

## Chapter 2: Probability and Statistics

Jonathan Roth

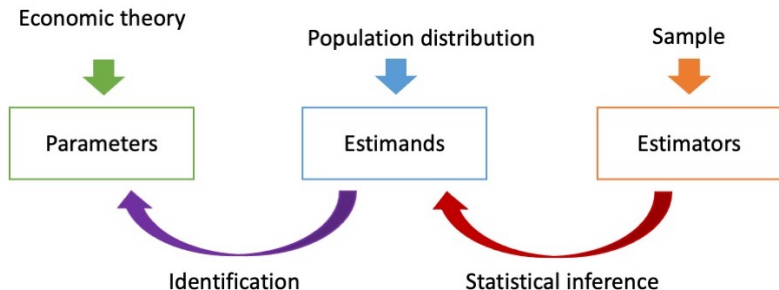
Mathematical Econometrics I  
Brown University  
Fall 2023

# Course Logistics

- Problem set 1 is posted. It is due on Friday September 22 at 4PM as a GradeScope submission
- TA sessions and OHs have started. See the announcement on Canvas for times/locations
- Any logistical questions?

# “Big Picture” Recap

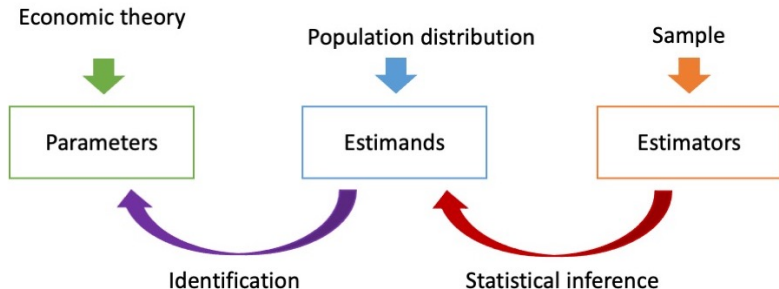
Recall our division of labor:



- **Statistics:** how does the sample data we observe relate to observable features of the population we're interested in?
- **Identification:** how do observable features of population relate to target parameters of interest?

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- **Statistics:** how does the sample data we observe relate to observable features of the population we're interested in?
- **Identification:** how do observable features of population relate to target parameters of interest?
- For both these tasks, we need a mathematical language for talking about how data is generated. Enter **probability and statistics**



I love  
Statis-sticks

# Outline

1. Random Variables and Probability Distributions
2. Means and Variances
3. Identification in Experiments
4. Random Sampling and Sample Means
5. Hypothesis Testing and Inference

# Random Variables

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- What are some examples of a random process?
  - Flip a coin – is it heads or tails?
  - Survey a random household in US – what is their income?
- The realization of a random process is called a **random variable**.

## Some Terminology

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Example: I toss two fair coins in the air

- What are the possible outcomes (sample space)?
- What is the probability of seeing at least one head (an event)?

# Random Variables and CDFs

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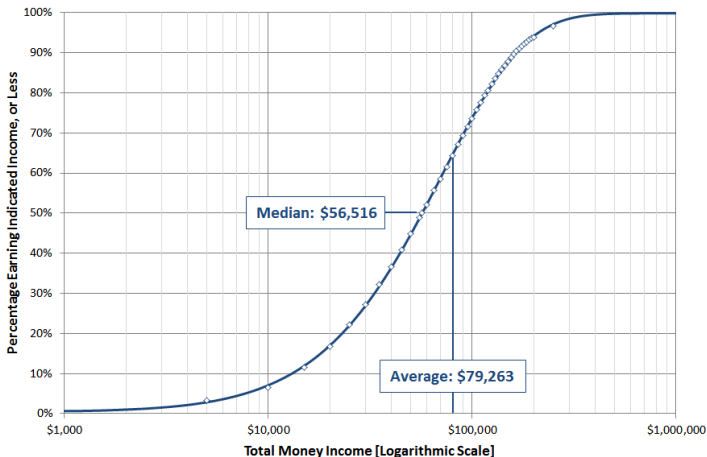
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- Note: we'll typically use lower-case letters like " $x$ " to denote realizations (i.e. non-random numbers) of random variables like " $X$ " ...

Cumulative Distribution of Total Money Income for U.S. Households, 2015

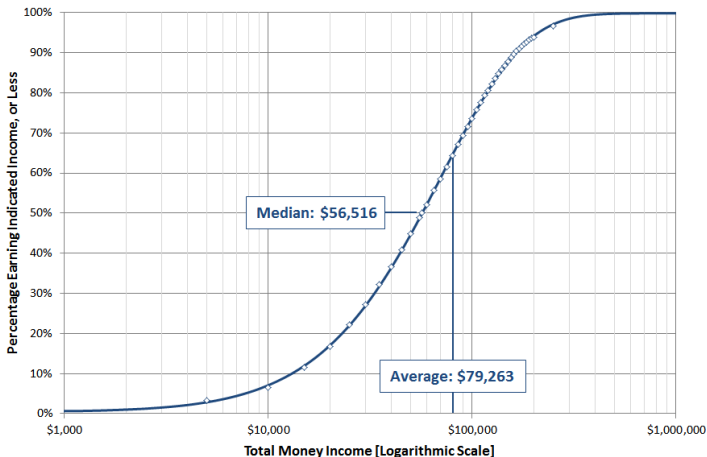


Source: U.S. Census, Current Population Survey, Annual Social and Economic Supplement, 2016

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- In 2016, around half of US households earned \$56K or less
- Formally:  $F(56,516) = 0.5$

# We All Need Some Support...

- The **support** of a random variable  $X$ , denoted  $\mathbb{X}$ , is the set of values that  $X$  can take
  - If  $X$  is months in the year you were employed,  $\mathbb{X} = \{0, 1, \dots, 12\}$
  - If  $X$  is your income, then  $\mathbb{X} = \mathbb{R}_{\geq 0}$  (approximately)

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- If the support of  $X$  is finite (e.g.  $\{0, 1\}$ ), we say  $X$  is **discrete**
- If the support of  $X$  is a continuum (e.g.  $\mathbb{R}$  or  $[0, 1]$ ), we say  $X$  is **continuously distributed**



# Density and Mass Functions

- If  $X$  is discrete, we define the probability mass function (PMF) as the probability that  $X$  takes on each value in the support:

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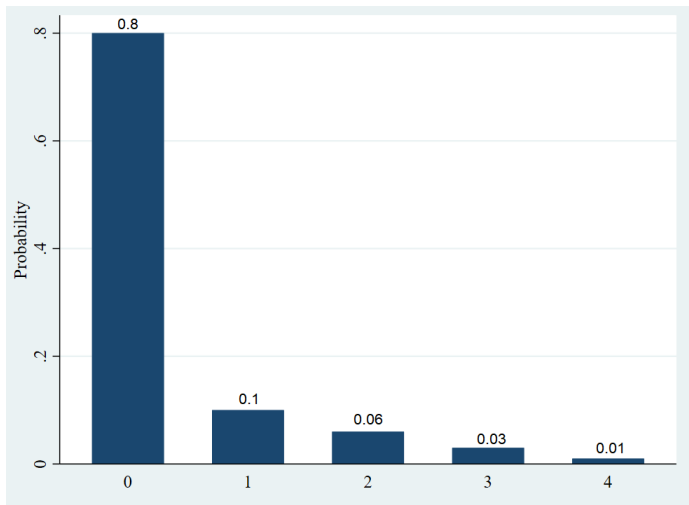
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- Notational note: both  $p(x)$  and  $f(x)$  are used for PDFs/PMFs

Example of a discrete random variable: number of wifi connection failures

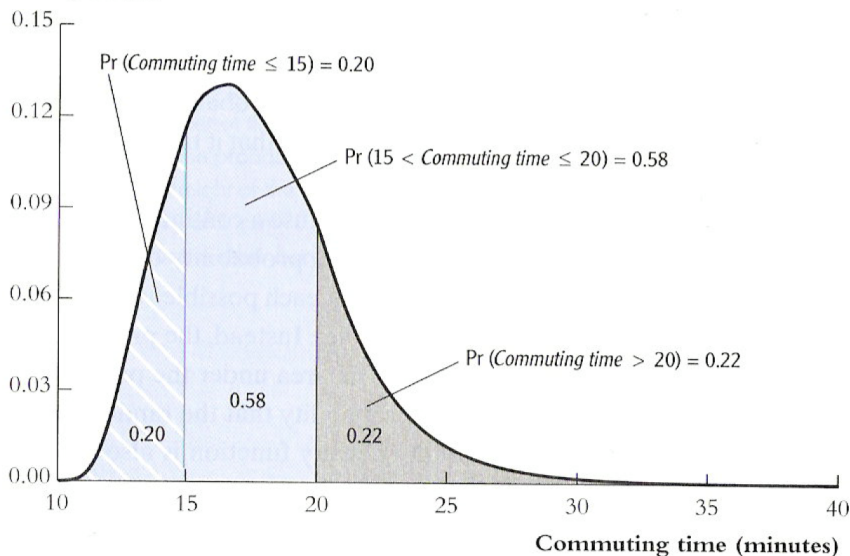
- Here's the PMF; what is the CDF?



Example of a continuous random variable: commuting time

- Here's the PDF; what is the CDF?

Probability density



# Properties of PDFs/CDFs

- Key properties of CDFs  $F(x) = Pr(X \leq x)$ :
  - Non-decreasing:  $F(x) \geq F(x')$  if  $x > x'$
  - Satisfies  $\lim_{x \rightarrow -\infty} F(x) = 0$  and  $\lim_{x \rightarrow +\infty} F(x) = 1$

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- Corresponding properties of PDFs  $f(x) = \frac{\partial}{\partial x} F(x)$ :
  - Non-negative:  $f(x) \geq 0$  for all  $x \in \mathbb{X}$
  - Satisfies  $\int_{x \in \mathbb{X}} f(x) dx = 1$
  - For PDFs:  $\sum_{x \in \mathbb{X}} p(x) = 1$



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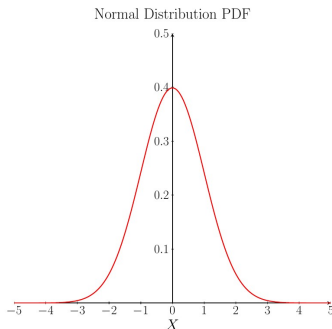
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- Note this is the *only* distribution of any binary  $X$

# Normal and Uniform Distributions

An important continuous distribution: Normal  $X \in \mathbb{R}$

- Example: the log of annual income (approximately)
- PDF:  $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2} \frac{(x-\mu)^2}{\sigma^2}\right)$  for  $x \in \mathbb{R}$  and  $\sigma > 0$
- Here  $\mu$  is the mean of  $X$  and  $\sigma^2$  is its variance (also defined soon)
- Written  $X \sim N(\mu, \sigma^2)$
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- PDF:  $f(x) = \frac{1}{b-a}$  for  $x \in (a, b)$
- Written  $X \sim U(a, b)$ ; also closed under linear transformations

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- Linked to the joint distribution by, e.g.,  $p(x) = \sum_{y \in \mathbb{Y}} p(x, y)$

# Conditional Distributions

Combining joint and marginal distributions gives us the **conditional distribution** of one random variable given another

- Intuitively, the conditional distribution  $Y|X = x$  is the distribution of  $Y$  among the sub-population with  $X = x$
- Cond'l PMF  $p(y | x) = Pr(Y = y | X = x) = \frac{Pr(Y=y, X=x)}{Pr(X=x)} = \frac{p(y,x)}{p(x)}$

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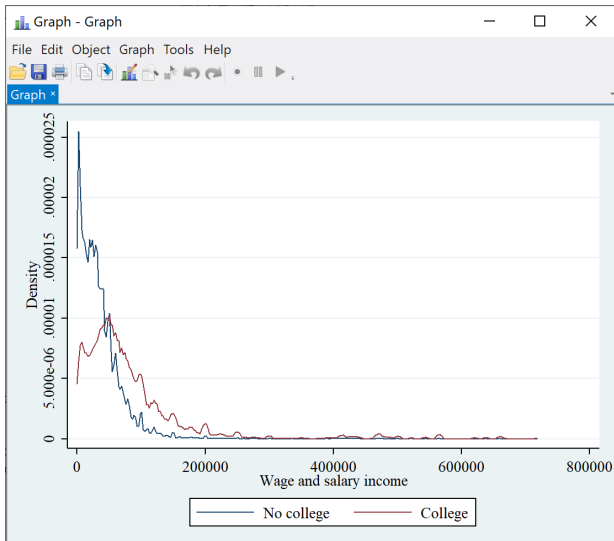
Leads immediately to *Bayes' rule*:

$$p(y | x) = p(x | y) \frac{p(y)}{p(x)}$$

# Random Variables and Probability Distributions XI

Conditional PDFs of annual income given college completion:

```
1 twoway (kdensity incwage if educ<10) (kdensity incwage if educ>=10), ///  
2 xtitle("Wage and salary income") ytitle("Density") ///  
3 legend(label(1 "No college") label(2 "College"))
```



# Independence

- An important concept in this course will be **independence**.
- Intuitively, independence says that knowing the value of  $X$  tells us nothing about the value of  $Y$
- Formally,  $X, Y$  are independent ( $X \perp\!\!\!\perp Y$ ) if the conditional PDF/PMF of  $Y|X = x$  is the same as the unconditional one:

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  - Understanding check: does this imply that  $D \perp\!\!\!\perp Y$ ?
- **Conditional independence** is defined similarly.  $Y \perp\!\!\!\perp X | W$  if

$$p(y | x, w) = p(y|w), \forall (y, x, w)$$

- Intuitively,  $X$  tells us nothing about  $Y$  once we know  $W$ .

# Multivariate Normals

An important multivariate distribution: joint normal  $\mathbf{X} \in \mathbb{R}^K$

- Parameterized by a (mean) vector  $\boldsymbol{\mu} \in \mathbb{R}^K$  and a positive-definite (variance-covariance) matrix  $\boldsymbol{\Sigma} \in \mathbb{R}^K \times \mathbb{R}^K$
- Note: In general I will be using **boldface** to indicate vectors/matrices

# Multivariate Normals

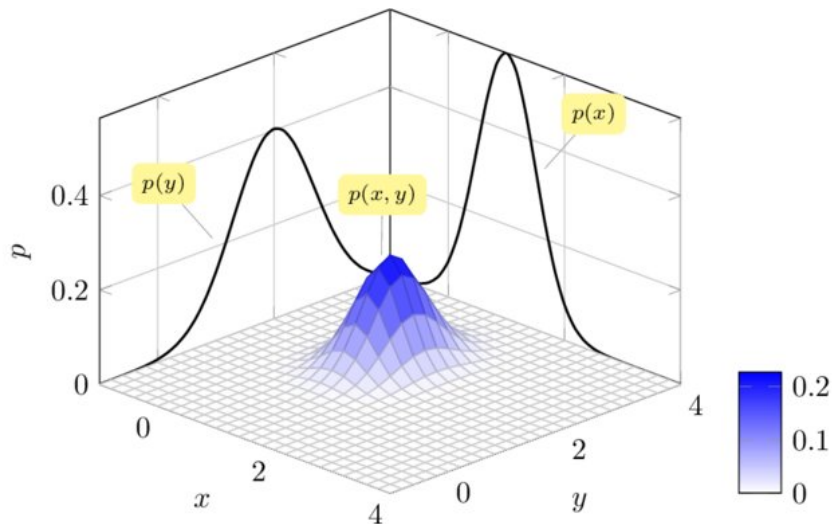
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Many useful facts; here's a few. If  $(X, Y)'$  is joint-normally distributed:

- The marginal distributions of  $X$  and  $Y$  are normal
- The conditional distributions of  $X | Y$  and  $Y | X$  are normal
- Any fixed linear combination  $aX + bY + c$  is normally distributed

# Bivariate Normal PDF



# Outline

1. Random Variables and Probability Distributions✓
2. Means and Variances
3. Identification in Experiments
4. Random Sampling and Sample Means
5. Hypothesis Testing and Inference

# Means

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- **Important fact:** The expectation operator is *linear*:

$$E[a + bX] = a + bE[X] \text{ for constants } (a, b)$$

- Easily proved from the above definitions (make sure you can!)

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- If the die is fair,  $Pr(X = 1) = \dots = Pr(X = 6) = \frac{1}{6}$
- Plugging this in, we have

$$E[X] = \frac{1}{6}(1 + \dots + 6) = 3.5$$

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This implies that  $Std(a + bX) = b \cdot Std(X)$ .

- Intuitively, if I measure income in cents, the standard deviation should be 100 times if I measure it in dollars



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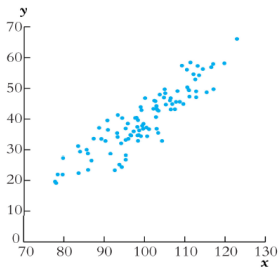
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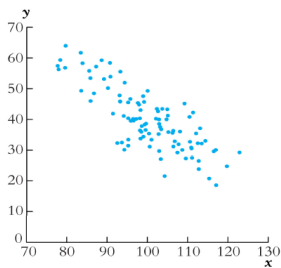
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- But not vice-versa! Independence is a stronger notion of association

# Examples of Correlations



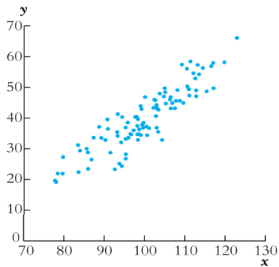
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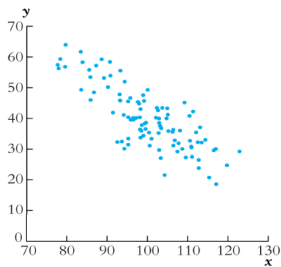
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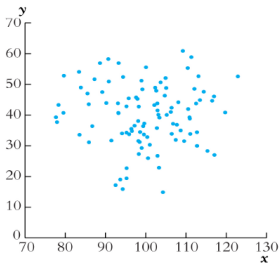
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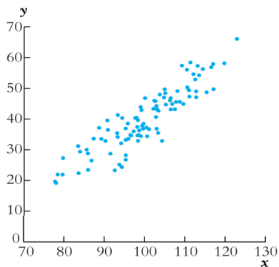


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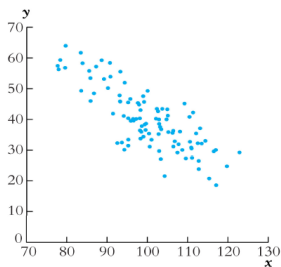


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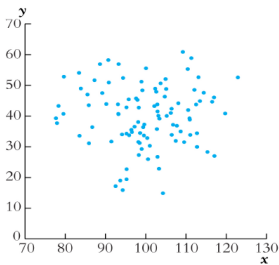
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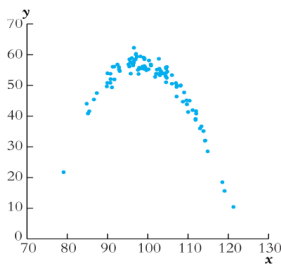
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(d) Correlation = 0.0 (quadratic)

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and  $\text{Cov}(X + Z, Y) = \text{Cov}(X, Y) + \text{Cov}(Z, Y)$

# Conditional Expectations

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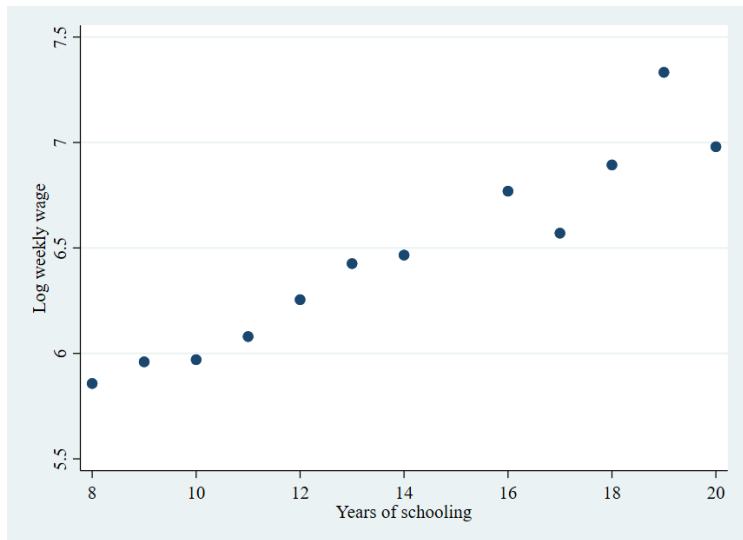
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- Conditioning on  $X$  makes functions of it constant: e.g.  
 $E[f(X) + g(X)Y | X = x] = f(x) + g(x)E[Y | X = x]$  for any  $f(\cdot)$ ,  $g(\cdot)$

# Conditional Expectation Example

CEF of (log) annual income given years of schooling



# The Big LIE

A very important result for us: the **Law of Iterated Expectations** (LIE)

Let's start with an example. Suppose I want to calculate the average height of people in the United States. The LIE says I can:

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Note that the expectation on the LHS uses  $p(y)$ , while the outer expectation on the RHS uses  $p(x)$  and the inner expectation uses  $p(y | x)$



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- Also, of course, independent  $\implies$  mean independent (but not  $\impliedby$ )



## Quick Aside on Vector/Matrix Notation

Often it will be useful to work with random vectors  $\mathbf{X} = [X_1, \dots, X_N]'$

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Expectations are elementwise: e.g.  $E \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} = \begin{bmatrix} E[X_{11}] & E[X_{12}] \\ E[X_{21}] & E[X_{22}] \end{bmatrix}$

- Define  $Var \left( \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \right) = \begin{bmatrix} Var(X_1) & Cov(X_1, X_2) \\ Cov(X_1, X_2) & Var(X_2) \end{bmatrix}$  and  
 $Cov \left( \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}, \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \right) = \begin{bmatrix} Cov(X_1, Y_1) & Cov(X_1, Y_2) \\ Cov(X_2, Y_1) & Cov(X_2, Y_2) \end{bmatrix}$ , etc

# Outline

1. Random Variables and Probability Distributions✓
2. Means and Variances ✓
3. Identification in Experiments
4. Random Sampling and Sample Means
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- Suppose that for each person we assign  $D_i$  by flipping a coin. This implies that  $D_i \perp\!\!\!\perp (Y_i(1), Y_i(0))$ . Why?



## Using Randomization

- By virtue of the experiment,  $D_i \perp\!\!\!\perp (Y_i(1), Y_i(0))$ .

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- Combining these results, we can see that

$$\underbrace{E[Y_i|D_i = 1]}_{\text{Pop mean for treated}} - \underbrace{E[Y_i|D_i = 0]}_{\text{Pop mean for control}} = \underbrace{E[Y_i(1) - Y_i(0)]}_{\text{Avg treatment effect}} = \tau$$

Thus, the difference in treated/control population means in an experiment identifies the ATE!

## Identification under Conditional Unconfoundedness

- Now suppose that  $D_i \perp\!\!\!\perp (Y_i(1), Y_i(0)) | \mathbf{X}_i$ , where  $\mathbf{X}_i$  is a vector of observable characteristic
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- Park et al (2021) study the impact of hot days ( $D_i$ ) during the school year on test scores ( $Y_i$ )
  - Note Their  $D_i$  is not binary, although we could imagine a binarized treatment, e.g.  $D_i = 1[Hotdays > 10]$
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- Does this seem reasonable to you?

## Park et al. (2021) Abstract

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Human capital generally, and cognitive skills specifically, play a crucial role in determining economic mobility and macroeconomic growth. While elevated temperatures have been shown to impair short-run cognitive performance, much less is known about whether heat exposure affects the rate of skill formation. We combine standardized achievement data for 58 countries and 12,000 US school districts with detailed weather and academic calendar information to show that the rate of learning decreases with an increase in the number of hot school days. These results provide evidence that climatic differences may contribute to differences in educational achievement both across countries and within countries by socioeconomic status and that may have important implications for the magnitude and functional form of climate damages in coupled human–natural systems.

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What is the effect on earnings of attending a selective college?
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- Essentially, they argue that once we know what colleges you applied/were admitted to, where you choose to go is effectively random
- Do you believe this? Why might this assumption go wrong?
  - Students who choose to go to selective college may still differ in family background, motivation, career plans, etc.

## Abstract

Estimates of the effect of college selectivity on earnings may be biased because elite colleges admit students, in part, based on characteristics that are related to future earnings. We matched students who applied to, and were accepted by, similar colleges to try to eliminate this bias. Using the College and Beyond data set and National Longitudinal Survey of the High School Class of 1972, we find that students who attended more selective colleges earned about the same as students of seemingly comparable ability who attended less selective schools. Children from low-income families, however, earned more if they attended selective colleges.

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- $E[Y_i(1) - Y_i(0) | \mathbf{X}_i = \mathbf{x}]$  is often called the *conditional average treatment effect*, written  $CATE(\mathbf{x})$ .

## Using Conditional Unconfoundedness (cont.)

- We showed that under conditional unconfoundedness,  $CATE(\mathbf{x}) = E[Y_i(1) - Y_i(0) | \mathbf{X}_i = \mathbf{x}]$  is identified.
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- Requires  $0 < Pr(D_i = 1|\mathbf{X}_i = \mathbf{x}) < 1$ : called an **overlap** condition
- Intuitively, we need there to be some treated and some control units for each value of  $X_i$ , in order to learn about the overall  $ATE$

# Learning about Population Means

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- We need to learn about these **estimands** from the observed sample
- Enter **statistical inference...**

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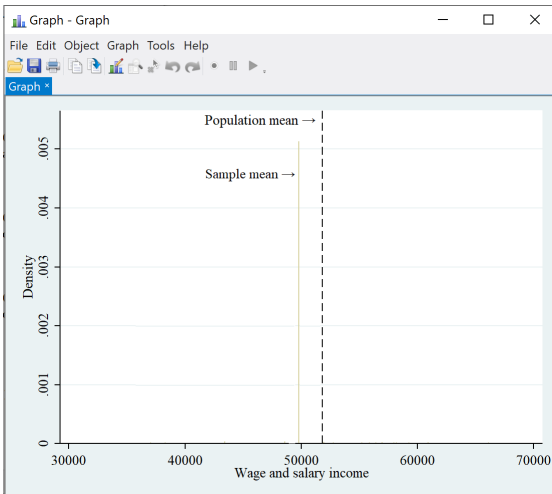
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- Equation (1) says that  $\hat{\mu}$  is *unbiased*: its average value is  $\mu$
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# Random Sampling and Sample Means

Simulating unbiasedness and consistency for estimating mean income:

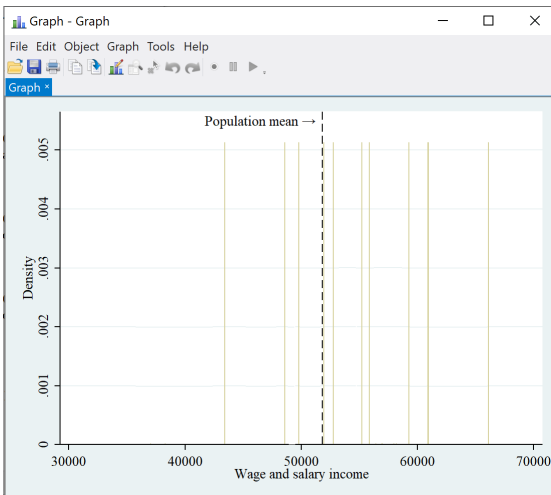
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Do-file Editor - Lecture2
File Edit View Project Tools
Lecture2 *
1  summ incwage
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3  matrix samp_means=J(50,1,.)
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13 svmat samp_means
14 hist samp_means1, xline(`incwage_mean')
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# Simulations of Random Sampling

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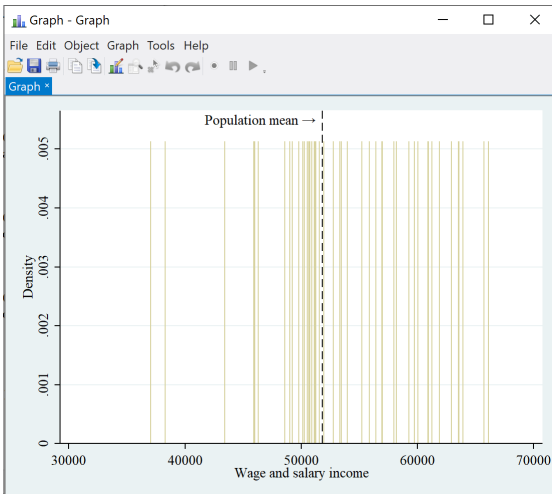
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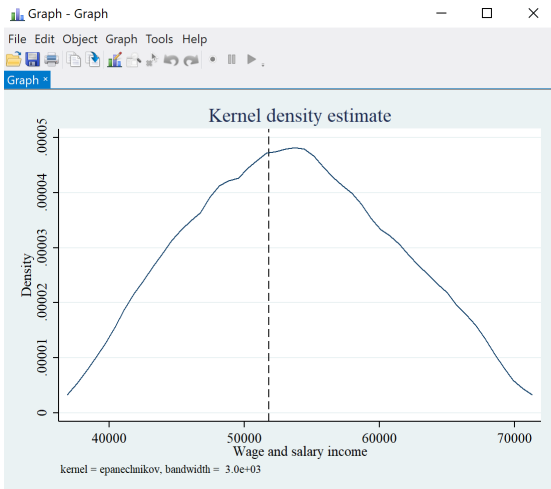
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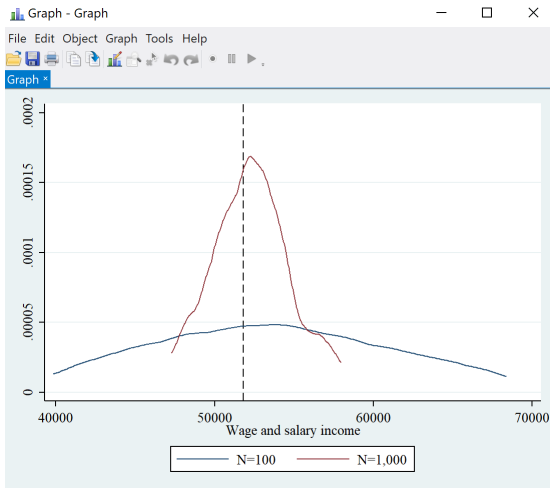
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So: two reasons why  $\hat{\mu} = \frac{1}{N} \sum_i Y_i$  is a good estimator of  $\mu = E[Y_i]$ :

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In the next chapter we'll see another nice property of  $\hat{\mu}$ : when  $N$  is large, its *distribution* is approximately normal

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- What if  $X_i$  is not discrete, or  $N_x \not\rightarrow \infty$ ? Coming soon...

# Outline

1. Random Variables and Probability Distributions ✓
2. Means and Variances ✓
3. Identification in Experiments ✓
4. Random Sampling and Sample Means ✓
5. Hypothesis Testing and Inference



# Hypothesis Testing – an Introduction

- We've shown that when  $N$  gets large, the sample mean  $\hat{\mu}$  gets close to the population mean  $\mu$
- But what does “close” mean?
- If the sample mean of income in our data is \$50,000, is it reasonable to think the population mean could be \$55,000? What about \$70,000?

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- **Hypothesis testing** helps us formalize the notion of “close.”
- It tells us whether it is likely to see a sample mean of \$50,000 if the truth is \$55,000, \$70,000, etc.

# Overview of Hypothesis Testing

- 1 Specify a **null hypothesis** that the population mean is a particular value,  $H_0 : \mu = \mu_0$ .
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  - The CI, by construction, contains the true value  $\mu$  in 95% of the realizations of the data when  $\alpha = 0.05$

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## Hypothesis Testing with Normally Distributed $\hat{\mu}$ (cont.)

- We've shown that under the null,  $H_0 : \mu = \mu_0$ ,  $\hat{t} \sim N(0, 1)$ .
- What is  $Pr(|\hat{t}| > t)$  for some  $t \geq 0$ ?

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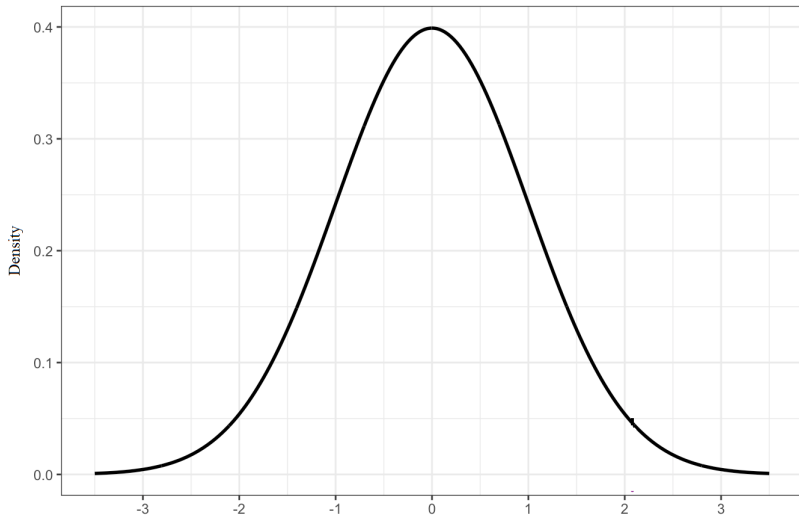
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$$\begin{aligned} p(\hat{t}) &= 1 - (\Phi(|\hat{t}|) - \Phi(-|\hat{t}|)) = 1 - \left( \Phi\left(\frac{|\hat{\mu} - \mu_0|}{\sigma/\sqrt{N}}\right) - \Phi\left(\frac{-|\hat{\mu} - \mu_0|}{\sigma/\sqrt{N}}\right) \right) \\ &= 2 \left( 1 - \Phi\left(\frac{|\hat{\mu} - \mu_0|}{\sigma/\sqrt{N}}\right) \right) \end{aligned}$$

- Intuitively,  $p$  is the probability we would see a  $|\hat{t}|$  at least this big if the null is true.

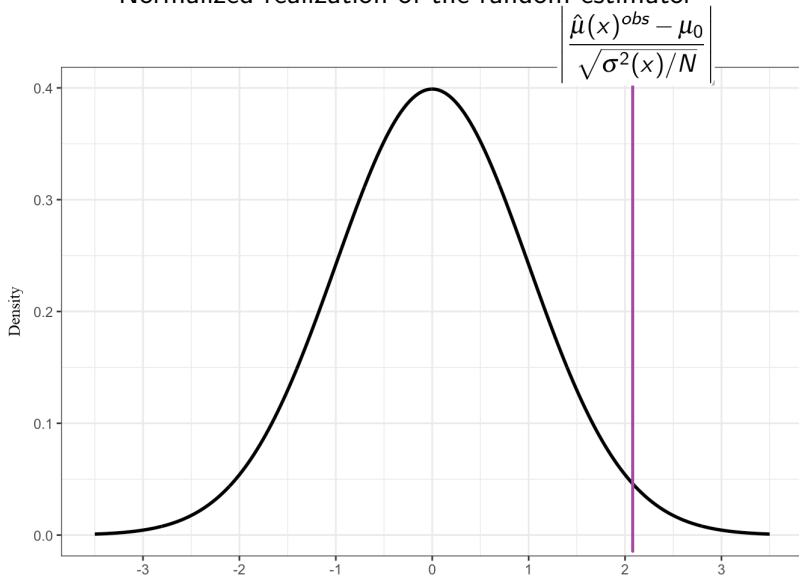
# Illustration of P-Value Construction

Standard Normal PDF (mean zero, unit std. dev.)



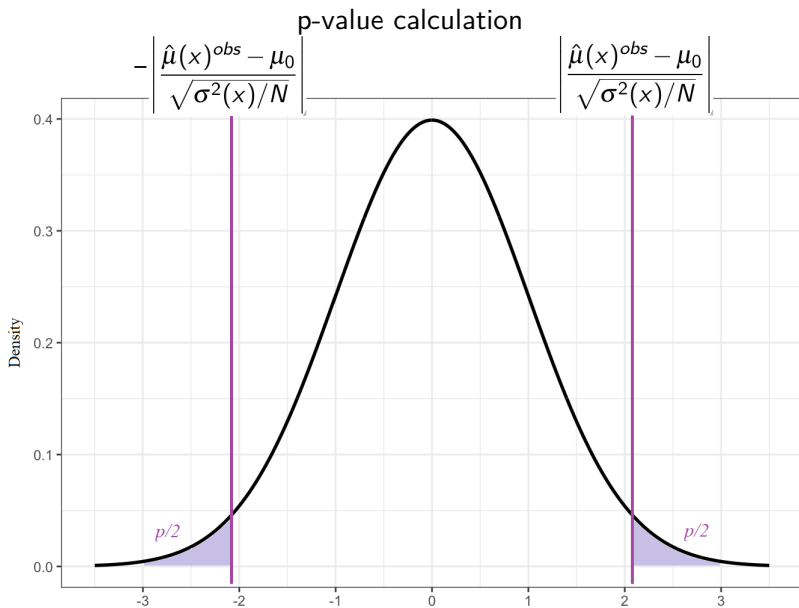
# Illustration of P-Value Construction

Normalized realization of the random estimator





# Illustration of P-Value Construction



## When Do We Reject the Null?

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$$1 - (\Phi(|\hat{t}|) - \Phi(-|\hat{t}|))$$

- It turns out that  $\Phi(1.96) - \Phi(-1.96) \approx 0.95$ . Thus,  $p < 0.05$  if and only if  $|\hat{t}| > 1.96$ . So we reject at the 5% level if  $|\hat{t}| > 1.96$ .

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- The interval  $\hat{\mu} \pm 1.96\sigma/\sqrt{N}$  is thus the 95% confidence interval (CI)
  - It has the property that  $Pr(\mu_0 \in CI) = 0.95$  when  $H_0 : \mu = \mu_0$  is true

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  - E.g. a 5% level test rejects when  $p < 0.05$ .
- The *power* of a test is the probability of correctly rejecting the null when it is false (1 - type-II error rate)
  - The power is a function of the *alternative* hypothesis. I.e., the probability that we reject  $H_0 : \mu = \mu_0$  when in fact  $\mu = \mu_A$

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  - By Bayes' rule,  $P(H_0|\text{Data}) = P(\text{Data}|H_0) * P(H_0) / P(\text{Data})$ . But to formalize this, we need to take a stand on our *prior* belief that  $H_0$  is true,  $P(H_0)$ .

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  - It's better to view the  $p$ -value as a spectrum indicating how likely is the observed data given the null.
  - Moreover,  $p$ -values can be large even if the null is false (low power!)

[nature](#) > [social selection](#) > [article](#)

Published: 26 February 2015

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**09 March 2015** This story originally asserted that “The closer to zero the *P* value gets, the greater the chance the null hypothesis is false.” *P* values do not give the probability that a null hypothesis is false, they give the probability of obtaining data at least as extreme as those observed, if the null hypothesis was true. It is by convention that smaller *P* values are interpreted as stronger evidence that the null hypothesis is false. The text has been changed to reflect this.



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  - Even when they are, why would we know their variance?!
  - Have I just been wasting your time?! No.
- We will next review powerful **asymptotic** results. These will allow us to apply similar inference tools if the sample is “large” even when  $Y_i$  is not normally distributed.