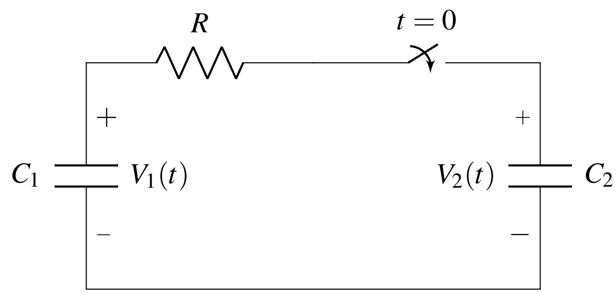
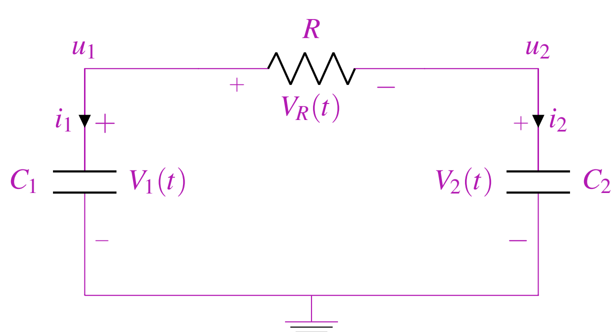
**1. Two capacitors**




(a) Using nodal analysis, find the differential equation describing $V_2(t)$ after the switch closes ($t \geq 0$).

**Answer:**

After the switch closes, If we make the bottom node ground, the circuit looks like the following:




$$u_1 = V_1$$
$$u_2 = V_2$$

We know the current flowing through a capacitor is $i_c = C\frac{dV_c}{dt}$, so:

$$i_1 = C_1\frac{dV_1}{dt}$$
$$i_2 = C_2\frac{dV_2}{dt}$$

Using KCL at node $u_1$:

$$i_1 + i_2 = 0$$
$$C_1\frac{dV_1}{dt} + C_2\frac{dV_2}{dt} = 0 \quad (1)$$

In order to make this a differential equation for $V_2$, we need to find an expression for $\frac{dV_1}{dt}$ in terms of $V_2$. We find a relationship by looking at the voltage drop across the resistor:

$$V_R = u_1 - u_2 = V_1 - V_2$$

$$V_1 = V_2 + V_R$$

Using Ohm's law:

$$V_1 = V_2 + i_2 R$$

$$V_1 = V_2 + RC_2 \frac{dV_2}{dt}$$

If we take the derivative of both sides:

$$\frac{dV_1}{dt} = \frac{dV_2}{dt} + RC_2 \frac{d^2V_2}{dt^2}$$

Plugging back into (1):

$$C_1 \left( \frac{dV_2}{dt} + RC_2 \frac{d^2V_2}{dt^2} \right) + C_2 \frac{dV_2}{dt} = 0$$

$$RC_1C_2 \frac{d^2V_2}{dt^2} + (C_1 + C_2) \frac{dV_2}{dt} = 0$$

$$\frac{d^2V_2}{dt^2} + \frac{(C_1 + C_2)}{RC_1C_2} \frac{dV_2}{dt} = 0$$

Since there is no $V_2(t)$ term in this equation, this is not actually a 2nd order differential equation. We can integrate both sides with respect to $t$ to get the equation into the standard 1st order diff. eq. form. Don't forget the generic constant $D$ when integrating:

$$\int \left( \frac{d^2V_2}{dt^2} + \frac{(C_1 + C_2)}{RC_1C_2} \frac{dV_2}{dt} \right) dt = \int 0 dt$$

$$\frac{dV_2}{dt} + \frac{(C_1 + C_2)}{RC_1C_2} V_2 = D$$

(b) Assuming $V_1(0) = 10\text{V}$ and $V_2(0) = 5\text{V}$, find the solution to the differential equation found in part (a). Use component values $C_1 = 1\text{fF}$, $C_2 = 4\text{fF}$, and $R = 10\text{k}\Omega$.

**Answer:**

This is a non-homogeneous differential equation, so we can use substitution to make it homogeneous:

$$x = V_2 - \frac{RC_1C_2}{C_1 + C_2} D$$

$$V_2 = x + \frac{RC_1C_2}{C_1 + C_2} D$$

$$\frac{dV_2}{dt} = \frac{dx}{dt}$$

Substituting back into the differential equation from part (a):

$$\frac{dx}{dt} + \frac{C_1 + C_2}{RC_1C_2} x + \left( \frac{C_1 + C_2}{RC_1C_2} \right) \left( \frac{RC_1C_2}{C_1 + C_2} \right) D = D$$

$$\frac{dx}{dt} + \frac{C_1+C_2}{RC_1C_2}x = 0$$

$$\frac{dx}{dt} = -\frac{C_1+C_2}{RC_1C_2}x$$

Using the general solution $x(t) = Ae^{\lambda t}$ where $\lambda = -\frac{C_1+C_2}{RC_1C_2}$, we get:

$$x = Ae^{-\frac{C_1+C_2}{RC_1C_2}t}$$

Now we substitute $V_2(t)$ back into the equation:

$$V_2 - \frac{RC_1C_2}{C_1+C_2}D = Ae^{-\frac{C_1+C_2}{RC_1C_2}t}$$

$$V_2(t) = \frac{RC_1C_2}{C_1+C_2}D + Ae^{-\frac{C_1+C_2}{RC_1C_2}t}$$

Because D is a generic constant, we can absorb the $\frac{RC_1C_2}{C_1+C_2}$ into $D$.

$$V_2(t) = D + Ae^{-\frac{C_1+C_2}{RC_1C_2}t}$$

Plugging in our initial condition for $V_2(0)$:

$$V_2(0) = 5 = D + A \tag{2}$$

Usually, we only need to look at the initial condition for $V_2(t)$ to solve for $A$. However, this time we have $D$ which is also an unknown constant. In this case, we can solve for $A$ and $D$ by also using the initial condition of $V_1(t)$:

$$i_2(0) = \frac{V_R(0)}{R} = \frac{V_1(0) - V_2(0)}{R}$$

$$i_2(0) = C_2\frac{dV_2(0)}{dt} = C_2 * -A\frac{C_1+C_2}{RC_1C_2} = -A\frac{C_1+C_2}{RC_1}$$

$$-A\frac{C_1+C_2}{RC_1} = \frac{V_1(0) - V_2(0)}{R}$$

$$A = -\frac{C_1}{C_1+C_2}(V_1(0) - V_2(0)) = -1$$

Plugging A back into (2):

$$5 = D - 1$$

$$D = 6$$

Plugging $D$ and $A$ back into the solution we get:

$$V_2(t) = 6 - e^{-\frac{C_1+C_2}{RC_1C_2}t} = 6 - e^{-\frac{5}{4*10^{-11}}t}$$

(c) How does $V_2(t=\infty)$ compare with what we expect from 16A (charge sharing)?

**Answer:**

If we look at our solution to the differential equation, we can see as $t$ approaches infinity:

$$V_2(\infty) = 6$$

Now let's look at it from the charge sharing perspective. First we need to calculate the total charge in the system before the switch closes. We can find the charge stored on each capacitor using $Q = VC$:

$$Q_1(0) = V_1(0)C_1$$

$$Q_2(0) = V_2(0)C_2$$

$$Q_{tot} = Q_1(0) + Q_2(0) = V_1(0)C_1 + V_2(0)C_2$$

After the switch closes, eventually the circuit will balance such that both capacitors will have the same voltage drop ($V_1(\infty) = V_2(\infty) = V_f$). Using conservation of charge, we can say:

$$Q_{tot} = V_f C_1 + V_f C_2$$

$$V_f = \frac{Q_{tot}}{C_1 + C_2}$$

$$V_f = \frac{V_1(0)C_1 + V_2(0)C_2}{C_1 + C_2}$$

$$V_f = \frac{(10+20)10^{-15}}{(1+4)10^{-15}} = 6$$

As you can see, the answers match up.

(d) Calculate the energy stored in $C_1$ and $C_2$ versus time.

**Answer:**

The energy stored in a capacitor is $\frac{1}{2}CV^2$

We already solved for $V_2(t)$, so we can plug in values:

$$U_2(t) = \frac{1}{2}(4*10^{-15})\left(6 - e^{-\frac{5}{4*10^{-11}}t}\right)^2 = \frac{1}{2}(4*10^{-15})\left(36 - 12e^{-\frac{5}{4*10^{-11}}t} + e^{-\frac{5}{2*10^{-11}}t}\right)$$

To find the energy stored in $C_1$, we need to solve for $V_1(t)$. We know:

$$V_1(t) = V_2(t) + V_R(t)$$

$$V_1(t) = V_2(t) + Ri_2(t)$$

$$V_1(t) = V_2(t) + RC_2\frac{dV_2}{dt}$$

$$V_1(t) = 6 - e^{-\frac{5}{4*10^{-11}}t} + (4*10^{-11})\left(\frac{5}{4*10^{-11}}\right)e^{-\frac{5}{4*10^{-11}}t}$$

$$V_1(t) = 6 + 4e^{-\frac{5}{4*10^{-11}}t}$$

$$U_1(t) = \frac{1}{2}C_1V_1^2(t)$$

$$U_1(t) = \frac{1}{2}(10^{-15})\left(6 + 4e^{-\frac{5}{4*10^{-11}}t}\right)^2 = \frac{1}{2}(10^{-15})\left(36 + 48e^{-\frac{5}{4*10^{-11}}t} + 16e^{-\frac{5}{2*10^{-11}}t}\right)$$

(e) Find the difference in total energy stored in the capacitors at $t = \infty$ and $t = 0$.

**Answer:**

$$U_{tot}(t) = U_1(t) + U_2(t)$$

$$U_{tot}(t) = (1 * 10^{-15})\left(90 + 10e^{-\frac{5}{2*10^{-11}}t}\right)$$

$$U_{tot}(0) = (90 + 10) * 10^{-15} = 100 * 10^{-15}\text{J}$$

$$U_{tot}(\infty) = 90 * 10^{-15}\text{J}$$

$$U_{tot}(0) - U_{tot}(\infty) = 10 * 10^{-15}\text{J}$$

(f) Calculate the energy dissipated by the resistor. Compare it to the difference in the total energy stored in the capacitors at $t = 0$ and $t = \infty$.

**Answer:**

To find the energy dissipated by the resistor, we take the integral of its instantaneous power:

$$U_R = \int_0^\infty V_R(t) i_R(t) dt$$

$$V_R(t) = V_1(t) - V_2(t) = 5e^{-\frac{5}{4*10^{-11}}t}$$

$$i_R(t) = i_2(t) = C_2 \frac{dV_2(t)}{dt} = (5 * 10^{-4})e^{-\frac{5}{4*10^{-11}}t}$$

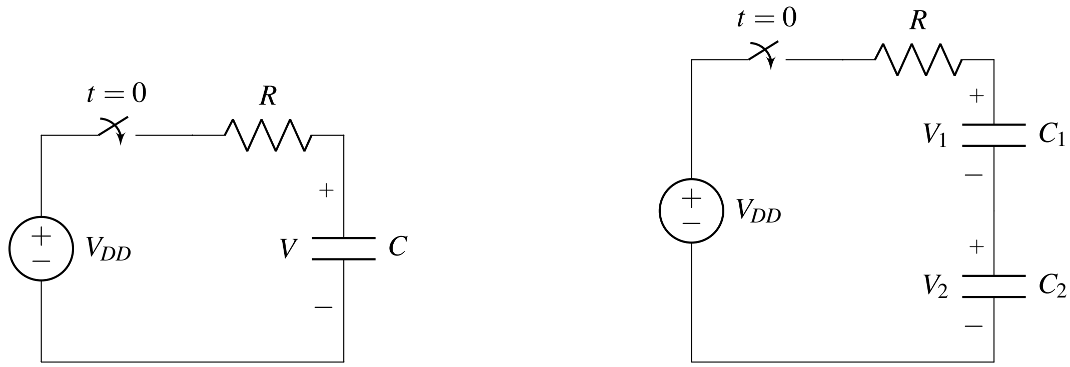
$$U_R = \int_0^\infty (25 * 10^{-4})e^{-\frac{5}{2*10^{-11}}t} dt$$

$$U_R = (25 * 10^{-4})\left(\frac{-2 * 10^{-11}}{5}\right) e^{-\frac{5}{4*10^{-11}}t}\Big|_0^\infty$$

$$U_R = 10^{-14}\text{J}$$

The energy disspated in the resistor is the same as the change in energy stored in the capacitors.

## 2. RC Circuit Variants




Circuit A on left, circuit B on right

(a) Using nodal analysis, find and solve the differential equation describing $V$ after the switch closes ($t \geq 0$) in circuit A. Assume the capacitor is initially discharged ($V(t \leq 0) = 0\text{V}$).

**Answer:** Let $u$ be the node between the resistor and capacitor. Using nodal analysis, we find

$$\begin{aligned}
\frac{u - V_{DD}}{R} + C\frac{du}{dt} &= 0 \\
\frac{1}{RC}(V_{DD} - V) &= \frac{dV}{dt} \\
V &= V_{DD} - ke^{-\frac{t}{RC}} \\
V(0) = 0 &= V_{DD} - ke^{-\frac{(0)}{RC}} \\
k &= V_{DD} \\
V &= V_{DD}(1 - e^{\frac{-t}{RC}})
\end{aligned}$$

(b) Using nodal analysis, find and solve the differential equation describing $V_1 + V_2$ (the total voltage across the capacitors) after the switch closes ($t \geq 0$) in circuit B. Assume both capacitors are initially discharged ($V_1(t \leq 0) = V_2(t \leq 0) = 0$).

**Answer:** Let $u_1$ be the node between the resistor and $C_1$ and $u_2$ be the node between $C_1, C_2$. Using nodal analysis, we find

$$\begin{aligned}
\frac{u_1 - V_{DD}}{R} + C_1\frac{d(u_1 - u_2)}{dt} &= 0 \\
C_1\frac{d(u_1 - u_2)}{dt} &= C_2\frac{du_2}{dt}
\end{aligned}$$

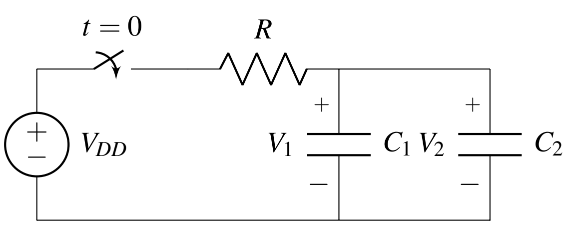
If we solve for $\frac{du_2}{dt}$ in the second equation and substitute in the first equation, we can get a differential equation for $u_1$, which is the quantity we desire.

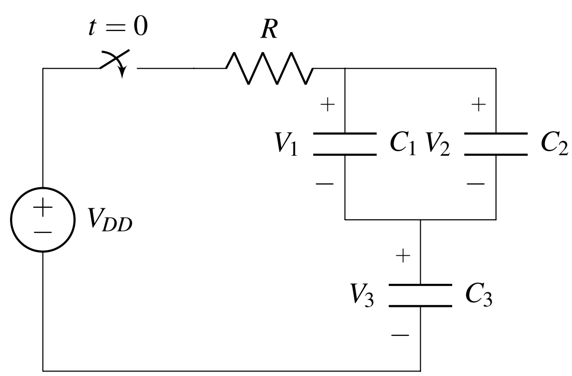
$$\begin{aligned}
C_1\frac{du_1}{dt} &= (C_1+C_2)\frac{du_2}{dt}\\
\frac{du_2}{dt} &= \frac{C_1}{C_1+C_2}\frac{du_1}{dt}\\
0 &= \frac{u_1-V_{DD}}{R}+C_1\frac{du_1}{dt}-C_1\left(\frac{C_1}{C_1+C_2}\frac{du_1}{dt}\right)\\
\frac{V_{DD}-u_1}{R} &= \frac{C_1C_2}{C_1+C_2}\frac{du_1}{dt}\\
\frac{1}{RC_1||C_2}(V_{DD}-u_1) &= \frac{du_1}{dt}
\end{aligned}$$

This looks exactly like the expression we arrived at previously with a different capacitor. We even have the same initial condition since both capcitors are intially discharged. Thus, our solution is

$$u_1 = V_{DD}(1-e^{\frac{-t}{RC_1||C_2}})$$

Which is exactly the expression we would've arrived at if we had treated the two capacitors as if they were in series.




Circuit C on left, circuit D on right

(c) Using nodal analysis, find and solve the differential equation describing $V_1$ (the total voltage across the capacitors) after the switch closes ($t \geq 0$) in circuit C. Assume both capacitors are initially discharged ($V_1(t \leq 0) = V_2(t \leq 0) = 0$).

**Answer:** Once again, let $u$ be the node between the resistor and the capacitor (either one, same node). Nodal analysis gives us

$$\begin{aligned}
\frac{u-V_{DD}}{R}+C_1\frac{du}{dt}+C_2\frac{du}{dt} &= 0\\
\frac{1}{R(C_1+C_2)}(V_{DD}-u) &= \frac{du}{dt}
\end{aligned}$$

The initial condition on $u$ is $u(0)=0$, again, so the final solution is

$$V_1 = V_{DD}(1-e^{\frac{-t}{R(C_1+C_2)}})$$

(d) Using nodal analysis, find and solve the differential equation describing $V_1 + V_3$ (the total voltage across the capacitors) after the switch closes ($t \geq 0$) in circuit D. Assume all capacitors are initially discharged ($V_1(t \leq 0) = V_2(t \leq 0) = V_3(t \leq 0) = 0$).

**Answer:** At this point, we know that parallel and series capcitor equivalents work for RC circuits. The equivalent capacitence for circuit D is $(C_1 + C_2)||C_3$, so the desired voltage is

$$V_{DD}(1 - e^{\frac{-t}{R((C_1+C_2)||C_3)}})$$

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