Closure properties:

Domain operation

(D, o) is closed

iff

$\boxed{\forall L_1, L_2 \in D \text{ such that } L_1 o L_2 \in D}$

(D, o) is not closed

iff

$\boxed{\exists L_1, L_2 \in D \text{ such that } L_1 o L_2 \notin D}$

D :

- set of languages
- set of finite languages
- set of infinite "
- " " Regular "
- " " DCFLs
- " " CFLs
- " " CSLs
- ⋮

O :

- $\cup$
- $\cap$
- Complement
- Difference
- Reversal
- Concatenation
- Subset
- ⋮

	D = Set of finite languages		D = Set of infinite languages
1) Union	Fin ∪ Fin ⇒ Fin	Closed!	Closed
2) Intersection	Fin ∩ Fin ⇒ Fin	Closed	Not closed $\quad a^* \cap b^* = \{\epsilon\}$
3) Complement	$\overline{\text{Fin}} \Rightarrow \Sigma^* - \text{Fin} \Rightarrow \text{Inf}$	Not closed	Not closed
4) Difference	Fin − Fin ⇒ Fin	closed	Not closed
5) Symmetric Difference	$\text{Fin}_1 \Delta \text{Fin}_2 \Rightarrow \text{Fin} = (\text{Fin}_1 \cup \text{Fin}_2) - (\text{Fin}_1 \cap \text{Fin}_2)$	closed	Not closed
6) Reversal	$\text{Fin}^{Rev} \Rightarrow \text{Fin}$	closed	Closed
7) Concatenation	Fin.Fin ⇒ Fin	closed	Closed
8) Subset	Subset(Fin) ⇒ Fin	Closed	Not closed
9) Kleene star	$(\text{Fin})^* \Rightarrow$ Need not be Fin	Not closed	Closed

Kleene star:

$(\text{Fin})^* \Rightarrow \text{Fin or Inf}$

$\underset{\text{fin}}{\phi}^* = \{\varepsilon\}$

$\underset{\text{fin}}{a}^* = a^*$

$(\text{Inf})^* \Rightarrow \text{Inf}$

$\text{Inf}^0 \cup \text{Inf}^1 \cup \cdots$

$$\boxed{\overline{L} = \Sigma^* - L}$$

$$\overline{\text{Fin}} = \underbrace{\Sigma^*}_{\text{Inf}} - \text{Fin}$$

$$= \text{Inf}$$

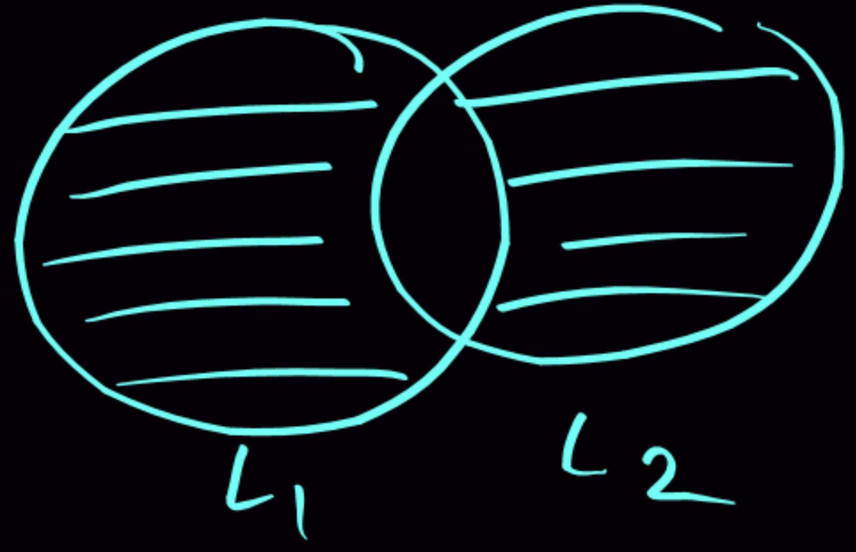
$$\overline{\text{Inf}} = \Sigma^* - \text{Inf}$$

$$= \text{fin or Inf}$$

$$\overline{\Sigma^*} = \emptyset$$

$$\overline{a\Sigma^*} = b\Sigma^* + \varepsilon$$

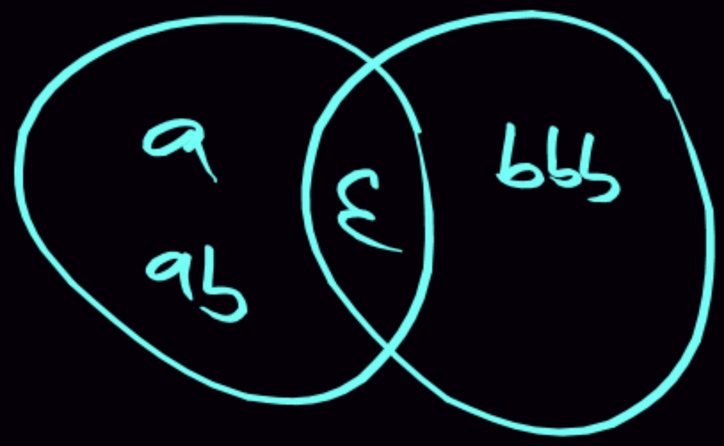
$L_1 \Delta L_2$



$L_1 = \{\varepsilon, a, ab\}$

$L_2 = \{\varepsilon, bbb\}$

$L_1 \Delta L_2 = \{a, ab, bbb\}$



$$L_1 \Delta L_2 = (L_1 - L_2) \cup (L_2 - L_1)$$
$$= (L_1 \cup L_2) - (L_1 \cap L_2)$$

Closure properties for Regular Languages:

1) $L_1 \cup L_2$
2) $L_1 \cap L_2$
3) $\overline{L}$
4) $L_1 - L_2$
5) $L_1 \Delta L_2$
6) $L_1 \cdot L_2$
7) L^{Rev}
8) $\boxed{\text{Subset}(L)}$
9) L^*
10) L^+

11) L_1 / L_2 Quotient
12) prefix(L)
13) Suffix(L)
14) Substring(L)
15) $f(L)$: Substitution
16) $h(L)$: Homomorphism
17) $h^{-1}(L)$: Inverse Homom.
18) $\text{Half}(L) = \frac{1}{2}(L) = \text{First Half}(L)$
19) Second $\frac{1}{2}(L)$

20) $\frac{1}{3}(L)$
21) Middle $\frac{1}{3}(L)$
22) Last $\frac{1}{3}(L)$

23) Finite Union: $L_1 \cup L_2 \cup L_3 \cup \ldots \cup L_k$
24) Finite Intersection: $L_1 \cap L_2 \cap \ldots \cap L_k$
25) Finite Difference: $L_1 - L_2 - L_3 - \ldots - L_k$
26) Finite Concatenation: $L_1 \cdot L_2 \cdot L_3 \cdots L_k$
27) Finite subset(L)
28) Finite Substitution(L)

29) Infinite Union: $L_1 \cup L_2 \cup L_3 \cup \ldots$
30) Inf $\cap$
31) Inf $-$
32) Inf $\cdot$
33) Inf $\subseteq$
34) Inf substitution

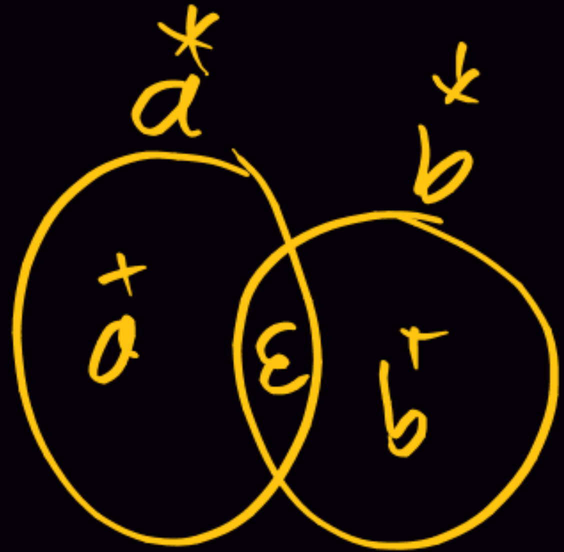
1) $\left.\begin{aligned} L_1 &= a(a+b)^* \\ L_2 &= b(a+b)^* \end{aligned}\right\} \Rightarrow L_1 \cup L_2 = a\Sigma^* + b\Sigma^* = \Sigma^+$

2) $\left.\begin{aligned} L_1 &= a^* \\ L_2 &= b^* \end{aligned}\right\} \Rightarrow L_1 \cap L_2 = \{\varepsilon\}$

$$\boxed{\begin{aligned} L \cup \overline{L} &= \Sigma^* \\ L \cap \overline{L} &= \phi \end{aligned}}$$

3) $L = a(a+b)^* \Rightarrow \overline{L} = b\Sigma^* + \varepsilon$

4) $\left.\begin{aligned} L_1 &= a^* \\ L_2 &= b^* \end{aligned}\right\} \Rightarrow \begin{aligned} L_1 - L_2 &= a^+ \\ L_1 \Delta L_2 &= a^+ + b^+ \end{aligned}$



5) $L_1 = a^*$, $L_2 = b^*$ $\Rightarrow L_1 L_2 = a^* b^*$

6) $L = a(a+b)^* \Rightarrow L^R = (a+b)^* a$

7) $L = \{\varepsilon, a\} \Rightarrow L^* = (\varepsilon + a)^* = a^*$

$L^+ = (\varepsilon + a)^+ = a^*$

8) $L = \{aa, \varepsilon\}$

Find subsets of L : $\{\,\}$

$\{aa\}$

$\{\varepsilon\}$

$\{\varepsilon, aa\}$

4 subsets

9) $L_1 = \{\varepsilon, ab\}$

$L_2 = \{a, b, ab\}$

$L_1/L_2 = \text{Quotient}(L_1, L_2) = \{\varepsilon/\boxed{a}, \varepsilon/\boxed{b}, \varepsilon/\boxed{ab}, ab/\boxed{a}, a\boxed{b}/\boxed{b}, \boxed{ab}/\boxed{ab}\}$

$\times \quad \times \quad \times \quad \times \quad a \quad \varepsilon$

$= \{\varepsilon, a\}$ //

$L_1/L_2 = \{u \mid uv \in L_1, v \in L_2\}$

$uv/v = a\boxed{b}/\boxed{b} = a$

$u\boxed{v}/\boxed{v}$

$ab/a = \times$

10)

$$a/\varepsilon = a\boxed{\varepsilon}/\boxed{\varepsilon} = a$$

$$a\boxed{b}/\boxed{b} = a$$

$$a\boxed{bc}/\boxed{bc} = a$$

$$abc/\boxed{ab} = X$$

$$abc/\varepsilon = abc\cdot\varepsilon/\varepsilon = abc$$

$$L/\varepsilon = L$$

11) $L = \{ab, acb\}$

i) $\text{Prefix}(L) = \text{First}(L) = \{u \mid uv \in L\}$

$= \{\varepsilon, a, ab, ac, acb\}$

prefix(ab)
- ε
- a
- ab

pref(acb)
- ε
- a
- ac
- acb

ii) $\text{Suffix}(L) = \text{Last}(L) = \{v \mid uv \in L\}$

$= \{\varepsilon, b, ab, cb, acb\}$

Suffix(ab)
- ε
- b
- ab

Suffix(acb)
- ε
- b
- cb
- acb

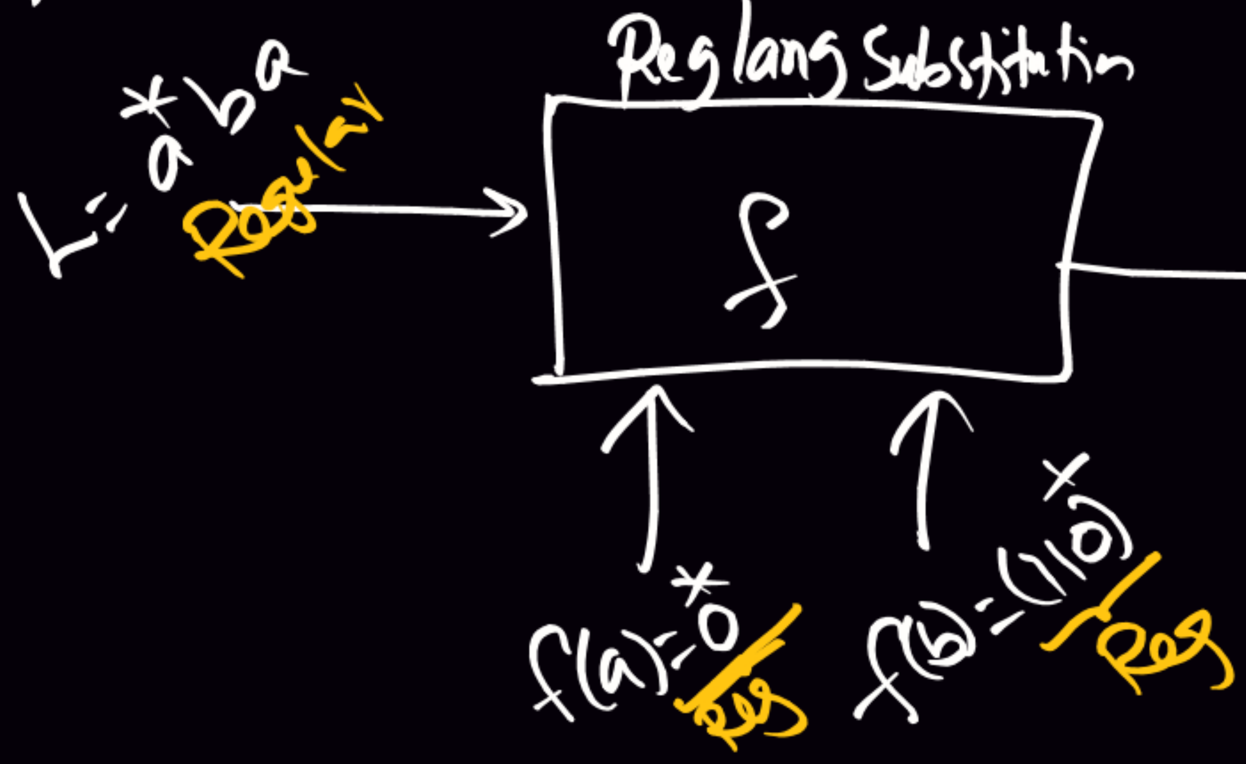
iii) $\text{Substring}(L) = \text{Subword}(L) = \text{Middle}(L) = \{y \mid xyz \in L\}$

$= \{\varepsilon, a, b, ab, c, ac, cb, acb\}$

Substring(ab)
- ε
- a
- b
- ab

Substring(acb)
- ε
- a, c, b
- ac, cb
- acb

12)

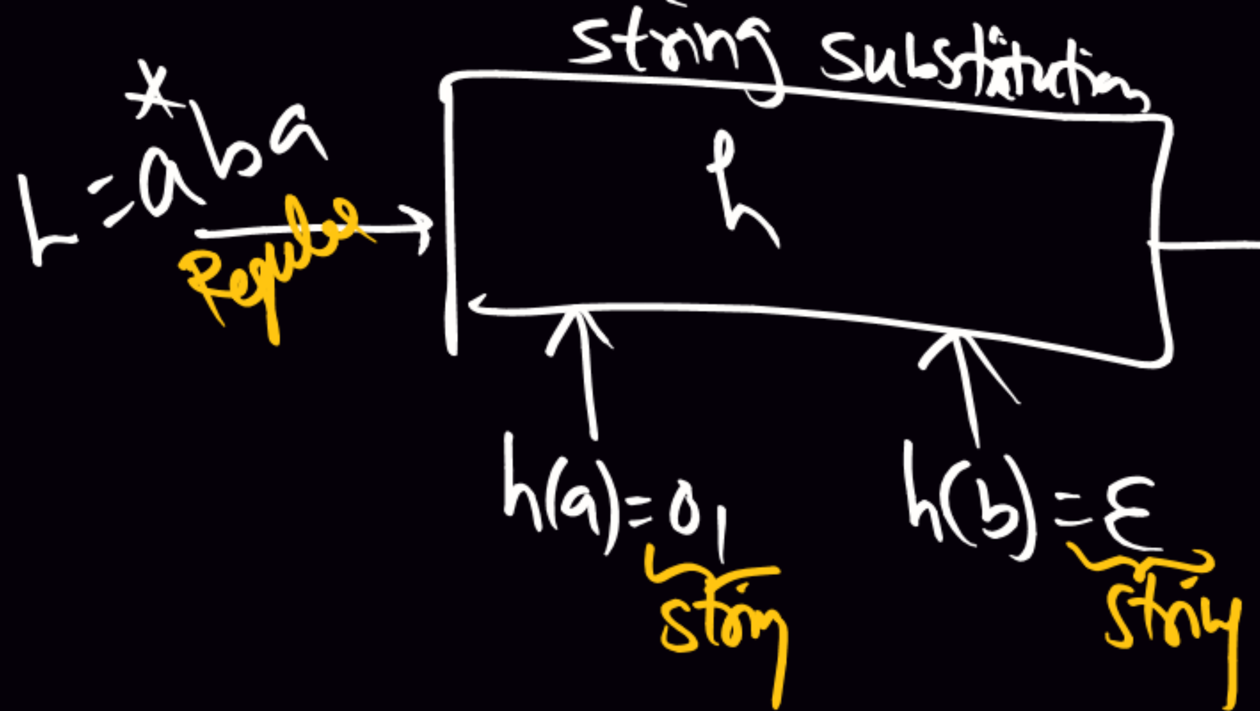


$f(L) = f(a^*ba)$

$= [f(a)]^* . f(b) . f(a)$

$= (0^*)^* (110)^* \, 0^*$

13)



$h(L) = [h(a)]^* . h(b) . h(a)$

$= (01)^* . \varepsilon . 01 = (01)^+$

I) Reg ∪ Reg ⟹ Regular

II) Reg ∪ NonReg ⟹ either Reg or Non-reg

$\Sigma^* \cup$ Nonreg

$\phi \cup$ Nonreg

III) Non-Reg ∪ NonReg ⟹ either Reg or Non-reg

Nonreg ∪ $\overline{\text{Nonreg}} = \Sigma^*$

$a^n b^n \cup a^n b^n = a^n b^n$

IV) Reg ∩ Reg ⟹ Reg

V) Reg ∩ Non-Reg ⟹ either Reg or Not reg

$\Sigma^* \cap$ Non-reg ⟹ Non-reg

$\phi \cap$ Non-reg ⟹ ϕ

VI) Non-Reg ∩ Non-Reg ⟹ either Reg or Not reg

Next : CFG