# Regular Languages:

- Regular Expressions
- Finite Automata
  - DFA
  - $\cong$
  - NFA
- Regular Grammar
  - LLG
  - RLG

# Regular Expression

→ It uses 4 operators to describe a Regular language.

- Binary
  - → OR (+) — $a+b$
  - → Concatenation (·) — $a \cdot b$
- Unary
  - → Kleene star (*) — $a^*$
  - → Kleene plus (+) — $a^+$

## OR

1) $+ \,, \cup \,, |$

2) $R_1 + R_2$
   $R_1 \cup R_2$
   $R_1 | R_2$

3) either $R_1$ or $R_2$

4) $a+b$

5)
- → a ✓
- → b ✓
- → ab ✗
- → ba ✗

## Concatenation

1) $\cdot$

2) $R_1 \cdot R_2$
   $R_1 R_2$

3) $R_1$ followed by $R_2$

4) $a \cdot b$

5)
- → ab ✓
- → a ✗
- → b ✗
- → ba ✗

## Kleene Star

1) $*$

2) $R^*$

$R^* = R^0 + R^1 + R^2 + \dots$

3) zero or more repetitions of R

4) $a^* = a^0 + a^1 + a^2 + a^3 + \dots$

5)
$$\left.\begin{array}{l} \to a^0 = \varepsilon \checkmark \\ \to a^1 = a \checkmark \\ \to a^2 = aa \checkmark \\ \quad \vdots \end{array}\right\} a^{\geq 0}$$

## Kleene plus

1) $+$

2) $R^+$

$R^+ = R^1 + R^2 + R^3 + \dots$

3) one or more repetitions of R

4) $a^+ = a^1 + a^2 + a^3 + \dots$

5)
- → $a^0$ ✗
- → $a^1$ ✓
- → $a^2$ ✓
- ⋮

| Regular Exp $R$ | Regular Language (Regular set) |
| --- | --- |
| $\varepsilon$ | $L(\varepsilon) = \{\varepsilon\}$ |
| $\varepsilon + a$ | $L(\varepsilon + a) = \{\varepsilon, a\}$ |
| $a^*$ | $L(a^*) = \{a^n \mid n \geq 0\} = \{\varepsilon, a, a^2, a^3, \ldots\}$ |
| $\phi$ ↳ empty exp | $L(\phi) = \{\ \} = \phi$ ↳ empty set |
| $a+b$ | $L(a+b) = \{a, b\}$ |

$\Sigma = \{a, b\}$

$\varepsilon$
- Symbol ✗
- string ✓
- RegExp ✓
- set ✗

$\{\varepsilon\}$
- symbol ✗
- string ✗
- RegExp ✗
- Set ✓

$\phi$
- symbol ✗
- string ✗
- RegExp ✓
- Set ✓

$a$
- Symbol ✓
- string ✓
- RegExp ✓
- set ✗

$L(\varepsilon) = \{\varepsilon\}$

$L(\phi) = \phi = \{\ \}$

1) $\phi + \phi = \phi$

2) $\varepsilon + \varepsilon = \varepsilon$

3) $a + a = a$

4) $\boxed{R + R = R}$

5) $\phi + \varepsilon = \varepsilon$

6) $\varepsilon + \phi = \varepsilon$

7) $\phi + a = a$

8) $a + \phi = a$

same $\begin{cases} 9)\ a + \varepsilon = \text{same} \\ 10)\ \varepsilon + a = \text{same} \end{cases}$

$\boxed{a \neq a + \varepsilon}$

11) $\phi \cdot \phi = \phi$

12) $\varepsilon \cdot \varepsilon = \varepsilon$

13) $a \cdot a = aa = a^2$

14) $R \cdot R = RR = R^2$

15) $\phi \cdot \varepsilon = \phi$

16) $\varepsilon \cdot \phi = \phi$

17) $\phi \cdot a = \phi$

18) $a \cdot \phi = \phi$

19) $\varepsilon \cdot a = a$

20) $a \cdot \varepsilon = a$

I) $R + \phi = R$

II) $R \cdot \phi = \phi$

III) $R \cdot \varepsilon = R$

21) $\phi^* = \phi^0 + \phi^1 + \phi^2 + \cdots = \varepsilon + \underbrace{\phi + \phi + \phi + \cdots}_{\phi} = \varepsilon + \phi = \boxed{\varepsilon}$

22) $\varepsilon^* = \varepsilon$

23) $a^* = a^0 + a^1 + a^2 + \cdots$

24) $R^* = R^0 + R^1 + R^2 + R^3 + \cdots$

$$\boxed{R^0 = \varepsilon}$$

25) $\phi^+ = \phi$

26) $\varepsilon^+ = \varepsilon$

27) $a^+ = a + aa + aaa + \cdots$

28) $R^+ = R^1 + R^2 + R^3 + \cdots$

$\varepsilon^0 = \varepsilon$

$a^0 = \varepsilon$

$\phi^0 = \varepsilon$

$b^0 = \varepsilon$

$(ab)^0 = \varepsilon$

$(a+b)^0 = \varepsilon$

29) $a.a^* = a.(\varepsilon + a + aa + \cdots) = a.\varepsilon + a.a + a.aa + \cdots$

$= a + a^2 + a^3 + \cdots = \boxed{a^+}$

30) $a^*.a = a^+$

31) $a + a^* = a + [\varepsilon + a + aa + \cdots] = \underline{a} + \varepsilon + \underline{a} + a^2 + a^3 + \cdots = a^*$

32) $a^* + a = a^*$

33) $a^+ + a^+ = a^+$

34) $a^+ \cdot a^+ = (a^1 + a^2 + a^3 + \cdots) \cdot (a^1 + a^2 + a^3 + \cdots)$

$= (a^+)^2$

$$= a.a + a\,a^2 + a.a^3 + \cdots + \underbrace{a^2.a}_{a^3 \checkmark} + \underbrace{a^2.a^2}_{a^4 \checkmark} + \underbrace{a^2.a^3}_{a^5 \checkmark} + \cdots$$

$$= a^2 + a^3 + a^4 + \cdots$$

$$= a^2 a^* = a\,a\,a^* = a\,a^+ = a^+ a$$

35) $(a^*)^{100} = a^* a^* a^* \dots$ 100 times $= a^*$

36) $(a^{100})^* =$ same

37) $(a^*)^* = (a^*)^0 + (a^*)^1 + (a^*)^2 + \dots = a^*$

38) $(a^*)^+ = (a^*)^1 + (a^*)^2 + \dots = a^*$

39) $(a^+)^* = \underbrace{(a^+)^0 + (a^+)^1} + (a^+)^2 + \dots = a^*$

40) $(a^+)^+ = a^+$

I) $(a^*)^2 = a^* \cdot a^* = a^*$

II) $(a^2)^* = (aa)^* = \varepsilon + aa + a^4 + a^6 + \cdots$

$(a^*)^2 \neq (a^2)^*$

Note:

I) $a^* = \varepsilon + a^+ = (a+\varepsilon)^* = (a+\varepsilon)^+ = (a^*)^2 = (a^*)^* = (a^*)^+ = (a^+)^*$

II) $(a+b)^* = \varepsilon + (a+b)^+ \quad = (a^*b)^*a^* \quad = (a+b)^*a^*$

$= (\varepsilon+a+b)^+ \quad = (b^*a)^*b^* \quad = a^*(a+b)^*$

$= (\varepsilon+a+b)^* \quad = a^*(ba^*)^*$

$= (a^*b^*)^* \quad = b^*(ab^*)^*$

$= (b^*a^*)^*$

$= (a^*+b^*)^* \quad = (a^*+b^*)^+ = (a^*+b)^* = (a^*+b)^+$

$(a^* b^*)^* \Rightarrow$ It will generate every string

$$\varepsilon = (\quad)^0 = (a^0 b^0)^1 = (a^0 b^0)^2 = \cdots$$

$$a = (a^1 b^0)^1 = (a\varepsilon)^1 = a$$

$$b = (a^0 b^1)^1$$

$$aa = (a^2 b^0)^1$$

$$ab = (a^1 b^1)^1$$

$$ba = (a^0 b^1)^1 (a^1 b^0)^1$$

$$bb = (a^0 b^2)^1$$

$\vdots$

$$\underline{a}\,\underline{b}\,\underline{a}\,\underline{a}\,\underline{b}\,\underline{b}\,\underline{b}\,\underline{a} \mapsto (a^1 b^0)^1 (a^0 b^1)^1 (a^1 b^0)^1 (a^1 b^0)^1 (a^0 b^1)^1 (a^0 b^1)^1 (a^0 b^1)^1 (a^1 b^0)^1$$

$$\boxed{(a^* b^*)^* \neq a^* b^*}$$

$$(a+\varepsilon)^+ = (a+\varepsilon)^1 + (a+\varepsilon)^2 + (a+\varepsilon)^3 + \cdots$$

$$R^+ = R^1 + R^2 + R^3 + \cdots$$

$$= a+\varepsilon + \varepsilon + a + aa + \varepsilon + a + a^2 + a^3 + \cdots$$

$$= \varepsilon + a + a^2 + a^3 + \cdots$$

$$= a^*$$

$a+\varepsilon \to \varepsilon$ ✓

$a+\varepsilon \to a$ ✓

$$(a+\varepsilon)^2 = (a+\varepsilon)(a+\varepsilon)$$

$\to \varepsilon$ ✓

$\to a$ ✓

$\to aa$ ✓

Note:

I) $a.b \neq b.a$ [$\cdot$ is not commutative]

II) $a+b = b+a$ [+ is commutative]

III) $R+\phi = \phi+R = R$ [$\phi$ is Identity exp for +]

IV) $R\cdot\varepsilon = \varepsilon\cdot R = R$ [$\varepsilon$ is Identity exp for $\cdot$]

V) $R\cdot\phi = \phi\cdot R = \phi$ [$\phi$ is Dominator for $\cdot$ (annihilator)]

VI) $R+\Sigma^* = \Sigma^*+R = \Sigma^*$ [$\Sigma^*$ is dominator for +]

VII) $a.(b+c) = ab + ac$

VIII) $(b+c).a = ba + ca$

$\left.\begin{matrix} \\ \end{matrix}\right\}$ $\cdot$ can be distributed over $+$

IX) $a+(b.c) \neq (a+b).(a+c)$

$$\Sigma^* = \Sigma^0 + \Sigma^1 + \Sigma^2 + \Sigma^3 + \cdots$$

$\Sigma = \{a\}$

$$a^* = a^0 + a^1 + a^2 + a^3 + \cdots$$

$$= \varepsilon + a + aa + aaa + \cdots$$

generates all strings over $\Sigma = \{a\}$

$$(a+b)(a+b)$$

$$aa + ab + ba + bb$$

$\Sigma = \{a, b\}$

$$\Sigma^* = \{a,b\}^* = (a+b)^* = \underbrace{(a+b)^0}_{\varepsilon} + \underbrace{(a+b)^1}_{a+b} + \underbrace{(a+b)^2}_{aa+ab+ba+bb} + \cdots$$

$$= \varepsilon + a + b + aa + ab + ba + bb$$

Write Regular Expression for following Regular Languages.

1) $L = \{ \}$ over $\Sigma = \{a\}$

$R = \phi$

$= \phi . a$

$= \phi . a^*$

$= \phi . aa$

$\vdots$

2) $L = \{ \}$ over $\Sigma = \{a, b\}$

$R = \phi$

$= \phi . a$

$= b . \phi$

$= a . \phi . b$

$= a^* . \phi$

$= b^* . \phi$

$\vdots$

3) $L = \{ \}$ over $\Sigma = \{a, b, c\}$

$R = \phi$

4) $L=\{\varepsilon\}$ over $\Sigma=\{a,b\}$ $\Longrightarrow$ $R=\varepsilon=\phi^*=a^0=(ab)^0$

5) $L=\{w \mid w\in(a+b)^*, |w|\geq 1\}$ $\Longrightarrow$ $(a+b)^+$

6) $L=\{w \mid w\in(a+b)^*, |w|\geq 0\}$ $\Longrightarrow$ $(a+b)^*$

7) $L=\{w \mid w\in(a+b)^*, |w|=2\}$ $\Longrightarrow$ $(a+b)^2=aa+ab+ba+bb$

8) $L=\{w \mid w\in(a+b)^*, |w|\leq 2\}$ $\Longrightarrow$ $\varepsilon+(a+b)+(a+b)^2=(\varepsilon+a+b)^2$

9) $L=\{w \mid w\in(a+b)^*, |w|\geq 2\}$ $\Longrightarrow$ $(a+b)^2(a+b)^*$

10) $L=\{w \mid w\in(a+b)^*, |w|\leq 100\}$ $\Longrightarrow$ $(\varepsilon+a+b)^{100}$

$\varepsilon \longrightarrow$ Zero length

$a+b \longrightarrow$ 1 len

$\varepsilon+a+b \longrightarrow \underbrace{0 \text{ or } 1 \text{ length}}_{\text{atmost 1 len}}$

$$(\varepsilon+a+b)^2 = (\varepsilon+a+b)(\varepsilon+a+b)$$

$$= \underbrace{\varepsilon+a+b+aa+ab+ba+bb}_{\text{atmost 2 length}}$$

$(\varepsilon+a+b)^{100}$ = All strings upto 100 length

11) $L = \{w \mid w \in (a+b)^*, \text{ w starts with 'a'}\} \Rightarrow a(a+b)^* = a\Sigma^*$

12) $L = \{w \mid w \in (a+b)^*, \text{ w starts with aa or bb}\} = (aa+bb)\Sigma^*$

13) $L = \{w \mid w \in (a+b)^*, \text{ w ends with 'a'}\} = \Sigma^* a = (a+b)^* a$

14) $L = \{w \mid w \in (a+b)^*, \text{ w ends with aa or bb}\} = \Sigma^*(aa+bb)$

15) $L = \{w \mid w \in (a+b)^*, \text{ w contains 'a'}\} = \Sigma^* a \Sigma^* = (a+b)^* a (a+b)^*$

16) $L = \{w \mid w \in (a+b)^*, \text{ w contains aa or bb}\} = \Sigma^*(aa+bb)\Sigma^*$

17) $\{a^m b^n \mid m,n \geq 0\} \Rightarrow a^* b^*$

18) $\{a^m b^n \mid m \geq 0, n \geq 1\} \Rightarrow a^* b^+$

19) $\{a^m b^n \mid m \geq 1, n \geq 0\} \Rightarrow a^+ b^*$

20) $\{a^m b^n \mid m \geq 1, n \geq 1\} \Rightarrow a^+ b^+$

21) $\{a^m \mid m = \text{even}\} \Rightarrow (aa)^*$

22) $\{a^m \mid m = \text{odd}\} \Rightarrow a(aa)^* = (aa)^* a$

23) $\{a^m b^n \mid m = \text{even}, n = \text{odd}\} \Rightarrow (aa)^* b(bb)^*$

24) $\{a^m b^n \mid m+n = \text{even}\} \Rightarrow a^{\text{even}} b^{\text{even}} + a^{\text{odd}} b^{\text{odd}} \Rightarrow (aa)^*(bb)^* + a(aa)^* b(bb)^*$

25) $\{a^m b^n \mid m+n = \text{odd}\} \Rightarrow a^{\text{odd}} b^{\text{even}} + a^{\text{even}} b^{\text{odd}} = a(aa)^*(bb)^* + (aa)^* b(bb)^*$

$\boxed{\begin{array}{c} m+n = \text{even} \\ \Downarrow \\ \text{Both } m \,\&\, n \text{ are even} \\ \text{OR} \\ \text{Both } m \,\&\, n \text{ are odd} \end{array}}$

26) $\{w \mid w \in (a+b)^*,\ \#_a(w) = 2\} \Rightarrow b^* a b^* a b^*$

27) $\{w \mid w \in (a+b)^*,\ \#_a(w) \le 2\} \Rightarrow b^* + b^* a b^* + b^* a b^* a b^*$

$\Rightarrow b^* (\varepsilon + a)\ b^*\ (\varepsilon + a) b^*$

28) $\{w \mid w \in (a+b)^*,\ \#_a(w) \ge 2\} \Rightarrow \Sigma^* a \Sigma^* a \Sigma^*$

29) $\{w \mid w \in (a+b)^*,\ |w| = \text{even}\} \Rightarrow [(a+b)^2]^*$

Length of w is even

30) $\{w \mid w \in (a+b)^*,\ |w| = \text{odd}\} \Rightarrow [(a+b)^2]^* (a+b)$

no. of a's in w

$\#_a(w)$

$n_a(w)$

H.W.

31) $\{w \mid w \in (a+b)^*,\ \#a(w) = \text{even}\}$

32) $\{w \mid w \in (a+b)^*,\ \#_a(w) = \text{odd}\}$