Notes

Supplemental Video: `https://www.youtube.com/watch?v=3twLwF2F6CY`

Non-homogeneous Differential Equations

The following differential equation is a non-homogeneous differential equation:

$$\frac{d^2y}{dt^2} + a_1\frac{dy}{dt} + a_0 y = b$$

where b is a constant.

Even though this expression isn't equal to 0, we can still solve it using our method for homogeneous differential equations. If we substitute y with $\tilde{y} = y - \frac{b}{a_0}$, then we end up with a new differential equation that is homogeneous:

$$\frac{d^2\tilde{y}}{dt^2} + a_1\frac{d\tilde{y}}{dt} + a_0\tilde{y} = 0$$

Now we can solve for $\tilde{y}$ and then reverse our substitution to get y.

Questions

1. **Differential Equations**

   Solve the following second-order differential equation.

   (a) $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 13y = 13$, where $y(0) = 3$ and $\frac{dy}{dt}(0) = 7$

   Answer:

   Since this is a non-homongeneous differential equation, define $\tilde{y} = y - 1$. Then, $\frac{d\tilde{y}}{dt} = \frac{dy}{dt}$ and $\frac{d^2\tilde{y}}{dt^2} = \frac{d^2y}{dt^2}$.

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 13y - 13 = 0$$

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 13(y-1) = 0$$

$$\frac{d^2\tilde{y}}{dt^2} - 4\frac{d\tilde{y}}{dt} + 13\tilde{y} = 0$$

   We set up the matrix differential equation with the state vector $\vec{x}(t) = \begin{bmatrix} \tilde{y}(t) \\ \frac{d\tilde{y}}{dt}(t) \end{bmatrix}$.

$$\begin{bmatrix} \frac{d\tilde{y}}{dt}(t) \\ \frac{d^2\tilde{y}}{dt^2}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -13 & 4 \end{bmatrix} \begin{bmatrix} \tilde{y}(t) \\ \frac{d\tilde{y}}{dt}(t) \end{bmatrix}$$

We then find the eigenvalues of the $\mathbf{A}$ matrix.

$$\det\left(\begin{bmatrix} -\lambda & 1 \\ -13 & 4-\lambda \end{bmatrix}\right) = \lambda^2 - 4\lambda + 13 = 0$$

We use the quadratic formula to solve for λ.

$$\lambda = \frac{4 \pm \sqrt{16 - 4 \cdot 13}}{2} = 2 \pm \sqrt{-9} = 2 \pm 3j$$

Since we have two complex eigenvalues, the general solution is given by:

$$\tilde{y}(t) = e^{2t}(c_1 \cos(3t) + c_2 \sin(3t))$$

We reverse the substitution, so we get:

$$y(t) = \tilde{y}(t) + 1 = e^{2t}(c_1 \cos(3t) + c_2 \sin(3t)) + 1$$

$$\frac{dy}{dt}(t) = 2e^{2t}(c_1 \cos(3t) + c_2 \sin(3t)) + e^{2t}(-3c_1 \sin(3t) + 3c_2 \cos(3t))$$

To find c_1 and c_2, we plug in the initial conditions.

$$\begin{cases} c_1 + 1 = 3 \\ 2c_1 + 3c_2 = 7 \end{cases}$$

Solving this system of equations gives:

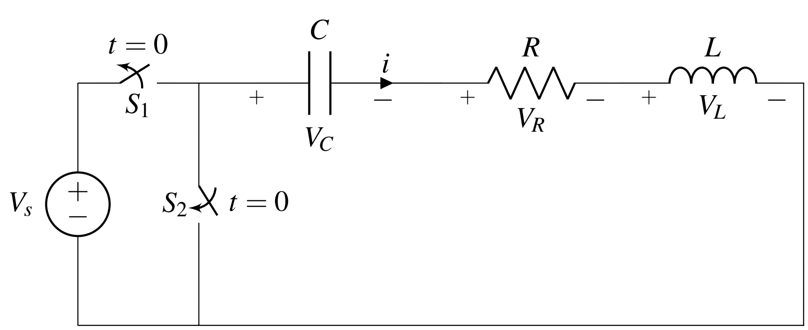
$$c_1 = 2 \qquad c_2 = 1$$

Therefore,

$$y(t) = e^{2t}(2\cos(3t) + \sin(3t)) + 1$$

2. RLC circuit

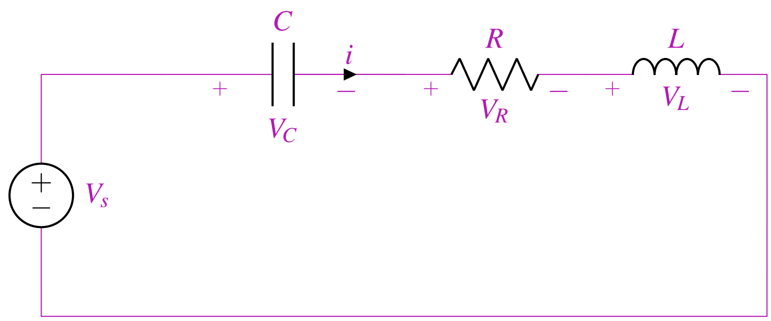
Consider the following circuit. Before $t = 0$, switch S_1 is on while S_2 is off. At $t = 0$, both switches flip state (S_1 turns off and S_2 turns on):



(a) Draw the circuit corresponding to $t < 0$. What are the values of V_C, V_R, V_L, and i at $t = 0_-$, the time right before the switches close. Assume this circuit has been in this state for a long time.

Answer:

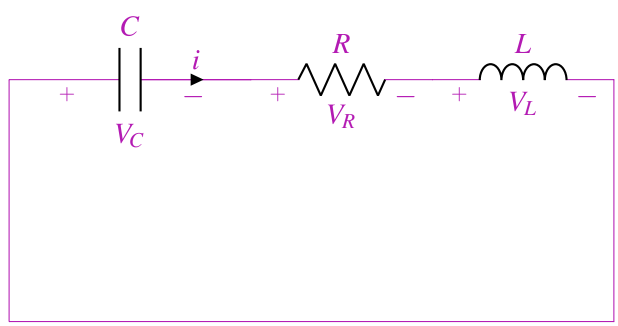
$V_C = V_s$, $V_R = 0\text{V}$, $V_L = 0\text{V}$, and $i = 0$



(b) Now draw the circuit corresponding to $t \geq 0$. Using your results from the previous part, what are V_C, V_R, V_L, and i at $t = 0_+$.

Answer:

Voltage across the capacitor cannot change instantaneously so $V_C = V_s$. Current through the inductor cannot change instantaneously so $i = 0$, which means $V_R = 0$ also. Finally, by KVL $V_L = -V_s$



(c) Define your state variables as $V_c(t)$ and $i_c(t)$. Find the equation for $V_c(t)$ for $t \geq 0$. Use component values $V_s = 4\text{V}$, $C = 2\text{fF}$, $R = 60\text{k}\Omega$, and $L = 1\mu\text{H}$.

Answer:

First we need to find $\frac{dV_c}{dt}$ and $\frac{di_c}{dt}$ in terms of V_c and i_c:

$$\frac{dV_c}{dt} = \frac{1}{C} i_c$$

Since the capacitor and inductor are in series, we can say:

$$i_c = i_L$$

$$\frac{di_c}{dt} = \frac{di_L}{dt} = \frac{1}{L} V_L$$

$$V_L = -(V_c + V_R) = -(V_c + R i_c)$$

$$\frac{di_c}{dt} = -\frac{1}{L} V_c - \frac{R}{L} i_c$$

Putting this into matrix form, we get:

$$\begin{bmatrix} \frac{dV_c}{dt} \\ \frac{di_c}{dt} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} V_C \\ i_c \end{bmatrix}$$

Next, we need to find the eigenvalues of A.

$$\det\left(\begin{bmatrix} -\lambda & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L}-\lambda \end{bmatrix}\right) = \lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC}$$

$$\lambda = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$\lambda = -\frac{60\times 10^3}{2\times 10^{-6}} \pm \sqrt{\left(\frac{60\times 10^3}{2\times 10^{-6}}\right)^2 - \frac{1}{10^{-6}\times 2\times 10^{-15}}}$$

$$\lambda = -3\times 10^{10} \pm 2\times 10^{10}$$

$$\lambda_1 = -1\times 10^{10}$$

$$\lambda_2 = -5\times 10^{10}$$

The two eigenvalues are unique and purely real, so the solution is in the form of:

$$V_c = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

To find c_1 and c_2, we use the initial conditions:

$$V_c(0) = V_s = 4 = c_1 + c_2 \tag{1}$$

$$C\frac{dV_c}{dt}(0) = i_c(0) = 0$$

$$c_1\lambda_1 + c_2\lambda_2 = 0$$

$$-c_1 - 5c_2 = 0 \tag{2}$$

Using equations (1) and (2), we can solve for c_1 and c_2:

$$c_1 = 5$$

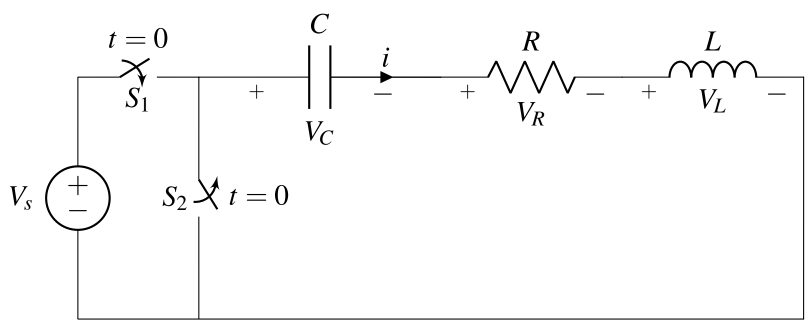
$$c_2 = -1$$

This gives us:

$$V_c(t) = 5e^{-10^{10}t} - e^{-5\times 10^{10}t}$$

3. Charging RLC Circuit

Consider the following circuit. Before $t = 0$, switch S_1 is off while S_2 is on. At $t = 0$, both switches flip state (S_1 turns on and S_2 turns off):



(a) Write out the differential equation describing this circuit for $t \geq 0$ in the form:

$$\frac{d^2V_c}{dt^2} + a_1\frac{dV_c}{dt} + a_0V_c = b$$

Answer:

$$\frac{d^2V_c}{dt^2} + \frac{R}{L}\frac{dV_c}{dt} + \frac{1}{LC}V_c = \frac{V_s}{LC}$$

(b) Find a $\tilde{V}_c$ and substitute it to the previous equation such that

$$\frac{d^2\tilde{V}_c}{dt^2} + a_1\frac{d\tilde{V}_c}{dt} + a_0\tilde{V}_c = 0$$

Answer: $\tilde{V}_c = V_c - V_s$

(c) Solve for $V_c(t)$ for $t \geq 0$. Use component values $V_s = 4\text{V}$, $C = 2\text{fF}$, $R = 60\text{k}\Omega$, and $L = 1\mu\text{H}$.

Answer:

$$\frac{d}{dt}\begin{bmatrix} \tilde{V}_c \\ \frac{d\tilde{V}_c}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} \tilde{V}_c \\ \frac{d\tilde{V}_c}{dt} \end{bmatrix}$$

Then, we need to solve the following quadratic equation for the eigenvalues:

$$\lambda^2 + \lambda\frac{R}{L} + \frac{1}{LC} = 0$$

This characteristic polynomial is the same as the previous problem:

$$\lambda_1 = -1 \times 10^{10}$$
$$\lambda_2 = -5 \times 10^{10}$$
$$\tilde{V}_c = c_1e^{\lambda_1 t} + c_2e^{\lambda_2 t}$$

Substituting back in $V_c(t)$:

$$V_c - V_s = c_1e^{\lambda_1 t} + c_2e^{\lambda_2 t}$$
$$V_c = c_1e^{\lambda_1 t} + c_2e^{\lambda_2 t} + V_s$$

Now we can use initial conditions to solve for c_1 and c_2. Before the switches change states, all voltages and currents are 0. Immediately after the switch closes, the voltage across the capacitor cannot change instantaneously, so:

$$V_c(0) = 0 = c_1 + c_2 + 4 \tag{1}$$

Just like the voltage across the capacitor, the current through the inductor cannot change instantaneously, so:

$$C\frac{dV_c(0)}{dt} = i_c(0) = i_L(0) = 0$$
$$-c_1 - 5c_2 = 0 \tag{2}$$

Using equations 1 and 2, we can solve for c_1 and c_2:

$$c_1 = -5$$

$$c_2 = 1$$

This gives us:

$$V_c(t) = 4 - 5e^{-10^{10}t} + e^{-5\times10^{10}t}$$

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