EECS 16B Designing Information Devices and Systems II
Fall 2018 Elad Alon and Miki Lustig

Discussion 6A

Linearization of systems

One dimensional linear approximation

Consider a differentiable function f of one variable.

$$f : \mathbb{R} \to \mathbb{R}$$

Say we are interested in f in a small, open neighborhood about a particular point t^*. t^* is often called the fixed point of the system. Let's call this open neighborhood U. In this case, we can construct a linear approximation of f about the neighborhood U. Recall Taylor Approximation,

$$\frac{\mathrm{d}f}{\mathrm{d}t}(t^*) \approx \frac{f(t) - f(t^*)}{t - t^*} \text{ for } t \in U$$

We can use the above to construct a linear approximation of f. Let f_l denote the linear approximation of f about U.

$$f_l(t) = \frac{\mathrm{d}f}{\mathrm{d}t}(t^*)(t - t^*) + f(t^*) \tag{1}$$

Strictly speaking, f_l is an affine approximation unless $t^* = 0$, but the process of obtaining f_l is colloquially called the linearization of f.

For example, consider $f(t) = t^2$. We will set the fixed point of the system to be $t^* = 1$. Then,

$$\begin{aligned} f_l(t) &= \tfrac{\mathrm{d}f}{\mathrm{d}t}(t^*)(t - t^*) + f(t^*) \\ &= 2(t-1) + 1 \\ &= 2t - 1 \end{aligned}$$

Let $\varepsilon = 10^{-2}$. Consider the open neighborhood $U = (1 - \varepsilon, 1 + \varepsilon)$. Let's plot f and f_l when their respective domains are restricted to U. This is seen in Figure **??** on the next page.

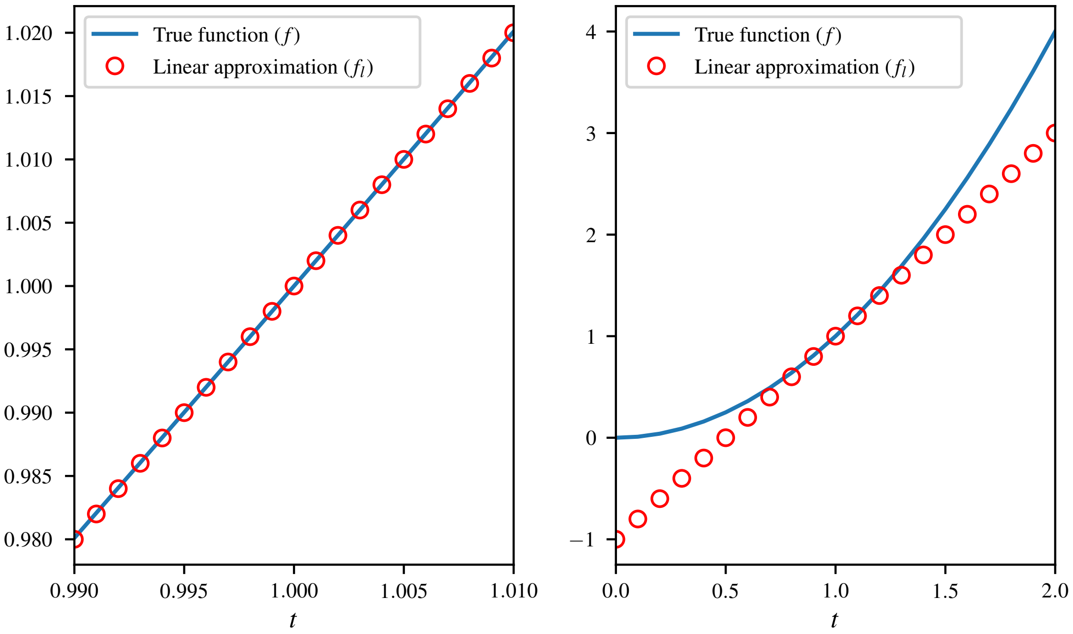


Figure 1: *Left*: The linear approximation of the function $f(t) = t^2$ about $t = 1$ is fairly accurate in the range $U = [0.99, 1.01]$. *Right*: Outside of this range, the approximation does not track the function very well.

Linearization of a system

Consider a continuous, non-linear system with state $\vec{x}(t)$ (which is n dimensional) and input $\vec{u}(t)$ (which is m dimensional) of the form,

$$\frac{\mathrm{d}\vec{x}}{\mathrm{d}t}(t) = f(\vec{x}(t), \vec{u}(t))$$

To clarify and establish notation, f is a function that takes in $\vec{x}(t)$ and $\vec{u}(t)$ and outputs an n dimensional vector. $f_k(\vec{x}(t), \vec{u}(t))$ refers to the function that is the k^{th} coordinate of the output of $f(\vec{x}(t), \vec{u}(t))$.

Let $\vec{x}^*(t)$ be the desired state trajectory and $\vec{u}^*(t)$ be the desired input. Analogous to (**??**), we will construct an affine estimate about a neighborhood around the fixed points.

$$f(\vec{x}(t), \vec{u}(t)) \approx f(\vec{x}^*(t), \vec{u}^*(t)) + \nabla_{\vec{x}} f(\vec{x}^*(t), \vec{u}^*(t))(\vec{x}(t) - \vec{x}^*(t)) + \nabla_{\vec{u}} f(\vec{x}^*(t), \vec{u}^*(t))(\vec{u}(t) - \vec{u}^*(t)) \qquad (2)$$

Let's define $\tilde{x} = \vec{x} - \vec{x}^*, \tilde{u} = \vec{u} - \vec{u}^*$. ∇f is the multidimensional generalization of the derivative, which is constructed as follows.

$$\nabla_{\mathbf{x}} f(\mathbf{x}, \mathbf{u}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{x}, \mathbf{u}) & \frac{\partial f_1}{\partial x_2}(\mathbf{x}, \mathbf{u}) & \dots & \frac{\partial f_1}{\partial x_n}(\mathbf{x}, \mathbf{u}) \\ \frac{\partial f_2}{\partial x_1}(\mathbf{x}, \mathbf{u}) & \frac{\partial f_2}{\partial x_2}(\mathbf{x}, \mathbf{u}) & \dots & \frac{\partial f_2}{\partial x_n}(\mathbf{x}, \mathbf{u}) \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_n}{\partial x_1}(\mathbf{x}, \mathbf{u}) & \frac{\partial f_n}{\partial x_2}(\mathbf{x}, \mathbf{u}) & \dots & \frac{\partial f_n}{\partial x_n}(\mathbf{x}, \mathbf{u}) \end{bmatrix} \text{ and } \nabla_{\mathbf{u}} f(\mathbf{x}, \mathbf{u}) = \begin{bmatrix} \frac{\partial f_1}{\partial u_1}(\mathbf{x}, \mathbf{u}) & \frac{\partial f_1}{\partial u_2}(\mathbf{x}, \mathbf{u}) & \dots & \frac{\partial f_1}{\partial u_m}(\mathbf{x}, \mathbf{u}) \\ \frac{\partial f_2}{\partial u_1}(\mathbf{x}, \mathbf{u}) & \frac{\partial f_2}{\partial u_2}(\mathbf{x}, \mathbf{u}) & \dots & \frac{\partial f_2}{\partial u_m}(\mathbf{x}, \mathbf{u}) \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_n}{\partial u_1}(\mathbf{x}, \mathbf{u}) & \frac{\partial f_n}{\partial u_2}(\mathbf{x}, \mathbf{u}) & \dots & \frac{\partial f_n}{\partial u_m}(\mathbf{x}, \mathbf{u}) \end{bmatrix}$$

Note the dimensions of the matrices. It must be notated that, when calculating $\nabla_x f$, $\vec{u}$ is considered constant. Similarly, when calculating $\nabla_u f$, $\vec{x}$ is considered constant.

To continue from (**??**), note that,

$$\frac{\mathrm{d}x}{\mathrm{d}t}(t) = f(x(t),u(t)), \frac{\mathrm{d}x^*}{\mathrm{d}t}(t) = f(x^*(t),u^*(t)) \text{ and } \frac{\mathrm{d}x}{\mathrm{d}t}(t) = \frac{\mathrm{d}x^*(t)}{\mathrm{d}t} + \frac{\mathrm{d}\tilde{x}}{\mathrm{d}t}(t)$$

Plugging this back into (**??**), we get,

$$\cancel{\frac{\mathrm{d}x^*}{\mathrm{d}t}(t)} + \frac{\mathrm{d}\tilde{x}}{\mathrm{d}t}(t) \approx \cancel{\frac{\mathrm{d}x^*}{\mathrm{d}t}(t)} + \nabla_{\vec{x}} f(\vec{x}^*(t),\vec{u}^*(t))\tilde{x}(t) + \nabla_{\vec{u}} f(\vec{x}^*(t),\vec{u}^*(t))\tilde{u}(t)$$

Thus, we get linearized version of our system.

$$\frac{\mathrm{d}\tilde{x}}{\mathrm{d}t}(t) \approx \nabla_{\vec{x}} f(\vec{x}^*(t),\vec{u}^*(t))\tilde{x}(t) + \nabla_{\vec{u}} f(\vec{x}^*(t),\vec{u}^*(t))\tilde{u}(t) \tag{3}$$

Note that, unlike the one dimensional linearization example, we are **linearizing with respect to $\vec{x}$ and $\vec{u}$**. Also, observe that our state variables are now the pertubations $\tilde{x}(t)$ and $\tilde{u}(t)$.

Questions

1. **Jacobian Warm-Up**

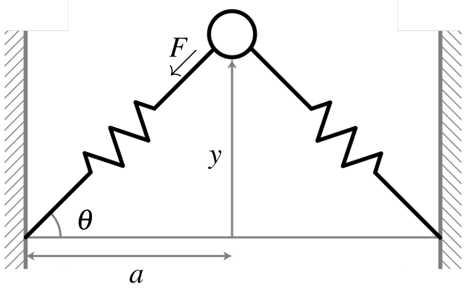
   Consider the following function $f : \mathbb{R}^2 \mapsto \mathbb{R}^3$

$$f(x_1,x_2) = \begin{pmatrix} f_1(x_1,x_2) \\ f_2(x_1,x_2) \\ f_3(x_1,x_2) \end{pmatrix} = \begin{pmatrix} x_1^2 - x_2^2 \\ x_1^2 + x_1x_2^2 \\ x_1 \end{pmatrix}$$

   Calculate its Jacobian.

2. **Linearization**

   Consider a mass attached to two springs:



We assume that each spring is linear with spring constant k and resting length X_0. We want to build a state space model that describes how the displacement y of the mass from the spring base evolves. The differential equation modeling this system is $\frac{d^2y}{dt} = -\frac{2k}{m}(y - X_0\frac{y}{\sqrt{y^2+a^2}})$.

   (a) Write this model in state space form $\dot{x} = f(x)$.

   (b) Find the equilibrium of the state-space model. You can assume $X_0 < a$.

   (c) Linearize your model about the equilibrium.