EECS 16B Designing Information Devices and Systems II
Fall 2018 Elad Alon and Miki Lustig

Discussion 8B

Notes

Complex Inner Product

The inner product of two complex vectors v and w is,

$$\langle v, w \rangle = \sum_i^n \overline{v}_i w_i = \overline{v}^\top w$$

$\overline{v}$ means that we take the complex conjugate of each element of v. We define the following notation.

$$\overline{v}^\top w = v^* w = v^H w$$

H and $*$ are often used to denote the same operation of taking the transpose of a vector after complex conjugating each element.

Self-Adjoint/ Hermitian Matrices

A matrix T is called Hermitian or self-adjoint if $T = T^*$. One such example is

$$T = \begin{bmatrix} 4 & 1+2j \\ 1-2j & 21 \end{bmatrix} = T^*$$

Positive (Semi-) Definite Matrices

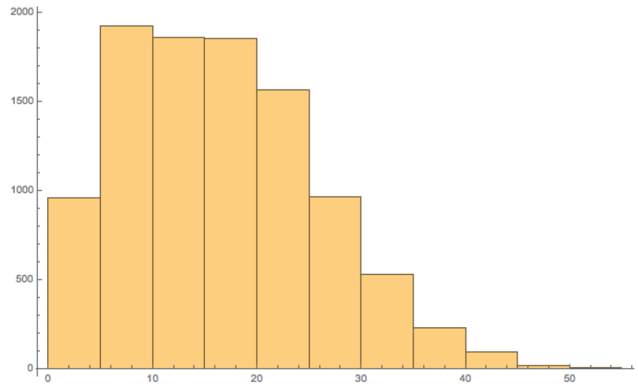
A matrix T is called positive semi-definite if

$$v^* T v \geq 0 \text{ for all } v \in \mathbb{C}^n$$

and positive-definite if in addition

$$v^* T v = 0 \implies v = 0$$

The above matrix actually is also positive semi-definite. We sampled ten thousand random complex vectors and evaluated $v^* T v$. The histogram is below



Complex Spectral Theorem: Statement

Let T be a self-adjoint matrix from $\mathbb{C}^n$ to $\mathbb{C}^n$. Then,

(a) There exists n linearly independent eigenvectors of T that form a basis for $\mathbb{C}^n$. Further more, the eigenvectors are orthonormal.

(b) The eigenvalues of T are real.

Questions

1. Eigenvalues are Real

Prove the following: For any self-adjoint matrix A, any eigenvalue of A is real.

Hint: Use the definition of an eigenvalue to show that $\lambda^*(\vec{v}^*\vec{v}) = \lambda(\vec{v}^*\vec{v})$.

Answer: Let λ be an eigenvalue of A with corresponding eigenvector $\vec{v}$. Since A is self adjoint, we have

$$\lambda^*(\vec{v}^*\vec{v}) = (\lambda\vec{v})^*\vec{v} = (A\vec{v})^*\vec{v} = \vec{v}^*A^*\vec{v} = \vec{v}^*A\vec{v} = \vec{v}^*(\lambda\vec{v}) = \lambda(\vec{v}^*\vec{v}).$$

Since $\vec{v}^*\vec{v}$ is nonzero this implies that $\lambda = \lambda^*$, so λ is real.

2. Eigenvectors are Orthogonal

Prove the following: For any symmetric matrix A, any two eigenvectors corresponding to distinct eigenvalues of A are orthogonal.

Hint: Use the definition of an eigenvalue to show that $\lambda_1(\vec{v}_1^*\vec{v}_2) = \lambda_2(\vec{v}_1^*\vec{v}_2)$.

Answer: Let λ_1, λ_2 be eigenvalues of A with corresponding eigenvectors $\vec{v}_1, \vec{v}_2$. Since A is self-adjoint, we have

$$\lambda_1\vec{v}_1^*\vec{v}_2 = (\lambda_1\vec{v}_1)^*\vec{v}_2 = (A\vec{v}_1)^*\vec{v}_2 = \vec{v}_1^*A^*\vec{v}_2 = \vec{v}_1^*Av_2 = \vec{v}^*(\lambda_2\vec{v}_2) = \lambda_2\vec{v}_1^*\vec{v}_2$$

This implies that,

$$(\lambda_1 - \lambda_2)\vec{v}_1^*\vec{v}_2 = 0$$

The only way this equation can be satisfied when $\lambda_1 \neq \lambda_2$ is for $\vec{v}_1^*\vec{v}_2$ to be zero. Therefore $\vec{v}_1$ and $\vec{v}_2$ must be orthogonal.

3. Power Iteration

Power iteration is a method for approximating eigenvectors of a matrix A numerically. It's particularly effective when A is very large but very sparse. For example Google's PageRank algorithm, used to determine the ranking of search results, essentially attempts to perform power iteration on the adjacency matrix of links between all web pages on the internet.

The method starts with any vector x_0 and then iterates the following update:

$$\vec{x}_{k+1} = \frac{A\vec{x}_k}{||A\vec{x}_k||}.$$

Here, $\vec{x}_k$ denotes the value of $\vec{x}$ in the kth iteration. You will show that this algorithm converges for symmetric A.

(a) Show that if A is a diagonal matrix $\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$ with λ_1 strictly greater than the other λ_i then the power iteration method converges to $\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, which is the eigenvector corresponding to the largest eigenvalue of A.

Answer: Note that each $\vec{x}_k$ has magnitude 1. Unrolling the recursion we get

$$\vec{x}_k = \frac{1}{||A^k\vec{x}_0||} \begin{bmatrix} \lambda_1^k x_1^0 \\ \vdots \\ \lambda_n^k x_n^0 \end{bmatrix}$$

Since $\lambda_1 > \lambda_i$ the first entry will eventually dominate the others so the iteration converges to $\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$.

(b) Now use the spectral decomposition to show that for any symmetric matrix whose largest eigenvalue is strictly greater than its other eigenvalues, the power iteration method converges to the eigenvector corresponding to the largest eigenvalue.

Answer: In the coordinate system, $\vec{y} = V^*\vec{x}$, the representation of A is diagonal. So $\vec{y}_k \to \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ as $k \to \infty$. Thus $\vec{x}_k \to V \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \vec{v}_1$ as $k \to \infty$, where $\vec{v}_1$ is the eigenvector corresponding to the largest eigenvalue.