EECS 16B — Designing Information Devices and Systems II
Fall 2018 — Elad Alon and Miki Lustig
Homework 11

This homework is due Wednesday, November 21, 2018, at 11:59 pm. Self grades are due Monday, November 26, 2018, at 11:59 pm.

1. **Complex Transpose (Mechanical)**

   Find the complex transpose A^* of the following matrices:

   (a) $A = \begin{bmatrix} 1-3j & 7+j \\ j & 1+j \\ 12 & 9-6j \end{bmatrix}$

   Solution: First, take the complex conjugate of each component:

   $$\overline{A} = \begin{bmatrix} 1+3j & 7-j \\ -j & 1-j \\ 12 & 9+6j \end{bmatrix}$$

   Then, transpose it:

   $$A^* = \begin{bmatrix} 1+3j & -j & 12 \\ 7-j & 1-j & 9+6j \end{bmatrix}$$

   (b) $A = \begin{bmatrix} 13 & 4+j \\ 4-j & 2 \end{bmatrix}$

   Solution:

   $$\overline{A} = \begin{bmatrix} 13 & 4-j \\ 4+j & 2 \end{bmatrix}$$

   $$A^* = \begin{bmatrix} 13 & 4+j \\ 4-j & 2 \end{bmatrix}$$

   It turns out that for this case, $A = A^*$, which means A is self-adjoint.

2. **Complex Inner Product (Mechanical)**

   Calculate the complex inner product $\langle \vec{u}, \vec{v} \rangle$ for the following set of vectors:

   (a) $\vec{u} = \begin{bmatrix} 1 \\ 1-j \\ -2j \\ 1+1j \end{bmatrix}, \vec{v} = \begin{bmatrix} 1-3j \\ 2j \\ -2+j \\ 1+4j \end{bmatrix}$

   Solution:

$$\langle \vec{u}, \vec{v} \rangle = \vec{u}^* \vec{v}$$

$$\langle \vec{u}, \vec{v} \rangle = \begin{bmatrix} 1 & 1+j & 2j & 1-1j \end{bmatrix} \begin{bmatrix} 1-3j \\ 2j \\ -2+j \\ 1+4j \end{bmatrix}$$

$$\langle \vec{u}, \vec{v} \rangle = (1-3j)+(1+j)(2j)+(2j)(-2+j)+(1-1j)(1+4j)$$

$$\langle \vec{u}, \vec{v} \rangle = 1-3j+2j-2-4j-2+1+3j+4 = 2-2j$$

(b) $\vec{u} = \begin{bmatrix} 1-3j \\ 2j \\ -2+j \\ 1+4j \end{bmatrix}, \vec{v} = \begin{bmatrix} 1 \\ 1-j \\ -2j \\ 1+1j \end{bmatrix}$

Solution:

$$\langle \vec{u}, \vec{v} \rangle = \vec{u}^* \vec{v}$$

$$\langle \vec{u}, \vec{v} \rangle = \begin{bmatrix} 1+3j & -2j & -2-j & 1-4j \end{bmatrix} \begin{bmatrix} 1 \\ 1-j \\ -2j \\ 1+1j \end{bmatrix}$$

$$\langle \vec{u}, \vec{v} \rangle = (1+3j)+(-2j)(1-j)+(-2-j)(-2j)+(1-4j)(1+1j)$$

$$\langle \vec{u}, \vec{v} \rangle = 1+3j-2j-2+4j-2+1-3j+4 = 2+2j$$

Alternatively, since all we did was switch the order of the vectors from part (a), we can get the same result using the answer from part (a) and the property $\langle \vec{v}, \vec{u} \rangle = \overline{\langle \vec{u}, \vec{v} \rangle}$.

$$\overline{2-2j} = 2+2j$$

(c) $\vec{u} = \begin{bmatrix} 1 \\ 1-j \\ -2j \\ 1+1j \end{bmatrix}, \vec{v} = \begin{bmatrix} 1 \\ 1-j \\ -2j \\ 1+1j \end{bmatrix}$

Solution:

$$\langle \vec{u}, \vec{v} \rangle = \vec{u}^* \vec{v}$$

$$\langle \vec{u}, \vec{v} \rangle = \begin{bmatrix} 1 & 1+j & 2j & 1-1j \end{bmatrix} \begin{bmatrix} 1 \\ 1-j \\ -2j \\ 1+1j \end{bmatrix}$$

$$\langle \vec{u}, \vec{v} \rangle = (1)+(1+j)(1-j)+(2j)(-2j)+(1-1j)(1+1j)$$

$$\langle \vec{u}, \vec{v} \rangle = 1+1+1+4+1+1 = 9$$

3. Change of Basis (Mechanical)

We have a vector $\vec{x} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$. Perform a change of basis on $\vec{x}$ using the orthonormal basis $\hat{b}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ and

$\hat{b}_2 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$.

Solution:

$$\vec{x} = (\hat{b}_1^* \vec{x})\hat{b}_1 + (\hat{b}_2^* \vec{x})\hat{b}_2$$

$$\vec{x} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 5 \\ 1 \end{bmatrix} \hat{b}_1 + \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 5 \\ 1 \end{bmatrix} \hat{b}_2$$

$$\vec{x} = \frac{6}{\sqrt{2}}\hat{b}_1 - \frac{4}{\sqrt{2}}\hat{b}_2$$

4. LTI Response Length (Mechanical)

(a) You have a discrete-time LTI system with input $u[n]$ and output $y[n]$. The system has a finite impulse response $h[n]$ of length 4. If we input a signal that has a length of 10, then what will the length of the output be?

Solution:

If you have an LTI system, and the impulse response of the system is length N, while the length of the input signal is length M, the then output should have length $M + N - 1$. So in this case, the output length should be $10 + 4 - 1 = 13$.

(b) Find the convolution between $x[n]$ and $h[n]$ where:

$$x[n] = \delta[n] + 4\delta[n-1] - 2\delta[n-3]$$
$$h[n] = \delta[n] - \frac{1}{2}\delta[n-1]$$

Solution:

$x[n]$ is length 4 while $h[n]$ has length 2. That means the convolution should have a length of 5. Putting the values $x[n]$ and $h[n]$ into an array:

$$\begin{bmatrix} x[0] & x[1] & x[2] & x[3] & x[4] \end{bmatrix} = \begin{bmatrix} 1 & 4 & 0 & -2 \end{bmatrix}$$

$$\begin{bmatrix} h[0] & h[1] \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{2} \end{bmatrix}$$

Let $y[n] = x[n] \circledast h[n]$. The value of y at time step i is:

$$y[i] = \sum_{m=0}^{i} x[m]h[i-m]$$
$$y[0] = x[0]h[0] = 1$$
$$y[1] = x[0]h[1] + x[1]h[0] = -0.5 + 4 = 3.5$$
$$y[2] = x[0]h[2] + x[1]h[1] + x[2]h[0] = 0 - 2 + 0 = -2$$

since $h[n] = 0$ for $n > 1$, we're going to omit any terms that include any $h[t > 1]$

$$y[3] = x[2]h[1] + x[3]h[0] = 0 - 2 = -2$$
$$y[4] = x[3]h[1] + x[4]h[0] = 1 + 0 = 1$$

The overall convolution is:

$$\begin{bmatrix} y[0] & y[1] & y[2] & y[3] & y[4] \end{bmatrix} = \begin{bmatrix} 1 & 3.5 & -2 & -2 & 1 \end{bmatrix}$$

5. Roots of Unity

The DFT is a coordinate transformation to a basis made up of roots of unity. In this problem we explore some properties of the roots of unity. An Nth root of unity is a complex number z satisfying the equation $z^N = 1$ (or equivalently $z^N - 1 = 0$).

(a) Show that $z^N - 1$ factors as

$$z^N - 1 = (z - 1)\left(\sum_{k=0}^{N-1} z^k \right).$$

Solution:

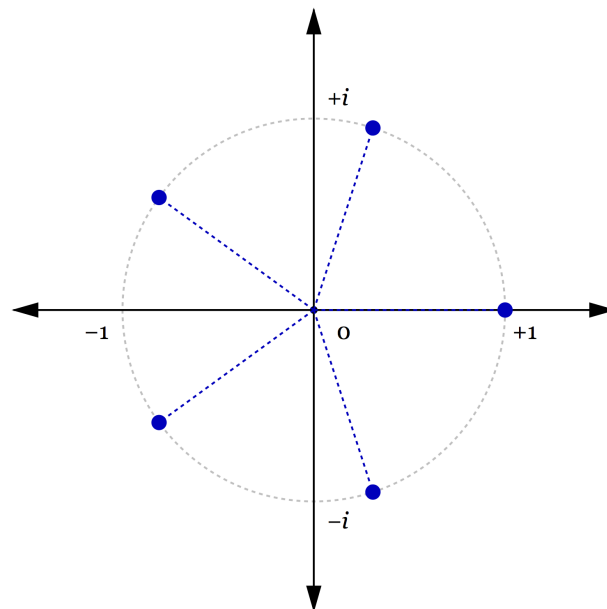$$(z - 1)\left(\sum_{k=0}^{N-1} z^k \right) = \sum_{k=1}^{N} z^k - \sum_{k=0}^{N-1} z^k = z^N - z^0 = z^N - 1$$

(b) Show that any complex number of the form $W_N^k = e^{j \frac{2\pi}{N} k}$ for $k \in \mathbb{Z}$ is an N-th root of unity.

Solution:

$$W_N^{kN} = \left(e^{j \frac{2\pi}{N} k} \right)^N = e^{j 2\pi k} = e^0 = 1$$

(c) Draw the fifth roots of unity in the complex plane. How many unique fifth roots of unity are there?

Solution: There are 5 fifth roots of unity (and in general there are N Nth roots of unity).

(d) Let $W_5 = e^{j\frac{2\pi}{5}}$. What is another expression for W_5^{42}?

Solution:
$$W_5^{42} = W_5^2$$

(e) What is the complex conjugate of W_5? What is the complex conjugate of W_5^{42}? What is the complex conjugate of W_5^4?

Solution:

$$\bar{W}_5 = W_5^{-1} = W_5^4$$
$$\bar{W}_5^{42} = W_5^{-42} = W_5^3$$
$$\bar{W}_5^4 = W_5^{-4} = W_5$$

6. LTI Low Pass Filters

Given a sequence of discrete samples with high frequency noise, we can de-noise our signal with a discrete low-pass filter. Two examples are given below:

$$y[n] = 0.5y[n-1] + x[n] \tag{1}$$
$$y[n] = 0.25x[n] + 0.25x[n-1] + 0.25x[n-2] + 0.25x[n-3] \tag{2}$$

(a) Write the impulse responses $h[n]$ for (1) and (2). Are they IIR or FIR?

Solution: for (1)
$$h[n] = \begin{cases} h[n] = 0.5^n & n \geq 0 \\ h[n] = 0 & n < 0 \end{cases}$$

This has an infinite impulse response (IIR).

for (2)
$$h[n] = \begin{cases} h[n] = 0.25 & n = 0,1,2,3 \\ h[n] = 0 & else \end{cases}$$

This has a finite impulse response (FIR).

(b) Are either of these filters causal? Are either of these filters stable?

Solution: Yes, $y[n]$ depends only on inputs from $x[n]$ and before.

Yes, both filters are stable.

(c) Give the output y for each filter given the input sequence $x[n] = 2cos(\pi n) + n$ from $n = 0$ to $n = 7$. Assume that $y[n] = 0$ for $n < 0$.

Solution: for (1)
$$y = [2, 0, 4, 3, 7.5, 6.75, 11.375, 10.6875]$$
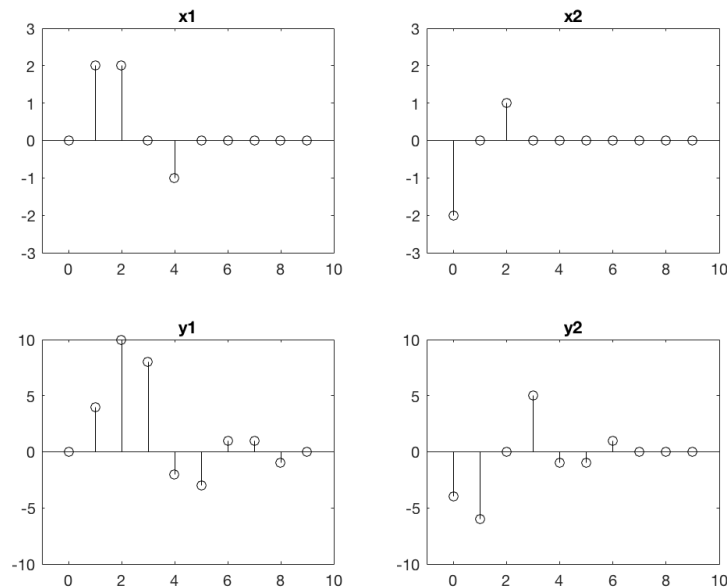
for (2)
$$y = [0.5, 0.25, 1.25, 1.5, 2.5, 3.5, 4.5, 5.5]$$

7. LTI Inputs

We have an LTI system whose exact characteristics we do not know. However, we know that it has a finite impulse response that isn't longer than 5 samples. We also observed two sequences, x_1 and x_2, pass through the system and observed the system's responses y_1 and y_2.

x_1	0	2	2	0	-1	0	0	0	0	0
y_1	0	4	10	8	-2	-3	1	1	-1	0

x_1	-2	0	1	0	0	0	0	0	0	0
y_1	-4	-6	0	5	-1	-1	1	0	0	0

(a) Given the above sequences, what would be the output for the input:

x_3	0	0	-2	0	1	0	0	0	0	0

Solution: This is a shifted version of x_2. Since the system is time-invariant, the output is:

y_3	0	0	-4	-6	0	5	-1	-1	1	0

(b) What is the output of the system for the input $x_1 - x_2$?

Solution: Since the system is linear, the output is $y_1 - y_2$:

$y_1 - y_2$	4	10	10	3	-1	-2	0	1	-1	0

(c) Given the above information, how could you find the impulse response of this system? What is the impulse response?

Solution: Since $\frac{1}{2}(x_1 + x_3)$ is an impulse at the second sample, $\frac{1}{2}(y_1 + y_3)$ is the system's impulse response starting from the second sample!

$y_1 + y_3$	0	4	6	2	-2	2	0	0	0	0

so

n	$n < 0$	0	1	2	3	4	$n > 4$
$h[n]$	0		2	3	1	-1	1	0

(d) What is the output of this system for the following input:

x_4	1	1	1	0	0	0	0	0	0	0

Solution:

y_4	2	5	6	3	1	0	1	0	0	0

8. LTI Exam Question

This question is from spring 2017's final exam.

(a) Prove that the composition of two LTI systems is LTI. In other words, that if each block in the figure below is LTI, then $u(t) \to y(t)$ is LTI.
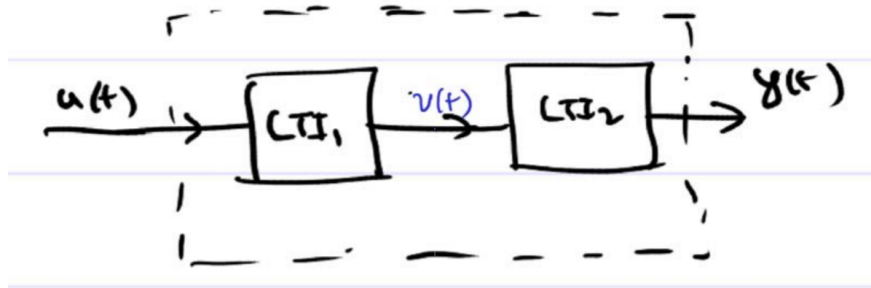
Solution:

First, let's prove it is time invariant:

Since the first block is LTI, if we feed in a signal $u[t + \tau]$:

$$u[t + \tau] \to v[t + \tau]$$

Feeding $v[t + \tau]$ into the 2nd block gives:

$$v[t + \tau] \to y[t + \tau]$$

Which means:

$$u[t + \tau] \to y[t + \tau]$$

The system is time invariant.

Next we need to prove linearity by showing scaling and superposition hold:

Scaling:

$$\alpha u[t] \xrightarrow{\text{LTI}_1} \alpha v[t]$$
$$\alpha v[t] \xrightarrow{\text{LTI}_2} \alpha y[t]$$
$$\alpha u[t] \xrightarrow{\text{LTI}_1 + \text{LTI}_2} \alpha y[t]$$

Scaling holds for $u[t] \to y[t]$

Superposition: Let:

$$u_1[t] \xrightarrow{\text{LTI}_1} v_1[t] \xrightarrow{\text{LTI}_2} y_1[t]$$

And:

$$u_2[t] \xrightarrow{\text{LTI}_1} v_2[t] \xrightarrow{\text{LTI}_2} y_2[t]$$

Then:

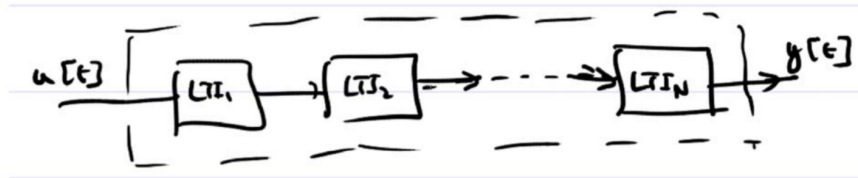$$u_1[t] + u_2[t] \xrightarrow{\text{LTI}_1} v_1[t] + v_2[t]$$
$$v_1[t] + v_2[t] \xrightarrow{\text{LTI}_2} y_1[t] + y_2[t]$$
$$u_1[t] + u_2[t] \xrightarrow{\text{LTI}_1 + \text{LTI}_2} y_1[t] + y_2[t]$$

Superposition holds.

Since both superposition and scaling hold, the system is linear. The system is both linear and time invariant, so the overall system is LTI.

(b) Using reasoning similar to part (a), it is easy to show that the composition of N LTI systems, as depicted below, is LTI.

Suppose all the internal LTI blocks LTI_1 to LTI_N are identical, with impulse response

$$h[t] = \begin{cases} 1, & t = 1 \\ 0, & \text{otherwise} \end{cases}$$

Find the impulse response $h_c[t]$ of the composed system, i.e. from $u[t] \to y[t]$.

Solution:

$h[t]$ delays the input by 1 time step.

$$\delta[t] \xrightarrow[h[t]]{\text{LTI}_1} \delta[t-1] \xrightarrow[h[t]]{\text{LTI}_2} \delta[t-2] \to \cdots \xrightarrow[h[t]]{\text{LTI}_N} \delta[t-N]$$

$$h_c[t] = \delta[t-N]$$

(c) You are a discrete-time, causal, LTI system with impulse response $h[t]$.
You are not told what $h[t]$ is.
However, you **are** told that if the input $u[t]$ to the system is chosen to be $h[t]$, then the first five samples of the output $y[t]$ are: $y[0] = 1$, $y[1] = 0$, $y[2] = -2$, $y[3] = 0$, $y[4] = 3$.
Find **two** different possible solutions for $h[t]$ (only for $t = 0, \cdots, 4$) that satisfy the above condition.
Are other solutions for $t = 0, \cdots, 4$ also possible?

Solution:

$$y[t] = h[t] \circledast h[t] = \sum_{i=0}^{t} h[i]h[t-i]$$

$$y[0] = h^2[0] = 1 \qquad\qquad \Rightarrow h[0] = \pm 1$$

$$y[1] = h[1]h[0] + h[0]h[1] = 2h[0]h[1] = 0 \qquad\qquad \Rightarrow h[1] = 0$$

$$y[2] = h[0]h[2] + h^2[1] + h[2]h[0] = 2h[0]h[2] = -2 \qquad\qquad \Rightarrow h[2] = \frac{-1}{h[0]} = \mp 1$$

$$y[3] = h[0]h[3] + h[1]h[2] + h[2]h[1] + h[3]h[0] = 2h[0]h[3] = 0 \qquad\qquad \Rightarrow h[3] = 0$$
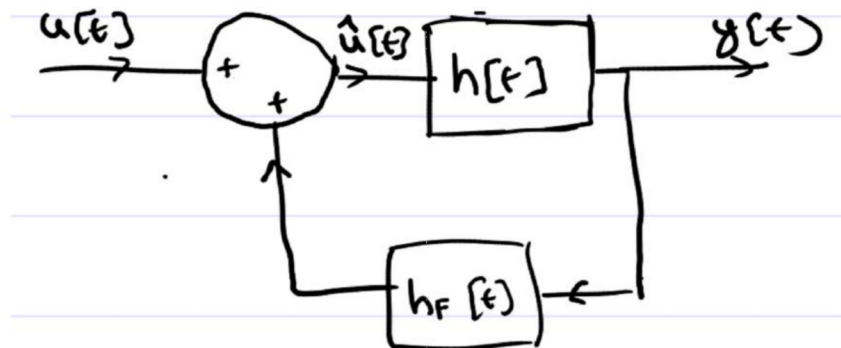
$$y[4] = 2h[0]h[4] + 2h[1]h[3] + h^2[2] = 3 \qquad\qquad \Rightarrow h[4] = \frac{1}{h[0]} = \pm 1$$

Hence the two solutions are:

$$\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$$

No other solutions are possible.

(d) Given two discrete-time LTI systems in a feedback loop:

with $h[t] = \begin{cases} 1, & t = 0 \\ -1, & t = 1 \\ 0, & \text{otherwise} \end{cases}$, and $h_F[t] = \begin{cases} 1, & t = 1 \\ 0, & \text{otherwise} \end{cases}$

Please read the expressions above for $h[t]$ and $h_F[t]$ **very carefully** and make sure you understand them right. Note also that the feedback **adds** to the input.

Assuming that $y[t] = 0$ for all $t < 0$, find the impulse response $h_c[t]$ of the closed-loop system (i.e. from $u[t] \to y[t]$).

Hint: write out $y[t]$ for $t = 0, \cdots, 15$ at least (not required, but highly recommended), examine the values, and use it to devise a general formula for $y[t]$. Be very careful to avoid mistakes in your calculations.

Solution:

$$\hat{u}[t] = u[t] + y[t] \circledast h_F[t] = u[t] + y[t-1]$$
$$y[t] = \hat{u}[t] \circledast h[t] = \hat{u}[t] - \hat{u}[t-1] = u[t] + y[t-1] - u[t-1] - y[t-2]$$
$$y[t] = (y[t-1] - y[t-2]) + (u[t] - u[t-1])$$

Set $u[t] = \delta[t]$:

$$y[0] = 1$$
$$y[1] = 0$$
$$y[2] = -1$$
$$y[3] = -1$$
$$y[4] = 0$$
$$y[5] = 1$$
$$y[6] = 1$$
$$y[7] = 0$$
$$y[8] = -1$$
$$y[9] = -1$$

For $t \geq 2$, $y[t]$ depends only on the previous two values $y[t-1]$ and $y[t-2]$ (since $u[t]$ only has a nonzero value at $t = 0$).

Since $y[8] = y[2]$ and $y[9] = y[3]$, the sequence must repeat, with $y[10] = y[4]$, $y[11] = y[5]$, and so on:

$$h_c[t] = \begin{cases} 0, & t < 0 \\ 1, & t = 0 \\ 0, & t-1 \\ -1, & t-2 \\ -1, & t-3 \\ 0, & t-4 \\ 1, & t-5 \\ 1, & t-6 \\ 0, & t-7 \\ h_c[t-6], & t \geq 8 \end{cases}$$

(e) Is the system in part (d) BIBO stable or unstable? If stable, explain why. If unstable, write an input $u[t]$ that would make $y[\infty] \to \infty$

Solution:

The system is BIBO unstable:

Try $u[t] = h_c[t]$ (max of $h_c[t]$ is 1, so $h_c[t]$ is bounded):

$$y[t] = h_c[t] \circledast h_c[t] = \sum_{i=0}^{t} h_c[i] h_c[t-i]$$

If we look at the first 12 values of $h_c[t]$, we have:

$$\underbrace{1,0,-1,-1,0,1,}_{}\underbrace{1,0,-1,-1,0,1}_{}$$

Notice that $h_c[5-i] = h_c[i]$ for $i = 0, \cdots, 5$. This means we can say:

$$y[5] = \sum_{i=0}^{5} h_c^2[i] = 4$$

Also notice that $h_c[i]$ repeats with period 6, which means that $y[6k-1] = k \cdot y[5] = 4k$. So, as $t \to \infty$, $y[t]$ will blow up for the bounded input $h_c[t]$.

9. **Write Your Own Question And Provide a Thorough Solution.**

Writing your own problems is a very important way to really learn material. The famous "Bloom's Taxonomy" that lists the levels of learning is: Remember, Understand, Apply, Analyze, Evaluate, and Create. Using what you know to create is the top level. We rarely ask you any homework questions about the lowest level of straight-up remembering, expecting you to be able to do that yourself (e.g. making flashcards). But we don't want the same to be true about the highest level. As a practical matter, having some practice at trying to create problems helps you study for exams much better than simply counting on solving existing practice problems. This is because thinking about how to create an interesting problem forces you to really look at the material from the perspective of those who are going to create the exams. Besides, this is fun. If you want to make a boring problem, go ahead. That is your prerogative. But it is more fun to really engage with the material, discover something interesting, and then come up with a problem that walks others down a journey that lets them share your discovery. You don't have to achieve this every week. But unless you try every week, it probably won't ever happen.