Chapter 5: Multivariate Regression

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Mathematical Econometrics I
Brown University
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Outline

1. Deriving Multivariate Regression and OLS
2. Regression and Causality
3. Regression Odds and Ends

Moving Beyond One "Regressor"

- So far we've talked about regression as a way of approximating the CEF $E[Y_i|X_i = x] \approx \alpha + x\beta$ for a single scalar X_i
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- Next we'll see how this can be generalized to approximate/estimate $E[Y_i|\mathbf{X}_i = \mathbf{x}] \approx \mathbf{x}'\boldsymbol{\beta}$ for a vector $\mathbf{X_i} = (1, X_{i1}, ..., X_{iK})'$
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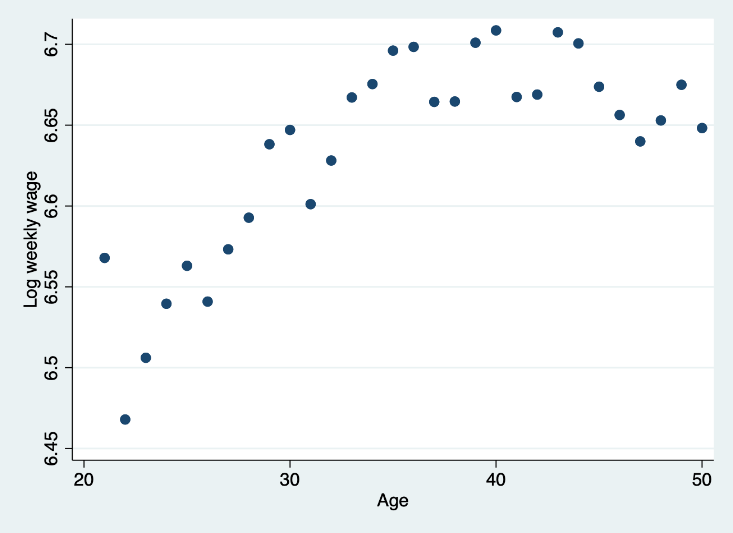
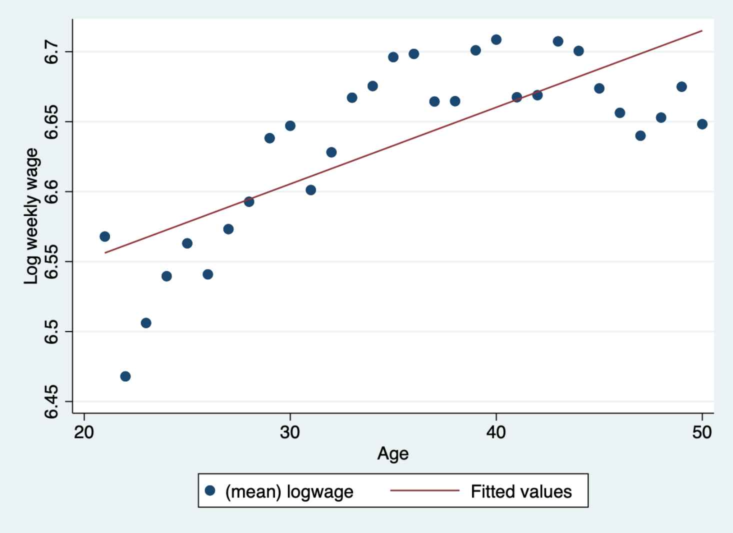
2. We want a *nonlinear* CEF approx.: e.g. $E[Y_i \mid X_i] \approx \alpha + X_i\beta + X_i^2\gamma$
   - We can "trick" regression into doing this by setting $\mathbf{X_i} = (1, X_i, X_i^2)'$

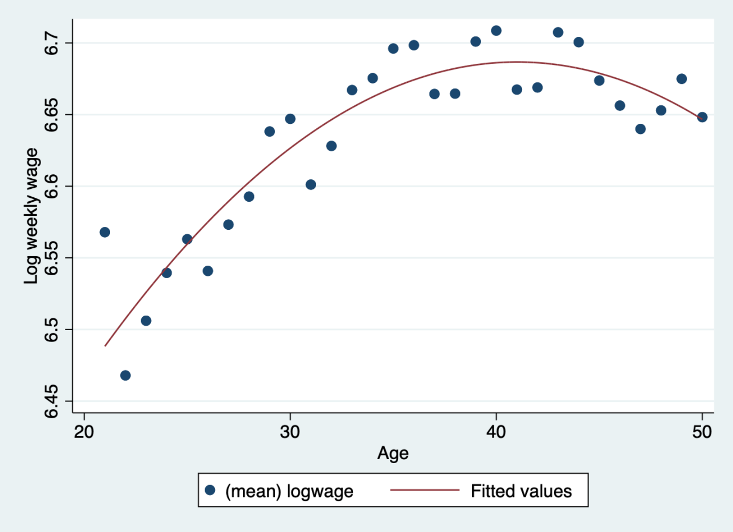
Log Wages by Age



OLS Regression (Linear Fit)



OLS Regression (Quadratic Fit)



Multivariate Regression as a Least-Squares Problem

- Recall with univariate OLS we solved for

$$(\alpha, \beta) = \arg\min_{a,b} E[(Y_i - (a + bX_i))^2]$$

We showed if the CEF is linear, then $E[Y|X] = \alpha + \beta X$; while if not, $\alpha + \beta X$ gave the best non-linear approximation

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- We will now consider the multi-variate analog:

$$\boldsymbol{\beta} = \arg\min_{\mathbf{b}} E\left[(Y_i - \mathbf{X}_i'\boldsymbol{b})^2\right]$$

- Using similar steps for the univariate case, we can show that if the CEF is linear in $\mathbf{X}$, then $E[Y|\mathbf{X}] = \mathbf{X}'\boldsymbol{\beta}$; if not, then $\mathbf{X}'\boldsymbol{\beta}$ is the MSE-minimizing approximation to the CEF.

Solving for Regression Coefficients

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\Rightarrow &E[\mathbf{X}_i Y_i] = E[\mathbf{X}_i\mathbf{X}_i']\boldsymbol{\beta}
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Multivariate Regression, in the Population and Sample

- Solving for $\boldsymbol{\beta}$, we obtain an expression involving population means:

$$\boldsymbol{\beta} = E[\boldsymbol{X}_i \boldsymbol{X}_i']^{-1} E[\boldsymbol{X}_i Y_i]$$

  - With a bit of algebra, you can show that this reduces to the bivariate formulas $\beta = \frac{Cov(X_i, Y_i)}{Var(X_i)}$ and $\alpha = E[Y_i] - E[X_i]\beta$ when $\mathbf{X}_i = (1, X_i)'$

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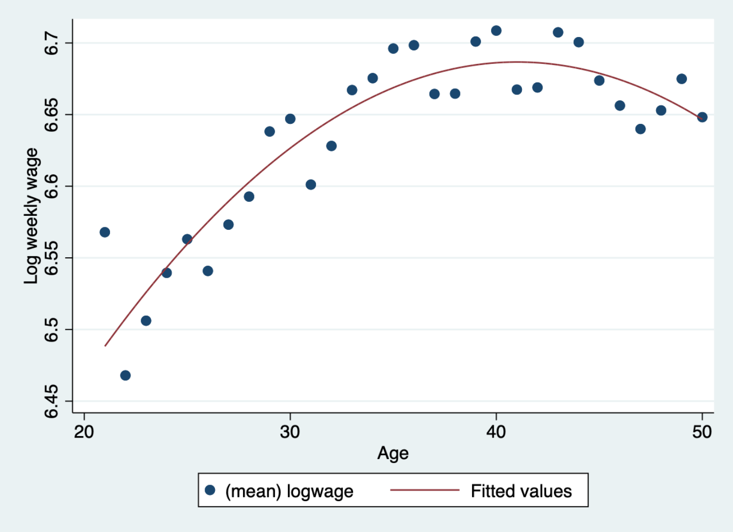
- We thus now have a general way of estimating $E[Y_i \mid \mathbf{X}_i] \approx \mathbf{X}_i' \boldsymbol{\beta}$ for any vector $\mathbf{X}_i = (1, X_{i1}, \ldots, X_{iK})'$

Quadratic Regression of Log Wages on Age



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Interpreting Quadratic Regression Coefficients

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- The estimated derivative from a multivariate regression, in this case $\hat{\boldsymbol{\beta}}_1 + 2\hat{\boldsymbol{\beta}}_2 x$, is sometimes called the "marginal effect" at x
  - This terminology is a bit unfortunate: this need not be a *causal* effect, just an estimated derivative of the CEF

Interpreting Our Example Coefficients

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Re-Writing Multivariate OLS with Matrix Algebra

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$$\hat{\boldsymbol{\beta}} = \left(\frac{1}{N} \sum_{i=1}^{N} \boldsymbol{X}_i \boldsymbol{X}_i' \right)^{-1} \left(\frac{1}{N} \sum_{i=1}^{N} \boldsymbol{X}_i Y_i \right)$$

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- Using this notation, one can show that $\hat{\boldsymbol{\beta}} = (\boldsymbol{X'X})^{-1} \boldsymbol{X}' \boldsymbol{Y}$

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- **Consistency:** $\hat{\boldsymbol{\beta}} \to_p \boldsymbol{\beta}$.

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- **Asymptotic normality**:

$$\sqrt{N}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}) \rightarrow_d \mathrm{N}(0, \boldsymbol{\Sigma}),$$

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where $\hat{\varepsilon}_i = Y_i - \boldsymbol{X}_i' \hat{\boldsymbol{\beta}}$.

- Note: $\hat{\boldsymbol{\Sigma}}$ is a *matrix*.
  - The standard error for $\hat{\boldsymbol{\beta}}_j$ is $\sqrt{\hat{\boldsymbol{\Sigma}}_{jj}} / \sqrt{N}$
  - The off-diagonal elements correspond with covariances between $\hat{\boldsymbol{\beta}}_j, \hat{\boldsymbol{\beta}}_k$

Example - Log Earnings by Age

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- We estimate the regression

$$Y_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \varepsilon_i$$

Controlling for Multiple Variables

- In addition to allowing for more flexible functional forms (e.g. quadratic), multivariate OLS allows us to approximate the CEF conditional on multiple variables at once

- Example: we have data from Texas on each county's presidential vote over the last three elections (2012,2016,2020)

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- This says that

$$E\left[\text{Biden vote} \middle| \text{Clinton vote}, \text{Obama vote}\right] \approx \beta_0 + \beta_1 \times \text{Clinton vote} + \beta_2 \times \text{Obama vote}$$

OLS Estimates

Variable	Coefficient	SE
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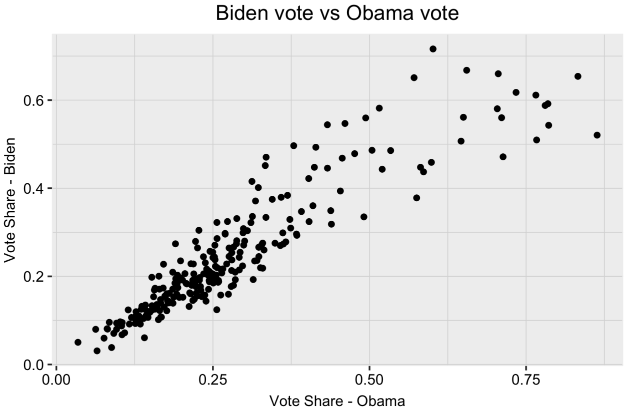
- Notice that the coefficient on Obama vote share is *negative*

- Wait, does this mean Biden did worse in places that Obama did well?!



- If we look at the data, we see that Biden vote share is highly positively correlated with Obama vote share.
- So what's going on?!

Interpreting Regression Coefficients

- Remember that multivariate OLS is approximating the CEF as

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- Thus, β_2 is an estimate of a *partial derivative*,

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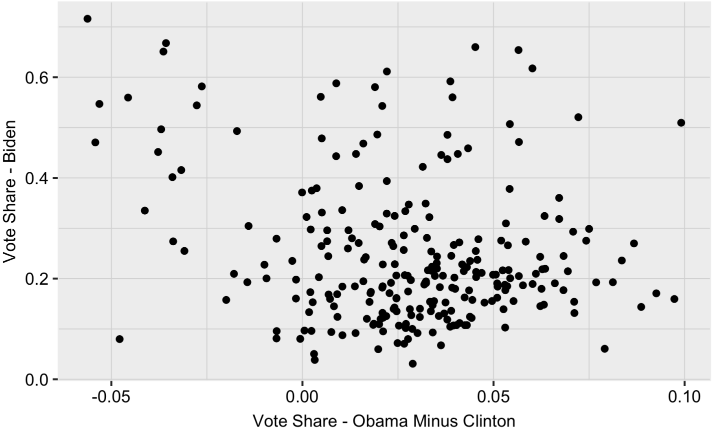
- If $\beta_2 < 0$, this means that among places where Clinton had the same vote share, Biden did better in places with lower Obama vote share.

- In other words, Biden did better in places where Democratic vote share was increasing between 2012 and 2016!

Obama vote vs Clinton vote



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- 3) Obtain $\hat{\beta}_2$ by regressing Y_i on the OLS residual $X_{i2} - \hat{X}_{i2}$:

$$Y_i = \alpha + (X_{i2} - \hat{X}_{i2})\beta_2 + v_i$$

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- Regress Obama vote share on Clinton vote share

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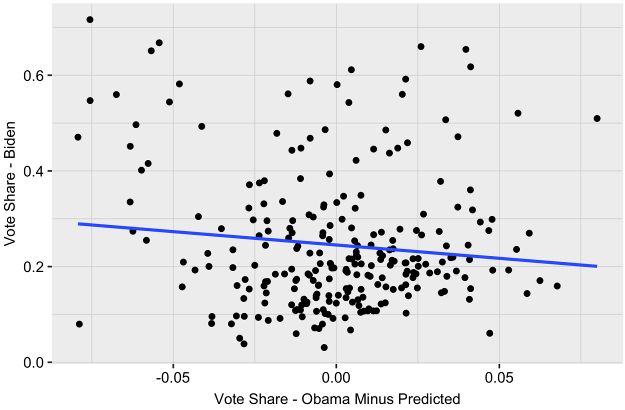
- Predict Obama vote share using Clinton vote share:

$$\hat{X}_{i2} = 0.03 + 0.98 X_{i1}$$

- Regress Biden vote share on $X_{i2} - \hat{X}_{i2}$:

Intercept ($\hat{\alpha}$)	0.25
Obama minus predicted ($\hat{\beta}_2$)	-0.56

- The estimate $\hat{\beta}_2$, -0.56, is exactly what we got before!



- The slope of the best-fit line is precisely $\hat{\beta}_2 = -0.56$.
- FWL generally gives us an easy way to visualize/interpret multivariate regression coefficients

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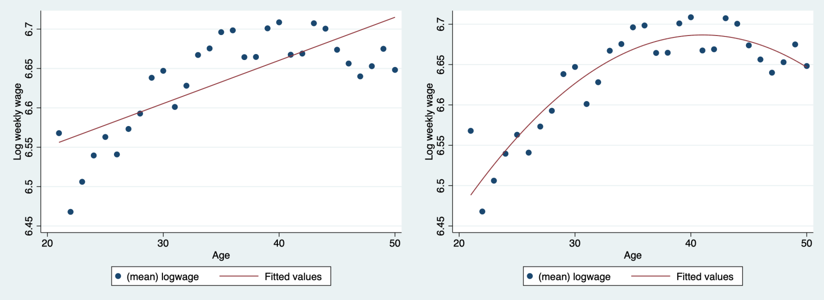
- To estimate R^2, we replace population values with sample analogs

$$\hat{R}^2 = \frac{\frac{1}{N}\sum_i(\mathbf{X}_i'\hat{\boldsymbol{\beta}} - \bar{\mathbf{X}}_i'\hat{\boldsymbol{\beta}})^2}{\frac{1}{N}\sum_i(Y_i - \bar{Y})^2} = 1 - \frac{\frac{1}{N}\sum_i \hat{\varepsilon}_i^2}{\frac{1}{N}\sum_i(Y_i - \bar{Y})^2}$$

R^2 in the Wage-Age Example



(a) $\hat{R}^2 = 0.44$ (b) $\hat{R}^2 = 0.73$

- The linear fit explains 44% of the variation in average earnings across ages, whereas the quadratic fit explains 73%

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  While the coefficients in a quadratic fit minimize

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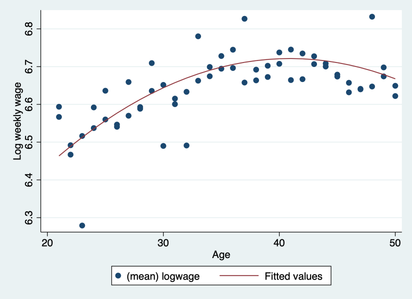
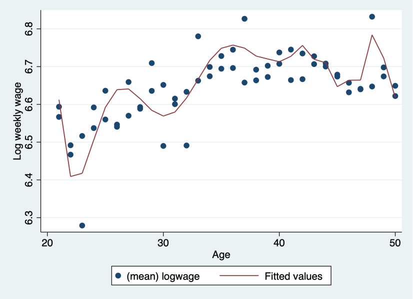
- But is a more complicated model always better?

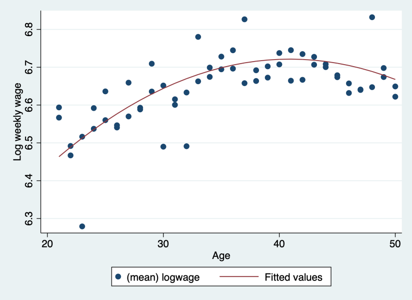


(c) Quadratic, $\hat{R}^2 = 0.44$



(d) 20th order poly, $\hat{R}^2 = 0.70$

- Suppose we take a sample of size 10,000 and fit a quadratic and a 20th order polynomial



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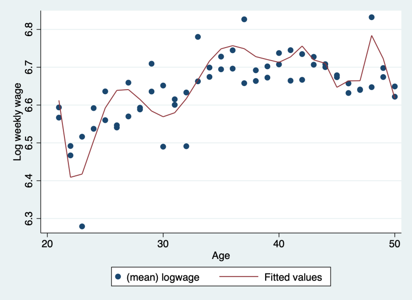

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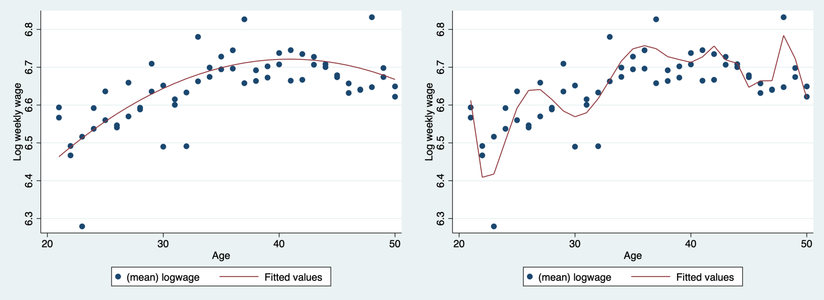
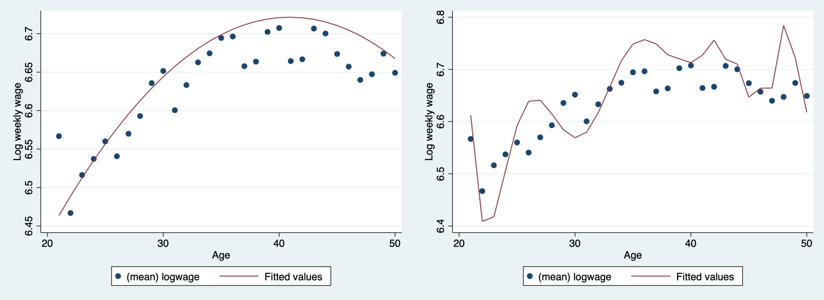


(g) Quadratic, $\hat{R}^2 = 0.44$ (h) 20th order poly, $\hat{R}^2 = 0.70$

- Suppose we take a sample of size 10,000 and fit a quadratic and a 20th order polynomial
- The 20th order poly has higher R^2, does it look reasonable to you?
- No, it looks too "squiggly" – it has adapted to fit the exact points in the sample

- Suppose we draw a new sample and test the prediction of our model trained on the first data-set



(i) Quadratic

(j) 20th order poly

- Suppose we draw a new sample and test the prediction of our model trained on the first data-set

- The quadratic fit generalizes pretty well to the new data.

- But the 20th-order polynomial does very poorly. It “overfit” the features of the specific previous sample. This doesn’t generalize well to a new sample

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  - Choose the complexity of the model based on how well it predicts out of sample
  - Cross-validation is the basis of modern *machine learning* (ML) methods. ML is very powerful, but how to use ML for causal inference is still being worked out

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- Then, they will assess the "robustness" of the conclusions to adding/subtracting variables and/or higher-order terms.
- Generally, we will be more confident if the model conclusions are not sensitive to tweaks in the model specification.

Outline

1. Deriving Multivariate Regression and OLS✓
2. Regression and Causality
3. Regression Odds and Ends

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- Then conditional unconfoundedness implies that $CATE(\boldsymbol{x}) \approx \beta$.
  - Doesn't depend on $\mathbf{x}$, so also have $\beta \approx ATE$

- So if we estimate the multivariate regression

$$Y_i = D_i\beta + \boldsymbol{X}_i'\boldsymbol{\gamma} + \varepsilon_i,$$

we can interpret $\hat{\boldsymbol{\beta}}$ as an estimate of the ATE.

Dale and Krueger

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- **Earnings** in 1996

- **College application and demographic variables** including SAT scores, class rank, family income, race, etc

- **College application decisions** — i.e. the set of schools students applied to and were admitted

Institution	School-average SAT score in 1978
Barnard College	1210
Bryn Mawr College	1370
Columbia University	1330
Denison University	1020
Duke University	1226
Emory University	1150
Georgetown University	1225
Hamilton College	1246
Kenyon College	1155
Miami University (Ohio)	1073
Northwestern University	1240
Oberlin College	1227
Pennsylvania State University	1038
Princeton University	1308
Rice University	1316
Smith College	1210
Stanford University	1270
Swarthmore College	1340
Tufts University	1200
Tulane University	1080
University of Michigan (Ann Arbor)	1110
University of North Carolina (Chapel Hill)	1080
University of Notre Dame	1200
University of Pennsylvania	1280
Vanderbilt University	1162
Washington University	1180
Wellesley College	1220
Wesleyan University	1260
Williams College	1255
Yale University	1360

Dealing with Selection

- Dale & Krueger assume **conditional unconfoundedness**, i.e. $D_i \perp\!\!\!\perp (Y_i(\cdot))|X_i$ where D_i is the average SAT score for students at your college and X_i is a set of controls

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- They then estimate regressions of the form

$$ln(Y_i) = D_i\beta + \boldsymbol{X}_i'\boldsymbol{\gamma} + \varepsilon_i$$

where Y_i is 1996 earnings

Dealing with Selection

- Dale & Krueger assume **conditional unconfoundedness**, i.e. $D_i \perp\!\!\!\perp (Y_i(\cdot))|X_i$ where D_i is the average SAT score for students at your college and X_i is a set of controls

- They then estimate regressions of the form

$$ln(Y_i) = D_i\beta + \boldsymbol{X}_i'\boldsymbol{\gamma} + \varepsilon_i$$

  where Y_i is 1996 earnings

- If conditional unconfoundedness holds & the regression approx. to the CEF is decent, then β should (approximately) equal the treatment effect of attending a college with higher average SAT scores.

First Pass: SAT Scores and Demographics in $\boldsymbol{X}_i$

Variable	Full sample 1
School-average SAT score/100	0.076 (0.016)
Predicted log(parental income)	0.187 (0.024)
Own SAT score/100	0.018 (0.006)
Female	−0.403 (0.015)
Black	−0.023 (0.035)
Hispanic	0.015 (0.052)
Asian	0.173 (0.036)
Other/missing race	−0.188 (0.119)
High school top 10 percent	0.061 (0.018)
High school rank missing	0.001 (0.024)
Athlete	0.102 (0.025)

- $\hat{\beta} = 0.076$ indicates about an increase in log wages of 7.6 from attending a school with 100 higher SAT points

Do You Buy This Estimate?

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- To address this concern, Dale and Kruger have a second analysis: control for the set of colleges that a student applied to / was admitted
  - Are able to see this because of the C&B data

Introduction to “Fixed Effects”

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  Case 1: Admitted to all schools
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- The variables $X_{i1}, ..., X_{i4}$ are often called "fixed effects" for the set of schools you were admitted to.

Introduction to "Fixed Effects"

- We can then approximate the CEF as

$$E[Y_i|\boldsymbol{X}_i, D_i] = D_i\beta_D + X_{i1}\beta_1 + X_{i2}\beta_2 + X_{i3}\beta_3 + X_{i4}\beta_4$$

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- Intuitively, β_D represents the average difference from going to an elite school *among* students who got into the same set of schools

Applicant Group	Student	Private			Public			1996 Earnings
		School I	School II	School III	School IV	School V	School VI	
	1		Reject	**Admit**		Admit		110,000
A	2		Reject	**Admit**		Admit		100,000
	3		Reject	Admit		**Admit**		110,000
B	4	**Admit**			Admit		Admit	60,000
	5	Admit			Admit		**Admit**	30,000
C	6		**Admit**					115,000
	7		**Admit**					75,000
D	8	Reject			**Admit**	Admit		90,000
	9	Reject			Admit	**Admit**		60,000

- "Applicant groups" all applied + admitted to the same set of schools
- The school a student actually attended is highlighted

Controlling for Application Decisions in Practice

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- As an approximation, they group colleges into bins based on average-SAT rounded to nearest 25 points
- They then control for the set of colleges you applied/were admitted to based on these bins (e.g., X_{i1} might correspond to being rejected at a school w/SAT 1350 and accepted at 2 schools w/SAT 1250)

	Basic model: no selection controls		Matched-applicant model
	Full sample	Restricted sample	Similar school-SAT matches*
Variable	1	2	3
School-average SAT score/100	0.076	0.082	−0.016
	(0.016)	(0.014)	(0.022)
Predicted log(parental income)	0.187	0.190	0.163
	(0.024)	(0.033)	(0.033)
Own SAT score/100	0.018	0.006	−0.011
	(0.006)	(0.007)	(0.007)
Female	−0.403	−0.410	−0.395
	(0.015)	(0.018)	(0.024)
Black	−0.023	−0.026	−0.057
	(0.035)	(0.053)	(0.053)
Hispanic	0.015	0.070	0.020
	(0.052)	(0.076)	(0.099)
Asian	0.173	0.245	0.241
	(0.036)	(0.054)	(0.064)
Other/missing race	−0.188	−0.048	0.060
	(0.119)	(0.143)	(0.180)
High school top 10 percent	0.061	0.091	0.079
	(0.018)	(0.022)	(0.026)
High school rank missing	0.001	0.040	0.016
	(0.024)	(0.026)	(0.038)
Athlete	0.102	0.088	0.104
	(0.025)	(0.030)	(0.039)

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Own SAT score/100	0.018 (0.006)	0.006 (0.007)	−0.011 (0.007)
Female	−0.403 (0.015)	−0.410 (0.018)	−0.395 (0.024)
Black	−0.023 (0.035)	−0.026 (0.053)	−0.057 (0.053)
Hispanic	0.015 (0.052)	0.070 (0.076)	0.020 (0.099)
Asian	0.173 (0.036)	0.245 (0.054)	0.241 (0.064)
Other/missing race	−0.188 (0.119)	−0.048 (0.143)	0.060 (0.180)
High school top 10 percent	0.061 (0.018)	0.091 (0.022)	0.079 (0.026)
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Athlete	0.102 (0.025)	0.088 (0.030)	0.104 (0.039)

- With application controls, we get $\hat{\beta} = -0.016$.
- This indicates about a -1.6 log wage return to attending a school with 100 higher SAT points (but not significant!)

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  - Students may choose to go to a lower-ranked school only if it has a particularly good program in what they're interested in
- It's also important to realize that the schools in the C&B study tend to be selective. These results can at best be interpreted as the causal effect between attending a selective school and highly-selective school

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- In the C&B example (and many other applications), we might be worried that we didn't condition on all of the necessary variables for conditional unconfoundedness to hold
- How will the coefficients we estimate be biased if we forget to include some variables?
- To answer this question, we will derive what is called the **omitted variable bias** (OVB) formula

Omitted Variable Bias – Simplest Case

- Suppose that conditional unconfoundedness holds conditional on X_{i1} (e.g. SAT score).

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$$Y_i = \beta_0 + \beta_D D_i + \beta_1 X_{i1} + e_i$$

  The population coefficient β_D approximates the ATE (assuming this is a good approx to the CEF).

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$$\tilde{\beta}_D = \frac{Cov(Y_i, D_i)}{Var(D_i)} = \frac{Cov(\beta_0 + \beta_D D_i + \beta_1 X_{i1} + e_i, D_i)}{Var(D_i)}$$

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$$\begin{aligned}
\tilde{\beta}_D &= \frac{Cov(Y_i, D_i)}{Var(D_i)} = \frac{Cov(\beta_0 + \beta_D D_i + \beta_1 X_{i1} + e_i, D_i)}{Var(D_i)} \\
&= \frac{\beta_D Cov(D_i, D_i) + \beta_1 Cov(X_{i1}, D_i) + Cov(e_i, D_i)}{Var(D_i)} \\
&= \beta_D + \beta_1 \frac{Cov(X_{i1}, D_i)}{Var(D_i)}
\end{aligned}$$

where we use $E[e_i] = E[D_i e_i] = 0$ from the FOCs for regression

Evaluating OVB

- Thus, the (population) regression $Y_i = \tilde{\beta}_0 + \tilde{\beta}_D D_i + \varepsilon_i$ yields

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- Hence $\tilde{\beta}$ will be very biased for β_D if the omitted variable X_{i1} is both highly correlated with Y_i and highly correlated with D_i
  - On the flip side, if either $\beta_1 = 0$ or $\gamma_D =$ then we have no OVB!

OVB Formula in Finite Samples

- We just showed that the coefficients from the population regressions

$$Y_i = \beta_0 + \beta_D D_i + \beta_1 X_i + e_i \tag{1}$$

$$Y_i = \tilde{\beta}_0 + \tilde{\beta}_D D_i + \varepsilon_i \tag{2}$$

are related by the OVB formula

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$$\tilde{\beta} = \beta_D + \beta_1 \frac{Cov(X_{i1}, D_i)}{Var(D_i)}$$

- It turns out that the OLS estimates for these two regressions have the same relationship

$$\hat{\tilde{\beta}} = \hat{\beta}_D + \hat{\beta}_1 \frac{\widehat{Cov}(X_{i1}, D_i)}{\widehat{Var}(D_i)}$$

OVB Illustration

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- Let's think about what would happen if we forgot to control for SAT

- Here are the results that A&P get when controlling for SAT:

Variable	Coefficient	SE
Private school ($\hat{\beta}_D$)	0.095	0.052
SAT score /100 ($\hat{\beta}_1$)	0.048	0.009
Constant	[...]	[...]

- What would happen if we omitted the control for SAT score from the regression? How would $\hat{\beta}_D$ change?

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Variable	Coefficient
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Constant	[...]

- Thus, if we omitted X_{i1} our estimated coefficient on private school would be $\hat{\gamma}_D \times \hat{\beta}_1 = 0.83 \times 0.048 \approx 0.04$ larger.

- Indeed, if we actually run the regression omitting SAT scores, we see that the coefficient on private school is 0.04 larger.

- Results including SAT score:

Variable	Coefficient	SE
Private school (β_D)	0.095	0.052
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SAT score /100 (β_1)	0.048	0.009
Constant	[...]	[...]

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Private school ($\tilde{\beta}$)	0.135	0.055
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- Answer:

$$\tilde{\beta}_D = \beta_D + \beta_2 \gamma_D$$

  where γ_D is the coefficient from the regression

$$X_{i2} = \gamma_0 + \gamma_D D_i + \gamma_1 X_{i1} + u_i$$

- This is similar to the OVB formula we had from before!

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- Let's think about what happens if we forgot the control for SAT score

- Here are the results that A&P get when controlling for both SAT score and set of schools you're admitted to:

Variable	Coefficient	SE
Private school ($\hat{\beta}_D$)	0.003	0.039
SAT score /100 ($\hat{\beta}_2$)	0.033	0.007
College admitted ($\hat{\boldsymbol{\beta}}_1$)	[...]	[...]

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Dale and Krueger - Heterogeneous Effects

- DK estimate a regression of the form:

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OF MALE AND FEMALE FULL-TIME WORKERS

Variable	Parameter estimates: Basic model: no selection controls	Parameter estimates: Matched-applicant model*
	1	2
School-average SAT score/100	0.701	0.537
	(0.185)	(0.224)
Predicted log(parental income)	0.915	0.819
	(0.212)	(0.247)
Predicted log of parental income ∗ school SAT score/100	−0.063	−0.056
	(0.019)	(0.023)
Own SAT score/100	0.018	−0.011
	(0.006)	(0.007)
High school top 10 percent	0.062	0.080
	(0.019)	(0.026)
High school rank missing	0.005	0.018
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Predicted log(parental income)	0.915	0.819
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Predicted log of parental income * school SAT score/100	−0.063	−0.056
	(0.019)	(0.023)
Own SAT score/100	0.018	−0.011
	(0.006)	(0.007)
High school top 10 percent	0.062	0.080
	(0.019)	(0.026)
High school rank missing	0.005	0.018
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- In DK, students at the 50th percentile of family earnings have log parental income of 10.39 ($\approx \$33,000$)
- What is the estimated effect of going to a school w/avg SAT 100 points higher at the median?

$$\hat{\beta}_D + \hat{\beta}_{DX} x =$$

OF MALE AND FEMALE FULL-TIME WORKERS

	Parameter estimates	
	Basic model: no selection controls	Matched-applicant model*
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- Not entirely clear... networking more important for students who don't have as many family connections?

Linear Combinations of Coefficients

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- So we can form a CI for $\beta_1 + \beta_2 x$ using $\hat{\beta}_1 + \hat{\beta}_2 x \pm 1.96\hat{\sigma}_x/\sqrt{N}$, where $\hat{\sigma}_x^2 = \hat{\Sigma}_{11} + x^2\hat{\Sigma}_{22} + 2x\hat{\Sigma}_{12}$.

Example

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- Similar (but slightly more complicated) asymptotic arguments can be used to test hypotheses on non-linear combinations of coefficients, e.g. $\beta_1\beta_2 + \beta_3^2$. This can be done using the `nlcom` command in Stata.

Outline

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- This can lead to what's called a **multiple hypothesis testing** problem

Multiple Hypothesis Testing (Cont.)

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- But suppose we first test for a significant effect among men. And we then also test for a significant effect among women.
- If there is no significant effect among either group, what is the probability that we find at least one significant effect?

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- Let N_{sig} be the number of significant results.

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Probability of finding at least one subgroup with a signficant effect with k independent groups (and zero treatment effect):

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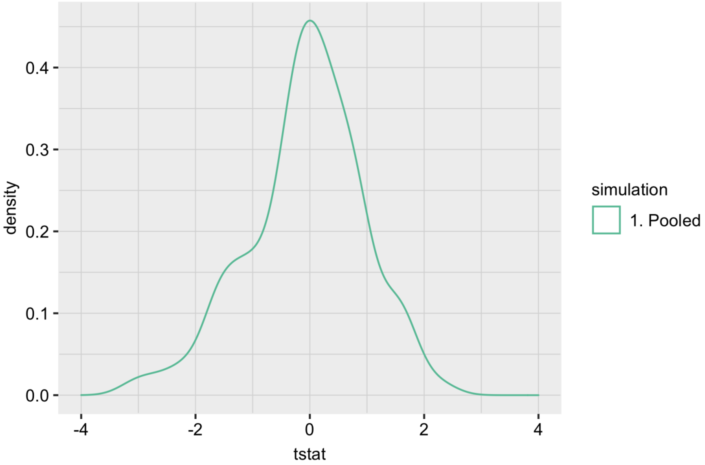
  3. The individual treatment effect for all 50 states in the data
- I then calculate the fraction of simulations in which we get at least one significant estimate



0.4
0.3
0.2
0.1
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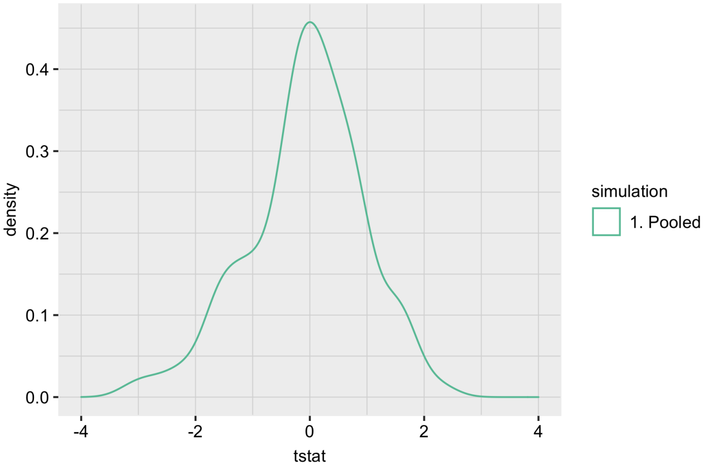
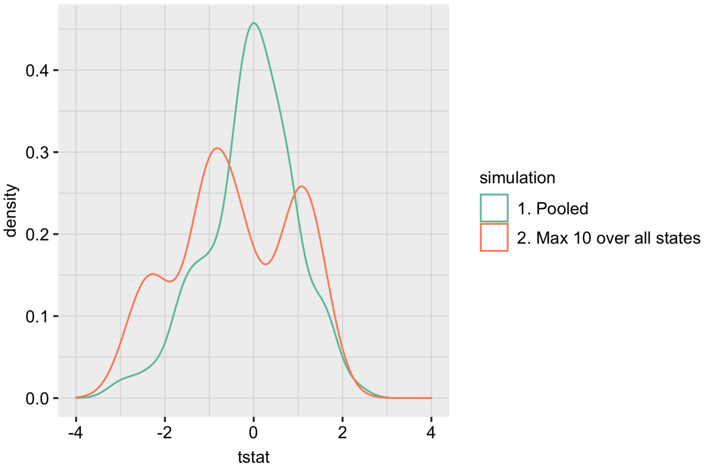
density
-4
-2
0
2
4
tstat
simulation
1. Pooled



- When testing the pooled effect across all states, we find a significant effect 5% of the time



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density
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0
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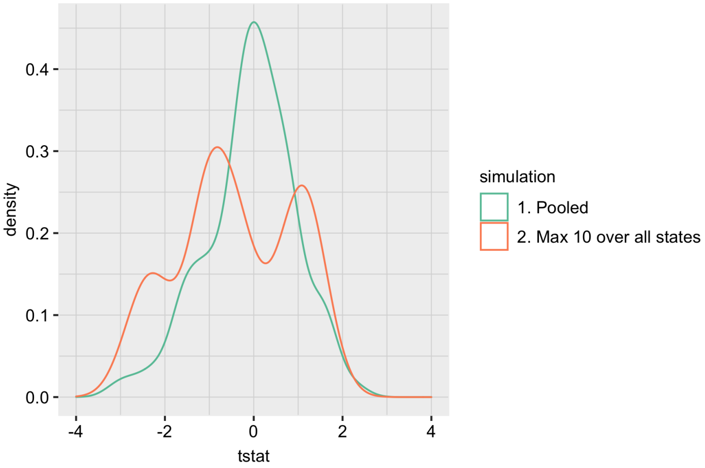
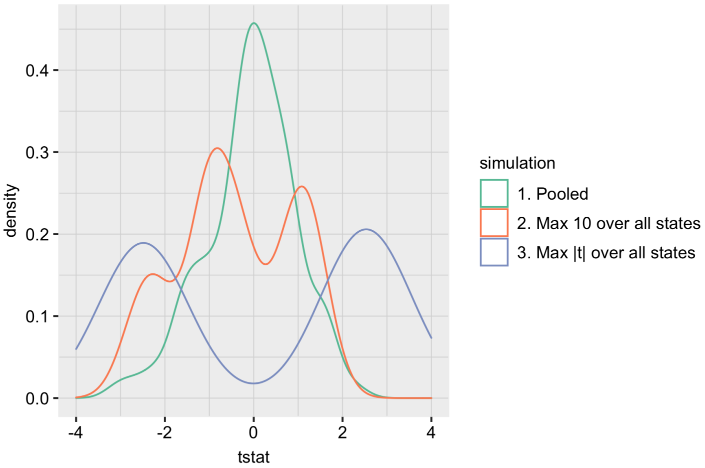
tstat
simulation
1. Pooled
2. Max 10 over all states



- When testing the pooled effect across all states, we find a significant effect 19% of the time


density
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1. Pooled
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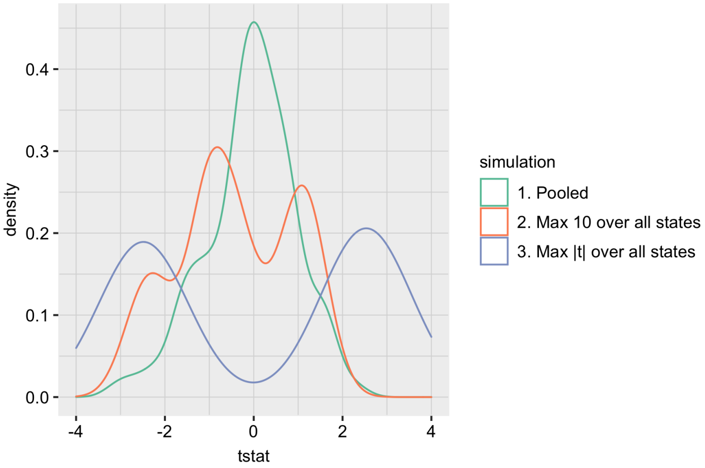
3. Max |t| over all states



- When testing the pooled effect across all states, we find a significant effect 90% of the time

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$$P(N_{sig} > 0) \leq \sum_k P(\text{k is signficant}) = k(0.05/k) = 0.05$$

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- Downside: if you have lots of hypotheses, power against any one hypothesis can be low (e.g., 99.9% confidence intervals are very wide). And the test is generally conservative in the sense that we find any significant effect $< 5\%$ of the time

Probability of rejecting at least one hypothesis without and with Bonferroni correction

Simulation	Uncorrected	Corrected
Pooled	0.05	0.05
10 States	0.19	0.00
50 States	0.90	0.04

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- How do we do this?

- We showed that $\sqrt{N}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}) \rightarrow_d \mathrm{N}(0, \boldsymbol{\Sigma})$

- From the continuous mapping theorem, we get that $X = \sqrt{N}\hat{\boldsymbol{\Sigma}}^{-\frac{1}{2}}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}) \rightarrow_d$

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$$F = \sum_k X_k^2 \to \sum_k Z_k^2, \text{ where } Z \sim \mathrm{N}(0, \boldsymbol{I})$$

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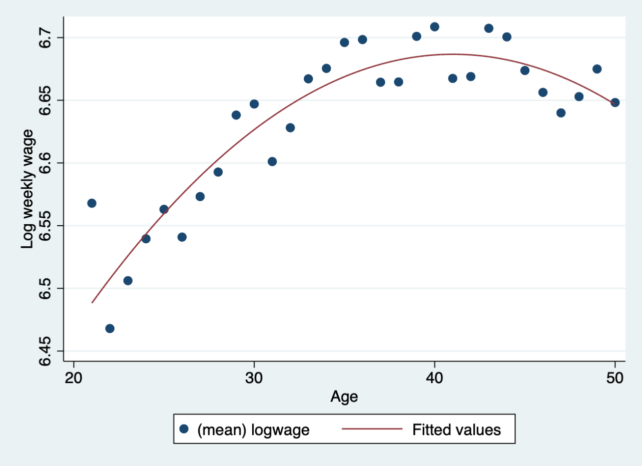
- This is often called an *F-test*.
  - Sometimes the critical values use what's called an F-distribution, which is a slight modification to the $\chi^2(k)$ that corrects for small sample sizes


Log weekly wage
6.45
6.5
6.55
6.6
6.65
6.7
20
30
40
50
Age
(mean) logwage
Fitted values

```
. reg logwage age agesq, r

Linear regression                               Number of obs     =         30
                                                F(2, 27)          =      30.81
                                                Prob > F          =     0.0000
                                                R-squared         =     0.8412
                                                Root MSE          =     .02622

------------------------------------------------------------------------------
             |               Robust
     logwage | Coefficient  std. err.      t    P>|t|     [95% conf. interval]
-------------+----------------------------------------------------------------
         age |   .0403444   .0076871     5.25   0.000     .0245718     .056117
       agesq |   -.000492   .0001011    -4.87   0.000    -.0006994   -.0002846
       _cons |   5.859108   .1409008    41.58   0.000     5.570003    6.148212
------------------------------------------------------------------------------

.
. test age agesq

 ( 1)  age = 0
 ( 2)  agesq = 0

       F(  2,    27) =   30.81
            Prob > F =    0.0000
```