

This homework is due on Wednesday, October 10, 2018, at 11:59PM.

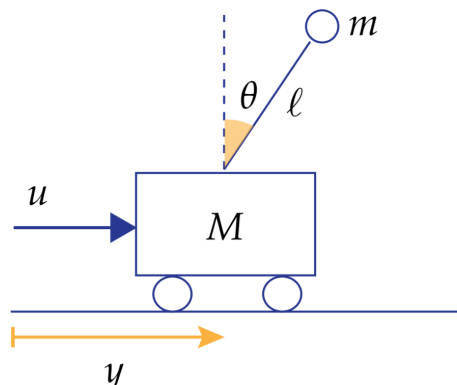
Self-grades are due on Monday, October 15, 2018, at 11:59PM.

1. Inverted Pendulum on a Rolling Cart (Mechanical)

Consider the inverted pendulum depicted below, which is placed on a rolling cart and whose equations of motion are given by:

$$\ddot{y} = \frac{1}{\frac{M}{m} + \sin^2 \theta} \left(\frac{u}{m} + \dot{\theta}^2 \ell \sin \theta - g \sin \theta \cos \theta \right)$$

$$\ddot{\theta} = \frac{1}{\ell \left(\frac{M}{m} + \sin^2 \theta \right)} \left(-\frac{u}{m} \cos \theta - \dot{\theta}^2 \ell \cos \theta \sin \theta + \frac{M+m}{m} g \sin \theta \right).$$



- (a) Write the state model using the variables $x_1(t) = \theta(t)$, $x_2(t) = \dot{\theta}(t)$, and $x_3(t) = \dot{y}(t)$. We do not include $y(t)$ as a state variable because we are interested in stabilizing at the point $\theta = 0$, $\dot{\theta} = 0$, $\dot{y} = 0$, and we are not concerned about the final value of the position $y(t)$.

Solution:

We have

$$\begin{aligned} \dot{x}_1 &= x_2 && \triangleq f_1(x_1, x_2, x_3, u) \\ \dot{x}_2 &= \left(\frac{1}{\ell \left(\frac{M}{m} + \sin^2(x_1) \right)} \right) \left(-\frac{u}{m} \cos(x_1) - x_2^2 \ell \cos(x_1) \sin(x_1) + \frac{M+m}{m} g \sin(x_1) \right) && \triangleq f_2(x_1, x_2, x_3, u) \\ \dot{x}_3 &= \left(\frac{1}{\frac{M}{m} + \sin^2(x_1)} \right) \left(\frac{u}{m} + x_2^2 \ell \sin(x_1) - g \sin(x_1) \cos(x_1) \right) && \triangleq f_3(x_1, x_2, x_3, u) \end{aligned}$$

- (b) Linearize this model at the equilibrium $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, and $u = 0$, and indicate the resulting A and B matrices.

Solution:

We can keep in mind that $x_1 = x_2 = x_3 = 0$ to make the derivative much easier. Since we aren't asked to linearize about a particular input, we can linearize about $u^* = 0$. This is fine because f_2 and f_3 are affine (linear plus a constant term) with respect to u .

$$\begin{aligned} \frac{\partial f_1}{\partial x_1}(0,0,0,0) &= 0 & \frac{\partial f_1}{\partial x_2}(0,0,0,0) &= 1 & \frac{\partial f_1}{\partial x_3}(0,0,0,0) &= 0 \\ \frac{\partial f_2}{\partial x_1}(0,0,0,0) &= \frac{M+m}{lM}g & \frac{\partial f_2}{\partial x_2}(0,0,0,0) &= 0 & \frac{\partial f_2}{\partial x_3}(0,0,0,0) &= 0 \\ \frac{\partial f_3}{\partial x_1}(0,0,0,0) &= -\frac{m}{M}g & \frac{\partial f_3}{\partial x_2}(0,0,0,0) &= 0 & \frac{\partial f_3}{\partial x_3}(0,0,0,0) &= 0 \end{aligned}$$

And,

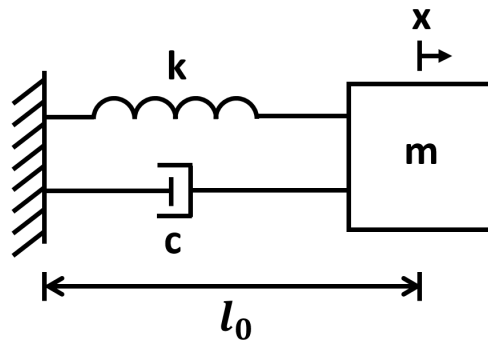
$$\frac{\partial f_1}{\partial u}(0,0,0,0) = 0 \quad \frac{\partial f_2}{\partial u}(0,0,0,0) = -\frac{1}{lM} \quad \frac{\partial f_3}{\partial u}(0,0,0,0) = \frac{1}{M}$$

Since $x^* = 0$ and $u^* = 0$, we can use the same state variables x and u . Then,

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ \frac{M+m}{lM}g & 0 & 0 \\ -\frac{m}{M}g & 0 & 0 \end{bmatrix}}_A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ -\frac{1}{lM} \\ \frac{1}{M} \end{bmatrix}}_B u$$

2. Spring and mass (Mechanical)

Lets look at a mechanical spring-mass system governed by differential equations similar to those of electrical circuits.



The total force F acting on a mass can be expressed as $F = ma$, where $a = \frac{dv}{dt}$ and $v = \frac{dx}{dt}$. Springs generate force according to $F_k = -k\Delta x$ where k is the spring's stiffness and Δx is the displacement of mass from its resting position. We also have a damper which creates a force $F_c = -cv$. We set x to be 0 when the spring is at its rest length l_0 so that $\Delta x = x$. Ignoring gravity, the differential equation describing the motion of the mass is:

$$m \frac{d^2 x}{dt^2} + c \frac{dx}{dt} + kx = 0$$

- (a) Write the state space model for this system as $\frac{d\vec{x}}{dt} = A\vec{x}$. What is your state vector?

Solution: There are multiple options for choosing the state variables of the system. In this case, we chose the state vector:

$$\vec{x} = \begin{bmatrix} x \\ \frac{dx}{dt} \end{bmatrix}$$

The equations are:

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{d^2x}{dt^2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \begin{bmatrix} x \\ \frac{dx}{dt} \end{bmatrix}$$

- (b) Find the eigenvalues of this system. Is this system stable? Use values $k = 30\text{N/m}$, $c = 40\text{kg/s}$, and $m = 10\text{kg}$. Remember that the standard unit of mass is kg (use the value 10 when plugging in for m, not 10×10^3).

Solution:

$$\det(A - \lambda I) = \lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0$$

$$\lambda = \frac{-\frac{c}{m} \pm \sqrt{\left(\frac{c}{m}\right)^2 - 4\frac{k}{m}}}{2}$$

Plugging in values:

$$\lambda = \frac{-4 \pm \sqrt{(4)^2 - 4 \cdot 3}}{2} = -2 \pm 1$$

$$\lambda_1 = -3$$

$$\lambda_2 = -1$$

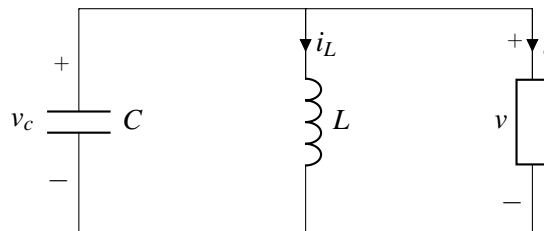
Since this is a continuous system, the condition for stability is that for all eigenvalues, $\text{Re}\{\lambda\} < 0$. Both eigenvalues meet the stability criterion, so the system is stable.

3. Nonlinear circuit component

This is a problem adapted from a past midterm problem (Spring 2017 midterm 2).

Consider the circuit below that consists of a capacitor, inductor, and a third element with a nonlinear voltage-current characteristic:

$$i = 2v - v^2 + 4v^3$$



- (a) Write a state space model of the form

$$\frac{dx_1(t)}{dt} = f_1(x_1(t), x_2(t))$$

$$\frac{dx_2(t)}{dt} = f_2(x_1(t), x_2(t))$$

Where $x_1(t) = v_c(t)$ and $x_2(t) = i_L(t)$.

Solution:

We need to get $\frac{dv_c}{dt}$ and $\frac{di_L}{dt}$ in terms of v_c and i_L .

All the components are in parallel, so:

$$v_c = v_L = v$$

Using the relation of an inductor's current and voltage:

$$v_c = L \frac{di_L}{dt}$$

$$\frac{di_L}{dt} = \frac{1}{L} v_c \quad (1)$$

Using KCL, we can say:

$$i_c + i_L + i = 0$$

$$C \frac{dv_c}{dt} + i_L + 2v - v^2 + 4v^3 = 0$$

$$\frac{dv_c}{dt} = \frac{1}{C} (-i_L - 2v_c + v_c^2 - 4v_c^3) \quad (2)$$

Taking equations (1) and (2) and substituting in x_1 and x_2 gives us our answer:

$$\frac{dx_1}{dt} = f_1(x_1, x_2) = \frac{1}{C} (-x_2 - 2x_1 + x_1^2 - 4x_1^3)$$

$$\frac{dx_2}{dt} = f_2(x_1, x_2) = \frac{1}{L} x_1$$

- (b) Linearize the state model at the equilibrium point $x_1 = x_2 = 0$ and specify the resulting A matrix.

Solution:

$$A = \nabla f(\vec{x})|_{x_1=x_2=0} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} \bigg|_{x_1=x_2=0} = \begin{bmatrix} \frac{1}{C} (-2 + 2x_1 - 12x_1^2) & -\frac{1}{C} \\ \frac{1}{L} & 0 \end{bmatrix} \bigg|_{x_1=x_2=0}$$

$$A = \begin{bmatrix} -\frac{2}{C} & -\frac{1}{C} \\ \frac{1}{L} & 0 \end{bmatrix}$$

- (c) Is the linearized system stable?

Solution:

$$\det(A - \lambda I) = \lambda^2 + \frac{2}{C}\lambda + \frac{1}{LC} = 0$$

$$\lambda = \frac{-\frac{2}{C} \pm \sqrt{\left(\frac{2}{C}\right)^2 - \frac{4}{LC}}}{2}$$

For a continuous system to be stable, all eigenvalues must have $\text{Re}\{\lambda\} < 0$.

Since both L and C can only take positive values, the square root term will always have a real part smaller than $\frac{2}{C}$, which means both eigenvalues will have negative real parts. The system is stable.

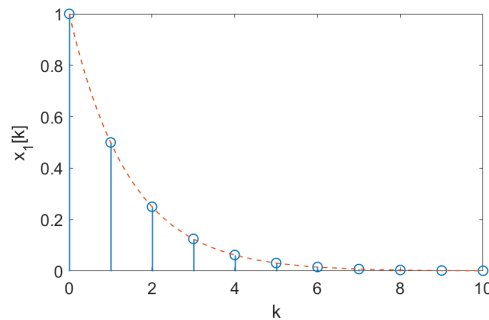
4. Discrete system responses

In many problems, we have an unknown system, and would like to characterize it. One of the ways of doing so is to observe the system response with different initial conditions (or inputs). This problem is also called system identification.

As an example, we have an unknown discrete system. The general form of the system can be written as:

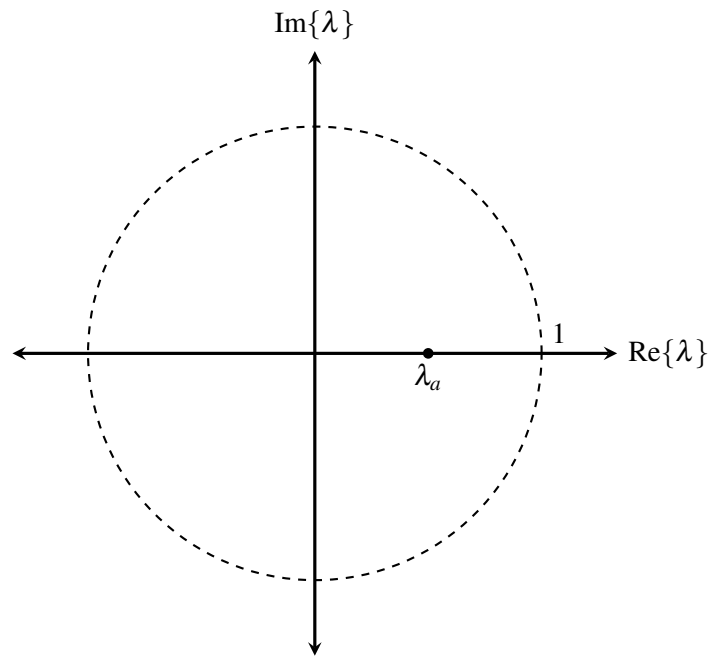
$$\vec{x}[k+1] = A\vec{x}$$

We apply an initial condition $\vec{x}[0] = \vec{x}_a$, where \vec{x}_a is some constant vector. We measure the response of $x_1[k]$ for $k > 0$ which is the first element in $\vec{x}[k]$ and get the following graph:

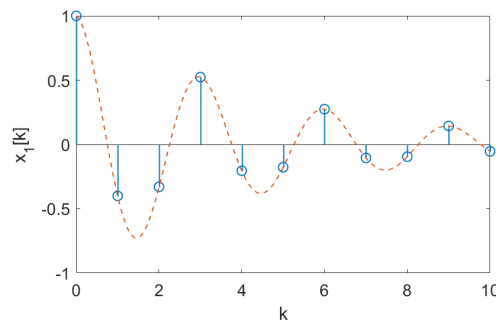


Based on this response, we can estimate the minimum order of the system and approximate the location of the eigenvalue(s) on the real-imaginary plane.

The response shows an exponential decay without oscillation. The minimum order system that could explain this response could be first order. For this case, we can say that the eigenvalue must be real (pure exponential behavior) and has magnitude less than 1 (decaying). Since the decay does not oscillate between positive and negative values, the eigenvalue is also positive. The eigenvalue has been marked at location λ_a



- (a) We have a different unknown system, and we apply an initial condition $\vec{x}[0] = \vec{x}_b$. We measure the response $x_1[k]$ for $k > 0$, where $x_1[k]$ is the first element of $\vec{x}[k]$:

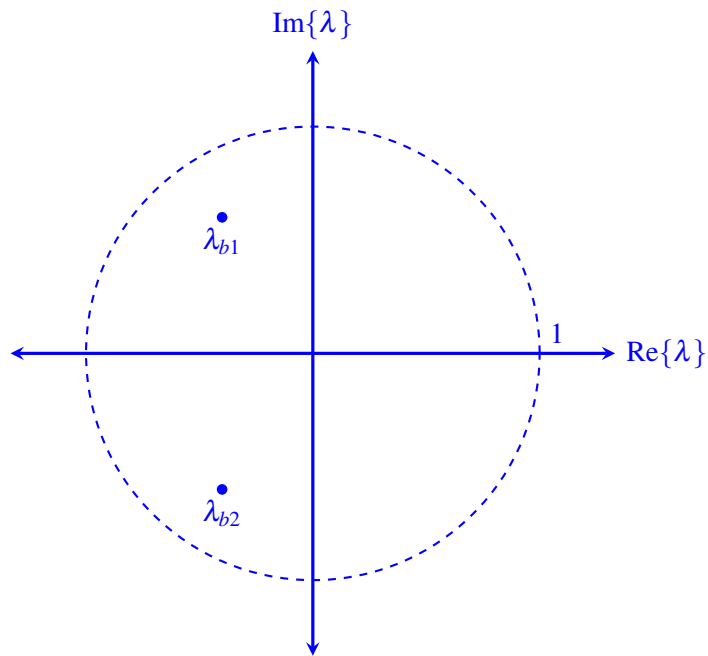


Based on this response, what is the minimum order the system can be? On the real-imaginary plane, plot the approximate location of the eigenvalues that correspond to this response. Your approximation does not need to be exact.

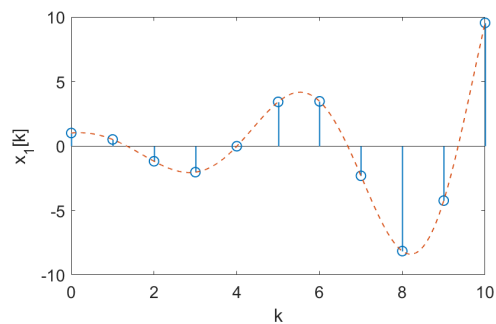
Solution:

The response is oscillatory. Since the period of the oscillations is greater than 2, that means the eigenvalue is complex. Complex eigenvalues can only occur in 2nd order and higher systems since they come in complex-conjugate pairs. That means the minimum order the overall system can be is 2nd order.

Since the response is decaying to zero, we know the eigenvalue has to be within the unit circle on the real-imaginary plane. The last thing to figure out is whether the eigenvalues have positive or negative real parts. If the period of the response is greater than 4, then the eigenvalues have positive real parts. The period of this response is less than 4, so the real part is negative. The eigenvalues have been marked at location λ_{b1} and λ_{b2} .



- (b) We have an unknown system, and we apply an initial condition $\vec{x}[0] = \vec{x}_c$. We measure the response of $x_1[k]$ for $k > 0$ and get the following graph:

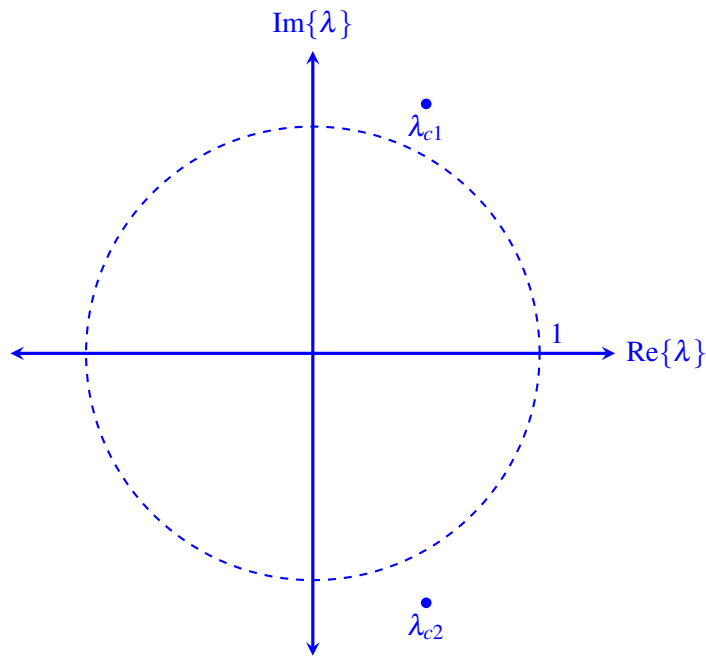


Based on this response, what is the minimum order the system can be? On the real-imaginary plane, plot the approximate location of the eigenvalues that correspond to this response. Your approximation does not need to be exact.

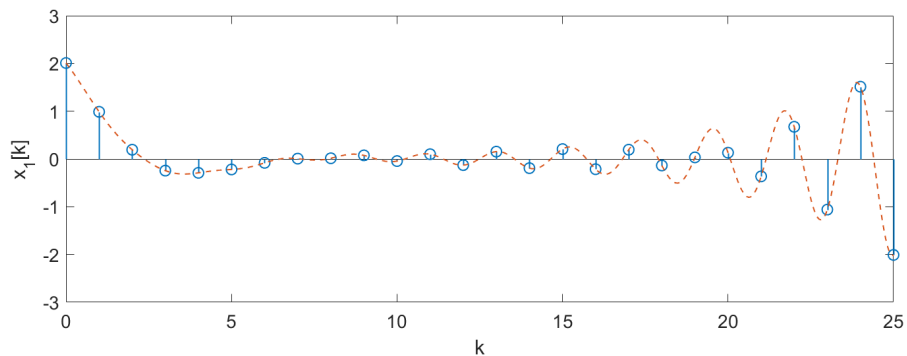
Solution:

The response is oscillatory. Since the period of the oscillations is greater than 2, that means the eigenvalue is complex, and thus at least 2nd order.

Since the response is growing over time, the response is unstable, so the eigenvalue has to be outside the unit circle on the real-imaginary plane. The period of this response is greater than 4, so the real part of the eigenvalue is positive. The eigenvalues have been marked at location λ_{c1} and λ_{c2} .



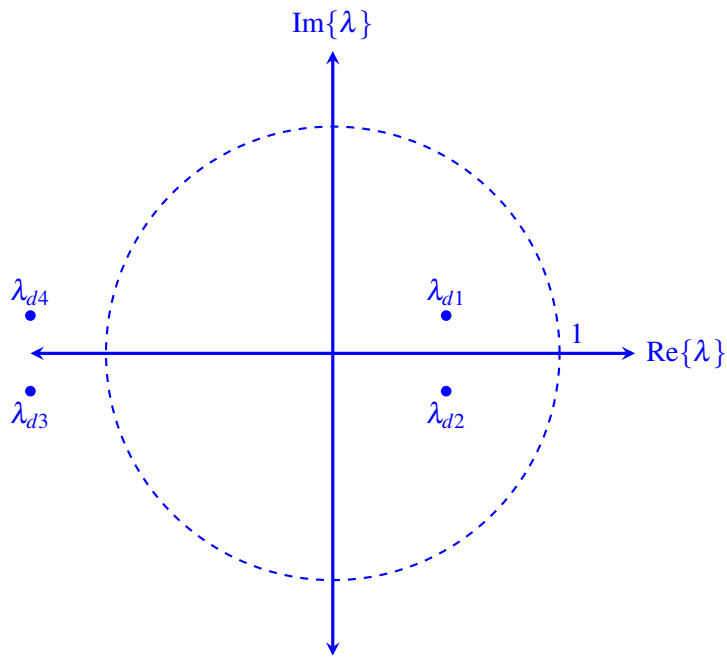
- (c) We have an unknown system, and we apply an initial condition $\vec{x}[0] = \vec{x}_d$. We measure the response of $x_1[k]$ for $k > 0$ and get the following graph:



Based on this response, what is the minimum order the system can be? On the real-imaginary plane, plot the approximate location of the eigenvalues that correspond to this response. Your approximation does not need to be exact.

Solution:

The response initially is decaying while oscillating at a low frequency. This corresponds to a complex eigenvalue pair that has a positive real part and within the unit circle. However, after a bit, the response starts to blow up and oscillates faster. This means there is another complex pair of eigenvalues that has a negative real part and is outside the unit circle. Since there are two complex pairs of eigenvalues, the minimum order of this system is 4. The eigenvalues have been marked at location λ_{d1} , λ_{d2} , λ_{d3} , and λ_{d4} .



5. Write Your Own Question And Provide a Thorough Solution.

Writing your own problems is a very important way to really learn material. The famous “Bloom’s Taxonomy” that lists the levels of learning is: Remember, Understand, Apply, Analyze, Evaluate, and Create. Using what you know to create is the top level. We rarely ask you any homework questions about the lowest level of straight-up remembering, expecting you to be able to do that yourself (e.g. making flashcards). But we don’t want the same to be true about the highest level. As a practical matter, having some practice at trying to create problems helps you study for exams much better than simply counting on solving existing practice problems. This is because thinking about how to create an interesting problem forces you to really look at the material from the perspective of those who are going to create the exams. Besides, this is fun. If you want to make a boring problem, go ahead. That is your prerogative. But it is more fun to really engage with the material, discover something interesting, and then come up with a problem that walks others down a journey that lets them share your discovery. You don’t have to achieve this every week. But unless you try every week, it probably won’t ever happen.