EECS 16B Designing Information Devices and Systems II
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Discussion 10B

1 Polynomial Interpolation

Given n distinct points, we can find a unique degree $n-1$ polynomial that passes through these points. Let the polynomial p be

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1}.$$

Let the n points be

$$p(x_1) = y_1, p(x_2) = y_2, \cdots, p(x_n) = y_n,$$

where $x_1 \neq x_2 \neq \cdots \neq x_n$.

We can construct a matrix-vector equation as follows to recover the polynomial p.

$$\underbrace{\begin{bmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \dots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^{n-1} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{bmatrix}}_{\vec{a}} = \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}}_{\vec{y}}$$

We can solve for the a values by setting:

$$\vec{a} = A^{-1}\vec{y}$$

Note that the matrix A is known as a Vandermonde matrix whose determinant is given by

$$\det(A) = \prod_{1 \le i < j \le n} (x_j - x_i)$$

Since $x_1 \neq x_2 \neq \cdots \neq x_n$, the determinant is non-zero and A is always invertible.

2 Polynomial Regression

Sometimes we may want to fit our data to a polynomial with an order less than $n-1$. If we fit the data to a polynomial of order $m < n$ we get:

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{m-1} x^{m-1}$$

Now when we construct the matrix-vector equation to recover polynomial p, we get:

$$\underbrace{\begin{bmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{m-1} \\ 1 & x_2 & x_2^2 & \dots & x_2^{m-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_m & x_m^2 & \dots & x_m^{m-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^{m-1} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{m-1} \end{bmatrix}}_{\vec{a}} = \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \\ \vdots \\ y_n \end{bmatrix}}_{\vec{y}}$$

With this matrix equation, we have n equations with m unknowns, which means our system is over-defined (since $m < n$). One way to find the best fitting a values for this polynomial is to use least-squares, where you set:

$$\vec{a} = (A^T A)^{-1} A^T \vec{y}$$

1. **Interpolation Example**

   Use polynomial interpolation to find the polynomial that passes through the points $(1,5)$, $(2,15)$ and $(3,33)$

2. **Regression Example**

   Using least-squares, find the best-fit quadratic equation for the data set: $(-2,28)$, $(-1,-14)$, $(0,0)$, $(1,-42)$, and $(2,56)$.

3. **Minimum Norm Polynomial Interpolation**

   We have two data points: (0,0) and (1,1).

   (a) Find a linear fit curve for the two data points.

   (b) Find the second order polynomial for which the coefficients have the smallest norm. Compare the norm of the result to the first order polynomial found in part (a). Note that a first order polynomial is also a (degenerate) second order polynomial.