Counting

Permutations and Combinations

Counting describes the methods to find the number of elements in a finite set of objects. Instead of enumerating every object in the set, we can utilize general patterns to more efficiently count.

Theorem 1 (First Rule of Counting). *Suppose an object is of length k, and is constructed as follows. There are n_1 choices for the first position in the object, and for every choice of the first position, there are n_2 choices for the second position, and so on. The total number of such objects is $n_1 n_2 \cdots n_k$.*

One common way that the first rule of counting is used is for permutations of objects. A permutation is an ordering of the elements of a set.

Example 1. How many anagrams are there of the word COINS?
The letters are distinct, so the number of anagrams is the number of permutations of the set $\{$C, O, I, N, S$\}$. There are 5 choices for the first letter, and for each choice of the first letter, there are 4 choices for the second letter, and so on. Therefore, the total number of permutations is $5 \cdot 4 \cdots 1 = 5!$.

So the first rule of counting is to used preserve order. If we do not care about order, we must use another rule of counting.

Theorem 2 (Second Rule of Counting). *Suppose A is a collection of ordered objects. Let B be the collection of the objects in A, but unordered. If there exists a k to 1 function from A to B, then the size of B is*

$$|B| = \frac{|A|}{k}$$

Example 2. How many anagrams are there of PROBABILITY?
The number of ordered arrangments is 11!, if we considered each character to be distinguishable. To find the number of anagrams, we need to consider characters of the same letter to be indistinguishable. Fixing all other letters except the B's and I's, there are 2! ways to order the B's, and 2! ways to order the I's. Thus we have $k = 2!2!$, so the number of anagrams is $11!/(2!2!)$.

Example 3. How many ways are there to choose 5 distinct letters from the English alphabet?
First calculate the number of arrangments where order matters. This is $26 \cdot 25 \cdots 22 = \frac{26!}{21!}$. If we choose any 5 letters, there will be 5! ways to arrange them where order matters. Thus, if we want to calculate the number of unordered arrangments, we have $k = 5!$. So our answer is $\frac{26!}{5!21!} = \binom{26}{5}$.

In general, the number of ways to choose k elements from a set of n objects is $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. These are called combinations. Note that combinations are symmetrical in the sense that

$$\binom{n}{k} = \binom{n}{n-k}$$

Intuitively, this is because if we choose k elements from a set of size n, we are also choosing to leave out $n-k$ elements. Combinations are also called binomial coefficients. This is because they appear in the binomial theorem

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$$

Example 4 (Number of subsets)**.** How many subsets are there of a set of size n?
The number of subsets of size k is the same as choosing k elements from n, which is $\binom{n}{k}$. Thus, the total number of subsets is

$$\sum_{k=0}^{n} \binom{n}{k}$$

One way to simplify this expression is by considering a bitstring of length n. Let's say we order the objects in the set. If we choose an element to be in a subset, we denote the position in the string corresponding to the index in the set as a 1, otherwise we denote it as a 0. The number of subsets is the number of possible bitstrings, as each subset has a unique representation in this bitstring format. The number of bitstrings is 2^n, so the number of subsets of a set of size n is 2^n. This sort of argument, where we use two perspectives of the same combinatorial process, is called a combinatorial proof.

Stars and Bars

One of the important methods in counting is called stars and bars, which involves putting objects into bins.

Theorem 3. *Suppose we want to place n indistinguishable objects into k distinguishable bins. Each bin must contain at least one object. The number of ways to do so is*

$$\binom{n-1}{k-1}$$

Proof. *Let the indistinguishable objects be denoted by 0s, and the bin seperators be denoted by 1s. Suppose $n = 5, k = 3$. For example, one way to place the n objects is*

$$0010010$$

Notice that the number of seperators is 2. If we fix the 0s, there are 4 possible positions for these seperators. Thus there are $\binom{4}{2}$ ways of placing the objects. In general, we have $k-1$ seperators, and $n-1$ positions for those seperators, so the total number of ways to place the n objects is $\binom{n-1}{k-1}$. ■

Theorem 4. *Suppose we want to place n indistinguishable objects into k bins. The number of objects in any particular bin could be 0. The number of ways to do so is*

$$\binom{n+k-1}{k-1} = \binom{n+k-1}{n}$$

Proof. *Since the number of objects in any particular bin can be equal to 0, we cannot fix the objects and the select the positions for the bins as we did earlier. For example, we could have*

$$0110000$$

The total number of 0s and 1s is $n+k-1$. Since we can have the objects and seperators in any order, we have the number of ways to place the objects is just the number of ways of choosing positions for the separators, so we get $\binom{n+k-1}{k-1}$. Alternatively, we can choose the positions of the objects to get $\binom{n+k-1}{n}$. ■

Example 5 (Number of integer solutions)**.** Suppose we have positive integer x_i's such that

$$x_1 + x_2 + x_3 + x_4 = 16$$

How many solutions of the form $(x_1, \ldots, x_4)$ are there?
This is the positive form of stars and bars. We can consider the problem as throwing 16 objects into 4 bins. We let x_1 be the number of objects in the first bin, x_2 the number of objects in the second bin, etc... The answer is $\binom{16-1}{4-1} = \binom{15}{3}$.

Example 6 (Number of multisets)**.** Suppose we have a set $\{1, 2, \ldots 27\}$. We want to choose 6 objects from this set with replacement, meaning we can choose the same object multiple times. How many ways can we choose these 6 objects?
This is the nonnegative form of stars and bars. We can think of each element in the set as a bin. To choose an object from the set, we can think of it as throwing a ball into a bin. Since we are sampling with replacement, we can throw any nonnegative number of balls into a bin. Since we have 6 balls to throw, we have the answer is $\binom{6+27-1}{27-1} = \binom{32}{26}$. This is an important problem, because it generalizes to choosing n objects with replacement from a set of size k. We call this choosing a multiset of size n from a set of size k. A multiset differs from an ordinary set, because we allow duplicate objects. We call this process multichoosing, and we denote it as $\left(\binom{k}{n}\right)$.