Computer Science 70: Distributing Quantifiers

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1 The Problem

I will solve a particularly tough question which involves distributing quantifiers over an implication. The goal is to illustrate the reasoning behind proving complicated logical equivalences, as well as demonstrating that quantifier distribution is not at all obvious (and therefore should not be assumed without careful thought).

The logical equivalence in question is:

$$\exists x(P(x) \implies Q(x)) \equiv (\forall x P(x)) \implies (\exists x Q(x))$$

2 The Solution

The solution is to prove that the left side implies the right side, and the right side implies the left side. In other words, we split the proof into two parts:

2.1 Forward Direction ($\Rightarrow$)

Let us assume that the statement $\exists x(P(x) \implies Q(x))$ is true. Then, we know that for some x_0, we have $P(x_0) \implies Q(x_0)$. The implication is true in exactly two cases:

1. The implication is true if $P(x_0)$ is false. In this case, then we know that $\forall x P(x)$ is false. Hence, the implication $(\forall x P(x)) \implies (\exists x Q(x))$ is true (remember that an implication is true if the premise is false!).

2. Otherwise, we must have that $P(x_0)$ is true and $Q(x_0)$ is true. Now suppose that $\forall x P(x)$ is true (if it is false, then the implication is vacuously true, just as in the above part). Then $(\forall x P(x)) \implies (\exists x Q(x))$ is true if $\exists x Q(x)$ is also true, which it is (because $Q(x_0)$ is true).

Hence, we have shown that if the left hand side is true, the right hand side is also true:

$$[\exists x(P(x) \implies Q(x))] \implies [(\forall x P(x)) \implies (\exists x Q(x))]$$

2.2 Backward Direction ($\Leftarrow$)

Now, let us assume that $(\forall x P(x)) \implies (\exists x Q(x))$ is true. The implication is true in one of two cases:

1. The first case is that $\forall x P(x)$ is false. If so, then we know that there exists some x_0 for which $P(x_0)$ fails to hold. For this choice of x_0 we have that the implication $P(x_0) \implies Q(x_0)$ is true (since the premise is false), so $\exists x(P(x) \implies Q(x))$ is true.

2. The next case is that $\forall x P(x)$ is true and $\exists x Q(x)$ is true. If so, then there is some x_0 for which $Q(x_0)$ holds. Furthermore, since we know $P(x)$ holds true for every x, in particular it must hold for x_0, so we have $P(x_0)$ is true as well. These two imply that $P(x_0) \implies Q(x_0)$ is true, so we know that $\exists x(P(x) \implies Q(x))$ is true.

We have now shown that if the right hand side is true, so is the left hand side:

$$[(\forall x P(x)) \implies (\exists x Q(x))] \implies [\exists x(P(x) \implies Q(x))]$$

This completes the proof of logical equivalence.

3 Remarks

Perhaps it is strange that when we "distribute the existential quantifier" over the implication, the first existential quantifier is flipped into a universal quantifier. Is there a more intuitive reason for this? Indeed there is: we can write

$$\exists x(P(x) \implies Q(x)) \equiv \exists x(\neg P(x) \vee Q(x))$$

In fact, we **can** distribute the existential quantifier over a disjunction (just as we can distribute the universal quantifier over a conjunction). (Exercise: Prove this to yourself.) Therefore, we have

$$\exists x(P(x) \implies Q(x)) \equiv (\exists x \neg P(x)) \vee (\exists x Q(x)) \equiv \neg(\forall x P(x)) \vee (\exists x Q(x)) \equiv (\forall x P(x)) \implies (\exists x Q(x))$$

Was it worth the effort to go through the laborious steps of the first proof? If it helped you understand more clearly what a formal proof entails, then yes, I believe so.