EECS 16B Designing Information Devices and Systems II
Fall 2018 Elad Alon and Miki Lustig **Discussion 3A**

1 Second-Order Differential Equations

Second-order differential equations are differential equations of the form:

$$\frac{d^2y}{dt^2} + a_1\frac{dy}{dt} + a_0 y = b$$

If $b = 0$, we consider the differential equation to be homogeneous. Otherwise, the differential equation is said to be non-homogeneous.

1.1 Homogeneous Case

Let's first consider the homogeneous case.

Recall the solution for a first-order homogeneous differential equation.

$$\frac{dy}{dt} = \lambda y$$

Remember that differentiation is a linear operator, so this differential equation actually looks like an eigenvector/eigenvalue equation. Therefore, solving this equation is actually equivalent to finding an "eigenfunction" that corresponds to the eigenvalue λ.

The solution is:

$$y(t) = ce^{\lambda t}, c \in \mathbb{R}$$

In general, instead of solving a second order differential equation, we want to instead exploit what we know about first-order differential equations in order to find these "eigenfunctions" $ce^{\lambda t}$.

$$\frac{d^2y}{dt^2} + a_1\frac{dy}{dt} + a_0 y = 0$$

First, we rewrite the differential equation as a **matrix differential** equation $\frac{d}{dt}\vec{x}(t) = \mathbf{A}\vec{x}(t)$, where the state vector $\vec{x}(t)$ is a vector-valued function.

For a second-order differential equation, we define $\vec{x}(t) = \begin{bmatrix} y(t) \\ \frac{dy}{dt}(t) \end{bmatrix}$ and $\frac{d}{dt}\vec{x}(t) = \begin{bmatrix} \frac{dy}{dt}(t) \\ \frac{d^2y}{dt^2}(t) \end{bmatrix}$, where $\vec{x}(t)$ and $\frac{d}{dt}\vec{x}(t)$ are 2-dimensional.

We then have the following matrix differential equation:

$$\begin{bmatrix} \frac{dy}{dt}(t) \\ \frac{d^2y}{dt^2}(t) \end{bmatrix} = \mathbf{A} \begin{bmatrix} y(t) \\ \frac{dy}{dt}(t) \end{bmatrix}$$

We can fill in the entries of $\mathbf{A}$ by rewriting the original differential equation $\frac{d^2y}{dt^2} = -a_0 y - a_1\frac{dy}{dt}$ for the second row of $\mathbf{A}$. For the first row of $\mathbf{A}$, we observe that $\frac{dy}{dt}(t)$ appears in $\vec{x}(t)$ and in its derivative $\frac{d}{dt}\vec{x}(t)$.

Therefore,

$$\begin{bmatrix} \frac{dy}{dt}(t) \\ \frac{d^2y}{dt^2}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} \begin{bmatrix} y(t) \\ \frac{dy}{dt}(t) \end{bmatrix}$$

To transform this matrix differential equation into a system of first-order differential equations, we can diagonalize $\mathbf{A}$. Assuming that $\mathbf{A} = \mathbf{PDP}^{-1}$ is diagonalizble, where $\mathbf{D}$ is a diagonal matrix with the eigenvalues of $\mathbf{A}$ and $\mathbf{P}$ is the matrix with the corresponding eigenvectors, we can rewrite our matrix differential equation.

$$\frac{d}{dt}\vec{x}(t) = \mathbf{PDP}^{-1}\vec{x}(t)$$

$$\mathbf{P}^{-1}\frac{d}{dt}\vec{x}(t) = \mathbf{DP}^{-1}\vec{x}(t)$$

$$\frac{d}{dt}\mathbf{P}^{-1}\vec{x}(t) = \mathbf{DP}^{-1}\vec{x}(t)$$

We can then perform a change of variables. Let $\vec{z}(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \mathbf{P}^{-1}\vec{x}(t)$.

$$\frac{d}{dt}\vec{z}(t) = \mathbf{D}\vec{z}(t) = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \vec{z}(t)$$

$$\begin{bmatrix} \frac{dz_1}{dt}(t) \\ \frac{dz_2}{dt}(t) \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}$$

$$\begin{cases} \dfrac{dz_1}{dt}(t) = \lambda_1 z_1(t) \\ \dfrac{dz_2}{dt}(t) = \lambda_2 z_2(t) \end{cases}$$

We can now easily solve the system of first-order differential equations.

$$z_1(t) = k_1 e^{\lambda_1 t} \qquad z_2(t) = k_2 e^{\lambda_2 t}$$

To find $y(t)$, note that $\vec{x}(t) = \mathbf{P}\vec{z}(t)$.

$$\begin{bmatrix} y(t) \\ \frac{dy}{dt}(t) \end{bmatrix} = \mathbf{P} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}$$

Recall that $\mathbf{P}$ contains the eigenvectors of $\mathbf{A}$ and is so just a matrix of scalars. Therefore, $y(t)$ is simply a linear combination of $z_1(t)$ and $z_2(t)$, which are in turn linear combinations of $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$, respectively.

There are 3 cases for λ_1 and λ_2:

(a) λ_1 and λ_2 are both real. The solution is given by $y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$.

(b) $\lambda_1 = a + jb$ and $\lambda_2 = a - jb$ are complex conjugates of each other. The general solution is given by $y(t) = k_1 e^{\lambda_1 t} + k_2 e^{\lambda_2 t}$. However, using Euler's formula $e^{j\theta} = \cos(\theta) + j\sin(\theta)$, we can eliminate the complex exponentials and rewrite the solution as $y(t) = e^{at}(c_1 \cos(bt) + c_2 \sin(bt))$. The proof is left as an exercise for the reader.

(c) $\lambda_1 = \lambda_2 = \lambda$. The solution is given by $y(t) = c_1 e^{\lambda t} + c_2 t e^{\lambda t}$. We will not prove this in class.

Simply stated, to solve a second-order differential equation, we need to find the eigenvalues of the $\mathbf{A}$ matrix, i.e., finding the roots of the characteristic equation and determine the general solution based on the three cases of λ_1 and λ_2.

1.2 Non-homogeneous Case

We can also solve non-homogeneous second-order differential equations where $b \neq 0$ is a constant term, i.e.

$$\frac{d^2y}{dt^2} + a_1 \frac{dy}{dt} + a_0 y = b$$

We can still solve these equations using the method for homogeneous differential equations. If we substitute $\tilde{y} = y - \frac{b}{a_0}$, we note that $\frac{d\tilde{y}}{dt}(t) = \frac{dy}{dt}(t)$ and that $\frac{d^2\tilde{y}}{dt^2}(t) = \frac{d^2}{dt^2}(t)$.

$$\frac{d^2\tilde{y}}{dt^2}(t) + a_1 \frac{d\tilde{y}}{dt} + a_0 \tilde{y} = 0$$

Once we have solved for $\tilde{y}(t)$, we can reverse the substitution to get $y(t)$.

1.3 Initial Conditions

To solve for the scalars c_1 and c_2, we can set up a system of linear equations by plugging in the initial conditions into the general solution.

2 Inductors

2.1 Basics

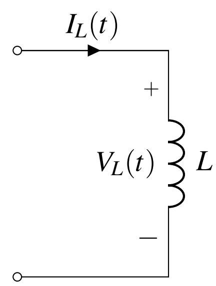

Figure 1: Example Inductor Circuit

The voltage across the inductor is related to its current as follows:

$$V_L(t) = L \frac{dI_L(t)}{dt},$$

where L is the *inductance* of the inductor. The SI unit of inductance is Henry (H).

At steady state, when the current flowing through an inductor is constant, there is no voltage drop across the inductor. Similarly, if the current through the inductor is changing, there will be a voltage drop across the inductor.

2.2 Equivalence Relations

Now that we have the basics, let's derive the equivalence relations for series and parallel combinations of inductors. We will find that there are similar to those of resistors.

2.2.1 Series Equivalence

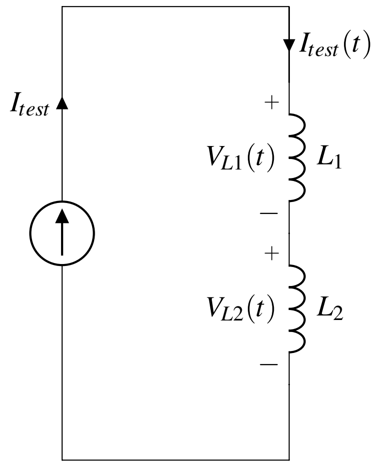

Figure 2: Series Inductor Circuit

Let's apply a $\frac{dI_{test}}{dt}$ through the two inductors, then

$$V_{L1}(t) + V_{L2}(t) = V_L$$

where, V_L is the voltage across the two inductors. From VI relationship for inductors, we get

$$L_1 \frac{dI_{test}}{dt} + L_2 \frac{dI_{test}}{dt} = V_L$$

$$(L_1 + L_2) \frac{dI_{test}}{dt} = V_L$$

$$L_{eq} \frac{dI_{test}}{dt} = V_L$$

where, $L_{eq} = L_1 + L_2$

2.2.2 Parallel Equivalence

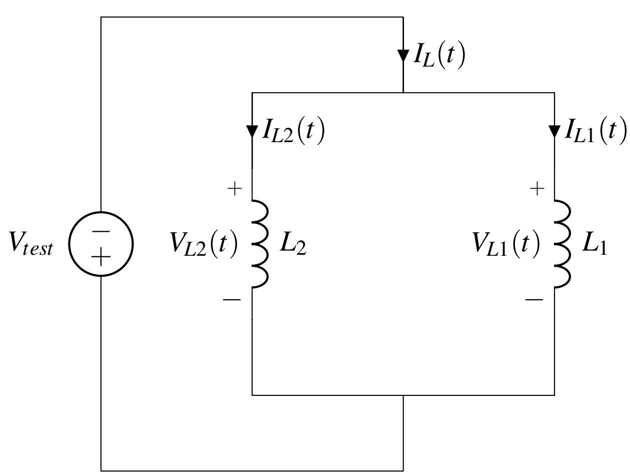

Figure 3: Parallel Inductor Circuit

We apply at V_{test} across the parallel combination. We have

$$V_{L1} = V_{L2} = V_{test}$$

$$L_1 \frac{dI_{L1}}{dt} = L_2 \frac{dI_{L2}}{dt} = L_{eq} \frac{dI_L}{dt}$$

and from KCL, we have

$$I_L(t) = I_{L1}(t) + I_{L2}(t)$$

Differentiating wrt time, and substituting from above equality,

$$\frac{dI_L}{dt} = \frac{dI_{L1}}{dt} + \frac{dI_{L2}}{dt}$$

$$\frac{dI_L}{dt} = \frac{L_{eq}}{L_1}\frac{dI_L}{dt} + \frac{L_{eq}}{L_2}\frac{dI_L}{dt}$$

$$\frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2}$$

1. **Solutions of Second-Order Differential Equations**

In this question, we are going to be converting complex exponentials into a sum of sines and cosines. In future lectures, we'll see how this relationship becomes useful for frequency analysis.

Consider a differential equation of the form,

$$\frac{\mathrm{d}^2 f}{\mathrm{d}t^2}(t) + a_1 \frac{\mathrm{d}f}{\mathrm{d}t}(t) + a_0 f(t) = 0,$$

such that

$$f(t) = c_1 e^{\lambda t} + c_2 e^{\lambda^* t},$$

where $f(\cdot)$ is a real valued function from $\mathbb{R}$ to $\mathbb{R}$ and λ^* is the complex conjugate of λ.

(a) Use the fact that f is real to prove that c_1 and c_2 are complex conjugates of each other.
Note: the complex conjugate of $z = x + jy$ is $\bar{z} = x - jy$, and recall the Euler's formula is

$$\mathrm{e}^{i\theta} = \cos\theta + j\sin\theta$$

Hint. Let $c_1 = a_1 + jb_1, c_2 = a_2 + jb_2$ and $\lambda = \sigma + j\omega$.

Answer:

Let $\lambda = \sigma + j\omega, c_1 = a_1 + jb_1$ and $c_2 = a_2 + jb_2$. Using Euler's formula, we can expand $f(t)$ as follows:

$$\begin{aligned} f(t) &= e^{\sigma t}\left((a_1 + jb_1)e^{j\omega t} + (a_2 + jb_2)e^{-j\omega t}\right) \\ &= e^{\sigma t}\left((a_1 + jb_1)\left(\cos(\omega t) + j\sin(\omega t)\right) + (a_2 + jb_2)\left(\cos(\omega t) - j\sin(\omega t)\right)\right) \end{aligned}$$

Particularly, observe that,

$$\mathrm{Im}\left\{f(t)\right\} = e^{\sigma t}\left(a_1 \sin(\omega t) + b_1 \cos(\omega t) - a_2 \sin(\omega t) + b_2 \cos(\omega t)\right)$$

Since $f(t)$ is a real function, it must be the case that,

$$\mathrm{Im}\left\{f(t)\right\} = 0 \text{ for all } t$$

This is the case when $a_1 = a_2$ and $b_1 = -b_2$. Thus $c_2 = c_1^*$.

(b) Let $c_1 = a + jb, c_2 = a - jb$ and $\lambda = \sigma + j\omega$. Show that you can reduce $f(t)$ to the following form:

$$f(t) = \left(2a\cos(\omega t) - 2b\sin(\omega t)\right)e^{\sigma t}$$

Answer:

Similar to part (a), we use Euler's to expand $f(t)$ as follows:

$$\begin{aligned} f(t) &= e^{\sigma t}\left((a+jb)e^{j\omega t} + (a-jb)e^{-j\omega t}\right) \\ &= e^{\sigma t}\left((a+jb)\left(\cos(\omega t) + j\sin(\omega t)\right) + (a-jb)\left(\cos(\omega t) - j\sin(\omega t)\right)\right) \\ &= e^{\sigma t}\left(2a\cos(\omega t) - 2b\sin(\omega t)\right) \end{aligned}$$

2. Differential Equations

Solve the following second-order differential equations.

(a) $\frac{d^2y}{dt^2} + \frac{dy}{dt} - 6y = 12$, where $y(0) = 1$ and $\frac{dy}{dt}(0) = 1$

Answer:

Since this is a non-homongeneous differential equation, define $\tilde{y} = y + 2$. Then, $\frac{d\tilde{y}}{dt} = \frac{dy}{dt}$ and $\frac{d^2\tilde{y}}{dt^2} = \frac{d^2y}{dt^2}$.

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} - 6y - 12 = 0$$

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} - 6(y+2) = 0$$

$$\frac{d^2\tilde{y}}{dt^2} + \frac{d\tilde{y}}{dt} - 6\tilde{y} = 0$$

We set up the matrix differential equation with the state vector $\vec{x}(t) = \begin{bmatrix} \tilde{y}(t) \\ \frac{d\tilde{y}}{dt}(t) \end{bmatrix}$.

$$\begin{bmatrix} \frac{d\tilde{y}}{dt}(t) \\ \frac{d^2\tilde{y}}{dt^2}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 6 & -1 \end{bmatrix} \begin{bmatrix} \tilde{y}(t) \\ \frac{d\tilde{y}}{dt}(t) \end{bmatrix}$$

We then find the eigenvalues of the $\mathbf{A}$ matrix.

$$\det\left(\begin{bmatrix} -\lambda & 1 \\ 6 & -1-\lambda \end{bmatrix}\right) = \lambda^2 + \lambda - 6 = (\lambda+3)(\lambda-2) = 0$$

Since $\lambda_1 = -3$ and $\lambda_2 = 2$, the general solution is given by:

$$\tilde{y}(t) = c_1 e^{-3t} + c_2 e^{2t}$$

We reverse the substitution, so we get:

$$y(t) = \tilde{y}(t) - 2 = c_1 e^{-3t} + c_2 e^{2t} - 2$$

$$\frac{dy}{dt}(t) = -3c_1 e^{-3t} + 2c_2 e^{2t}$$

To find c_1 and c_2, we plug in the initial conditions.

$$\begin{cases} c_1 + c_2 - 2 = 1 \\ -3c_1 + 2c_2 = 1 \end{cases}$$

Solving this system of equations gives:

$$c_1 = 1 \qquad c_2 = 2$$

Therefore,

$$y(t) = e^{-3t} + 2e^{2t} - 2$$

(b) $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = 0$, where $y(0) = 2$ and $\frac{dy}{dt}(0) = -1$

Answer:

We set up the matrix differential equation with the state vector $\vec{x}(t) = \begin{bmatrix} y(t) \\ \frac{dy}{dt}(t) \end{bmatrix}$.

$$\begin{bmatrix} \frac{dy}{dt}(t) \\ \frac{d^2y}{dt^2}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix} \begin{bmatrix} y(t) \\ \frac{dy}{dt}(t) \end{bmatrix}$$

We then find the eigenvalues of the **A** matrix.

$$\det\left(\begin{bmatrix} -\lambda & 1 \\ -4 & -4-\lambda \end{bmatrix}\right) = \lambda^2 + 4\lambda + 4 = (\lambda + 2)^2 = 0$$

Since we have only have one eigenvalue $\lambda = -2$, the general solution is given by:

$$y(t) = c_1 e^{-2t} + c_2 t e^{-2t}$$

$$\frac{dy}{dt}(t) = -2c_1 e^{-2t} + c_2 e^{-2t} - 2c_2 t e^{-2t}$$

To find c_1 and c_2, we plug in the initial conditions.

$$\begin{cases} c_1 = 2 \\ -2c_1 + c_2 = -1 \end{cases}$$

Solving this system of equations gives:

$$c_1 = 2 \qquad c_2 = 3$$

Therefore,

$$y(t) = 2e^{-2t} + 3te^{-2t}$$

(Let's believe this for now, we will prove this later on in the class)

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