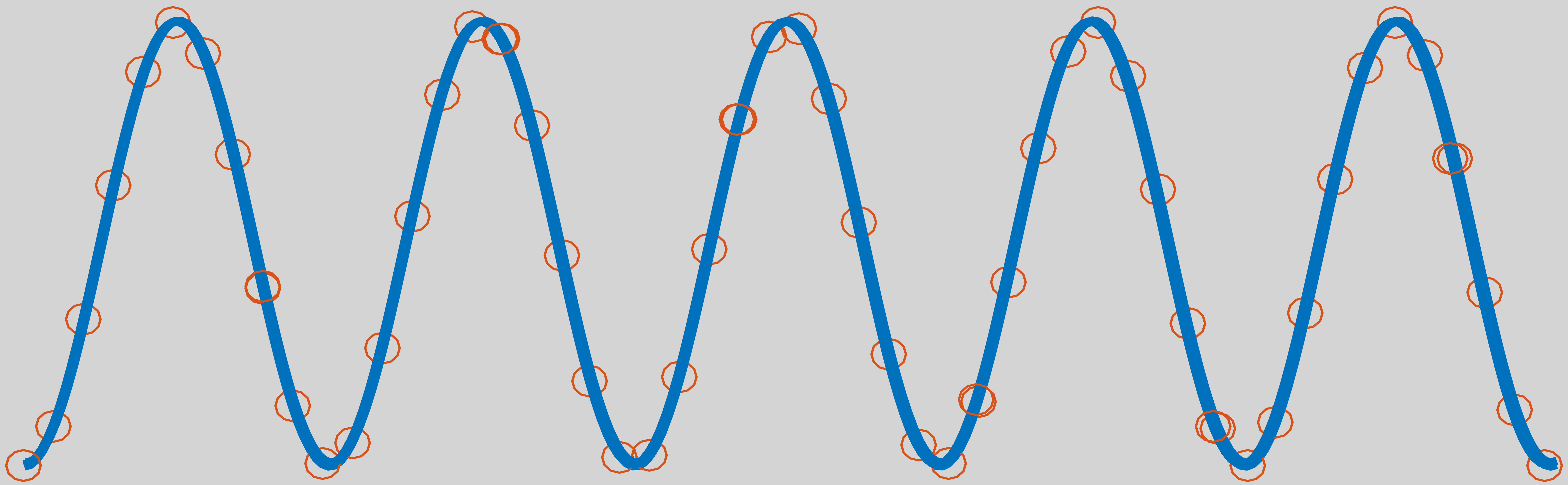
Interpolate:

$$\hat{f}(x) = \sum_{n=-\infty}^{\infty} y[n]\Phi(x - n\Delta)$$

$$=? f(x)$$

Sampling a sinusoid

- What rate should you be sampling a sinusoid?



Sampling Theorem

- If f(x) is bandlimited by frequency w_{max}, then

$$f(x) = \hat{f}(x) = \sum_{n=-\infty}^{\infty} y[n]\Phi(x - n\Delta) \qquad \Phi(x) = \text{sinc}\left(\frac{x}{\Delta}\right)$$

As long as,

$$\omega_{\max} < \frac{\pi}{\Delta}$$

$$\frac{\omega_{\max}}{\pi} < \frac{1}{\Delta}$$

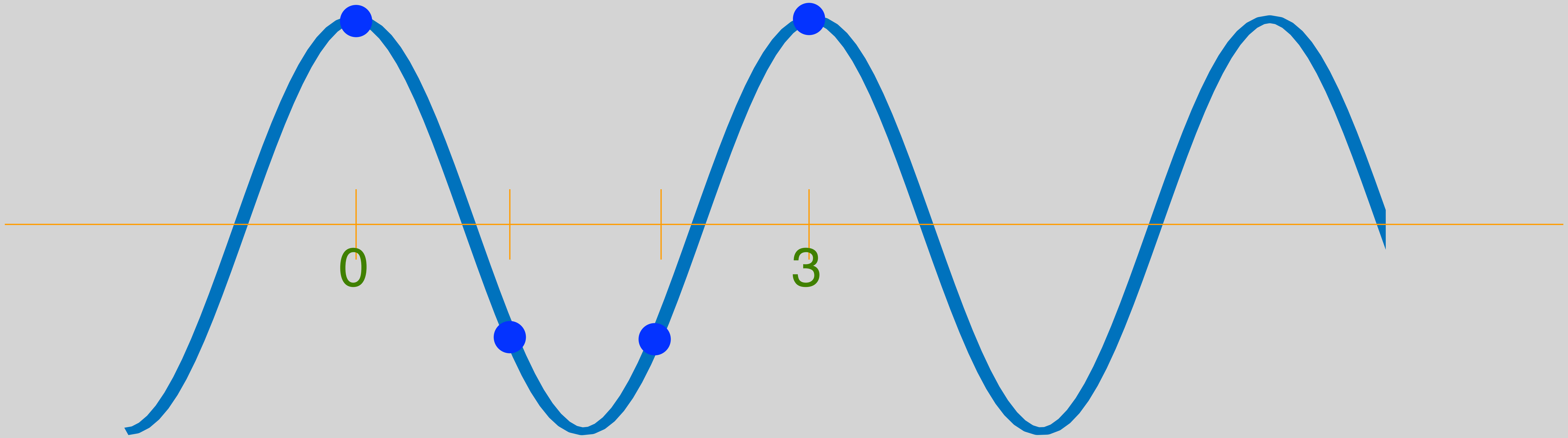
$$2\frac{\omega_{\max}}{2\pi} < \frac{1}{\Delta}$$

$$2f_{\max} < f_s \qquad \omega_s > 2\omega_{\max}$$

Proof: EE120, EE123

Example 1

$$f(x) = \cos(\frac{2\pi}{3}x) \qquad \Delta = 1 \qquad \omega_{\max} <? \frac{\pi}{\Delta}$$



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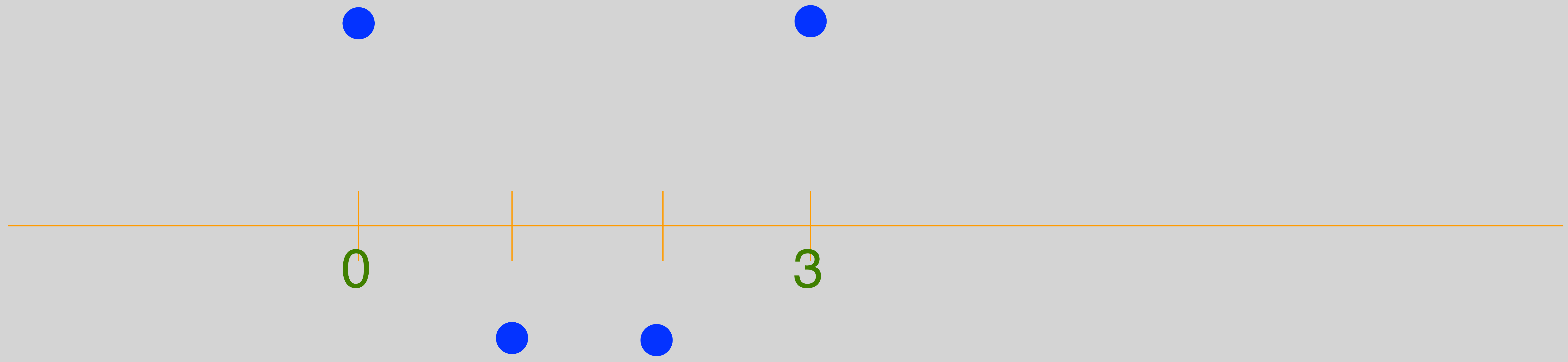

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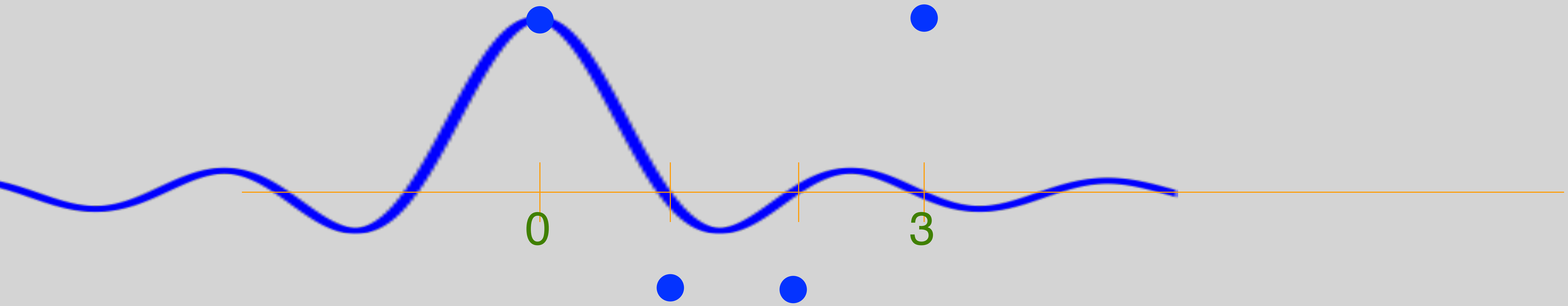

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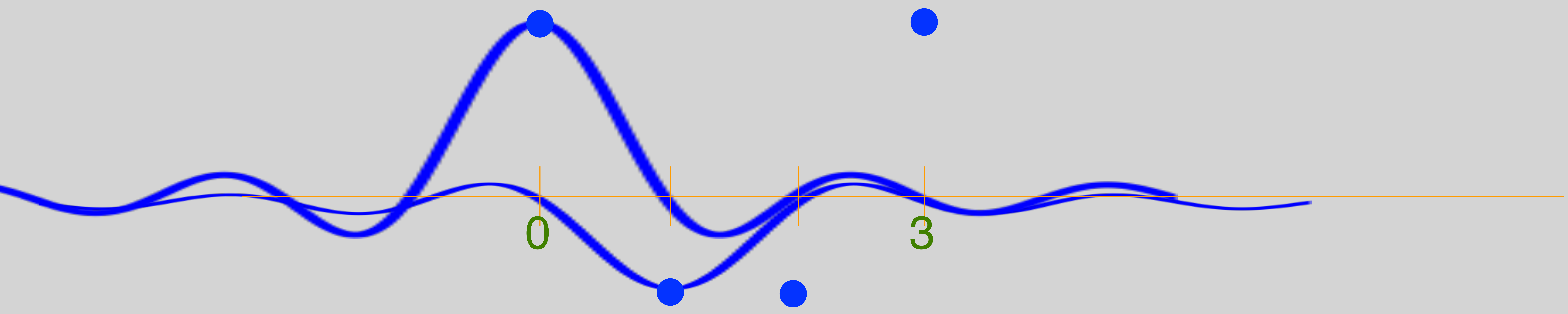

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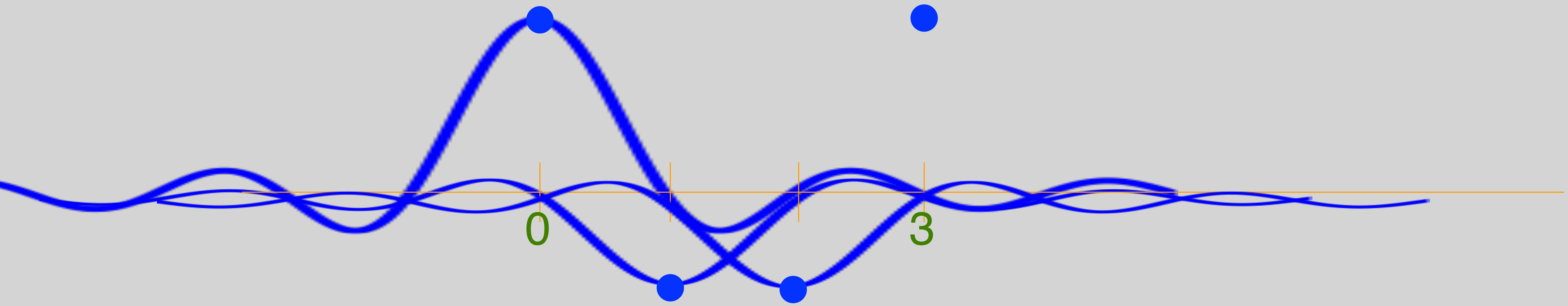

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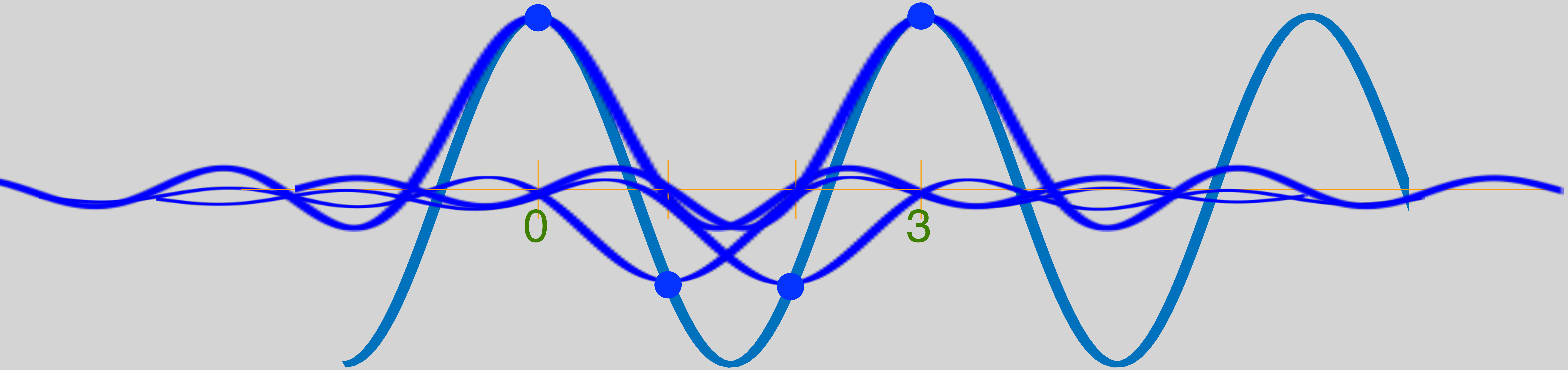

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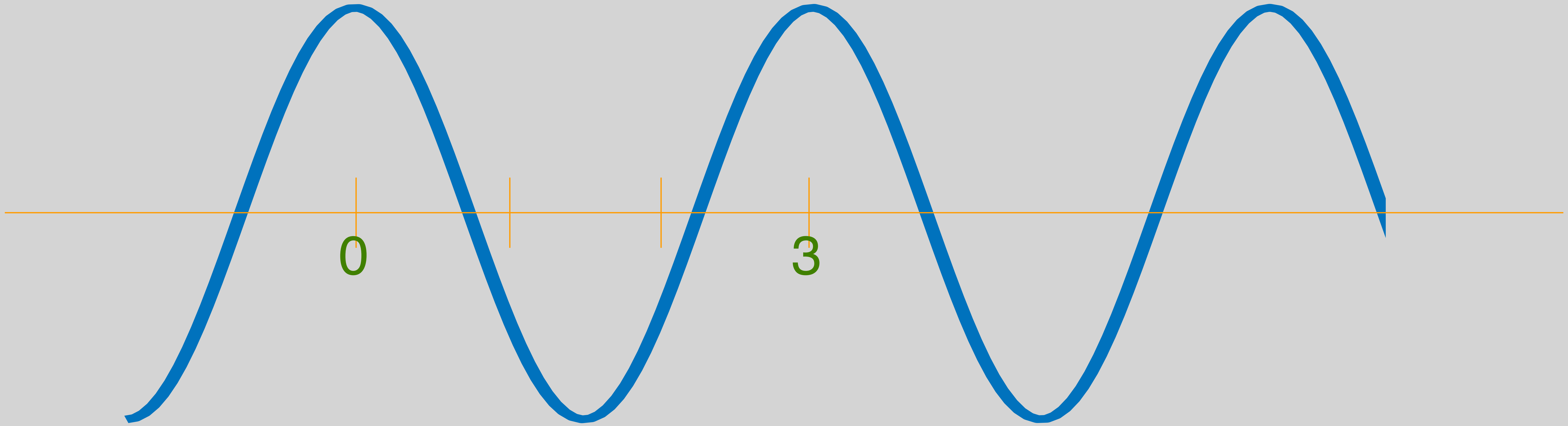
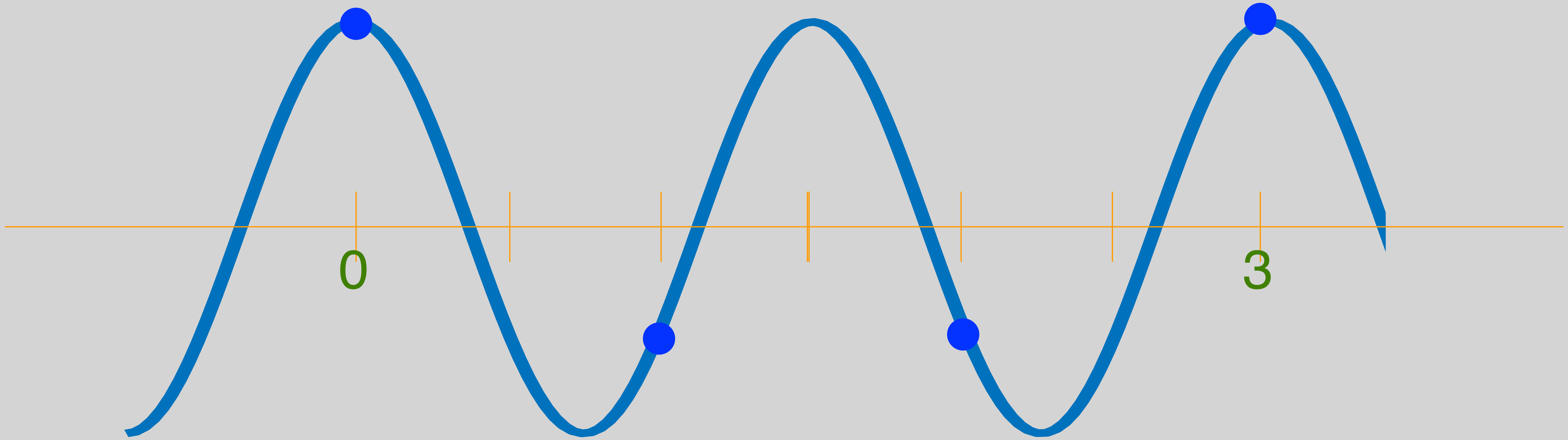


Example 2

$$f(x) = \cos(\frac{4\pi}{3}x) \qquad \Delta = 1 \qquad \omega_{\max} <? \frac{\pi}{\Delta}$$



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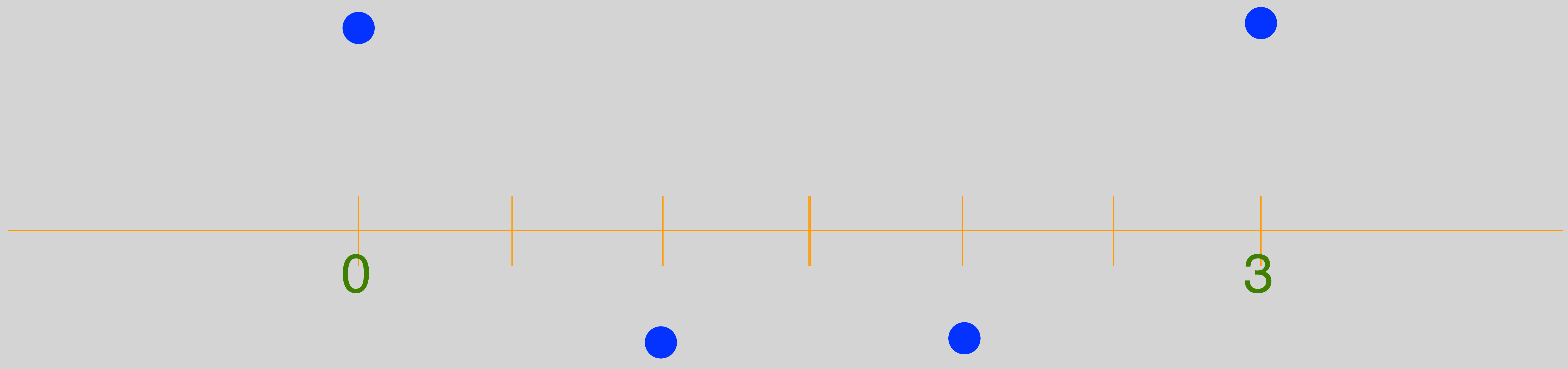

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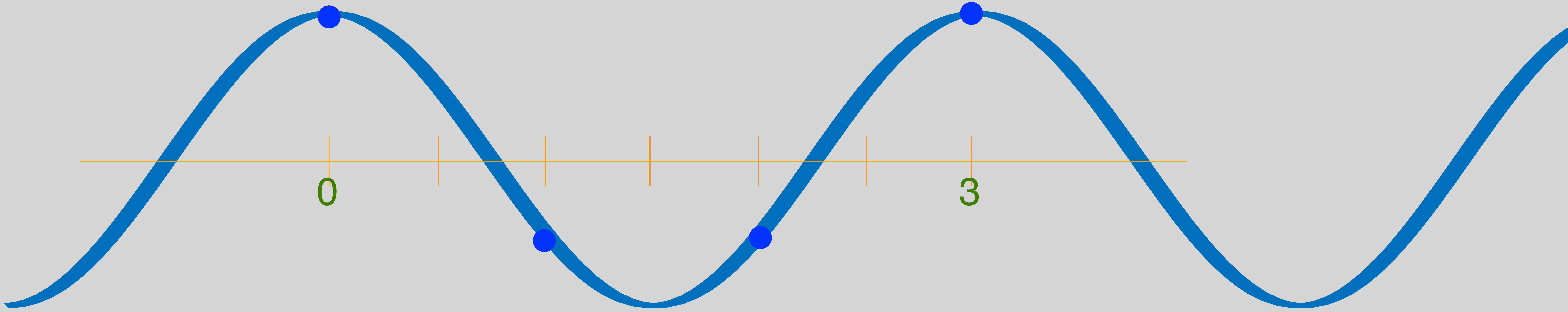
$$f(x) = \cos(\frac{4\pi}{3}x) \qquad \Delta = 1 \qquad \omega_{\max} <? \frac{\pi}{\Delta}$$



- Sinc interpolation gives: $\hat{f}(x) = \cos(\frac{2\pi}{3}x)$

Aliasing of high frequencies into lower ones!

Aliasing and Phase Reversal

$$f(x) = \cos(\omega x + \phi) \qquad \Delta = 1$$

$$y[n] = \cos(\omega n + \phi)$$

- Highest interpolated frequency will not be higher than π

$$y[n] = \cos(\omega n + \phi) = \cos(2\pi n - (\omega n + \phi)) = \cos((2\pi - \omega)n - \phi)$$

$$\cos(2\pi n - \theta) = \cos(\theta)$$

If $\pi < w < 2\pi$ and $\Delta=1$, there's an equivalent lower frequency signal with the same samples!

$$\hat{f}(x) = \cos((2\pi - \omega)x - \phi)$$

Example 2

$$f(x) = \cos(\omega x + \phi) \qquad \Delta = 1 \quad \omega = \frac{4\pi}{3} \;\; \phi = 0$$

$$\hat{f}(x) = \cos((2\pi - \omega)x - \phi)$$

$$= \cos(\frac{2\pi}{3}x)$$

Example 3

$$f(x) = \sin(1.9\pi x) \qquad \Delta = 1$$

$$= \cos(1.9\pi x - \frac{\pi}{2})$$

$$\hat{f}(x) = \cos(0.1\pi x + \frac{\pi}{2})$$

$$= -\sin(0.1\pi x)$$