Que 1:

l_2 problem:

$$\gamma = \min \left(\frac{\|\underset{\sim}{w}\|^2}{2} + \frac{C}{2} \sum_{i=1}^{n} \xi_i^2 \right)$$

$$\text{Subj}: \; y_i [\underset{\sim}{w}^T \underset{\sim}{x}_i + w_0] \geq 1 - \xi_i \;, \; i = 1(1)n$$

(a):

<u>claim</u>: Let $(\underset{\sim}{w}^*, w_0^*, \underset{\sim}{\xi}^*)$ be the point where the optimal solution exists.

then $\underset{\sim}{\xi}^* \geq_e 0$.

<u>proof</u>: Let $\underset{\sim}{\xi}^* \not\geq_e 0$ and let $\xi_j < 0$

$$\therefore \gamma = \frac{\|\underset{\sim}{w}^*\|^2}{2} + \frac{C}{2} \sum_{i=1}^{n} \xi_i^{*2}$$

Now, consider the point $(\underset{\sim}{w}^*, \underset{\sim}{w_0}^*, \underset{\sim}{\xi}^{1*})$

where

$$\underset{\sim}{\xi}_i^{1*} = \begin{cases} \xi_i^* & i \neq j \\ 0 & i = j \end{cases}$$

<u>Subclaim</u>: Let S = convex set defined by $y_i [\underset{\sim}{w}^T \underset{\sim}{x}_i + w_0] \geq 1 - \xi_i, i=1(1)n$

then $(\underset{\sim}{w}^*, w_0^*, \underset{\sim}{\xi}^{1*}) \in S$.

proof: As: $(\underset{\sim}{w}^*, w_0^*, \underset{\sim}{\xi}^*) \in S$ (solution of the optimization problem).

$\Rightarrow y_i [\underset{\sim}{w}^{*T} \underset{\sim}{x}_i + w_0^*] \geq 1 - \xi_i^* \quad i = 1(1)n$.

$\Rightarrow y_i [\underset{\sim}{w}^{*T} \underset{\sim}{x}_i + w_0^*] \geq 1 - \xi_i^*, \; i = 1(1)n \setminus j \;\; \& \;\; y_j [\underset{\sim}{w}^{*T} \underset{\sim}{x}_j + w_0^*] \geq 1 - \xi_j^*$

$\Rightarrow y_i [\underset{\sim}{w}^{*T} \underset{\sim}{x}_i + w_0^*] \geq 1 - \xi_i^{1*}, \; i = 1(1)n \setminus j \;\; \& \;\; y_j [\underset{\sim}{w}^{*T} \underset{\sim}{x}_j + w_0^*] \geq 1$

(Since $\xi_j^* < 0$)

$\Rightarrow y_i [\underset{\sim}{w}^{*T} \underset{\sim}{x}_i + w_0^*] \geq 1 - \xi_i^{1*}, \; i = 1(1)n \setminus j \;\; \& \;\; y_j [\underset{\sim}{w}^{*T} \underset{\sim}{x}_j + w_0^*] \geq 1 + \xi_j^{1*}$

$(\xi_j^{1*} = 0)$

$\Rightarrow y_i [\underset{\sim}{w}^{*T} \underset{\sim}{x}_i + w_0^*] \geq 1 - \xi_i^{1*}, \; i = 1(1)n$

$\Rightarrow (\underset{\sim}{w}^*, w_0^*, \underset{\sim}{\xi}^{1*}) \in S$

$\gamma' =$ Value of fn at $(\underset{\sim}{W}^*, w_0^*, \underset{\sim}{\xi}'^*)$

$$\therefore \gamma' = \frac{\|\underset{\sim}{W}^*\|^2}{2} + \frac{C}{2}\sum_{i=1}^{n} (\xi_i')^2$$

$$= \frac{\|\underset{\sim}{W}^*\|^2}{2} + \frac{C}{2}\sum_{\substack{i=1 \\ i\neq j}}^{n} (\xi_i')^2 + \frac{C}{2}(\xi_j')^2$$

$$= \frac{\|\underset{\sim}{W}^*\|^2}{2} + \frac{C}{2}\sum_{\substack{i=1 \\ i\neq j}}^{n} (\xi_i)^2 + 0$$

$$= \frac{\|\underset{\sim}{W}^*\|^2}{2} + \frac{C}{2}\sum_{\substack{i=1 \\ i\neq j}}^{n} (\xi_i)^2 + \frac{C}{2}\xi_j^2 - \frac{C}{2}\xi_j^2$$

$$\Rightarrow \gamma' = \gamma - \frac{C}{2}\xi_j^2$$

$\therefore$ γ is not the optimal solution as $\gamma' < \gamma$ & γ' is a feasible value

$\therefore$ This contradicts that $\xi_j < 0 \Rightarrow \underset{\sim}{\xi} \geq_e 0$

$\therefore$ We do not need to explicitly provide the constraint $\xi_i \geq 0 \ \forall i \in 1..n$

(b) $$\min \left(\frac{\|\underset{\sim}{W}\|^2}{2} + \frac{C}{2}\sum_{i=1}^{n} \xi_i^2 \right)$$

Subject to:
$y_i [\underset{\sim}{W}^T \underset{\sim}{x}_i + w_0] \geq 1 - \xi_i$, $i = 1..n$

$$\text{Lagrangian} = \ell(\underset{\sim}{w}, w_0, \underset{\sim}{\xi}, \underset{\sim}{\alpha}) = \frac{\|\underset{\sim}{W}\|^2}{2} + \frac{C}{2}\sum_{i=1}^{n} \xi_i^2 - \sum_{i=1}^{n} \alpha_i \left[y_i(\underset{\sim}{w}^T \underset{\sim}{x}_i + w_0) - 1 + \xi_i \right]$$

(C) KKT Stationary Conditions (Both necessary & sufficient):

① stationarity $\Rightarrow \frac{\partial l}{\partial \underset{\sim}{w}} = 0, \quad \frac{\partial l}{\partial w_0} = 0, \quad \frac{\partial l}{\partial \xi_i} = 0 \qquad i = 1..n$

② Dual - feasibility $\Rightarrow \bar{\alpha}_i \geq 0$

③ complementary slackness $\Rightarrow \alpha_i \left[y_i (\underset{\sim}{w}^T \underset{\sim}{x}_i + w_0) - 1 + \xi_i \right] = 0$

Using the KKT conditions:

$$\frac{\partial l}{\partial \underset{\sim}{w}} = 0 \Rightarrow \tilde{w} - \sum_{i=1}^{n} \alpha_i y_i \underset{\sim}{x}_i = 0 \Rightarrow \boxed{\tilde{w} = \sum_{i=1}^{n} \alpha_i y_i \underset{\sim}{x}_i}$$

$$\frac{\partial l}{\partial w_0} = 0 \Rightarrow \boxed{\sum_{i=1}^{n} \alpha_i y_i = 0}$$

$$\frac{\partial l}{\partial \xi_i} = 0 \Rightarrow C \xi_i - \alpha_i = 0 \Rightarrow \boxed{\xi_i = \frac{\alpha_i}{C}} \longrightarrow$$

since, $\alpha_i \geq 0 \Rightarrow \xi_i \geq 0$ (this re-verifies the claim in part 1).

$$\boxed{\bar{\alpha}_i \geq 0}$$

$$l(\bar{\alpha}) = \frac{1}{2} \left\langle \sum_{i=1}^{n} \alpha_i y_i \underset{\sim}{x}_i, \sum_{j=1}^{n} \alpha_j y_j \underset{\sim}{x}_j \right\rangle + \frac{1}{2C} \sum_{i=1}^{n} \alpha_i^2 + \sum_{i=1}^{n} \alpha_i - \sum \frac{\alpha_i^2}{C} - \sum_{i=1}^{n} \alpha_i y_i \underset{\sim}{w}^T x_i$$

$$l(\bar{\alpha}) = \sum_{i=1}^{n} \alpha_i - \frac{1}{2C} \sum_{i=1}^{n} \alpha_i^2 - \frac{1}{2} \sum_{i,j=1}^{n} \alpha_i y_i \alpha_j y_j \langle \bar{x}_i, \bar{x}_j \rangle$$

$$\text{dual} = \max_{\substack{\alpha_i \geq 0 \\ \sum_{i=1}^{n} \alpha_i y_i = 0}} (l(\bar{\alpha})) = \max_{\substack{\alpha_i \geq 0 \\ \sum_{i=1}^{n} \alpha_i y_i = 0}} \left(\sum_{i=1}^{n} \alpha_i - \frac{1}{2C} \sum_{i=1}^{n} \alpha_i^2 - \frac{1}{2} \sum_{i,j=1}^{n} \alpha_i y_i \alpha_j y_j \langle \bar{x}_i, \bar{x}_j \rangle \right)$$

We maximize the dual & using standard convex optimizations find the $\underset{\sim}{\alpha}$ which leads to optimal solution.

(d)

Calculation of $\underset{\sim}{w} = \sum_{i=1}^{n} \alpha_i y_i \underset{\sim}{x_i}$

Calculation of $\xi_i = \frac{\alpha_i}{C}$

Calculation of w_0 :

We use complementarity slackness to calculate w_0, once $\underset{\sim}{w}$ and $\underset{\sim}{\xi}$ are found.

$$\alpha_i [y_i w^T \underset{\sim}{x_i} + 1 - \xi_i] = -\alpha_i y_i w_0 . \quad \text{for } i = 1..n$$

We can solve these system of linear equations to find w_0.

The final value of w_0 is the average of these w_{0i} obtained from each equation.

Que-2 :

ν - SVM problem,

margin planes = $w^T x + w_0 = \pm\rho$.

$$\gamma = \min_{\underset{\sim}{w}, w_0, \xi, \rho} \frac{\|\tilde{w}\|^2}{2} - \nu\rho + \frac{1}{n}\sum_{i=1}^{n} \xi_i$$

Subject to: $y_i [\underset{\sim}{w}^T \underset{\sim}{x}_i + w_0] \geq \rho - \xi_i \qquad i = 1..n$

$\xi_i \geq 0 \qquad i = 1..n$

$\rho \geq 0$

Setting up the lagrangian

$$l(\underset{\sim}{w}, w_0, \xi, \rho, \underset{\sim}{\alpha}, \underset{\sim}{\beta}, \gamma) = \frac{\|\tilde{w}\|^2}{2} - \nu\rho + \frac{1}{n}\sum_{i=1}^{n}\xi_i - \sum_{i=1}^{n}\alpha_i\left[y_i(w^T x_i + w_0) + \xi_i - \rho\right]$$

$$- \sum_{i=1}^{n}\beta_i\xi_i - \gamma\rho.$$

KKT conditions :

① stationarity : $\frac{\partial l}{\partial \underset{\sim}{w}} = 0$, $\frac{\partial l}{\partial w_0} = 0$, $\frac{\partial l}{\partial \xi_i} = 0$, $\frac{\partial l}{\partial \rho} = 0$. for $i = 1..n$

② Dual feasibility : $\alpha_i \geq 0$, $\beta_i \geq 0$, $\gamma \geq 0$ for $i = 1..n$

③ Complimentarity slackness:

$$\left.\begin{aligned} &\alpha_i\left[y_i(w^T x_i + w_0) + \xi_i - \rho\right] = 0 \\ &\beta_i \xi_i = 0 \\ &\gamma\rho = 0 \end{aligned}\right\} \; i = 1..n$$

Using stationarity:

$$\frac{\partial l}{\partial \underset{\sim}{w}} = 0 \Rightarrow \underset{\sim}{w} - \sum_{i=1}^{n} \alpha_i y_i \underset{\sim}{x_i} = 0 \Rightarrow \boxed{\underset{\sim}{w} = \sum_{i=1}^{n} \alpha_i y_i \underset{\sim}{x_i}} \quad — ①$$

$$\frac{\partial l}{\partial w_0} = 0 \Rightarrow \boxed{\sum \alpha_i y_i = 0} \quad — ②$$

$$\frac{\partial l}{\partial \rho} = 0 \Rightarrow \boxed{r + \gamma = \sum_{i=1}^{n} \alpha_i} \quad — ③$$

$$\frac{\partial l}{\partial \xi_i} = 0 \Rightarrow \boxed{\alpha_i + \beta_i = \frac{1}{n}} \quad — ③ \text{ for } i = 1..n$$

$$\Rightarrow \alpha_i \in [0, 1/n], \ \beta_i \in [0, 1/n]$$

Using ①, ②, ③, ④ in l, we have

$$l = \frac{1}{2} \left\langle \sum_{i=1}^{n} \alpha_i y_i \underset{\sim}{x_i}, \sum_{i=1}^{n} \alpha_i y_i \underset{\sim}{x_i} \right\rangle + \rho\Big(\underbrace{\sum_{i=1}^{n} \alpha_i - r - \gamma}_{0}\Big)$$

$$+ \underbrace{\cancel{\frac{1}{n} \sum_{i=1}^{n} \xi_i - \sum_{i=1}^{n} \alpha_i \xi_i - \sum_{i=1}^{n} \beta_i \xi_i}}_{0} - \sum_{i=1}^{n} \alpha_i y_i \left\langle \sum_{j=1}^{n} \alpha_j y_j \underset{\sim}{x_j}, \underset{\sim}{x_i} \right\rangle$$

$$= -\frac{1}{2} \left\langle \sum_{i=1}^{n} \alpha_i y_i \underset{\sim}{x_i}, \sum_{i=1}^{n} \alpha_i y_i \underset{\sim}{x_i} \right\rangle$$

$$\boxed{\therefore \text{dual} = \max_{\substack{\alpha_i \geq 0 \\ r + \gamma = \sum_{i=1}^{n} \alpha_i \\ \sum \alpha_i y_i = 0 \\ \alpha_i + \beta_i = \frac{1}{n} \\ \gamma \geq 0}} \left(-\frac{1}{2} \sum_{i,j=1}^{n} \alpha_i \alpha_j y_i y_j \langle \underset{\sim}{x_i}, \underset{\sim}{x_j} \rangle \right)}$$

Once $\underset{\sim}{\alpha}$ has been found,

calculation of $\underset{\sim}{w} = \sum_{i=1}^{n} \alpha_i y_i \underset{\sim}{x_i}$.

Calculation of w_0, ρ = consider the system of linear equations in w_0, ρ =

$$\alpha_i [y_i (w^T x_i + w_0) + \xi_i - \rho] = 0$$

multiplying them by β_i, we have

$$\alpha_i \left[\beta_i y_i (w^T x_i + w_0) + \beta_i \xi_i - \beta_i \rho\right] = 0 \qquad \left[\beta_i = 1/n - \alpha_i\right]$$

$$(\text{using } \beta_i \xi_i = 0)$$

$$\alpha_i \beta_i y_i [w^T x_i + w_0) - \beta_i \rho = 0$$

We have a system of n linear eqn in w_0, ρ, we can pick any two of them & then solve for w_0, ρ. The value of w_0, ρ to be reported is ~~are~~ the average of these ${}^nC_2$ values.

Que 4(a):

To show $\theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_K x_K \in S$, where $x_i \in S$, $i = 1..K$, $\sum_{i=1}^{K} \theta_i = 1$, $\theta_i \geq 0$

Let us try to prove this using induction over K.

$K = 2$ (Base Case):

If $x_1, x_2 \in S \Rightarrow \theta_1 x_1 + \theta_2 x_2 \in S$ where $\sum_{i=1}^{2} \theta_i = 1$, $\theta_i \geq 0$.

(trivial from the definition of the convex sets).

Let us assume that this holds for $K = n$ i.e.

for all $x_1, x_2, \ldots, x_n \in S \Rightarrow \theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_n x_n \in S$, where $\sum_{i=1}^{n} \theta_i = 1$, $\theta_i \geq 0$

Let us prove this for $K = n+1$,

Let us assume $x_1, x_2, \ldots, x_{n+1} \in S$ & $\sum_{i=1}^{n+1} \theta_i = 1$, $\theta_i \geq 0$.

$$\Rightarrow \vec{a} = \theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_n x_n + \theta_{n+1} x_{n+1}.$$
$$= \theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_{n-1} x_{n-1} + (\theta_n + \theta_{n+1}) \left(\frac{\theta_n}{\theta_n + \theta_{n+1}} x_n + \frac{\theta_{n+1} x_{n+1}}{\theta_n + \theta_{n+1}} \right)$$

[Since, from induction, $\frac{\theta_n x_n}{\theta_n + \theta_{n+1}} + \frac{\theta_{n+1}}{\theta_n + \theta_{n+1}} x_{n+1} \in S$; here sum of coefficients $= 1$

$\therefore$ Let $\frac{\theta_n x_n}{\theta_n + \theta_{n+1}} + \frac{\theta_{n+1}}{\theta_n + \theta_{n+1}} x_{n+1} = y_n$].

$$\vec{a} = \theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_{n-1} x_{n-1} + (\theta_n + \theta_{n+1}) y_n.$$

where $x_1, x_2, \ldots, x_{n-1}, y_n \in S$.

~~Do the~~

Do the following transformation

$x_i = y_i$ for $i = 1..(n-1)$

$\theta_i = \alpha_i$ for $i = 1..(n-1)$

$\alpha_n = \theta_n + \theta_{n+1}$

$\therefore \vec{a} = \alpha_1 y_1 + \alpha_2 y_2 + \dots + \alpha_{n-1} y_{n-1} + \alpha_n y_n$

where $y_i \in S$ for $i = 1..n$, $\sum_{i=1}^{n} \alpha_i = \sum_{i=1}^{n+1} \theta_i = 1$, $\alpha_i \geq 0$

$\therefore$ By assumed case i.e. the condn holds true for $K = n$ we have

$\vec{a} \in S$.

$\therefore$ it holds true for $K = n+1$

$\therefore$ By Induction, we can show that

$\forall K$, $\theta_1 x_1 + \theta_2 x_2 + \dots + \theta_K x_K \in S$, where $x_1, x_2, \dots, x_K \in S$, $\sum_{i=1}^{K} \theta_i = 1$, $\theta_i \geq 0$ for $i = 1..n$

Que 4(b):

$S = \{ X = (x_1, x_2) \in \mathbb{R}^{+2} \mid x_1 x_2 \geq 1 \}$

Let $X_1, X_2 \in S$ $\Rightarrow$ $X_1 = (x_{11}, x_{12})$, $x_{11} * x_{12} \geq 1$; $x_{11}, x_{12} > 0$

$X_2 = (x_{21}, x_{22})$, $x_{21} * x_{22} \geq 1$; $x_{21}, x_{22} > 0$

Consider, $\alpha = \theta X_1 + (1-\theta) X_2 = (\theta x_{11} + (1-\theta) x_{21}, \theta x_{12} + (1-\theta) x_{22})$

$= (\alpha_1, \alpha_2)$

claim $\alpha_1 * \alpha_2 \geq 1$

$\alpha_1 * \alpha_2 = (\theta x_{11} + (1-\theta) x_{21}) * (\theta x_{12} + (1-\theta) x_{22})$

$= \theta^2 x_{11} * x_{12} + (1-\theta)^2 x_{21} * x_{22} + \theta(1-\theta) x_{11} * x_{22}$
$+ \theta(1-\theta) x_{21} * x_{12}$

$\geq \theta^2 + (1-\theta)^2 + \theta(1-\theta) x_{11} * x_{22} + \theta(1-\theta) x_{21} * x_{12}$

(as $x_{11} * x_{12} \geq 1$, $x_{21} * x_{22} \geq 1$)
$\Rightarrow x_{11} \geq \frac{1}{x_{12}}$, $x_{21} \geq \frac{1}{x_{22}}$

$\geq \theta^2 + (1-\theta)^2 + \theta(1-\theta) \left(\frac{x_{22}}{x_{12}} + \frac{x_{12}}{x_{22}} \right)$

$\geq \theta^2 + (1-\theta)^2 + \theta(1-\theta) \times 2$ (using AM-GM, $x_{12}, x_{22} \in \mathbb{R}^+$)

$\geq (\theta + 1 - \theta)^2 \geq 1$ $\therefore$ $\alpha \in S$

$\Rightarrow$ $\boxed{S \text{ is a convex Set}}$

Que 4(b): (continue).

$$S' = \{X = (x_1, \dots, x_n) \in \mathbb{R}^{+n} \mid \prod_{i=1}^{n} x_i \geq 1\}.$$

Let $X_1, X_2 \in S'$

$$X_1 = (x_{11}, \dots, x_{1n}), \quad \prod_{i=1}^{n} x_{1i} \geq 1, \quad x_{1i} \geq 0 \text{ for } i \in 1..n$$

$$X_2 = (x_{21}, \dots, x_{2n}), \quad \prod_{i=1}^{n} x_{2i} \geq 1, \quad x_{2i} \geq 0 \text{ for } i \in 1..n.$$

Let $\alpha = \theta X_1 + (1-\theta) X_2$.

$$\alpha = \left(\theta x_{11} + (1-\theta) x_{21}, \ \theta x_{12} + (1-\theta) x_{22}, \dots, \theta x_{1n} + (1-\theta) x_{2n}\right)$$

$$= (\alpha_1, \alpha_2, \dots, \alpha_n)$$

$$\prod_{i=1}^{n} \alpha_i = \prod_{i=1}^{n} (\theta x_{1i} + (1-\theta) x_{2i}) \geq \prod_{i=1}^{n} x_{1i}^{\theta} \, x_{2i}^{(1-\theta)}$$

using weighted AM-GM inequality, $x_{1i}, x_{2i} \geq 0$, $\theta + 1 - \theta = 1$.

$$\geq \left(\prod_{i=1}^{n} x_{1i}\right)^{\theta} \left(\prod_{i=1}^{n} x_{2i}\right)^{1-\theta} \geq 1.$$

Que 4c:

~~Lemma~~ claim used: If A & B are convex sets $\Rightarrow A \cap B$ is convex. — (I)

Forward Implication:

S is a convex set,

Anyline is a convex set

$\therefore S \cap$ Any line $=$ convex set (using ~~lem~~ claim (I))

Reverse Implication:

Lets say that intersection of S with any line is convex.

consider any 2 points $x_1, x_2 \in S$.

Let $l =$ line through x_1, x_2 $l = \{x \mid x = \theta x_1 + (1-\theta)x_2\}$

$A = S \cap l =$ convex $\therefore x_1, x_2 \in S \cap l \Rightarrow x_1, x_2 \in A$

But because ~~x_1, x_2~~ A is convex, $\therefore \theta x_1 + (1-\theta)x_2 \in$ for $\theta \geq 0$,

~~But θx_1~~ $\Rightarrow \theta x_1 + (1-\theta)x_2 \in A$ for $\theta \in [0,1]$

$\Rightarrow \theta x_1 + (1-\theta)x_2 \in S$ (as $A \subseteq S$)

$\therefore$ if $x_1, x_2 \in S \Rightarrow \theta x_1 + (1-\theta)x_2 \in S$ for $\theta \in [0,1]$.

$\Rightarrow S$ is convex.

---

claim: If A & B are affine sets, then $A \cap B$ is affine

Let $C = A \cap B$, Let $x_1, x_2 \in C \Rightarrow x_1, x_2 \in A \Rightarrow L(x_1, x_2) \in A$ — (1)

line through x_1, x_2; A is affine

$\Rightarrow x_1, x_2 \in B \Rightarrow L(x_1, x_2) \in B$ — (2)

using (1) & (2) $\Rightarrow L(x_1, x_2) \in A \cap B$

$\Rightarrow L(x_1, x_2) \in C$.

$\therefore$ If $x_1, x_2 \in C \Rightarrow L(x_1, x_2) \in C$.

$\therefore C$ is affine. $\Rightarrow A \cap B$ is affine.

Que 4c) :

Forward implication:

- S is affine
- Any line is an affine set

$\therefore$ $S \cap$ Any line = affine.

Backward implication :

if intersection of S with any line is affine $\Rightarrow$ S is affine.

Let $x_1, x_2 \in S$ and $l = L(x_1, x_2)$

$\therefore$ $A = S \cap l$ $\Rightarrow$ $x_1, x_2 \in A$ as $x_1, x_2 \in S$ & $x_1, x_2 \in l$

Since A is affine (using preassumptions). $\Rightarrow$ If $x_1, x_2 \in A \Rightarrow L(x_1, x_2) \in A$ $\Rightarrow L(x_1, x_2) \in S$

$\Rightarrow$ $\boxed{\text{If } x_1, x_2 \in S \Rightarrow L(x_1, x_2) \in S}$ $\Rightarrow$ S is affine.

Que 4d) :

convex hull of set S = $\operatorname{cnvsh}[S] = \{ \theta_1 \bar{x}_1 + \theta_2 \bar{x}_2 + \dots + \theta_K \bar{x}_K \mid \bar{x}_i \in S,\ i = 1..K,\ \sum_{i=1}^{K} \theta_i = 1,\ \theta_i \geq 0 \}$

i.e. convex hull of S contains all convex combinations of elements of S

claim 1 : ① Cnvxh(S) is Convex.

Let $y_1, y_2 \in \operatorname{cnvxh}(S)$

$$\Rightarrow y_1 = \theta_1 \tilde{x}_1 + \theta_2 \tilde{x}_2 + \dots + \theta_{K_1} \tilde{x}_{K_1} \mid \tilde{x}_i \in S,\ i = 1..K_1,\ \sum_{i=1}^{K_1} \theta_i = 1,\ \theta_i \geq 0 \ \ i = 1..K_1 \}$$

$$\Rightarrow y_2 = \theta_1' \tilde{x}_1' + \theta_2' \tilde{x}_2' + \dots + \theta_{K_2}' \tilde{x}_{K_2}' \mid \tilde{x}_i' \in S,\ i = 1..K_2,\ \sum_{i=1}^{K_2} \theta_i' = 1,\ \theta_i' \geq 0 \ \ i = 1..K_2 \}$$

Consider:

$$\tilde{z} = \alpha \tilde{y}_1 + (1-\alpha)\tilde{y}_2 \quad , \alpha \geq 0$$

$$= \alpha \sum_{i=1}^{K_1} \theta_1 \tilde{x}_1 + (1-\alpha) \sum_{i=1}^{K_2} \theta_1' \tilde{x}_1'$$

$$= \sum_{i=1}^{K_1} \alpha \theta_i \tilde{x}_i + \sum_{i=1}^{K_2} (1-\alpha)\theta_i' \tilde{x}_i'$$

$$= \sum_{i=1}^{K_1+K_2} \theta_i'' \tilde{x}_i'' \quad \text{where}$$

$$\theta_i'' = \begin{cases} \alpha \theta_i & i \leq K_1 \\ (1-\alpha)\theta_i' & K_1 \leq i \leq K_2 \end{cases} , \quad \tilde{x}_i'' = \begin{cases} \tilde{x}_1 & i \leq K_1 \\ \tilde{x}_1' & K_1 < i \leq K_2 \end{cases}$$

$$\therefore \tilde{z} = \sum_{i=1}^{K_1+K_2} \theta_i'' \tilde{x}_i'' \quad , \tilde{x}_i'' \in S \text{ for } i = 1..(K_1+K_2) ,$$

$$\sum_{i=1}^{K_1+K_2} \theta_i'' = \sum_{j=1}^{K_1} \alpha \theta_j + \sum_{k=1}^{K_2} (1-\alpha)\theta_k'$$

$$= \alpha \sum_{j=1}^{K_1} \theta_j + (1-\alpha) \sum_{j=1}^{K_2} \theta_j'$$

$$= \alpha + 1 - \alpha = 1.$$

$\therefore \tilde{z} \in \text{conv}[S]$ by its definition.

Claim 2: Let A = a convex set which containts S, then A contains Convsh[S].

Let $y \in$ convsh[S] $\Rightarrow$ By definition of convsh[S],

$$y = \sum_{i=1}^{K} \theta_i \tilde{x}_i \text{ for some } K, \quad \theta_i \geq 0, \quad \sum_{i=1}^{K} \theta_i = 1, \quad \tilde{x}_i \in S \text{ for } i = 1..K.$$

Now, $A \supseteq S \Rightarrow$ if $\tilde{x}_i \in S$ for $i = 1..K \Rightarrow \tilde{x}_i \in A$ for $i = 1..K$.

$$\therefore y = \sum_{i=1}^{K} \theta_i \tilde{x}_i \text{ for some } K, \ \theta_i \geq 0, \ \sum_{i=1}^{K} \theta_i = 1, \ \tilde{x}_i \in A \text{ for } i = 1..K$$

$\therefore y \in A$ (as A is convex, proved in Que 4(a))

$\therefore$ Convsh[S] $\subseteq$ any convex set which containts S.

$\therefore$ from claim 1 and claim 2

Intersection of all convex sets which contain S = Convsh[S] $\bigcap_{i=1}^{\infty} S_i$ where S_i = a convex set that contains S

= Cnvsh[S].

$\Rightarrow$ Intersection of all convex sets that contain S is the convex hull of S.

Using similar arguments, we can claim that

1. affine hull is an affine set on S
2. Any affine set on S contains affine hull of S.
   (using similar proof as above except we ignore $\theta_i \geq 0$)

~~1. conic hull is an conic affine set on S,~~
~~2. Any conic~~

Similarly for conic hulls.

Que 4(d):

In the case of conic hull, any point $x \in$ conhil hull $= \sum \theta_i x_i$ where $\theta_i \geq 0$
Thus, we can argue similarly as for convex hull apart from definition of the combination of points to form a conic hull changes.
Similary in case of affine hull, if $x \in$ affine hull $\Rightarrow x = \sum \theta_i x_i$ where $\theta_i \in \mathbb{R}$ and $\sum \theta_i = 1$.

Bascically, the ~~proof lies~~ important part of the proof lies in the fact that if $x \in S$ then $x \in A$ which would hold for all cases.

Que 4(e):

$S = \{x \mid \|\tilde{x} - \tilde{a}\|_2 \leq \theta \|\tilde{x} - \tilde{b}\|_2\}$, a, b are fixed points, $\theta \in [0,1]$.

Let $\tilde{x}$ be $d \times 1$ vector.

$\therefore \|\tilde{x} - \tilde{a}\|_2 \leq \theta \|\tilde{x} - \tilde{b}\|_2 \Rightarrow \|\tilde{x} - \tilde{a}\|_2^2 \leq \theta^2 \|\tilde{x} - \tilde{b}\|_2^2$

$$\Rightarrow \sum_{i=1}^{d} (x_i - a_i)^2 \leq \theta^2 \sum_{i=1}^{d} (x_i - b_i)^2$$

$$\Rightarrow \sum_{i=1}^{d} (x_i^2 + a_i^2 - 2a_ix_i) \leq \theta^2 \sum_{i=1}^{d} (x_i^2 + b_i^2 - 2b_ix_i)$$

$$\Rightarrow \sum_{i=1}^{d} \left(x_i^2(1-\theta^2) - 2x_i(a_i - \theta^2 b_i) + a_i^2 - \theta^2 b_i^2\right) \leq 0$$

$$\Rightarrow \sum_{i=1}^{d} \left(x_i^2 - 2x_i(a_i - \theta^2 b_i)/(1-\theta^2) + (a_i^2 - \theta^2 b_i^2)/(1-\theta^2)\right) \leq 0$$

[$1-\theta^2 \geq 0$ as $\theta \in [0,1]$, I will handle $\theta = 0, 1$ later on, now $\theta \in (0,1)$]

$$\Rightarrow \sum_{i=1}^{d} \left(x_i^2 - 2x_i\left(\frac{a_i - \theta^2 b_i}{1-\theta^2}\right) + \frac{(a_i - \theta^2 b_i)^2}{(1-\theta^2)^2} - \frac{(a_i - \theta^2 b_i)^2}{(1-\theta^2)^2} + \frac{a_i^2 - \theta^2 b_i^2}{1-\theta^2}\right) \leq 0$$

$$\Rightarrow \sum_{i=1}^{d} \left(\left(x_i - \frac{a_i - \theta^2 b_i}{(1-\theta^2)}\right)^2 - \frac{\cancel{a_i^2} - a_i^2\theta^2 - \theta^2 b_i^2 + \cancel{\theta^4 b_i^2} - \cancel{a_i^2} - \cancel{\theta^4 b_i^2} + 2a_ib_i\theta^2}{(1-\theta^2)^2}\right) \leq 0$$

$$\Rightarrow \sum_{i=1}^{d} \left(\left(x_i - \frac{a_i - \theta^2 b_i}{(1-\theta^2)}\right)^2 - \frac{(a_i - b_i)^2\theta^2}{(1-\theta^2)^2}\right) \leq 0$$

$$\Rightarrow \sum_{i=1}^{d} \left(x_i - \frac{a_i - \theta^2 b_i}{(1-\theta^2)} \right)^2 \leq \sum_{i=1}^{d} \frac{(a_i - b_i)^2 \theta^2}{(1-\theta^2)^2}.$$

This is a sphere centered at $\frac{\vec{a} - \theta^2 \vec{b}}{(1-\theta^2)}$ with raidius =

$$\frac{\theta}{1-\theta^2} \|\vec{a} - \vec{b}\| = \sqrt[2]{\sum_{i=1}^{d} \frac{(a_i - b_i)^2 \theta^2}{(1-\theta^2)^2}}$$

& sphere is a convex set (trivial)

$\therefore$ S defines a convex set.

Consider trivial cases, $\theta = 0$

$\|x - a\|_2 \leq 0 \Rightarrow x = \tilde{a} \Rightarrow S = \{\tilde{a}\}$ and a singleton set is a convex set.

$\theta = 1$

$$\Rightarrow \|x - a\|_2 \leq \|x - b\|_2$$

$$\Rightarrow (x-a)^T (x-a) \leq (x-b)^T (x-b)$$

$$\cancel{x^T x} - a^T x - x^T a + a^T a \leq \cancel{x^T x} - b^T x - x^T b + b^T b.$$

$$2(b-a)^T x \leq b^T b - a^T a.$$

$\Rightarrow$ Region or set $S =$ half-~~hyperpt~~space. which is a convex set.