CS 70 Fall 2018 — Discrete Mathematics and Probability Theory — Alistair Sinclair and Yun Song — HW 3

1 Short Answer: Graphs

(a) Bob removed a degree 3 node in an n-vertex tree, how many connected components are in the resulting graph? (An expression that may contain n.)

(b) Given an n-vertex tree, Bob added 10 edges to it, then Alice removed 5 edges and the resulting graph has 3 connected components. How many edges must be removed to remove all cycles in the resulting graph? (An expression that may contain n.)

(c) True or False: For all $n \geq 3$, the complete graph on n vertices, K_n has more edges than the n-dimensional hypercube. Justify your answer.

(d) A complete graph with n vertices where n is an odd prime can have all its edges covered with x edge-disjoint Hamiltonian cycles (a Hamiltonian cycle is a cycle where each vertex appears exactly once). What is the number, x, of such cycles required to cover the a complete graph? (Answer should be an expression that depends on n.)

(e) Give a set of edge-disjoint Hamiltonian cycles that covers the edges of K_5, the complete graph on 5 vertices. (Each path should be a sequence (or list) of edges in K_5, where an edge is written as a pair of vertices from the set $\{0,1,2,3,4\}$ - e.g: $(0,1),(1,2)$.)

Solution:

(a) **3**.
Each neighbor must be in a different connected component. This follows from a tree having a unique path between each neighbor in the tree as it is acyclic. The removed vertex broke that path, so each neighbor is in a separate component. Moreover, every other node is connected to one of the neighbors as every other vertex has a path to the removed node which must go through a neighbor.

(b) **7**
The problem is asking you to make each component into a tree. The components should have $n_1 - 1$, $n_2 - 1$ and $n_3 - 1$ edges each or a total of $n-3$ edges. The total number of edges after Bob and Alice did their work was $n-1+10-5 = n+4$, thus one needs to remove 7 edges to ensure there are no cycles.

(c) **False**
This is just an exercise in definitions. The complete graph has $n(n-1)/2$ edges where the hypercube has $n2^{n-1}$ edges. For $n \geq 3$, $2^{n-1} \geq (n-1)/2$.

(d) $(n-1)/2$.
Each cycle removes degree 2 from each vertex. As the degree of each vertex is $n-1$, we require a total of $\frac{n-1}{2}$ disjoint cycles. This is also sufficient. For a construction in the case of n being an odd prime, see explanations below.

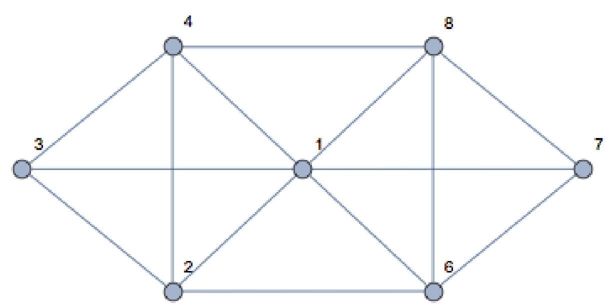
(e) $(0,1),(1,2),(2,3),(3,4),(4,0)$
$(0,2),(2,4),(4,1),(1,3),(3,0)$

The following details a procedure for generating the paths using ideas from modular arithmetic. Note that modular arithmetic is not necessary for the solution, but it provides a clean solution.

The idea is that we can generate disjoint Hamiltonian cycles by repeatedly adding an element a to the current node. This produces the sequence of edges $(0,a),(a,2a),\ldots,((p-1)a,0)$ which are disjoint for different a, as long as $a \neq -a \pmod p$, as that would simply be subtracting a everytime. (In other words, there exists no integer k such that $-a+pk=a$.)

We use primality to say that inside a sequence the edges are disjoint since the elements $\{0a,\ldots,(p-1)a\}$ are distinct $\pmod p$.

2 Eulerian Tour and Eulerian Walk



(a) Is there an Eulerian tour in the graph above?

(b) Is there an Eulerian walk in the graph above?

(c) What is the condition that there is an Eulerian walk in an undirected graph? Briefly justfy your answer.

Solution:

(a) No. Two vertices have odd degree.

(b) Yes. One of the two vertices with odd degree must be the starting vertex, and the other one must be the ending vertex. For example: $3,4,2,1,3,2,6,1,4,8,1,7,8,6,7$ will be an Eulerian walk (the numbers are the vertices visited in order). Note that there are 14 edges in the graph.

(c) An undirected graph has an Eulerian walk if and only if it is connected (except for isolated vertices) and at most two vertices have odd degree.

Note: There is no graph with only one odd degree vertex.

Justifications: *Only if.* Given an Eulerian walk, say starting at u and ending at v (possibly the same), clearly the graph is connected. Moreover, every other intermediate visit to a vertex is being paired with two edges, and therefore, except for u and v, every other vertices must be of even degree.

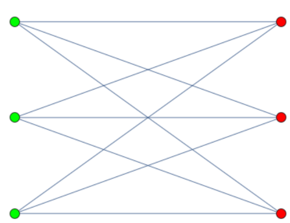
If. For a connected graph with no odd degree vertices, we have shown in lectures that there is an Eulerian tour.

Solution 1: If it has two odd degree vertices, say u and v, then one can first find a path from u to v, and remove the edges of the path from the graph. Next for each connected component of the residual graph, we find an Eulerian tour. Observe that an Eulerian walk is simply an edge-disjoint walk that covers all the edges. What we just did is decomposing all the edges into a path from u to v and a bunch of edge-disjoint Eulerian tours. A path is clearly an edge-disjoint walk. Then, given an edge-disjoint walk and an edge-disjoint tour such that they share at least one common vertex, one can combine them into an edge-disjoint walk simply by augmenting the walk with the tour at the common vertex. Therefore we can combine all the edge-disjoint Eulerian tours into the path from u to v to make up an Eulerian walk from u to v.

Solution 2: Alternatively, take the two odd degree vertices u and v, and add a vertex w with two edges (u,w) and (w,v).The resulting graph G' has only vertices of even degree (we added one to the degree of u and v and introduced a vertex of degree 2) and is still connected. So, we can find an Eulerian tour on G'. Now, delete the component of the tour that uses edges (u,w) and (w,v). The part of the tour that is left is now an Eulerian walk from u to v on the original graph, since it traverses every edge on the original graph.

3 Bipartite Graph

A bipartite graph consists of 2 disjoint sets of vertices (say L and R), such that no 2 vertices in the same set have an edge between them. For example, here is a bipartite graph (with $L = \{$green vertices$\}$ and $R = \{$red vertices$\}$), and a non-bipartite graph.



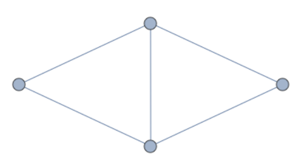


Figure 1: A bipartite graph (left) and a non-bipartite graph (right).

Prove that a graph is bipartite if and only if it has no tours of odd length.

Solution:

Begin by proving the forward direction: an undirected bipartite graph has no tours of odd length.

Suppose there is a tour in the bipartite graph. Let us start traveling the tour from a node n_0 in L. Since each edge in the graph connects a vertex in L to one in R, the 1st edge in the tour connects our start node n_0 to a node n_1 in R. The 2nd edge in the tour must connect n_1 to a node n_2 in L. Continuing on, the $(2k+1)$-th edge connects node n_{2k} in L to node n_{2k+1} in R, and the $2k$-th edge connects node n_{2k-1} in R to node n_{2k} in L. Since only even numbered edges connect to vertices in L, and we started our tour in L, the tour must end with an even number of edges.

Prove the reverse direction: A undirected graph with no tours of odd length is bipartite.

Take some vertex v. Add all vertices where the shortest path to v is odd, to R. Add all vertices where the shortest path to v is even, to L. If any of the vertices in $u_1, u_2 \in R$ are adjacent, then we have a tour of odd length formed by appending: the shortest path between v and u_1 (odd), the edge (u_1, u_2) (odd), and the shortest path between u_2 and v (odd). This means no two vertices in R are adjacent. Similarly, if any two vertices $v_1, v_2 \in L$ are adjacent, we get a tour of odd length by appending: the shortest path between v and v_1 (even), the edge (v_1, v_2) (odd), and the shortest path between v_2 and v (even). This means no two vertices in L are adjacent either. If there are other connected componenets, we can proceed by choosing a new vertex in each component and repeating this process. Then we will have disjoint L, R which include all vertices.

4 Hypercubes

The vertex set of the n-dimensional hypercube $G = (V, E)$ is given by $V = \{0,1\}^n$ (recall that $\{0,1\}^n$ denotes the set of all n-bit strings). There is an edge between two vertices x and y if and only if x and y differ in exactly one bit position. These problems will help you understand hypercubes.

(a) Draw 1-, 2-, and 3-dimensional hypercubes and label the vertices using the corresponding bit strings.

(b) Show that for any $n \geq 1$, the n-dimensional hypercube is *bipartite*: the vertices can be partitioned into two groups so that every edge goes between the two groups.

Solution:

(a) The three hypercubes are a line, a square, and a cube, respectively. See also p12 on lecture notes 5.

(b) Consider the vertices with an even number of 0 bits and the vertices with an odd number of 0 bits. Each vertex with an even number of 0 bits is adjacent only to vertices with an odd number of 0 bits, since each edge represents a single bit change (either a 0 bit is added by flipping a 1 bit, or a 0 bit is removed by flipping a 0 bit). Let L be the set of the vertices with an even number of 0 bits and let R be the vertices with an odd number of 0 bits, then no two adjacent vertices will belong to the same set.

5 Triangulated Planar Graph

In this problem you will prove that every triangulated planar graph (every face has 3 sides; that is, every face has three edges bordering it, including the unbounded face) contains either (1) a vertex of degree 1, 2, 3, 4, (2) two degree 5 vertices which are adjacent, or (3) a degree 5 and a degree 6 vertices which are adjacent. Justify your answers.

(a) Place a "charge" on each vertex v of value $6-\text{degree}(v)$. What is the sum of the charges on all the vertices? (*Hint*: Use Euler's formula and the fact that the planar graph is triangulated.)

(b) What is the charge of a degree 5 vertex and of a degree 6 vertex?

(c) Suppose now that we shift $1/5$ of the charge of a degree 5 vertex to each of its neighbors that has a negative charge. (We refer to this as "discharging" the degree 5 vertex.) Conclude the proof under the assumption that, after discharging all degree 5 vertices, there is a degree 5 vertex with positive remaining charge.

(d) If no degree 5 vertices have positive charge after discharging the degree 5 vertices, does there exist any vertex with positive charge after discharging? If there is such a vertex, what are the possible degrees of that vertex?

(e) Suppose there exists a degree 7 vertex with positive charge after discharging the degree 5 vertices. How many neighbors of degree 5 might it have?

(f) Continuing from Part (e). Since the graph is triangulated, are two of these degree 5 vertices adjacent?

(g) Finish the proof from the facts you obtained from the previous parts.

Solution:

(a) Let V be the vertex set , E be the edge set, F be the faces in the graph, we have

$$\begin{aligned}\sum_{v\in V} 6-\text{degree}(v) &= 6|V|-\sum_{v\in V}\text{degree}(v) &(1)\\ &= 6|V|-2|E|. &(2)\end{aligned}$$

The last step is because that we count each edge twice as degree for each end vertex. And since the graph is triangulated, each face uses exactly three edges and each edge is shared by two faces, so we can substitute $|F|=2|E|/3$ in Euler's formula to get

$$\begin{aligned}|V|+|F| &= |E|+2 &(3)\\ |V|+\frac{2|E|}{3} &= |E|+2 &(4)\\ 3|V|+2|E| &= 3|E|+6 &(5)\\ |E| &= 3|V|-6. &(6)\end{aligned}$$

Substitute (6) into (2) to get that the sum of charge is 12.

(b) The charge is 1 for degree 5 vertex, and 0 for degree 6 vertex.

(c) If there is a degree 5 vertex with positive remaining charge, that means at least one of its neighbors is not negatively charged. In other words, at least one of its neighbors has degree 1, 2, 3, 4, 5, or 6, which proves the statement.

(d) Yes, there exists a vertex with positive charge. Since we know that the sum of charge of the entire graph is 12, it is impossible to have no positively charged vertex.

The possible degrees of vertex that have positive charge after discharging are $1,2,3,4,7$. Vertices with degree at least 8 will have initial charge -2, and will not have positive charge even if all their neighbors are of degree 5.

Side note: one can use Part (e,f) to rule out the possibility of degree-7 vertices. But for simplicity we only require students to rule out degrees 5 and 8 and beyond. The idea is that, we can't have two adjacent degree-5 vertices to have zero-charge after discharging. Suppose that there exists a degree-7 vertex with positive charges after discharging, then by Part (e,f), out of the 7 neighbors, 6 of them have to be of degree 5, and two of these degree-5 vertices have to be adjacent. Therefore, if none of the degree-5 vertices have positive charges, then we get two adjacent degree-5 vertices that have zero-charge, which is a contradiction.

(e) At least 6 out of the 7 are degree 5.

(f) Since the graph is triangulated, observe that fixing a drawing of the planar graph, we can order neighbors of the degree 7 node clockwise. And every two consecutive neighbors (defined by the ordering) form a triangle with the degree 7 vertex. From Part (e) we know that at least 6 out of the 7 are of degree 5. Therefore, it is impossible that none of these degree-5 vertices are adjacent to another degree-5 node.

(g) We split the proof into several cases. First note that there is always a vertex with degree at most 5. Suppose the contrary that every vertex is of degree at least 6, then the total charge would not have been positive, contradicting Part (a), where we showed the total charge is always 12.

If there are no vertices with degree at least 5, then we see that this is case (1). Therefore we consider the case where there is always a vertex of degree 5 from now on.

When there is a degree-5 vertex with a positive remaining charge, the statement is true by Part (c). When there is no degree-5 vertex with positive remaining charge, we know from Part (d) that either there is a positively charged vertex with degree $1,2,3,4$ or with degree $6,7$. For a degree-6 vertex to have a positive charge after discharging, it must be adjacent to a degree-5 vertex. For a degree-7 vertex, we know that at least two degree 5 vertices are adjacent from Part (f) which concludes our proof.

Side note: alternatively, one could simply rule out the possibility of a degree-7 vertex as explained in Part (d).