EECS 16B Designing Information Devices and Systems II
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Discussion 8A

Primer to Complex Linear Algebra

Complex Conjugates and Adjoints

The complex conjugate of a complex number $z = a + bj = re^{j\theta}$ is defined as

$$z^* = z^H = a - bj = re^{-j\theta}$$

Let $\vec{z} \in \mathbb{C}^n$.

$$\vec{z} = \begin{bmatrix} z_1 & z_2 & \dots & z_n \end{bmatrix}^T$$

The conjugate transpose (or adjoint, or Hermitian transpose) of $\vec{z}$ is defined as

$$\vec{z}^* = \vec{z}^H = \begin{bmatrix} z_1^* & z_2^* & \dots & z_n^* \end{bmatrix}$$

Inner Product Properties

An inner product on a complex vector space $\mathbb{C}^n$ is a function such that the following all hold for $\vec{u}, \vec{v}, \vec{w} \in \mathbb{C}^n$

$$\begin{aligned} \langle \vec{u}, \vec{v} \rangle &= \langle \vec{v}, \vec{u} \rangle^* \\ \langle \vec{u} + \vec{v}, \vec{w} \rangle &= \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle \\ \langle \alpha \vec{u}, \vec{v} \rangle &= \alpha \langle \vec{u}, \vec{v} \rangle \\ \langle \vec{u}, \vec{u} \rangle &\geq 0 \\ \langle \vec{u}, \vec{u} \rangle &= 0 \implies \vec{u} = \vec{0} \end{aligned}$$

Questions

1. Controls

Consider the following system:

$$\begin{aligned} \frac{dx_1(t)}{dt} &= -x_1(t)^2 + x_2(t)u(t) \\ \frac{dx_2(t)}{dt} &= 2x_1(t) - 2x_2(t)u(t) \end{aligned}$$

(a) Choose states and write a state space model for the system in the form $\frac{d\vec{x}(t)}{dt} = f(\vec{x}(t), u(t))$.

Answer:

Because the states should be related to their derivatives, we choose $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$.

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} f_1(\vec{x}(t), u(t)) \\ f_2(\vec{x}(t), u(t)) \end{bmatrix} = \begin{bmatrix} -x_1(t)^2 + x_2(t)u(t) \\ 2x_1(t) - 2x_2(t)u(t) \end{bmatrix}$$

(b) Find the equilibrium $\vec{x}^*$ and input u^* when $x_2^* = 1$ and $u^* = 1$.

Answer:

Plugging in $x_2^* = 1$ and $u^* = 1$, we solve the system of equations for x_1^*.
Looking at the second equation, we get:

$$0 = 2x_1^* - 2 \implies x_1^* = 1$$

$$\vec{x}^* = \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$u^* = 1$$

(c) Linearize the system around the equilibrium state and input from the previous part. Your answer should be in the form $\frac{d\tilde{\vec{x}}(t)}{dt} = A\tilde{\vec{x}}(t) + B\tilde{u}(t)$.

Answer:

Recall that $\tilde{\vec{x}}(t) \triangleq \vec{x}(t) - \vec{x}^*$ and $\tilde{u}(t) \triangleq u(t) - u^*$.
From linearization, we get:

$$\frac{d\tilde{\vec{x}}(t)}{dt} = \underbrace{[\nabla_{\vec{x}} f(\vec{x}^*(t), u^*(t))]}_{A}\tilde{\vec{x}}(t) + \underbrace{[\nabla_{\vec{u}} f(\vec{x}^*(t), u^*(t))]}_{B}\tilde{u}(t)$$

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} -2x_1^* & u^* \\ 2 & -2u^* \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 2 & -2 \end{bmatrix}$$

$$B = \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{bmatrix} = \begin{bmatrix} x_2^* \\ -2x_2^* \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

(d) Is this system controllable? Is it stable?

Answer:

$$R_2 = \begin{bmatrix} AB & B \end{bmatrix} = \begin{bmatrix} -4 & 1 \\ 6 & -2 \end{bmatrix}$$

$$\det(A - \lambda I) = (-2-\lambda)^2 - 2 = \lambda^2 + 4\lambda + 2 = 0$$

$$\lambda = \frac{-4 \pm \sqrt{4^2 - 8}}{2} = -2 \pm \sqrt{2}$$

The controllability matrix has rank 2, so this system is controllable. This system also only has negative eigenvalues, so it is stable.

(e) Find a state feedback controller K to place both system eigenvalues at $\lambda = -1$, where $\tilde{u}(t) = K\vec{\tilde{x}}(t)$.

Answer:

$$
\begin{aligned}
K &= \begin{bmatrix} k_1 & k_2 \end{bmatrix} \\
A + BK &= \begin{bmatrix} -2 & 1 \\ 2 & -2 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} \\
&= \begin{bmatrix} -2 & 1 \\ 2 & -2 \end{bmatrix} + \begin{bmatrix} k_1 & k_2 \\ -2k_1 & -2k_2 \end{bmatrix} \\
&= \begin{bmatrix} -2 + k_1 & 1 + k_2 \\ 2 - 2k_1 & -2k_2 - 2 \end{bmatrix}
\end{aligned}
$$

Now, let's find the characterstic polynomial for the closed-loop matrix:

$$
\begin{aligned}
&\lambda^2 + \lambda(4 - k_1 + 2k_2) + (2 - k_1)(2k_2 + 2) - (1 + k_2)(2 - 2k_1) = 0 \\
&\lambda^2 + \lambda(4 - k_1 + 2k_2) + (2k_2 + 2) = 0
\end{aligned}
$$

Next, comparing the coefficients of our required characteristic equation, $\lambda^2 + 2\lambda + 1 = 0$, we get:

$$
\begin{aligned}
4 + 2k_2 - k_1 &= 2 \\
2k_2 + 2 &= 1
\end{aligned}
$$

Solving the above equations, we get, $k_1 = 1$ and $k_2 = -\frac{1}{2}$.

Extra Practice

1. Feedback Design

Consider the following system:

$$
\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}
$$

$$
\vec{f}(\vec{x}, u) = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}
$$

$$
\begin{aligned}
\frac{dx_1(t)}{dt} &= f_1(\vec{x}, u) = x_1(t)^2 x_2(t) - 4x_2(t) + u(t)x_2(t) \\
\frac{dx_2(t)}{dt} &= f_2(\vec{x}, u) = 2x_2(t) - 3x_1(t) - x_1(t)u(t)
\end{aligned}
$$

(a) Find the equilibrium points of $\vec{x}$ when $u(t) = 0$.

Answer:

To find the equilibrium points, we set the derivatives and $u(t)$ to 0:

$$
0 = (x_1(t)^2 - 4)x_2(t) \tag{1}
$$

$$0 = 2x_2(t) - 3x_1(t) \tag{2}$$

From Equation (2):

$$x_2(t) = \frac{3}{2}x_1(t)$$

Plugging into Equation (1):

$$0 = (x_1(t)^2 - 4)\frac{3}{2}x_1(t)$$

$$x_1(t) = -2,\ 0,\ 2$$

With the $x_1(t)$ values, we can find the corresponding $x_2(t)$ values for each equilibrium point (in the form (x_1, x_2)):

$$\text{equilibrium points} = (-2,-3), (0,0), (2,3)$$

(b) Linearize the system around $\vec{x^*} = \begin{bmatrix} 2 & 3 \end{bmatrix}^T$, and $u^*(t) = 0$.

Answer: Let, $\vec{\tilde{x}} = \vec{x} - \vec{x^*}$ and $\tilde{u}(t) = u(t) - u^*(t) = u(t)$.
The linearized system will have the form:

$$\frac{d\vec{\tilde{x}}}{dt} = A\vec{\tilde{x}} + B\tilde{u}$$

$$\frac{d\vec{\tilde{x}}}{dt} = \nabla_x \vec{f}(x,u)\vec{\tilde{x}} + \nabla_u \vec{f}(x,u)\tilde{u}\Big|_{x_1=2,x_2=3,u=0}$$

$$\frac{d\vec{\tilde{x}}}{dt} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} \vec{\tilde{x}} + \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_1}{\partial x_1} \end{bmatrix} \tilde{u}\Bigg|_{x_1=2,x_2=3,u=0}$$

$$\frac{d\vec{\tilde{x}}}{dt} = \begin{bmatrix} 2x_1x_2 & x_1^2 - 4 + u \\ -3 - u & 2 \end{bmatrix} \vec{\tilde{x}} + \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} \tilde{u}\Bigg|_{x_1=2,x_2=3,u=0}$$

$$\frac{d\vec{\tilde{x}}}{dt} = \begin{bmatrix} 12 & 0 \\ -3 & 2 \end{bmatrix} \vec{\tilde{x}} + \begin{bmatrix} 3 \\ -2 \end{bmatrix} \tilde{u}$$

(c) Is the linearized system stable?

Answer:

To check stability, we need to find the eigenvalues of A. Since A is lower triangular, the eigenvalues are $\lambda = 2, 12$. The condition for stability of a continuous system is $\text{Re}\{\lambda\} < 0$, so this system is unstable.

(d) Is the linearized system controllable?

Answer:

Since this is a 2×2 system, the controllability matrix is:

$$\mathscr{R}_2 = \begin{bmatrix} AB & B \end{bmatrix} = \begin{bmatrix} 36 & 3 \\ -13 & -2 \end{bmatrix}$$

The controllability matrix $\mathscr{R}_2$ is full rank, so the system is controllable.

(e) Using state feedback with $\tilde{u} = \begin{bmatrix} k_1 & k_2 \end{bmatrix} \vec{\tilde{x}}$, find k_1 and k_2 to make the system stable with $\lambda = -1, -9$.

Answer:

Plugging in $\tilde{u}(t)$:

$$\frac{d\vec{\tilde{x}}}{dt} = \begin{bmatrix} 12 & 0 \\ -3 & 2 \end{bmatrix} \vec{\tilde{x}} + \begin{bmatrix} 3 \\ -2 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} \vec{\tilde{x}}$$

$$\frac{d\vec{\tilde{x}}}{dt} = \begin{bmatrix} 12+3k_1 & 3k_2 \\ -3-2k_1 & 2-2k_2 \end{bmatrix} \vec{\tilde{x}}$$

$$\det(A - \lambda I) = (12 + 3k_1 - \lambda)(2 - 2k_2 - \lambda) + 3k_2(3 + 2k_2) = 0$$

$$\lambda^2 - (3k_1 - 2k_2 + 14)\lambda + (6k_1 - 15k_2 + 24) = 0$$

We need to set the characteristic polynomial to the characteristic polynomial with the desired eigenvalues:

$$(\lambda + 9)(\lambda + 1) = \lambda^2 + 10\lambda + 9 = \lambda^2 - (3k_1 - 2k_2 + 14)\lambda + (6k_1 - 15k_2 + 24) = 0$$

Coefficient matching:

$$-10 = 3k_1 - 2k_2 + 14 \tag{3}$$

$$9 = 24 + 6k_1 - 15k_2 \tag{4}$$

From Equation (3):

$$3k_1 = -24 + 2k_2$$

Plugging into Equation (4):

$$-48 + 24 + 4k_2 - 15k_2 = 9$$

$$k_2 = -3$$

$$k_1 = -\frac{24+6}{3} = -10$$