Chapter 3: Asymptotic Statistics

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Mathematical Econometrics I
Brown University
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Outline

Motivation

- We've seen how we can test hypotheses about population means using information from the sample mean $\hat{\mu}$ when it is **normally distributed** with a known variance
- This situation arises when we know that $Y_i \sim \mathrm{N}(\mu, \sigma^2)$ with known σ
- But this situation is rare... how do we "do inference" more generally?

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- But this situation is rare... how do we "do inference" more generally?
- Fortunately, the assumption of normally distributed sample means turns out to be a good **approximation** when samples are large
- What we mean by a "good approximation" is formalized by asymptotic statistics, which considers the distribution of $\hat{\mu}$ in the limit as $N \to \infty$

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- The **Central Limit Theorem** (CLT) says that when N is large, the distribution of $\hat{\mu}$ is approximately normally distributed with mean μ and variance σ^2/n
- The **Continuous Mapping Theorem** says that when N is large, continuous functions of $\hat{\mu}$, say $g(\hat{\mu})$, are also close to $g(\mu)$

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1. Overview ✓
2. LLN, CLT, and CMT
3. Putting Asymptotics into Practice

Convergence in Probability

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- Typically x is a constant, although we will sometimes also say $X_N \to X$ for X a random variable (using the same definition as above)

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- **Proof** (you won't be responsible for this):
  By the law of iterated expectations,

$$
\begin{aligned}
E[(X_N - x)^2] = & P(|X_n - x| > \varepsilon) E[(X_N - x)^2 || X_n - x| > \varepsilon] + \\
& P(|X_n - x| \leq \varepsilon) E[(X_N - x)^2 || X_n - x| \leq \varepsilon]
\end{aligned}
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  This implies that

$$P(|X_N - x| > \varepsilon) \leq E[(X_N - x)^2]/\varepsilon^2 \text{ (Chebychev's Inequality)}$$

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  This implies that

$$P(|X_N - x| > \varepsilon) \leq E[(X_N - x)^2] / \varepsilon^2 \text{ (Chebychev's Inequality)}$$

  Hence, $E[(X_N - x)^2] \to 0$ implies $P(|X_N - x| > \varepsilon) \to 0$

Law of Large Numbers

- **Law of Large Numbers**. Suppose that $Y_1, ..., Y_N$ are drawn *iid* from a distribution with $Var(Y_i) = \sigma^2 < \infty$. Then

$$\hat{\mu}_N = \frac{1}{N}\sum_{i=1}^{N} Y_i \rightarrow_p \mu = E[Y_i]$$

- In words: as the sample gets large, the sample mean will be close to the population mean with high probability.

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- In words: as the sample gets large, the sample mean will be close to the population mean with high probability.

- **Proof:** We saw last chapter that $E[\hat{\mu}_N] = \mu$ and $Var(\hat{\mu}_N) = \sigma^2/N$. Thus,

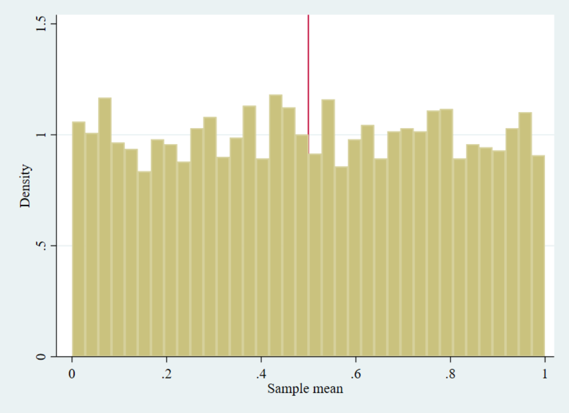
$$Var(\hat{\mu}_N) = E[(\hat{\mu}_N - \mu)^2] = \sigma^2/N \rightarrow 0$$

  Hence, $\hat{\mu}_N \rightarrow_p \mu$ by our "useful fact".

Laws of Large Numbers Illustration

Distribution and mean of $\frac{1}{N}\sum_i Z_i$ when $Z_i \sim \mathrm{U}(0,1)$, $\mathbf{N} = \mathbf{1}$

```
matrix sims=J(5000,1,.)
foreach N in 1 10 100 1000 {
    forval j=1/5000 {
        qui {
            clear
            set obs `N'
            gen X = runiform()
            summ X
            matrix sims[`j',1]=r(mean)
        }
    }
    clear
    svmat sims
    summ sims
    local mean = r(mean)
    hist sims, xlabel(0(0.2)1) xline(`mean')
    graph export sims`N'_2.png, replace
}
```



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Distribution and mean of $\frac{1}{N}\sum_i Z_i$ when $Z_i \sim \mathrm{U}(0,1)$, $\mathbf{N} = \mathbf{10}$

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matrix sims=J(5000,1,.)
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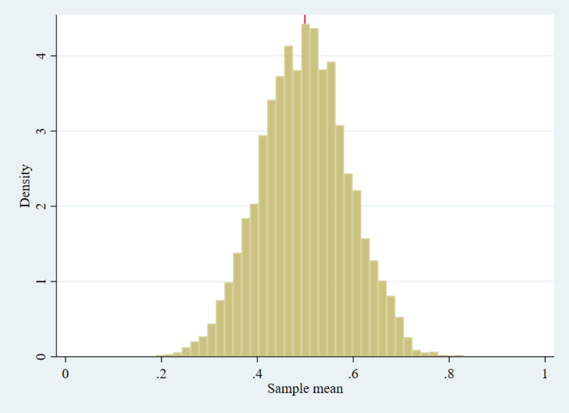

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matrix sims=J(5000,1,.)
foreach N in 1 10 100 1000 {
    forval j=1/5000 {
        qui {
            clear
            set obs `N'
            gen X = runiform()
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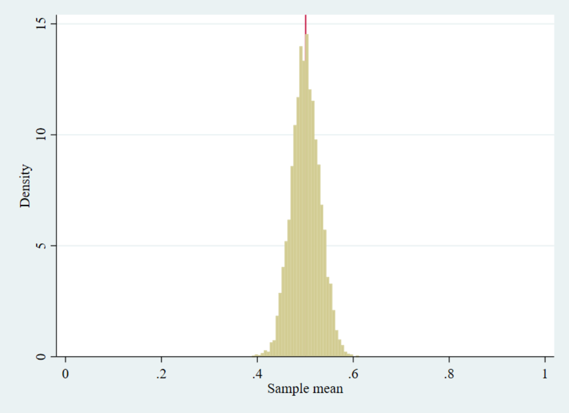

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Distribution and mean of $\frac{1}{N}\sum_i Z_i$ when $Z_i \sim \mathrm{U}(0,1)$, $\mathbf{N = 1000}$

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foreach N in 1 10 100 1000 {
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            clear
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            gen X = runiform()
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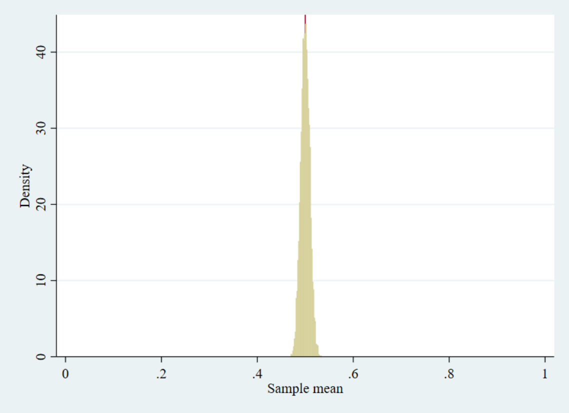

Convergence in Distribution

- You might have noticed that the distribution of $\hat{\mu}$ in the simulations looks close to a normal distribution as N gets large
- The notion of **convergence in distribution** formalizes what it means for one distribution to be close to another distribution

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- The notion of **convergence in distribution** formalizes what it means for one distribution to be close to another distribution
- Definition: We say that X_N converges in distribution to a continuously distributed variable X, denoted $X_n \to_d X$ or $X_n \Rightarrow X$, if the CDF of X_N converges (pointwise) to the CDF of X,

$$F_{X_N}(x) \to F_X(x) \text{ for all } x$$

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- Theorem: Suppose that $Y_1, ..., Y_N$ are drawn *iid* from a distribution with mean $\mu = E[Y_i]$ and variance $Var(Y_i) = \sigma^2 < \infty$. Then the sample mean $\hat{\mu} = \frac{1}{N}\sum_{i=1}^{N} Y_i$ satisfies

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- In words, the theorem says the following:
  1. We can start with any distribution Y_i, possibly non-normal
  2. If we take the average of the $Y_1, ..., Y_N$ in a sample sufficiently large, the distribution of $\hat{\mu} = \frac{1}{N}\sum_i Y_i$ is (approximately) normal!

CLT Illustration

Distributions of $\hat{\mu} = \frac{1}{N}\sum_i X_i$ vs. $\mathrm{N}(E[\hat{\mu}], Var(\hat{\mu}))$: $X_i \sim \mathrm{U}(0,1)$, $\mathbf{N = 1}$

```
clear all
set seed 42
set more off
set matsize 5000

matrix sims=J(5000,1,.)
foreach N in 1 2 5 10 {
    forval j=1/5000 {
        qui {
            clear
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            matrix sims[`j',1]=r(mean)
        }
    }
    clear
    svmat sims
    hist sims, normal xtitle("Sample mean")
    graph export sims`N'.png, replace
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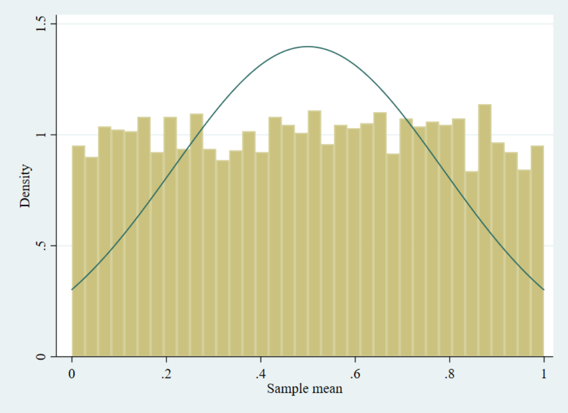

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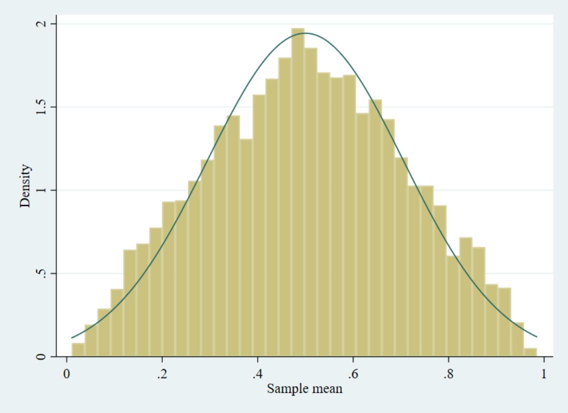

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clear all
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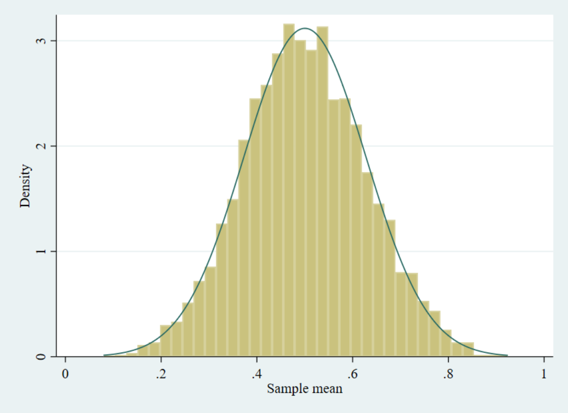

CLT Illustration

Distributions of $\hat{\mu} = \frac{1}{N}\sum_i X_i$ vs. $\mathrm{N}(E[\hat{\mu}], Var(\hat{\mu}))$: $X_i \sim \mathrm{U}(0,1)$, $\mathbf{N = 10}$

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clear all
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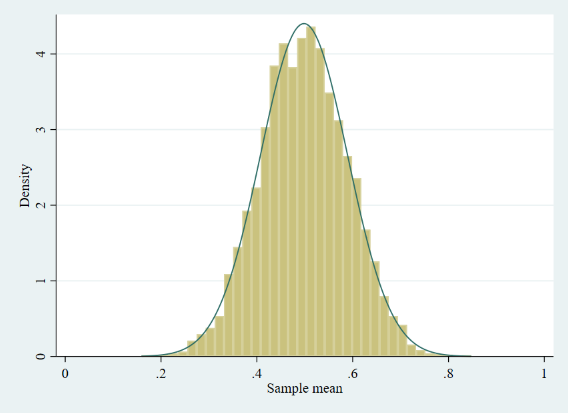

CLT Illustration II

https://www.youtube.com/watch?v=EvHiee7gs9Y

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- **LLN**: For $\hat{\boldsymbol{\mu}}_N$, the sample mean of *iid* vectors $\mathbf{Y_1}, \ldots \mathbf{Y_N}$ with mean $\boldsymbol{\mu}$ and finite variance, $\hat{\boldsymbol{\mu}}_N \to_p \boldsymbol{\mu}$

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- For a vector $\mathbf{X_N} \in \mathbb{R}^k$, we say $\mathbf{X_N} \rightarrow_d \mathbf{X}$ for $\mathbf{X}$ continuously distributed if $F_{\mathbf{X_N}}(\mathbf{x}) \rightarrow F_{\mathbf{X}}(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^k$.
- **CLT**: For $\hat{\boldsymbol{\mu}}_N$, the sample mean of *iid* vectors $\mathbf{Y_1}, \ldots \mathbf{Y_N}$ with mean $\boldsymbol{\mu}$ and finite variance $\boldsymbol{\Sigma}$, $\sqrt{N}(\hat{\boldsymbol{\mu}}_N - \boldsymbol{\mu}) \rightarrow_d \mathrm{N}(\mathbf{0}, \boldsymbol{\Sigma})$

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  If $X_N \rightarrow_d X$, then $g(X_N) \rightarrow_d g(X)$

  Multivariate versions here too: If $\mathbf{X_N} \rightarrow_p \mathbf{X}$, then $g(\mathbf{X_N}) \rightarrow_p g(\mathbf{X})$ and if $\mathbf{X_N} \rightarrow_d \mathbf{X}$, then $g(\mathbf{X_N}) \rightarrow_d g(\mathbf{X})$

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- Claim: if $Y_1, ..., Y_N$ are *iid* and $Var(Y_i^2)$ is finite, then $\hat{\sigma}^2 \rightarrow_p \sigma^2 = Var(Y_i)$.

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- Proof:
  We can write the sample variance as $\hat{\sigma}^2 = \frac{1}{N}\sum_{i=1}^{N} Y_i^2 - \hat{\mu}^2$.

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- Proof:
  We can write the sample variance as $\hat{\sigma}^2 = \frac{1}{N}\sum_{i=1}^{N} Y_i^2 - \hat{\mu}^2$.
  First term: by the LLN, $\frac{1}{N}\sum_{i=1}^{N} Y_i^2 \to_p E[Y_i^2]$.

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- Proof:
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  First term: by the LLN, $\frac{1}{N}\sum_{i=1}^{N} Y_i^2 \rightarrow_p E[Y_i^2]$.
  Second term: by the LLN, $\hat{\mu} \rightarrow_p \mu = E[Y_i]$. Thus, by the CMT, $\hat{\mu}^2 \rightarrow_p E[Y_i]^2$.

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  First term: by the LLN, $\frac{1}{N}\sum_{i=1}^{N} Y_i^2 \rightarrow_p E[Y_i^2]$.
  Second term: by the LLN, $\hat{\mu} \rightarrow_p \mu = E[Y_i]$. Thus, by the CMT, $\hat{\mu}^2 \rightarrow_p E[Y_i]^2$.
  Thus, by the CMT again, $\frac{1}{N}\sum_{i=1}^{N} Y_i^2 - \hat{\mu}^2 \rightarrow_p E[Y_i^2] - E[Y_i]^2 = \sigma^2$.

Slutsky's Lemma

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- Analogous versions apply for vector-valued random variables.

Asymptotic Hypothesis Testing

- Recall that when $Y_i \sim \mathrm{N}(\mu, \sigma^2)$, we showed that the t-statistic $\hat{t} = \frac{\hat{\mu} - \mu_0}{\sigma/\sqrt{n}} \sim \mathrm{N}(0,1)$ under $H_0 : \mu = \mu_0$.

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- Hence, asymptotically $Pr(|\hat{t}| > 1.96) \rightarrow 0.05$, even though Y_i is not normal and $\hat{\sigma}$ is estimated! We can hypothesis test just like before.

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- Analogously, when Y_i is non-normal with unknown variance, $\hat{\mu} \pm 1.96\hat{\sigma}/\sqrt{N}$ contains the true μ with probability approaching 95% as N grows large.

Outline

1. Overview ✓
2. LLN, CLT, and CMT ✓
3. Putting Asymptotics into Practice

Example – Oregon Health Insurance Experiment

In 2008, a group of uninsured low-income adults in Oregon was selected by lottery to be given the chance to apply for Medicaid. This lottery provides an opportunity to gauge the effects of expanding access to public health insurance on the health care use, financial strain, and health of low-income adults using a randomized controlled design. In the year after random assignment, the treatment group selected by the lottery was about 25 percentage points more likely to have insurance than the control group that was not selected. We find that in this first year, the treatment group had substantively and statistically significantly higher health care utilization (including primary and preventive care as well as hospitalizations), lower out-of-pocket medical expenditures and medical debt (including fewer bills sent to collection), and better self-reported physical and mental health than the control group. *JEL* Codes: H51, H75, I1.

Sample Means for Depression Outcome

	Control Group	Treated Group
Mean	0.329	0.306
SD	0.470	0.461
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CIs for Treatment Effects in Experiments

- We showed previously that in an experiment, the average treatment effect is given by

$$\tau = E[Y_i(1) - Y_i(0)] = E[Y_i|D_i = 1] - E[Y_i|D_i = 0].$$

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- How can we form confidence intervals (or test hypotheses) about the treatment effect?

Mean and variance of the difference-in-means

- Let $\bar{Y}_1 = \frac{1}{N_1}\sum_{i:D_i=1} Y_i$ be the sample mean for the treated group.
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  where $\mu_d = E[Y_i \mid D_i = d]$ and $\sigma_d^2 = Var(Y_i \mid D_i = d)$.

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  where the fact that the samples are independent implies that $Cov(\bar{Y}_1, \bar{Y}_0) = 0$.

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- As with sample means, we do not know that $\hat{\tau}$ is normally distributed, but we can show that for N large, it is *approximately* normally distributed, which allows us to use CIs of the form

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for $\hat{\sigma}_d^2$ the estimated conditional variance.

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where $\sigma^2 = \frac{1}{E[D_i]} Var(Y_i(1)) + \frac{1}{E[1-D_i]} Var(Y_i(0))$

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- We can thus form a 95% confidence interval for $\tau = E[Y_i(1) - Y_i(0)]$,

$$\bar{Y}_1 - \bar{Y}_0 \pm 1.96\hat{\sigma}/\sqrt{N},$$

where $\hat{\sigma}^2 = \frac{N}{N_1}\hat{\sigma}_1^2 + \frac{N}{N_0}\hat{\sigma}_0^2$, where $\hat{\sigma}_d^2$ is the sample variance for treatment group $d \in \{0, 1\}$

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- By averaging $CATE(x)$, we can do hypothesis testing / form CIs for ATE.

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- We've shown thus far how we can estimate $CATE(x)$ when the number of observations with $X_i = x$ is large.

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- We thus need a different way of estimating conditional means when X_i is continuously distributed.
- The next part of the course will focus on achieving this task using linear regression as an approximation to the CEF.