1 Transfer Functions

When we analyzed circuits in the phasor domain, we always told you what the input voltage, or the input sinusoid to the circuit was. However, sometimes we have many input sinusoids, and we want to look at how a circuit (or system) generically responds to a sinusoid input of frequency ω. We want to see how an input sinusoid "transfers" into an output sinusoid. How do we do this?

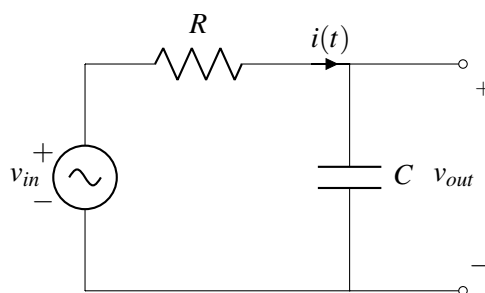Let's start with a simple RC circuit.

Figure 1: First Order RC Low Pass Filter

In the phasor domain, the impedance of the capacitor is $Z_C = \frac{1}{j\omega C}$ and the impedance of the resistor is $Z_R = R$. Because we treat impedances the same as resistances, this circuit looks like a voltage divider in the phasor domain. Remember we must also represent v_{in} as a phasor $\tilde{V}_{in}$; transfer functions are in the phasor domain only, not the time domain.

$$\tilde{V}_{out} = \frac{Z_C}{Z_R + Z_C}\tilde{V}_{in} = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}}\tilde{V}_{in} = \frac{1}{j\omega RC + 1}\tilde{V}_{in}$$

We define the frequency response as

$$H(\omega) = \frac{\tilde{V}_{out}}{\tilde{V}_{in}} = \frac{1}{j\omega RC + 1}$$

Now, given an arbitrary input sinusoid, if we multiply it by the frequency response, we can get the output sinusoid. What this allows us to do is to model any arbitrary circuit as a single-input-single-output black box. The transfer function completely defines how our circuit works.

The transfer function of a circuit provides deeper insight than the DC response of a circuit. Let's consider the transfer function above. How does the circuit respond to high frequencies? What about low frequencies?

$$\lim_{\omega \to \infty} H(\omega) = \lim_{\omega \to \infty} \frac{1}{j\omega RC + 1} = 0$$

$$\lim_{\omega \to 0} H(\omega) = \lim_{\omega \to 0} \frac{1}{j\omega RC + 1} = 1$$

This tells us that the RC circuit above passes sinusoids of lower frequencies (frequencies close to 0), and stops sinusoids of high frequencies (frequencies closer to ∞). For this reason we call it a "low pass" filter. If we input a combination of sinusoids of different frequencies into this filter, we can see that it will respond differently to all of them.

Note: If we cascade transfer circuits to make more complex transfer functions, we can't treat the transfer functions separately. To illustrate this, let's take the example above and cascade it.
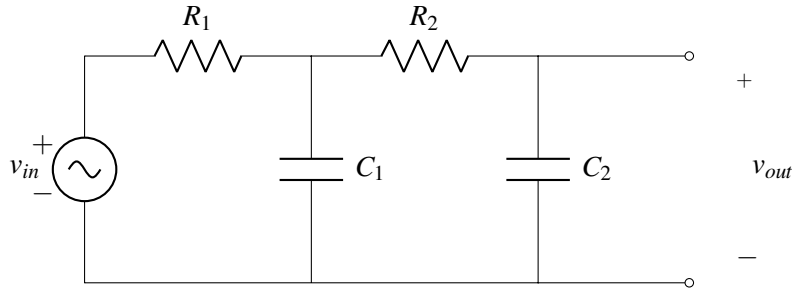

Figure 2: Second Order RC Filter

If we set the intermediate node to $\tilde{V}_x$, we get the following equations to solve for the transfer function. Performing KCL at the $\tilde{V}_x$ node and the $\tilde{V}_{out}$ node,

$$\frac{\tilde{V}_x - \tilde{V}_{in}}{R_1} + \frac{\tilde{V}_x}{\frac{1}{j\omega C_1}} + \frac{\tilde{V}_x - \tilde{V}_{out}}{R_2} = 0$$

$$\frac{\tilde{V}_{out} - \tilde{V}_x}{R_2} + \frac{\tilde{V}_{out}}{\frac{1}{j\omega C_2}} = 0$$

Putting the equations together and solving for the transfer function, we get:

$$H(\omega)_{RC,Actual} = \frac{\tilde{V}_{out}}{\tilde{V}_{in}}$$

$$= \frac{1}{(j\omega)^2 R_1 R_2 C_1 C_2 + j\omega(C_2 R_1 + C_1 R_1 + R_2 C_2) + 1}$$

$$= \frac{1}{(j\omega R_1 C_1 + 1)(j\omega R_2 C_2 + 1) + j\omega C_2 R_1}$$

If we had thought of the circuit in Figure 2 as two separate RC low pass filters (as shown in Figure 1), then our frequency response would have been

$$H(\omega)_{RC,Predicted} = \frac{1}{(j\omega R_1 C_1 + 1)(j\omega R_2 C_2 + 1)}$$

The predicted solution that we have is incorrect for the above circuit (Figure 2). Because there is coupling between the two RC filters, we can't model the it as two separate RC filters. If we wanted to do that, we would have to find a way to de-couple the two circuits from each other. Luckily, we learned how to do this! If we put an op-amp in buffer configuration between the two RC filters, they are no longer electrically coupled to one another and act as two separate RC low pass filters.
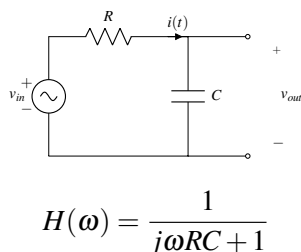

$$H(\omega) = \frac{1}{j\omega RC + 1}$$
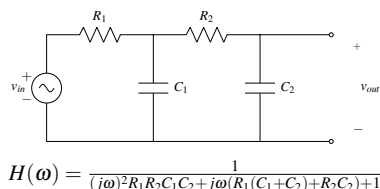
Figure 3: Simple RC Low Pass Filter

$$H(\omega) = \frac{1}{(j\omega)^2 R_1 R_2 C_1 C_2 + j\omega(R_1(C_1+C_2)+R_2C_2)+1}$$

Figure 4: Cascaded RC Filter with no buffer stage

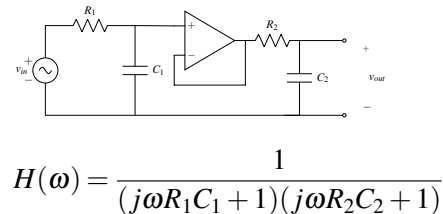$$H(\omega) = \frac{1}{(j\omega R_1 C_1 + 1)(j\omega R_2 C_2 + 1)}$$

Figure 5: Cascaded RC Filter with intermediate buffer stage

2 Plotting Transfer Functions

It is often useful to be able to plot the transfer function of a circuit as a function of frequency. This allows us to visualize how our circuit reacts to any particular frequency. Because the frequency response is a complex number, we plot the magnitude and phase of the frequency response on separate plots. Transfer functions can be very complicated, so being able to plot them allows us to look at how the circuit responds at intermediate frequencies, not just the extremes of 0 and ∞.

For example, let's plot the transfer function for the RC Low pass filter above. First, we find the magnitude and phase response from the transfer function.

$$|H(\omega)| = \left|\frac{1}{j\omega RC + 1}\right| = \frac{|1|}{|j\omega RC + 1|} = \frac{1}{\sqrt{(\omega RC)^2 + 1}}$$

$$\angle H(\omega) = \angle\frac{1}{j\omega RC + 1} = \angle 1 - \angle(j\omega RC + 1) = -\tan^{-1}(\omega RC)$$

Plotting using numerical methods and $R = 1\,\text{k}\Omega, C = 1\,\mu\text{F}$, we get the following:
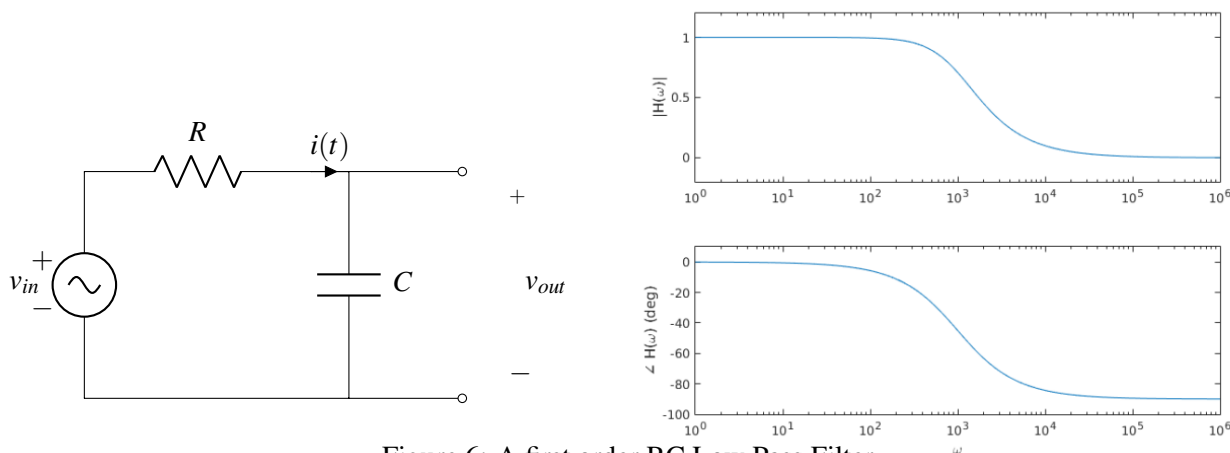

Figure 6: A first order RC Low Pass Filter

There are a few things to notice about the plot. We plotted the frequency on a logarithmic scale so that we can see the circuit response over a wide range. One important thing to notice is that at the pole frequency ($\omega_p = \frac{1}{RC} = 1000\,\frac{\text{rad}}{\text{s}}$), the phase is precisely $-45°$ and the magnitude is precisely $\frac{1}{\sqrt{2}} \approx 0.707$. We refer to this point as the "cutoff frequency" for this low pass filter. Every frequency below the cutoff frequency is "passed" (its amplitude remains about the same), and everything above the cutoff frequency is stopped (its amplitude goes to ≈ 0). This is an approximation, but as long as we design our circuit to have minimal frequencies near the cutoff, it is a pretty good one.

3 Log Properties

In the coming lectures, you will see how to analyze transfer functions graphically using Bode plots. Since these plots are on a log-log scale, the following properties are good to know:

Product Rule:
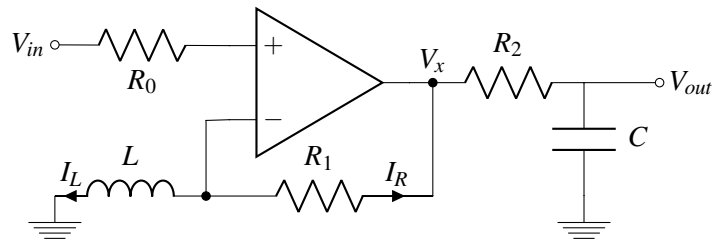$$\log(AB) = \log(A) + \log(B)$$

Quotient Rule:
$$\log \frac{A}{B} = \log A - \log B$$

Power Rule:
$$\log(A^p) = p\log(A)$$

1. Transfer Functions

(a) Assume that the op-amp is ideal. Derive the transfer function for the following circuit:
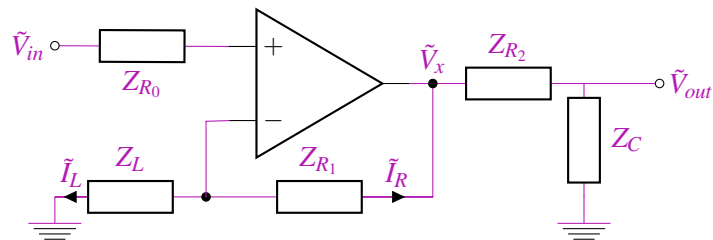

Use the following values:

$$R_0 = 100\,\Omega, R_1 = 1\,k\Omega, L = 1\,\mu H, R_2 = 100\,k\Omega, C = 1\,pF$$

Answer:
First, transform the circuit to phasor domain. Now we can cast KCL/KVL equations like what we did before with ideal op amps and resistors in time domain.

With the Golden Rules, we know

$$\tilde{V}_+ = \tilde{V}_- = \tilde{V}_{in}$$
$$\tilde{I}_+ = \tilde{I}_- = 0$$

Perform KCL at negative terminal of the op amp, labeling the output node of op amp as v_x

$$0 = \tilde{I}_L + \tilde{I}_R$$
$$0 = \frac{\tilde{V}_{in}}{Z_L} + \frac{\tilde{V}_{in} - \tilde{V}_x}{Z_{R_1}}$$
$$\tilde{V}_x = \tilde{V}_{in}\left(\frac{Z_{R_1}}{Z_L} + 1\right) = \tilde{V}_{in}\frac{Z_{R_1} + Z_L}{Z_L}$$

Now let's look at the second half of the circuit, which looks like a voltage divider. So

$$\tilde{V}_{out} = \tilde{V}_x \frac{Z_C}{Z_{R_2} + Z_C}$$
$$= \tilde{V}_{in}\frac{Z_{R_1} + Z_L}{Z_L}\frac{Z_C}{Z_{R_2} + Z_C}$$

And now we can find the transfer function:

$$H(\omega) = \frac{\tilde{V}_{out}}{\tilde{V}_{in}}$$

$$= \frac{Z_{R_1} + Z_L}{Z_L} \frac{Z_C}{Z_{R_2} + Z_C}$$

$$= \frac{R_1 + j\omega L}{j\omega L} \frac{\frac{1}{j\omega C}}{R_2 + \frac{1}{j\omega C}}$$

$$= \frac{R_1 + j\omega L}{(j\omega L)(1 + j\omega R_2 C)}$$

$$= \frac{R_1}{L} \frac{1 + j\omega \frac{L}{R_1}}{(j\omega)(1 + j\omega R_2 C)}$$

Plugging in the values, we have

$$H(\omega) = 10^9 \left(\frac{1 + j\omega 10^{-9}}{(j\omega)(1 + j\omega 10^{-7})} \right)$$

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