EE16B
Designing Information Devices and Systems II

Lecture 6B
Cont. stability of Linear State Models
Controllability

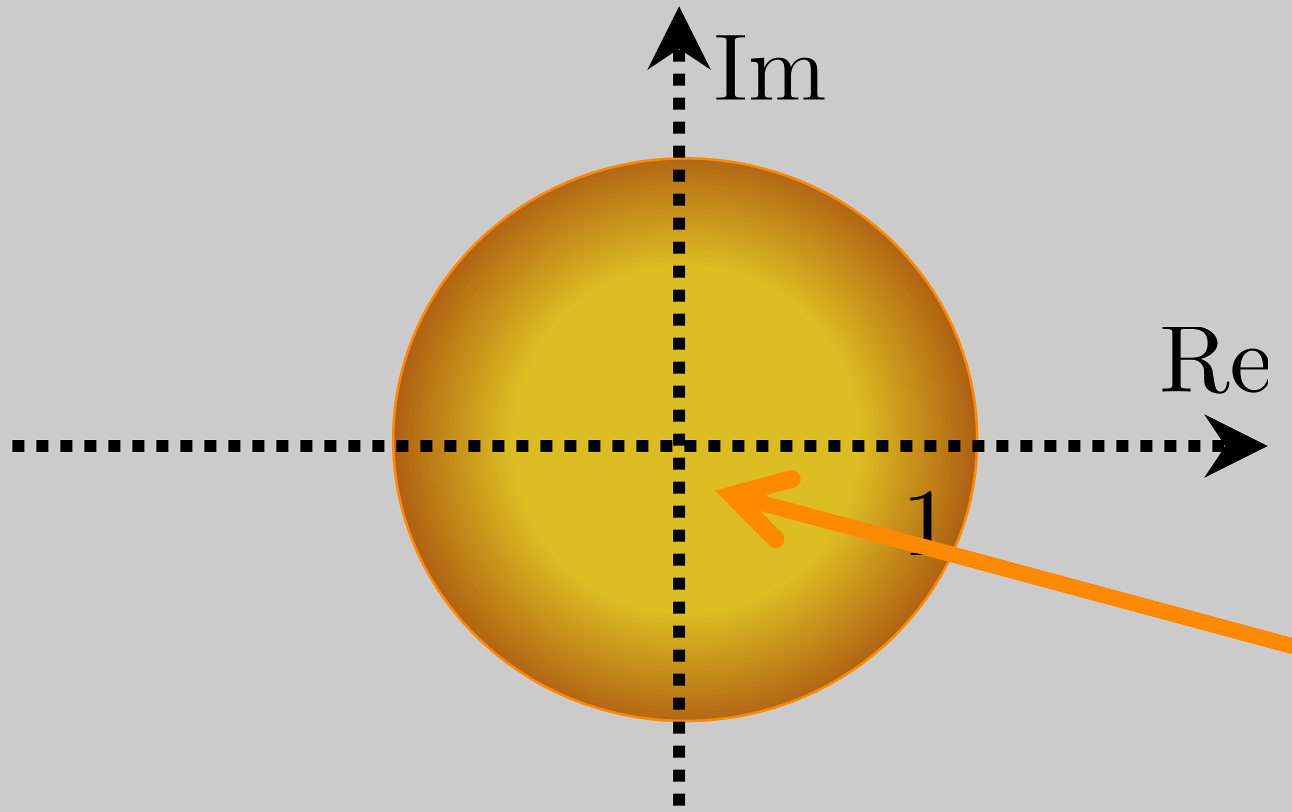
Today

- Last time:
  - Derived stability conditions for disc. and cont. systems
  - Easy to analyze using eigenvalues
- Today:
  - Eigenvalues can predict system behaviour
    - Envelope (decay) and Oscillation (frequency)
  - Controllability of systems

Stability -- Summary

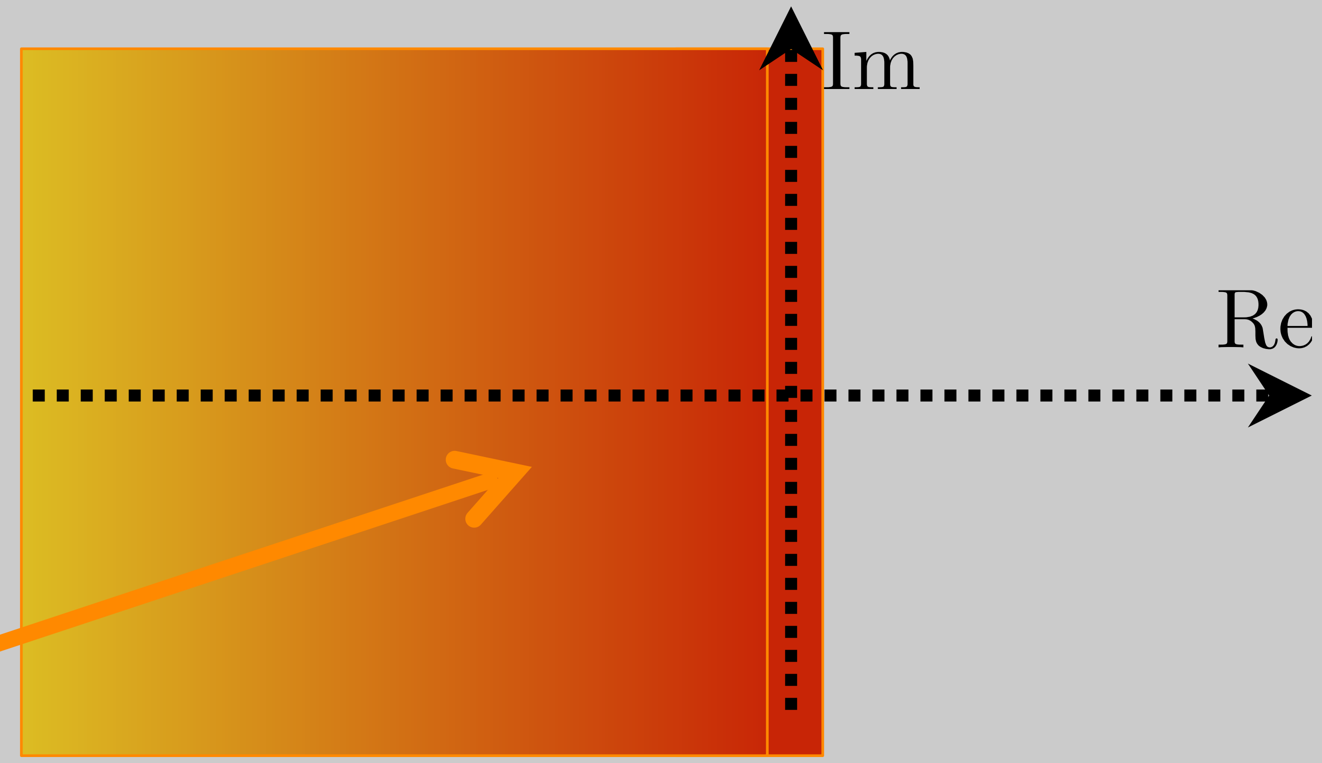
Discrete-Time

$$|\lambda_i(A)| < 1$$



Continuous-Time

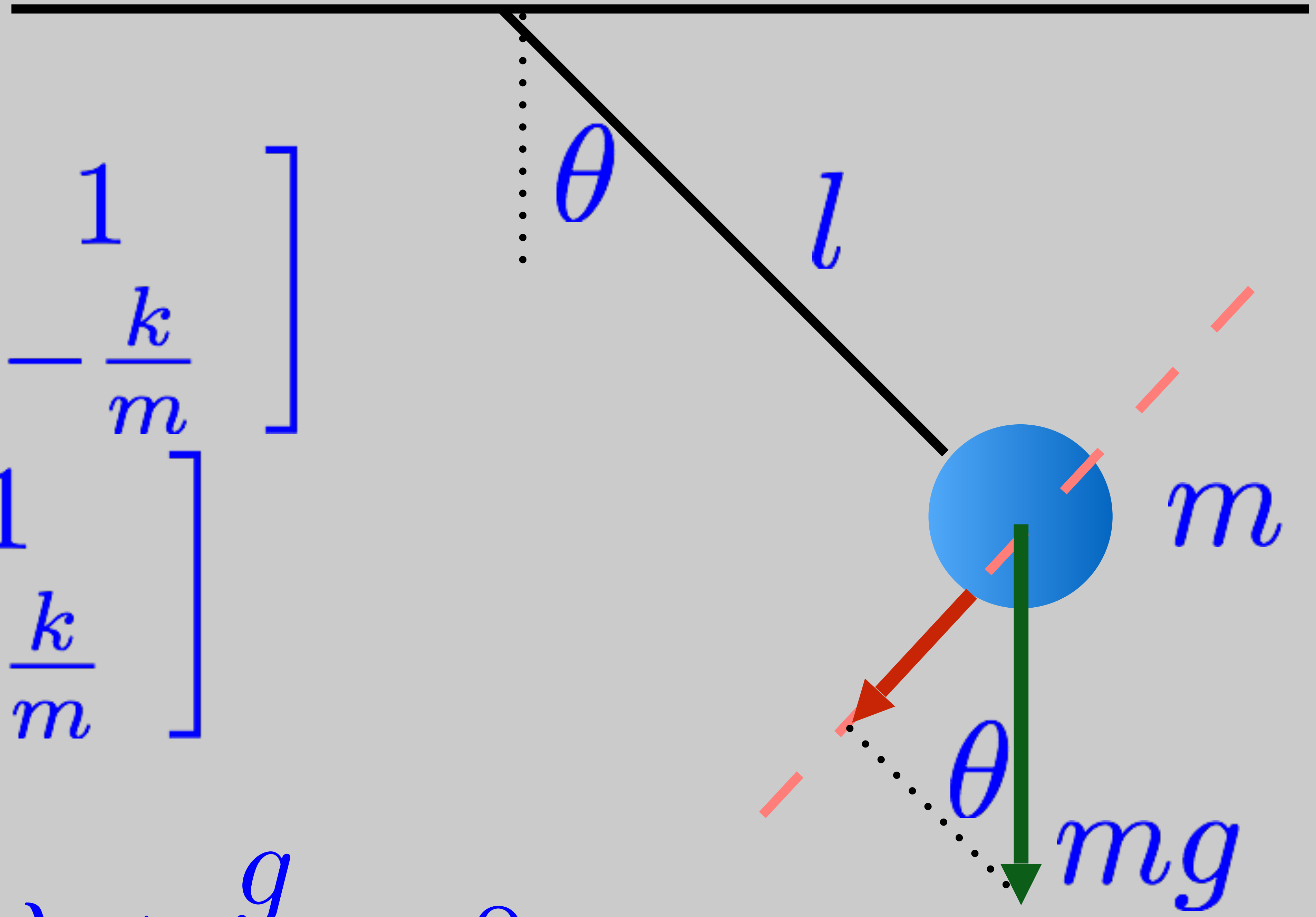
$$\text{Real}\{\lambda_i(A)\} < 0$$



Stable regions

Stay away from boundaries! System uncertainty can Move you over to unstable region

Back to the Pendulum



$$A_{\text{down}} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & -\frac{k}{m} \end{bmatrix}$$

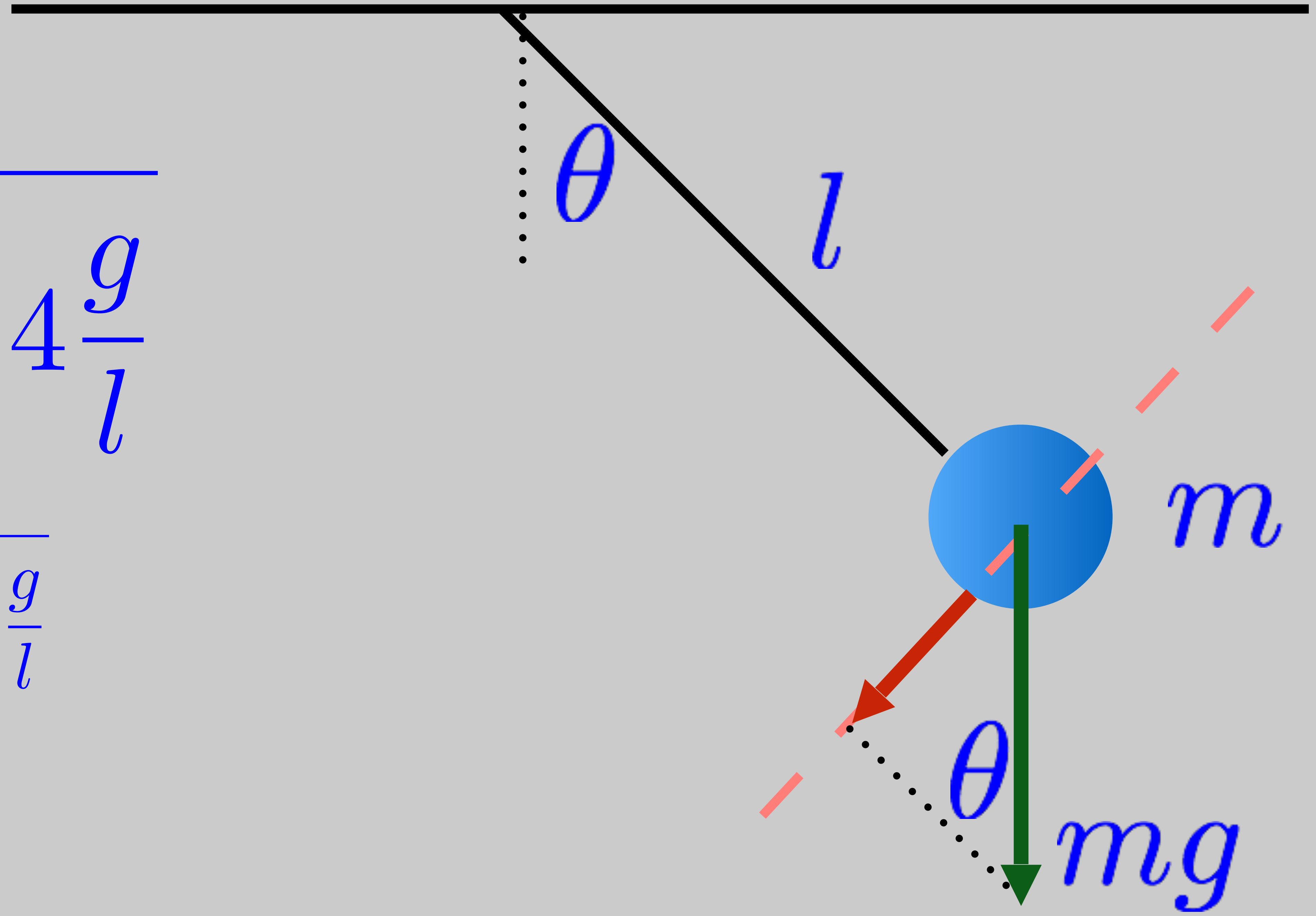
$$A_{\text{up}} = \begin{bmatrix} 0 & 1 \\ \frac{g}{l} & -\frac{k}{m} \end{bmatrix}$$

$$|\lambda I - A_{\text{down}}| = \begin{bmatrix} \lambda & -1 \\ \frac{g}{l} & \lambda + \frac{k}{m} \end{bmatrix} = \lambda^2 + \frac{k}{m}\lambda + \frac{g}{l} = 0$$

$$\lambda_{1,2} = -\frac{k}{2m} \pm \frac{1}{2}\sqrt{\frac{k^2}{m^2} - 4\frac{g}{l}}$$

Back to the Pendulum

$$\lambda_{1,2} = -\frac{k}{2m} \pm \frac{1}{2}\sqrt{\frac{k^2}{m^2} - 4\frac{g}{l}}$$



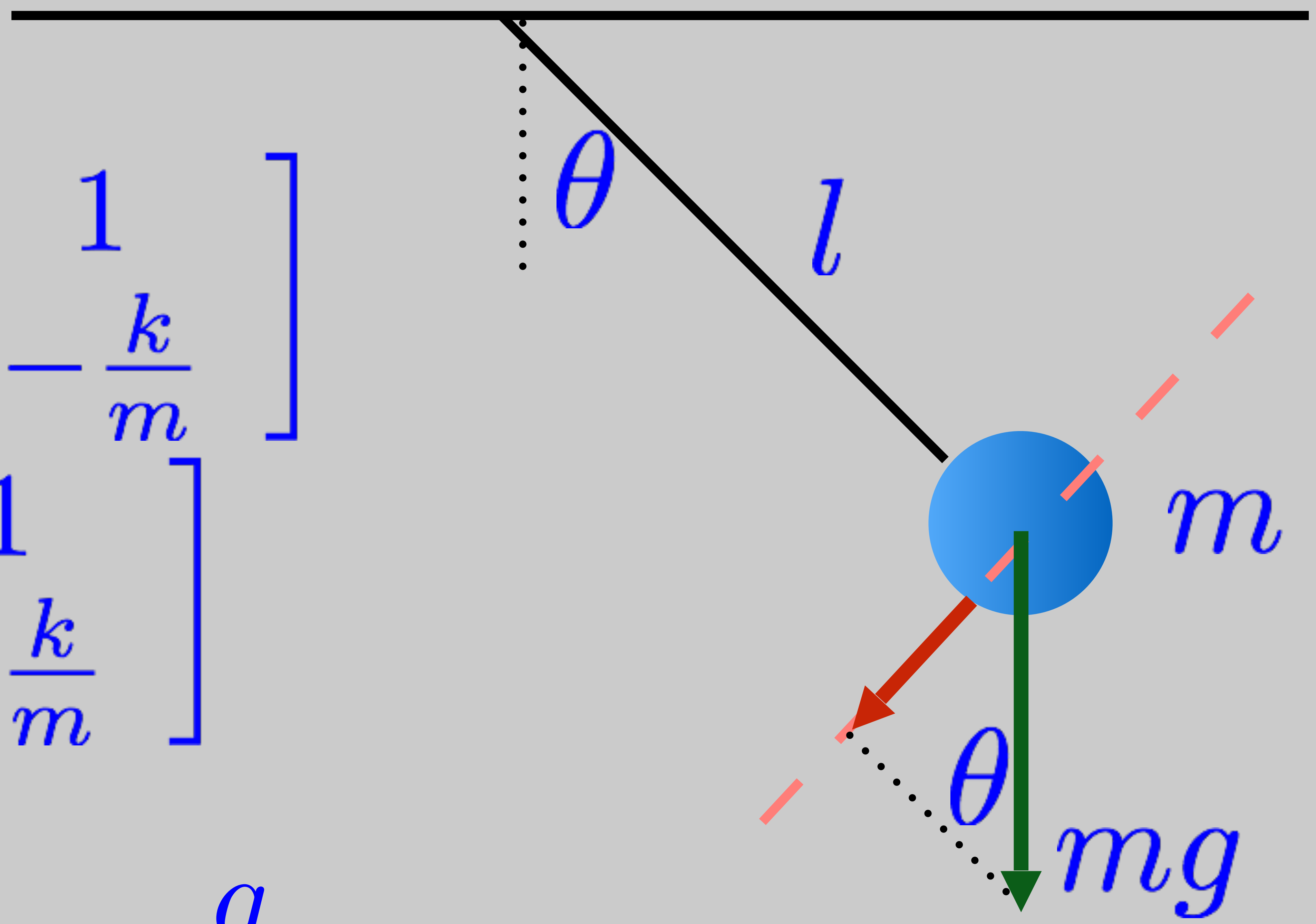
If $\frac{k^2}{m^2} \geq 4\frac{g}{l}$, i.e, sqrt is real, then $\frac{k}{2m} \geq \frac{1}{2}\sqrt{\frac{k^2}{m^2} - 4\frac{g}{l}}$

So, $\lambda_{1,2}$ always negative -- stable!

If $\frac{k^2}{m^2} < 4\frac{g}{l}$, i.e, sqrt is imaginary, then $\text{Re}\{\lambda_{1,2}\} = -\frac{k}{2m}$

So, $\text{Re}\{\lambda_{1,2}\}$ always negative -- stable!

Back to the Pendulum



$$A_{\text{down}} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & -\frac{k}{m} \end{bmatrix}$$

$$A_{\text{up}} = \begin{bmatrix} 0 & 1 \\ \frac{g}{l} & -\frac{k}{m} \end{bmatrix}$$

$$|\lambda I - A_{\text{up}}| = \begin{bmatrix} \lambda & -1 \\ -\frac{g}{l} & \lambda + \frac{k}{m} \end{bmatrix} = \lambda^2 + \frac{k}{m}\lambda - \frac{g}{l} = 0$$

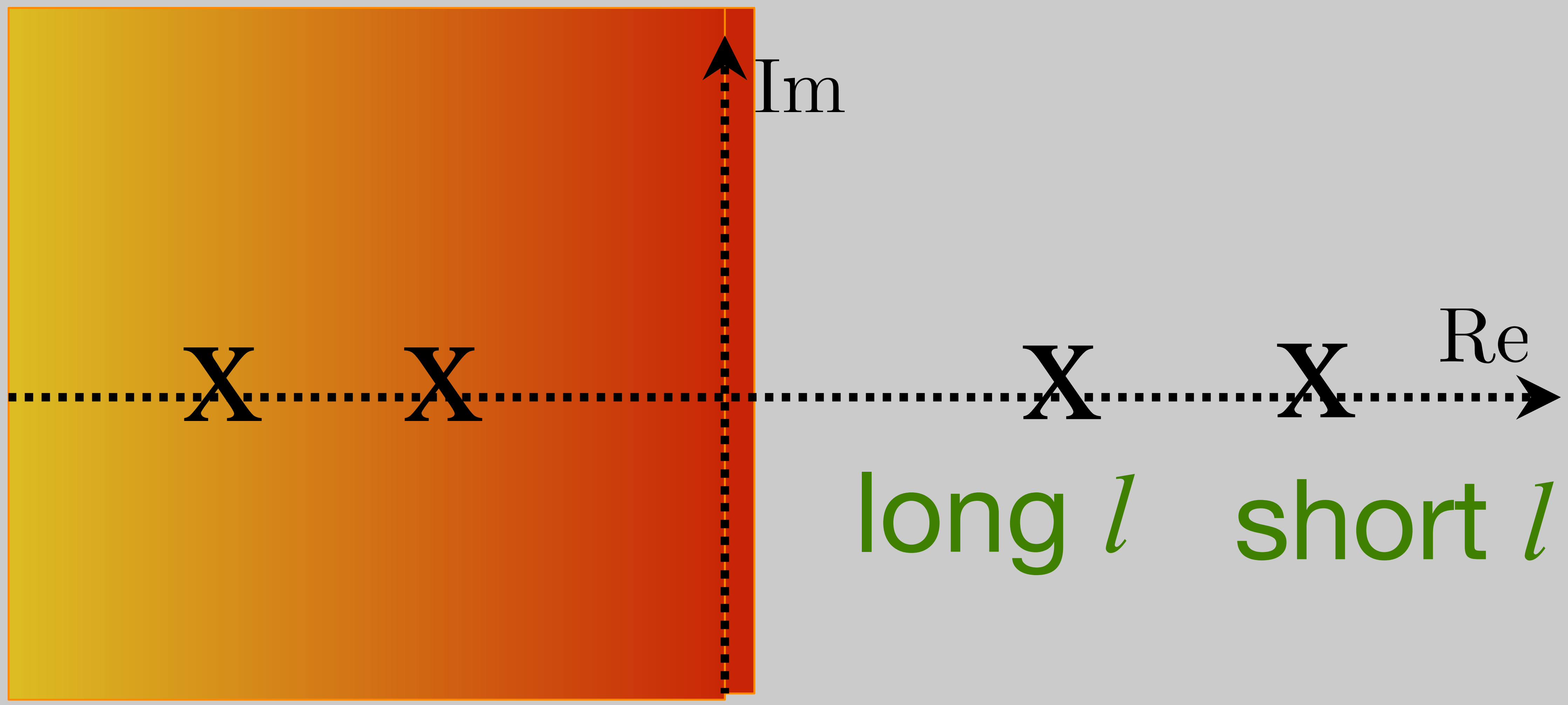
$$\lambda_{1,2} = -\frac{k}{2m} \pm \frac{1}{2}\sqrt{\frac{k^2}{m^2} + 4\frac{g}{l}}$$

$$\lambda_1 > 0$$

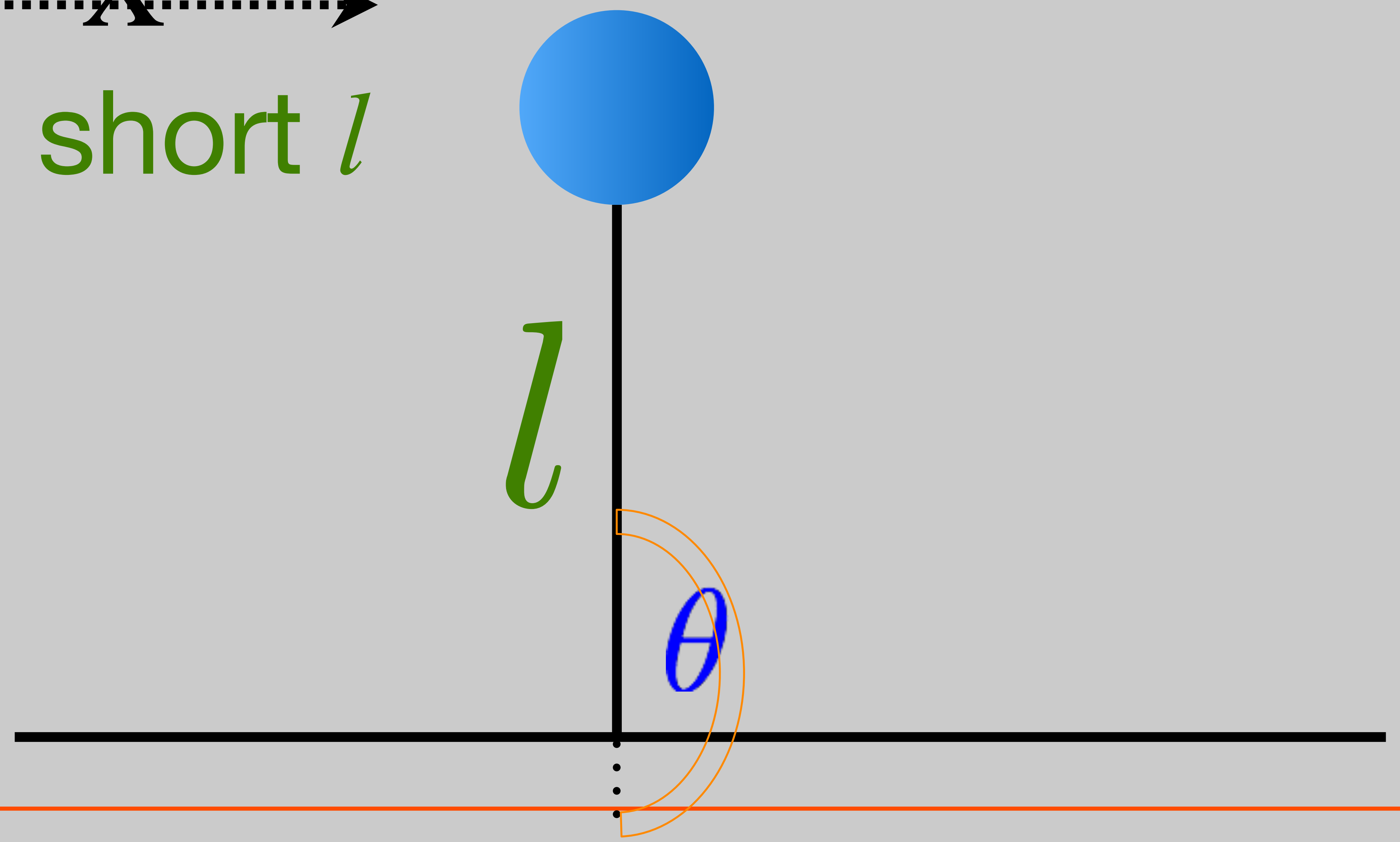
$$\lambda_2 < 0$$

Back to the Pendulum

$$\lambda_{1,2} = -\frac{k}{2m} \pm \frac{1}{2}\sqrt{\frac{k^2}{m^2} + 4\frac{g}{l}}$$



$\lambda_1 > 0$
$\lambda_2 < 0$



Predicting System Behavior

$$z(t+1) = \lambda_i z(t)$$

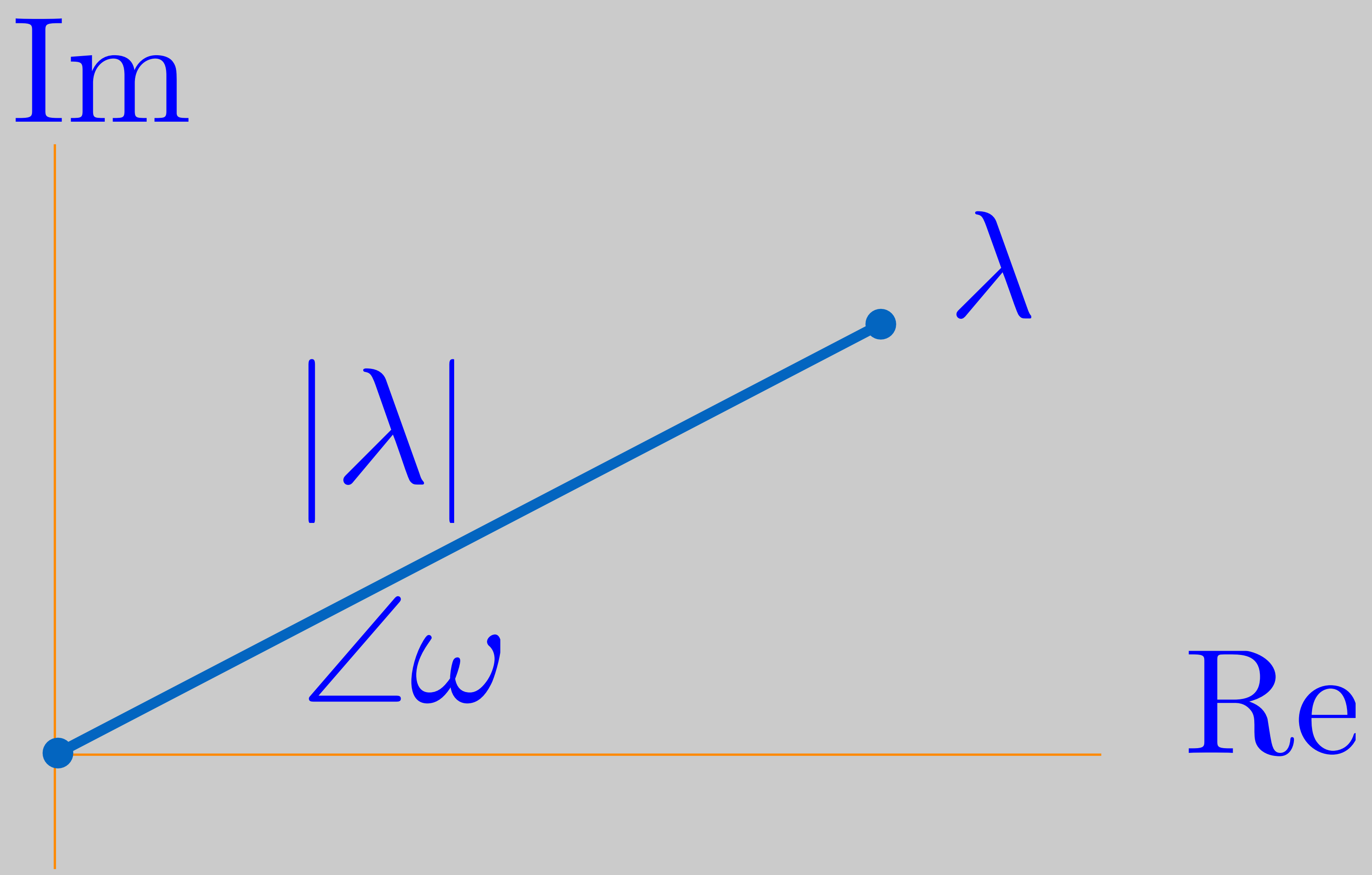
$$\text{Soln}: \quad \lambda_i^t z(0)$$

Discrete Time

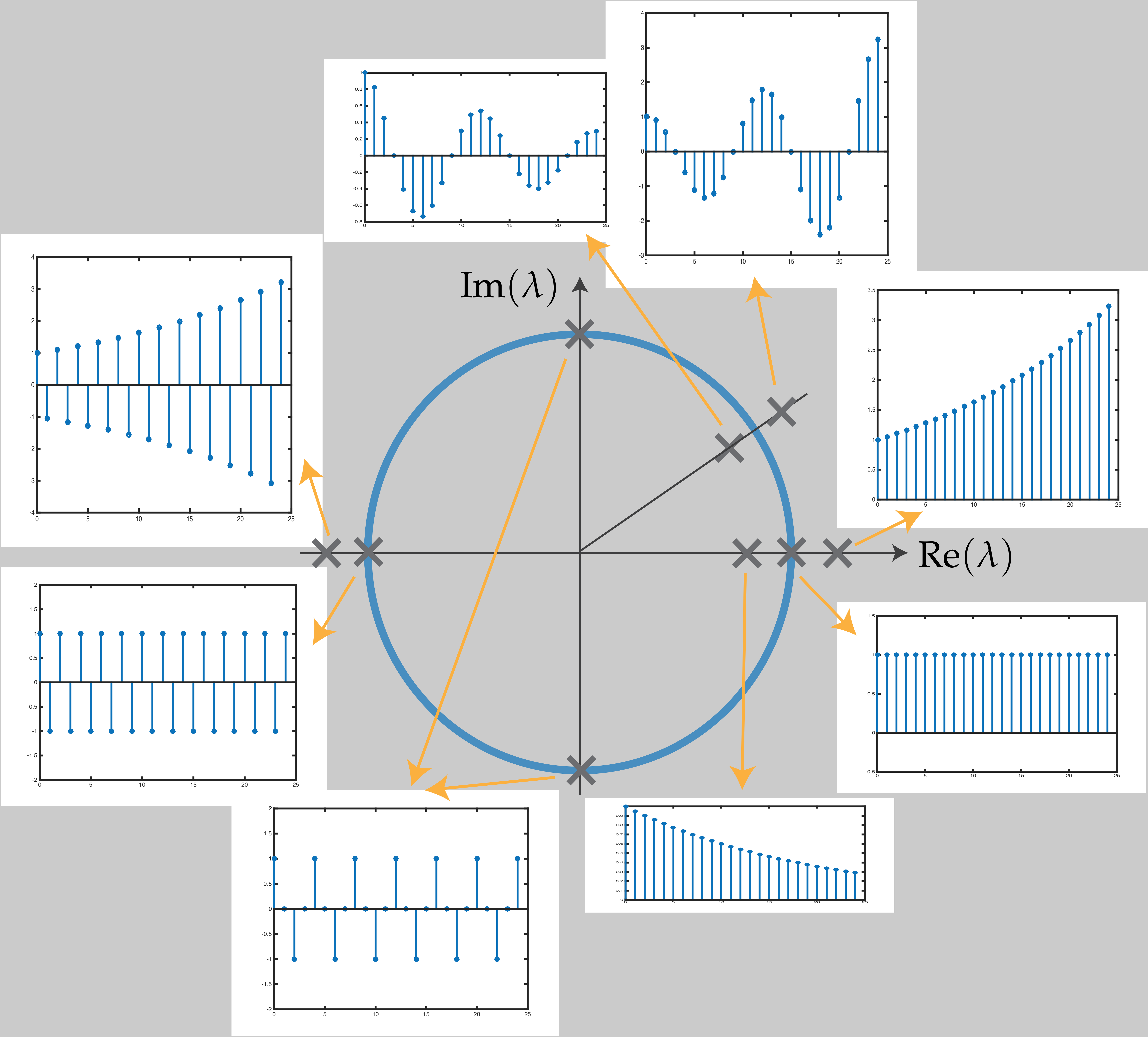
$$\lambda^t$$

- If λ is complex

$$\lambda^t = \left(|\lambda| e^{j\omega}\right)^t$$
$$= |\lambda|^t e^{j\omega t}$$



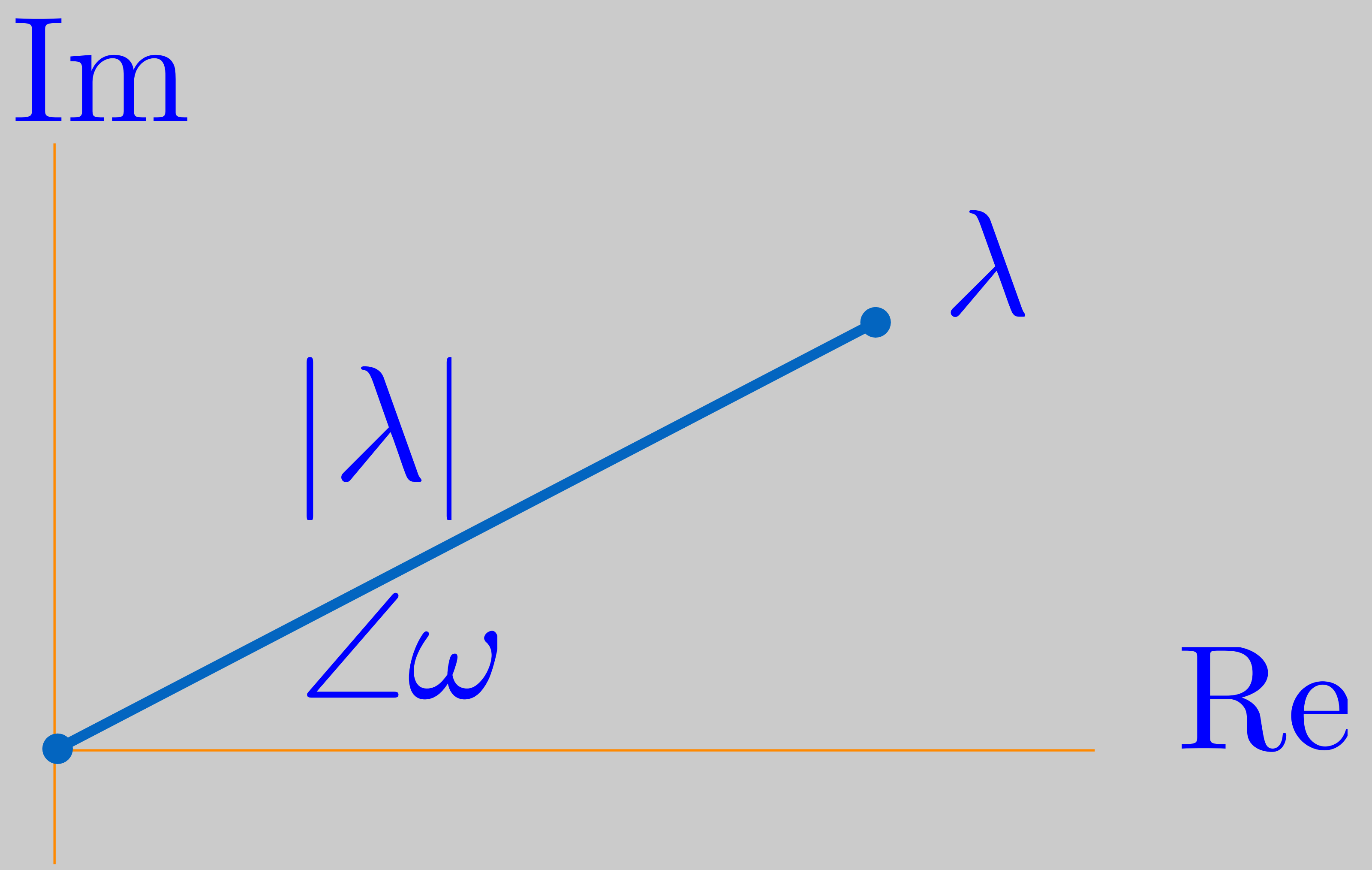
$$\lambda = |\lambda| e^{j\omega}$$ (Euler)



$\mathrm{Im}(\lambda)$
$\mathrm{Re}(\lambda)$

- If λ is complex

$$\lambda^t = \left(|\lambda| e^{j\omega}\right)^t$$
$$= |\lambda|^t e^{j\omega t}$$



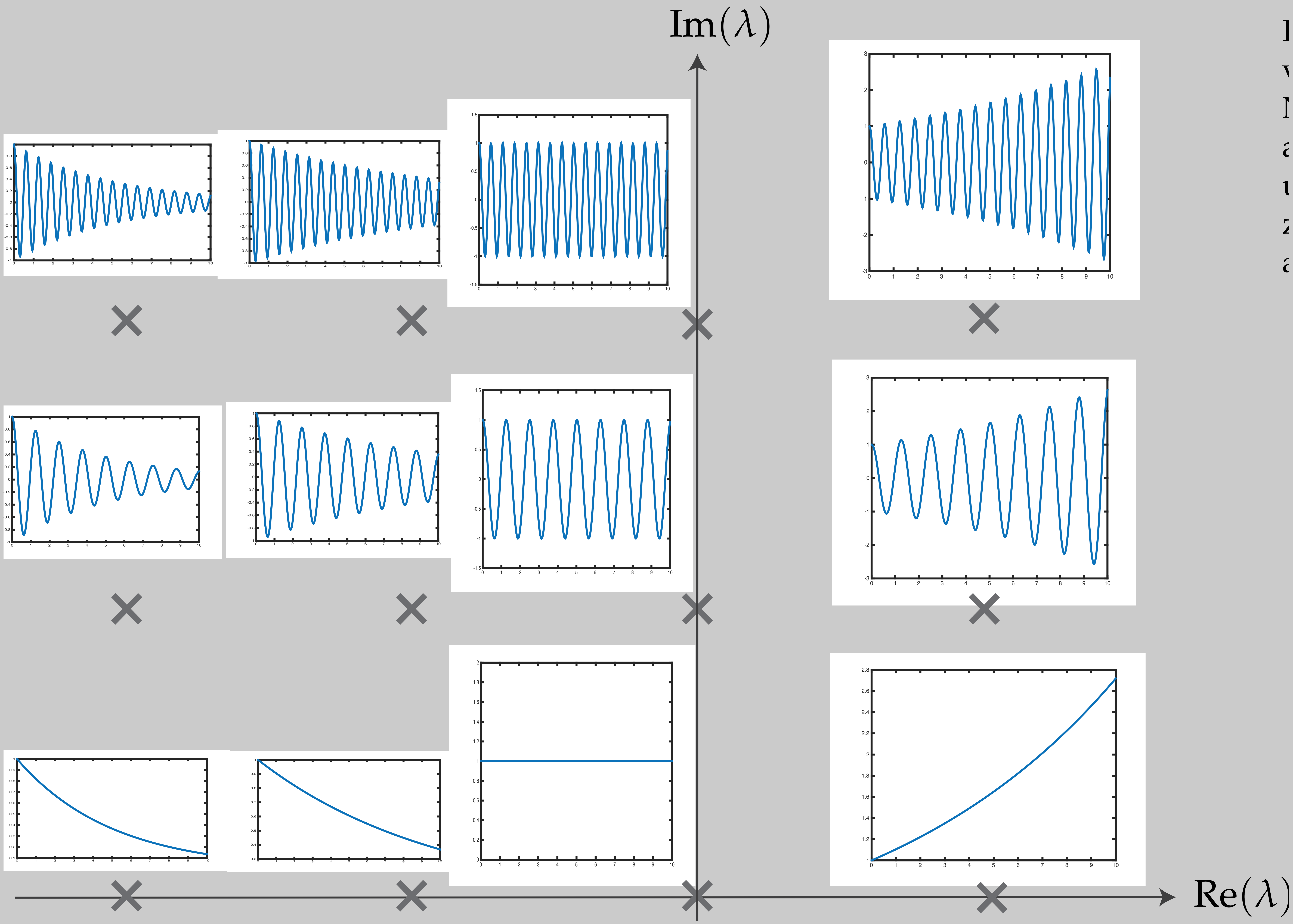
$\lambda = |\lambda| e^{j\omega}$ (Euler)

- Continuous time:

$$\frac{d}{dt} Z_i(t) = \lambda_i Z_i(t) \Rightarrow e^{\lambda_i t} Z_i(0)$$

Q) What does $e^{\lambda t}$ look like for different choices of λ?

A) $\lambda = v + j\omega \qquad \Rightarrow e^{\lambda t} = e^{vt} e^{j\omega t}$


$\mathrm{Im}(\lambda)$
$\mathrm{Re}(\lambda)$

Example

Big Picture

- State space modeling is powerful
- Linear state-space are awesome
  - We can say a lot about them!
  - We can approximate non-linear as linear at equilibrium points
- What can we say about linear systems?
  - We can tell if they are stable – we have a test!
  - We can can predict system behaviour for initial conditions!
- What about controls?
  - Can test if the system can be controlled to reach all states (controllability)
  - We can control a system to move to a certain state (open loop)
  - We can control a system to stay around a state (feedback)

Controllability

Discrete-time: $$\vec{x}(t+1) = A\vec{x}(t) + Bu(t)$$

From last time:

$$\vec{x}(t) = A^t\vec{x}(0) + \sum_{k=0}^{t-1} A^{t-1-k}Bu(k)$$

$$= A^t\vec{x}(0) + A^{t-1}Bu(0) + A^{t-2}Bu(1) + \cdots + Bu(t-1)$$

$$\vec{x}(t) - A^t\vec{x}(0) = \begin{bmatrix} & & \\ & ? & \\ & & \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(t-1) \end{bmatrix}$$

Controllability

$$\vec{x}(t) - A^t\vec{x}(0) = A^{t-1}Bu(0) + A^{t-2}Bu(1) + \cdots + Bu(t-1)$$

$$\vec{x}(t) - A^t\vec{x}(0) = \begin{bmatrix} \quad & \quad & \quad \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(t-1) \end{bmatrix}$$

Controllability

$$\vec{x}(t) - A^t\vec{x}(0) = A^{t-1}Bu(0) + A^{t-2}Bu(1) + \cdots + Bu(t-1)$$

$$\vec{x}(t) - A^t\vec{x}(0) = \underbrace{\begin{bmatrix} A^{t-1}B & A^{t-2}B & \cdots & AB & B \end{bmatrix}}_{R_t} \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(t-1) \end{bmatrix}$$

Q) Given any x(0), can we find u(t) s.t. x(t) = x_{target} for some t?

A) Depends if it is in the span of R_t

Controllability

$$R_t = \left[\begin{array}{cccccc} A^{t-1}B & A^{t-2}B & \cdots & AB & B \end{array} \right]$$

Q) For t < n ? A) No

Q) At t=n, If columns are independent? A) Absolutely!

Q) If not independent, does increasing t helps? A) No!

Cayley-Hamilton Theorem: If A is n x n, then A^n can be written as a linear combination of A^{n-1},…A, 1

$$A^n = \alpha_{n-1}A^{n-1} + \cdots + \alpha_1 A + \alpha_0 1$$

So does: $$A^n B = \alpha_{n-1}A^{n-1}B + \cdots + \alpha_1 AB + \alpha_0 B$$

Controllability

$$R_n = \begin{bmatrix} A^{n-1}B & A^{n-2}B & \cdots & AB & B \end{bmatrix}$$

What about R_{n+1}?

$$R_{n+1} = \begin{bmatrix} A^{n}B & A^{n-1}B & A^{n-2}B & \cdots & AB & B \end{bmatrix}$$

$$A^n B = \alpha_{n-1} A^{n-1} B + \cdots + \alpha_1 AB + \alpha_0 B$$

Controllability Test

If R_t doesn't have n independent columns at t=n, it never will for t > n either!

Therefore, we need only to examine R_n for controllability:

$$R_n = \begin{bmatrix} A^{n-1}B & A^{n-2}B & \cdots & AB & B \end{bmatrix}$$

Conclusion: $\vec{x}(t+1) = A\vec{x}(t) + Bu(t)$

is controllable if and only if

$$\text{rank}\{R_n\} = n$$

Example 1:

$$\vec{x}(t+1) = \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}}_{A} \vec{x}(t) + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{B} u(t)$$

$$R_2 = [AB \quad B] = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$R_3 = [A^2B \quad AB \quad B] = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Rank {R2} = 1, < n=2 ⇒ Not controllable!

State equations: $x_1(t+1) = x_1(t) + x_2(t) + u(t)$

$$x_2(t+1) = 2x_2(t)$$ (not stable)

Can not control x_2, not with u and not with x_1

Example 2:

$$p(t+1) = p(t) + Tv(t) + \frac{1}{2}T^2 u(t)$$

$$v(t+1) = v(t) + Tu(t)$$

$$\begin{bmatrix} p(t+1) \\ v(t+1) \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}}_{A} \begin{bmatrix} p(t) \\ v(t) \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{1}{2}T^2 \\ T \end{bmatrix}}_{B} u(t)$$

$$R_2 = [AB \quad B] = \begin{bmatrix} \frac{3}{2}T^2 & \frac{1}{2}T^2 \\ T & T \end{bmatrix}$$

Rank = 2 $\Rightarrow$ Controllable!

Continuous Time (no derivation here)

- The continuous-time system

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + Bu(t)$$

is controllable if and only if

$$R_n = \begin{bmatrix} A^{n-1}B & A^{n-2}B & \cdots & AB & B \end{bmatrix}$$

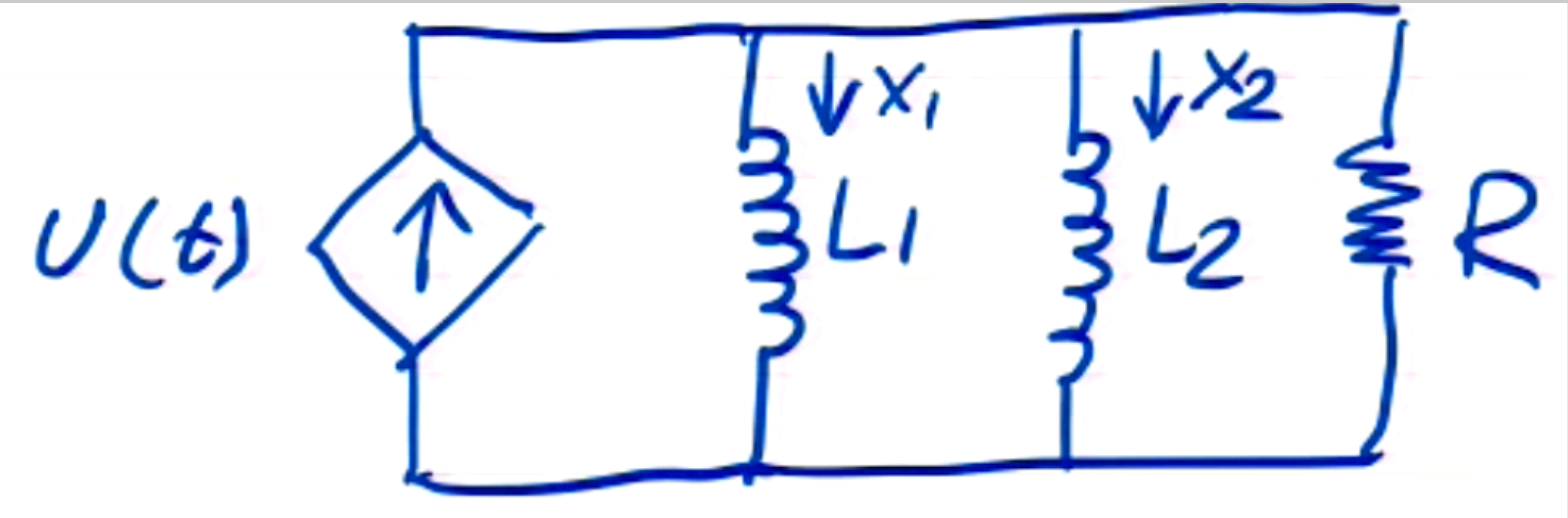
has rank = n

Example 3 + Quiz

- Write the state model:

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} \quad \\ \quad \end{bmatrix} u(t)$$

For:

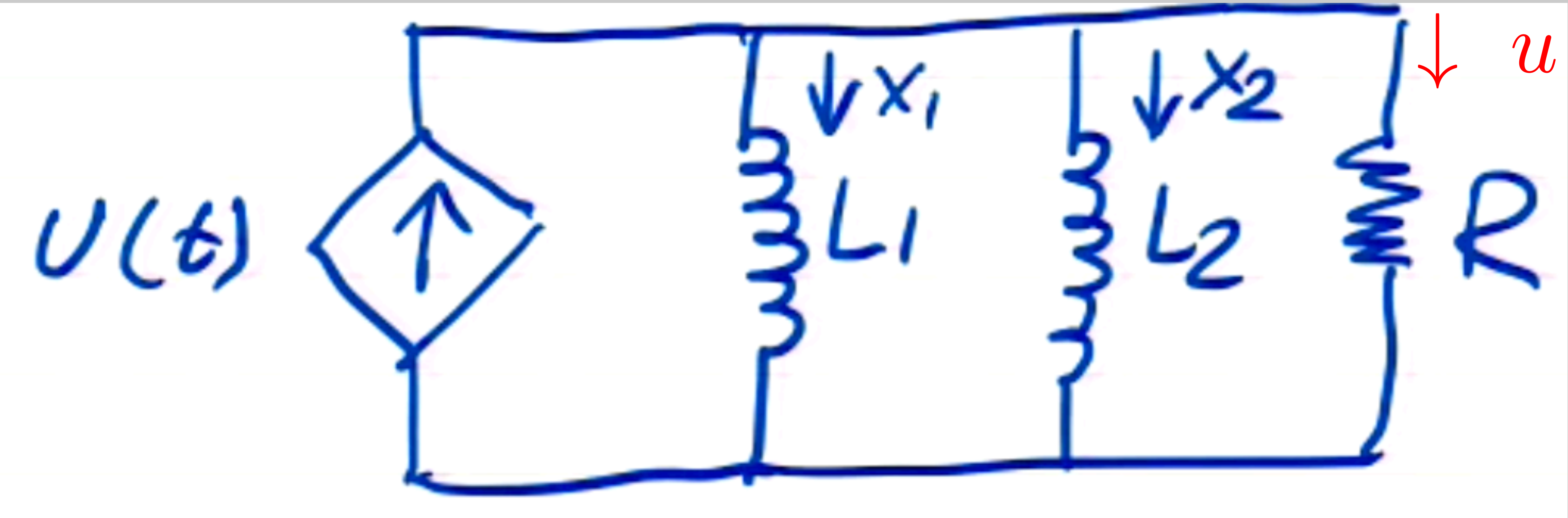


Quiz

- Write the state model:

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -\frac{R}{L_1} & -\frac{R}{L_1} \\ -\frac{R}{L_2} & -\frac{R}{L_2} \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} \frac{R}{L_1} \\ \frac{R}{L_2} \end{bmatrix} u(t)$$

For:



$\downarrow\ u - x_1 - x_2$

$$V_r = R(u - x_1 - x_2) = L_1\dot{x}_1 = L_2\dot{x}_2$$

Example 3

- Controllability:

$$B = \begin{bmatrix} \frac{R}{L_1} \\ \frac{R}{L_2} \end{bmatrix}$$

$$AB = \begin{bmatrix} -\frac{R}{L_1}\left(\frac{R}{L_1}+\frac{R}{L_2}\right) \\ -\frac{R}{L_2}\left(\frac{R}{L_1}+\frac{R}{L_2}\right) \end{bmatrix}$$

$$R = [AB \quad B]$$

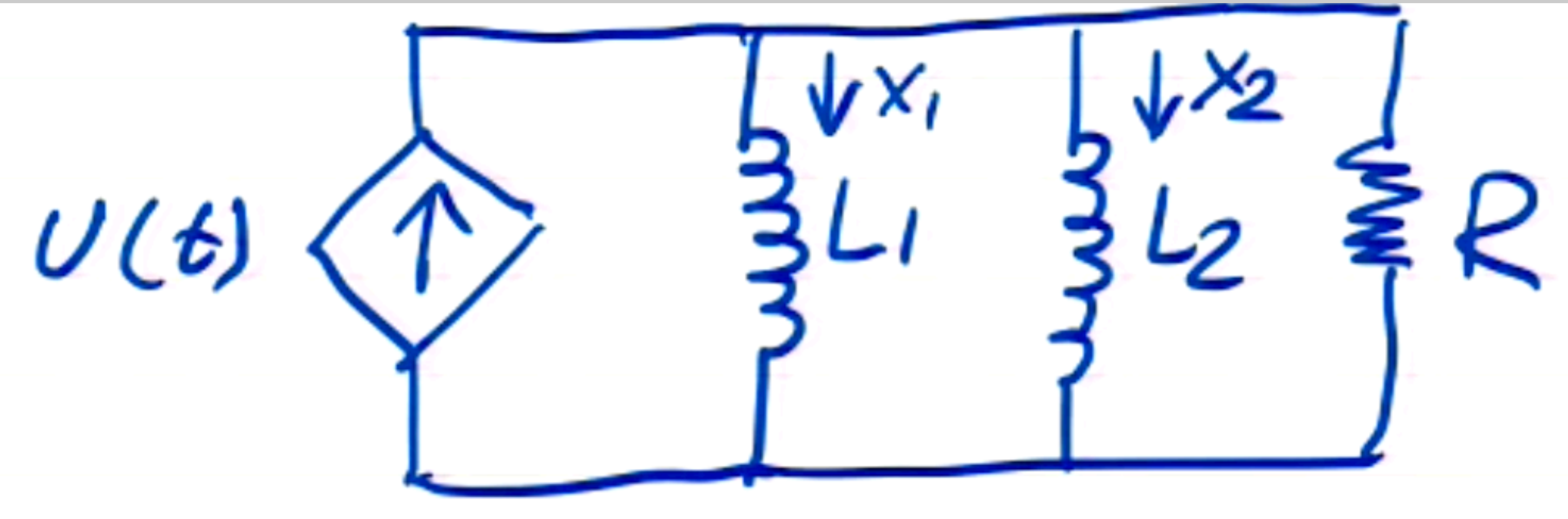
$$AB = \left(\frac{R}{L_1}+\frac{R}{L_2}\right) B$$

Rank = 1 !

Not controllable

Physical explanation

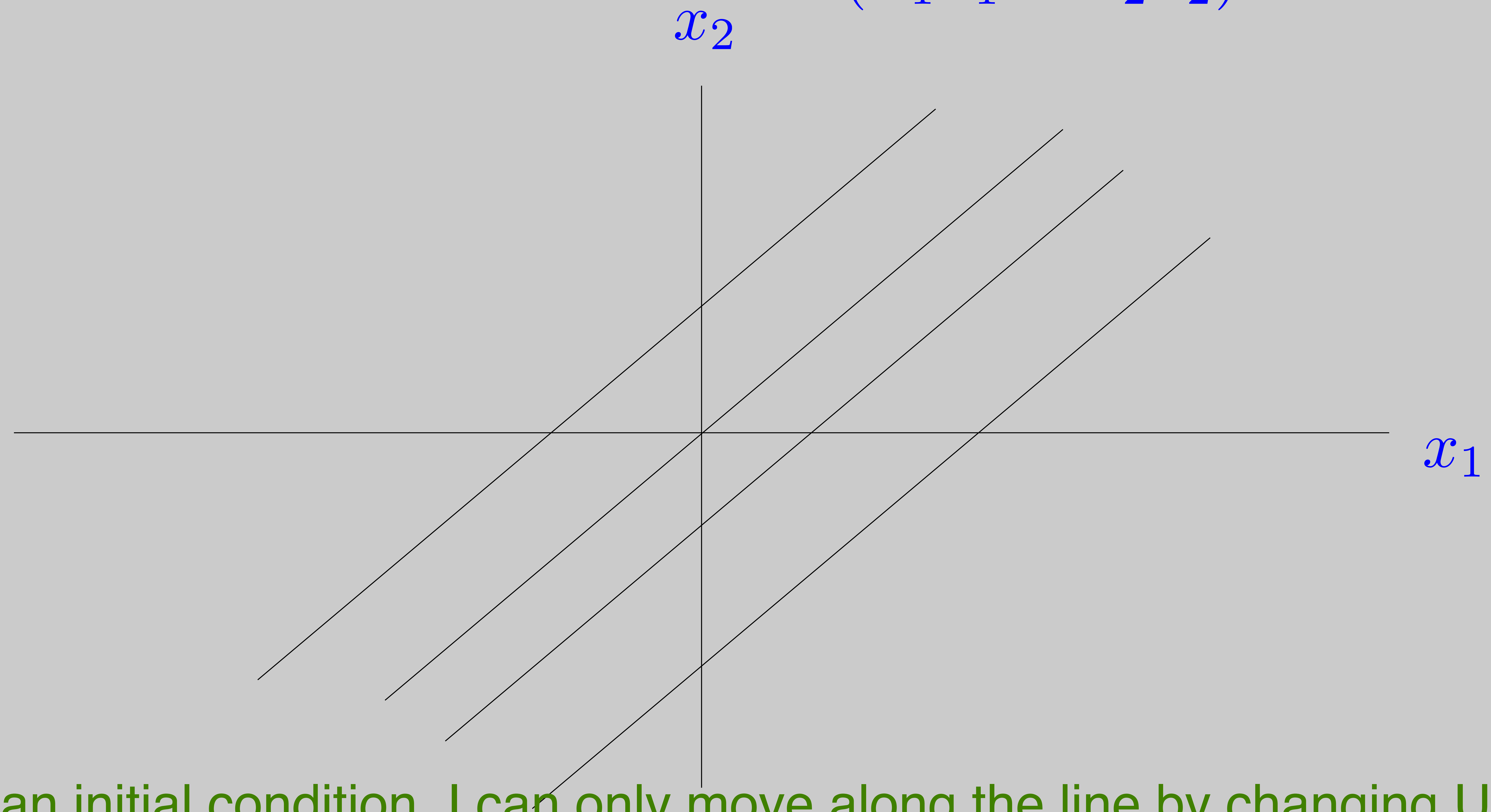
- Why can't I drive the currents x1 and x2 freely using U(t)?



$$L_1 \frac{dx_1}{dt} = L_2 \frac{dx_2}{dt} = R \cdot i_{\mathrm{R}} = V_{\mathrm{R}} \quad \Rightarrow L_1 \frac{dx_1}{dt} - L_2 \frac{dx_2}{dt} = 0$$

$$\frac{d}{dt}\left(L_1 x_1 - L_2 x_2\right) = 0 \qquad \Rightarrow \left(L_1 x_1 - L_2 x_2\right) = \mathrm{Const}$$

$$(L_1 x_1 - L_2 x_2) = \text{Const}$$



Given an initial condition, I can only move along the line by changing U

Q) What if A = 0 ? Can the system be controllable?

$$\frac{d}{dt}\vec{x}(t) = Bu(t)$$

$$R = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & B \end{bmatrix}$$

A) Only if u(t) is a vector with the same number of elements as the number of states

Summary

- Described and derived conditions for controllability of linear state models.
  – Rank of R_n for both discrete and continuouse
- Showed how to discretize continuous systems
- Showed examples of controllable and non-controllable systems

- Next time:
  – Open loop and state feedback control
  – Controllers to make systems do what we want!