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MATHEMATICS FOR JEE (MAIN & ADVANCED)

GEOMETRY

Volume 4

Dr. G S N MURTI

Mathematics
for JEE (Main & Advanced)

GEOMETRY

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Dedication

Dedicated to

Sri. Poojya Ganapati Sachchidananda Swamyjee,
Dattanagar, Mysore, Karnataka

Dr. G. S. N. Murti

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Dr. G. S. N. Murti

Features and Benefits at a Glance

Feature	Benefit to student
Chapter Opener	Peaks the student's interest with the chapter opening vignette, definitions of the topic, and contents of the chapter.
Clear, Concise, and Inviting Writing Style, Tone and Layout	Students are able to **Read** this book, which reduces math anxiety and encourages student success.
Theory and Applications	Unlike other books that provide very less or no theory, here theory is well matched with solved examples.
Theorems	Relevant theorems are provided along with proofs to emphasize conceptual understanding.
Solved Examples	Topics are followed by solved examples for students to practice and understand the concept learned.
Examples	Wherever required, examples are provided to aid understanding of definitions and theorems.
Quick Look	Formulae/concepts that do not require extensive thought but can be looked at the last moment.
Try It Out	Practice problems for students in between the chapter.
Subjective Problems	Solved subjective problems for the preceding sections.
Summary	Key formulae, ideas and theorems are presented in this section in each chapter.
Worked Out Problems	The problems are presented in the form of **Single Correct Choice Type Questions** **Multiple Correct Choice Type Questions** **Matrix-Match Type Questions** **Comprehension-Type Questions** **Integer Answer Type Questions** In-depth solutions are provided to all problems for students to understand the logic behind.
Exercises	Offer self-assessment. The questions are divided into subsections which include large number of Multiple Choice Questions as per requirements of JEE (Main & Advanced).
Answers	Answers are provided for all exercise questions for students to validate their solution.

Note to the Students

The JEE (Main & Advanced) is one of the hardest exams to crack for students, for a very simple reason – concepts cannot be learned by rote, they have to be absorbed, and IIT believes in strong concepts. Each question in the JEE (Main & Advanced) entrance exam is meant to push the analytical ability of the student to its limit. That is why the questions are called brainteasers!

Students find Mathematics the most difficult part of JEE (Main & Advanced). We understand that it is difficult to get students to love mathematics, but one can get students to love succeeding at mathematics. In order to accomplish this goal, the book has been written in clear, concise, and inviting writing style. It can be used as a self-study text as theory is well supplemented with examples and solved examples. Wherever required, figures have been provided for clear understanding.

If you take full advantage of the unique features and elements of this textbook, we believe that your experience will be fulfilling and enjoyable. Let's walk through some of the special book features that will help you in your efforts to crack JEE (Main & Advanced).

To crack mathematics paper for JEE (Main & Advanced) the five things to remember are:

1. Understanding the concepts

2. Proper applications of concepts

3. Practice

4. Speed

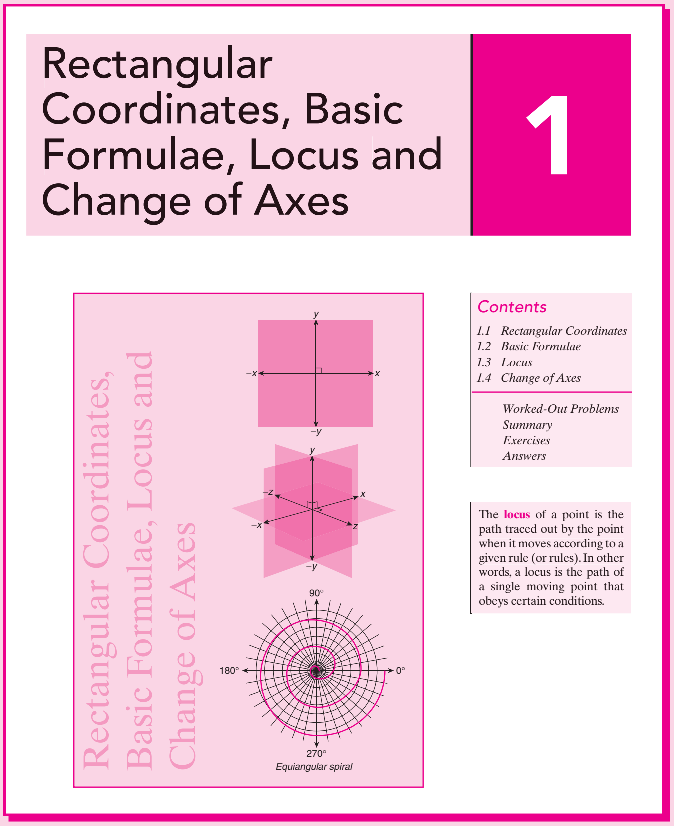
5. Accuracy

About the Cover Picture

The picture on the cover is *Jatiyo Smriti Soudho* or National Martyrs' Memorial – a monument located in Savar, Bangladesh – which is built to commemorate the valour and the sacrifice of those killed in the Bangladesh Liberation War of 1971. The monument is composed of seven isosceles triangular pyramid shaped towers, with the middle one being the tallest, that is, 150 feet. The arrangement of the seven towers is unique. The planes are arranged uniquely so that one can see its distinctive patterns when looking at it from different angles.

A. PEDAGOGY

CHAPTER OPENER


Rectangular Coordinates, Basic Formulae, Locus and Change of Axes

1

Contents

The **locus** of a point is the path traced out by the point when it moves according to a given rule (or rules). In other words, a locus is the path of a single moving point that obeys certain conditions.

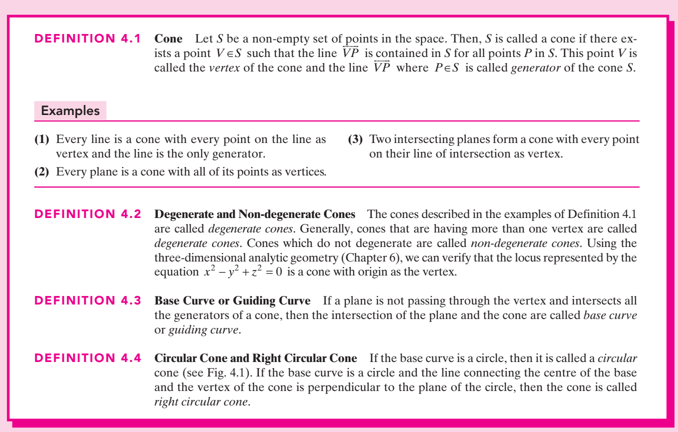
Each chapter starts with an opening vignette, definition of the topic, and contents of the chapter that give you an overview of the chapter to help you see the big picture.

CLEAR, CONCISE, AND INVITING WRITING

Special attention has been paid to present an engaging, clear, precise narrative in the layout that is easy to use and designed to reduce math anxiety students may have.

DEFINITIONS

Every new topic or concept starts with defining the concept for students. Related examples to aid the understanding follow the definition.


DEFINITION 4.1 **Cone** Let S be a non-empty set of points in the space. Then, S is called a cone if there exists a point $V \in S$ such that the line $\overline{VP}$ is contained in S for all points P in S. This point V is called the *vertex* of the cone and the line $\overline{VP}$ where $P \in S$ is called *generator* of the cone S.

Examples

(1) Every line is a cone with every point on the line as vertex and the line is the only generator.

(2) Every plane is a cone with all of its points as vertices.

(3) Two intersecting planes form a cone with every point on their line of intersection as vertex.

DEFINITION 4.2 **Degenerate and Non-degenerate Cones** The cones described in the examples of Definition 4.1 are called *degenerate cones*. Generally, cones that are having more than one vertex are called *degenerate cones*. Cones which do not degenerate are called *non-degenerate cones*. Using the three-dimensional analytic geometry (Chapter 6), we can verify that the locus represented by the equation $x^2 - y^2 + z^2 = 0$ is a cone with origin as the vertex.

DEFINITION 4.3 **Base Curve or Guiding Curve** If a plane is not passing through the vertex and intersects all the generators of a cone, then the intersection of the plane and the cone are called *base curve* or *guiding curve*.

DEFINITION 4.4 **Circular Cone and Right Circular Cone** If the base curve is a circle, then it is called a *circular* cone (see Fig. 4.1). If the base curve is a circle and the line connecting the centre of the base and the vertex of the cone is perpendicular to the plane of the circle, then the cone is called *right circular cone*.

EXAMPLES

Examples pose a specific problem using concepts already presented and then work through the solution. These serve to enhance the students' understanding of the subject matter.

Example 2.13

Write $2x + 3y + 5 = 0$ in the normal form. What is the distance of the line from origin?

Solution: In $2x + 3y + 5 = 0$, the constant 5 should be taken to the RHS of the equation, that is

$$2x + 3y = -5$$

Dividing both sides with $\sqrt{a^2+b^2} = \sqrt{2^2+3^2} = \sqrt{13}$, we get

$$\frac{2}{\sqrt{13}}x + \frac{3}{\sqrt{13}}y = \frac{-5}{\sqrt{13}}$$

To make the RHS positive, we multiply both sides with (-1). Thus, the normal form is

$$\left(\frac{-2}{\sqrt{13}}\right)x + \left(\frac{-3}{\sqrt{13}}\right)y = \frac{5}{\sqrt{13}}$$

where

$$\cos\alpha = \frac{-2}{\sqrt{13}} \quad \text{and} \quad \sin\alpha = \frac{-3}{\sqrt{13}}$$

Now, the distance of the line from the origin is

$$p = \frac{|c|}{\sqrt{a^2+b^2}} = \frac{|-5|}{\sqrt{13}} = \frac{5}{\sqrt{13}}$$

Example 2.14

Find the normal form of the line $3x + 4y - 10 = 0$ and its distance from the origin.

Solution: The equation $3x + 4y - 10 = 0$ can be written as $3x + 4y = 10$. Dividing both sides with $\sqrt{a^2+b^2} = \sqrt{3^2+4^2} = 5$, we get

$$\frac{3}{5}x + \frac{4}{5}y = 2$$

where

$$\cos\alpha = \frac{3}{5} \quad \text{and} \quad \sin\alpha = \frac{4}{5}$$

Now the distance of the line from the origin is

$$p = \frac{|10|}{5} = \frac{10}{5} = 2$$

THEOREMS

Relevant theorems are provided along with proofs to emphasize conceptual understanding rather than rote learning.

THEOREM 2.11 Let $ax + by + c = 0$ be a straight line. Then

1. $\left(\frac{a}{\sqrt{a^2+b^2}}\right)x + \left(\frac{b}{\sqrt{a^2+b^2}}\right)y = \frac{-c}{\sqrt{a^2+b^2}}$ is the normal form of the given line if $c < 0$.
2. $\left(\frac{-a}{\sqrt{a^2+b^2}}\right)x + \left(\frac{-b}{\sqrt{a^2+b^2}}\right)y = \frac{c}{\sqrt{a^2+b^2}}$ is the normal form of the given line if $c > 0$.

PROOF Suppose $x\cos\alpha + y\sin\alpha = p$ is the normal form of $ax + by + c = 0$. Therefore, by Theorem 2.10, there exists a real $\lambda \neq 0$ such that $\cos\alpha = \lambda a$, $\sin\alpha = \lambda b$ and $-p = \lambda c$. Now,

$$\cos^2\alpha + \sin^2\alpha = 1$$

implies

$$\lambda = \pm\frac{1}{\sqrt{a^2+b^2}}$$

Also $-\lambda c = p > 0$ (since p is the distance of the line from origin) implies that

$$\lambda = \begin{cases} \dfrac{1}{\sqrt{a^2+b^2}} & \text{if } c < 0 \\ \dfrac{-1}{\sqrt{a^2+b^2}} & \text{if } c > 0 \end{cases}$$

Therefore, if $c < 0$, then the normal form of the line is

$$x\left(\frac{a}{\sqrt{a^2+b^2}}\right) + y\left(\frac{b}{\sqrt{a^2+b^2}}\right) = \frac{-c}{\sqrt{a^2+b^2}}$$

or if $c > 0$, the normal form of the line is

$$x\left(\frac{-a}{\sqrt{a^2+b^2}}\right) + y\left(\frac{-b}{\sqrt{a^2+b^2}}\right) = \frac{c}{\sqrt{a^2+b^2}}$$

■

QUICK LOOK 1

The properties of the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are as follows:

1. The curve is symmetric about both axes.
2. For any point (x, y) on the curve, we have $-a \le x \le a$ and $-b \le y \le b$.
3. The x-axis meets the curve at $A(a, 0)$ and $A'(-a, 0)$. The y-axis meets the curve at $B(0, b)$ and $B'(0, -b)$.
4. For each value of x,
$$y = \pm b\sqrt{1 - \frac{x^2}{a^2}}$$
and for each value of y,
$$x = \pm a\sqrt{1 - \frac{y^2}{b^2}}$$
5. $b < a$.
6. If $P(x, y)$ is a point on the ellipse
$$\frac{x^2}{a^2} + \frac{y^2}{b^2}$$
then we have
$$SP = e(PM) = e(NZ) = e(CZ - CN)$$
$$= e\left(\frac{a}{e} - x\right) = a - ex$$
7. Since the curve is symmetric about both axes, there must be second focus and directrix. Another focus $S'(-ae, 0)$ and its corresponding directrix is
$$x = \frac{-a}{e}$$

QUICK LOOK

Some important formulae and concepts that do not require exhaustive explanation, but their mention is important, are presented in this section. These are marked with a **magnifying glass.**

TRY IT OUT

Within each chapter the students would find problems to reinforce and check their understanding. This would help build confidence as one progresses in the chapter. These are marked with a **pointed finger**.

Try it out Try Example 5.15 for the hyperbola $x^2 - y^2 = a^2$ whose asymptotes are $x \pm y = 0$.

SUBJECTIVE PROBLEMS

Since geometry requires a lot of practice, some chapters in addition provide numerous solved examples in the pattern of Subjective Problems. We have provided such problems within the chapter, near to the concept.

Subjective Problems

1. If p_1 and p_2 are the distances between the opposite sides of a parallelogram and α is one of its angles, then show that the area of the parallelogram is p_1p_2 cosecα.

Solution: $ABCD$ is a parallelogram (see Fig. 2.18). $AM = p_1$, $DN = p_2$ and $\angle BAD = \alpha$.

Area of the parallelogram = $(AB)\, p_2$ (2.17)

Now from ΔAMP, $\sin\alpha = p_1/AB$ and hence we have $AB = p_1$ cosecα. Therefore, from Eq. (2.17), the area of the parallelogram = p_1p_2 cosecα.

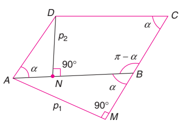


FIGURE 2.18

2. Show that the area of parallelogram whose sides are $a_1x+b_1y+c_1=0, a_1x+b_1y+d_1=0, a_2x+b_2y+c_2=0$ and $a_2x+b_2y+d_2=0$ is

$$\left|\frac{(d_1-c_1)(d_2-c_2)}{a_1b_2-a_2b_1}\right|$$

Solution: Consider Fig. 2.18. Let the equations of the sides be $a_1x+b_1y+c_1=0$, $a_1x+b_1y+d_1=0$, $a_2x+b_2y+c_2=0$ and $a_2x+b_2y+d_2=0$. Therefore,

$$p_1 = \frac{|d_1-c_1|}{\sqrt{a_1^2+b_1^2}}$$

$$p_2 = \frac{|d_2-c_2|}{\sqrt{a_2^2+b_2^2}}$$

Also

$$\cos\alpha = \frac{|a_1a_2+b_1b_2|}{\sqrt{(a_1^2+b_1^2)(a_2^2+b_2^2)}}$$

Therefore

$$\sin^2\alpha = 1-\cos^2\alpha = 1-\frac{(a_1a_2+b_1b_2)^2}{(a_1^2+b_1^2)(a_2^2+b_2^2)}$$

$$= \frac{(a_1b_2-a_2b_1)^2}{(a_1^2+b_1^2)(a_2^2+b_2^2)} \quad (2.18)$$

Now the area of the parallelogram (by Problem 1) is

$$p_1p_2(\operatorname{cosec}\alpha) = \frac{|d_1-c_1|}{\sqrt{a_1^2+b_1^2}}\cdot\frac{|d_2-c_2|}{\sqrt{a_2^2+b_2^2}}\cdot\frac{\sqrt{(a_1^2+b_1^2)(a_2^2+b_2^2)}}{|a_1b_2-a_2b_1|}$$

[from Eq. (2.18)]

$$= \left|\frac{(d_1-c_1)(d_2-c_2)}{a_1b_2-a_2b_1}\right|$$

3. Prove that the area of the parallelogram formed by the lines $4y-3x-a=0, 3y-4x+a=0, 4y-3x-3a=0$ and $3y-4x+2a=0$ is $2a^2/7$.

Solution: Rewriting the equations of the sides of the parallelogram, we have

$$3x-4y+a=0$$
$$4x-3y-a=0$$
$$3x-4y+3a=0$$
and
$$4x-3y-2a=0$$

Here, $c_1=a, d_1=3a, c_2=-a, d_2=-2a, a_1=3, b_1=-4, a_2=4$ and $b_2=-3$. Therefore, by Problem 2,

$$\text{Area} = \left|\frac{(d_1-c_1)(d_2-c_2)}{a_1b_2-a_2b_1}\right|$$

SUMMARY

At the end of every chapter, a summary is presented that organizes the key formulae and theorems in an easy to use layout. The related topics are indicated so that one can quickly summarize a chapter.

SUMMARY

2.1. ***Slope of line:*** Let l be a non-vertical line (i.e., l is not parallel to y-axis) making an angle θ with the positive direction of x-axis. Then, tan θ is called the slope of the line l. Generally, the slope of a line is denoted by m.

Caution: The concept of slope is followed only for non-vertical lines.

Note: Slope of a horizontal line (which is parallel to x-axis) is always zero.

2.2. If $A(x_1, y_1)$ and $B(x_2, y_2)$ are two points on a non-vertical line, then the slope of the line $\overline{AB}$ is $\dfrac{y_2-y_1}{x_2-x_1}$.

2.3. ***Intercepts on the axes:*** If a line l meets x-axis at $(a, 0)$ and y-axis at $(0, b)$, then a is called x-intercept and b is called y-intercept of the line l.

2.4. ***Equations of the axis:*** The equation of x-axis is $y = 0$ and the equation of y-axis is $x = 0$.

2.5. ***Various forms of straight line equations:***

1. ***Two-point form:*** Equation of the line passing through two points (x_1, y_1) and (x_2, y_2) is
$$(x-x_1)(y_1-y_2) = (y-y_1)(x_1-x_2)$$
2. ***Point–slope form:*** Equation of the line which is having slope m and passing through the point (x_1, y_1) is
$$y-y_1 = m(x-x_1)$$
3. ***Symmetric form:*** If a non-vertical makes an angle θ with the positive direction of x-axis and passes through a point (x_1, y_1), then its equation is
$$\frac{x-x_1}{\cos\theta} = \frac{y-y_1}{\sin\theta}$$
Note: In the above relation, if we consider that each ratio is equal to r (real number), then every point on the line is of the form $(x_1 + r\cos\theta, y_1 + r\sin\theta)$. Also $|r|$ gives the distance of the point (x, y) on the given line from the fixed point (x_1, y_1).
4. ***Intercept form:*** If a and b are x and y intercepts of a line $(ab \neq 0)$, then the line equation is $\dfrac{x}{a}+\dfrac{y}{b}=1$.
Note: Area of the triangle formed by the coordinate axis and the line $\dfrac{x}{a}+\dfrac{y}{b}=1$ is $\dfrac{1}{2}|ab|$ sq. unit.
5. ***Slope–intercept form:*** The equation of a non-vertical line which is having slope m and y-intercept c is
$$y = mx + c$$
Note: Equation of any line (except the y-axis) passing through origin is the form $y = mx$.
6. ***Normal form:*** Let l be a line whose distance from the origin is ON $(= p)$ and $\overline{ON}$ make an angle α with the positive direction of the x-axis. Then, the equation of the line l is $x\cos\alpha + y\sin\alpha = p$.

2.6. ***Definition (first-degree equation):*** If a, b and c are real and either a or b is not zero, then $ax + by + c$ is called first-degree expression in x and y and $ax + by + c = 0$ is called first-degree equation in x and y.

2.7. ***Theorem:*** Every first-degree equation in x and y represents a straight line and the equation of any line in the coordinate plane is a first-degree equation in x and y.

2.8. ***General equation of a straight line:*** First-degree equation in x and y is called the general equation of a straight line.

2.9. ***Various forms of $ax + by + c = 0$, where $abc \neq 0$:***

1. ***Slope–intercept form:***
$$y = \left(\frac{-a}{b}\right)x + \left(\frac{-c}{b}\right)$$
2. ***Intercept form:***
$$\frac{x}{(-c/a)} + \frac{y}{(-c/b)} = 1$$

B. WORKED-OUT PROBLEMS AND ASSESSMENT

Mere theory is not enough. It is also important to practice and test what has been proved theoretically. The worked-out problems and exercise at the end of each chapter will enhance the concept building of students. The worked-out problems and exercises have been divided into:

1. **Single Correct Choice Type Questions**
2. **Multiple Correct Choice Type Questions**
3. **Matrix-Match Type Questions**
4. **Comprehension-Type Questions**
5. **Integer Answer Type Questions**

WORKED-OUT PROBLEMS

In-depth solutions are provided to all worked-out problems for students to understand the logic behind and formula used.

WORKED-OUT PROBLEMS

Single Correct Choice Type Questions

1. If the line $3ax + 5y + a - 2 = 0$ passes through the point $(-1, 4)$, then value of a is

(A) 9 (B) 7 (C) −9 (D) −7

Solution: Since the line passes through (−1, 4), we have

$$3a(-1) + 5(4) + a - 2 = 0$$
$$\Rightarrow -2a + 18 = 0$$

Hence, $a = 9$ and the line is $27x + 5y + 7 = 0$.

Answer: (A)

2. A line has slope $-3/4$, positive y-intercept and forms a triangle of area 24 sq. units with coordinate axes. Then, the equation of the line is

(A) $3x + 4y + 24 = 0$ (B) $3x + 4y - 24 = 0$
(C) $3x + 4y - 25 = 0$ (D) $3x + 4y + 25 = 0$

Solution: Let the line be

$$y = \left(-\frac{3}{4}\right)x + c$$

so that the intercepts on the x and y axes, respectively, are $4c/3$ and c. Therefore, the area of the triangle (by Quick Look 4) is

$$\frac{1}{2}\left|\frac{4c}{3}\cdot c\right| = 24 \Rightarrow c^2 = 36 \Rightarrow c = \pm 6$$

Since y-intercept is positive, the value c is 6 and the equation of the line is $3x + 4y - 24 = 0$.

Answer: (B)

3. A non-horizontal line passing through the point $(4, -2)$ and whose distance from the origin is 2 units is

(A) $3x + 4y - 10 = 0$ (B) $x + y - 2 = 0$
(C) $4x + 3y - 10 = 0$ (D) $2x + 3y - 2 = 0$

Solution: Let the slope of the line be m. Now the equation of the line, by Theorem 2.2, is

$$y + 2 = m(x - 4)$$

Therefore, by Theorem 2.14,

$$2 = \frac{|m(0-4) - 0 - 2|}{\sqrt{m^2+1}}$$
$$\Rightarrow (2m+1)^2 = m^2 + 1$$
$$\Rightarrow 3m^2 + 4m = 0$$
$$\Rightarrow m = 0, -\frac{4}{3}$$

SINGLE CORRECT CHOICE TYPE QUESTIONS

These are the regular multiple choice questions with four choices provided as asked in JEE (Main & Advanced). Only one among the four choices will be the correct answer.

Multiple Correct Choice Type Questions

1. For the hyperbola $9x^2 - 16y^2 - 18x + 32y - 151 = 0$, which of the following are true?

(A) Eccentricity is $\frac{5}{4}$
(B) Foci are $(-4, 1)$ and $(6, 1)$
(C) Centre is $(1, -1)$
(D) Length of the latus rectum is $\frac{9}{2}$

Solution: The given equation is

$$9(x^2 - 2x) - 16(y^2 + 2y) - 151 = 0$$
$$\Rightarrow 9(x-1)^2 - 16(y+1)^2 = 151 + 9 - 16 = 144$$
$$\Rightarrow \frac{(x-1)^2}{16} - \frac{(y+1)^2}{9} = 1$$
$$\Rightarrow \frac{X^2}{16} - \frac{Y^2}{9} = 1$$

where $X = x - 1$, $Y = y + 1$. Here $a^2 = 16, b^2 = 9$. The eccentricity e is given by

$$9 = 16(e^2 - 1) \text{ or } e^2 = 1 + \frac{9}{16} = \frac{25}{16}$$

so that

$$e = \frac{5}{4}$$

The centre is given by

$$(X = 0, Y = 0) = (x - 1 = 0, y + 1 = 0) = (1, -1)$$

The foci is given by

$$(X = \pm ae,\ Y = 0) = (x - 1 = \pm 5,\ -1) = (6, -1) \text{ and } (-4, -1)$$

The latus rectum is given by

$$\frac{2b^2}{a} = \frac{2(9)}{4} = \frac{9}{2}$$

Answers: (A), (C), (D)

2. If the circle $x^2 + y^2 = a^2$ cuts the hyperbola $xy = c^2$ at four points (x_k, y_k) (where $k = 1, 2, 3$ and 4), then

(A) $x_1 + x_2 + x_3 + x_4 = 0$ (B) $y_1 + y_2 + y_3 + y_4 = 0$
(C) $x_1 x_2 x_3 x_4 = c^4$ (D) $y_1 y_2 y_3 y_4 = c^4$

Solution: The abscissa x_k (where $k = 1, 2, 3$ and 4) are the roots of the equation

$$x^2 + \frac{c^4}{x^2} = a^2$$
$$\Rightarrow x^4 - a^2x^2 + c^4 = 0$$

Therefore

$$x_1 + x_2 + x_3 + x_4 = 0$$

Since the coefficient of x_3 is zero, we have

$$\Sigma x_1x_2 = -a^2,\ \Sigma x_1x_2x_3 = 0,\ \Sigma x_1x_2x_3x_4 = c^4$$

Now,

$$y_1 + y_2 + y_3 + y_4 = c^2\left(\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4}\right)$$
$$= \frac{c^2(\Sigma x_2x_3x_4)}{x_1x_2x_3x_4}$$
$$= \frac{c^2(0)}{c^4} = 0$$

Finally

$$y_1y_2y_3y_4 = \frac{c^2}{x_1}\cdot\frac{c^2}{x_2}\cdot\frac{c^2}{x_3}\cdot\frac{c^2}{x_4} = \frac{c^8}{c^4} = c^4$$

Answers: (A), (B), (C), (D)

3. On the ellipse $4x^2 + 9y^2 = 1$, the points at which the tangents are parallel to the line $9y = 8x$ are

(A) $\left(\frac{2}{5}, \frac{1}{5}\right)$ (B) $\left(\frac{-2}{5}, \frac{1}{5}\right)$

MULTIPLE CORRECT CHOICE TYPE QUESTIONS

Multiple correct choice type questions have four choices provided, but one or more of the choices provided may be correct.

MATRIX-MATCH TYPE QUESTIONS

These questions are the regular "Match the Following" variety. Two columns each containing 4 subdivisions or first column with four subdivisions and second column with more subdivisions are given and the student should match elements of column I to that of column II. There can be one or more matches.

Matrix-Match Type Questions

1. Match items of Column I with those of Column II.

Column I	Column II
(A) If x-axis bisected each of two chords drawn from the point $(a, b/2)$ on the circle $2x(x-a)+y(2y-b)=0$ $(ab \neq 0)$, then a/b belongs to	(p) $(-\infty,-2)\cup(2,\infty)$
	(q) $(-2, 2)$
(B) If the circles $x^2+y^2-10x+16=0$ and $x^2+y^2=r^2$ intersect in two distinct points, then r lies in the interval	(r) $(-\infty,-\sqrt{2})\cup(\sqrt{2},\infty)$
(C) If the line $y+x=0$ bisects chords drawn from the point $(1+a\sqrt{2}/2, 1-a\sqrt{2}/2)$ to the circle $2x^2+2y^2-(1+a\sqrt{2})x-(1-a\sqrt{2})y=0$, then a belongs to	(s) $(-3, 3)$
(D) Point $(2, \lambda)$ lies inside the circle $x^2+y^2=13$ if and only if λ belongs to	(t) $(2, 8)$

Solution:

(A) The given circle equation is

$$S \equiv 2x^2+2y^2-2ax-by=0$$

$$S \equiv x^2+y^2-ax-\frac{b}{2}y=0$$

Let $(x_1, 0)$ be the midpoint of a chord of the circle. Therefore, the equation of the chord is

$$xx_1+y(0)-\frac{a}{2}(x+x_1)-\frac{b}{4}(y+0)=x_1^2-ax_1$$

$$\Rightarrow 4xx_1-2ax-2ax_1-by=4x_1^2-4ax_1$$

$$\Rightarrow 2(2x_1-a)x-by+2ax_1-4x_1^2=0$$

This passes through the point $(a, b/2)$. This implies that

$$2(2x_1-a)a-b\left(\frac{b}{2}\right)+2ax_1-4x_1^2=0$$

$$\Rightarrow -4x_1^2+6ax_1-\left(2a^2+\frac{b^2}{2}\right)=0$$

(which has two distinct real roots)

$$\Rightarrow 4x_1^2-6ax_1+\left(2a^2+\frac{b^2}{2}\right)=0$$

(which has two distinct real roots)

$$\Rightarrow (6a)^2>4(4)\left(2a^2+\frac{b^2}{2}\right)$$

$$\Rightarrow 9a^2>2(4a^2+b^2)$$

$$\Rightarrow a^2>2b^2$$

$$\left|\frac{a}{b}\right|>\sqrt{2}$$

Therefore

$$\frac{a}{b}<-\sqrt{2} \quad \text{or} \quad \frac{a}{b}>\sqrt{2}$$

Answer: (A) → (r)

(B) $O=(0,0)$ and $A=(5,0)$ are the centres and r, 3 are the radii of the circles. The two circles intersect in two distinct points. So

$$|r-3|<OA<r+3$$

$$\Leftrightarrow |r-3|<5<r+3$$

COMPREHENSION-TYPE QUESTIONS

Comprehension-type questions consist of a small passage, followed by three multiple choice questions. The questions are of single correct answer type.

Comprehension-Type Questions

1. **Passage:** Consider the straight line $3x+y+4=0$. Answer the following questions.

   (i) The point on the line $3x+y+4=0$ which is equidistant from the points $(-5, 6)$ and $(3, 2)$ is

   (A) $(-1,-1)$ (B) $(-2, 2)$
   (C) $(-3, 5)$ (D) $(-\frac{1}{3},-3)$

   (ii) Equation of the line passing through the point $(1, 1)$ and perpendicular to the given line is

   (A) $x-3y+4=0$ (B) $x-3y+5=0$
   (C) $x-3y-4=0$ (D) $x-3y+2=0$

   (iii) If the line $y+5=k(x-3)$ is parallel to the given line then the area of the triangle formed by this line and the coordinate axes (in sq. units) is

   (A) $\frac{8}{3}$ (B) $\frac{16}{3}$ (C) 4 (D) 5

Solution:

(i) Let $A=(-5, 6)$ and $B=(3, 2)$. The slope of AB is

$$\frac{6-2}{-5-3}=\frac{-1}{2}$$

and the midpoint of $AB=(-1, 4)$. Hence, the perpendicular bisector of the segment $\overline{AB}$ is $y-4=2(x+1)$ or $2x-y+6=0$. Solving this equation and the given line equations, we have $x=-2$ and $y=2$. Thus, $(-2, 2)$ is the point on the given line which is equidistant from both $A(-5,6)$ and $B(3,2)$.

Answer: (B)

(ii) Line perpendicular to the given line is of the form

$$y=\frac{1}{3}x+c$$

This line passes through $(1, 1)$. It implies that

$$1=\frac{1}{3}+c \Rightarrow c=\frac{2}{3}$$

Thus, the required line is

$$y=\frac{x}{3}+\frac{2}{3} \text{ or } x-3y+2=0$$

Answer: (D)

INTEGER-TYPE QUESTIONS

The questions in this section are numerical problems for which no choices are provided. The students are required to find the exact answers to numerical problems and enter the same in OMR sheets. Answers can be one-digit or two-digit numerals.

Integer Answer Type Questions

1. The area of the quadrilateral formed by the lines $|x|+|y|=1$ is __________ sq. unit.

Solution: The given quadrilateral is a square with vertices $(1, 0)$, $(0, 1)$, $(-1,-1)$ and $(0,-1)$, and hence its area is $(\sqrt{2})^2=2$.

Answer: 2

2. Two rays in the first quadrant, $x+y=|a|$ and $ax-y=1$, intersect each other in the interval $a\in(a_0,\infty)$. The value of a_0 is __________. **(IIT-JEE 2006)**

Solution: Solving the given two equations, we have

$$x=\frac{1+|a|}{1+a} \text{ and } y=ax-1=\frac{a(1+|a|)}{1+a}-1=\frac{a|a|-1}{1+a}$$

Since the two rays intersect each other in the first quadrant, we have $x>0$ and $y>0$ which implies that

$$1+a>0 \text{ and } a|a|-1>0$$

Therefore, if $-1<a<0$, then the $a(-a)-1>0$ which is not sensible. Hence, $a\notin(-1,0)$. If $a=0$, then the lines $x+y=0$ and $y=-1$ intersect in fourth quadrant. Thus, $a\neq 0$. Hence, $a>0$ and $a^2-1>0 \Rightarrow a>1$. Therefore, $a_0=1$.

Answer: 1

3. The orthocentre of the triangle formed by the lines $x+y=1$, $2x+3y=6$ and $4x-y+4=0$ lies in the quadrant whose number is __________.

Solution: Solving the above equations taken two by two, the vertices of the triangle are

$$A\left(\frac{-3}{5},\frac{8}{5}\right),\ B(-3, 4), \text{ and } C\left(\frac{-3}{7},\frac{16}{7}\right)$$

The equation of the altitude drawn from A to the side BC is

$$y-\frac{8}{5}=\frac{3}{2}\left(x+\frac{3}{5}\right)$$

$$\Rightarrow 3x-2y=-5 \quad (2.126)$$

Again the equation of the altitude from B onto CA is

$$y-4=\frac{-1}{4}(x+3)$$

$$\Rightarrow x+4y=13 \quad (2.127)$$

Solving Eqs. (2.126) and (2.127), the coordinates of the orthocentre are

EXERCISES

For self-assessment, each chapter has adequate number of exercise problems where the questions have been subdivided into various categories which include Multiple Choice Questions as asked in JEE (Main & Advanced).

EXERCISES

Single Correct Choice Type Questions

1. Equation of the line through (0, –3) and having slope –2 is
   (A) $y-2x+3=0$ (B) $y+2x-3=0$
   (C) $y+2x+3=0$ (D) $y-2x-3=0$

2. Equation of the line passing through (–5, 2) and (3, 2) is
   (A) $x-2=0$ (B) $y-2=0$
   (C) $x+2=0$ (D) $y+2=0$

   (A) $\left(\frac{-19}{8}, \frac{5}{4}\right)$ (B) $\left(\frac{19}{8}, \frac{5}{4}\right)$
   (C) $\left(\frac{-5}{4}, \frac{19}{8}\right)$ (D) $\left(\frac{5}{4}, \frac{-19}{8}\right)$

5. If the area of the triangle formed by the line $2x + 3y + c = 0$ with coordinate axes is 27 sq. units, then c is equal to
   (A) ±16 (B) ±15 (C) ±8 (D) ±18

Multiple Correct Choice Type Questions

1. If the distance of the line $8x + 15y + \lambda = 0$ from the point (2, 3) is equal to 5 units, then the value of λ is
   (A) 24 (B) –24 (C) 146 (D) –146

2. If the line $\sqrt{3}x+y-9=0$ is reduced to the form $x\cos\alpha + y\sin\alpha = p$, then
   (A) $\alpha = 60°$ (B) $\alpha = 30°$
   (C) $p=\frac{9}{2}$ (D) $p=9$

3. If l is the line passing through the point (–2, 3) and perpendicular to the line $2x - 3y + 6 = 0$, then
   (A) the slope of $l = -\frac{3}{2}$
   (B) the line l passes through (0, 0)
   (C) the intercept on the axes are 2, 3
   (D) the line l forms a triangle of area 5 sq. units with the coordinate axes

4. If l is the line passing through the point (2, –3) and is parallel to the line joining the points (4, 1) and (–2, 3), then
   (A) (–10, –1) is a point on l
   (B) the slope of l is 6

Matrix-Match Type Questions

In each of the following questions, statements are given in two columns, which have to be matched. The statements in *column I* are labeled as **(A)**, **(B)**, **(C)** and **(D)**, while those in *column II* are labeled as **(p)**, **(q)**, **(r)**, **(s)** and **(t)**. Any given statement in *column I* can have correct matching with *one or more* statements in *column II*. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following example.

Example: If the correct matches are (A) → (p), (s), (B) → (q), (s), (t), (C) → (r), (D) → (r), (t), that is if the matches are (A) → (p) and (s); (B) → (q), (s) and (t); (C) → (r); and (D) → (r), then the correct darkening of bubbles will look as follows:

1. Let σ be the system of lines passing through the intersection of the lines $x + y - 1 = 0$ and $x - y - 1 = 0$. Match the items of Column I with those of Column II.

Column I	*Column II*
(A) Equation of the line belonging to σ and passing through the point (2, 3) is	(p) $2x - y - 2 = 0$
(B) Equation of the line belonging to σ and parallel to the line $y =$	(q) $x + y - 1 = 0$

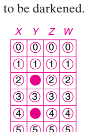
Comprehension Type Questions

1. **Passage:** Let $u \equiv x + y = 0$, $A = (1, 2)$ and $B = (3, –1)$. Answer the following questions.
   (i) If M is a point on the line $u = 0$ such that $AM + BM$ is minimum, then the reflection of M on the line $y = x$ is
   (A) (2, –2) (B) (–2, 2)
   (C) (1, –1) (D) (–1, 1)
   (ii) If M is a point on $u = 0$ such that $|AM - BM|$ is maximum, then the distance between M and the point $N(1, 1)$ is
   (A) $3\sqrt{5}$ (B) $5\sqrt{2}$ (C) 7 (D) 10

   (i) Area of the triangle in square units is
   (A) $\frac{1}{\sqrt{3}}$ (B) $\frac{2}{\sqrt{3}}$ (C) $\frac{1}{2\sqrt{3}}$ (D) $\frac{1}{2\sqrt{2}}$
   (ii) The gradients of the two sides AB and AC are
   (A) $\sqrt{3}, \frac{1}{\sqrt{3}}$ (B) $\sqrt{2}, \frac{1}{\sqrt{2}}$
   (C) $\sqrt{2}+1, \sqrt{2}-1$ (D) $2+\sqrt{3}, 2-\sqrt{3}$
   (iii) The circumradius of the triangle is
   (A) $\frac{1}{3}$ (B) $\frac{\sqrt{2}}{3}$ (C) $\frac{1}{\sqrt{3}}$ (D) $\frac{1}{\sqrt{2}}$

Integer Answer Type Questions

The answer to each of the questions in this section is a non-negative integer. The appropriate bubbles below the respective question numbers have to be darkened. For example, as shown in the figure, if the correct answer to the question number Y is 246, then the bubbles under Y labeled as 2, 4, 6 are to be darkened.



4. In ΔABC, the equations of the madians AD and BE, respectively, are $2x + 3y - 6 = 0$ and $3x - 2y - 10 = 0$. If $AD = 6$, $BE = 11$, then $\frac{1}{11}$(Area of ΔABC) is ______.

5. $P(1, 2)$, $Q(4, 6)$, $R(5, 7)$ and $S(a, b)$ are the vertices of the parallogram $PQRS$. Then, $a + b$ is equal to ______.

6. The area of the triangle formed by the line $x + y = 3$ and the angle bisectors of the pair of lines $x^2 - y^2 + 2y - 1 = 0$ is ______ sq. unit.

7. A straight line through the origin O meets the par-

ANSWERS

The Answer key at the end of each chapter contains answers to all exercise problems.

ANSWERS

Single Correct Choice Type Questions

1. (A)
2. (B)
3. (D)
4. (B)
5. (A)
6. (D)
7. (B)
8. (A)
9. (D)
10. (B)
11. (D)
12. (D)
13. (A)
14. (B)
15. (C)
16. (B)
17. (D)
18. (C)
19. (B)
20. (C)

Multiple Correct Choice Type Questions

1. (B), (C)
2. (A), (C)
3. (A), (C)
4. (A), (B), (C)
5. (A), (B), (C)

Matrix-Match Type Questions

1. (A) → (q); (B) → (t); (C) → (p), (r); (D) → (s)
2. (A) → (r); (B) → (p); (C) → (s); (D) → (t)
3. (A) → (t); (B) → (p); (C) → (q); (D) → (r)

Comprehension Type Questions

1. (i) →(A); (ii) → (C); (iii) → (A)
2. (i) →(D); (ii) → (A); (iii) → (D)
3. (i) →(C); (ii) → (B); (iii) → (A)

Integer Answer Type Questions

1. 4
2. 5
3. 3
4. 1
5. 3
6. 3

Cut from here

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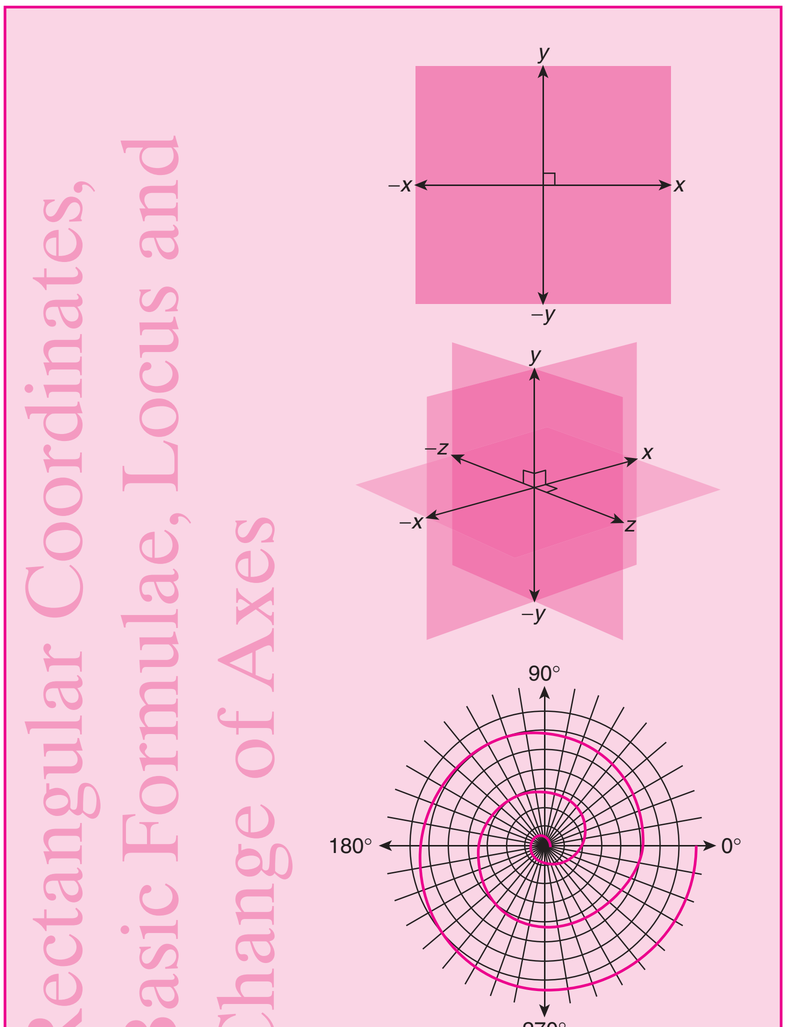
Contents

Rectangular Coordinates, Basic Formulae, Locus and Change of Axes

1



Equiangular spiral

Contents

The **locus** of a point is the path traced out by the point when it moves according to a given rule (or rules). In other words, a locus is the path of a single moving point that obeys certain conditions.

1.1 | Rectangular Coordinates

Geometry is a thought-provoking subject for any genuine mathematics student. Geometry was initially pursued by the Indians and Greeks. That kind of geometry is called *pure geometry*. Even though pure geometry is very interesting, sometimes the proofs needed constructions and also they were cumbersome. At this stage, the concept of studying geometry by using algebra was introduced by Rene Descartes (1596–1650 AD). Thus, the modern analytic geometry emerged and is called "Cartesian geometry" named after Rene Descartes. In the following section, we discuss the rectangular Cartesian coordinates.

1.1.1 Rectangular Cartesian Coordinates

Select a plane and in that plane, let $\overrightarrow{X'OX}$ and $\overrightarrow{Y'OY}$ be two perpendicularly intersecting lines (intersecting at O). $\overrightarrow{X'OX}$ is called x-axis, $\overrightarrow{Y'OY}$ is called y-axis and O is called the origin. Further, $\overrightarrow{OX}$ and $\overrightarrow{OY}$ are called positive directions, and $\overrightarrow{OX'}$ and $\overrightarrow{OY'}$ are called negative directions. Let P be a point in the plane. From P, draw perpendicular PL to x-axis and PM perpendicular to y-axis (see Fig. 1.1).

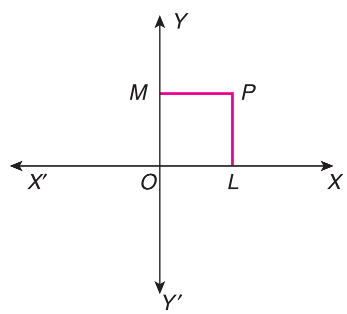


FIGURE 1.1

Let the magnitudes of OL and OM be x and y, respectively.

1. If $\overrightarrow{OL}$ and $\overrightarrow{OM}$ are in the directions of $\overrightarrow{OX}$ and $\overrightarrow{OY}$, then we say that x is the x-coordinate of P and y is the y-coordinate of P and we write $P = (x, y)$.
2. If $\overrightarrow{OL}$ is in the direction of $\overrightarrow{OX'}$ and $\overrightarrow{OM}$ is in the direction of $\overrightarrow{OY}$, then we write $P = (-x, y)$.
3. If both $\overrightarrow{OL}$ and $\overrightarrow{OM}$ are in the directions of $\overrightarrow{OX'}$ and $\overrightarrow{OY'}$, then we write $P = (-x, -y)$.
4. If $\overrightarrow{OL}$ is in the direction of $\overrightarrow{OX}$ and $\overrightarrow{OM}$ is in the direction of $\overrightarrow{OY'}$, then we write $P = (x, -y)$.

QUICK LOOK 1

$O = (0, 0)$, $P = (x, 0)$ lies on the x-axis, $Q = (0, y)$ lies on the y-axis.

DEFINITION 1.1 **Quadrants** The regions bounded by $(\overrightarrow{OX}, \overrightarrow{OY})$, $(\overrightarrow{OX'}, \overrightarrow{OY})$, $(\overrightarrow{OX'}, \overrightarrow{OY'})$ and $(\overrightarrow{OX}, \overrightarrow{OY'})$ are the first, second, third and fourth quadrants, respectively.

Sign of the Coordinates

1. (x, y) lies in the first quadrant $\Leftrightarrow x > 0, y > 0$.
2. (x, y) lies in the second quadrant $\Leftrightarrow x < 0, y > 0$.
3. (x, y) lies in the third quadrant $\Leftrightarrow x < 0, y < 0$.
4. (x, y) lies in the fourth quadrant $\Leftrightarrow x > 0, y < 0$.

From the above list one can conclude that $(\sqrt{2}, 1)$, $(-3, \sqrt{2})$, $(-2, -3/2)$ and $(1/2, -\sqrt{3})$ belong to the first, second, third and fourth quadrants, respectively.

1.2 | Basic Formulae

In this section, we recall some of the basic formulae which have been discussed in your earlier mathematics classes in school. We state such formulae without proofs.

1.2.1 Distance Between Two Points

1. The distance between two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by

$$AB = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

2. Distance between origin O and the point $P(x, y)$ is

$$OP = \sqrt{(x-0)^2 + (y^2 - 0)^2} = \sqrt{x^2 + y^2}$$

3. If $A = (x_1, 0)$ and $B = (x_2, 0)$, then

$$AB = |x_1 - x_2|$$

Also if $A = (0, y_1)$ and $B = (0, y_2)$ then

$$AB = |y_1 - y_2|$$

1.2.2 Notation

Let A and B be two points. Thus, the line segment connecting A and B is denoted by $\overline{AB}$ and the line through A and B by $\overleftrightarrow{AB}$. The ray from A to B is denoted by $\overrightarrow{AB}$ (readers please observe the arrowheads in all cases). In vectors $\overrightarrow{AB}$ means a line segment $\overline{AB}$ having direction from A to B.

1.2.3 Section Formulae

1. Let A and B be two points and P be a point on $\overline{AB}$ lying between A and B. Then we say that P divides $\overline{AB}$ internally in the ratio $AP{:}PB$ (see Fig. 1.2).

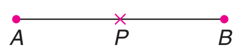


FIGURE 1.2

2. If P lies on the line $\overleftrightarrow{AB}$ (not in between A and B), then we say that P divides externally the segment $\overline{AB}$ and we write the ratio as $-(AP){:}PB$ or $AP{:}-PB$. The minus sign indicates external division.

3. The coordinates of a point P which divides the segment joining $A(x_1, y_2)$ and $B(x_2, y_2)$ in the ratio $m{:}n$ $(m + n \neq 0)$ are

$$P = \left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right)$$

If m/n is positive, then division is internal division and if m/n is negative, the division is external division.

QUICK LOOK 2

The coordinates of the midpoint of the segment joining $A(x_1, y_1)$ and $B(x_2, y_2)$ are

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Examples

1. The distance between the points $A(2,3)$ and $B(-2,2)$ is $\sqrt{(2+2)^2+(3-2)^2}=\sqrt{17}$.
2. The distance between the points $P(\cos\alpha, \cos\beta)$ and $Q(\sin\alpha, \sin\beta)$ is
$$\sqrt{(\cos\alpha-\sin\alpha)^2+(\cos\beta-\sin\beta)^2}=\sqrt{2-\sin 2\alpha-\sin 2\beta}$$
3. The distance between the points $A(at^2, 2at)$ and $B(a/t^2, -2a/t)$ is

$$\begin{aligned}\sqrt{a^2\left(t^2-\frac{1}{t^2}\right)^2+4a^2\left(t+\frac{1}{t}\right)^2} &= |a|\left|t+\frac{1}{t}\right|\sqrt{\left(t-\frac{1}{t}\right)^2+4}\\ &= |a|\left|t+\frac{1}{t}\right|\sqrt{\left(t+\frac{1}{t}\right)^2}\\ &= |a|\left(t+\frac{1}{t}\right)^2\\ &= a\left(t+\frac{1}{t}\right)^2 \quad \text{if } a>0\end{aligned}$$

Examples

1. The coordinates of the point which divides the segment joining $A(2, -3)$ and $B(3, 2)$ in the ratio 1:2 are
$$\left(\frac{2\times 2+1\times 3}{1+2}, \frac{2\times -3+1\times 2}{1+2}\right)=\left(\frac{7}{3}, \frac{-4}{3}\right)$$
2. If P divides $\overline{AB}$ in the ratio 2:1, then
$$P=\left(\frac{8}{3}, \frac{1}{3}\right)$$

Convention: If a point P divides the line joining A and B internally in the ratio 1:2 or 2:1, then P is called point of *trisection* of $\overline{AB}$.

1.2.4 Area of a Triangle

The area of the triangle whose vertices are $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ is
$$\frac{1}{2}|x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)|=\frac{1}{2}|\det A|$$

where $A=\begin{bmatrix}x_1 & x_2 & x_3\\ y_1 & y_2 & y_3\\ 1 & 1 & 1\end{bmatrix}$ is a 3×3 matrix.

Generally, area of a triangle is denoted by Δ (see Vol. 2).

QUICK LOOK 3

Three points $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ are collinear if and only if
$$\begin{vmatrix}x_1 & x_2 & x_3\\ y_1 & y_2 & y_3\\ 1 & 1 & 1\end{vmatrix}=0$$

QUICK LOOK 4

Area of quadrilateral $ABCD$ = Area of ΔABC + Area of ΔACD

1.2.5 Some Points and Circles Associated with a Triangle

In this section, we will discuss some important terms associated with a triangle. The vertices of the triangle would be considered as A, B, C.

Centroid

The point of concurrence of the medians of a triangle is called the *centroid* of the triangle and is denoted by G. The coordinates of the centroid of the triangle with vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3) are

$$\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$$

Note that centroid trisects each median.

Incentre

The internal bisectors of the angles of a triangle are concurrent. This point is called the *incentre* of the triangle and is denoted by I. The incentre is equidistant from three sides and this equal distance r is called the *inradius* of the triangle. If a circle is drawn with centre at I and radius r, then this circle touches the sides of the triangle internally. (For more details, see Chapter 4, Vol. 2.)

Incentre formulae

Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of a triangle and $BC = a$, $CA = b$ and $AB = c$. Thus, the incentre is

$$I = \left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}\right)$$

Excentre

The point where the external bisectors of two angles and the internal bisector of one angle are concurrent is called the *excentre*. Thus, there are three excentres, namely, excentre opposite to the vertex A (denoted by I_1), excentre opposite to the vertex B (denoted by I_2) and excentre opposite to the vertex C (denote by I_3). Also

$$I_1 = \left(\frac{-ax_1+bx_2+cx_3}{-a+b+c}, \frac{-ay_1+by_2+cy_3}{-a+b+c}\right)$$
$$I_2 = \left(\frac{ax_1-bx_2+cx_3}{a-b+c}, \frac{ay_1-by_2+cy_3}{a-b+c}\right)$$
$$I_3 = \left(\frac{ax_1+bx_2-cx_3}{a+b-c}, \frac{ay_1+by_2-cy_3}{a+b-c}\right)$$

Note: For inradius and three exradii, see Chapter 4, Vol. 2.

Circumcentre, Circumradius and Circumcircle

The point where the perpendicular bisectors of the sides of a triangle are concurrent is called the *circumcentre* of the triangle. It is equidistant from the vertices of the triangle. This equal distance is denoted by R and is called the *circumradius*. Thus, if a circle is drawn with the circumcentre as centre and circumradius R as radius, then that circle will pass through the vertices of the triangle. Such a circle is called the *circumcircle* of the triangle.

Orthocentre

The point of concurrence of the three altitudes of a triangle is called the *orthocentre* of the triangle.

Note: The circumcentre and orthocentre lie inside the triangle if the triangle is acute.

Nine-Point Circle and Nine-Point Centre

In a triangle, the feet of the altitudes, the midpoints of the three sides and the midpoints of the segments joining the orthocentre and the vertices are concyclic. Such a circle is called the *nine-point circle* and its centre is called the *nine-point centre* of the triangle. Also the nine-point centre N is the midpoint of the segment joining the circumcentre and the orthocentre. The radius of the nine-point circle is half of the circumradius of the triangle because if ΔDEF is the pedal triangle of ΔABC, then the angles of ΔDEF are $180^\circ - 2A$, $180^\circ - 2B$ and $180^\circ - 2C$ and the sides are $a\cos A$, $b\cos B$ and $c\cos C$ (see Theorem 4.23, page 223, Chapter 4, Vol. 2). If we use sine rule to ΔDEF, then we obtain that $R/2$ is the circumradius of ΔDEF which is the radius of the nine-point circle.

QUICK LOOK 5

Nine-point circle of ΔABC is the circumcircle of the pedal triangle of ΔABC as well as the circle passing through the midpoints of the sides.

IMPORTANT NOTE

In a triangle ABC,

1. The circumcentre, the centroid, the nine-point centre and the orthocentre are collinear in the given order (see Definition 4.7, page 228, Chapter 4, Vol. 2).
2. The centroid G divides the segment joining the circumcentre and orthocentre in the ratio 1:2.
3. The nine-point centre is the midpoint of the segment joining the circumcentre and the orthocentre.

Pedal Line (or Simson's Line)

THEOREM 1.1 The feet of the perpendiculars drawn from a point on the circumcircle of a triangle onto its sides are collinear. This line is called *Pedal line or Simson's line* of the triangle. The converse of this theorem is also true. That is, if from any point in the plane of a triangle, the feet of the perpendiculars onto the sides are collinear, then the point lies on the circumcircle.

We can prove these two results by using plane geometry or what is called pure geometry. The line LMN is the Pedal line (see Fig. 1.3).

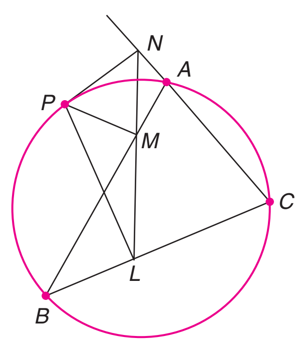


FIGURE 1.3

1.3 | Locus

In pure geometry, using congruent triangles property, it was proved that all the points on a line bisecting perpendicularly the segment joining two given points are equidistant from the two given points and this line is called the perpendicular bisector of the segment joining the two points. Of course, any point which is equidistant from these two given points lies on this line. Thus, describing a set of points satisfying a certain geometrical condition(s) is difficult in pure geometry. That is why mathematicians introduced analytical geometry (a combination of algebra and pure geometry) and described the locus by algebraic equations. In this section, we introduce the concept of locus, equation of a locus and few examples.

DEFINITION 1.2 **Locus** Let P be a geometrical condition(s) and S be the set of all points in the plane which satisfy P. Then S is called a locus.

QUICK LOOK 6

S is the locus of a geometrical condition(s) $P \Leftrightarrow$ every point of S satisfies the condition P and every point satisfying P belongs to S.

DEFINITION 1.3 **Equation of the Locus** Let S be a locus and $f(x, y) = 0$ be an algebraic equation in x and y. If every point $P(x, y)$ belonging to S satisfies the equation $f(x, y) = 0$ and any point in the plane satisfying the equation $f(x, y) = 0$ belongs to S, then $f(x, y) = 0$ is called the equation of the locus S.

Here afterwards, we will describe a locus by its algebraic equation.

1.4 | Change of Axes

In analytical plane, selection of the rectangular coordinate axes is arbitrary. When the axes change, the coordinates of the point also change. The study of the relations between the original coordinates and the changed coordinates is called change of axes. This change of axes is sometimes necessary to make the equation of a curve as simple as possible to prove certain properties. Not that the same properties could not be proved otherwise, but the working out of the proof would be more complicated. It is by experience that the student will learn the best method for change of axes. In the following, we discuss three types of change of axes. We begin with shifting of the origin.

1.4.1 Shifting of Origin Without Changing the Directions of the Axes

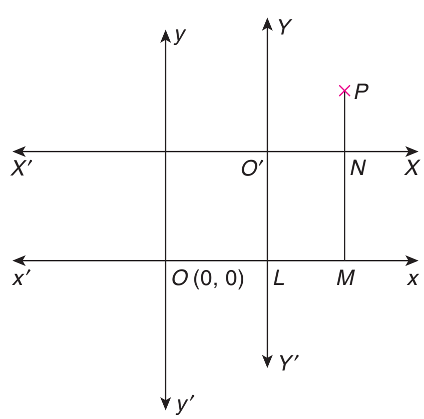


FIGURE 1.4

Let $\overleftrightarrow{x'Ox}$ and $\overleftrightarrow{y'Oy}$ be the coordinate axes. Let $O'(x_1, y_1)$ be a point. Through O' draw lines $\overleftrightarrow{X'O'X}$ parallel to $\overleftrightarrow{x'Ox}$ and $\overleftrightarrow{Y'O'Y}$ parallel to $\overleftrightarrow{y'Oy}$. Suppose $\overleftrightarrow{Y'O'Y}$ meets $\overleftrightarrow{x'Ox}$ in L. We call ($\overleftrightarrow{x'Ox}$, $\overleftrightarrow{y'Oy}$) as the old system of axes and ($\overleftrightarrow{X'O'X}$, $\overleftrightarrow{Y'O'Y}$) as the new system of axes. Now, every point in the coordinate plane will have two systems of coordinates, namely old coordinates (with respect to old axes) and new coordinates (with respect to new axes).

Suppose P is a point whose old and new coordinates are (x, y) and (X, Y), respectively. Draw PM perpendicular to old x-axes meeting the new X-axes in N. Now

$$x = OM = OL + LM = OL + O'N = x_1 + X = X + x_1$$

and

$$y = PM = PN + NM = PN + O'L = Y + y_1$$

Thus, the relations between old and new coordinates of the point P are

$$\begin{aligned} x &= X + x_1 \\ y &= Y + y_1 \end{aligned} \tag{1.1}$$

QUICK LOOK 7

1. Shifting of the origin is also called *TRANSLATION* of axes. The effect on the coordinates is

   Old coordinate = New coordinate + Corresponding coordinate of the new origin

2. Equation $f(x, y) = 0$ of a curve will be changed to $f(X + x_1, Y + y_1) = 0$.

Note: The new origin O' may be in any quadrant. Still the relation between old and new coordinates is the same.

1.4.2 Rotation of Axes (Without Changing the Origin)

Let $\overleftrightarrow{x'Ox}$ and $\overleftrightarrow{y'Oy}$ be original axes (old axes) (see Fig. 1.5). Rotate $\overrightarrow{Ox}$ about O through an angle θ in the anticlockwise sense. Let the new axes be $\overleftrightarrow{X'OX}$ and $\overleftrightarrow{Y'OY}$ ($\overrightarrow{OX'}$ and $\overrightarrow{OY'}$ are not shown in the figure). Let P be a point in the plane and let its old and new coordinates be (x, y) and (X, Y), respectively. Draw PL perpendicular to $\overleftrightarrow{x'Ox}$ (old x-axis) and PM perpendicular to $\overleftrightarrow{X'OX}$ (new X-axis). Draw MQ perpendicular to old x-axis and MN perpendicular PL. Now

$$\underline{|QOM} = \theta = \underline{|NMO} \Rightarrow \underline{|PMN} = 90^\circ - \theta$$

so that $\underline{|NPM} = \theta$. Also

$$x = OL = OQ - LQ - MN = OQ - MN \tag{1.2}$$

From ΔMOQ,

$$\cos\theta = \frac{OQ}{OM}$$

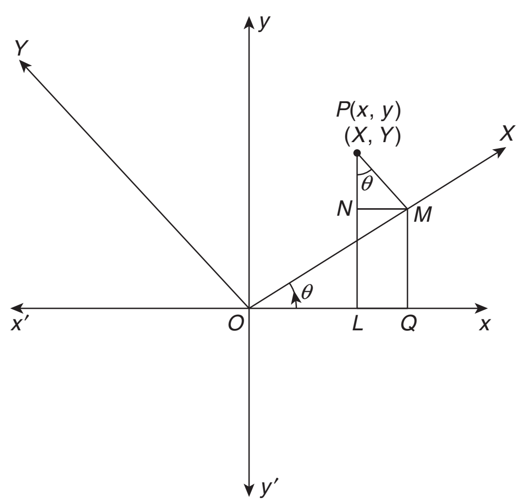


FIGURE 1.5

Therefore

$$OQ = OM\cos\theta = X\cos\theta \tag{1.3}$$

Also, from ΔPNM,

$$\sin\theta = \frac{MN}{PM}$$

Therefore

$$MN = PM\sin\theta = Y\sin\theta \tag{1.4}$$

Hence from Eqs. (1.2), (1.3) and (1.4), we have

$$x = X\cos\theta - Y\sin\theta$$

Again,

$$\begin{aligned} y &= PL \\ &= PN + NL \\ &= PN + MQ \\ &= MQ + PN \\ &= X\sin\theta + Y\cos\theta \end{aligned}$$

Therefore

$$y = X\sin\theta + Y\cos\theta$$

The above-mentioned relations (in color screen) can be written in the form of a matrix equation as follows:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

Notes:

1. If the origin is shifted to the point (x_1, y_1) and the axes are rotated through θ in the anticlockwise sense, then the old coordinates (x, y) will be transformed to

$$x = X\cos\theta - Y\sin\theta + x_1, \quad y = X\sin\theta + Y\cos\theta + y_1$$

2. If the rotation is clockwise, then we have to replace θ with $-\theta$.

DEFINITION 1.4 If a, b, h are real and at least one of a, h, b is not zero, then $ax^2 + 2hxy + by^2$ is called *second degree homogeneous expression* and $ax^2 + 2hxy + by^2 = 0$ is called *second degree homogeneous equation*.

DEFINITION 1.5 If a, h, b, g, f, c are real and at least one of a, h, b is not zero, then $ax^2 + 2hxy + by^2 + 2gx + 2fy + c$ is called *second degree general expression* and $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is called *second degree general equation*.

Examples

1. $2x^2 + xy + y^2$

2. $x^2 + \sqrt{2}xy + y^2$

3. $2x^2 + 3xy + y^2 + x - y + 1$

4. $x^2 + 2xy + y^2 - x + y + 1$

DEFINITION 1.6 In the second degree general expression, ax^2, $2hxy$, by^2 are called second degree terms; gx, fy are called first degree terms; and c is called constant term.

THEOREM 1.2 If $h^2 \neq ab$, then to remove the first degree terms of the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, the origin is to be shifted to the point $\left(\dfrac{hf-bg}{ab-h^2}, \dfrac{gh-af}{ab-h^2}\right)$.

PROOF Suppose that the origin is shifted to the point (x_1, y_1) and let $x = X + x_1$, $y = Y + y_1$. Therefore, the given equation is transformed to

$$a(X+x_1)^2 + 2h(X+x_1)(Y+y_1) + b(Y+y_1)^2 + 2g(X+x_1) + 2f(Y+y_1) + c = 0$$

This implies

$$\begin{aligned} &aX^2 + 2hXY + bY^2 + 2(ax_1 + hy_1 + g)X + 2(hx_1 + by_1 + f)Y \\ &\quad + ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c = 0 \end{aligned}$$

This further implies

$$\begin{aligned} &aX^2 + 2hXY + bY^2 + 2(ax_1 + hy_1 + g)X + 2(hx_1 + by_1 + f)Y \\ &\quad + (ax_1 + hy_1 + g)x_1 + (hx_1 + by_1 + f)y_1 + (gx_1 + fy_1 + c) = 0 \end{aligned} \tag{1.5}$$

Now, put $ax_1 + hy_1 + g = 0$ and $hx_1 + by_1 + f = 0$. Solving we get

$$x_1 = \frac{hf-bg}{ab-h^2} \quad \text{and} \quad y_1 = \frac{gh-af}{ab-h^2}$$

Therefore, the origin is to be shifted to the point $\left(\dfrac{hf-bg}{ab-h^2}, \dfrac{gh-af}{ab-h^2}\right)$ so that Eq. (1.5) will be

$$aX^2 + 2hXY + bY^2 + gx_1 + fy_1 + c = 0$$

where x_1 and y_1 are as defined above. ■

IMPORTANT NOTE

1. Under shifting of origin, the second degree terms of a second degree equation will not change.
2. Under change of axes, only the coordinates of the points will change, but the distance between two points as well as the areas will not change.

THEOREM 1.3 To remove the xy-term from the equation $ax^2 + 2hxy + by^2 = 0$, the axes are to be rotated through the angle $(1/2)\,\mathrm{Tan}^{-1}[2h/(a-b)]$ when $a \neq b$ and through the angle $\pi/4$ when $a = b$.

PROOF Suppose axes are rotated through an angle θ in the anticlockwise sense. Let $x = X\cos\theta - Y\sin\theta$ and $y = X\sin\theta + Y\cos\theta$. Therefore, the given equation will be transformed to

$$a(X\cos\theta - Y\sin\theta)^2 + 2h(X\cos\theta - Y\sin\theta)(X\sin\theta + Y\cos\theta) + b(X\sin\theta + Y\cos\theta)^2 = 0$$

In this equation, the coefficient of xy is

$$\begin{aligned} &-2a\sin\theta\cos\theta + 2h(\cos^2\theta - \sin^2\theta) + 2b\sin\theta = 0 \\ &\Rightarrow (b-a)\sin 2\theta + 2h\cos 2\theta = 0 \\ &\Rightarrow \tan 2\theta = \frac{2h}{a-b}, \text{ if } a \neq b \\ &\Rightarrow \theta = \frac{1}{2}\mathrm{Tan}^{-1}\left(\frac{2h}{a-b}\right), \text{ if } a \neq b \end{aligned}$$

If $a = b$, then $\cos 2\theta = 0 \Rightarrow 2\theta = \pi/2$ or $\theta = \pi/4$. ■

WORKED-OUT PROBLEMS

Since this chapter is only to recall what the students have learnt in their junior classes and practice the important formulae, we give a combination of both subjective and objective type questions here. Students should practice all questions.

Questions Based on Basic Formulae

1. Find the area of the triangle whose vertices are $A(2,-3)$, $B(4,2)$ and $C(-5,-2)$.

Solution: We have

$$\begin{aligned}\text{Area} &= \frac{1}{2}|x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)|\\ &= \frac{1}{2}|2[2-(-2)]+4[-2-(-3)]-5(-3-2)|\\ &= \frac{1}{2}|8+4+25|\\ &= \frac{37}{2}\text{ sq. units}\end{aligned}$$

2. Show that the area of the triangle with vertices $A(-3,4)$, $B(6,2)$ and $C(4,-3)$ is 24.5 sq. units.

Solution: We have

$$\begin{aligned}\text{Area} &= \frac{1}{2}|x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)|\\ &= \frac{1}{2}|-3(2+3)+6(-3-4)+4(4-2)|\\ &= \frac{1}{2}|-15-42+8|\\ &= \frac{1}{2}|-49|\\ &= 24.5\text{ sq. units}\end{aligned}$$

3. Prove that the points $A(x, x-2)$, $B(x+3, x)$ and $C(x+2, x+2)$ form a triangle whose area is independent of x.

Solution: We have

$$\begin{aligned}\text{Area of }\Delta ABC &= \frac{1}{2}|x[x-(x+2)]+(x+3)[x+2-(x-2)]\\ &\qquad +(x+2)(x-2-x)|\\ &= \frac{1}{2}|-2x+4x+12-2x-4|\\ &= \frac{8}{2}\\ &= 4\text{ sq. units which is independent of } x\end{aligned}$$

4. Show that the three points $P(a, b+c)$, $Q(b, c+a)$ and $R(c, a+b)$ are collinear.

Solution: We have

$$\begin{aligned}\text{Area of }\Delta PQR &= \frac{1}{2}\left|\begin{matrix}a(c+a-a-b)+b(a+b-b-c)\\ +c(b+c-c-a)\end{matrix}\right|\\ &= \frac{1}{2}|ac-ab+ba-bc-cb-ca|\\ &= \frac{1}{2}|0|\\ &= 0\end{aligned}$$

Therefore P, Q, R are collinear (see Quick Look 3).

5. If the three points $A(3,1)$, $B(2\lambda, 3\lambda)$ and $C(\lambda, 2\lambda)$ are collinear, find the value of λ.

Solution: A, B and C are collinear implies

$$\begin{aligned}&\text{Area of }\Delta ABC = 0\\ &\Rightarrow \frac{1}{2}|3(3\lambda-2\lambda)+2\lambda(2\lambda-1)+\lambda(1-3\lambda)| = 0\\ &\Rightarrow \frac{1}{2}\left|3\lambda+4\lambda^2-2\lambda+\lambda-3\lambda^2\right| = 0\\ &\Rightarrow \left|\lambda^2+2\lambda\right| = 0\\ &\Rightarrow \lambda = 0 \text{ or } \lambda = -2\end{aligned}$$

Now

$\lambda = 0 \Rightarrow B = C$ so that A, B and C are collinear.
$\lambda = -2 \Rightarrow A = (3,1)$, $B = (-4,-6)$ and $C = (-2,-4)$

6. If the three points $(a, 0)$, $(0, b)$ and $(2, 2)$ are collinear, then show that

$$\frac{1}{a}+\frac{1}{b}=\frac{1}{2}$$

Solution: By hypothesis

$$\begin{aligned}&\frac{1}{2}|a(b-2)+0(2-0)+2(0-b)| = 0\\ &\Rightarrow ab-2a-2b = 0\\ &\Rightarrow 2a+2b = ab\\ &\Rightarrow \frac{1}{a}+\frac{1}{b}=\frac{1}{2}\end{aligned}$$

7. The points $(1,2)$, $(2,4)$ and $(t,6)$ are collinear. Find t.

Solution: By hypothesis the area of the triangle is zero. Therefore

$$\frac{1}{2}|1(4-6)+2(6-2)+t(2-4)|=0$$
$$\Rightarrow \frac{1}{2}|-2+8-2t|=0$$
$$\Rightarrow 6-2t=0$$
$$\Rightarrow t=4$$

8. If O is the origin and $Q(-2,-4)$ is a point on OP such that $OQ=(1/3)OP$, find the coordinates of P.

Solution: Let $P=(x, y)$. See Fig. 1.6. Now,

$$OQ=(1/3)OP \Rightarrow OP=3OQ$$
$$\Rightarrow OQ:QP=1:2$$

Therefore

$$(-2,-4)=Q=\left(\frac{x}{3}, \frac{y}{3}\right)$$
$$\Rightarrow \frac{x}{3}=-2, \quad \frac{y}{3}=-4$$
$$\Rightarrow x=-6, \quad y=-12$$
$$\Rightarrow P=(-6,-12)$$

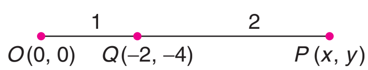


FIGURE 1.6

9. If x_1, x_2, x_3 as well as y_1, y_2, y_3 are in GP with the same common ratio, then show that the points (x_1, y_1), (x_2, y_2) and (x_3, y_3) are collinear. **(IIT-JEE 1999)**

Solution: Let $x_2=x_1k, x_3=x_1k^2$ and $y_2=y_1k, y_3=y_1k^2$. Then

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix}=\begin{vmatrix} x_1 & x_1k & x_1k^2 \\ y_1 & y_1k & y_1k^2 \\ 1 & 1 & 1 \end{vmatrix}$$
$$=x_1y_1\begin{vmatrix} 1 & k & k^2 \\ 1 & k & k^2 \\ 1 & 1 & 1 \end{vmatrix}$$
$$=0 \text{ (since two rows are identical)}$$

Hence, the points are collinear (by Quick Look 3).

10. In a parallelogram $ABCD$ if (x_1, y_1), (x_2, y_2) and (x_3, y_3) are the coordinates of A, B, C, respectively, then show that the coordinates of D are $(x_1+x_3-x_2, y_1+y_3-y_2)$.

Solution: Suppose the coordinates of D are (x, y). See Fig. 1.7. It is known that in a parallelogram the diagonals bisect each other. Hence, AC and BD bisect each other. That is, $\overline{AC}$ and $\overline{BD}$ have the same point as their midpoint.

Therefore

$$\left(\frac{x_1+x_3}{2}, \frac{y_1+y_3}{2}\right)=\left(\frac{x_2+x}{2}, \frac{y_2+y}{2}\right)$$
$$\Rightarrow x_1+x_3=x_2+x, \quad y_1+y_3=y_2+y$$
$$\Rightarrow x=x_1-x_2+x_3, \quad y=y_1-y_2+y_3$$

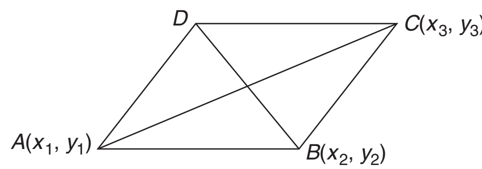


FIGURE 1.7

Note: The student can remember it easily.

11. If $P(1, 2)$, $Q(4, 6)$, $R(5, 7)$ and $S(a, b)$ are the vertices of a parallelogram $PQRS$, then find the values of a and b. **(IIT-JEE 1998)**

Solution: According to the above formula (Problem 10), we have

$$a=1+5-4=2$$
$$b=2+7-6=3$$

12. Show that the four points $A(-a,-b)$, $O(0, 0)$, $B(a, b)$ and $C(a^2, ab)$ are collinear.

Solution: Since $O(0, 0)$ is the midpoint of $\overline{AB}$ it follows that

$$\text{Points } A, O \text{ and } B \text{ are collinear} \tag{1.6}$$

Now,

$$\begin{vmatrix} -a & 0 & a^2 \\ -b & 0 & ab \\ 1 & 1 & 1 \end{vmatrix}=ab\begin{vmatrix} -1 & 0 & a \\ -1 & 0 & a \\ 1 & 1 & 1 \end{vmatrix}=0 \ (\because \text{two rows are identical})$$

$$\text{Points } A, O \text{ and } C \text{ are collinear} \tag{1.7}$$

Statements (1.6) and (1.7) $\Rightarrow A, O, B, C$ are collinear.

13. Let $O(0, 0)$, $P(3, 4)$, $Q(6, 0)$ be the vertices of the triangle OPQ. The point R lies inside the ΔOPQ such that the triangles OPR, PQR, OQR are of equal area. The coordinates of R are

(A) $(4/3, 3)$ (B) $(3, 2/3)$

(C) $(3, 4/3)$ (D) $(4/3, 2/3)$

(IIT-JEE 2007)

Solution: A point inside a triangle divides the triangle into three triangles of equal areas if and only if the point is the centroid of the triangle. Hence, R must be the centroid of ΔOPQ. Therefore

$$R=\left(\frac{0+3+6}{3},\frac{0+4+0}{3}\right)=\left(3,\frac{4}{3}\right)$$

Answer: (C)

14. An integral point means that both coordinates of the point are integers. The number of integral points exactly in the interior of the triangle with vertices $(0,0)$, $(0,21)$ and $(21,0)$ (see Fig. 1.8) is

(A) 133 (B) 190 (C) 233 (D) 105

Solution: The integral points must be on the vertical lines $x = 1, 2, 3, \ldots, 20$. The number of integral points on $x = 1$ inside the triangle are $(1, 1), (1, 2), (1, 3), \ldots, (1, 19)$ (total number is 19). Similarly, the number of points on $x = 2$ is 18, on $x = 3$ is 17, etc. Finally, the number of points on $x = 19$ is 1 and on $x = 20$ is 0.

Therefore, the total number of integral points inside the triangle is

$$19+18+17+\cdots+1+0=\frac{19\times 20}{2}=190$$

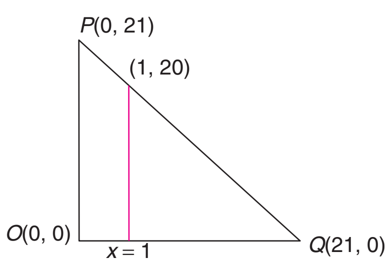


FIGURE 1.8

Answer: (B)

15. A line segment $\overline{AB}$ is of length 10 units and $A = (2,-3)$. If the abscissa of B is 10, then there will be two values for the coordinate of B whose sum is equal to

(A) 3 (B) –3 (C) 6 (D) –6

Solution: Suppose $B = (10, y)$. Then

$$AB = 10 \Rightarrow (10-2)^2 + (y+3)^2 = 10^2$$
$$\Rightarrow (y+3)^2 = 36$$
$$\Rightarrow y+3 = \pm 6$$
$$\Rightarrow y = 3 \text{ or } -9$$

Therefore, the sum of the values of $y = 3 - 9 = -6$.

Answer: (D)

16. Origin is the centroid of a triangle ABC. If $A = (4,-3)$ and $B = (-5, 2)$, then find the coordinates of C.

Solution: Suppose $C = (x, y)$. Then

$$(0,0)=\left(\frac{4-5+x}{3},\frac{-3+2+y}{3}\right)$$
$$\Rightarrow \frac{4-5+x}{3}=0 \text{ and } \frac{-3+2+y}{3}=0$$
$$\Rightarrow x=1,\ y=1$$

So the coordinates of C are $(1, 1)$.

17. If $A(a,b)$, $B(a+r\cos\alpha, b+r\sin\alpha)$ and $C(a+r\cos\beta, b+r\sin\beta)$ are the vertices of an equilateral triangle, then

(A) $|\alpha-\beta| = \pi/4$ (B) $|\alpha-\beta| = \pi/2$

(C) $|\alpha-\beta| = \pi/6$ (D) $|\alpha-\beta| = \pi/3$

Solution: ΔABC is an equilateral triangle implies

$$AB = BC = CA$$
$$\Rightarrow r^2(\cos^2\alpha+\sin^2\alpha) = r^2(\cos^2\beta+\sin^2\beta) = r^2(\cos\alpha-\cos\beta)^2 + r^2(\sin\alpha+\sin\beta)^2$$
$$\Rightarrow 2r^2[1-\cos(\alpha-\beta)] = r^2$$
$$\Rightarrow \cos(\alpha-\beta)=\frac{1}{2}$$
$$\Rightarrow |\alpha-\beta|=\frac{\pi}{3}$$

Answer: (D)

18. The centroid of ΔABC is $(2,7)$. If $A = (4,8)$, $B = (a,0)$ and $C = (0,b)$, then

(A) $a = 2, b = -13$ (B) $a = 2, b = 13$

(C) $a = 13, b = 2$ (D) $a = -2, b = -13$

Solution: By hypothesis

$$\frac{4+a+0}{3}=2 \text{ and } \frac{8+0+b}{3}=7$$

This implies $a = 2$, $b = 13$.

Answer: (B)

19. If the point $P(x, y)$ is equidistant from the points A $(6, -1)$ and $B(2, 3)$, then find a relation between x and y.

Solution: By hypothesis

$$PA = PB$$
$$\Rightarrow PA^2 = PB^2$$
$$\Rightarrow (x-6)^2 + (y+1)^2 = (x-2)^2 + (y-3)^2$$
$$\Rightarrow x^2 - 12x + 36 + y^2 + 2y + 1 = x^2 - 4x + 4 + y^2 - 6y + 9$$
$$\Rightarrow 8x - 8y = 24$$
$$\Rightarrow x - y = 3$$

20. If the area of the triangle whose vertices are $(a, 0)$, $(3, 4)$ and $(5, -2)$ is 10, then

(A) $a = 1$ or $22/3$ (B) $a = 1$ or $13/3$
(C) $a = 1$ or $23/3$ (D) $a = 2$ or $23/3$

Solution: By hypothesis

$$6a - 26 = \pm 20$$
$$\Rightarrow a = 46/6 \text{ or } 6/6$$
$$\Rightarrow a = 1 \text{ or } 23/3$$

Answer: (C)

21. If the points $(x, 2-2x)$, $(1-x, 2x)$ and $(-4-x, 6-2x)$ are collinear, find x.

Solution: By hypothesis,

$$\frac{1}{2}\left|\begin{matrix} x(2x-6+2x)+(1-x)(6-2x-2+2x) \\ -(4+x)(2-2x-2x) \end{matrix}\right| = 0$$
$$\Rightarrow x(4x-6)+4(1-x)+(4+x)(4x-2)=0$$
$$\Rightarrow 4x^2-6x+4-4x+4x^2+14x-8=0$$
$$\Rightarrow 8x^2+4x-4=0$$
$$\Rightarrow 2x^2+x-1=0$$
$$\Rightarrow (2x-1)(x+1)=0$$
$$\Rightarrow x = \frac{1}{2}, -1$$

Note:

$$x = \frac{1}{2} \Rightarrow (x, 2-2x) = (1-x, 2x) = \left(\frac{1}{2}, 1\right)$$

22. Show that the area of the triangle whose vertices are $(am_1m_2, a(m_1+m_2))$, $(am_2m_3, a(m_2+m_3))$ and $(am_3m_1, a(m_3+m_1))$ is

$$\frac{1}{2}a^2\,|(m_1-m_2)(m_2-m_3)(m_3-m_1)|$$

Solution: Let Δ be the area of the determinant so that the value of Δ is the numerical value (i.e., absolute value) of the determinant

$$\frac{1}{2}\begin{vmatrix} am_1m_2 & am_2m_3 & am_3m_1 \\ a(m_1+m_2) & a(m_2+m_3) & a(m_3+m_1) \\ 1 & 1 & 1 \end{vmatrix}$$

$$= \text{Absolute value of } \frac{a^2}{2}\begin{vmatrix} m_1m_2 & m_2m_3 & m_3m_1 \\ m_1+m_2 & m_2+m_3 & m_3+m_1 \\ 1 & 1 & 1 \end{vmatrix}$$

$= \text{Absolute value of}$

$$\frac{a^2}{2}\begin{vmatrix} m_1m_2 & m_2(m_3-m_1) & m_1(m_3-m_2) \\ m_1+m_2 & m_3-m_1 & m_3-m_2 \\ 1 & 0 & 0 \end{vmatrix}$$

(By $C_2 - C_1$ and $C_3 - C_1$)

$= \text{Absolute value of}$

$$\frac{a^2}{2}(m_3-m_1)(m_3-m_2)\begin{vmatrix} m_1m_2 & m_2 & m_1 \\ m_1+m_2 & 1 & 1 \\ 1 & 0 & 0 \end{vmatrix}$$

$$= \frac{1}{2}a^2\,|(m_1-m_2)(m_2-m_3)(m_3-m_1)|$$

23. If a, b, c are the roots of the equation $x^3 - 6x^2 + 11x - 6 = 0$, then find the centroid of the triangle whose vertices are $(ab, 1/ab)$, $(bc, 1/bc)$, $(ca, 1/ca)$.

Solution: By hypothesis

$$a+b+c=6$$
$$ab+bc+ca=11$$

and $$abc=6$$

Therefore

$$\frac{ab+bc+ca}{3} = \frac{11}{3}$$

and

$$\frac{1}{3}\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right) = \frac{a+b+c}{3abc} = \frac{6}{3\times 6} = \frac{1}{3}$$

Hence, the centroid of the triangle is

$$\left(\frac{ab+bc+ca}{3}, \frac{\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}}{3}\right) = \left(\frac{11}{3}, \frac{1}{3}\right)$$

Caution: First, we have to check whether a, b, c are real or not. In the present case, the roots of the given equation are 1, 2 and 3.

24. Find the area of the triangle having midpoints of its sides at $(2, 1)$, $(-1, -3)$ and $(4, 5)$.

Solution: Area of the triangle is four times the area of the triangle formed by the midpoints of its sides. Therefore

$$\text{Area of the triangle} = 4\times\frac{1}{2}|2(-3-5)-1(5-1)+4(1+3)|$$
$$= 2|-16-4+16|$$
$$= 8 \text{ sq. units}$$

25. $O(0, 0)$ is one of the vertices of triangle whose circumcentre is $S(3, 4)$ and centroid $G(6, 8)$. Then, the triangle

(A) is right angled
(B) must be equilateral
(C) must be right-angled isosceles
(D) is isosceles

Solution: Clearly $S(3, 4)$ is the midpoint of $\overline{OG}$ (see Fig. 1.9). Hence OG is the median through as well as the perpendicular bisector of the side opposite to the vertex O. Hence the triangle is isosceles.

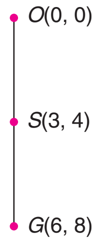


FIGURE 1.9

Answer: (D)

26. Find the circumcentre and circumradius of the triangle whose vertices are $A(1, 1)$, $B(2, -1)$ and $C(3, 2)$.

Solution: Let $S(x, y)$ be the circumcentre of ΔABC so that $SA = SB = SC$. Now

$$SA = SB \Rightarrow (x-1)^2 + (y-1)^2 = (x-2)^2 + (y+1)^2$$
$$\Rightarrow -2x - 2y + 2 = -4x + 2y + 5$$
$$\Rightarrow 2x - 4y = 3 \quad (1.8)$$

$$SB = SC \Rightarrow (x-2)^2 + (y+1)^2 = (x-3)^2 + (y-2)^2$$
$$\Rightarrow -4x + 2y + 5 = -6x - 4y + 13$$
$$\Rightarrow 2x + 6y = 8 \quad (1.9)$$

$$SC = SA \Rightarrow (x-3)^2 + (y-2)^2 = (x-1)^2 + (y-1)^2$$
$$\Rightarrow -6x - 4y + 13 = -2x - 2y + 2$$
$$\Rightarrow 4x + 2y = 11 \quad (1.10)$$

Solving Eqs. (1.8) and (1.9), we have $x = 5/2$ and $y = 1/2$ which also satisfy Eq. (1.10). Hence

$$\text{Circumcentre of the triangle} = \left(\frac{5}{2}, \frac{1}{2}\right)$$

$$\text{Circumradius, } SA = \sqrt{\left(\frac{5}{2}-1\right)^2 + \left(\frac{1}{2}-1\right)^2} = \sqrt{\frac{9}{4}+\frac{1}{4}} = \sqrt{\frac{5}{2}}$$

27. Find the incentre of the triangle whose vertices are $A\ (2, 3)$, $B(-2, -5)$ and $C(-4, 6)$.

Solution: We have

$$a = BC = \sqrt{(-2+4)^2 + (-5-6)^2} = \sqrt{4+121} = 5\sqrt{5}$$
$$b = CA = \sqrt{(2+4)^2 + (3-6)^2} = \sqrt{36+9} = 3\sqrt{5}$$
$$c = AB = \sqrt{(2+2)^2 + (3+5)^2} = \sqrt{16+64} = 4\sqrt{5}$$

Therefore

$$a + b + c = 12\sqrt{5}$$

Now,

$$\text{Incentre} = \left(\frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1 + by_2 + cy_3}{a+b+c}\right)$$

where

$$\frac{ax_1 + bx_2 + cx_3}{a+b+c} = \frac{5\sqrt{5}(2) + 3\sqrt{5}(-2) + 4\sqrt{5}(-4)}{12\sqrt{5}}$$
$$= \frac{10\sqrt{5} - 22\sqrt{5}}{12\sqrt{5}} = -1$$
$$\frac{ay_1 + by_2 + cy_3}{a+b+c} = \frac{5\sqrt{5}(3) + 3\sqrt{5}(-5) + 4\sqrt{5}(6)}{12\sqrt{5}}$$
$$= \frac{39\sqrt{5} - 15\sqrt{5}}{12\sqrt{5}} = 2$$

Hence, the incentre of the triangle is $(-1, 2)$.

28. Find the ratio in which the point $(-2, -9)$ divides the segment joining the points $A(1, 3)$ and $B(2, 7)$.

Solution: Suppose $P = (-2, -9)$ and $AP{:}PB = m{:}1$. Then using section formula we have

$$(-2, -9) = \left(\frac{2m+1}{m+1}, \frac{7m+3}{m+1}\right)$$

This implies

$$-2 = \frac{2m+1}{m+1} \quad \text{and} \quad -9 = \frac{7m+3}{m+1} \quad (1.11)$$

Solving the first equality in Eq. (1.11), we get

$$-2m - 2 = 2m + 1$$
$$\Rightarrow 4m = -3$$
$$\Rightarrow m = -3/4$$

Solving the second equality in Eq. (1.11), we get

$$-9m - 9 = 7m + 3$$
$$\Rightarrow 16m = -12$$
$$\Rightarrow m = -3/4$$

We can see from both cases that the ratio is –3:4 or 3:–4. The division is external division.

29. Show that the points $(-3, -4)$, $(2, 6)$ and $(-6, 10)$ form a right-angled triangle.

Solution: Let the vertices be A, B and C, respectively. Then

$$(AB)^2 = (-3-2)^2 + (-4-6)^2 = 25 + 100 = 125$$
$$(BC)^2 = (2+6)^2 + (6-10)^2 = 64 + 16 = 80$$
$$(CA)^2 = (-6+3)^2 + (10+4)^2 = 9 + 196 = 205$$

Now, by Pythagoras theorem, we have

$$(AB)^2 + (BC) = (AC)^2$$

$$\Rightarrow \underline{|B} = 90^{\circ}$$

Hence, $\Delta\, ABC$ is a right-angled triangle.

30. The circumcentre of a triangle lies at the origin and the centroid is the midpoint of the segment joining $(2, 2)$ and $(2, -2)$. Find the orthocentre.

Solution: In a triangle, circumcentre, centroid and orthocentre are collinear and the centroid divides the line joining the circumcentre and orthocentre in the ratio 1:2.

Suppose $H(x, y)$ is the orthocentre. Then

$$OG{:}GH = 1{:}2$$

Now, $O = (0, 0)$, $G = (2, 0)$ and $H = (x, y)$ (see Fig. 1.10). Therefore

$$OG{:}GH = 1{:}2 \Rightarrow (2, 0) = G = (x/3, y/3)$$
$$\Rightarrow x = 6, y = 0$$

So $H = (6, 0)$.

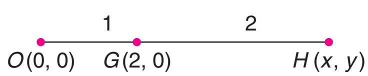


FIGURE 1.10

Locus

31. Find the locus of the point which is equidistant from the points $(-3, 1)$ and $(7, 5)$.

Solution: Let $A = (-3, 1)$ and $B = (7, 5)$. Let $P = (x, y)$. Then

$$\begin{aligned} AP = PB &\Leftrightarrow (AP)^2 = (PB)^2 \\ &\Leftrightarrow (x+3)^2 + (y-1)^2 = (x-7)^2 + (y-5)^2 \\ &\Leftrightarrow 6x - 2y + 10 = -14x - 10y + 74 \\ &\Leftrightarrow 20x + 8y - 64 = 0 \\ &\Leftrightarrow 5x + 2y - 16 = 0 \end{aligned}$$

Hence, the equation of the locus is $5x + 2y - 16 = 0$.

32. Let $A(5, -4)$ and $B(7, 6)$ be two points. Find the locus of the point P such that $PA{:}PB = 2{:}3$.

Solution: Let P be (x, y). Then

$$\begin{aligned} PA{:}PB = 2{:}3 &\Leftrightarrow 3PA = 2PB \\ &\Leftrightarrow 9(PA)^2 = 4(PB)^2 \\ &\Leftrightarrow 9[(x-5)^2 + (y+4)^2] = 4[(x-7)^2 \\ &\qquad + (y-6)^2] \\ &\Leftrightarrow 5x^2 + 5y^2 - 34x + 120y + 29 = 0 \end{aligned}$$

Hence, the equation of the locus is $5x^2 + 5y^2 - 34x + 120y + 29 = 0$. This equation represents circle (which will be discussed in Chapter 3).

33. Find the equation of the locus of a point P which moves such that its distance from the origin is twice its distance from the point $A(1, 2)$.

Solution: We have

$$\begin{aligned} OP = 2PA &\Leftrightarrow (OP)^2 = 4(PA)^2 \\ &\Leftrightarrow x^2 + y^2 = 4[(x-1)^2 + (y-2)^2] \\ &\Leftrightarrow x^2 + y^2 = 4x^2 + 4y^2 - 8x - 16y + 20 \\ &\Leftrightarrow 3x^2 + 3y^2 - 8x - 16y + 20 = 0 \end{aligned}$$

Hence, the equation of the locus is $3x^2 + 3y^2 - 8x - 16y + 20 = 0$.

34. Find the locus of the point P such that the distance of P from the point $A(4, 0)$ is twice the distance of P from the x-axis.

Solution: Let $P = (x, y)$. Distance of P from the x-axis is $|y|$. Therefore

$$\begin{aligned} AP = 2|y| &\Leftrightarrow (AP)^2 = 4y^2 \\ &\Leftrightarrow (x-4)^2 + y^2 = 4y^2 \\ &\Leftrightarrow x^2 - 3y^2 - 8x + 16 = 0 \end{aligned}$$

Hence the equation of the locus is $x^2 - 3y^2 - 8x + 16 = 0$.

35. Let $A = (2, 3)$ and $B = (-3, 4)$. If P is a moving point such that the area of ΔPAB is 17/2 sq. units, then find the locus of P.

Solution: Let $P = (x, y)$. Then

$$\begin{aligned} \text{Area of } \Delta PAB &= \frac{17}{2} \\ \Leftrightarrow \frac{1}{2}|x(3-4) + 2(4-y) - 3(y-3)| &= \frac{17}{2} \\ \Leftrightarrow |-x - 5y + 17| &= 17 \\ \Leftrightarrow x + 5y - 17 &= \pm 17 \end{aligned}$$

Therefore the equation of the locus is

$$(x + 5y)\,(x + 5y - 34) = 0$$

36. Find the equation of the locus of the point which is at a constant distance of 5 units from the fixed point $(-2, 3)$.

Solution: Let $A = (-2, 3)$ and let $P = (x, y)$. Now

$$\begin{aligned} AP = 5 &\Leftrightarrow (AP)^2 = 25 \\ &\Leftrightarrow (x+2)^2 + (y-3)^2 = 25 \\ &\Leftrightarrow x^2 + y^2 + 4x - 6y - 12 = 0 \end{aligned}$$

The equation of the locus is $x^2 + y^2 + 4x - 6y - 12 = 0$. Later in Chapter 3, we will see that this equation represents circle with centre at the point $(-2, 3)$ and radius 5 units.

Change of Axes

37. Suppose the origin is shifted to the point $(-1, 2)$. Find the new coordinates of the point $(2, 3)$.

Solution: Let (X, Y) be the new coordinates. Therefore by the formula [Eq. (1.1)], we have

$$2 = x - 1, 3 = y + 2$$
$$\Rightarrow x = 3, y = 1$$

Hence, $(3, 1)$ are new coordinates of the point $(2, 3)$.

38. Find the transformed form of the equation $2x^2 + 4xy + 3y^2 = 0$ if the origin is shifted to the point $(1, 1)$.

Solution: Put $x = X + 1$, $y = Y + 1$ in the given equation. We get

$$2(X+1)^2 + 4(X+1)(Y+1) + 3(Y+1)^2 = 0$$
$$\Rightarrow 2X^2 + 4XY + 3Y^2 + 8X + 10Y + 9 = 0$$

39. When the axes are rotated through $30°$ in the anti-clockwise sense without changing the origin, find the new coordinates of the point $(-2, 4)$.

Solution: Let (x, y) be the old coordinates and (X, Y) be the new coordinates. Therefore

$$x = X\cos\theta - Y\sin\theta \qquad (1.12)$$

and
$$y = X\sin\theta + Y\cos\theta \qquad (1.13)$$

Solving Eq. (1.12), we get

$$-2 = X\cos 30^\circ - Y\sin 30^\circ$$
$$\Rightarrow -2 = \frac{\sqrt{3}X - Y}{2}$$
$$\Rightarrow \sqrt{3}X - Y = -4 \qquad (1.14)$$

Solving Eq. (1.13), we get

$$4 = X\sin 30^\circ + Y\cos 30^\circ$$
$$\Rightarrow 4 = \frac{X + \sqrt{3}Y}{2}$$
$$\Rightarrow X + \sqrt{3}Y = 8 \qquad (1.15)$$

Solving Eqs. (1.14) and (1.15), we get $X = 2 - \sqrt{3}$ and $Y = 1 + 2\sqrt{3}$ are the new coordinates of $(-2, 4)$.

40. If the axes are rotated through $45°$ in the clockwise sense, then find the transformed form of the equation $x^2 - y^2 = a^2$.

Solution: Let (x, y) and (X, Y) be the old and the new coordinates, respectively. Therefore

$$x = X\cos(-45^\circ) - Y\sin(-45^\circ) = \frac{X+Y}{\sqrt{2}}$$

and
$$y = X\sin(-45^\circ) + Y\cos(45^\circ) = \frac{-X+Y}{\sqrt{2}}$$

Now,

$$x^2 - y^2 = a^2 \Rightarrow \left(\frac{X+Y}{\sqrt{2}}\right)^2 - \left(\frac{Y-X}{\sqrt{2}}\right)^2 = a^2$$
$$\Rightarrow 4XY = 2a^2$$
$$\Rightarrow XY = \frac{a^2}{2} \text{ or } XY = c^2 \text{ where } c = \frac{a}{\sqrt{2}}$$

Note: When we deal with hyperbola (conic section), we see that $x^2 - y^2 = a^2$ is the standard equation of rectangular hyperbola and $xy = c^2$ is the rectangular hyperbola in the simplest form.

41. Find the point to which origin is to be shifted so as to remove the first degree terms of the equation $2x^2 + 4xy - 5y^2 + 20x - 22y - 14 = 0$.

Solution: In the given equation, $a = 2$, $h = 2$, $b = -5$, $g = 10$, $f = -11$, $c = -14$. Therefore

$$ab - h^2 = -10 - 4 = -14$$
$$hf - bg = 2(-11) - (-5)(10) = -22 + 50 = 28$$
$$gh - af = 10(2) - 2(-11) = 42$$

Therefore

$$\text{New origin} = \left(\frac{hf - bg}{ab - h^2}, \frac{gh - af}{ab - h^2}\right)$$
$$= \left(\frac{28}{-14}, \frac{42}{-14}\right)$$
$$= (-2, -3)$$

42. Find the angle through which the axes are to be rotated so as to remove the xy-term from the equation $x^2 + 2\sqrt{3}xy - y^2 - 2a^2 = 0$.

Solution: In the given equation, $a = 1$, $h = \sqrt{3}$ and $b = -1$ so that $a \neq b$. Therefore

$$\text{Required angle of rotation} = \frac{1}{2}\text{Tan}^{-1}\left(\frac{2h}{a-b}\right)$$
$$= \frac{1}{2}\text{Tan}^{-1}\left(\frac{2\sqrt{3}}{2}\right)$$
$$= \frac{1}{2}\text{Tan}^{-1}(\sqrt{3})$$
$$= \frac{1}{2}\left(\frac{\pi}{3}\right)$$
$$= \frac{\pi}{6}$$

SUMMARY

1.1 ***Distance between two points:*** If $A(x_1, y_1)$ and $B(x_2, y_2)$ are two points, then the distance between the two points is given by

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

1.2 ***Section formula:*** If $A(x_1, y_1)$ and $B(x_2, y_2)$ are two points and $P(x, y)$ is a point on the line AB dividing the segment $\overline{AB}$ in the ratio $l{:}m$, where $l + m \neq 0$, then

$$x = \frac{lx_2 + mx_1}{l+m} \text{ and } y = \frac{ly_2 + my_1}{l+m}$$

This formula is valid for both internal and external divisions.

1.3 ***Midpoint:*** If $A(x_1, y_1)$ and $B(x_2, y_2)$ are two points, then the coordinates of the midpoint of $\overline{AB}$ are given by

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

1.4 ***Centroid coordinates:*** If $G(x, y)$ is the centroid of a triangle whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3), then

$$x = \frac{x_1 + x_2 + x_3}{3} \text{ and } y = \frac{y_1 + y_2 + y_3}{3}$$

1.5 ***Area of a triangle:*** The area of the triangle whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3) is the absolute value of the determinant

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

1.6 ***Condition for collinearity of three points:*** Three points (x_1, y_1), (x_2, y_2) and (x_3, y_3) are collinear if and only if

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

1.7 ***Coordinates of incentres and excentres:*** Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of a triangle and suppose the lengths BC, CA, and AB are a, b, c, respectively, then

(1) Incentre $I = \left(\dfrac{ax_1 + bx_2 + cx_3}{a+b+c}, \dfrac{ay_1 + by_2 + cy_3}{a+b+c}\right)$

(2) Excentre

$$I_1 = \left(\frac{-ax_1 + bx_2 + cx_3}{-a+b+c}, \frac{-ay_1 + by_2 + cy_3}{-a+b+c}\right)$$

(3) Excentre

$$I_2 = \left(\frac{ax_1 - bx_2 + cx_3}{a-b+c}, \frac{ay_1 - by_2 + cy_3}{a-b+c}\right)$$

(4) Excentre

$$I_3 = \left(\frac{ax_1 + bx_2 - cx_3}{a+b-c}, \frac{ay_1 + by_2 - cy_3}{a+b-c}\right)$$

1.8 ***Nine-point centre:*** In any triangle, the midpoints of the sides, the feet of the altitudes and the midpoints of the segments joining the vertices with the orthocentre are concyclic. This circle is called the nine-point circle whose centre is known as the nine-point centre which is denoted by N.

1.9 In any triangle, the circumcentre S, the centroid G, the nine-point centre N and the orthocentre H are collinear in the given order (that is, $SGNH$). Further, G divides $\overline{SH}$ in the ratio 1:2 and N is the midpoint of $\overline{SH}$. Further, the radius of the nine-point circle is half of the circumradius of the triangle.

1.9* ***Pedal line:*** Let ABC be a triangle and P be a point on the circumcircle of the triangle other than the vertices. Then the feet of the perpendiculars drawn from P on to the sides of the triangle are collinear. This line is called the Pedal line of the point P or Simson's line.

1.10 If $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are three consecutive vertices of a parallelogram $ABCD$, then the fourth vertex D is given by $(x_1 + x_3 - x_2, y_1 + y_3 - y_2)$.

1.11 ***Locus:*** Let P be a geometrical condition(s) and S be the set of all points in the plane which satisfy P. Then S is called a locus.

1.12 ***Equation of the locus:*** Let S be a locus and $f(x, y) = 0$ be an algebraic equation in x and y. If every point (x, y) belonging to S satisfies the equation $f(x, y) = 0$, and any point in the plane satisfying the equation $f(x, y) = 0$ belongs to S, then $f(x, y) = 0$ is called the equation of the locus S. The locus is generally given by its equation.

1.13 ***Shifting of origin without changing the direction of the axes:*** Suppose the origin $O\,(0,0)$ is shifted to the point $O'(h, k)$. Let the old and the new coordinates of a point be (x, y) and (X, Y), respectively. Then $x = X + h$ and $y = Y + k$ give the relations between the old and the new coordinates.

1.14 ***Rotation of the axes without changing the origin:*** Let $\overleftrightarrow{x'Ox}$ and $\overleftrightarrow{y'Oy}$ be the original axes. Rotate these axes through an angle θ in the anticlockwise direction about the origin O. Let the new position of the axes be $\overleftrightarrow{X'OX}$ and $\overleftrightarrow{Y'OY}$. If (x, y) and (X, Y) be the coordinates of a point with respect to the old and the new axes, then

$$x = X\cos\theta - Y\sin\theta \text{ and } y = X\sin\theta + Y\cos\theta$$

If the rotation is in the clockwise sense, then in the above relations, replace θ with $-\theta$.

1.15 ***Second degree homogeneous and general equations:***

(1) ***Homogeneous equation:*** If a, h, b are real and atleast one of them is not zero, then $ax^2 + 2hxy + by^2 = 0$ is called second degree homogeneous equation in x and y.

(2) ***General equation:*** If a, h, b, g, f and c are real and atleast one of a, h, b is not zero, then the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is called second degree general equation in x and y.

1.16 In the second degree general equation, ax^2, $2hxy$, by^2 are called the second degree terms, $2gx$, $2fy$ are called the first degree terms and c is called the constant term.

1.17 ***Theorem:*** To remove the first degree terms from the second degree general equation, the origin is to be shifted to the point

$$\left(\frac{hf - bg}{ab - h^2}, \frac{gh - af}{ab - h^2}\right)$$

provided $h^2 \neq ab$.

1.18 To remove the xy term from the second degree general equation, the axes are to be rotated through the angle

$$\frac{1}{2}\mathrm{Tan}^{-1}\left(\frac{2h}{a - b}\right)$$

provided $a \neq b$. When $a = b$, the angle of rotation is $\pi/4$.

1.19 When the origin is shifted, the second degree terms of the second degree general equation will not change.

1.20 In the change of axes, only the coordinates of a point will change, but the distance between two points and the areas will not change.

EXERCISES

1. Show that the points $(1, -1)$, $(\sqrt{3}, \sqrt{3})$ and $(0, \sqrt{3} - 1)$ form the vertices of an isosceles right-angled triangle.

2. Let $ABCD$ be a rectangle and P any point in the plane of the rectangle. Then, prove that
$$(PA)^2 + (PC)^2 = (PB)^2 + (PD)^2$$
(***Hint:*** Take A as origin, $\overrightarrow{AB}$ and $\overrightarrow{AD}$ as axes.)

3. Prove that the points $(3, 6)$, $(2, 1)$ and $(1, -4)$ are collinear.

4. Show that the points $(1, 4)$, $(3, -2)$ and $(-3, 16)$ are collinear.

5. Show that the points (a, a), $(-a, -a)$ and $(-a\sqrt{3}, a\sqrt{3})$ are the vertices of an equilateral triangle.

6. Let $A = (3, -5)$, $B = (-5, -4)$, $C = (7, 10)$ and $D = (15, 9)$. In the given order, show that the points form a parallelogram.

7. If $A = (2, -3)$, $B = (6, 5)$, $C = (-2, 1)$ and $D = (-6, -7)$, then show that $ABCD$ is a rhombus.

8. Show that $(1, 6)$, $(5, 2)$, $(12, 9)$, $(8, 13)$, taken in this order, form a rectangle.

9. If $A = (5, 3)$, $B = (11, -5)$ and $C = (12, \lambda)$ are such that $\underline{|ABC} = 90°$, then show that λ equals either -4 or 2.

10. Show that the four points $(0, -1)$, $(2, 1)$, $(0, 3)$, $(-2, 1)$, taken in this order, form a square.

11. Find the area of the triangle whose vertices are $(5, 2)$ $(-9, -3)$ and $(-3, -5)$.

12. Show that the area of the triangle whose vertices are $(a\cos\alpha, b\sin\alpha)$, $(a\cos\beta, b\sin\beta)$ and $(a\cos\gamma, b\sin\gamma)$, where a, b are positive, is
$$2ab\left|\sin\left(\frac{\alpha - \beta}{2}\right)\sin\left(\frac{\beta - \gamma}{2}\right)\sin\left(\frac{\gamma - \alpha}{2}\right)\right|$$

13. O is the origin, $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2)$ and $\underline{|P_1OP_2} = \theta$. Show that $OP_1 \cdot OP_2 \cdot \cos\theta = x_1x_2 + y_1y_2$.

14. Find the incentre of the triangle whose vertices are $(7, 9)$, $(3, -7)$ and $(-3, 3)$.

15. Find the centroid of the triangle with vertices $(2, 7)$, $(3, -1)$ and $(-5, 6)$.

16. Find the incentre of the triangle whose vertices are $(3, 2)$, $(7, 2)$ and $(7, 5)$.

17. Let $A = (2, 3)$ and $B = (-1, 5)$. If P is a variable point such that the segment $\overline{AB}$ subtends right angle at P, then find the equation of the locus of P.

18. Let $A = (1, 1)$ and $B = (-2, 3)$. If P is a variable point such that the area of ΔPAB is 2 sq. units, then show

that the equation of the locus of P is $(2x + 3y - 1)(2x + 3y - 9) = 0$.

19. $O(0, 0)$, $A(6, 0)$ and $B(0, 4)$ are three points. P is a variable point such that the area of ΔPOB is twice that of ΔPOA. Show that the equation of the locus of P is $x^2 - 9y^2 = 0$.

20. $A(2, 3)$, $B(1, 5)$ and $C(-1, 2)$ are three points. P is a variable point such that $(PA)^2 + (PB)^2 = 2(PC)^2$. Show that the locus of P is $10x + 8y - 29 = 0$.

21. Let $A = (2, 3)$ and $B = (2, -3)$. Find the locus of the point P such that $PA + PB = 8$.

22. Show that the equation of the locus of the point equidistant from the points $(a + b, a - b)$ and $(a - b, a + b)$ is $x - y = 0$.

23. Find the incentre of the triangle with vertices $(1, \sqrt{3})$, $(0, 0)$ and $(2, 0)$.

(***Hint:*** The triangle is equilateral.)

24. A point moves such that the sum of its distances from two fixed points $(ae, 0)$ and $(-ae, 0)$ is always $2a$. Prove that the equation of the locus is $x^2/a^2 + y^2/b^2 = 1$ where $b^2 = a^2(1 - e^2)$ or $a^2(e^2 - 1)$ according as $0 < e < 1$ or $e > 1$.

25. Show that the equation of the locus of the point which is equidistant from the points $(a + b, b - a)$ and $(a - b, a + b)$ is $bx - ay = 0$.

26. A bar of length $a + b$ is moving such that its extremities lie on the coordinate axes. Show that the locus of the point dividing the bar in the ratio $a{:}b$ (the direction is from y-axis tip towards x-axis tip) is $x^2/a^2 + y^2/b^2 = 1$.

27. (α, β), $(\bar{x}, \bar{y})$ and (p, q) are the circumcentre, centroid and orthocentre of a triangle. Prove that $3\bar{x} = 2\alpha + p$ and $3\bar{y} = 2\beta + q$.

(***Hint:*** See the note given under Quick Look 4.)

28. $(2, 3)$, $(3, 4)$ and $(6, 8)$ are the vertices of a triangle. Find its centroid, circumcentre and orthcentre.

(***Hint:*** To find the orthocentre, refer the note under Quick Look 4.)

29. If the origin is shifted to the point $(1, -1)$, then find the transformed equation of $x^2 + y^2 - 2x + 2y - 4 = 0$.

30. Show that the equation $2x^2 + y^2 - 8x + 4y + 1 = 0$ will be transferred to $2X^2 + Y = 11$ if the origin is shifted to the point $(2, -2)$.

31. Find the point to which the origin is to be shifted so as to remove the first degree terms of the following equations:

(i) $x^2 + y^2 + 8x - 6y - 25 = 0$

(ii) $4x^2 + 9y^2 - 8x + 36y + 4 = 0$

(iii) $14x^2 - 4xy + 11y^2 - 36x + 48y + 41 = 0$

32. When the axes are rotated through an angle 45° in the anticlockwise sense, then show that the equation $3x^2 + 10xy + 3y^2 - 9 = 0$ will be transformed to the form $8x^2 - 2y^2 - 9 = 0$.

33. When the axes are rotated through an angle $\pi/4$ in the anticlockwise sense, the transformed equation of a curve is $17x^2 - 16xy + 17y^2 = 225$. Find the original equation of the curve.

(***Hint:*** Solve the equation $x = X\cos\theta - Y\sin\theta$, $y = X\sin\theta + Y\cos\theta$ for X and Y after replacing θ with $\pi/4$ and substitute the values of X and Y in the given equation.)

34. Find the angle through which the axes are to be rotated so as to remove the xy term of the equation $x^2 + y^2 + 4xy - 2x + 2y - 6 = 0$.

35. Find the angle of rotation of the axes so as to remove the xy term from the equation $x^2 + 2\sqrt{3}xy - y^2 = 2a^2$ and also find the transformed form.

ANSWERS

11. 29 sq. units

14. $(13 - 8\sqrt{2},\ 2\sqrt{2} - 1)$

15. $(0, 4)$

16. $(6, 3)$

17. $x^2 + y^2 - x - 8y + 13 = 0$

21. $16x^2 + 7y^2 - 64x - 48 = 0$

23. $(1, 1/\sqrt{3})$

28. $G = (11/3, 5)$; Circumcentre $= (-27/2, 39/2)$; Orthocentre $= (38, -24)$

29. $x^2 + y^2 = 6$

31. (i) $(-4, 3)$; (ii) $(1, -2)$; (iii) $(1, -2)$

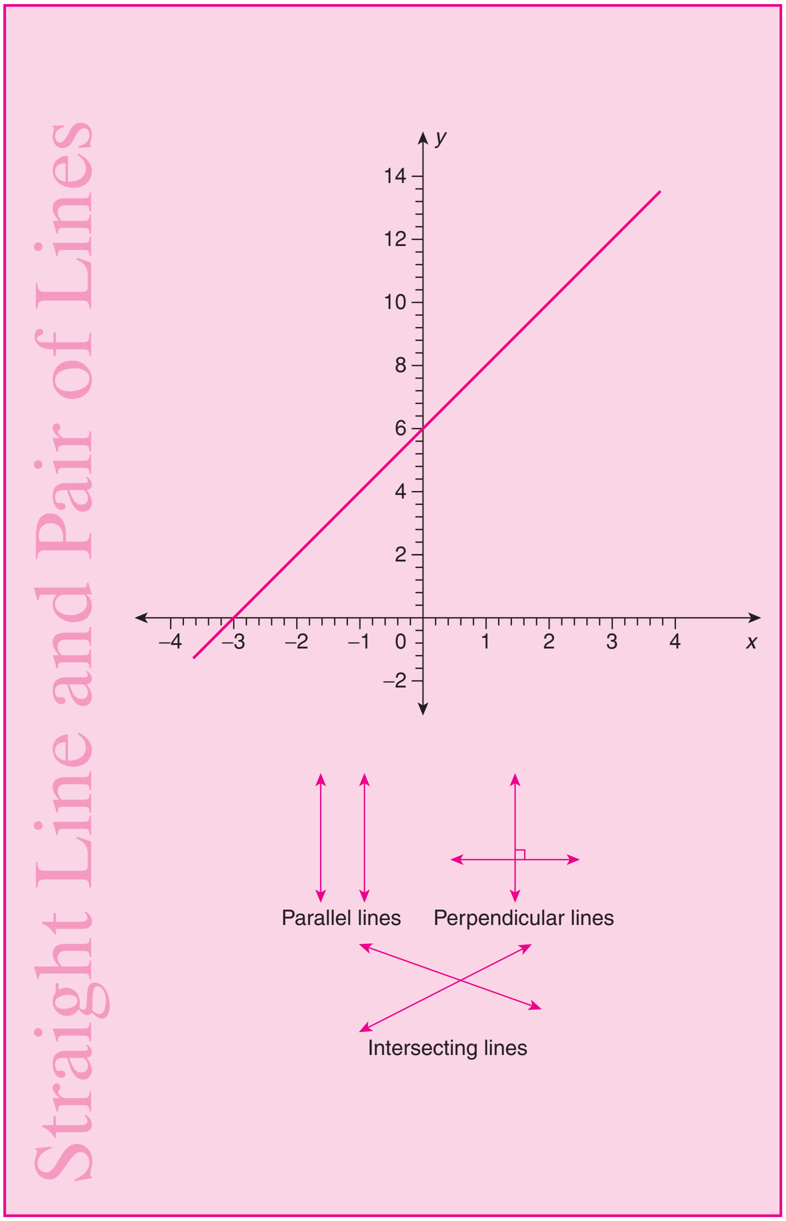
33. $25x^2 + 9y^2 = 225$

34. 45°

35. $30°, x^2 - y^2 = a^2$

Straight Line and Pair of Lines

2



Contents

The simplest geometric figure is a **straight line** which is a line segment joining any two points.

Two lines, that is, **pair of lines**, can be related each other by different ways such as intersecting lines, perpendicular lines and parallel lines.

In this chapter, we discuss various types (forms) of equations of a straight line, angle between two lines, conditions for two lines to be parallel and perpendicular, sides of straight lines, concurrency of lines, etc. 'Subjective Problems' section provides subjective worked-out problems for the preceding sections. Students are advised to solve each and every problem to grasp the topics.

2.1 Straight Line

When areas of coordinates are selected, any line parallel to x-axis (including x-axis) is called a horizontal line and any line perpendicular to x-axis is called a vertical line. First, we discuss the concept of an 'slope of a non-vertical line'.

DEFINITION 2.1 **Slope** If a non-vertical line l makes an angle θ with the positive direction of the x-axis (that is measured in counterclock sense), then the value of $\tan\theta$ is called the slope of line l. Generally, slope of a line is denoted by m.

Note: Slope is defined for non-vertical lines only. We do not talk about the slope of a vertical line.

QUICK LOOK 1

1. Since the angle made by a horizontal line with x-axis is 0 or π, its slope is always zero.
2. Two non-vertical parallel lines make the same angles with the x-axis implies that their slopes are equal; however, if the slopes of two lines are equal, then the lines are parallel lines.
3. The slope of a line is positive $\Leftrightarrow$ the line makes acute angle with the positive direction of the x-axis.
4. The slope of line is negative $\Leftrightarrow$ the line makes obtuse angle with the positive direction of the x-axis.

THEOREM 2.1 The slope of line passing through two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is

$$\frac{y_2 - y_1}{x_2 - x_1}\left(= \frac{y_1 - y_2}{x_1 - x_2}\right)$$

PROOF **Case 1:** If the line is horizontal, then

$$y_1 = y_2 \Rightarrow \frac{y_1 - y_2}{x_1 - x_2} = 0$$

and slope of the line is also zero.

Case 2: If the line is not horizontal, then we consider the following: Let the line make an angle θ with the positive direction of the x-axis (see Fig. 2.1). Draw AL and BM perpendicular to x-axis and AN perpendicular to BM. Clearly, $\angle NAB = \theta$. Therefore, it is clear that

$$\tan\theta = \frac{BN}{AN} = \frac{BN}{LM} = \frac{BM - MN}{OM - OL}$$

$$= \frac{BM - AL}{OM - OL} = \frac{y_2 - y_1}{x_2 - x_1}$$

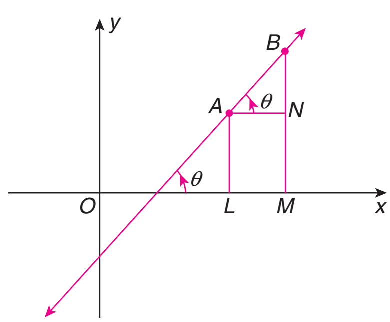


FIGURE 2.1

■

DEFINITION 2.2 **Intercepts of a Line** Suppose a line meets x-axis in the point $(a, 0)$ and y-axis in the point $(0, b)$. Then a is called x-intercept of the line and b is called the y-intercept of the line.

QUICK LOOK 2

1. A vertical line has x-intercept only.
2. A horizontal line has y-intercept only.
3. For a line through origin, both intercepts are zero.

THEOREM 2.2 (POINT–SLOPE FORM) Equation of a line passing through point $A(x_1, y_1)$ and having slope m (see Fig. 2.2) is

$$y - y_1 = m(x - x_1)$$

PROOF Suppose $P(x, y)$ is any point on the given line. Then, by Theorem 2.1, the slope of the line is

$$\frac{y - y_1}{x - x_1} = m$$

Therefore,

$$y - y_1 = m(x - x_1)$$

Conversely, let $Q(x', y')$ be a point such that

$$y' - y_1 = m(x - x_1)$$

$$\Rightarrow \frac{y' - y_1}{x - x_1} = m \qquad \text{(slope of the line)}$$

This implies that the line $\overleftrightarrow{AQ}$ coincides with the given line which, in turn, implies that Q lies on the given line. Therefore, equation of the line is $y - y_1 = m(x - x_1)$.

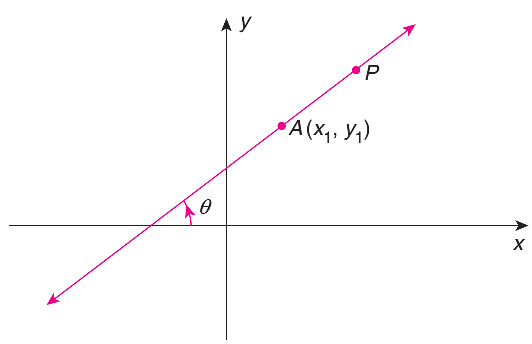


FIGURE 2.2

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Example 2.1

Write the equation of a line passing through $(\sqrt{2}, \sqrt{2})$ and having slope 1.

Solution: We have $m = 1$ and $x_1 = \sqrt{2} = y_1$. Therefore, the equation of the line is

$$y - \sqrt{2} = 1(x - \sqrt{2})$$

$$\Rightarrow x - y = 0$$

Example 2.2

Write the equation of the line which is inclined at an angle $\pi/3$ with the positive direction of the x-axis and passing through the point $(1, -2)$.

Solution: We have

$$m = \tan\left(\frac{\pi}{3}\right) = \sqrt{3}$$

Now given $x_1 = 1$ and $y_1 = -2$. Therefore, equation of the line is

$$y+2=\sqrt{3}(x-1)$$
$$\Rightarrow \sqrt{3}x-y-(2+\sqrt{3})=0$$

THEOREM 2.3 (SLOPE–INTERCEPT FORM) Equation of the line having slope m and y-intercept c is $y = mx + c$.

PROOF Since c is the y-intercept of the line, the line passes through the point $(0, c)$ and has slope m. Therefore, by Theorem 2.2, the equation of the line is

$$y-c=m(x-0)$$
$$\Rightarrow y=mx+c$$

■

QUICK LOOK 3

Equation of any line passing through origin (excluding y-axis) is of the form $y = mx$ (i.e., $c = 0$).

Example 2.3

Write the equation of the line having slope 1/2 and y-intercept 1.

Solution: We have $m = 1/2$ and $c = 1$ so that the equation of the line is

$$y=\frac{1}{2}x+1$$
$$\Rightarrow x-2y+2=0$$

Example 2.4

Write the equation of the line having slope 2 and y-intercept -3 and hence find the area of the triangle formed by this line and the two coordinate axes.

Solution: We have $m = 2$ and $c = -3$. Therefore, the equation of the line is $y = 2x - 3$. The line meets the axes at (3/2, 0) and (0, –3). Hence, the vertices of the triangle are (0, 0), (3/2, 0) and (0, –3). Therefore, area of the triangle is

$$\frac{1}{2}\left|0(0+3)+\frac{3}{2}(-3-0)+0(0-0)\right|=\frac{9}{4}\ \text{sq. unit}$$

THEOREM 2.4 (INTERCEPT FORM) If a and b are non-zero intercepts of a line on x- and y-axis, respectively, then the equation of these two intercepts is

$$\frac{x}{a}+\frac{y}{b}=1$$

PROOF This line passes through the points $(a, 0)$ and $(0, b)$ (see Fig. 2.3) so that its slope is

$$\frac{b-0}{0-a}=-\frac{b}{a}$$

Therefore, by Theorem 2.3, the equation of the line is

$$y=\frac{-b}{a}(x-a)$$

$$\Rightarrow \frac{y}{b} = -\frac{x}{a} + 1$$

$$\Rightarrow \frac{x}{a} + \frac{y}{b} = 1$$

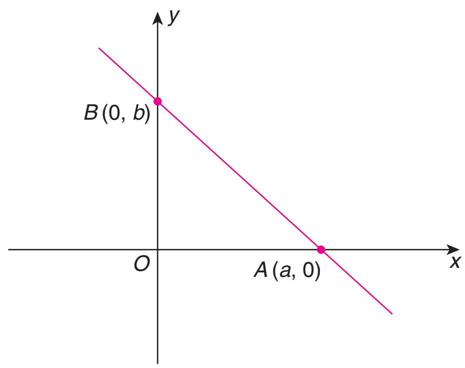

FIGURE 2.3

QUICK LOOK 4

Area of the triangle formed by the line

$$\frac{x}{a} + \frac{y}{b} = 1$$

and the coordinate axes is equal to

$$\frac{1}{2}(OA \cdot OB) = \frac{1}{2}|a||b| = \frac{1}{2}|ab| \text{ sq. unit}$$

Example 2.5

Find the equation of the line whose intercepts are numerically equal.

Solution: Suppose the intercepts are a and $-a$ or $-a$ and a. Then the equation is

$$\frac{x}{a} + \frac{y}{-a} = 1$$

or

$$\frac{x}{-a} + \frac{y}{a} = 1$$

That is,

$$x - y = a$$

or $$x - y = -a$$

Example 2.6

Find the equation of the line whose sum of the intercepts on the axes is 3 and their product is 2.

Solution: The equation of a line is

$$\frac{x}{a} + \frac{y}{b} = 1$$

Now,

$$a + b = 3$$

and $$ab = 2$$

Solving these equations, we get $a = 2$ and $b = 1$ or $a = 1$ and $b = 2$. Therefore, equation of the lines is

$$\frac{x}{2} + \frac{y}{1} = 1$$

or $$\frac{x}{1}+\frac{y}{2}=1$$

Therefore, the equation of the line is

$$x+2y-2=0$$

or $$2x+y-2=0$$

THEOREM 2.5 (TWO-POINT FORM) Equation of the line passing through two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is

$$(x-x_1)(y_1-y_2)-(y-y_1)(x_1-x_2)=0$$

PROOF **Case 1:** Suppose the line $\overleftrightarrow{AB}$ is vertical. Hence, $x_1 = x_2$ and $y_1 \neq y_2$. If $P(x, y)$ is any point on the line $\overleftrightarrow{AB}$, then $x_1 = x_2 = x$ so that

$$(x-x_1)(y_1-y_2)=(y-y_1)(x_1-x_2)$$

Case 2: $\overleftrightarrow{AB}$ is not a vertical line. Therefore, by Theorems 2.1 and 2.2, its equation is

$$y-y_1=\frac{(y_1-y_2)}{(x_1-x_2)}(x-x_1)$$

$$\Rightarrow (x-x_1)(y_1-y_2)=(y-y_1)(x_1-x_2)$$

■

Example 2.7

Write the equation of the line passing through the points $(a+b, a-b)$ and $(a-b, a+b)$.

Solution: We have $x_1 = a+b, y_1 = a-b, x_2 = a-b$ and $y_2 = a+b$. Therefore, equation of the line is

$$[x-(a+b)][a-b-(a+b)]=[y-(a-b)][a+b-(a-b)]$$
$$-2b[x-(a+b)]=2b[y-(a-b)]$$
$$-x+a+b=y-a+b$$

Therefore

$$x+y=2a$$

Example 2.8

Find the equation of the line joining the points $(a\cos\alpha, a\sin\alpha)$ and $(a\cos\beta, a\sin\beta)$ where $a>0$.

Solution: Equation of the line is

$$(x-a\cos\alpha)(a\sin\alpha-a\sin\beta)=(y-a\sin\alpha)(a\cos\alpha-a\cos\beta)$$

$$\Rightarrow (x-a\cos\alpha)(\sin\alpha-\sin\beta)=(y-a\sin\alpha)(\cos\alpha-\cos\beta)$$

$$\Rightarrow (x-a\cos\alpha)2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)=(y-a\sin\alpha)\left(-2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}\right)$$

$$\Rightarrow x\cos\frac{(\alpha+\beta)}{2}+y\sin\left(\frac{\alpha+\beta}{2}\right)=a\left[\cos\alpha\cos\frac{\alpha+\beta}{2}+\sin\alpha\sin\frac{\alpha+\beta}{2}\right]$$

$$\Rightarrow x\cos\frac{\alpha+\beta}{2}+y\sin\frac{\alpha+\beta}{2}=a\cos\left(\frac{\alpha-\beta}{2}\right)$$

Note: One can see in Example 2.8 that $(a\cos\alpha, a\sin\alpha)$ and $(a\cos\beta, a\sin\beta)$ are the points on a circle with centre at origin and radius a. In the above chord equation, if $\beta=\alpha$, then $x\cos\alpha+y\sin\alpha=a$ is the equation of the tangent to the circle with centre $(0, 0)$ and radius a.

Example 2.9

Show that the equation of the line joining two points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ is

$$y - 2at_1 = \frac{2}{t_1 + t_2}(x - at_1^2)$$

Solution: The slope of the line joining the two points is

$$\frac{2at_2 - 2at_1}{at_2^2 - at_1^2} = \frac{2}{t_1 + t_2}$$

Hence, the equation of the line is

$$y - 2at_1 = \frac{2}{t_1 + t_2}(x - at_1^2)$$

Note: In Chapter 4, we will discuss and show that for all real values of t, the locus of the point $(at^2, 2at)$ is the parabola $y^2 = 4ax$ and the equation of the tangent at $(at^2, 2at)$ is $ty = x + at^2$ which can be obtained by substituting $t_1 = t_2 = t$ in the equation of the line joining the points $(at_1^2, 2at)$ and $(at_2^2, 2at)$ which is also discussed in Example 4.3.

Theorem 2.6 (Equation of Straight Line in Normal Form)

A line l is such that its perpendicular distance ON (O is the origin) from the origin is p and the ray $\overrightarrow{ON}$ makes angle α with the positive direction of the x-axis (measured in counterclock sense). Then the equation of the line l is

$$x\cos\alpha + y\sin\alpha = p$$

Proof

Case 1: Suppose that the given line l is a vertical line meeting positive x-axis at point N [see Fig. 2.4(a)] so that $ON = p$ and $\alpha = 0$. Therefore, if $p(x, y)$ is any point on l, then $x = p$ and the equation of the line l is $x = p$ which is equivalent to

$$x\cos\alpha + y\sin\alpha = p \qquad (\because \alpha = 0)$$

Similarly, if l meets the negative x-axis at point N [see Fig. 2.4(b)], then the equation of the line l is $x = -p$ so that

$$x\cos\alpha + y\sin\alpha = p \qquad (\because \alpha = \pi)$$

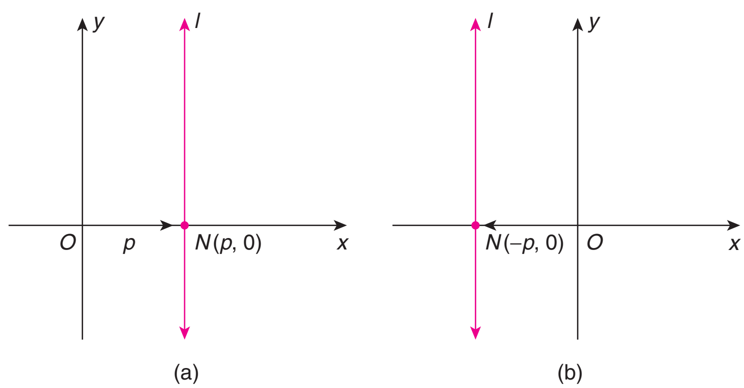

FIGURE 2.4

Case 2: Suppose the line l is horizontal and meets positive y-axis at point N so that $N = (0, p)$ and $\alpha = \pi/2$ [see Fig. 2.5(a)]. Since its equation is $y = p$, it is also given by

$$x\cos\alpha + y\sin\alpha = p \qquad (\because \alpha = \pi/2)$$

Similarly, if N lies on the negative y-axis, then $N = (0, -p)$ and $\alpha = -3\pi/2$ [see Fig. 2.5(b)] so that its equation is $y = -p$. The equation of the line can be written as

$$x\cos\alpha + y\sin\alpha = p$$

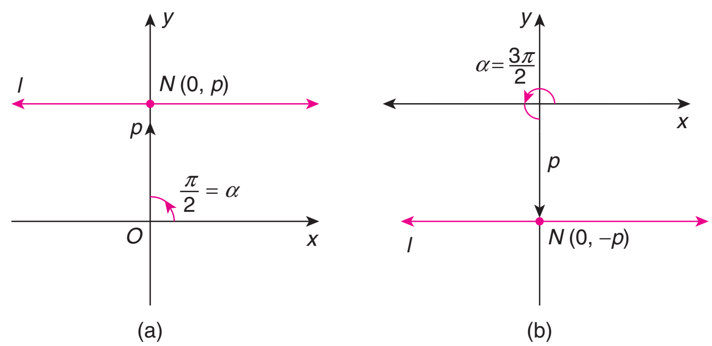

FIGURE 2.5

Case 3: Suppose l is an inclined line meeting positive coordinate axes at points A and B, respectively [see Fig. 2.6(a)]. So the coordinates are $A = (p\sec\alpha, 0)$ and $B = (0, p\ \text{cosec}\,\alpha)$. By Theorem 2.4, the equation of the line l is

$$\frac{x}{p\sec\alpha} + \frac{y}{p\,\text{cosec}\,\alpha} = 1$$

Hence

$$x\cos\alpha + y\sin\alpha = p$$

Similarly, for other cases, namely,

$$\frac{\pi}{2} < \alpha < \pi \ \text{ or } \ \pi < \alpha < \frac{3\pi}{2} \ \text{ or } \ \frac{3\pi}{2} < \alpha < 3\pi$$

we can show that its equation is

$$x\cos\alpha + y\sin\alpha = p$$

These three cases are shown in Figs. 2.6(b), 2.6(c) and 2.6(d), respectively.

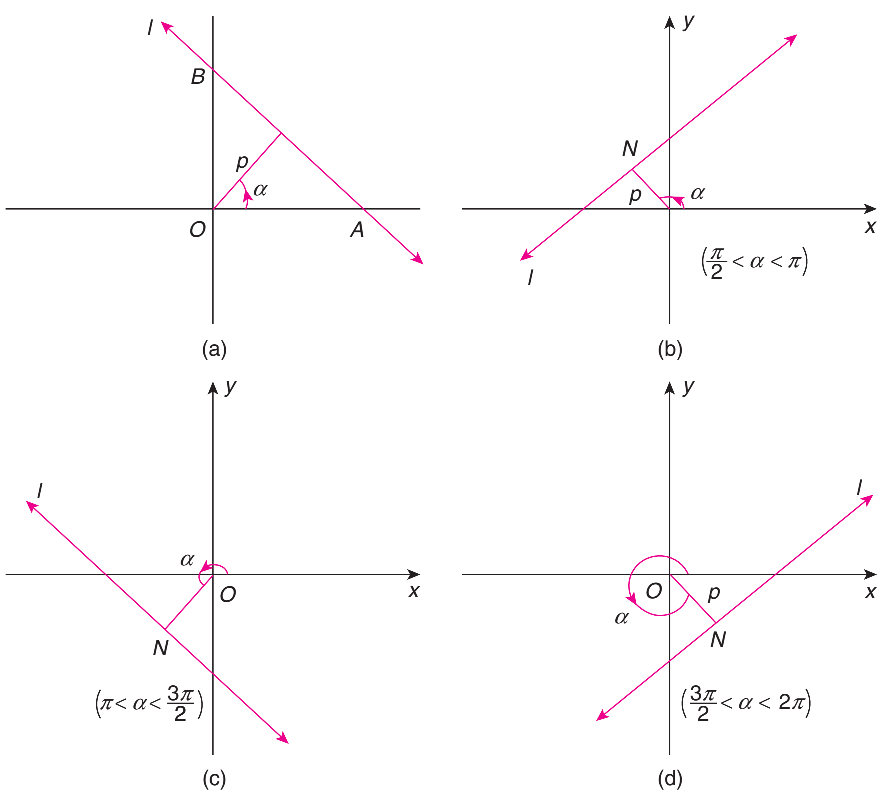

FIGURE 2.6

Example 2.10

Find the equation of the line whose distance from origin is 2 and the normal ray through origin makes an angle $\pi/4$ with the positive direction of the x-axis measured in counterclock sense.

Solution: We have $p = 2$ at $\alpha = \pi/4$. Hence, the equation of the line is

$$x\cos\frac{\pi}{4} + y\sin\frac{\pi}{4} = 2 \quad \text{or} \quad x + y = 2\sqrt{2}$$

Example 2.11

Find the equation of the line whose distance from the origin is 4 and the normal ray through origin makes an angle $(2\pi)/3$ with the positive direction of the x-axis measured in counterclock sense.

Solution: We have $p = 4$ and $\alpha = (2\pi)/3$. Hence the equation of the line is

$$x\cos\frac{2\pi}{3} + y\sin\frac{2\pi}{3} = 4 \quad \text{or} \quad x\left(-\frac{1}{2}\right) + y\left(\frac{\sqrt{3}}{2}\right) = 4$$

That is,

$$x - \sqrt{3}y = -8$$

Theorem 2.7 (Symmetric Form of Line) The equation of the straight line passing through (x_1, y_1) and making an angle θ with the positive direction of the x-axis measured in counterclock sense is

$$(x - x_1) : \cos\theta = (y - y_1) : \sin\theta$$

Proof Suppose the line is L which passes through (x_1, y_1).

Case 1: The line L is vertical so that $\theta = \pi/2$. Since the equation of L is $x = x_1$ which is written as

$$(x - x_1) : \cos\theta = (y - y_1) : \sin\theta$$

because $\cos\theta = 0$ and $\sin\theta = 1$.

Case 2: Suppose L is non-vertical so that $\theta \neq \pi/2$. Hence by Theorem 2.2, its equation is

$$(y - y_1) = \tan\theta(x - x_1)$$

$$\Rightarrow (x - x_1)\sin\theta = (y - y_1)\cos\theta$$

Thus

$$(x - x_1) : \cos\theta = (y - y_1) : \sin\theta$$

■

If $(x - x_1) = \gamma\cos\theta$ and $(y - y_1) = \gamma\sin\theta$, we have

$$\sqrt{(x - x_1)^2 + (y - y_1)^2} = |\gamma|$$

which shows that $|\gamma|$ represents the distance of the point (x, y) on the line from the given point (x_1, y_1). Therefore, if (x, y) is any point on the line and γ is any real parameter, then the locus of the point $(x_1 + \gamma\cos\theta, y_1 + \gamma\sin\theta)$ is the straight line

$$(x - x_1) : \cos\theta = (y - y_1) : \sin\theta$$

Also, γ takes positive values for points on the line on one side of (x_1, y_1) and takes negative values for the points on the other side of (x_1, y_1). For a given positive value of γ, there will be two points on the line which are equidistant from (x_1, y_1). Further, the equations

$$x = x_1 + \gamma\cos\theta \quad \text{and} \quad y = y_1 + \gamma\sin\theta$$

(θ is a fixed and γ is a parameter) are called the *parametric equations* of the line passing through (x_1, y_1) and making angle θ with the positive direction of the x-axis measured in counterclock sense.

Example 2.12

Write the parametric equation of the line passing through the point (3, 2) and making an angle 120° with the positive direction of the x-axis measured in counterclock sense and also find the coordinates of the points on the line which are at unit distance from the point (3, 2).

Solution: We have $(x_1, y_1) = (3,2), \cos\theta = \cos 120° = -1/2$ and $\sin\theta = \sin 120° = \sqrt{3}/2$. Therefore, the parametric equations of the line are

$$x = x_1 + \gamma\cos\theta = 3 + \gamma\left(-\frac{1}{2}\right)$$

and
$$y = y_1 + \gamma\sin\theta = 2 + \gamma\left(\frac{\sqrt{3}}{2}\right)$$

Solving these equations, we get

$$x = 3 - \gamma/2$$

and
$$y = 2 + \gamma\,(\sqrt{3}/2)$$

Substituting $\gamma = 1$ and $\gamma = -1$ in the above coordinates, the points on the line which are at a distance of 1 unit from the point (3, 2) are obtained, respectively, as

$$\left(\frac{5}{2}, \frac{4+\sqrt{3}}{2}\right) \text{ and } \left(\frac{7}{2}, \frac{4-\sqrt{3}}{2}\right)$$

DEFINITION 2.3 If a, b and c are real and either a or b is non-zero, then the expression $ax + by + c$ is called *first-degree expression in x and y* and the equation $ax + by + c = 0$ is called *first-degree equation in x and y*.

Examples

1. $2x - y + 3 = 0$
2. $x + 2 = 0$
3. $\sqrt{2}y - 1 = 0$
4. $\sqrt{3}x + 4y + 1 = 0$
5. $x\cos\alpha + y\sin\alpha = p$

THEOREM 2.8 Every first-degree equation in x and y represents a straight line and conversely, the equation of a straight line is a first-degree equation in x and y.

PROOF Suppose $ax + by + c = 0$ is a first-degree equation so that either a or b is non-zero. Now, either $b = 0$ or $b \neq 0$. If $b = 0$, then the equation reduces to $ax + c = 0$ where $a \neq 0$ which represents the vertical line $x = -c/a$. If $b \neq 0$, then the equation can be written as $y = (-a/b)x + (-c/b)$ which represents straight line having slope $(-a/b)$ and y-intercept $-c/b$. In any case, $ax + by + c = 0$ represents a straight line.

Conversely, let L be a straight line in the coordinate plane. Therefore, L is either a vertical line or a non-vertical line. Suppose L is a vertical line meeting x-axis at the point $(a, 0)$, the equation of L is $x-a = 0$ which is a first-degree equation in x and y. If L is non-vertical, then it meets y-axis at some point, say $(0, c)$, and has slope, say, m, then the equation of L is $y = mx + c$ (see Theorem 2.3) which is a first-degree equation as coefficient of y is -1 $(\neq 0)$. ■

In view of Theorem 2.8, the following definition is provided.

DEFINITION 2.4 **General Equation of a Straight Line** First-degree equation in x and y is called the *general equation of a straight line*. Suppose $ax + by + c = 0$ is a first-degree equation, then we have the following conditions:

1. If $c = 0$, then the line passes through (0, 0).
2. If $b \neq 0$, then the first-degree equation represents a line having slope $(-a/b)$ and y-intercept $(-c/b)$.
3. If $b = 0$, $ax + by + c = 0$ represents a vertical line.
4. If $abc \neq 0$, then the first-degree equation represents a line with x-intercept $(-c/a)$ and y-intercept $(-c/b)$.

THEOREM 2.9 Two first-degree equations $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ represent parallel lines if and only if $a_1b_2 = a_2b_1$ (i.e., $a_1{:}b_1 = a_2{:}b_2$).

Proof When two lines are parallel, both are either vertical or non-vertical. If both are vertical, then $b_1 = b_2 = 0$ [by point (3) below Definition 2.4] so that $a_1b_2 = a_2b_1 = 0$. If both are non-vertical, their slopes are equal [by point (2) below Definition 2.4]. Thus,

$$-\frac{a_1}{b_1} = -\frac{a_2}{b_2}$$

Hence, $$a_1b_2 = a_2b_1 \text{ or } a_1{:}b_1 = a_2{:}b_2$$

Conversely, when $a_1b_2 = a_2b_1$ and if $b_1 = 0$, then $a_1 \neq 0$ so that $b_2 = 0$. Therefore, $b_1 = 0 \Leftrightarrow b_2 = 0$ and hence if one of the lines is vertical, then the other is also vertical so that the given lines are parallel. If both b_1 and b_2 are non-zero, then

$$a_1b_2 = a_2b_1 \Rightarrow \frac{a_1}{b_1} = \frac{a_2}{b_2}$$
$$\Rightarrow -\frac{a_1}{b_1} = -\frac{a_2}{b_2}$$

Since the slopes are equal, the lines are parallel. ■

QUICK LOOK 5

1. Equation of any line parallel to the line $ax + by + c = 0$ is of the form $ax + by + c' = 0$.
2. In particular, equation of the line passing through the point (x_1, y_1) and parallel to the line $ax + by + c = 0$ is $a(x-x_1) + b(y-y_1) = 0$.

Theorem 2.10 Two first-degree equations $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ represent the same straight line if and only if $a_1{:}b_1{:}c_1 = a_2{:}b_2{:}c_2$.

Proof Suppose the two equations represent the same straight line. Since every line is parallel to itself, by Theorem 2.9, $a_1b_2 = a_2b_1$ and hence $a_1{:}b_1 = a_2{:}b_2$. If $b_1 = 0$, then $b_2 = 0$ so that a_1 and a_2 are non-zero. This implies that

$$-\frac{c_1}{a_1} = -\frac{c_2}{a_2}$$

Therefore

$$a_1 : b_1 : c_1 = a_2 : b_2 : c_2$$

If $b_1 \neq 0$, then $b_2 \neq 0$ (since $a_1b_2 = a_2b_1$). Let

$$\frac{b_1}{b_2} = \lambda$$

Therefore,

$$a_1b_2 = a_2b_1 \Rightarrow \frac{a_2}{a_1} = \frac{b_2}{b_1} = \frac{1}{\lambda}$$
$$\Rightarrow a_1 = \lambda a_2 \quad (2.1)$$

Also $(0, -c_1/b_1)$ is a point on $a_1x + b_1y + c_1 = 0$ which implies that $(0, -c_1/b_1)$ also lies on $a_2x + b_2y + c_2 = 0$. Therefore

$$b_2\left(\frac{-c_1}{b_1}\right) + c_2 = 0$$
$$\Rightarrow \frac{c_2}{c_1} = \frac{b_2}{b_1} = \frac{1}{\lambda} \quad (2.2)$$

Therefore, from Eqs. (2.1) and (2.2)

$$a_1{:}b_1{:}c_1 = a_2{:}b_2{:}c_2$$

Conversely, suppose $a_1{:}b_1{:}c_1 = a_2{:}b_2{:}c_2$. Therefore, for some real $\lambda \neq 0$, we have $a_1 = \lambda a_2$, $b_1 = \lambda b_2$, $c_1 = \lambda c_2$. Hence

$$\begin{aligned} a_1x + b_1y + c_1 = 0 &\Leftrightarrow \lambda(a_2x + b_2y + c_2) = 0 \\ &\Leftrightarrow a_2x + b_2y + c_2 = 0 \end{aligned}$$

Therefore, both equations represent the same straight line. ■

THEOREM 2.11 Let $ax + by + c = 0$ be a straight line. Then

1. $\left(\dfrac{a}{\sqrt{a^2+b^2}}\right)x + \left(\dfrac{b}{\sqrt{a^2+b^2}}\right)y = \dfrac{-c}{\sqrt{a^2+b^2}}$ is the normal form of the given line if $c < 0$.
2. $\left(\dfrac{-a}{\sqrt{a^2+b^2}}\right)x + \left(\dfrac{-b}{\sqrt{a^2+b^2}}\right)y = \dfrac{c}{\sqrt{a^2+b^2}}$ is the normal form of the given line if $c > 0$.

PROOF Suppose $x\cos\alpha + y\sin\alpha = p$ is the normal form of $ax + by + c = 0$. Therefore, by Theorem 2.10, there exists a real $\lambda \neq 0$ such that $\cos\alpha = \lambda a$, $\sin\alpha = \lambda b$ and $-p = \lambda c$. Now,

$$\cos^2\alpha + \sin^2\alpha = 1$$

implies

$$\lambda = \pm\frac{1}{\sqrt{a^2+b^2}}$$

Also $-\lambda c = p > 0$ (since p is the distance of the line from origin) implies that

$$\lambda = \begin{cases} \dfrac{1}{\sqrt{a^2+b^2}} & \text{if } c < 0 \\ \dfrac{-1}{\sqrt{a^2+b^2}} & \text{if } c > 0 \end{cases}$$

Therefore, if $c < 0$, then the normal form of the line is

$$x\left(\frac{a}{\sqrt{a^2+b^2}}\right) + y\left(\frac{b}{\sqrt{a^2+b^2}}\right) = \frac{-c}{\sqrt{a^2+b^2}}$$

or if $c > 0$, the normal form of the line is

$$x\left(\frac{-a}{\sqrt{a^2+b^2}}\right) + y\left(\frac{-b}{\sqrt{a^2+b^2}}\right) = \frac{c}{\sqrt{a^2+b^2}}$$

■

QUICK LOOK 6

1. To reduce $ax + by + c = 0$ to the normal form, take constant c to the right-hand side (RHS), divide both sides with $\sqrt{a^2+b^2}$ and then make RHS positive.

2. $p = \pm\dfrac{c}{\sqrt{a^2+b^2}}$ is the distance of the line $ax + by + c = 0$ from the origin. That is,

$$p = \frac{|c|}{\sqrt{a^2+b^2}}$$

Example 2.13

Write $2x + 3y + 5 = 0$ in the normal form. What is the distance of the line from origin?

Solution: In $2x + 3y + 5 = 0$, the constant 5 should be taken to the RHS of the equation, that is

$$2x + 3y = -5$$

Dividing both sides with $\sqrt{a^2+b^2} = \sqrt{2^2+3^2} = \sqrt{13}$, we get

$$\frac{2}{\sqrt{13}}x + \frac{3}{\sqrt{13}}y = \frac{-5}{\sqrt{13}}$$

To make the RHS positive, we multiply both sides with (-1). Thus, the normal form is

$$\left(\frac{-2}{\sqrt{13}}\right)x + \left(\frac{-3}{\sqrt{13}}\right)y = \frac{5}{\sqrt{13}}$$

where

$$\cos\alpha = \frac{-2}{\sqrt{13}} \quad \text{and} \quad \sin\alpha = \frac{-3}{\sqrt{13}}$$

Now, the distance of the line from the orgin is

$$p = \frac{|c|}{\sqrt{a^2+b^2}} = \frac{|-5|}{\sqrt{13}} = \frac{5}{\sqrt{13}}$$

Example 2.14

Find the normal form of the line $3x + 4y - 10 = 0$ and its distance from the origin.

Solution: The equation $3x + 4y - 10 = 0$ can be written as $3x + 4y = 10$. Dividing both sides with $\sqrt{a^2+b^2} = \sqrt{3^2+4^2} = 5$, we get

$$\frac{3}{5}x + \frac{4}{5}y = 2$$

where

$$\cos\alpha = \frac{3}{5} \quad \text{and} \quad \sin\alpha = \frac{4}{5}$$

Now the distance of the line from the origin is

$$p = \frac{|10|}{5} = \frac{10}{5} = 2$$

Theorem 2.12 (Angle between Two Lines) If $0 < \theta < \pi/2$ is the angle between the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, then

$$\cos\theta = \frac{|a_1a_2 + b_1b_2|}{\sqrt{(a_1^2+b_1^2)(a_2^2+b_2^2)}}$$

or equivalently

$$\theta = \text{Cos}^{-1}\left(\frac{|a_1a_2 + b_1b_2|}{\sqrt{(a_1^2+b_1^2)(a_2^2+b_2^2)}}\right)$$

Proof Let the lines represented by the given equations (in the order written) be l_1 and l_2, respectively. If l_1 and l_2 are parallel lines, then we consider that θ is equal to 0 or π and also

$$\frac{|a_1a_2 + b_1b_2|}{\sqrt{(a_1^2+b_1^2)(a_2^2+b_2^2)}} = 1$$

because $a_1{:}b_1 = a_2{:}b_2$ (Theorem 2.9). Hence

$$\text{Cos}^{-1}\left(\frac{|a_1a_2 + b_1b_2|}{\sqrt{(a_1^2+b_1^2)(a_2^2+b_2^2)}}\right) = \text{Cos}^{-1}(1) = 0$$

Therefore, without loss of generality, we assume that l_1 and l_2 are intersecting so that $a_1{:}b_1 \neq a_2{:}b_2$. Let $\overleftrightarrow{OL_1}$ and $\overleftrightarrow{OL_2}$ be the lines through origin O and parallel to the lines l_1 and l_2, respectively (see Fig. 2.7). Therefore, from Quick Look 5, part (2), the equations of $\overleftrightarrow{OL_1}$ and $\overleftrightarrow{OL_2}$ are, respectively, $a_1x + b_1y = 0$ and $a_2x + b_2y = 0$. Let θ_1 and θ_2 be the angles made by $\overleftrightarrow{OL_1}$ and $\overleftrightarrow{OL_2}$ with the positive direction of the x-axis so that $|\theta_1 - \theta_2|$ is the measure of the angle between the lines. We know that $P(b_1, -a_1)$ lies on $\overleftrightarrow{OL_1}$ and $Q(b_2, -a_2)$ lies on $\overleftrightarrow{OL_2}$. Draw PL and QM perpendicular to x-axis. Now,

$$\begin{aligned}\cos\theta = \cos(\theta_1 - \theta_2) &= \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\\ &= \left(\frac{b_1}{\sqrt{a_1^2+b_1^2}}\right)\left(\frac{b_2}{\sqrt{a_2^2+b_2^2}}\right)+\left(\frac{-a_1}{\sqrt{a_1^2+b_1^2}}\right)\left(\frac{-a_2}{\sqrt{a_1^2+b_1^2}}\right)\\ &= \frac{a_1a_2+b_1b_2}{\sqrt{(a_1^2+b_1^2)(a_2^2+b_2^2)}}\end{aligned}$$

$\theta = |\theta_1 - \theta_2|$ and $0 < \theta < \pi/2$.

$$\cos\theta = \frac{|a_1a_2+b_1b_2|}{\sqrt{(a_1^2+b_1^2)(a_2^2+b_2^2)}}$$

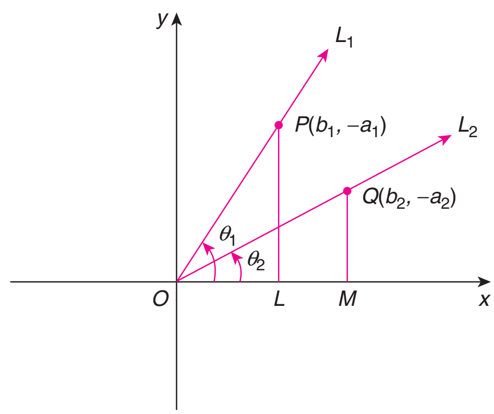

FIGURE 2.7

Note: If both lines are non-vertical and θ is the angle between them, then

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right|$$

where m_1 and m_2 are their slopes.

QUICK LOOK 7

1. **(a)** The lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are at right angles $\Leftrightarrow$ $a_1a_2 + b_1b_2 = 0$ (since $\theta = 90°$).
   (b) Product of the slopes is -1 provided the lines are at right angles.
2. The equation of any line perpendicular to the line $ax + by + c = 0$ is of the form $bx - ay + c' = 0$.
3. In particular, the equation of the line passing through the point (x_1, y_1) and perpendicular to the line $ax + by + c = 0$ is $b(x - x_1) - a(y - y_1) = 0$.

DEFINITION 2.5 **Image or a Reflection of Point** Let l be a straight line and P be a point on the plane of the line, but not on the line. A point Q in the same plane is called the image or reflection of P in the line l if l is the perpendicular bisector of the segment $\overline{PQ}$.

QUICK LOOK 8

To find the image of P on the line l, draw PM perpendicular to the line l and produce it to Q such that $PM = MQ$ (see Fig. 2.8).

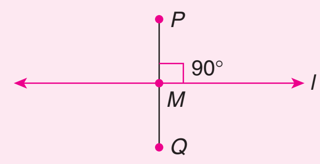

FIGURE 2.8

THEOREM 2.13 Let l be a line whose equation is $ax + by + c = 0$ and $P(x_1, y_1)$ be a point on the plane of the line, but not on l. Suppose $M(h, k)$ is the foot of the perpendicular drawn from point P onto l and $Q(x_1', y_1')$ is the image of point P in the line. Then

1. $$\frac{h - x_1}{a} = \frac{k - y_1}{b} = -\frac{(ax_1 + by_1 + c)}{a^2 + b^2}$$

2. $$\frac{x_1' - x_1}{a} + \frac{y_1' - y_1}{b} = -2\frac{(ax_1 + by_1 + c)}{a^2 + b^2}$$

PROOF 1. See Fig. 2.9. The line PM is perpendicular to the line $ax + by + c = 0$, so by Quick look 7, part (2), we have

$$\left(\frac{k - y_1}{h - x_1}\right)\left(\frac{-a}{b}\right) = -1$$

$$\Rightarrow \frac{h - x_1}{a} = \frac{k - y_1}{b} = \lambda \qquad \text{(say)}$$

$$\Rightarrow h = x_1 + \lambda a, \quad k = y_1 + \lambda b \tag{2.3}$$

Since $M(h, k)$ lies on the line, we have $ah + bk + c = 0$, which implies that

$$a(x_1 + \lambda a) + b(y_1 + \lambda b) + c = 0$$

$$\Rightarrow \lambda = -\frac{(ax_1 + by_1 + c)}{a^2 + b^2} \tag{2.4}$$

Therefore, from Eqs. (2.3) and (2.4), we get

$$\frac{h - x_1}{a} = \frac{k - y_1}{b} = -\frac{(ax_1 + by_1 + c)}{a^2 + b^2}$$

2. Since M is the midpoint of PQ, we can see that

$$\frac{x_1' - x_1}{a} = \frac{y_1' - y_1}{b} = -2\frac{(ax_1 + by_1 + c)}{a^2 + b^2}$$

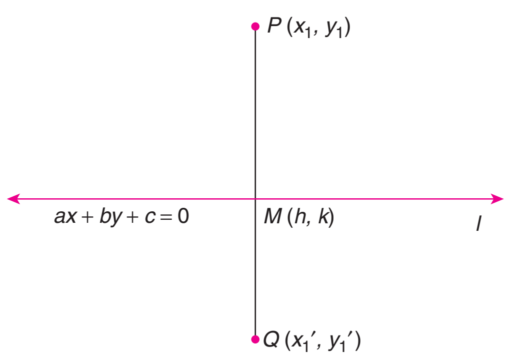

FIGURE 2.9 ■

***Note*:** Theorem 2.13 gives the formulae for the foot of the perpendicular drawn from a point onto the given line and the image of a point on the given line.

Example 2.15

Find the foot of the perpendicular drawn from the point $(1, 2)$ onto the line $3x + 4y - 1 = 0$ and also find the image of $(1, 2)$ on the given line.

Solution: We have $(x_1, y_1) = (1, 2)$ and the line is $3x + 4y - 1 = 0$.

1. Suppose (h, k) is the foot of the perpendicular. Therefore, from Theorem 2.13, we have

$$\frac{h-1}{3} = \frac{k-2}{4} = -\frac{[3(1)+4(2)-1]}{3^2+4^2} = -\frac{10}{25} = -\frac{2}{5}$$

Hence

$$h = 1 - \frac{6}{5} = -\frac{1}{5}$$
$$k = 2 - \frac{8}{5} = \frac{2}{5}$$

Therefore, foot of the perpendicular from $(1, 2)$ is

$$\left(-\frac{1}{5}, \frac{2}{5}\right)$$

2. Suppose (α, β) is the image of $(1, 2)$. From Theorem 2.13, we have

$$\frac{\alpha-1}{3} = \frac{\beta-2}{4} = -2\frac{[3(1)+4(2)-1]}{3^2+4^2} = \frac{-20}{25} = \frac{-4}{5}$$

Hence

$$\alpha = 1 - \frac{12}{5} = -\frac{7}{5}$$
$$\beta = 2 - \frac{16}{5} = -\frac{6}{5}$$

Thus, the image is

$$\left(-\frac{7}{5}, -\frac{6}{5}\right)$$

THEOREM 2.14 The distance of the line $ax + by + c = 0$ from the point $P(x_1, y_1)$ is

$$\frac{|ax_1 + by_1 + c|}{\sqrt{a^2+b^2}}$$

PROOF See Fig. 2.10. Shift the origin to the point $P(x_1, y_1)$ and let the new coordinates of any point be denoted by (X, Y). Hence, by Section 1.4.1, we have $x = X + x_1$ and $y = Y + y_1$ so that P becomes $(0, 0)$ and the equation of the given line is $a(X + x_1) + b(Y + y_1) + c = 0$. That is,

$$aX + bY + ax_1 + by_1 + c = 0 \tag{2.5}$$

Hence, by Quick Look 6, part (2), the distance of the line [Eq. (2.5)] from $P(0, 0)$ is equal to

$$\frac{|ax_1 + by_1 + c|}{\sqrt{a^2+b^2}}$$

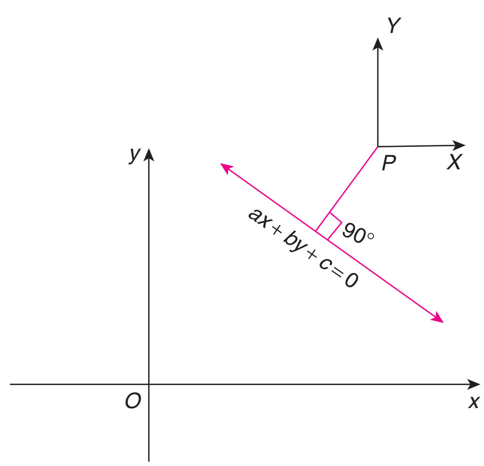

FIGURE 2.10 ■

THEOREM 2.15 The distance between the parallel lines $ax + by + c = 0$ and $ax + by + c' = 0$ is

$$\frac{|c - c'|}{\sqrt{a^2 + b^2}}$$

PROOF See Fig. 2.11. Let $P(x_1, y_1)$ be a point on $ax + by + c' = 0$ so that

$$ax_1 + by_1 + c' = 0 \tag{2.6}$$

Now, the distance between the two parallel lines is equal to the distance of the line $ax + by + c = 0$ from the point $P(x_1, y_1)$, which is given by

$$\frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}} \quad \text{(by Theorem 2.14)}$$

$$\Rightarrow \frac{|-c' + c|}{\sqrt{a^2 + b^2}} \quad \text{[from Eq. (2.6)]}$$

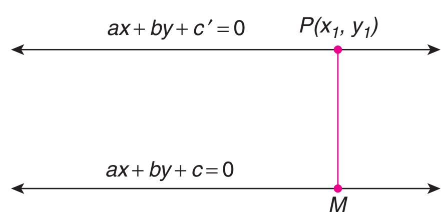

FIGURE 2.11

THEOREM 2.16 If $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are two intersecting lines, then their point of intersection is

$$\left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \frac{c_1a_2 - a_1c_2}{a_1b_2 - a_2b_1}\right)$$

PROOF Since the lines are not parallel, we have $a_1b_2 \neq a_2b_1$. Let (α, β) be the point of intersection of the lines. Therefore,

$$a_1\alpha - b_1\beta = -c_1 \tag{2.7}$$

and
$$a_2\alpha - b_2\beta = -c_2 \tag{2.8}$$

Solving Eqs. (2.7) and (2.8) for α and β, we get

$$\alpha = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}$$

$$\beta = \frac{c_1a_2 - a_1c_2}{a_1b_2 - a_2b_1}$$

Notation: The line $ax + by + c = 0$ is denoted by L so that $L = 0$ represents a straight line if either a or b is not zero.

$$L \equiv ax + by + c$$
$$L_{11} = ax_1 + by_1 + c$$
$$L_{22} = ax_2 + by_2 + c$$

THEOREM 2.17 Let $L \equiv ax + by + c$ be a line and $A(x_1, y_1)$ and $B(x_2, y_2)$ be the two points on the same plane, but not on the line $L = 0$, then $L = 0$ divides the line segment $\overline{AB}$ in the ratio $-L_{11}:L_{22}$.

PROOF Suppose $L = 0$ meets the line $\overleftrightarrow{AB}$ in point $P(x, y)$ and $AP{:}PB = \lambda{:}1$. Therefore

$$x = \frac{x_1 + \lambda x_2}{\lambda + 1}$$

$$y = \frac{y_1 + \lambda y_2}{\lambda + 1}$$

Since $P(x, y)$ lies on the line $L = 0$, we have

$$a\frac{(x_1 + \lambda x_2)}{\lambda + 1} + b\frac{(y_1 + \lambda y_2)}{\lambda + 1} + c = 0$$

That is, $ax_1 + by_1 + c + \lambda(ax_2 + by_2 + c) = 0$. That is, $L_{11} + \lambda L_{22} = 0$. Therefore,

$$\lambda = -\frac{L_{11}}{L_{22}}$$

Hence, $AP{:}PB = -L_{11}{:}L_{22}$. ■

QUICK LOOK 9

1. $L = 0$ divides $\overline{AB}$ internally

$\Leftrightarrow -L_{11}{:}L_{22}$ is positive

$\Leftrightarrow \frac{-L_{11}}{L_{22}} > 0$

$\Leftrightarrow L_{11}$ and L_{22} are opposite signs

Therefore, points A and B lie on the opposite sides of line $L = 0 \Leftrightarrow L_{11}$ and L_{22} are of opposite signs.

2. Points A and B lie on the same side of $L = 0$

$\Leftrightarrow$ the division is external

$\Leftrightarrow \frac{-L_{11}}{L_{22}} < 0$

$\Leftrightarrow \frac{L_{11}}{L_{22}} > 0$

Therefore, L_{11} and L_{22} have the same sign.

DEFINITION 2.6 **Origin and Non-Origin Sides** Let L be a straight line in the coordinate plane which is not passing through the origin. The side of the region in which the origin lies is called the origin side of the line and the other is called the non-origin side of the line L (see Fig. 2.12).

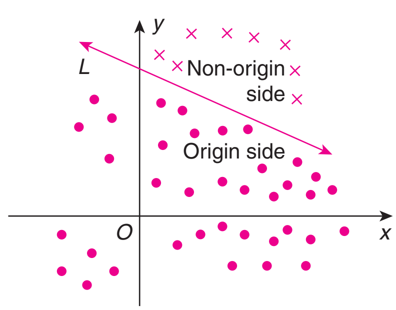

FIGURE 2.12

THEOREM 2.18 Let $L \equiv ax + by + c = 0$ be a line which is not passing through the origin. Then a point $A(x_1, y_1)$ (not on the line $L = 0$) lies

(a) on the non-origin side of $L = 0 \Leftrightarrow c$ and L_{11} are of opposite signs

(b) $A(x_1, y_1)$ lies on the origin side of $L = 0 \Leftrightarrow c$ and L_{11} have the same sign

PROOF In Quick Look 9, take $B = (x_2, y_2) = (0, 0)$. Hence, points A and B (which is equal to the origin) lie on the opposite sides of $L = 0$ so that L_{11} and $L_{22} = c$ have the opposite signs and points A and $(0, 0)$ are on the same side. Hence, L_{11} and $L_{22} = c$ have the same sign. ■

QUICK LOOK 10

Let $ax + by + c = 0$ be a line and $c \neq 0$.

1. If $c > 0$, then for all points on the origin side $L_{11} > 0$ and for all points on the non-origin side $L_{11} < 0$.
2. If $c < 0$, then for all points on the origin side $L_{11} < 0$ and $L_{11} > 0$ for all points on the non-origin side.

Example 2.16

Check whether the point (2, 1) lies on the origin side or non-origin side of the line $2x - 3y + 1 = 0$.

Solution: We have $L \equiv 2x - 3y + 1$, $A(x_1, y_1) = (2, 1)$ and $c = 1$. That is,

$$L_{11} = 2(2) - 3(1) + 1 = 2$$

and

$$c = 1$$

That is, L_{11} and c have the same origin. Hence, (2,1) lies on the origin side of $L = 0$.

Example 2.17

Find the ratio in which the line $L \equiv x + 2y - 3 = 0$ divides the line joining the points $A(1, 2)$ and $B(3, 2)$.

Solution: $A(1, 2) = (x_1, y_1)$ and $B(3, 2) = (x_2, y_2)$ so that

$$L_{11} = 1 + 2(2) - 3 = 2$$

and

$$L_{22} = 3 + 2(2) - 3 = 4$$

Therefore, $L = 0$ divides $\overline{AB}$ in the ratio

$$-L_{11}:L_{22} = -2:4 = -1:2$$

That is, $L = 0$ divides $\overline{AB}$ *externally* in the ratio 1:2.

THEOREM 2.19 Suppose $U_1 \equiv a_1x + b_1y + c_1 = 0$ and $U_2 \equiv a_2x + b_2y + c_2 = 0$ are two parallel lines. Then for any real values of λ_1 and λ_2 such that $|\lambda_1| + |\lambda_2| \neq 0$, the equation $\lambda_1 U_1 + \lambda_2 U_2 = 0$ represents the line parallel to each of the lines $U_1 = 0$ and $U_2 = 0$.

PROOF From Quick Look 5, $U_2 = 0$ can be written as $U_2 \equiv a_1x + b_1y + c_1' = 0$. Therefore,

$$\lambda_1 U_1 + \lambda_2 U_2 \equiv (\lambda_1 + \lambda_2)\,[a_1x + b_1y] + \lambda_1 c_1 + \lambda_2 c_1' = 0$$

which represents the line parallel to $U_1 \equiv a_1x + b_1y + c_1 = 0$. ■

THEOREM 2.20 If $U_1 \equiv a_1x + b_1y + c_1 = 0$ and $U_2 \equiv a_2x + b_2y + c_2 = 0$ are intersecting lines, then the equation $\lambda_1 U_1 + \lambda_2 U_2 = 0$ $(|\lambda_1| + |\lambda_2| \neq 0)$ represents the lines passing through the intersection of $U_1 = 0$ and $U_2 = 0$. Conversely, the equation of any line passing through the intersection of $U_1 = 0$ and $U_2 = 0$ is of the form $\lambda_1 U_1 + \lambda_2 U_2 = 0$ for some λ_1 and λ_2 such that $|\lambda_1| + |\lambda_2| \neq 0$.

PROOF Suppose $U_1 \equiv a_1x + b_1y + c_1 = 0$ and $U_2 \equiv a_2x + b_2y + c_2 = 0$ are intersecting lines so that $a_1b_2 - a_2b_1 \neq 0$. Let $p(\alpha, \beta)$ be the point of intersection of $U_1 = 0$ and $U_2 = 0$. Therefore,

$$a_1\alpha + b_1\beta + c_1 = 0 \tag{2.9}$$

and

$$a_2\alpha + b_2\beta + c_2 = 0 \tag{2.10}$$

Now, $|\lambda_1| + |\lambda_2| \neq 0$ implies that atleast one of λ_1 and λ_2 is not zero. Consider that

$$\lambda_1 U_1 + \lambda_2 U_2 = (\lambda_1 a_1 + \lambda_2 a_2)x + (\lambda_1 b_1 + \lambda_2 b_2)y + \lambda_1 c_1 + \lambda_2 c_2 = 0$$

If both $\lambda_1 a_1 + \lambda_2 a$ and $\lambda_1 b_1 + \lambda_2 b_2$ are zero, then

$$\frac{a_1}{a_2} = -\frac{\lambda_1}{\lambda_2} = \frac{b_1}{b_2}$$

which is a contradiction of the hypothesis $a_1b_2 \neq a_2b_1$. Therefore, $\lambda_1 U_1 + \lambda_2 U_2 = 0$ is a first-degree equation and hence it represents a straight line. Also, from Eqs. (2.9) and (2.10), it follows that $\lambda_1 U_1 + \lambda_2 U_2 = 0$ passes through (α, β). Conversely, suppose $U \equiv ax + by + c = 0$ is a line passing through (α, β), then

$$a\alpha + b\beta + c = 0 \tag{2.11}$$

Since either a or b is not zero [i.e., $(a, b) \neq (0, 0)$] and $a_1b_2 - a_2b_1 \neq 0$, it follows that the equations

$$a_1x + a_2y = a \quad \text{and} \quad b_1x + b_2y = b$$

have unique solution, say, $x = \lambda_1$ and $y = \lambda_2$ and $|\lambda_1| + |\lambda_2| \neq 0$. Therefore,

$$\lambda_1 a_1 + \lambda_2 a_2 = a \quad \text{and} \quad \lambda_1 b_1 + \lambda_2 b_2 = 0$$

Now, from Eq. (2.11), we get

$$\begin{aligned} c &= -a\alpha - \beta b = -(\lambda_1 a_1 + \lambda_2 a_2)\alpha - (\lambda_1 b_1 + \lambda_2 b_2)\beta \\ &= -\lambda_1(a_1\alpha + b_1\beta) - \lambda_2(a_2\alpha + b_2\beta) \\ &= -\lambda_1(-c_1) - \lambda_2(-c_2) \qquad \text{[from Eqs. (2.9) and (2.10)]} \\ &= \lambda_1 c_1 + \lambda_2 c_2 \end{aligned}$$

Therefore

$$\begin{aligned} ax + by + c &= (\lambda_1 a_1 + \lambda_2 a_2)x + (\lambda_1 b_1 + \lambda_2 b_2)y + \lambda_1 c_1 + \lambda_2 c_2 \\ &= \lambda_1(a_1x + b_1y + c_1) + \lambda_2(a_2x + b_2y + c_2) \\ &= \lambda_1 U_1 + \lambda_2 U_2 \end{aligned}$$

Thus, the equation of any line passing through the intersection of the lines $U_1 = 0$ and $U_2 = 0$ is of the form $\lambda_1 U_1 + \lambda_2 U_2 = 0$ where $|\lambda_1| + |\lambda_2| \neq 0$. ■

Note: If $U_1 = 0$ and $U_2 = 0$ are two intersecting lines, then $\lambda_1 U_1 + \lambda_2 U_2 = 0$, where either λ_1 or λ_2 is non-zero, represents all the lines passing through their point of intersection including $U_1 = 0$ and $U_2 = 0$. If $\lambda_1 \neq 0$, then the equation can be written as $U_1 + (\lambda_2/\lambda_1)\, U_2 = 0$ which is of the form $U_1 + \lambda U_2 = 0$, where λ is λ_2/λ_1. The equation $U_1 + \lambda U_2 = 0$ represents all lines passing through the intersection of $U_1 = 0$ and $U_2 = 0$ including $U_1 = 0$, but excluding $U_2 = 0$. Hence, for all practical purposes, we consider the equation $U_1 + \lambda U_2 = 0$, where λ is a real parameter in solving the problems.

QUICK LOOK 11

Suppose $U_1 = 0$ and $U_2 = 0$ are two lines, then

1. If $U_1 = 0$ and $U_2 = 0$ are parallel lines, then $U_1 + \lambda U_2 = 0$ represents a line parallel to each of $U_1 = 0$ and $U_2 = 0$.
2. If $U_1 = 0$ and $U_2 = 0$ are intersecting lines, then $U_1 + \lambda U_2 = 0$ represents a line passing through the intersection of $U_1 = 0$ and $U_2 = 0$.

THEOREM 2.21 If $U_1 = 0$ and $U_2 = 0$ are two intersecting lines, then every line in the plane of $U_1 = 0$ and $U_2 = 0$ is of the form $\lambda_1 U_1 + \lambda_2 U_2 + \lambda_3 = 0$.

PROOF Let $U = 0$ be a line in the plane of $U_1 = 0$ and $U_2 = 0$. Therefore, either $U = 0$ passes through the intersection of $U_1 = 0$ and $U_2 = 0$ or it does not. In the former case, $U = 0$ must be of the form $\lambda_1 U_1 + \lambda_2 U_2 = 0$ for some real λ_1 and λ_2, such that $|\lambda_1| + |\lambda_2| \neq 0$. In this case, $\lambda_3 = 0$. Suppose $U = 0$ does not pass through the intersection of $U_1 = 0$ and $U_2 = 0$. Hence, there is one and only one line L in the plane passing through the intersection of $U_1 = 0$ and $U_2 = 0$ and parallel to the line L. Hence, by Theorem 2.20, the equation of L must be of the form $\lambda_1 U_1 + \lambda_2 U_2 = 0$. Since L is parallel to $U = 0$, then $U = 0$ is of the form $\lambda_1 U_1 + \lambda_2 U_2 + \lambda_3 = 0$ for some real constant λ_3. Thus the theorem is proved. ■

Example 2.18

Find the equation of the line passing through the intersection of the lines $2x - y + 5 = 0$ and $x + y + 1 = 0$ and the origin.

Solution: By Theorem 2.20 and the Note mentioned below it, the required line is of the form

$$(2x - y + 5 = 0) + \lambda(x + y + 1) = 0$$

Since this also passes through (0, 0), we have $\lambda = -5$. Hence the required line is

$$(2x - y + 5) - 5\,(x + y + 1) = 0$$

$$\Rightarrow -3x - 6y = 0$$

$$\Rightarrow x + 2y = 0$$

Direct Method: Solving the equations $2x - y + 5 = 0$ and $x + y + 1 = 0$, we get the point of intersection (–2, 1). Therefore, the equation of the line joining (–2, 1) and (0, 0) is

$$y = \frac{0-1}{0+2}x \quad \text{or} \quad x + 2y = 0$$

Example 2.19

Find the equation of the line passing through the intersection of the lines $x - 2y - 3 = 0$ and $x + 3y - 6 = 0$ and parallel to the line $3x + 4y - 7 = 0$.

Solution: We have $U_1 \equiv x - 2y - 3 = 0$ and $U_2 \equiv x + 3y - 6 = 0$. The required line is

$$U_1 + \lambda U_2 = (1 + \lambda)\, x + (-2 + 3\lambda)\, y - 3 - 6\lambda = 0$$

Since this line is parallel to $3x + 4y - 7 = 0$, the slopes must be equal. Therefore

$$\frac{-(1+\lambda)}{-2+3\lambda} = \frac{-3}{4}$$

$$\Rightarrow 4 + 4\lambda = -6 + 9\lambda \Rightarrow \lambda = 2$$

Hence, the required line is $3x + 4y - 15 = 0$.

Direct Method: The point of intersection of the lines $x - 2y - 3 = 0$ and $x + 3y - 6 = 0$ is (21/5, 3/5). Hence, from Quick Look 5, part (2), the equation of the required line is

$$3\left(x - \frac{21}{5}\right) + 4\left(y - \frac{3}{5}\right) = 0$$

$$\Rightarrow 3x + 4y - 15 = 0$$

Example 2.20

Find the equation of the line passing through the intersection of the lines $x + 3y - 1 = 0$ and $x - 2y + 4 = 0$ and perpendicular to the line $2x + 3y = 0$.

Solution: We have $U_1 \equiv x + 3y - 1 = 0$ and $U_2 \equiv x - 2y + 4 = 0$. Equation of the required line is

$$U_1 + \lambda U_2 \equiv (1+\lambda)x + (3-2\lambda)y - 1 + 4\lambda = 0$$

Since this line is perpendicular to the line $2x + 3y = 0$, we have

$$\frac{-(1+\lambda)}{3-2\lambda} \times \left(\frac{-2}{3}\right) = -1$$

$$\Rightarrow -9 + 6\lambda = 2\lambda + 2$$

$$\Rightarrow \lambda = \frac{11}{4}$$

Thus, the required line equation is

$$\left(1 + \frac{11}{4}\right)x + \left(3 - \frac{22}{4}\right)y - 1 + \frac{44}{4} = 0$$

$$\Rightarrow 15x - 10y + 40 = 0$$

$$\Rightarrow 3x - 2y + 8 = 0$$

Direct Method: Let $U_1 \equiv x + 3y - 1 = 0$ and $U_2 \equiv x - 2y + 4 = 0$. Therefore, the point of intersection of the lines $U_1 = 0$ and $U_2 = 0$ is (–2, 1). Hence, from Quick Look 7, part (3), the equation of the required line is

$$3(x + 2) - 2(y - 1) = 0$$

$$\Rightarrow 3x - 2y + 8 = 0$$

THEOREM 2.22 Let $U_1 \equiv a_1x + b_1y + c_1 = 0$, $U_2 \equiv a_2x + b_2y + c_2 = 0$ and $U_3 \equiv a_3x + b_3y + c_3 = 0$ be three lines such that no two lines are parallel. Then, these lines are concurrent if and only if

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

PROOF Let P be the point of the intersection of the lines $U_2 = 0$ and $U_3 = 0$. Therefore, by Theorem 2.16, we have

$$P = \left(\frac{b_3c_2 - b_2c_3}{a_3b_2 - a_2b_3}, \frac{c_3a_2 - c_2b_3}{a_3b_2 - a_2b_3} \right)$$

Point P also lies on $U_1 = 0$. So

$$a_1\left(\frac{b_3c_2 - b_2c_3}{a_3b_2 - a_2b_3} \right) + b_1\left(\frac{c_3a_2 - c_2b_3}{a_3b_2 - a_2b_3} \right) + c_1 = 0$$

$$\Leftrightarrow a_1(b_3c_2 - b_2c_3) + b_1(c_3a_2 - c_2a_3) + c_1(a_3b_2 - a_2b_3) = 0$$

$$\Leftrightarrow a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2) = 0$$

$$\Leftrightarrow \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

■

THEOREM 2.23 Let $U_1 \equiv a_1x + b_1y + c_1 = 0$, $U_2 \equiv a_2x + b_2y + c_2 = 0$ and $U_3 \equiv a_3x + b_3y + c_3 = 0$ be three lines such that no two are parallel. If there exists non-zero real numbers λ_1, λ_2 and λ_3 such that $\lambda_1U_1 + \lambda_2U_2 + \lambda_3U_3 = 0$, then the three lines $U_1 = 0$, $U_2 = 0$ and $U_3 = 0$ are concurrent.

PROOF Suppose $\lambda_1U_1 + \lambda_2U_2 + \lambda_3U_3 = 0$ where λ_1, λ_2 and λ_3 are non-zero real numbers. Therefore

$$U_3 = \left(\frac{-\lambda_1}{\lambda_3} \right) + \left(\frac{-\lambda_2}{\lambda_3} \right) U_2$$

which is of the form $\lambda U_1 + \mu U_2 = 0$. Hence, by Theorem 2.20, the line $U_3 = 0$ passes through the point of intersection of the lines $U_1 = 0$ and $U_2 = 0$. Therefore, the three lines are concurrent. ■

Note: Direct method of showing three lines to be concurrent: (a) Find the point of intersection of two of the three given lines. (b) Verify whether the point lies on the remaining third line or not.

2.2 Pair of Lines

In this section, we obtain the equation of the angle bisectors of the angle between two intersecting lines. Also we will study the condition for a second-degree homogeneous equation and general equation to represent pair of lines, and the properties of these lines.

THEOREM 2.24 If $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are intersecting lines, then the equations of their angle bisectors are

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm \frac{(a_2x + b_2y + c_2)}{\sqrt{a_2^2 + b_2^2}}$$

PROOF Let the lines represented by the given equations be L_1 and L_2, respectively (see Fig. 2.13) and A be their point of intersection. Now

$P(x, y)$ is a point on the bisector of the lines

$\Leftrightarrow$ the perpendicular distances drawn from P on to the lines are equal

$$\Leftrightarrow \frac{|a_1x_1 + b_1y_1 + c_1|}{\sqrt{a_1^2 + b_1^2}} = \left|\frac{a_2x_1 + b_2y_1 + c_2}{\sqrt{a_1^2 + b_1^2}}\right| \quad \text{(by Theorem 2.14)}$$

$$\Leftrightarrow \frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm\frac{(a_2x + b_2y + c_2)}{\sqrt{a_2^2 + b_2^2}}$$

Therefore, the locus of $P(x_1, y_1)$ is

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm\frac{(a_2x + b_2y + c_2)}{\sqrt{a_2^2 + b_2^2}}$$

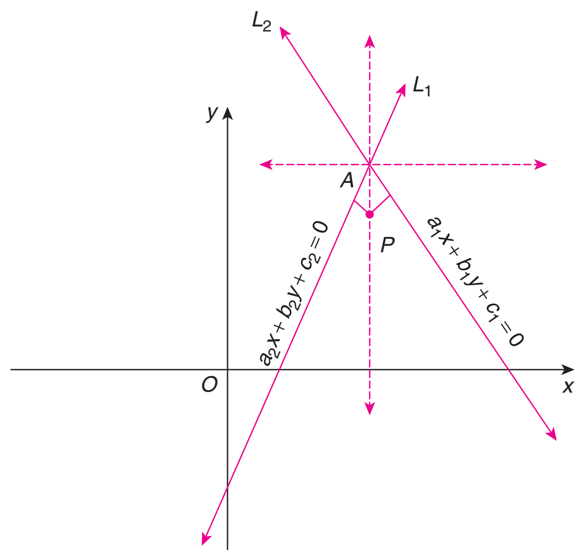

FIGURE 2.13

Note: **Identification of Acute Angle Bisector** Suppose L_1 and L_2 are not at right angles and let L_1' and L_2' be their angle bisectors. Let θ be the angle between L_1 and L_1'. If $0 < |\tan\theta| < 1$, then L_1' is the angle bisector; if $|\tan\theta| > 1$, then L_2' is the acute angle bisector.

The following theorems are also useful in identifying acute or obtuse angle bisectors of two lines.

THEOREM 2.25 Suppose $U_1 \equiv a_1x + b_1y + c_1 = 0$ and $U_2 \equiv a_2x + b_2y + c_2 = 0$, where $c_1 \neq 0$ and $c_2 \neq 0$, be two intersecting lines. If c_1 and c_2 are of same sign, then the equation

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = +\frac{(a_2x + b_2y + c_2)}{\sqrt{a_2^2 + b_2^2}}$$

gives the bisector of the angle in which the origin lies.

PROOF Suppose the lines represented by $U_1 = 0$ and $U_2 = 0$ are L_1 and L_2, respectively, and L_1' and L_2' are their angle bisectors (see Fig. 2.14). Point $P(x_1, y_1)$ lies on the origin angle bisector $L_1 \Leftrightarrow$ either both the origin and P are on the same sides for both L_1 and L_2 or lie on opposite sides for both L_1 and L_2. Since c_1 and c_2 are of same sign, it follows that (by Quick Look 10) $L_{11} = a_1x + b_1y + c_1$ and $L_{22} = a_2x + b_2y + c_2$ are of same sign. Therefore

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = +\frac{(a_2x + b_2y + c_2)}{\sqrt{a_2^2 + b_2^2}}$$

is the origin angle bisector.

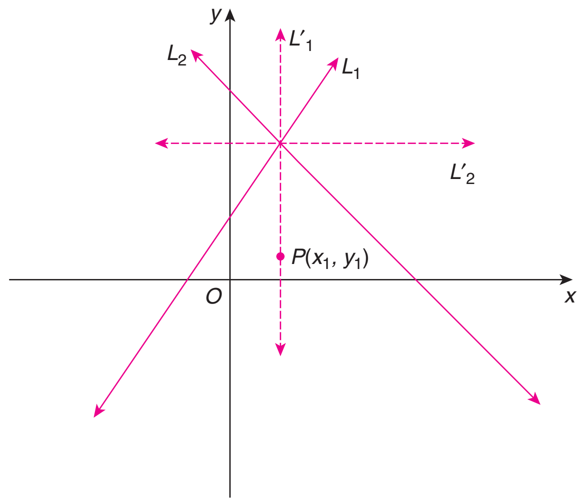

FIGURE 2.14

THEOREM 2.26 Suppose $U_1 \equiv a_1x + b_1y + c_1 = 0$ and $U_2 \equiv a_2x + b_2y + c_2 = 0$ are two lines that do not pass through origin and let c_1 and let c_2 have the same sign. Then

1. $\dfrac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = +\dfrac{(a_2x + b_2y + c_2)}{\sqrt{a_2^2 + b_2^2}}$ is the acute angle bisector if $a_1a_2 + b_1b_2 < 0$.

2. $\dfrac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = -\dfrac{(a_2x + b_2y + c_2)}{\sqrt{a_2^2 + b_2^2}}$ if $a_1a_2 + b_1b_2 > 0$.

PROOF Let L_1 and L_2 be the lines represented by $U_1 = 0$ and $U_2 = 0$, respectively. Draw OM and ON perpendicular to L_1 and L_2, respectively. Let $\angle NOM = \alpha$ so that $\angle MPN = \pi - \alpha$ (see Fig. 2.15). Now, α is acute or obtuse according to whether $\pi - \alpha$ is obtuse or acute. Also

$$\cos\alpha = \frac{a_1a_2 + b_1b_2}{\sqrt{(a_1^2 + b_1^2)}\sqrt{(a_2^2 + b_2^2)}} < 0 \text{ or } > 0$$

according to whether α is acute or obtuse. So,

$$a_1a_2 + b_1b_2 < 0 \text{ or } > 0$$
$$\Leftrightarrow \alpha \text{ is acute or obtuse}$$

Therefore,

1. If $a_1a_2 + b_1b_2 < 0$, then the origin angle bisector is the acute angle bisector and hence its equation is

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = +\frac{(a_2x + b_2y + c_2)}{\sqrt{a_2^2 + b_2^2}} \quad \text{(by Theorem 2.25)}$$

2. If $a_1a_2 + b_1b_2 > 0$, then $\pi - \alpha$ is obtuse and hence origin angle bisector is the obtuse angle bisector so that its equation is

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = -\frac{(a_2x + b_2y + c_2)}{\sqrt{a_2^2 + b_2^2}} \quad \text{(by Theorem 2.25)}$$

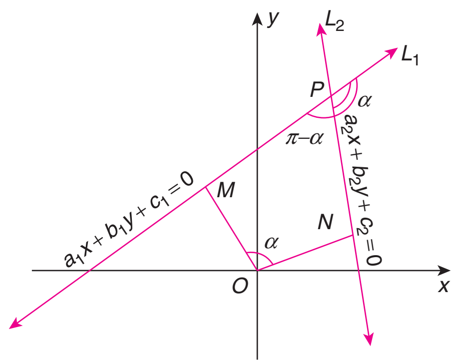

FIGURE 2.15

QUICK LOOK 12 IDENTIFYING ACUTE ANGLE BISECTOR

Let $U_1 \equiv a_1x+b_1y+c_1=0$ and $U_2 \equiv a_2x+b_2y+c_2=0$ be intersecting lines and $c_1 \neq 0 \neq c_2$. First, make both c_1 and c_2 of same sign (or c_1 and c_2 are positive). Then

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}=+\frac{(a_2x+b_2y+c_2)}{\sqrt{a_2^2+b_2^2}}$$

represents origin angle bisector (Theorem 2.25). The origin angle bisector (that is with + sign) is the acute angle or obtuse angle bisector according to whether $a_1a_2+b_1b_2$ is negative or positive.

Example 2.21

Find the angle bisectors of the angle between the lines $x-y+1=0$ and $7x+y+3=0$ and identify the acute angle bisector.

Solution: We have $U_1 \equiv x-y+1=0$ and $U_2 \equiv 7x+y+3=0$. Here, $c_1=1$ and $c_2=3$ are of the same sign. Therefore, the angle bisectors are

$$\frac{x-y+1}{\sqrt{1^2+1^2}}=\pm\frac{7x+y+3}{\sqrt{7^2+1^2}}$$

$$\Rightarrow \frac{x-y+1}{1}=\pm\frac{7x+y+3}{5}$$

Since $a_1a_2+b_1b_2=1(7)+(-1)(1)=6>0$, the bisector with + sign is obtuse angle and hence

$$\frac{x-y+1}{1}=-\frac{7x+y+3}{5}$$

$$\Rightarrow 5x-5y+5=-7x-y-3$$

$$\Rightarrow 12x-4y+8=0$$

$$\Rightarrow 3x-y+2=0$$

is the acute angle bisector.

Aliter: Consider the line $x-y+1=0$ and the bisector $3x-y+2=0$ and let θ be the angle between them. Hence

$$|\tan\theta|=\left|\frac{m_1-m_2}{1+m_1m_2}\right|=\left|\frac{1-3}{1+3}\right|=\frac{1}{2}<1$$

Therefore, $3x-y+2=0$ is the acute angle bisector.

DEFINITION 2.7 Let a, h, b, g, f and c be real numbers and atleast one of the real numbers a, h and b be non-zero. Then, the expression $ax^2+2hxy+by^2$ is called *second-degree homogeneous expression* and $ax^2+2hxy+by^2+2gx+2fy+c$ is called *second-degree general expression* in x and y. At the same time, $ax^2+2hxy+by^2=0$ is called *second-degree homogeneous equation* and $ax^2+2hxy+by^2+2gx+2fy+c=0$ is called *second-degree general equation*.

Examples

1. $2x^2+xy+by^2=0$
2. $x^2-y^2=0$
3. $x^2+\sqrt{2}\,xy+y^2=0$
4. $3x^2-6xy+y^2+20x-y+3=0$
5. $x^2+xy+y^2-x+y-1=0$

THEOREM 2.27 If $h^2 \geq ab$, then the locus represented by the equation $ax^2 + 2hxy + by^2 = 0$ is a pair of lines passing through the origin.

PROOF Since a is real, either $a = 0$ or $a \neq 0$.

Case 1: $a = 0$. In this case, the given equation becomes

$$y(2hx + by) = 0 \tag{2.12}$$

Since $a = 0$, atleast one of the values of h and b is not zero so that $2hx + by = 0$ is a first-degree equation and hence it represents a straight line. Also, $y = 0$ represents x-axis. Hence, Eq. (2.12) represents pair of lines, namely, $y = 0$ (i.e., x-axis) and the line $2hx + by = 0$.

Case 2: $a \neq 0$. In this case, the equation is written as

$$a^2x^2 + 2ahxy + aby^2 = 0$$

$$\Rightarrow (ax + hy)^2 - (h^2 - ab)y^2 = 0$$

Therefore

$$(ax + hy + \sqrt{h^2 - ab}\,y)(ax + hy - \sqrt{h^2 - ab}\,y) = 0$$

The locus represented by the given equation is the pair of lines $ax + (h \pm \sqrt{h^2 - ab})y = 0$ and both these lines pass through origin. When $h^2 = ab$ or $h = b = 0$, these two lines are identical (coincide) with each other. ■

Note: This is very useful in problem solving.

1. When $h^2 \geq ab$, let the lines represented by $ax^2 + 2hxy + by^2 = 0$ be $l_1x + m_1y = 0$ and $l_2x + m_2y = 0$ so that $ax^2 + 2hxy + by^2 \equiv (l_1x + m_1y)\,(l_2x + m_2y)$. Equating the corresponding coefficients on both sides, we have

   $$l_1l_2 = a, l_1m_2 + l_2m_1 = 2h$$

   and $$m_1m_2 = b$$

2. If both the lines represented by $ax^2 + 2hxy + by^2 = 0$ are non-vertical having slopes m_1 and m_2, then

   $$y^2 + \left(\frac{2h}{b}\right)xy + \left(\frac{a}{b}\right)x^2 \equiv (y - m_1x)(y - m_2x)$$

   so that $m_1m_2 = a/b$ and $m_1 + m_2 = -(2h)/b$.

QUICK LOOK 13

To find the lines represented by the second-degree homogeneous equation $ax^2 + 2hxy + by^2 = 0$, factorise the expression $ax^2 + 2hxy + by^2$ into two linear factors so that the corresponding linear equations are the required equations.

Example 2.22

Find the lines represented by the equation $x^2 - 5xy + 6y^2 = 0$.

Solution: We have $a = 1$, $h = -5/2$ and $b = 6$. Also $h^2 = 25/4 > 6 = ab$. Therefore, $x^2 - 5xy + 6y^2 = 0$ represents a pair of lines passing through origin. Further,

$$x^2 - 5xy + 6y^2 \equiv (x - 2y)(x - 3y)$$

$\Rightarrow$ The lines are $x - 2y = 0$ and $x - 3y = 0$

THEOREM 2.28 If $ax^2 + 2hxy + by^2 = 0$ represents a pair of lines and α is the angle between them, then

$$\cos\alpha = \frac{a+b}{\sqrt{(a-b)^2+4h^2}}$$

PROOF Suppose the lines are $l_1x + m_1y = 0$ and $l_2x + m_2y = 0$. Therefore, from Theorem 2.27, Note (1), $l_1l_2 = a$, $l_1m_2 + l_2m_1 = 2h$ and $m_1m_2 = b$. From Theorem 2.12, we have

$$\begin{aligned}\cos\alpha &= \frac{l_1l_2+m_1m_2}{\sqrt{(l_1^2+m_1^2)(l_2^2+m_2^2)}}\\ &= \frac{a+b}{\sqrt{l_1^2l_2^2+m_1^2m_2^2+l_1^2m_2^2+l_2^2m_1^2}}\\ &= \frac{a+b}{\sqrt{(l_1l_2-m_1m_2)^2+(l_1m_2+l_2m_1)^2}}\\ &= \frac{a+b}{\sqrt{(a-b)^2+4h^2}}\end{aligned}$$

where α is the acute or obtuse according to whether $a + b > 0$ or < 0. ■

Note:

1. $\dfrac{|a+b|}{\sqrt{(a-b)^2+4h^2}}$ gives the acute angle.
2. $\tan\alpha = \dfrac{2\sqrt{h^2-ab}}{|a+b|}$ also gives the acute angle between the lines.
3. $ax^2 + 2hxy + by^2 = 0$ represents a pair of perpendicular lines $\Leftrightarrow a + b = 0$ (i.e., coefficient of x^2 + coefficient of $y^2 = 0$).

THEOREM 2.29 Suppose $ax^2 + 2hxy + by^2 = 0$ represents a pair of lines and (x_1, y_1) is a point in the plane. Then,

1. The equation of the pair of lines passing through (x_1, y_1) and parallel to these lines is $a(x-x_1)^2 + 2h(x-x_1)(y-y_1) + b(y-y_1)^2 = 0$.
2. The equation of the pair of lines passing through (x_1, y_1) and perpendicular to the given lines is $b(x-x_1)^2 - 2h(x-x_1)(y-y_1) + a(y-y_1)^2 = 0$.

PROOF Suppose the lines represented by $ax^2 + 2hxy + by^2 = 0$ are $l_1x + m_1y = 0$ and $l_2x + m_2y = 0$. Hence, from Theorem 2.27, Note (1), $l_1l_2 = a$, $l_1m_2 + l_2m_1 = 2h$ and $m_1m_2 = b$.

1. From Quick Look 5, part (2), the equations of the lines through (x_1, y_1) and parallel to the lines $l_1x + m_1y = 0$ and $l_2x + m_2y = 0$ are $l_1(x - x_1) + m_1(y - y_1) = 0$ and $l_2(x - x_1) + m_2(y - y_1) = 0$. Hence, their combined equation is

$$\begin{aligned}&[l_1(x-x_1)+m_1(y-y_1)][l_2(x-x_1)+m_2(y-y_1)] = 0\\ &\Rightarrow l_1l_2(x-x_1)^2 + (l_1m_2+l_2m_1)(x-x_1)(y-y_1) + m_1m_2(y-y_1)^2 = 0\\ &\Rightarrow a(x-x_1)^2 + 2h(x-x_1)(y-y_1) + b(y-y_1)^2 = 0\end{aligned}$$

2. From Quick Look 7, part (3), the equations of the lines through (x_1, y_1) and perpendicular to the lines are $m_1(x - x_1) - l_1(y - y_1) = 0$ and $m_2(x - x_1) - l_2(y - y_1) = 0$. Hence, their combined equation is

$$\begin{aligned}&[m_1(x-x_1)-l_1(y-y_1)][m_2(x-x_1)-l_2(y-y_1)] = 0\\ &\Rightarrow m_1m_2(x-x_1)^2 - (l_1m_2+l_2m_1)(x-x_1)(y-y_1) + l_1l_2(y-y_1)^2 = 0\\ &\Rightarrow b(x-x_1)^2 - 2h(x-x_1)(y-y_1) + a(y-y_1)^2 = 0\end{aligned}$$

■

Example 2.23

Find the equation of the pair of lines passing through the point $(1, 1)$ and

1. parallel to the lines $x^2 - 5xy + 6y^2 = 0$.
2. perpendicular to the lines $x^2 - 5xy + 6y^2 = 0$.

Solution: The lines represented by $x^2 - 5xy + 6y^2 = 0$ are $x - 2y = 0$ and $x - 3y = 0$. Therefore, from Theorem 2.29, we get the following:

1. Equation of the pair of lines passing through $(1, 1)$ and parallel to the lines $x - 2y = 0$ and $x - 3y = 0$ is

$$(x - 1)^2 - 5(x - 1)(y - 1) + 6(y - 1)^2 = 0$$
$$\Rightarrow x^2 - 5xy + 6y^2 + 3x - 7y + 2 = 0$$

2. Equation of the pair of lines passing through $(1, 1)$ and perpendicular to the lines is

$$6(x - 1)^2 + 5(x - 1)(y - 1) + (y - 1)^2 = 0$$
$$\Rightarrow 6x^2 + 5xy + y^2 - 17x - 7y + 12 = 0$$

THEOREM 2.30 If $ax^2 + 2hxy + by^2 = 0$ represents two intersecting lines, then the combined equation of the pair of angle bisectors of the angle between the lines is

$$h(x^2 - y^2) = (a - b)xy$$

PROOF Suppose the lines are $l_1x + m_1y = 0$ and $l_2x + m_2y = 0$ so that $l_1l_2 = a$, $l_1m_2 + l_2m_1 = 2h$ and $m_1m_2 = b$. By Theorem 2.24, the angle bisectors are

$$\frac{l_1x + m_1y}{\sqrt{l_1^2 + m_1^2}} = \pm\frac{l_2x + m_2y}{\sqrt{l_2^2 + m_2^2}}$$
$$\Rightarrow (l_2^2 + m_2^2)(l_1x + m_1y)^2 = (l_1^2 + m_1^2)(l_2x + m_2y)^2$$
$$\Rightarrow (l_1^2m_2^2 - l_2^2m_1^2)x^2 - (l_1^2m_2^2 - l_2^2m_1^2)y^2 = 2\left[l_2m_2(l_1^2 + m_1^2) - l_1m_1(l_2^2 + m_2^2)\right]xy$$
$$\Rightarrow (l_1^2m_2^2 - l_2^2m_1^2)(x^2 - y^2) = 2[(l_1m_2 - l_2m_1)(l_1l_2 - m_1m_2)]xy$$

Since the lines are intersecting, $l_1m_2 - l_2m_1 \neq 0$. Hence, cancelling $l_1m_2 - l_2m_1$ on both sides, we have

$$(l_1m_2 + l_2m_1)(x^2 - y^2) = 2(l_1l_2 - m_1m_2)xy$$
$$\Rightarrow 2h(x^2 - y^2) = 2(a - b)xy$$
$$\Rightarrow h(x^2 - y^2) = (a - b)xy$$

■

THEOREM 2.31 The second-degree general equation $S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of lines if and only if

1. $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$.
2. $h^2 \geq ab$, $g^2 \geq ac$ and $f^2 \geq bc$.

PROOF Suppose $S = 0$ represents pair of lines and let the lines be $l_1x + m_1y + n_1 = 0$ and $l_2x + m_2y + n_2 = 0$. Therefore

$$S \equiv (l_1x + m_1y + n_1)(l_2x + m_2y + n_2)$$

Equating the corresponding coefficients on both sides, we have $l_1l_2 = a$, $l_1m_2 + l_2m_1 = 2h$ and $m_1m_2 = b$, $l_1n_2 + l_2n_1 = 2g$, $m_1n_2 + m_2n_1 = 2f$, $n_1n_2 = c$.

1. $8fgh = (2f)(2g)(2h) = (m_1n_2 + m_2n_1)(l_1n_2 + l_2n_1)(l_1m_2 + l_2m_1)$

$$= l_1l_2(m_1^2n_2^2 + m_2^2n_1^2) + m_1m_2(l_1^2n_2^2 + l_2^2n_1^2) + n_1n_2(l_1^2m_2^2 + l_2^2m_1^2) + 2l_1l_2m_1m_2n_1n_2$$
$$= a[(m_1n_2 + m_2n_1)^2 - 2m_1m_2n_1n_2] + b[(l_1n_2 + l_2n_1)^2 - 2l_1l_2n_1n_2] +$$
$$c[(l_1m_2 + l_2m_1)^2 - 2l_1l_2m_1m_2] + 2abc$$
$$= a(4f^2 - 2bc) + b(4g^2 - 2ca) + c(4h^2 - 2ab) + 2abc$$
$$= 4(af^2 + bg^2 + ch^2 - abc)$$

Therefore

$$2fgh = af^2 + bg^2 + ch^2 - abc \text{ or } abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

Generally, the number of $abc + 2fgh - af^2 - bg^2 - ch^2$ is denoted by Δ. Therefore, $\Delta = 0$. Also

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & e \end{vmatrix}$$

2. $4(h^2 - ab) = 4h^2 - 4ab = (l_1m_2 + l_2m_1)^2 - 4l_1l_2m_1m_2$

$$= (l_1m_2 - l_2m_1)^2 \geq 0 \qquad \text{(equality holds if the lines are parallel).}$$

Similarly, $g^2 \geq ca$ and $f^2 \geq bc$.

The proof of the converse part is a bit lengthy and beyond the scope of this book. Hence, we assume the validity of the converse part. ■

THEOREM 2.32 If $S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of lines, then the homogeneous equation $ax^2 + 2hxy + by^2 = 0$ also represents pair of lines passing through origin and parallel to the lines $S = 0$.

PROOF Let the lines represented by $S = 0$ be $l_1x + m_1y + n_1 = 0$ and $l_2x + m_2y + n_2 = 0$. Therefore

$$l_1l_2 = a, l_1m_2 + l_2m_1 = 2h, m_1m_2 = b, l_1n_2 + l_2n_1 = 2g, m_1n_2 + m_2n_1 = 2f, n_1n_2 = c, h^2 \geq ab.$$

Now,

$$ax^2 + 2hxy + by^2 \equiv l_1l_2x^2 + (l_1m_2 + l_2m_1)\, xy + m_1m_2y^2$$
$$\equiv (l_1x + m_1y)(l_2x + m_2y)$$

Therefore, the lines represented by $ax^2 + 2hxy + by^2 = 0$ are $l_1x + m_1y = 0$ and $l_2x + m_2y = 0$ which are, respectively, parallel to the $l_1x + m_1y + n_1 = 0$ and $l_2x + m_2y + n_2 = 0$. ■

QUICK LOOK 14

1. If $h^2 = ab$, then $ax^2 + 2hxy + by^2 = 0$ represents pair of coincidental lines so that in this case, $S = ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of parallel lines.
2. If $S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of distinct intersecting lines, this pair together with the pair represented by $ax^2 + 2hxy + by^2 = 0$ form a parallelogram with the origin as one of the vertices.

Example 2.24

Prove that the equation $S \equiv x^2 + 4xy + 3y^2 - 4x - 10y + 3 = 0$ represents a pair of lines and find the equations of the lines.

Solution: We have $a = 1, h = 2, b = 3, g = -2, f = -5$ and $c = 3$. Now,

1.

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & e \end{vmatrix} = \begin{vmatrix} 1 & 2 & -2 \\ 2 & 3 & -5 \\ -2 & -5 & 3 \end{vmatrix}$$
$$= 1\,(9 - 25) - 2\,(6 - 10) - 2\,(-10 + 6)$$
$$= -16 + 8 + 8 = 0$$

2. $h^2 = 4, ab = 3, h^2 > ab, g^2 = 4, ac = 3, g^2 > ac$ and $f^2 = 25$, $bc = 9, f^2 > bc$.

Therefore, the given equation $S \equiv x^2 + 4xy + 3y^2 - 4x - 10y + 3 = 0$ represents a pair of intersecting lines (by Theorem 2.31). Consider the equation $x^2 + 4xy + 3y^2 = 0$. Therefore

$$x^2 + xy + 3xy + 3y^2 = 0$$
$$x(x + y) + 3y(x + y) = 0$$
$$(x + y)(x + 3y) = 0$$

Since $S = 0$ represents lines parallel to $x + y = 0$ and $x + 3y = 0$, we have

$$x^2+4xy+3y^2-4x-10y+3 \equiv (x+y+m)(x+3y+n)$$

Equating the coefficients of x and y and the constant terms, we have

$$m+n=-4 \tag{2.13}$$

$$3m+n=-10 \tag{2.14}$$

and
$$mn=3 \tag{2.15}$$

Solving Eqs. (2.13) and (2.14), we get $m=-3, n=-1$ so that $mn = 3$. Therefore, the lines represented by $S = 0$ are $x + y - 3 = 0$ and $x + 3y - 1 = 0$.

2.2.1 Procedure to Find the Lines Represented by the Second-Degree General Equation $S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

Step 1: Factorise the homogeneous part $ax^2 + 2hxy + by^2$ and suppose

$$ax^2 + 2hxy + by^2 = (l_1x + m_1y)\,(l_2x + m_2y)$$

Step 2: $S \equiv (l_1x + m_1y + n_1)(l_2x + m_2y + n_2)$

Step 3: Equate the corresponding coefficients of x and y and also the constant terms on both sides and solve for n_1 and n_2.

Theorem 2.33 is the last result which is very useful in solving some locus problems. This theorem is called *homogenising the second-degree curve equation with a straight line equation.*

THEOREM 2.33 Suppose that the straight line $lx + my = 1$ meets the curve represented by the second-degree general equation $S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ at two points A and B. If O is the origin, then the combined equation of the pair of lines $\overleftrightarrow{OA}$ and $\overleftrightarrow{OB}$ is

$$S' \equiv ax^2 + 2hxy + by^2 + 2(gx+fy)(lx+my) + c(lx+my)^2 = 0$$

PROOF Clearly the coordinates of both points A and B satisfy the line equation $lx + my = 1$ as well as $S = 0$ and hence points A and B satisfy $S' = 0$. Also $(0, 0)$ satisfies $S' = 0$. That is, $S' = 0$ passes through A, B and origin (see Fig. 2.16). On simplification, we can see that $S' = 0$ is a homogeneous equation of second degree representing pair of lines, which are nothing but the lines $\overleftrightarrow{OA}$ and $\overleftrightarrow{OB}$.

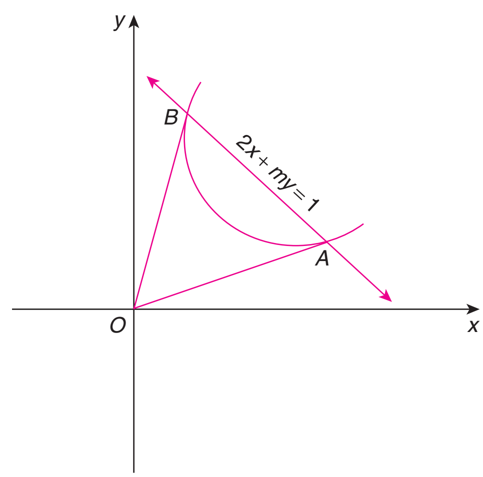

FIGURE 2.16 ■

Example 2.25

Show that the lines joining the origin to the points of intersection of the curve $x^2 - xy + y^2 + 3x + 3y - 2 = 0$ and the line $x - y - \sqrt{2} = 0$ are mutually perpendicular.

Solution: Suppose that the line $[(x-y)/\sqrt{2}] = 1$ meets the curve at points A and B (see Fig. 2.17). Therefore, the combined equation of the pair of lines $\overleftrightarrow{OA}$ and $\overleftrightarrow{OB}$ is

$$x^2 - xy + y^2 + (3x+3y)\left(\frac{x-y}{\sqrt{2}}\right) - 2\left(\frac{x-y}{\sqrt{2}}\right)^2 = 0 \quad (2.16)$$

From Eq. (2.16),

$$\text{Coefficient of } x^2 + \text{Coefficient of } y^2 = [1 + (3/\sqrt{2}) - 1] + [1 - (3/\sqrt{2}) - 1] = 0$$

Therefore, from Theorem 2.28, Note (3), it implies that $\angle AOB = 90°$.

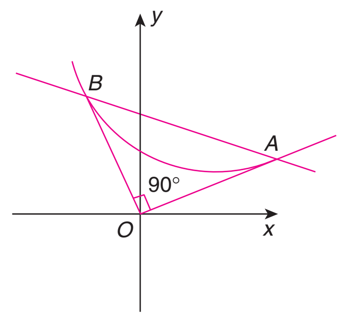

FIGURE 2.17

Subjective Problems

1. If p_1 and p_2 are the distances between the opposite sides of a parallelogram and α is one of its angles, then show that the area of the parallelogram is $p_1 p_2$ cosecα.

Solution: $ABCD$ is a parallelogram (see Fig. 2.18). $AM = p_1$, $DN = p_2$ and $\angle BAD = \alpha$.

Area of the parallelogram $= (AB)\, p_2$ (2.17)

Now from ΔAMP, $\sin\alpha = p_1/AB$ and hence we have $AB = p_1$ cosecα. Therefore, from Eq. (2.17), the area of the parallelogram $= p_1 p_2$ cosecα.

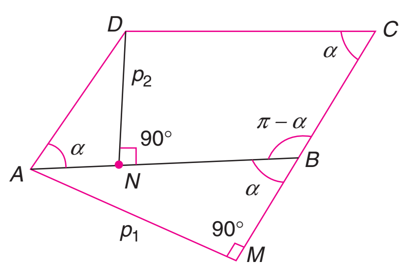

FIGURE 2.18

2. Show that the area of parallelogram whose sides are $a_1x + b_1y + c_1 = 0, a_1x + b_1y + d_1 = 0, a_2x + b_2y + c_2 = 0$ and $a_2x + b_2y + d_2 = 0$ is

$$\left|\frac{(d_1 - c_1)(d_2 - c_2)}{a_1b_2 - a_2b_1}\right|$$

Solution: Consider Fig. 2.18. Let the equations of the sides be $a_1x + b_1y + c_1 = 0$, $a_1x + b_1y + d_1 = 0$, $a_2x + b_2y + c_2 = 0$ and $a_2x + b_2y + d_2 = 0$. Therefore,

$$p_1 = \frac{|d_1 - c_1|}{\sqrt{a_1^2 + b_1^2}}$$

$$p_2 = \frac{|d_2 - c_2|}{\sqrt{a_2^2 + b_2^2}}$$

Also

$$\cos\alpha = \frac{|a_1a_2 + b_1b_2|}{\sqrt{(a_1^2 + b_1^2)(a_2^2 + b_2^2)}}$$

Therefore

$$\sin^2\alpha = 1 - \cos^2\alpha = 1 - \frac{(a_1a_2 + b_1b_2)^2}{(a_1^2 + b_1^2)(a_2^2 + b_2^2)}$$

$$= \frac{(a_1b_2 - a_2b_1)^2}{(a_1^2 + b_1^2)(a_2^2 + b_2^2)} \quad (2.18)$$

Now the area of the parallelogram (by Problem 1) is

$$p_1p_2(\operatorname{cosec}\alpha) = \frac{|d_1 - c_1|}{\sqrt{a_1^2 + b_1^2}} \cdot \frac{|d_2 - c_2|}{\sqrt{a_2^2 + b_2^2}} \cdot \frac{\sqrt{(a_1^2 + b_1^2)(a_2^2 + b_2^2)}}{|a_1b_2 - a_2b_1|}$$

[from Eq. (2.18)]

$$= \left|\frac{(d_1 - c_1)(d_2 - c_2)}{a_1b_2 - a_2b_1}\right|$$

3. Prove that the area of the parallelogram formed by the lines $4y - 3x - a = 0, 3y - 4x + a = 0, 4y - 3x - 3a = 0$ and $3y - 4x + 2a = 0$ is $2a^2/7$.

Solution: Rewriting the equations of the sides of the parallelogram, we have

$$3x - 4y + a = 0$$
$$4x - 3y - a = 0$$
$$3x - 4y + 3a = 0$$

and

$$4x - 3y - 2a = 0$$

Here, $c_1 = a, d_1 = 3a, c_2 = -a, d_2 = -2a, a_1 = 3, b_1 = -4, a_2 = 4$ and $b_2 = -3$. Therefore, by Problem 2,

$$\text{Area} = \left|\frac{(d_1 - c_1)(d_2 - c_2)}{a_1b_2 - a_2b_1}\right|$$

$$= \left| \frac{(3a-a)(-2a+a)}{3(-3)-(4)(-4)} \right| = \frac{2a^2}{7}$$

4. Let p and q be non-zero real numbers and x_1, x_2 and x_3 be non-zero real roots of the equation $x^3 - 3px^2 + 3qx - 1 = 0$. Then show that the centroid of the triangle whose vertices are $[x_1, (1/x_1)]$, $[x_2, (1/x_2)]$ and $[x_3, (1/x_3)]$ is (p, q).

Solution: Let $y_1 = 1/x_1$, $y_2 = 1/x_2$ and $y_3 = 1/x_3$. By hypothesis,

$$x_1 + x_2 + x_3 = 3p$$
$$x_1x_2 + x_2x_3 + x_3x_1 = 3q$$
$$x_1x_2x_3 = 1$$

Therefore

$$\frac{x_1 + x_2 + x_3}{3} = \frac{3p}{3} = p \tag{2.19}$$

Now,

$$\frac{y_1 + y_2 + y_3}{3} = \frac{1}{3}\left(\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}\right)$$
$$= \frac{1}{3}\left(\frac{x_2x_3 + x_3x_1 + x_1x_2}{x_1x_2x_3}\right)$$
$$= \frac{1}{3}\left(\frac{3q}{1}\right) = q \tag{2.20}$$

From Eqs. (2.19) and (2.20), the centroid of the triangle is (p, q).

5. (**Menelaus' Theorem**) Suppose a straight line meets the sides BC, CA and AB of a triangle at points D, E and F, respectively. Then show that

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = -1$$

Solution: Let the coordinates of A, B and C be (x_1, y_1), (x_2, y_2) and (x_3, y_3). See Fig. 2.19. Suppose L is the line whose equation is $L \equiv ax + by + c = 0$ meeting the sides in D, E and F. Therefore, by Theorem 2.17, we get

$$\frac{BD}{DC} = \frac{-L_{22}}{L_{33}}$$
$$\frac{CE}{EA} = \frac{-L_{33}}{L_{11}}$$
$$\frac{AF}{FB} = \frac{-L_{11}}{L_{22}}$$

Hence

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = -1$$

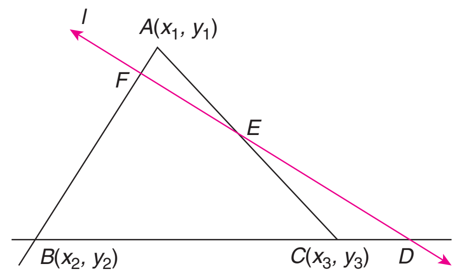

FIGURE 2.19

Note: Either the line l meets all three sides externally or two sides internally and one side externally. That is why product of the ratios is equal to -1.

Try it out The converse of Menelaus' theorem is also true.

6. (**Ceva's Theorem**) In the plane of ΔABC, let O be a point (not on any side). If the lines AO, BO and CO meet the opposite sides BC, CA and AB at points D, E and F, respectively, then show that

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = +1$$

Solution: Without loss of generality, we may assume that O is the origin and points A, B and C are (x_1, y_1), (x_2, y_2) and (x_3, y_3), respectively. See Fig. 2.20. Observe that either all the three lines $\overleftrightarrow{AO}$, $\overleftrightarrow{BO}$ and $\overleftrightarrow{CO}$ divide the sides BC, CA and AB internally or two of them divide two sides externally and one divides the third side internally. Now, the equations of the lines $\overleftrightarrow{AO}$, $\overleftrightarrow{BO}$ and $\overleftrightarrow{CO}$ are, respectively, $xy_1 - x_1y = 0$, $xy_2 - x_2y = 0$ and $xy_3 - x_3y = 0$. Therefore, by Theorem 2.17, we get

$$\frac{BD}{DC} = -\frac{(x_2y_1 - x_1y_2)}{x_3y_1 - x_1y_3}$$
$$\frac{CE}{EA} = -\frac{(x_3y_2 - x_2y_3)}{x_1y_2 - x_2y_1}$$
$$\frac{AF}{FB} = -\frac{(x_1y_3 - x_3y_1)}{x_2y_3 - x_3y_2}$$

Hence

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = +1$$

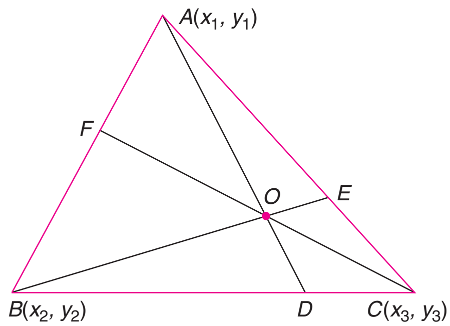

FIGURE 2.20

> **Try it out** Converse of Ceva's theorem is also true.

7. If $A(3, 4)$ and $C(1, -1)$ are the ends of a diagonal of a square $ABCD$, then find the equations of the sides of the square.

Solution: See Fig. 2.21. Equation of the line AC is

$$y-4=\frac{4+1}{3-1}(x-3)$$

$$\Rightarrow 2y - 8 = 5x - 15$$

$$\Rightarrow 5x - 2y - 7 = 0$$

Its slope is 5/2. Let the slope of side CD be m. Then

$$\angle DAC = 45° \Rightarrow 1 = \tan 45° = \left|\frac{m-(5/2)}{1+(5m/2)}\right|$$

$$\Rightarrow 1=\left|\frac{2m-5}{5m+2}\right|$$

$$\Rightarrow 5m+2=\pm(2m-5)$$

$$\Rightarrow m=-\frac{7}{3} \text{ or } \frac{3}{7}$$

Therefore, equations of CB and CD are

$$y+1=\frac{-7}{3}(x-1)\Rightarrow 7x+3y-4=0$$

$$y+1=\frac{3}{7}(x-1)\Rightarrow 3x-7y-10=0$$

Similarly, the equations of AB and BD, respectively, are

$$y-4=\frac{3}{7}(x-3)\Rightarrow 3x-7y-19=0$$

$$y-4=\frac{-7}{3}(x-3)\Rightarrow 7x+3y-33=0$$

Therefore, the equations of the sides are $3x - 7y - 10 = 0$, $7x + 3y - 4 = 0$, $3x - 7y - 19 = 0$ and $7x + 3y - 33 = 0$.

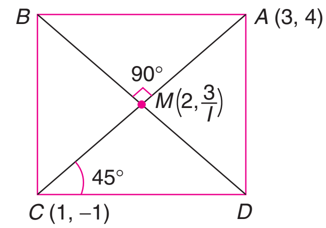

FIGURE 2.21

Aliter (Using Complex Numbers): Let $A = 3 + 4i$, $C = 1 - i$, M = midpoint of $AC = 2 + 3/2i$. Let B denote Z. Rotate $\overrightarrow{MA}$ about M through 90° in the anticlockwise sense so that

$$\frac{Z-[2+(3/2)i]}{(3+4i)-[2+(3/2)i]}=i$$

$$\Rightarrow \frac{2Z-4-3i}{2+5i}=i$$

$$\Rightarrow Z=\frac{-1+5i}{2}$$

$$\Rightarrow B=\left(-\frac{1}{2},\frac{5}{2}\right)$$

Similarly,

$$D=\left(3+1+\frac{1}{2},4-1-\frac{5}{2}\right)=\left(\frac{9}{2},\ \frac{1}{2}\right)$$

Now, we can write the sides equations because the vertices are obtained as $A(3, 4)$, $B(-1/2, 5/2)$, $C(1, -1)$, $D(9/2, 1/2)$.

8. Find the equations of the lines passing through the point $(2, 3)$ and making an angle 45° with the line $3x - y + 5 = 0$.

Solution: See Fig. 2.22. Let m be the slope of a side. $\angle ABC = 45°$ so that we have

$$1=\tan 45°=\left|\frac{m-3}{1+3m}\right|$$

Therefore,

$$3m+1=\pm(m-3)$$

$$m=-2 \text{ or } \frac{1}{2}$$

Therefore, the equations of the lines are

$$y - 3 = -2(x-2) \Rightarrow 2x + y - 7 = 0$$

$$y-3=\frac{1}{2}(x-2)\Rightarrow x-2y+4=0$$

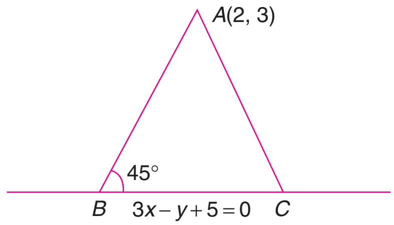

FIGURE 2.22

9. Two adjacent sides of a parallelogram are $4x + 5y = 0$ and $7x + 2y = 0$ and one diagonal is $11x + 7y - 9 = 0$. Find the equations of the other sides and the second diagonal.

Solution: Solving the equations $4x + 5y = 0$ and $11x + 7y - 9 = 0$, we have $x = 5/3$, $y = -4/3$. Let $A = (5/3, -4/3)$. Solving the equations $7x + 2y = 0$ and $11x + 7y - 9 = 0$, we have $x = -2/3$, $y = 7/3$. Let $C = (-2/3, 7/3)$. Therefore,

$$C\left(-\frac{2}{3}, \frac{7}{3}\right), O(0,\ 0) \quad \text{and} \quad A\left(\frac{5}{3}, -\frac{4}{3}\right)$$

are the three consecutive vertices of the parallelogram. So its fourth vertex B (see Fig. 2.23) is

$$\left(-\frac{2}{3} + \frac{5}{3}, \frac{7}{3} - \frac{4}{3}\right) = (1,\ 1)$$

Therefore, the vertices of the parallelogram $OABC$ are $O(0, 0)$, $A(5/3, -4/3)$, $B(1, 1)$ and $C(-2/3, 7/3)$. Since the side BC is parallel to OA and passes through $(1, 1)$, its equation is

$$y - 1 = \frac{-4}{5}(x - 1) \text{ or } 4x + 5y - 9 = 0$$

Also, the equation of the side AB is

$$y - 1 = \frac{-7}{2}(x - 1) \quad \text{or} \quad 7x + 2y - 9 = 0$$

and the second diagonal is $x - y = 0$.

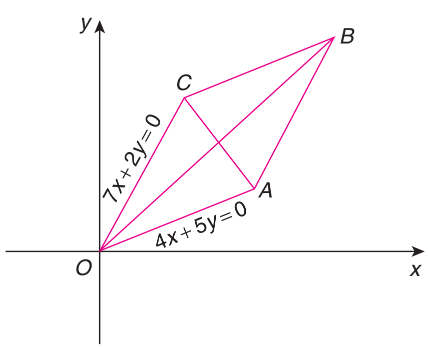

FIGURE 2.23

10. One side of a rectangle lies along the line $3x + 7y + 2 = 0$ and $(-3, 1)$ is a vertex on it. If $(1, 1)$ is another vertex of the rectangle, find the equations of the other three sides.

Solution: Let $ABCD$ be the rectangle (see Fig. 2.24) where A is $(-3, 1)$ and the equation of AD is $3x + 7y + 2 = 0$. Since the line joining $(-3, 1)$ and $(1, 1)$ is horizontal, the point $(1, 1)$ must be the end of the diagonal through A $(-3, 1)$. Therefore, C is $(1, 1)$ and the equation of BC is

$$y - 1 = \frac{-3}{7}(x - 1)$$
$$\Rightarrow 3x + 7y - 10 = 0$$

Equation of AB is

$$y - 1 = \frac{7}{3}(x + 3)$$
$$\Rightarrow 7x - 3y + 24 = 0$$

Equation of the side CD is

$$y - 1 = \frac{7}{3}(x - 1) \Rightarrow 7x - 3y - 4 = 0$$

Therefore, the other three sides are $3x + 7y - 10 = 0$, $7x - 3y + 24 = 0$ and $7x - 3y - 4 = 0$.

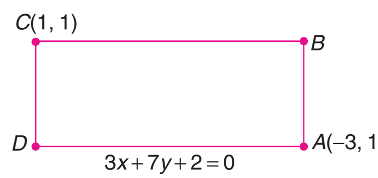

FIGURE 2.24

11. The three lines $x + 2y + 3 = 0$, $x + 2y - 7 = 0$ and $2x - y - 4 = 0$ form the three sides of two squares. Find the equations of fourth side of each square.

Solution: Let the squares be $ABCD$ and $ADEF$ with AD as common side (see Fig. 2.25). Solving $x + 2y + 3 = 0$ and $2x - y - 4 = 0$, we have $A = (1, -2)$. Solving the equations $x + 2y - 7 = 0$ and $2x - y - 4 = 0$, we have $D = (3, 2)$. The length of the sides of squares $= AD = \sqrt{(3-1)^2 + (2+2)^2} = \sqrt{4+16} = 2\sqrt{5}$. Let B $(h, -(3 + h)/2)$ be a point on the line $x + 2y + 3 = 0$ such that $AB = 2\sqrt{5}$. Therefore,

$$(h-1)^2 + \left(\frac{3+h}{2} - 2\right)^2 = 20$$
$$4(h-1)^2 + (h-1)^2 = 80$$
$$(h-1)^2 = 16$$
$$h - 1 = \pm 4$$
$$h = 5 \text{ or } -3$$

Therefore, $B = (5, -4)$ and $F = (-3, 0)$. Hence, the equation of the side BC is $y + 4 = 2(x - 5)$ or $2x - y - 14 = 0$ and the equation of the side FE is $y - 0 = 2(x + 3)$ or $2x - y - 6 = 0$. Equations of the fourth side of the square are $2x - y - 14 = 0$ and $2x - y - 6 = 0$.

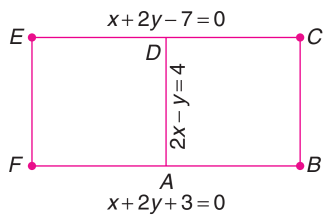

FIGURE 2.25

12. Each side of a square is of length 4. The centre of the square is (3, 7) and one of its diagonals is parallel to the line $y = x$. Find the coordinate of the vertices of square.

Solution: See Fig. 2.26. $ABCD$ is the square. $M(3, 7)$ is the centre of the square. AC is parallel to the line $y = x$. Therefore, the equation of the diagonal AC is

$$y - 7 = 1(x - 3)$$

$$\Rightarrow x - y + 4 = 0 \tag{2.21}$$

Hence, the equation of the diagonal BD is

$$y - 7 = -1\,(x - 3)$$

$$\Rightarrow x + y - 10 = 0 \tag{2.22}$$

Since the length of the side is 4, the lengths of the diagonals are $4\sqrt{2}$. Let $A = (h, h + 4)$ and $MA = 2\sqrt{2}$. This implies that

$$(h-3)^2 + (h+4-7)^2 = 8$$

$$\Rightarrow (h-3)^2 = 4$$

$$\Rightarrow h = 3 \pm 2$$

$$\Rightarrow h = 5 \text{ or } 1$$

Hence, $A = (5, 9)$ and $C = (1, 5)$. Let $B = (k, 10 - k)$ and $MD = 2\sqrt{2}$. This implies that

$$(k-3)^2 + (10-k-7)^2 = 8$$

$$\Rightarrow k - 3 = \pm 2$$

$$\Rightarrow k = 5 \text{ or } 1$$

Hence, $B = (5, 5)$ and $D = (1, 9)$. Therefore, the vertices of the square are $(5, 9), (5, 5), (1, 5), (1, 9)$.

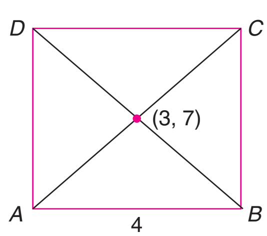

FIGURE 2.26

13. The vertices B and C of ΔABC lie on the line $4x - 3y = 0$ and x-axis, respectively, BC passes through (2/3, 2/3) and $ABOC$ is a rhombus where O is the origin. Find the equation of the line BC and the coordinates of the vertex A.

Solution: See Fig. 2.27. $OB = OC = CA = AB$ and the diagonals OA and BC are at right angles. Let $OC = a$. Suppose $B = [x_1, 4x_1/3]$. Equation of the side AB is $y = (4x_1)/3$ and the coordinates of A are $[a + x_1, (4x_1)/3]$. Hence

$$(CA)^2 = a^2$$

$$\Rightarrow x_1^2 + \frac{16x_1^2}{9} = a^2$$

$$\Rightarrow a = \pm\frac{5x_1}{3}$$

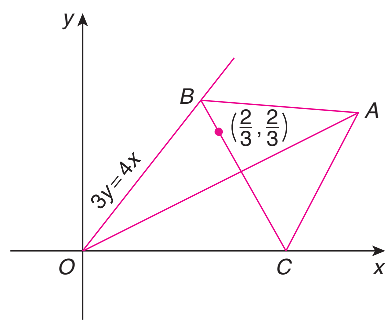

FIGURE 2.27

Case 1: If $a = 5x_1/3$, we have

$$C = \left(\frac{5x_1}{3},\ 0\right), A = \left(\frac{8x_1}{3}, \frac{4x_1}{3}\right), B = \left(x_1,\ \frac{4x_1}{3}\right) \text{and } O = (0, 0)$$

By hypothesis, points B, (2/3, 2/3) and C are collinear. This implies that

$$\begin{vmatrix} x_1 & \frac{4x_1}{3} & 1 \\ \frac{2}{3} & \frac{2}{3} & 1 \\ \frac{5x_1}{3} & 0 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \frac{5x_1}{3}\left(\frac{4x_1}{3} - \frac{2}{3}\right) + 1\left(\frac{2x_1}{3} - \frac{8x_1}{9}\right) = 0$$

$$\Rightarrow 5x_1(4x_1 - 2) + (6x_1 - 8x_1) = 0$$

$$\Rightarrow 20x_1^2 - 12x_1 = 0$$

$$\Rightarrow x_1 = 0, \frac{3}{5}$$

Now, $x_1 = 0 \Rightarrow B = (0, 0)$, which is actually not the origin. Hence, $x_1 = 3/5$. Therefore,

$$O = (0, 0), B = \left(\frac{3}{5},\ \frac{4}{5}\right), A = \left(\frac{8}{5},\ \frac{4}{5}\right) \text{ and } C = (1, 0)$$

Case 2: $a = -\dfrac{5x_1}{3}$

> **Try it out** Try case 2 mentioned above, that is, $a = -5x_1/3$.

14. In ΔABC, $A = (-4, 1)$. The internal bisectors the angles B and C are, respectively, $x - 1 = 0$ and $x - y - 1 = 0$. Find the coordinates of B and C and the equations of the sides AB and AC.

Solution: See Fig. 2.28. Let BE and CF be the bisectors of the angles B and C whose equations are, respectively, $x - 1 = 0$ and $x - y - 1 = 0$. Suppose M and N are the reflections of the vertex A in the bisectors BE and CF, respectively. Hence, M and N lie on the line BC. Let $M = (h, k)$. Therefore, by Theorem 2.13, we have

$$\frac{h-4}{1} = \frac{k+1}{0} = -2\frac{(4-1)}{1} \quad \left(\text{here } \frac{k+1}{0} \text{ means } k = -1\right)$$

$$\Rightarrow h = -2 \quad \text{and} \quad k = -1$$

Hence, $M = (-2, -1)$. Let $N = (h', k')$. Therefore,

$$\frac{h'-4}{1} = \frac{k'+1}{-1} = -2\frac{(4+1-1)}{1^2+1^2} = -4$$

$$\Rightarrow h' = 0 \quad \text{and} \quad k' = 3$$

Hence, $N = (0, 3)$. Therefore, equation of the side BC is

$$y - 3 = \left(\frac{3+1}{0+2}\right)(x - 0)$$

$$\Rightarrow 2x - y + 3 = 0$$

Equations of BE is

$$x = 1 \qquad (2.23)$$

Equation of CF is

$$x - y = 1 \qquad (2.24)$$

Equation of BC is

$$2x - y = -3 \qquad (2.25)$$

Solving Eqs. (2.23)–(2.25), we have $B = (1, 5)$ and $C = (-4, -5)$.

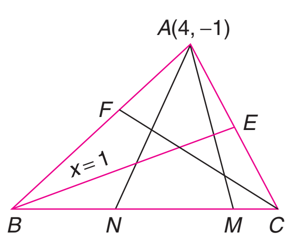

FIGURE 2.28

Note: To check whether our answers are correct or not, we need to verify the following:

1. The incentre $I(1, 0)$ is inside ΔABC.
2. The distance of $I(1, 0)$ from the three sides must be equal.

15. Determine all values of α for which the point (α, α^2) lies inside the triangle formed by the lines $2x + 3y - 1 = 0$, $x + 2y - 3 = 0$ and $5x - 6y - 1 = 0$.
(IIT-JEE 1992)

Solution: Let $L \equiv 2x+3y-1=0$, $L' \equiv x+2y-3=0$ and $L'' \equiv 5x - 6y - 1 = 0$. Vertices are A (5/4, 7/8), B (1/3, 1/9) and C (−7, 5). See Fig. 2.29.

1. For the side BC ($L = 0$), the points $A(5/4, 7/8)$ and (α, α^2) must be on the same side (by Quick Look 9).

$$L_{11} = 2\left(\frac{5}{4}\right) + 3\left(\frac{7}{8}\right) - 1 = \frac{20+21-8}{8} > 0.$$

$$\Rightarrow L_{22} = 2(\alpha) + 3\alpha^2 - 1 > 0$$

$$\Rightarrow 3\alpha^2 + 2\alpha - 1 > 0$$

$$\Rightarrow (3\alpha - 1)(\alpha + 1) > 0$$

Therefore,

$$\alpha < -1 \quad \text{or} \quad \alpha > \frac{1}{3} \qquad (2.26)$$

2. For the side CA ($L' = 0$), the points $B(1/3, 1/9)$ and (α, α^2) are on the same side. Therefore

$$L'_{11} = \frac{1}{3} + \frac{2}{9} - 3 = \frac{3+2-27}{9} < 0$$

and $\quad L'_{22} = \alpha + 2\alpha^2 - 3 < 0 \quad$ (by Quick Look 9)

$$\Rightarrow (2\alpha + 3)(\alpha - 1) < 0$$

$$\Rightarrow -\frac{3}{2} < \alpha < 1 \qquad (2.27)$$

3. For the side AB ($L'' = 0$), the points C and (α, α^2) are on the same side. Therefore

$$L''_{11} = 5(-7) - 6(-5) - 1 < 0$$

and $\quad L''_{22} = 5\alpha - 6\alpha^2 - 1 < 0$

$$\Rightarrow 6\alpha^2 - 5\alpha + 1 > 0$$

$$\Rightarrow (3\alpha - 1)(2\alpha - 1) > 0$$

$$\Rightarrow \alpha < \frac{1}{3} \text{ or } \alpha > \frac{1}{2} \qquad (2.28)$$

From Eqs. (2.26)–(2.28), $\alpha \in (-3/2, -1) \cup (1/2, 1)$.

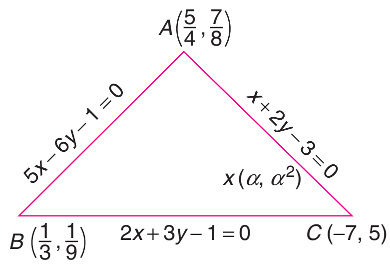

FIGURE 2.29

16. If the sum of the distances of a point from two perpendicular lines in a plane is 1, then prove that its locus is a square. **(IIT-JEE 1992)**

Solution: Take the two perpendicular lines as coordinate axes. $P(x, y)$ is a point on the locus $\Leftrightarrow |x|+|y|=1$. This implies that

$$x+y=1$$
$$x-y=1$$
$$-x+y=1$$
$$-x-y=1$$

These lines form a square.

17. A line meets the straight lines $5x - y - 4 = 0$ and $3x + 4y - 4 = 0$ at points P and Q. If $(1, 5)$ is the midpoint of PQ, find the equation of the line PQ.

Solution: See Fig. 2.30. Let the line PQ be

$$y-5=m(x-1) \tag{2.29}$$

Substituting $y=mx+5-m$ in the equation $5x-y-4=0$, we have

$$5x-mx-5+m-4=0$$
$$\Rightarrow (5-m)x+m-9=0$$

Therefore

$$x=\frac{9-m}{5-m} \quad \text{and} \quad y=m\left(\frac{9-m}{5-m}\right)+5-m=\frac{25-m}{5-m}$$

Hence

$$P=\left(\frac{9-m}{5-m}, \frac{25-m}{5-m}\right)$$

Substituting $y=mx+5-m$ in the equation $3x+4y-4=0$, we have

$$3x+4(mx+5-m)-4=0$$
$$\Rightarrow (3+4m)x+16-4m=0$$

Therefore

$$x=\frac{4m-16}{4m+3}$$

and
$$y=\frac{m(4m-16)}{4m+3}+5-m$$
$$=\frac{4m^2-16m-4m^2+17m+15}{4m+3}$$
$$=\frac{m+15}{4m+3}$$

Hence

$$Q=\left(\frac{4m-16}{4m+3}, \frac{m+15}{4m+3}\right)$$

Since $M(1, 5)$ is the midpoint of PQ, we have

$$1=\frac{1}{2}\left(\frac{9-m}{5-m}+\frac{4m-16}{4m+3}\right) \tag{2.30}$$

and
$$5=\frac{1}{2}\left(\frac{25-m}{5-m}+\frac{m+15}{4m+3}\right) \tag{2.31}$$

From Eq. (2.31), we get

$$2(5-m)(4m+3)=(9-m)(4m+3)+(5-m)(4m-16)$$
$$\Rightarrow 2(-4m^2+17m+15)=(-4m^2+33m+27)+(-4m^2+36m-80)$$
$$\Rightarrow -8m^2+34m+30=-8m^2+69m-53$$
$$\Rightarrow 35m=83$$
$$\Rightarrow m=\frac{83}{35}$$

Substituting the value of $m = 83/35$ in Eq. (2.29), equation of line PQ is obtained as

$$y-5=\frac{83}{35}(x-1)$$
$$\Rightarrow 83x-35y+92=0$$

The value of m obtained from Eq. (2.31) is also equal to 83/35.

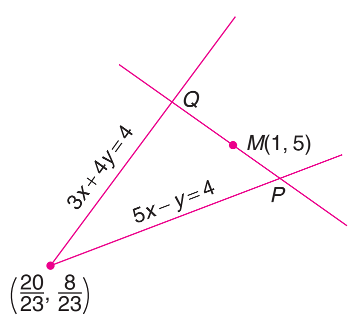

FIGURE 2.30

Try it out

We can take the equation of the line PQ as

$$\frac{x-1}{\cos\theta} = \frac{y-5}{\sin\theta} = \gamma$$

and substitute $x = 1 + \gamma\cos\theta$ and $y = 5 + \gamma\sin\theta$ in both the line equations and obtain the coordinates of P and Q. Finally, use that $(1, 5)$ is the midpoint of PQ.

18. The line joining two points $A(2, 0)$ and $B(3, 1)$ is rotated about A through an angle 15° in the counterclock sense. Find the equation of the line in the new position and the coordinates of the new position of B.

Solution: See Fig. 2.31. Let $\angle XAB = \alpha$ so that

$$\tan\alpha = \frac{1-0}{3-2} = 1 \text{ or } \alpha = 45°$$

Since $\angle XAC = \alpha + 15° = 60°$, the equation of the line AC (point C is the new position of point B) is

$$y - 0 = \tan 60°(x-2) = \sqrt{3}(x-2)$$

Therefore, equation of the line AB in its new position is

$$\sqrt{3}x - y - 2\sqrt{3} = 0$$

Since $C = (x, \sqrt{3}\,(x-2))$ and $AC = AB = \sqrt{2}$, we have

$$(x-2)^2 + 3(x-2)^2 = (AC)^2 = 2$$

$$\Rightarrow (x-2) = \pm\frac{1}{\sqrt{2}}$$

$$\Rightarrow x = 2 \pm \frac{1}{\sqrt{2}}$$

Hence

$$C = \left(2 + \frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}\right)$$

The value of $x = 2 - 1/\sqrt{2}$ gives the position of point B, when $\overline{AB}$ is rotated about point A through angle 15° in clockwise sense.

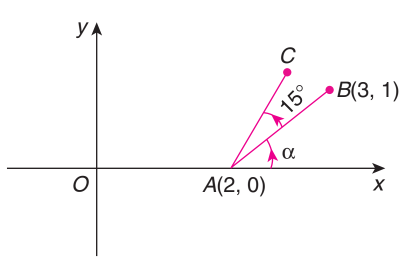

FIGURE 2.31

19. A line through $A(-5, -4)$ meets the lines $x + 3y + 2 = 0$, $2x + y + 4 = 0$ and $x - y - 5 = 0$ at the points B, C and D, respectively. If

$$\left(\frac{15}{AB}\right)^2 + \left(\frac{10}{AC}\right)^2 = \left(\frac{6}{AD}\right)^2$$

then find the equation of line.

(IIT-JEE 1993)

Solution: Let the line through $A(-5, -4)$ be

$$\frac{x+5}{\cos\theta} = \frac{y+4}{\sin\theta} = \gamma \qquad \text{(by Theorem 2.7)}$$

Therefore, every point on the line is of the form $x = -5 + \gamma\cos\theta$, $y = -4 + \gamma\sin\theta$. Let $AB = \gamma_1$, $AC = \gamma_2$ and $AD = \gamma_3$. Since $B = (-5 + \gamma_1\cos\theta, -4 + \gamma_1\sin\theta)$, we have

$$(-5 + \gamma_1\cos\theta) + 3(-4 + \gamma_1\sin\theta) + 2 = 0$$
$$(\because B \text{ lies on } x + 3y + 2 = 0)$$

Therefore

$$\gamma_1 = \frac{15}{\cos\theta + 3\sin\theta}$$

Hence

$$\frac{15}{AB} = \cos\theta + 3\sin\theta \qquad (\because \gamma_1 = AB) \quad (2.32)$$

Similarly, points C and D lie on $2x + y + 4 = 0$ and $x - y - 5 = 0$, respectively. We have

$$\frac{10}{AC} = 2\cos\theta + \sin\theta \qquad (2.33)$$

and

$$\frac{6}{AD} = \cos\theta - \sin\theta \qquad (2.34)$$

Now, by hypothesis,

$$\left(\frac{15}{AB}\right)^2 + \left(\frac{10}{AC}\right)^2 = \left(\frac{6}{AD}\right)^2$$

Hence, from Eqs. (2.32)–(2.34), we have

$$(\cos\theta + 3\sin\theta)^2 + (2\cos\theta + \sin\theta)^2 = (\cos\theta - \sin\theta)^2$$

$$\Rightarrow 5\cos^2\theta + 10\sin^2\theta + 10\sin\theta\cos\theta = \cos^2\theta + \sin^2\theta - 2\sin\theta\cos\theta$$

$$\Rightarrow 4\cos^2\theta + 9\sin^2\theta + 12\sin\theta\cos\theta = 0$$

$$\Rightarrow (2\cos\theta + 3\sin\theta)^2 = 0$$

Therefore,

$$2\cos\theta + 3\sin\theta = 0 \quad \text{or} \quad \tan\theta = \frac{-2}{3}$$

Hence, the equation of the line is

$$y+4=-\frac{2}{3}(x+5) \quad \text{or} \quad 2x+3y+22=0$$

20. One diagonal of a square is the portion of the line $7x + 5y = 35$ intercepted between the axes. Determine the extremities of the other diagonal.

Solution: See Fig. 2.32. $A(5, 0)$ and $B(0, 7)$ are the extremities of the given diagonal. Therefore, the slope of the diagonal AB is $-7/5$. Hence, the slope of the other diagonal, say, CD is 5/7. Therefore

$$\tan\theta = \frac{5}{7}$$

so that

$$\cos\theta = \frac{7}{\sqrt{74}} \quad \text{and} \quad \sin\theta = \frac{5}{\sqrt{74}}$$

The equation of the diagonal CD is

$$\frac{x-(5/2)}{7/\sqrt{74}} = \frac{y-(7/2)}{5/\sqrt{74}} = \gamma \qquad \text{(say)}$$

Since, $\gamma = \sqrt{74}/2$, the other vertices of the diagonal are

$$C = \left(\frac{5}{2}+\frac{7}{\sqrt{74}}\times\frac{\sqrt{74}}{2}, \frac{7}{2}+\frac{5}{\sqrt{74}}\times\frac{\sqrt{74}}{2}\right) = (6,6)$$

and
$$D = \left(\frac{5}{2}-\frac{\sqrt{74}}{2}\times\frac{7}{\sqrt{74}}, \frac{7}{2}-\frac{\sqrt{74}}{2}\times\frac{5}{\sqrt{74}}\right)$$
$$= \left(\frac{5}{2}-\frac{7}{2}, \frac{7}{2}-\frac{5}{2}\right) = (-1,1)$$

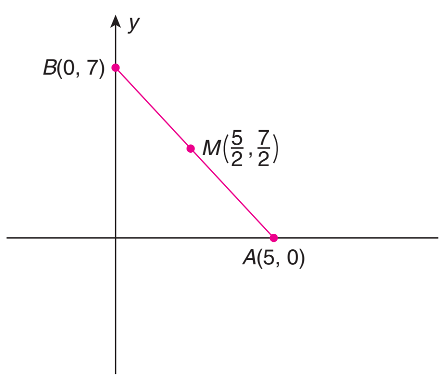

FIGURE 2.32

21. A variable line l passing the point $B(2, 5)$ intersects the lines $2x^2 - 5xy + 2y^2 = 0$ at P and Q. Find the locus of the point R such that the distances BP, BR and BQ are in harmonic progression (HP).

Solution: The given equation $2x^2 - 5xy + 2y^2 = 0$ represents the pair of lines $2x - y = 0$ and $x - 2y = 0$. Let the equation of the line through $B(2, 5)$ be

$$\frac{x-2}{\cos\theta} = \frac{y-2}{\sin\theta} = \gamma \qquad \text{(by Theorem 2.7)}$$

See Fig. 2.33. Let $BP = \gamma_1$, $BQ = \gamma_2$ and $BR = \gamma$. Therefore

$$P = (2+\gamma_1\cos\theta, 5+\gamma_1\sin\theta)$$

and
$$Q = (2+\gamma_2\cos\theta, 5+\gamma_2\sin\theta)$$

Since point P lies on $x - 2y = 0$, we have

$$(2+\gamma_1\cos\theta) - 2(5+\gamma_1\sin\theta) = 0$$

$$\Rightarrow \gamma_1 = \frac{8}{\cos\theta - 2\sin\theta} \qquad (2.35)$$

Similarly, Q lies on $2x - y = 0$, we have

$$\gamma_2 = \frac{1}{2\cos\theta - \sin\theta} \qquad (2.36)$$

By hypothesis,

$$\frac{1}{\gamma_1}+\frac{1}{\gamma_2} = \frac{2}{\gamma}$$

Therefore, from Eqs. (2.35) and (2.36), we get

$$\frac{\cos\theta-2\sin\theta}{8} + (2\cos\theta - \sin\theta) = \frac{2}{\gamma}$$

$$\Rightarrow 17\cos\theta - 10\sin\theta = \frac{16}{\gamma} \qquad (2.37)$$

Let $R = (x, y)$ so that

$$x = 2+\gamma\cos\theta \Rightarrow \gamma\cos\theta = x-2$$
$$y = 5+\gamma\sin\theta \Rightarrow \gamma\sin\theta = y-5$$

Therefore, from Eq. (2.37),

$$16 = 17\gamma\cos\theta - 10\gamma\sin\theta$$
$$= 17(x-2) - 10(y-5)$$

The locus of R is $17x - 10y = 0$.

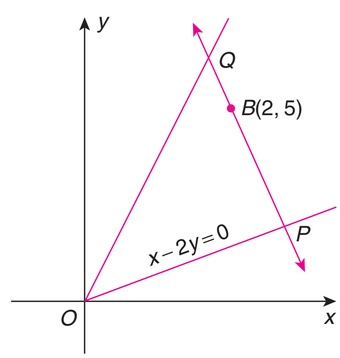

FIGURE 2.33

22. ABC is an equilateral triangle in which $B = (1, 3)$ and $C = (-2, 7)$. Find the coordinates of the vertex A.

Solution: See Fig. 2.34. The slope of BC is

$$\frac{7-3}{-2-1}=-\frac{4}{3}$$

Therefore, the slope of perpendicular bisector of BC is 3/4 and its equation is

$$y-5=\frac{3}{4}\left(x+\frac{1}{2}\right)$$

where $M(-1/2, 5)$ is the midpoint of BC. Since $BC=\sqrt{(7-3)^2+(-2-1)^2}=5,$ we have

$$\text{Altitude } AM = (\sin 60°)AB=\frac{5\sqrt{3}}{2}$$

The equation of the altitude AM is

$$y=\frac{3}{8}(2x+1)+5 \qquad (2.38)$$

Now,

$$\begin{aligned}\frac{75}{4}&=(AM)^2\\ &=\left(x+\frac{1}{2}\right)^2+(y-5)^2 \quad [\text{where } A=(x,\ y)]\\ &=\left(x+\frac{1}{2}\right)^2+\left[\frac{3}{8}(2x+1)+5-5\right] \quad [\text{by Eq. (2.38)}]\\ &=\frac{(2x+1)^2}{4}+\frac{9}{64}(2x+1)^2\\ &=\frac{25(2x+1)^2}{64}\end{aligned}$$

Therefore

$$\begin{aligned}&(2x+1)^2=48\\ &\Rightarrow 2x+1=\pm 4\sqrt{3}\\ &\Rightarrow x=\frac{-1\pm 4\sqrt{3}}{2}\\ &\Rightarrow x=2\sqrt{3}-\frac{1}{2} \quad \text{and} \quad -\left(2\sqrt{3}+\frac{1}{2}\right)\end{aligned}$$

When $x=2\sqrt{3}-(1/2)$ we have

$$\begin{aligned}y&=\frac{3}{4}\left(x+\frac{1}{2}\right)+5\\ &=\frac{3}{4}\left(2\sqrt{3}+\frac{1}{2}-\frac{1}{2}\right)+5\\ &=\frac{3\sqrt{3}}{2}+5\end{aligned}$$

and when $x=-2\sqrt{3}-(1/2)$ we have

$$y=5-\frac{3\sqrt{3}}{2}$$

Hence

$$A=\left(2\sqrt{3}-\frac{1}{2},\frac{3\sqrt{3}}{2}+5\right) \quad \text{or} \quad \left(-2\sqrt{3}-\frac{1}{2},5-\frac{3\sqrt{3}}{2}\right)$$

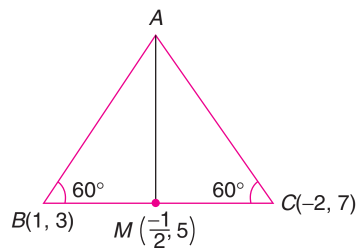

FIGURE 2.34

Aliter (Using Complex Numbers): $B=1+3i$ and $C=-2+7i$. Rotate CB about C through an angle 60° in anti-clockwise sense (see Fig. 2.35). Thus

$$\frac{Z-(-2+7i)}{(1+3i)-(-2+7i)}=\cos 60°+i\sin 60°$$

Therefore

$$Z+2-7i=\left(\frac{1+i\sqrt{3}}{2}\right)(3-4i)=\frac{3+4\sqrt{3}+i(3\sqrt{3}-4)}{2}$$

This gives

$$\begin{aligned}Z&=\frac{3+4\sqrt{3}+i(3\sqrt{3}-4)}{2}-2+7i\\ &=\frac{3+4\sqrt{3}+i(3\sqrt{3}-4)-4+14i}{2}\\ &=\frac{4\sqrt{3}+i(3\sqrt{3}+10)}{2}\\ &=\left(2\sqrt{3}-\frac{1}{2}\right)+i\left(\frac{3\sqrt{3}}{2}+5\right)\end{aligned}$$

Therefore

$$A=\left(2\sqrt{3}-\frac{1}{2},\frac{3\sqrt{3}}{2}+5\right)$$

Similarly, if we rotate CB in clockwise sense about C through an angle 60°, we get the second position of A.

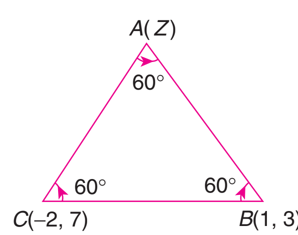

FIGURE 2.35

> **Try it out** Solve the same problem by taking the equation of the altitude AM as
>
> $$\frac{x+\frac{1}{2}}{\cos\theta} = \frac{y-5}{\sin\theta} = \gamma$$
>
> where $\cos\theta = 4/5$ and $\sin\theta = 3/5$.

***Note*:** With regard to geometric problems concerning equilateral triangles, squares, etc., the method of using complex numbers is easier than the method of using coordinates.

23. The sides of a rhombus are parallel to the lines $y + 2x = 3$ and $y = 7x + 2$. The diagonals intersect at $(1, 2)$. If one vertex lies on the y-axis, then find the coordinates of this vertex.

Solution: See Fig. 2.36. Let $ABCD$ be the rhombus where $A = (0, k)$. AB and CD are parallel to $y = 7x + 2$ whereas BC and AD are parallel to $y + 2x = 3$. Hence, the equation of AB is

$$y = 7x + k \tag{2.39}$$

Since $(1, 2)$ is the midpoint of AC and $A = (0, k)$, it follows that $C = (2, 4 - k)$. Also, BC is parallel to $2x + 3$ and passes through $C\ (2, 4 - k)$. Hence, the equation of BC is

$$y-(4-k) = 2(x-2)$$

$$\Rightarrow y = 2x - k \tag{2.40}$$

Solving Eqs. (2.39) and (2.40), we have

$$B = \left(-\frac{2k}{5}, -\frac{9k}{5}\right)$$

Since $A = (0, k), B = \left(-\frac{2k}{5}, -\frac{9k}{5}\right), C = (2, 4-k)$ and $AB = BC$, we have

$$\left(\frac{2k}{5}\right)^2 + \left(k+\frac{9k}{5}\right)^2 = \left(2+\frac{2k}{5}\right)^2 + \left(4-k+\frac{9k}{5}\right)^2$$

$$\Rightarrow 10k^2 = (k+5)^2$$

$$\Rightarrow k+5 = \pm\sqrt{10}k$$

$$\Rightarrow k = \frac{5}{\sqrt{10}-1} = \frac{5}{9}(\sqrt{10}+1), \frac{-5}{9}(\sqrt{10}-1)$$

Therefore

$$A = \left(0, \frac{5}{9}(\sqrt{10}+1)\right), \left(0, \frac{-5}{9}(\sqrt{10}-1)\right)$$

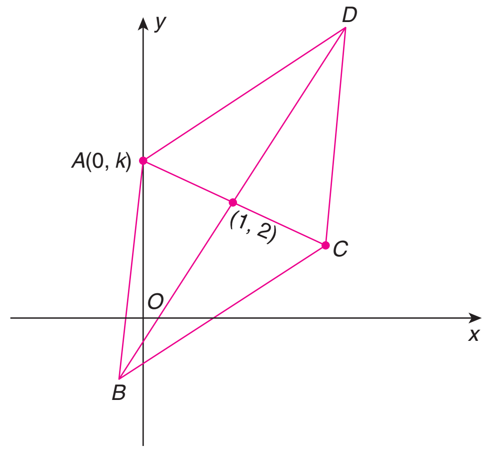

FIGURE 2.36

24. A ray of light is sent along the line $x - 2y + 5 = 0$. Upon reaching the line $3x - 2y + 7 = 0$, the ray is reflected from it. Find the equation of the reflected ray.

Solution: See Fig. 2.37. We have

$$x - 2y + 5 = 0 \quad \text{(Incident ray)} \tag{2.41}$$

$$3x - 2y + 7 = 0 \quad \text{(Surface line)} \tag{2.42}$$

$P = (-1, 2)$ is the point of incidence of the lines given in Eqs. (2.41) and (2.42). Hence, the equation of the normal at $P(-1, 2)$ is

$$y - 2 = \frac{-2}{3}(x+1)$$

$$\Rightarrow 2x + 3y - 4 = 0 \tag{2.43}$$

Let m be the slope of the reflected ray. Since the normal at P makes equal angles with the incident line and reflected line, we have

$$\left|\frac{m+\frac{2}{3}}{1-\frac{2m}{3}}\right| = \left|\frac{\frac{-2}{3}-\frac{1}{2}}{1+\left(\frac{-2}{3}\right)\left(\frac{1}{2}\right)}\right|$$

$$\Rightarrow \left|\frac{3m+2}{3-2m}\right| = \frac{7}{4}$$

$$\Rightarrow \frac{3m+2}{3-2m} = \pm\frac{7}{4}$$

$$\Rightarrow m = \frac{1}{2}, \frac{29}{2}$$

However, $m \neq 1/2$. So that the reflected line is

$$y - 2 = \frac{29}{2}(x+1)$$

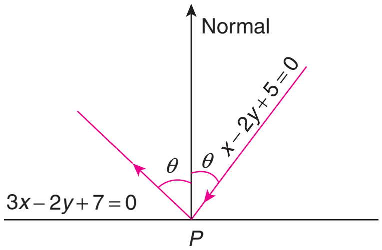

FIGURE 2.37

25. Straight lines $3x + 4y = 5$ and $4x - 3y = 15$ intersect at the point A. Points B and C are chosen on those lines such that $AB = AC$. Determine the possible equations of the line BC passing through the point BC.

Solution: See Fig. 2.38. The two lines intersect at $A(-3, 1)$. Clearly, the lines $3x+4y=5$ and $4x-3y=15$ are at right angles to each other and $AB=AC$ implies that ΔABC is a right-angled isosceles triangle with its right angle at vertex A. Let m be the slope of the line BC. Hence

$$1 = \tan 45° = \left|\frac{m+\frac{3}{4}}{1-\frac{3m}{4}}\right| = \left|\frac{4m+3}{4-3m}\right|$$

$$\Rightarrow 4m+3 = \pm(4-3m)$$

Two cases arise:

Case 1: $4m+3=(4-3m) \Rightarrow m = \frac{1}{7}$.

Case 2: $4m+3=-4+3m \Rightarrow m=-7$.

Therefore, the equation of the line BC is

$$y-2=\frac{1}{7}(x-1) \Rightarrow x-7y+13=0$$

or

$$y-2=-7(x-1) \Rightarrow 7x+y-9=0$$

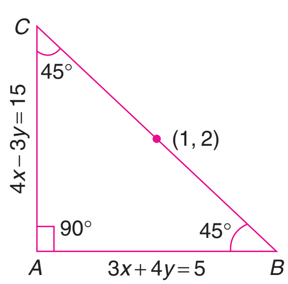

FIGURE 2.38

26. Show that the point $(3, -5)$ lies between the parallel lines $2x + 3y = 7$ and $2x + 3y = -12$ and find the equation of the lines through $(3, -5)$ cutting the above lines at an angle 45°.

Solution: See Fig. 2.39. The distance between the parallel lines is

$$\left|\frac{12+7}{\sqrt{2^2+3^2}}\right| = \frac{19}{\sqrt{13}}$$

Let $P = (3, -5)$. Let p_1 be the distance from the first line, which is given by

$$\left|\frac{2(3)+3(-5)-7}{\sqrt{2^2+3^2}}\right| = \frac{16}{\sqrt{13}}$$

and p_2 be the distance from the second line, which is given by

$$\left|\frac{2(3)+3(-5)+12}{\sqrt{2^2+3^2}}\right| = \frac{3}{\sqrt{13}}$$

Therefore

$$p_1+p_2=\frac{19}{\sqrt{13}}$$

This implies that $P(3, 5)$ lies in between the two lines. Let m be the slope of a line through P $(3, -5)$. Therefore, by hypothesis, we have

$$1 = \tan 45° = \left|\frac{m+\frac{2}{3}}{1-\frac{2m}{3}}\right|$$

$$\Rightarrow \frac{3m+2}{3-2m} = \pm 1$$

Two cases arise:

Case 1: $3m+2=3-2m \Rightarrow m=\frac{1}{5}$.

Case 2: $3m+2=-(3-2m) \Rightarrow m=-5$.

Therefore, the required lines are

$$y+5=\frac{1}{5}(x-3) \Rightarrow x-5y-28=0$$

and

$$y+5=-5(x-3) \Rightarrow 5x+y-10=0$$

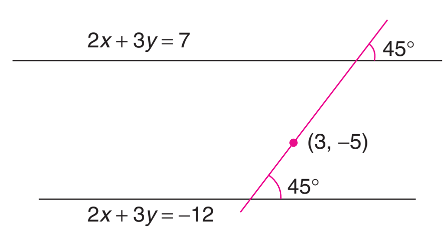

FIGURE 2.39

27. $A(10, 0)$ and $B(-5, -5)$ are two vertices of a triangle whose incentre is the origin. Find the coordinates of the third vertex.

Solution: See Fig. 2.40. Let $C(h, k)$ be the third vertex. Since $O(0, 0)$ is the incentre, BO and AO are the bisectors of the angles B and A, respectively. The equation of BO is

$$x - y = 0 \tag{2.43}$$

The equation of AO is

$$y = 0 \tag{2.44}$$

Equation of the side AB is

$$y = \frac{0+5}{10+5}(x - 10)$$
$$\Rightarrow x - 3y - 10 = 0 \tag{2.45}$$

Since BO is the angle bisector of $\angle B$, the image of $C(h, k)$ in the line BO lies on the line AB. Since the equation of BO is $y = x$, the image of $C(h, k)$ in BO is (k, h) and this lies on AB. Therefore

$$k - 3h = 10 \tag{2.46}$$

The image of $C(h, k)$ in the angle bisector AO lies on the side AB. That is, $(h, -k)$ lies on the side AB. Therefore, from Eq. (2.45), we get

$$h - 3(-k) = 10 \tag{2.47}$$

Solving Eqs. (2.46) and (2.47), we get $h = -2$ and $k = 4$. Hence, the third vertex is $(-2, 4)$.

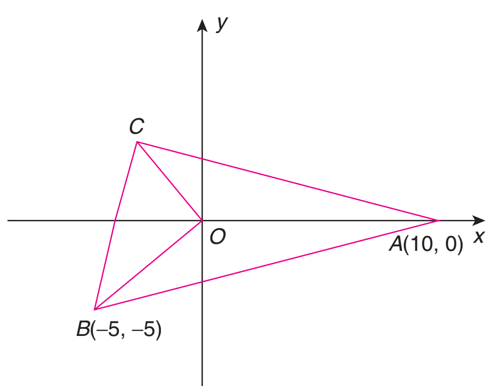

FIGURE 2.40

28. Find the equations of the sides of a right-angled isosceles triangle whose hypotenuse is the line $3x + 4y - 4 = 0$ and the right angle vertex is the point $(2, 2)$.

Solution: See Fig. 2.41. Let the slope of AB be m. Therefore

$$\angle CBA = 45^\circ$$
$$\Rightarrow 1 = \tan 45^\circ = \left| \frac{m - \left(\frac{-3}{4}\right)}{1 + m\left(\frac{-3}{4}\right)} \right|$$
$$\Rightarrow 4m + 3 = \pm(3m - 4)$$

Two cases arise:

Case 1: $4m + 3 = 3m - 4 \Rightarrow m = -7$.

Case 2: $4m + 3 = -(3m - 4) \Rightarrow m = \frac{1}{7}$.

Therefore, the equations of the sides AB and AC are

$$y - 2 = -7(x - 2) \Rightarrow 7x + y - 16 = 0$$

and $$y - 2 = \frac{1}{7}(x - 2) \Rightarrow x - 7y + 12 = 0$$

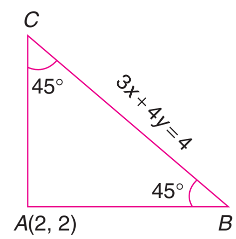

FIGURE 2.41

29. The points $(1, 3)$ and $(5, 1)$ are two opposite vertices of a rectangle. The other two vertices lie on the line $y = 2x + c$. Find c and the other remaining vertices.

Solution: See Fig. 2.42. $ABCD$ is the rectangle in which $A = (1, 3)$, $C = (5, 1)$. Points B and D lie on the line $y = 7x + c$. The diagonals intersect in $(3, 2)$ which lies on the line $y = 7x + c$. Therefore, $2 = 2(3) + c$ or $c = -4$. That is, the equation of the diagonal BD is

$$y = 2x - 4 \tag{2.48}$$

Suppose $M = (3, 2)$ is the midpoint of the diagonals. Therefore,

$$MD = MB = (1/2)AC = \sqrt{5}$$

Let B be $(x, 2x - 4)$. Therefore

$$MB = \sqrt{5}$$
$$\Rightarrow (MB)^2 = 5$$
$$\Rightarrow (3 - x)^2 + (2 - 2x + 4)^2 = 5$$
$$\Rightarrow (3 - x)^2 + (6 - 2x)^2 = 5$$
$$\Rightarrow 5x^2 - 30x + 40 = 0$$
$$\Rightarrow x^2 - 6x + 8 = 0$$
$$\Rightarrow x = 2, 4$$

Hence, $B = (2, 0)$ and $D = (4, 4)$.

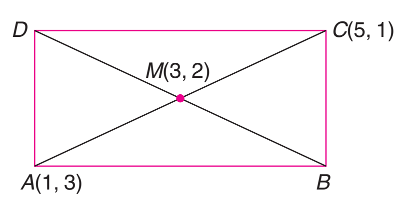

FIGURE 2.42

30. The ends of a straight line segment $\overline{AB}$ of constant length c move on two perpendicular lines OX and OY which are the coordinate axes. If the rectangle $OAPB$ is completed, then show that the locus of the foot of the perpendicular drawn from P on to AB is $x^{2/3}+y^{2/3}=c^{2/3}$.

Solution: See Fig. 2.43. Let $A=(a,0)$ and $B=(0,b)$ so that the equation of the line AB is

$$\frac{x}{a}+\frac{y}{b}=1 \tag{2.49}$$

and also

$$a^2+b^2=c^2 \tag{2.50}$$

Let $M(x_1, y_1)$ be the foot of the perpendicular drawn from P onto the line AB. Since $M(x_1, y_1)$ lies on AB, we have

$$\frac{x_1}{a}+\frac{y_1}{b}=1 \tag{2.52}$$

Since $P=(a, b)$ and PM is perpendicular to AB, we have

$$\text{Slope of } PM \times \text{Slope of } AB = -1$$

Therefore,

$$\left(\frac{b-y_1}{a-x_1}\right)\left(\frac{b-0}{0-a}\right)=-1$$

$$\Rightarrow ax_1-by_1=a^2-b^2 \tag{2.53}$$

From Eqs. (2.52) and (2.53), we get

$$x_1=\frac{a^3}{a^2+b^2}, y_1=\frac{b^3}{a^2+b^2}$$

Therefore

$$x_1^{2/3}+y_1^{2/3}=\frac{a^2+b^2}{(a^2+b^2)^{2/3}}=(a^2+b^2)^{1/3}=c^{2/3}$$

[by Eq. (2.50)]

Therefore, the locus of $M(x_1, y_1)$ is

$$x^{2/3}+y^{2/3}=c^{2/3}$$

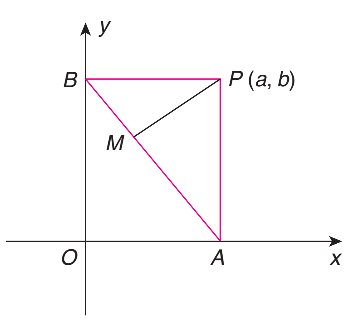

FIGURE 2.43

31. A variable straight line through the point of intersection of the lines

$$\frac{x}{a}+\frac{y}{b}=1 \text{ and } \frac{x}{b}+\frac{y}{a}=1$$

meets the coordinate axes at A and B. Show that the locus of the midpoint of AB is the curve

$$2\,(a+b)\,xy=ab(x+y)$$

Solution: Equation of any line through the intersection of the given lines is of the form

$$\frac{x}{a}+\frac{y}{b}-1+\lambda\left(\frac{x}{b}+\frac{y}{a}-1\right)=0$$

This line meets the x-axis at

$$A=\left(\frac{ab(1+\lambda)}{b+a\lambda}, 0\right) \quad\text{and}\quad B=\left(0, \frac{ab(1+\lambda)}{a+b\lambda}\right)$$

Let $M(x_1, y_1)$ be the midpoint of $\overline{AB}$. Therefore

$$2x_1=\frac{ab(1+\lambda)}{b+a\lambda} \quad\text{and}\quad 2y_1=\frac{ab(1+\lambda)}{a+b\lambda}$$

Therefore

$$\frac{1}{2x_1}+\frac{1}{2y_1}=\frac{(b+a\lambda)+(a+b\lambda)}{ab(1+\lambda)}=\frac{(a+b)(1+\lambda)}{ab(1+\lambda)}=\frac{a+b}{ab}$$

Hence

$$(a+b)2x_1y_1=ab(x_1+y_1)$$

Therefore, the locus of (x_1, y_1) is

$$2(a+b)xy=(ab)(x+y)$$

32. The equations of the perpendicular bisectors of the sides AB and AC of the triangle ABC are $x-y+5=0$ and $x+2y=0$, respectively. If the point A is $(1,-2)$, then find the equation of the side BC.

Solution: See Fig. 2.44. Perpendicular bisectors of the sides AB and AC, respectively, are

$$x-y+5=0 \tag{2.54}$$

and

$$x+2y=0 \tag{2.55}$$

Solving Eqs. (2.54) and (2.55), the circumcentres of ΔABC is $(-10/3, 5/3)$. Also, the equation of AB is

$$y+2=-1(x-1)$$

$$\Rightarrow x+y+1=0$$

Suppose $B=(x, -x-1)$. We have

$$SA=SB$$

$$\Rightarrow\left(\frac{-10}{3}-1\right)^2+\left(\frac{5}{3}+2\right)^2=\left(x+\frac{10}{3}\right)^2+\left(-x-1-\frac{5}{3}\right)^2$$

$$\Rightarrow 13^2 + 11^2 = (3x+10)^2 + (3x+8)^2$$
$$\Rightarrow 18x^2 + 108x - 126 = 0$$
$$\Rightarrow 2x^2 + 12x - 14 = 0$$
$$\Rightarrow x^2 + 6x - 7 = 0$$
$$\Rightarrow (x+7)(x-1) = 0$$
$$\Rightarrow x = 1, -7$$

Hence

$$x = 1 \Rightarrow (x, -x-1) = (1, -2) = A$$

and $$x = -7 \Rightarrow B = (-7, 6)$$

Therefore

$$B = (-7, 6)$$

Similarly, equation of AC is

$$y + 2 = 2(x-1)$$
$$\Rightarrow y = 2x - 4$$

Suppose $C = (h, 2h-4)$, we have

$$SC = SA$$
$$\Rightarrow \left(\frac{-10}{3} - h\right)^2 + \left(2h - 4 - \frac{5}{3}\right)^2 = \left(-\frac{10}{3} - 1\right)^2 + \left(\frac{5}{3} + 2\right)^2$$
$$\Rightarrow (3h+10)^2 + (6h-17)^2 = 13^2 + 11^2$$
$$\Rightarrow 45h^2 - 144h + 99 = 0$$
$$\Rightarrow 5h^2 - 16h + 11 = 0$$
$$\Rightarrow 5h^2 - 5h - 11h + 11 = 0$$
$$\Rightarrow 5h(h-1) - 11(h-1) = 0$$
$$\Rightarrow h = 1, \frac{11}{5}$$

Now

$$h = 1 \Rightarrow \text{the point } (h, 2h-4) = (1, -2) = A$$
$$h = \frac{11}{5} \Rightarrow C = \left(\frac{11}{5}, \frac{2}{5}\right)$$

Hence

$$B = (-7, 6) \text{ and } C = \left(\frac{11}{5}, \frac{2}{5}\right)$$

This implies that the equation of the side BC is

$$y - 6 = \frac{6 - (2/5)}{-7 - (11/5)}(x+7)$$
$$= \frac{28}{-46}(x+7)$$
$$= \frac{-14}{23}(x+7)$$
$$\Rightarrow 14x + 23y - 40 = 0$$

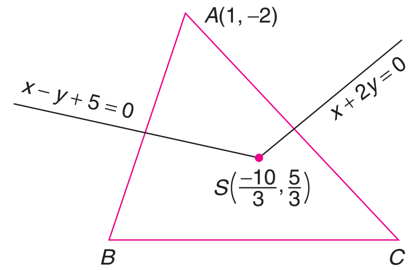

FIGURE 2.44

Aliter: $A = (1, -2)$ and the perpendicular bisector of AB is

$$x - y + 5 = 0 \tag{2.56}$$

Hence, B is the image of A in the given line in Eq. (2.56). If $B = (h, k)$, then by Theorem 2.13, we have

$$\frac{h-1}{1} = \frac{k+2}{-1} = \frac{-2[1-(-2)+5]}{1^2 + (-1)^2} = -8$$

Therefore

$$h = -7, k = 6$$

Hence

$$B = (-7, 6)$$

Similarly, $C\,(x_1, y_1)$ is the image of A in the line $x + 2y = 0$ which implies that

$$\frac{x_1 - 1}{1} = \frac{y_1 + 2}{2} = \frac{-2(1-4)}{1^2 + 2^2} = \frac{6}{5}$$
$$\Rightarrow x_1 = \frac{6}{5} + 1 = \frac{11}{5}$$

and $$y_1 = \frac{12}{5} - 2 = \frac{2}{5}$$

Therefore

$$C = \left(\frac{11}{5}, \frac{2}{5}\right)$$

Thus, the equation of the side BC is

$$14x + 23y - 40 = 0$$

33. A line cuts x-axis at $A(7, 0)$ and y-axis at $B(0, -5)$. A variable line PQ is drawn perpendicular to AB cutting the x-axis at P and the y-axis at Q. If AQ and BP intersect at R, then find the locus of R. **(IIT-JEE 1990)**

Solution: See Fig. 2.45. The slope of $\overleftrightarrow{AB}$ is

$$\frac{-5-0}{0-7} = \frac{5}{7}$$

Therefore, the slope of PQ is –7/5. Consider ΔABQ in which QP is the altitude from Q onto AB and AP is the altitude from A onto BQ. These two intersect at P. Hence, BP is the third altitude of ΔABQ. Therefore, BR is perpendicular to AR. Hence, if $R = (h, k)$, then

$$\text{Slope of } BR \times \text{Slope of } AR = -1$$

$$\Rightarrow \left(\frac{k+5}{h}\right)\left(\frac{k}{h-7}\right) = -1$$

$$\Rightarrow h^2 + k^2 - 7h + 5k = 0$$

Therefore, the locus of $R(h, k)$ is

$$x^2 + y^2 - 7x + 5y = 0$$

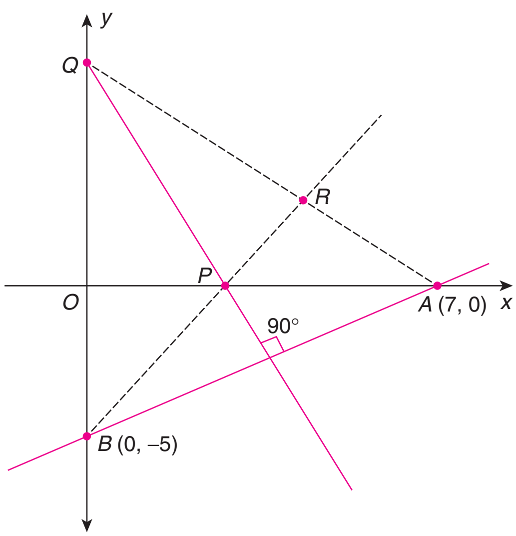

FIGURE 2.45

34. Let $S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ where $h^2 > ab$ represent a pair of lines both passing through origin. Prove that $g = f = c = 0$.

Solution: Since $S = 0$ passes through (0, 0) we have $c = 0$. Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be points on one of the lines other than the origin so that area of $\Delta OPQ \neq 0$. This implies that

$$|x_1y_2 - x_2y_1| \neq 0$$

$$\Rightarrow x_1y_2 - x_2y_1 \neq 0 \tag{2.57}$$

$P(x_1, y_1)$ lies on one line which passes through origin. This implies that $(-x_1, -y_1)$ also lies on the line. Therefore,

$$ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 = 0$$

and $$ax_1^2 + 2hx_1y_1 + by_1^2 - 2gx_1 - 2fy_1 = 0$$

imply that

$$gx_1 + fy_1 = 0 \tag{2.58}$$

Similarly,

$$gx_2 + fy_2 = 0 \tag{2.59}$$

Therefore, $g = f = 0$ because $x_1y_2 - x_2y_1 \neq 0$ or the matrix

$$\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}$$

is a non-singular matrix. Equations (2.58) and (2.95) have zero solution only so that $g = 0$ and $f = 0$.

35. If $S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of distinct lines ($h^2 > ab$), then prove that their point of intersection satisfies the equations $ax + hy + g = 0$, $hx + by + f = 0$ and $gx + fy + c = 0$, and the point of intersection is

$$\left(\frac{hf-bg}{ab-h^2}, \frac{gh-af}{ab-h^2}\right)$$

Solution: Since $S = 0$ represents a pair of lines, we have

$$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0 \quad \text{(by Theorem 2.31)}$$

and $h^2 > ab$ implies that the lines are distinct intersecting lines. Let $P(x_1, y_1)$ be the point of intersection (see Fig. 2.46). Shift the origin to the point $P(x_1, y_1)$ and let the new coordinates be (X, Y) so that by Section 1.4.1, $x = X + x_1$, $y = Y + y_1$. Therefore, $S = 0$ is transformed to

$$S \equiv a(X + x_1)^2 + 2h(X + x_1)(Y + y_1) + b(Y + y_1)^2 + 2g(X + x_1) + 2f(Y + y_1) + c = 0$$

$$\Rightarrow S \equiv aX^2 + 2hXY + bY^2 + 2(ax_1 + hy_1 + g)X + 2(hx_1 + by_1 + f)Y + ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c = 0 \tag{2.60}$$

Since Eq. (2.60) represents a pair of lines through origin P, by Problem 34, we have

$$ax_1 + hy_1 + g = 0 \tag{2.61}$$

$$hx_1 + by_1 + f = 0 \tag{2.62}$$

and $$S_{11} = ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c = 0 \tag{2.63}$$

Solving Eqs. (2.61) and (2.62) for x_1 and y_1, we get

$$x_1 = \frac{hf-bg}{ab-h^2}$$

$$y_1 = \frac{gh-af}{ab-h^2}$$

Therefore, the point of intersection is

$$\left(\frac{hf-bg}{ab-h^2}, \frac{gh-af}{ab-h^2}\right)$$

Also, from Eq. (2.63), we have

$$(ax_1 + hy_1 + g)x_1 + (hx_1 + by_1 + f)y_1 + gx_1 + fy_1 + c = 0$$

Since $ax_1 + hy_1 + g = 0$ and $hx_1 + by_1 + f = 0$, we have

$$gx_1 + fy_1 + c = 0$$

The point of intersection of the lines represented by $S = 0$ satisfy the following three equations:

$$\left.\begin{aligned} ax_1 + hy_1 + g &= 0 \\ hx_1 + by_1 + f &= 0 \\ gx_1 + fy_1 + c &= 0 \end{aligned}\right\} \qquad (2.64)$$

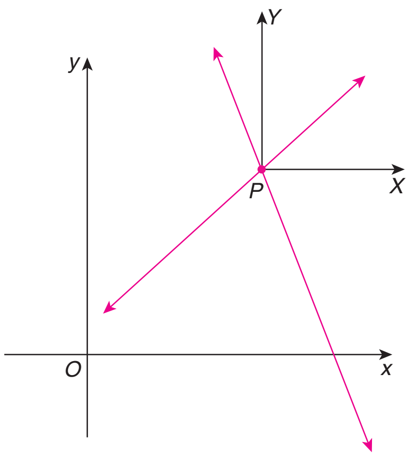

FIGURE 2.46

QUICK LOOK 15

Solving any two equations given in Eq. (2.64), we get the point of intersection of lines represented by $S = 0$.

36. Find the point of intersection of the lines represented by $S \equiv 3x^2 + xy - 4y^2 + 10x + 4y + 8 = 0$.

Solution: Comparing the given equation with the second-degree general equation, we have $a = 3$, $h = 1/2$, $b = -4$, $g = 5$, $f = 2$ and $c = 8$. So

$$hf - bg = \frac{1}{2}(2) - (-4)5 = 21$$

$$gh - af = 5\left(\frac{1}{2}\right) - (3)2 = \frac{-7}{2}$$

Therefore

$$\frac{hf - bg}{ab - h^2} = \frac{21}{-(49/4)} = -\frac{21 \times 4}{49} = -\frac{12}{7}$$

$$\frac{gh - af}{ab - h^2} = -\frac{7}{2} \times \frac{-4}{49} = \frac{2}{7}$$

Hence, the point of intersection is

$$\left(\frac{-12}{7}, \frac{2}{7}\right)$$

37. Show that the equation of the pair of lines passing through origin and perpendicular to the lines represented by the equation $ax^2 + 2hxy + by^2 = 0$ is $bx^2 - 2hxy + ay^2 = 0$.

Solution: Suppose the lines represented by $ax^2 + 2hxy + by^2 = 0$ are $l_1x + m_1y = 0$ and $l_2x + m_2y = 0$ so that we have

$$\left.\begin{aligned} l_1l_2 &= a \\ l_1m_2 + l_2m_1 &= 2h \\ \text{and} \qquad m_1m_2 &= b \end{aligned}\right\} \qquad (2.65)$$

We know that [see Quick Look 7, part (3)] the equations of the lines passing through the origin and perpendicular to the lines $l_1x + m_1y = 0$ and $l_2x + m_2y = 0$, respectively, are $m_1x - l_1y = 0$ and $m_2x - l_2y = 0$ and hence their combined equation is

$$(m_1x - l_1y)(m_2x - l_2y) = 0$$

$$\Rightarrow (m_1m_2)\,x^2 - (l_1m_2 + l_2m_1)\,xy + (l_1l_2)\,y^2 = 0$$

From Eq. (2.65), we have

$$bx^2 - 2hxy + ay^2 = 0$$

38. Show that the product of the perpendicular distances of the lines $ax^2 + 2hxy + by^2 = 0$ from a point (x_0, y_0) is

$$\left|\frac{ax_0^2 + 2hx_0y_0 + by_0^2}{\sqrt{(a-b)^2 + 4h^2}}\right|$$

Solution: Suppose the given lines are $l_1x + m_1y = 0$ and $l_2x + m_2y = 0$ so that

$$\begin{aligned} l_1l_2 &= a \\ l_1m_2 + l_2m_1 &= 2h \\ \text{and} \qquad m_1m_2 &= b \end{aligned}$$

See Fig. 2.47. Let d_1 and d_2 be the distances of the lines from $P(x_0, y_0)$. Therefore, by Theorem 2.14, we have

$$d_1 = \left|\frac{l_1x_0 + m_1y_0}{\sqrt{l_1^2 + m_1^2}}\right|$$

and

$$d_2 = \left|\frac{l_2x_0 + m_2y_0}{\sqrt{l_2^2 + m_2^2}}\right|$$

Hence

$$d_1 d_2 = \left|\frac{(l_1x_0 + m_1y_0)(l_2x_0 + m_2y_0)}{\sqrt{(l_1^2 + m_1^2)(l_2^2 + m_2^2)}}\right|$$

$$= \left|\frac{l_1l_2x_0^2 + (l_1m_2 + l_2m_1)x_0y_0 + m_1m_2y_0^2}{\sqrt{(l_1l_2 - m_1m_2)^2 + (l_1m_2 + l_2m_1)^2}}\right|$$

$$= \left|\frac{ax_0^2 + 2hx_0y_0 + by_0^2}{\sqrt{(a-b)^2 + 4h^2}}\right|$$

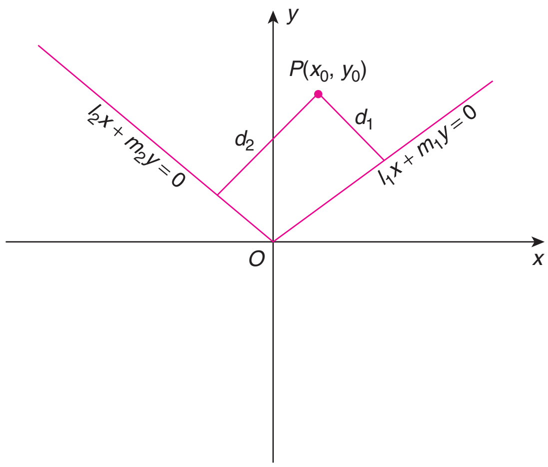

FIGURE 2.47

39. Find the area of the triangle formed by the lines $ax^2 + 2hxy + by^2 = 0$ and $lx + my = 1$

Solution: Let the lines be $l_1x + m_1y = 0$ and $l_2x + m_2y = 0$ so that

$$l_1l_2 = a$$
$$l_1m_2 + l_2m_1 = 2h$$

and
$$m_1m_2 = b$$

Suppose the line $lx + my = 1$ meets these lines at A and B (see Fig. 2.48). Substituting $y = -l_1/m_1$ in $lx + my = 1$, we have

$$x = \frac{m_1}{lm_1 - l_1m} \text{ and } y = \frac{-l_1}{lm_1 - l_1m}$$

Thus,

$$A = \left(\frac{m_1}{lm_1 - l_1m}, \frac{-l_1}{lm_1 - l_1m}\right)$$

and similarly,

$$B = \left(\frac{m_2}{lm_2 - l_2m}, \frac{-l_2}{lm_2 - l_2m}\right)$$

Therefore, the area of ΔOAB is

$$\frac{1}{2}\left|\frac{l_1m_2 - l_2m_1}{(lm_1 - l_1m)(lm_2 - l_2m)}\right| \quad \text{(by Section 1.2.4)}$$

$$= \frac{1}{2}\left|\frac{\sqrt{(l_1m_2 + l_2m_1)^2 - 4l_1l_2m_1m_2}}{l^2m_1m_2 - (l_1m_2 + l_2m_1)lm + l_1l_2m^2}\right|$$

$$= \frac{1}{2}\left|\frac{\sqrt{4h^2 - 4ab}}{bl^2 - 2hlm + am^2}\right|$$

$$= \left|\frac{\sqrt{h^2 - ab}}{bl^2 - 2hlm + am^2}\right|$$

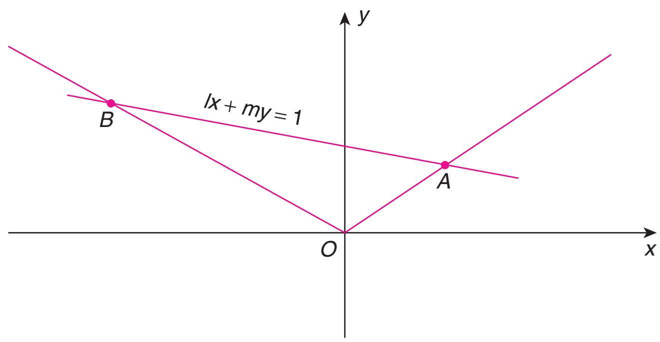

FIGURE 2.48

Note: If the given line is $lx + my + n = 0$ $(n \neq 0)$, then write

$$\left(\frac{-l}{n}\right)x + \left(\frac{-m}{n}\right)y = 1$$

and use the above formula so that the area is

$$\left|\frac{n^2\sqrt{h^2 - ab}}{bl^2 - 2hlm + am^2}\right|$$

40. If $n \neq 0$, then show that the triangle formed by the lines $(lx + my)^2 - 3(mx - ly)^2$ and $lx + my + n = 0$ is equilateral and find its area.

Solution: See Fig. 2.49. The equation of the sides passing through origin is

$$(l^2 - 3m^2)x^2 + 8lmxy + (m^2 - 3l^2)y^2 = 0 \quad (2.66)$$

Let θ be the angle between these sides. Therefore, by Theorem 2.28, Note (3), we have

$$\tan\theta = \frac{2\sqrt{16l^2m^2 - (l^2 - 3m^2)(m^2 - 3l^2)}}{|(l^2 - 3m^2) + (m^2 - 3l^2)|}$$

$$= \frac{2\sqrt{16l^2m^2 - 10l^2m^2 + 3(l^4 + m^4)}}{2(l^2 + m^2)}$$
$$= \frac{\sqrt{3}\sqrt{l^4 + m^4 + 2l^2m^2}}{l^2 + m^2}$$
$$= \frac{\sqrt{3}(l^2 + m^2)}{l^2 + m^2}$$
$$= \sqrt{3}$$

That is, the angle between the sides represented by the equation of the sides provided in Eq. (2.66) is 60°. Also, the combined equation of the pair of angle bisectors of equation of the sides passing through origin given in Eq. (2.66) is

$$4lm(x^2 - y^2) = [l^2 - 3m^2 - (m^2 - 3l^2)]xy \quad \text{(by Theorem 2.30)}$$
$$\Rightarrow lm\,(x^2 - y^2) = (l^2 - m^2)\,xy$$
$$\Rightarrow (lx + my)\,(mx - ly) = 0$$

The angle bisectors of the angle at the vertex origin are $lx + my = 0$ and $mx - ly = 0$ which, respectively, are parallel and perpendicular to the base line $lx + my + n = 0$. Hence, the triangle formed by the lines is equilateral. Suppose a is the length of the sides and p is the length of the altitude from origin onto the base $lx + my + n = 0$ so that

$$p = \frac{|n|}{\sqrt{l^2 + m^2}} \qquad \text{(by Theorem 2.14)}$$

and
$$\frac{\sqrt{3}}{2} = \sin 60° = \frac{p}{a}$$
$$\Rightarrow a = \frac{2p}{\sqrt{3}} = \frac{2}{\sqrt{3}}\frac{|n|}{\sqrt{l^2 + m^2}}$$

Therefore, area of the triangle is

$$\frac{\sqrt{3}}{4}a^2 = \frac{\sqrt{3}}{4}\frac{4n^2}{3(l^2 + m^2)}$$
$$= \frac{n^2}{\sqrt{3}(l^2 + m^2)}$$

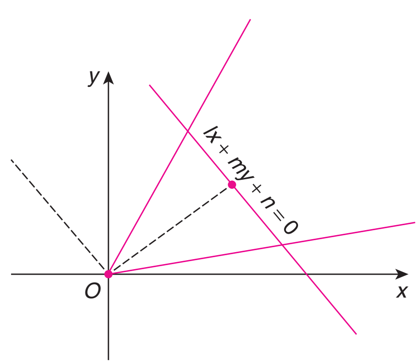

FIGURE 2.49

41. Show that the equation of the diagonal not passing through the origin of the parallelogram formed by the lines $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ and $ax^2 + 2hxy + by^2 = 0$ is $2gx + 2fy + c = 0$ and the diagonal passing through the origin is $y(hf - bg) = x(gh - af)$.

Solution: Suppose the lines

$$S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \qquad (2.67)$$

and the lines

$$ax^2 + 2hxy + by^2 = 0 \qquad (2.68)$$

intersect in $A(x_1, y_1)$ and $B(x_2, y_2)$. Hence, both A and B satisfy Eqs. (2.67) and (2.68) and hence they satisfy their difference $2gx + 2fy + c = 0$ which is first-degree equation in x and y. Hence, $2gx + 2fy + c = 0$ represents a straight line passing through points A and B. Thus, AB is the diagonal represented by $2gx + 2fy + c = 0$. Since

$$(0, 0) \quad \text{and} \quad \left(\frac{hf - bg}{ab - h^2}, \frac{gh - af}{ab - h^2}\right)$$

are the ends of the diagonal passing through origin, its equation is

$$y = \left(\frac{gh - af}{hf - bg}\right)x$$
$$\Rightarrow y(hf - bg) = (gh - af)x$$

42. Show that the area of parallelogram formed by the lines $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ and $ax^2 + 2hxy + by^2 = 0$ is

$$\frac{|c|}{2\sqrt{h^2 - ab}}$$

Solution: Suppose the sides of the parallelogram are (by Theorem 2. 32)

$$l_1x + m_1y + n_1 = 0$$
$$l_2x + m_2y + n_2 = 0$$
$$l_1x + m_1y = 0$$
and
$$l_2x + m_2y = 0$$

Therefore, $l_1l_2 = a, l_1m_2 + l_2m_1 = 2h, m_1m_2 = b, l_1n_2 + l_2n_1 = 2g, m_1n_2 + m_2n_1 = 2f$ and $n_1n_2 = c$. Now, by Problem 2, the area of the parallelogram is

$$\frac{|(n_1 - 0)(n_2 - 0)|}{|l_1m_2 - l_2m_1|} = \frac{|n_1n_2|}{\sqrt{(l_1m_2 + l_2m_1)^2 - 4l_1l_2m_1m_2}}$$
$$= \frac{|c|}{\sqrt{4h^2 - 4ab}} = \frac{|c|}{2\sqrt{h^2 - ab}}$$

43. Show that the triangle formed by the pair of lines $x^2 - 4xy + y^2 = 0$ and the line $x + y - 3 = 0$ is an equilateral triangle and find its area.

Solution: The equation of the lines $x^2 - 4xy + y^2 = 0$ can be written as

$$(x + y)^2 - 3(x - y)^2 = 0$$

Hence, by Problem 40, the pair of lines $x^2 - 4xy + y^2 = 0$ and the line $x + y - 3 = 0$ form an equilateral triangle whose area is

$$\frac{n^2}{\sqrt{3}(l^2+m^2)} = \frac{3\sqrt{3}}{2} \text{ sq. units}$$

44. If $S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of lines, then show that $h^2 = ab$, $af^2 = bg^2$ and the distance between these lines is

$$2\sqrt{\frac{g^2 - ac}{a(a+b)}}$$

which is also equal to

$$2\sqrt{\frac{f^2 - bc}{b(a+b)}}$$

Solution: Suppose the pair of parallel lines are $lx + my + n_1 = 0$ and $lx + my + n_2 = 0$. Therefore,

$$l^2 = a, 2lm = 2h, m^2 = b$$

$$l(n_1 + n_2) = 2g, m(n_1 + n_2) = 2f, c = n_1 n_2$$

Now, $$h^2 = l^2 m^2 = ab$$

and $$af^2 = l^2\left[\frac{m^2(n_1+n_2)^2}{4}\right]$$

$$= m^2\left[\frac{l(n_1+n_2)^2}{2}\right]^2 = bg^2$$

Also, the distance between the two parallel lines is (by Theorem 2.15)

$$\left|\frac{n_1 - n_2}{\sqrt{l^2+n^2}}\right| = \left|\frac{\sqrt{(n_1+n_2)^2 - 4n_1n_2}}{\sqrt{l^2+m^2}}\right|$$

$$= \left|\frac{\sqrt{\frac{4g^2}{l^2} - 4c}}{\sqrt{l^2+m^2}}\right|$$

$$= 2\left|\sqrt{\frac{g^2 - ac}{a(a+b)}}\right|$$

$$= 2\sqrt{\frac{g^2 - ac}{a(a+b)}}$$

If we use $n_1 + n_2 = 2f/m^2$, then the distance between the parallel lines is

$$2\sqrt{\frac{f^2 - bc}{b(a+b)}}$$

45. Show that the pair of lines $a^2x^2 + 2h(a+b)xy + b^2y^2 = 0$ are equally inclined to the pair of lines $ax^2 + 2hxy + by^2 = 0$.

Solution: Two pairs of lines ($\overrightarrow{PA}$, $\overrightarrow{PB}$) and ($\overrightarrow{PC}$, $\overrightarrow{PD}$) are said to be equally inclined to each other if both pairs have the same angle bisectors at point P. Now, for the given pairs of lines, origin is the common point. By Theorem 2.30, the equation of the pair of angle bisectors of the lines $a^2x^2 + 2h(a+b)xy + b^2y^2 = 0$ is

$$h(a+b)(x^2 - y^2) = (a^2 - b^2)xy$$

$$\Rightarrow h(x^2 - y^2) = (a - b)xy$$

which also represents the pair of angle bisectors of the line $ax^2 + 2hxy + by^2 = 0$. Hence, both pairs are equally inclined to each other.

46. Find the equation of the pair of lines passing through origin which are at a distance d units from a point $(x_1, y_1) \neq (0, 0)$.

Solution: See Fig. 2.50. Let P be (x_1, y_1) and $y = mx$ the line whose distance from P is equal to d. That is,

$$\left|\frac{y_1 - mx_1}{\sqrt{1+m^2}}\right| = d \qquad \text{(by Theorem 2.14)}$$

$$\Rightarrow (y_1 - mx_1)^2 = d^2(1+m^2)$$

$$\Rightarrow \left(y_1 - \frac{y}{x}x_1\right)^2 = d^2\left(1 + \frac{y^2}{x^2}\right) \qquad \left(\because m = \frac{y}{x}\right)$$

$$\Rightarrow (xy_1 - x_1y)^2 = d^2(x^2 + y^2)$$

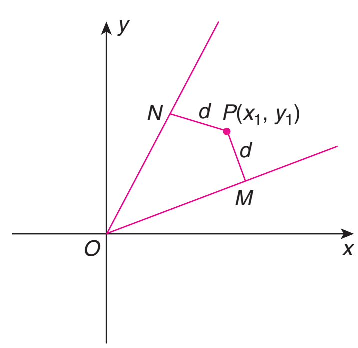

FIGURE 2.50

47. If the equation $x^2 + 5xy + 4y^2 + 3x + 2y + c = 0$ represents a pair of lines, then find the value of c and also the angle between the lines.

Solution: By Theorem 2.31, we have

$$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

This implies

$$\begin{vmatrix} 1 & \frac{5}{2} & \frac{3}{2} \\ \frac{5}{2} & 4 & 1 \\ \frac{3}{2} & 1 & c \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 2 & 5 & 3 \\ 5 & 8 & 2 \\ 3 & 2 & 2c \end{vmatrix} = 0$$

$$\Rightarrow 2(16c - 4) - 5(10c - 6) + 3(10 - 24) = 0$$

$$\Rightarrow -18c - 20 = 0$$

$$\Rightarrow c = -\frac{10}{9}$$

If α is the angle between the lines, then

$$\cos\alpha = \left| \frac{a+b}{\sqrt{(a-b)^2 + 4h^2}} \right| \quad \text{(by Theorem 2.28)}$$

$$= \left| \frac{1+4}{\sqrt{(1-4)^2 + 4\left(\frac{25}{4}\right)}} \right| = \frac{5}{\sqrt{9+25}}$$

Hence

$$\alpha = \mathrm{Cos}^{-1}\left(\frac{5}{\sqrt{34}}\right)$$

48. Show that the straight lines $ax^2 + 2hxy + by^2 = 0$ and the straight lines $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ form a rhombus if $(a - b)fg + h(f^2 - g^2) = 0$.

Solution: By Theorem 2.32, the given pairs of lines form a parallelogram. Also, by Problem 41, the diagonals of the parallelogram are

$$2gx + 2fy + c = 0$$

and $$y(hf - bg) = x(gh - af)$$

The parallelogram is a rhombus if the diagonals are at right angles (see Fig. 2.51). That is, if the product of their slopes is equal to -1:

$$-\frac{g}{f}\left(\frac{gh - af}{hf - bg}\right) = -1$$

$$\Rightarrow g^2h - afg = f^2h - bfg$$

$$\Rightarrow h(f^2 - g^2) + fg(a - b) = 0$$

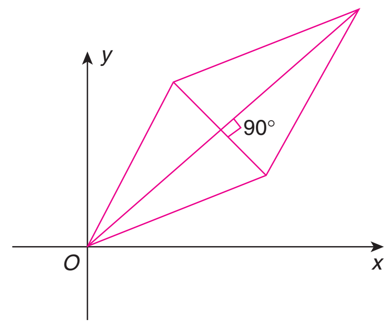

FIGURE 2.51

49. Show that the four lines given by the equations $2x^2 + 3xy - 2y^2 = 0$ and $2x^2 + 3xy - 2y^2 + 3x + y + 1 = 0$ form a square.

Solution: We know that the lines represented by the equations are

$$2x - y = 0$$

$$x + 2y = 0$$

$$2x - y + 1 = 0$$

and $$2x + 2y + 1 = 0$$

Solving $2x - y = 0$ and $x + 2y + 1 = 0$, we have

$$x = -\frac{3}{5} \text{ and } y = -\frac{1}{5}$$

so that the slope of diagonal through origin is 1/3. Since the other diagonal is $3x + y + 1 = 0$, the diagonals are at right angles. Therefore, the parallelogram is a square.

50. Find the centroid of the triangle formed by the lines $12x^2 - 20xy + 7y^2 = 0$ and $2x - 3y + 4 = 0$.

Solution: We have

$$12x^2 - 20xy + 7y^2 \equiv (2x - y)(6x - 7y)$$

Therefore, the sides of the triangle are

$$2x - y = 0 \tag{2.69}$$

$$6x - 7y = 0 \qquad (2.70)$$

$$2x - 3y = -4 \qquad (2.71)$$

Solving Eqs. (2.69) and (2.71), we get (1, 2) as the vertex. Solving Eqs. (2.70) and (2.71), we obtain (7, 6) as another vertex. By hypotheses, (0, 0) is the third vertex. Hence, the centroid of the triangle is

$$\left(\frac{0+1+7}{3}, \frac{0+2+6}{3}\right) = \left(\frac{8}{3}, \frac{8}{3}\right)$$

51. If $S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of lines equidistant from the origin, then show that $f^4 - g^4 = c(bf^2 - ag^2)$.

Solution: Suppose the lines are $l_1x + m_1y + n_1 = 0$ and $l_2x + m_2y + n_2 = 0$. Therefore

$$l_1l_2 = a$$

$$l_1m_2 + l_2m_1 = 2h$$

$$m_1m_2 = b$$

$$l_1n_2 + l_2n_1 = 2g$$

$$m_1n_2 + m_2n_1 = 2f$$

$$n_1n_2 = c$$

By hypothesis,

$$\frac{|n_1|}{\sqrt{l_1^2 + m_1^2}} = \frac{|n_2|}{\sqrt{l_2^2 + m_2^2}}$$

$$\Rightarrow n_1^2(l_2^2 + m_2^2) = n_2^2(l_1^2 + m_1^2)$$

(squaring and cross-multiplying)

$$\Rightarrow l_1^2n_2^2 - l_2^2m_1^2 = m_2^2n_1^2 - m_1^2n_2^2$$

$$\Rightarrow (l_1n_2 + l_2n_1)(l_1n_2 - l_2n_1) = (m_2n_1 + m_1n_2)(m_2n_1 - m_1n_2)$$

$$\Rightarrow 2g(l_1n_2 - l_2n_1) = 2f(m_2n_1 - m_1n_2)$$

$$\Rightarrow g^2(4g^2 - 4ac) = f^2(4f^2 - 4bc)$$

$$\Rightarrow f^4 - g^4 = bcf^2 - cag^2 = c(bf^2 - ag^2)$$

WORKED-OUT PROBLEMS

Single Correct Choice Type Questions

1. If the line $3ax + 5y + a - 2 = 0$ passes through the point $(-1, 4)$, then value of a is

(A) 9 (B) 7 (C) –9 (D) –7

Solution: Since the line passes through (–1, 4), we have

$$3a(-1) + 5(4) + a - 2 = 0$$

$$\Rightarrow -2a + 18 = 0$$

Hence, $a = 9$ and the line is $27x + 5y + 7 = 0$.

Answer: (A)

2. A line has slope $-3/4$, positive y-intercept and forms a triangle of area 24 sq. units with coordinate axes. Then, the equation of the line is

(A) $3x + 4y + 24 = 0$ (B) $3x + 4y - 24 = 0$
(C) $3x + 4y - 25 = 0$ (D) $3x + 4y + 25 = 0$

Solution: Let the line be

$$y = \left(-\frac{3}{4}\right)x + c$$

so that the intercepts on the x and y axes, respectively, are $4c/3$ and c. Therefore, the area of the triangle (by Quick Look 4) is

$$\frac{1}{2}\left|\frac{4c}{3}\cdot c\right| = 24 \Rightarrow c^2 = 36 \Rightarrow c = \pm 6$$

Since y-intercept is positive, the value c is 6 and the equation of the line is $3x + 4y - 24 = 0$.

Answer: (B)

3. A non-horizontal line passing through the point $(4, -2)$ and whose distance from the origin is 2 units is

(A) $3x + 4y - 10 = 0$ (B) $x + y - 2 = 0$
(C) $4x + 3y - 10 = 0$ (D) $2x + 3y - 2 = 0$

Solution: Let the slope of the line be m. Now the equation of the line, by Theorem 2.2, is

$$y + 2 = m(x - 4)$$

Therefore, by Theorem 2.14,

$$2 = \frac{|m(0-4) - 0 - 2|}{\sqrt{m^2 + 1}}$$

$$\Rightarrow (2m + 1)^2 = m^2 + 1$$

$$\Rightarrow 3m^2 + 4m = 0$$

$$\Rightarrow m = 0, -\frac{4}{3}$$

When $m = 0$, the line is

$$y + 2 = 0$$

which is horizontal. When $m = -4/3$, the line is

$$y + 2 = -\frac{4}{3}(x - 4)$$

$$\Rightarrow 4x + 3y - 10 = 0$$

Answer: (C)

4. The positive value of k such that the distance of the line $8x + 15y + k = 0$ from the point $(2, 3)$ is 5 units is

(A) 12 (B) 6 (C) 8 (D) 24

Solution: By Theorem 2.14, we have

$$\frac{|8(2) + 15(3) + k|}{\sqrt{8^2 + 15^2}} = 5$$

$$\Rightarrow 61 + k = \pm(5 \times 17) = \pm 85$$

$$\Rightarrow k = 24 \text{ or } -146$$

Since $k > 0$, its value is 24.

Answer: (D)

5. Equation of the line passing through the point $(2, -3)$ and parallel to the line joining the points $(4, 1)$ and $(-2, 2)$ is

(A) $x + 6y + 12 = 0$ (B) $x + 6y - 12 = 0$
(C) $x + 6y - 16 = 0$ (D) $x + 6y + 16 = 0$

Solution: Slope of the line joining the points $(4, 1)$ and $(-2, 2)$ is

$$\frac{2-1}{-2-4}\left(= -\frac{1}{6}\right)$$

Hence, by Theorem 2.2, the equation of the given line is

$$y + 3 = \frac{1}{6}(x - 2)$$

$$\Rightarrow x + 6y + 16 = 0$$

Answer: (D)

6. Equation of the line passing through the point $(-2, 3)$ and perpendicular to the line $2x - 3y + 6 = 0$ is

(A) $3x - 2y = 0$ (B) $2x + 3y = 0$
(C) $3x + 2y = 0$ (D) $2x - 3y = 0$

Solution: By Quick Look 7, part (3), the equation of the line is

$$-3(x + 2) - 2(y - 3) = 0$$

$$\Rightarrow 3x + 2y = 0$$

Answer: (C)

7. Let $A(h, k)$, $B(1, 1)$ and $C(2, 1)$ be the vertices of a right-angled triangle with AC as its hypotenuse. If the area of the triangle is 1, then the set of values of k is given by

(A) $\{0, 2\}$ (B) $\{-1, 3\}$ (C) $\{-3, -2\}$ (D) $\{1, 3\}$

Solution: See Fig. 2.52. Since BC is horizontal line ($y = 1$) and AB is perpendicular to BC, it follows that AB is a vertical line passing through $B(1, 1)$. Hence, $h = 1$. Therefore

$$1 = \text{Area of } \Delta ABC = (AB)(BC)$$

$$= \frac{1}{2}|k - 1|(1)$$

So, $k - 1 = \pm 2$ or $k = -1, 3$.

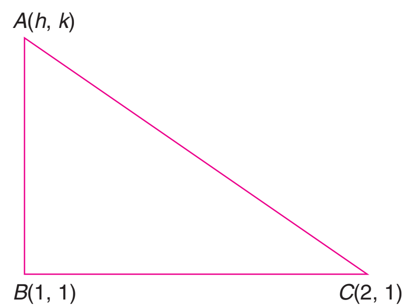

FIGURE 2.52

Answer: (B)

8. If the perpendicular bisector of the line segment joining $P(1, 4)$ and $Q(\alpha, 3)$ has y-intercept -4, then a possible value of k is

(A) -2 (B) -4 (C) 1 (D) 2

Solution: Since every point (x, y) on the perpendicular bisector of $\overline{PQ}$ is equidistant from both P and Q we have

$$(x-1)^2 + (y-4)^2 = (x-\alpha)^2 + (y-3)^2$$

$$\Rightarrow -2x - 8y + 17 = -2\alpha x - 6y + \alpha^2 + 9$$

Therefore, equation of the perpendicular bisector of $\overline{PQ}$ is

$$2(\alpha - 1)x + 8 = 2y + \alpha^2$$

Hence

$$-4 = y\text{-intercept} = \frac{8 - \alpha^2}{2}$$

Therefore, $\alpha^2 = 16$ or $\alpha = \pm 4$.

Answer: (B)

9. The lines

$$\lambda(\lambda^2 + 1)x - y + b = 0$$

and $(\lambda^2+1)^2x+(\lambda^2+1)y+2b=0$

are perpendicular to a common line for

(A) exactly one value of λ
(B) exactly two values of λ
(C) more than two values of λ
(D) no value of λ

Solution: By hypothesis, the given lines are parallel. Hence

$$\lambda(\lambda^2+1)=\frac{-(\lambda^2+1)^2}{\lambda^2+1}$$

Therefore $\lambda=-1$.

Answer: (A)

10. The number of integral values of m for which the x-coordinate of the point of intersection of the lines $3x+4y=9$ and $y=mx+1$ is also an integer is

(A) 2 (B) 0 (C) 4 (D) 1

(IIT-JEE 2001)

Solution: Solving the given equations, the x-coordinate of their point of intersection is

$$\frac{5}{3+4m}=-5,-1$$

when $m=-1$ and -2, respectively. That is for $m=-1$ and -2, the x-coordinate is also an integer. Hence, the number of integral values is 2.

Answer: (A)

11. The locus of the orthocentre of the triangle formed by the lines $y=0$ (i.e., x-axis) and the lines

$$(1+p)x-py+p(1+p)=0$$
$$(q+1)x-qy+q(1+q)=0$$

where $p\neq q$, is

(A) the line $x+y=0$
(B) the line $x-y=0$
(C) the curve $y^2=4pq(p+q)(-pq)q$
(D) the curve $\dfrac{x^2}{p(p+q)}+\dfrac{y^2}{q(p+q)}=1$

(IIT-JEE 2009)

Solution: The given lines are

$$y=0 \tag{2.72}$$
$$(1+p)x-py+p(1+p)=0 \tag{2.73}$$
$$(1+q)x-qy+q(1+q)=0 \tag{2.74}$$

Solving the equations which are taken pairwise, the vertices of the triangle are

$$A(-p,0), B(-q,0) \text{ and } C[pq,(p+1)(q+1)]$$

Since the side AB is along the x-axis, altitude CN, drawn from C to the side, is a vertical line so that the equation of CN is

$$x=pq \tag{2.75}$$

Equation of altitude AM is

$$y-0=-\frac{(pq+q)}{(p+1)(q+1)}(x+p)$$

$$\Rightarrow y=\frac{-q}{q+1}(x+p) \tag{2.76}$$

Solving Eqs. (2.75) and (2.76), we get the orthocentre as

$$y=\frac{-q}{q+1}(pq+p)=-pq$$

If (x,y) is the orthocentre, then we have $x=pq$ and $y=-pq$ so that the locus of the orthocentre is $y=-x$ or $x+y=0$

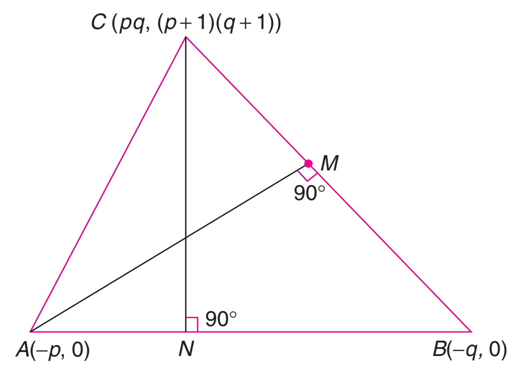

FIGURE 2.53

Answer: (A)

12. Two adjacent sides of a parallelogram are $4x+5y=0$ and $7x+2y=0$. If the equation to one diagonal is $11x+7y-9=0$, then the equation of the other diagonal is

(A) $x+y=0$ (B) $x-y=0$
(C) $2x-11y+9=0$ (D) $x-y-9=0$

Solution: See Fig. 2.54. We have

$$4x+5y=0 \tag{2.77}$$
$$7x+2y=0 \tag{2.78}$$
$$11x+7y=9 \tag{2.79}$$

Solving Eqs. (2.77) and (2.79), we get

$$A=\left(\frac{5}{3},-\frac{4}{3}\right)$$

Also solving Eqs. (2.78) and (2.79), we get

$$C = \left(-\frac{2}{3}, \frac{7}{3}\right)$$

Now

$$A\left(\frac{5}{3}, -\frac{4}{3}\right), O(0,0) \text{ and } C\left(-\frac{2}{3}, \frac{7}{3}\right)$$

are three consecutive vertices of a parallelogram. The fourth vertex B is

$$\left(\frac{5}{3} - 0 - \frac{2}{3}, -\frac{4}{3} - 0 + \frac{7}{3}\right) = (1, 1)$$

Hence, the other diagonal is $y = x$.

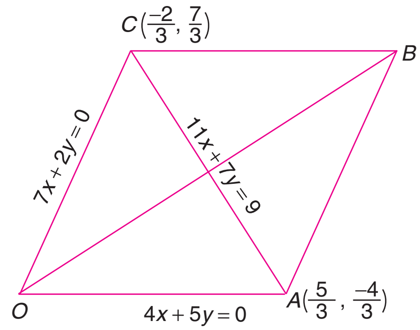

FIGURE 2.54

Answer: (B)

13. The vertices B and C of ΔABC lie on the lines $4y - 3x = 0$ and $y = 0$, respectively, and the side BC passes through the point $P(0, 5)$. If $ABOC$ is a rhombus where O is the origin and P is an internal point to the rhombus, then the vertex A is

(A) $(3, 9)$ (B) $(9, 3)$

(C) $(3, -9)$ (D) $(-3, 9)$

Solution: See Fig. 2.55. Let $B = (x, 3x/4)$ and $C = (h, 0)$. Since B, P and C are collinear, from Quick Look 2, Chapter 1, we have

$$\begin{vmatrix} x & 0 & h \\ \frac{3x}{4} & 5 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

$$\Rightarrow 5x + h\left(\frac{3x}{4} - 5\right) = 0$$

$$\Rightarrow 20x + h(3x - 20) = 0$$

$$\Rightarrow x(20 + 3h) = 20h$$

$$\Rightarrow x = \frac{20h}{20 + 3h} \tag{2.80}$$

Also

$$OB^2 = OC^2$$

$$\Rightarrow x^2 + \frac{9x^2}{16} = h^2$$

$$\Rightarrow \frac{25x^2}{16} = h^2$$

$$\Rightarrow \frac{25}{16} \cdot \frac{400h^2}{(20 + 3h)^2} = h^2$$

$$\Rightarrow 25^2 = (20 + 3h)^2$$

$$\Rightarrow 20 + 3h = \pm 25$$

$$\Rightarrow h = \frac{5}{3}, -15$$

When $h = 5/3$

$$x = \frac{20h}{20 + 3h} = \frac{20\left(\frac{5}{3}\right)}{20 + 3\left(\frac{5}{3}\right)} = \frac{100}{75} = \frac{4}{3}$$

When $h = -15$

$$x = \frac{20(-15)}{20 + 3(-15)} = \frac{-300}{-25} = 12$$

Hence, the coordinates of B and C, respectively, are either $\left(\frac{4}{3}, 1\right)$ and $\left(\frac{5}{3}, 0\right)$ or $(12, 9)$ and $(-15, 0)$.

Two cases arise:

Case 1: $B = \left(\frac{4}{3}, 1\right)$, $P = (0, 5)$ and $C = \left(\frac{5}{3}, 0\right)$

P is an inside point and B, P and C are collinear. This implies

$$BP + PC = BC$$

$$BP + PC = \sqrt{\frac{16}{9} + 16} + \sqrt{\frac{25}{9} + 25}$$

$$= \frac{4\sqrt{10}}{3} + \frac{5\sqrt{10}}{3} = 3\sqrt{10}$$

and $$BC = \sqrt{\frac{1}{9} + 1} = \frac{\sqrt{10}}{3}$$

Therefore, $B \neq \left(\frac{4}{3}, 1\right)$ and $C \neq \left(\frac{5}{3}, 0\right)$

Case 2: $B = (12, 9)$, $P = (0, 5)$ and $C = (-15, 0)$. Now

$$BP = \sqrt{12^2 + 4^2} = \sqrt{160} = 4\sqrt{10}$$

$$PC = \sqrt{225 + 25} = \sqrt{250} = 5\sqrt{10}$$

$$BC = \sqrt{27^2 + 9^2} = \sqrt{729 + 81} = \sqrt{810} = 9\sqrt{10}$$

Therefore,

$$BP + PC = 9\sqrt{10} = BC$$

Hence

$$A = (12 - 0 - 15, 9 - 0 - 0) = (-3, 9)$$

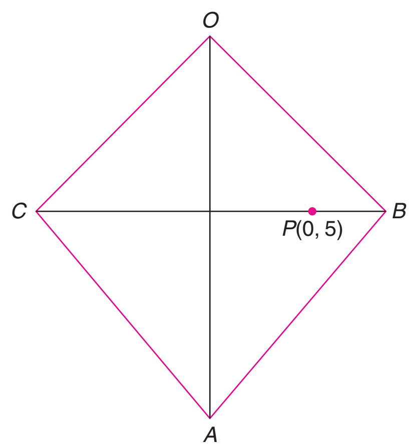

FIGURE 2.55

Answer: (D)

14. Consider the points

$$P = [-\sin(\beta - \alpha), -\cos\beta]$$
$$Q = [\cos(\beta - \alpha), \sin\beta]$$

and $$R = [\cos(\beta - \alpha + \theta), \sin(\beta - \theta)]$$

where $0 < \alpha, \beta, \theta < \pi/4$. Then

(A) P lies on the segment RQ
(B) Q lies on the segment PR
(C) R lies in the segment QP
(D) P, Q and R are non-collinear

(IIT-JEE 2008)

Solution: It is known that three points (x_1, y_1), (x_2, y_2) and (x_3, y_3) are collinear if and only if

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

Take $P = (x_1, y_1)$, $Q = (x_2, y_2)$ and $R = (x_3, y_3)$. Therefore,

$$x_3 = \cos(\beta - \alpha + \theta) = \cos\theta\cos(\beta - \alpha) - \sin\theta\sin(\beta - \alpha)$$
$$= x_2\cos\theta + x_1\sin\theta$$

and

$$y_3 = \sin(\beta - \theta) = \sin\beta\cos\theta - \sin\theta\cos\beta$$
$$= y_2\cos\theta + y_1\sin\theta$$

Therefore, from $R_3 - R_1\sin\theta - R_2\cos\theta$, we have

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ 0 & 0 & 1-(\sin\theta+\cos\theta) \end{vmatrix}$$
$$= (1 - \sin\theta - \cos\theta)(x_1y_2 - x_2y_1) = 0$$

This implies that $x_1y_2 - x_2y_1 = 0$ or $\sin\theta + \cos\theta = 1$. Since $0 < \theta < \pi/4$, $\sin\theta + \cos\theta \neq 1$. Therefore, $x_1y_2 - x_2y_1 = 0$. This implies that

$$-\sin(\beta - \alpha)\sin\beta + \cos\beta\cos(\beta - \alpha) = 0$$
$$\Rightarrow \cos(\beta - \alpha + \beta) = 0$$

Hence, $2\beta - \alpha = \pi/2$ which is impossible because $0 < \alpha$ and $\beta < \pi/4$. Therefore

$$2\beta - \alpha \neq \pi/2$$

Thus $x_1y_2 - x_2y_1 \neq 0$. Hence, the points P, Q and R are non-collinear.

Answer: (D)

15. Let $O(0, 0)$, $P(3, 4)$ and $Q(6, 0)$ be the vertices of a triangle OPQ. A point R is inside the triangle such that the triangles OPR, PQR and OQR are of equal areas. Then, the coordinates of R are

(A) $\left(\frac{4}{3}, 3\right)$ (B) $\left(3, \frac{2}{3}\right)$
(C) $\left(3, \frac{4}{3}\right)$ (D) $\left(\frac{2}{3}, \frac{4}{3}\right)$

(IIT-JEE 2007)

Solution: Let ΔABC be any triangle. P is an inside point such that the triangles PBC, PCA and PAB are of equal areas if and only if P is the centroid of ΔABC. Hence, R must be the centroid of ΔOPQ and so

$$R = \left(3, \frac{4}{3}\right)$$

Answer: (C)

16. If none of two lines among the following three lines

$$ax + by - (a + b) = 0$$
$$bx - (a + b)y + a = 0$$

and $$(a + b)x - ay - b = 0$$

are parallel then the lines

(A) are concurrent
(B) form a right-angled triangle

(C) form an equilateral triangle

(D) form a triangle with circumcentre $(a+b, a+b)$

Solution: By adding C_1 and C_2 to C_3, we get

$$\begin{vmatrix} a & b & -(a+b) \\ b & -(a+b) & a \\ a+b & -a & -b \end{vmatrix} = \begin{vmatrix} a & b & 0 \\ b & -(a+b) & 0 \\ a+b & -a & 0 \end{vmatrix} = 0$$

Hence by Theorem 2.22, the lines are concurrent.

Answer: (A)

17. If the line $2x+3y+4+\lambda(6x-y+12)=0$ is perpendicular to the line $7x+5y-4=0$, then the value of λ is

(A) $\frac{29}{37}$ (B) $\frac{25}{27}$ (C) $\frac{-25}{27}$ (D) $\frac{-29}{37}$

Solution: Slope of the given line is

$$-\frac{(2+6\lambda)}{3-\lambda}$$

This line is perpendicular to the line $7x+5y-4=0$. This implies that

$$\frac{-(2+6\lambda)}{3-\lambda} \times \left(\frac{-7}{5}\right) = -1$$

$$\Rightarrow 14+42\lambda = -15+5\lambda$$

$$\Rightarrow 37\lambda = -29$$

$$\Rightarrow \lambda = -\frac{29}{37}$$

Answer: (D)

18. The points $A(1,3)$ and $C(5,1)$ are extremities of a diagonal of a rectangle $ABCD$. The other two vertices B and D lie on the line $y=2x+c$. Then, the value of c is

(A) 2 (B) 4 (C) –4 (D) –2

Solution: See Fig. 2.56. Let $M=(3,2)$ be the intersection of the diagonals. Since $M(3,2)$ lies on the diagonal BD whose equation is given as $y=2x+c$, we have

$$2 = 2(3)+c$$

$$\Rightarrow c = -4$$

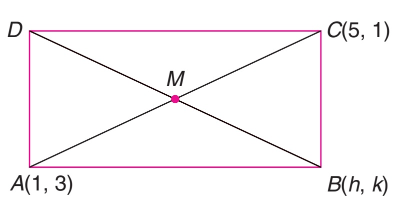

FIGURE 2.56

Answer: (C)

19. Sum of the slopes of the lines which make 45° with the line $3x-y+5=0$ is

(A) $\frac{3}{2}$ (B) $-\frac{3}{2}$ (C) -2 (D) $\frac{1}{2}$

Solution: Let m be the slope of the required line. Therefore

$$1 = \tan 45° = \left|\frac{m-3}{1+m(3)}\right|$$

$$\Rightarrow 3m+1 = \pm(m-3)$$

Hence, $m=-2,\ 1/2$ so that their sum is

$$-2+\frac{1}{2} = -\frac{3}{2}$$

Answer: (B)

20. Let PS be the median of the triangle with vertices $P(2,2)$, $Q(6,-1)$ and $R(7,3)$. The equation of the line passing through $(1,-1)$ and parallel to PS is

(A) $2x-9y-7=0$ (B) $2x-9y-11=0$

(C) $2x+9y-11=0$ (D) $2x+9y+7=0$

(IIT-JEE 2000)

Solution: See Fig. 2.57. $P=(2,2)$ and $S=(13/2,1)$. Therefore, the slope of the median PS is

$$\frac{2-1}{2-(13/2)} = \frac{-2}{9}$$

Hence, the equation of the line through the point $(1,-1)$ and having slope $-2/9$ is

$$y+1 = \frac{-2}{9}(x-1)$$

$$\Rightarrow 2x+9y+7=0$$

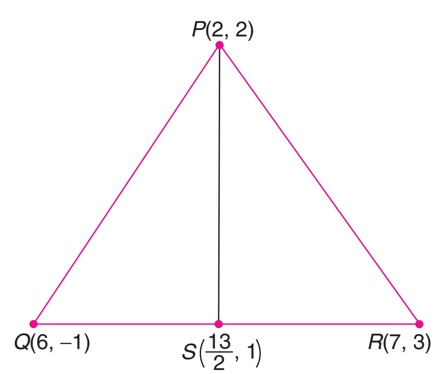

FIGURE 2.57

Answer: (D)

21. The orthocentre of the triangle formed by the lines $xy = 0$ and $x + y = 1$ is

(A) $\left(\frac{1}{2}, \frac{1}{2}\right)$ (B) $\left(\frac{1}{3}, \frac{1}{3}\right)$

(C) $(0, 0)$ (D) $\left(\frac{1}{4}, \frac{1}{4}\right)$

(IIT-JEE 1995)

Solution: See Fig. 2.58. By hypothesis, the sides of the triangle are $y = 0, x = 0$ and $x + y = 1$. Therefore, the triangle is a right-angled triangle at the origin. Now for a right-angled triangle, the right-angled vertex is its orthocentre. Hence, origin is the orthocentre of the given triangle.

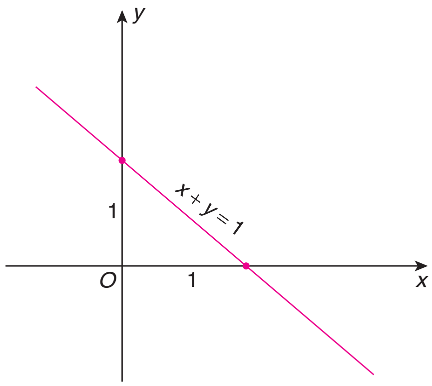

FIGURE 2.58

Answer: (C)

22. In ΔABC, $\angle C = 90°$ and the side AB is of length 60 units. The equations of the medians AD and BE, respectively, are $y = x + 3$ and $y = 2x + 4$. Then the area of ΔABC (in square units) is

(A) 200 (B) 300 (C) 400 (D) 500

Solution: See Fig. 2.59. By hypothesis, we have

$$a^2 + b^2 = 60^2 \tag{2.81}$$

Let θ be the acute angle between the medians AD and BE. Therefore

$$\tan\theta = \left|\frac{2-1}{1+2(1)}\right| = \frac{1}{3} \tag{2.82}$$

From the quadrilateral $DGEC$, we have

$$\underline{|DGE} + \underline{|GDC} + 90° + \underline{|GEC} = 360°$$

Hence

$$(\pi - \theta) + \underline{|GDC} + \frac{\pi}{2} + \underline{|GEC} = 2\pi$$

$$\Rightarrow \underline{|GDC} + \underline{|GEC} = \frac{\pi}{2} + \theta \tag{2.83}$$

Thus, from Eq. (2.82), we have

$$\begin{aligned} -3 &= -\cot\theta \\ &= \tan\left(\frac{\pi}{2} + \theta\right) \\ &= \tan(\underline{|D} + \underline{|E}) \qquad \text{[by Eq.(2.83)]} \\ &= \frac{\tan\underline{|D} + \tan\underline{|E}}{1 - \tan\underline{|D}\tan\underline{|E}} \\ &= \frac{(AC/DC) + (BC/EC)}{1 - (AC/DC)\cdot(BC/EC)} \\ &= \frac{(2b/a) + (2a/b)}{(1-4)} \\ &= \frac{2(a^2+b^2)}{-3ab} = \frac{2(60)^2}{-3ab} \end{aligned} \tag{2.84}$$

Therefore, from Eq. (2.84), the area of ΔABC is

$$\frac{1}{2}ab = \frac{60^2}{9} = 400$$

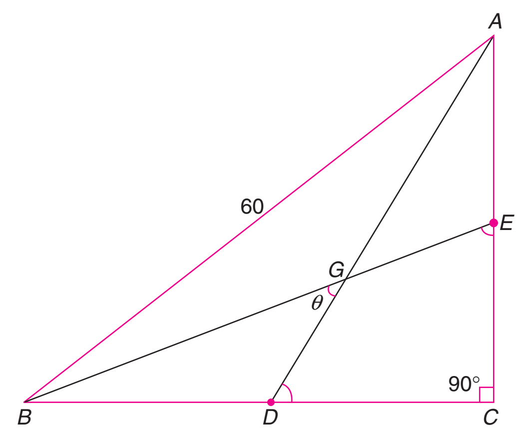

FIGURE 2.59

Answer: (C)

23. A straight line L with negative slope passes through the point (8, 2) and cuts the positive coordinate axes at points P and Q. Then the absolute minimum value of $OP + OQ$ as L varies (where O is the origin) is

(A) 6 (B) 9 (C) 12 (D) 18

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Solution: Let the equation of the line L, by hypothesis, be

$$y - 2 = m(x - 8)$$

where $m < 0$. Therefore

$$p = \left(8 - \frac{2}{m}, 0\right) \text{ and } Q = (0, 2 - 8m)$$

Now,

$$OP + OQ = \left(8 - \frac{2}{m}\right) + (2 - 8m) \quad (\because m < 0)$$
$$= 10 - \left(\frac{2}{m} + 8m\right) \geq 10 + 2\sqrt{\frac{-2}{m} \times (-8m)}$$
$$= 10 + 8 \quad (\because AM \geq GM)$$

and equality occurs if and only if

$$-\frac{2}{m} = -8m \text{ or } m = -\frac{1}{2}$$

Hence, the absolute minimum of $OP + OQ$ is $12 + 6 = 18$.

Answer: (D)

24. A straight line through the origin O meets the parallel lines $4x + 2y = 9$ and $2x + y + 6 = 0$ at points P and Q, respectively. Then the point O divides the segment PQ in the ratio

(A) 1:2 (B) 3:4 (C) 2:1 (D) 4:3

Solution: See Fig. 2.60. The line $4x + 2y = 9$ has positive interception coordinates while the line $2x + y + 6 = 0$ has negative interception on the axes. Hence, origin O lies in between the axes. Suppose OM and ON, respectively, are drawn perpendicular to the given two lines. Observe O, M and N are collinear (see Fig. 2.60). Now,

$$OM = \frac{9}{2\sqrt{5}} \quad \text{and} \quad ON = \frac{6}{\sqrt{5}}$$

If any line through O meets the parallel lines in P and Q, then by pure geometry, we have

$$OP : OQ = OH : ON = \frac{9}{2\sqrt{5}} : \frac{6}{\sqrt{5}} = 3:4$$

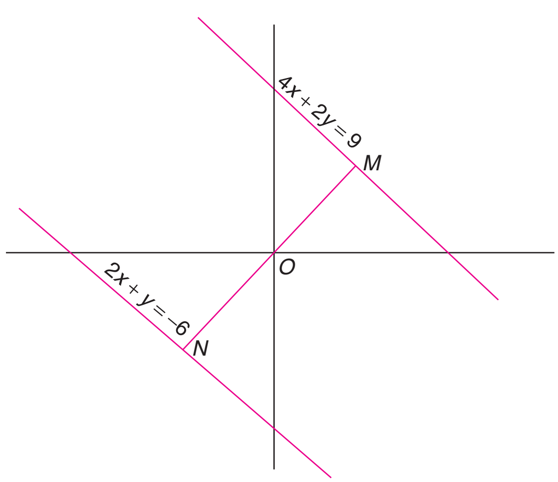

FIGURE 2.60

Answer: (B)

25. The equations of a pair of opposite sides of a parallelogram are $x^2 - 5x + 6 = 0$ and $y^2 - 6y + 5 = 0$. The equations of its diagonals are

(A) $x + 4y = 13, y = 4x - 17$
(B) $4x + y = 13, 4y = x - 7$
(C) $4x + y = 13, y = 4x - 7$
(D) $y - 4x = 13, y + 4x = 7$

Solution: See Fig. 2.61. We have

$$x^2 - 5x + 6 \equiv (x - 2)(x - 3)$$
$$y^2 - 6y + 5 \equiv (y - 1)(y - 5)$$

Therefore, the sides of the parallelogram are

$$x = 2, x = 3 \text{ and } y = 1, y = 5$$

Therefore, the vertices of the rectangle are $A(2, 1), B(3, 1)$, $C(3, 5)$ and $D(2, 5)$. Hence, the equation of AC is

$$y - 1 = \frac{5 - 1}{3 - 2}(x - 2) = 4(x - 2)$$
$$\Rightarrow 4x - y - 7 = 0$$

Equation of BD is

$$y - 1 = \frac{5 - 1}{2 - 3}(x - 3) = -4(x - 3)$$
$$\Rightarrow 4x + y = 13$$

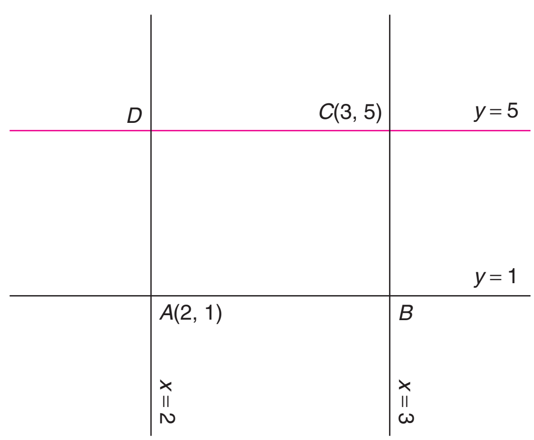

FIGURE 2.61

Answer: (C)

26. The line $x - y - 2 = 0$ divides the segment joining the points $A(3, -1)$ and $B(8, 9)$ in the ratio

(A) 2:3 (B) 2:1 (C) 1:2 (D) 3:4

Solution: See Fig. 2.62. Suppose the line $x - y - 2 = 0$ meets the line joining A and B in P and $AP:PB = \lambda:1$. Therefore,

$$P = \left(\frac{8\lambda + 3}{\lambda + 1}, \frac{9\lambda - 1}{\lambda + 1}\right)$$

Since P lies on the line $x - y - 2 = 0$, we have

$$\frac{8\lambda+3}{\lambda+1} - \frac{9\lambda-1}{\lambda+1} - 2 = 0$$

$$\Rightarrow -3\lambda + 3 + 1 - 2 = 0$$

$$\Rightarrow \lambda = \frac{2}{3}$$

Hence, $AP:PB = 2:3$.

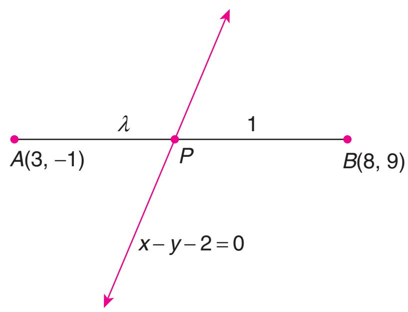

FIGURE 2.62

Answer: (A)

Note: Using Theorem 2.17, the ratio $AP:PB$ is $-L_{11}:L_{22}$ where

$$L \equiv x - y - 2 = 0, A = (3, -1) \text{ and } B = (8, 9)$$

Now,

$$L_{11} = 3 - (-1) - 2 = 2$$

and

$$L_{22} = 8 - 9 - 2 = -3$$

so that

$$AP:PB = -L_{11}:L_{22}$$

$$\Rightarrow AP:PB = (-2):(-3) = 2:3$$

27. The ratio in which the line $3x - y + 6 = 0$ divides the line joining the points $A(3, 4)$ and $B(-2, 1)$ is

(A) 2:11 (B) 9:2 (C) 11:1 (D) 1:12

Solution: $L \equiv 3x - y + 6 = 0, A = (3, 4), B = (-2, 1)$

$$L_{11} = 3(3) - 4 + 6 = 11, L_{22} = 3(-2) - 1 + 6 = -1$$

Hence, by Theorem 2.17, the ratio $AP:PB$ is

$$-L_{11}:L_{22} = (-11):(-1) = 11:1$$

Answer: (C)

28. One of the diameters of the circle circumscribing the rectangle $ABCD$ is $4y = x + 7$. If A and B are the points $(-3, 4)$ and $(5, 4)$, respectively, then the area of the rectangle in square units is

(A) 22 (B) 32 (C) 42 (D) 26

Solution: See Fig. 2.63. Let M be the midpoint of AB so that $M = (1, 4)$. Clearly, A and B do not lie on the diameter $4y - x - 7 = 0$. If O is the centre of the circle, then we know that

$$AD = 2(OM) \quad (2.85)$$

Since $\overleftrightarrow{AB}$ is a horizontal line, $\overleftrightarrow{OM}$ is a vertical line and hence the equation of $\overleftrightarrow{OM}$ is $x = 1$. Therefore, the line OM and the diameter intersect at O and we have $O = (1, 2)$. Now

$$AD = 2(OM) = 2(2) = 4$$

By hypothesis, AB is 8. Thus, the area of the rectangle $ABCD = 8 \times 4 = 32$.

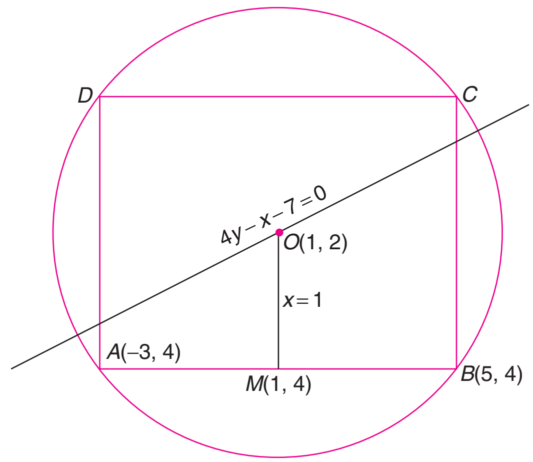

FIGURE 2.63

Answer: (B)

29. The vertices of a triangle are $A(-1, -7)$, $B(5, 1)$ and $C(1, 4)$. The equation of the bisector of the angle $\underline{|ABC}$ is

(A) $2y = x + 7$ (B) $7y = x + 2$
(C) $3y = x + 7$ (D) $7y = x + 3$

Solution: We have

$$AB = \sqrt{(5+1)^2 + (1+7)^2} = \sqrt{100} = 10$$

and

$$BC = \sqrt{(5-1)^2 + (1-4)^2} = \sqrt{25} = 5$$

Suppose the bisector of $\underline{|ABC}$ meets the side AC at D (see Fig. 2.64). Therefore, $CD:DA = BC:BA = 5:10 = 1:2$. Hence

$$D = \left(\frac{2-1}{3}, \frac{-7+8}{3}\right) = \left(\frac{1}{3}, \frac{1}{3}\right)$$

Since $B = (5, 1)$ and $D = (1/3, 1/3)$, the equation of the bisector BD of $\underline{|ABC}$ is

$$y - 1 = \frac{1-(1/3)}{5-(1/3)}(x - 5)$$

$$\Rightarrow y - 1 = \frac{2}{14}(x-5) = \frac{1}{7}(x-5)$$

Hence, $7y = x + 2$.

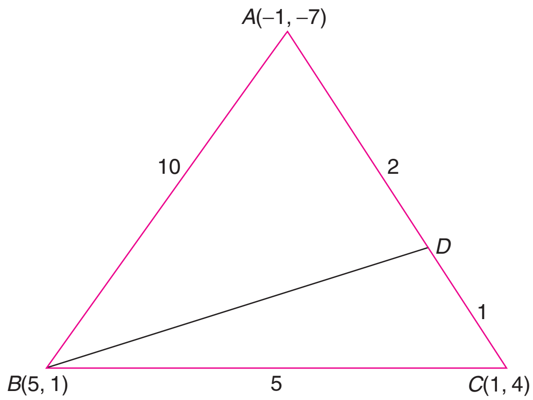

FIGURE 2.64

Answer: (B)

30. Let the algebraic sum of the perpendicular distances from the points $(2, 0)$, $(0, 2)$ and $(1, 1)$ onto a variable line be zero. Then, the line passes through a fixed point whose coordinates are

(A) $(1, 1)$ (B) $(-1, -1)$

(C) $\left(\frac{2}{3}, \frac{1}{3}\right)$ (D) $\left(\frac{1}{3}, \frac{2}{3}\right)$

Solution: Algebraic distance of a point (x_1, y_1) from a line $ax + by + c = 0$ is

$$\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}$$

Suppose the variable line is $ax + by + c = 0$. Therefore, by hypothesis,

$$\frac{2a + 0 + c}{\sqrt{a^2 + b^2}} + \frac{0 + 2b + c}{\sqrt{a^2 + b^2}} + \frac{a + b + c}{\sqrt{a^2 + b^2}} = 0$$

$$\Rightarrow 3a + 3b + 3c = 0$$

$$\Rightarrow a + b + c = 0$$

Hence, the line $ax + by + c = 0$ passes through the point $(1, 1)$.

Answer: (A)

Note: If the algebraic distances of the points (x_r, y_r) $(r = 1, 2, ..., n)$ from a straight line l are zero, then the line l passes through the point $(\bar{x}, \bar{y})$ where

$$\bar{x} = \frac{x_1 + x_2 + \cdots + x_n}{n} \quad \text{and} \quad \bar{y} = \frac{y_1 + y_2 + \cdots + y_n}{n}$$

31. Lines $L_1: y - x = 0$ and $L_2: 2x + y = 0$ intersect the line $L_3: y + 2 = 0$ at P and Q, respectively. The bisector of the acute angle between L_1 and L_2 intersects the line L_3 at R. Consider the following two statements:

1. S_1: The ratio $PR{:}RQ = 2\sqrt{2}{:}\sqrt{5}$.
2. S_2: In any triangle, bisector of an angle divides the triangle into two similar triangles.

Then

(A) both S_1 and S_2 are true

(B) both S_1 and S_2 are false

(C) S_1 is true while S_2 is false

(D) S_2 is true while S_1 is before

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Solution: Solving $L_1 = 0$ and $L_3 = 0$, we get $P = (-2, -2)$. Solving $L_2 = 0$ and $L_3 = 0$, we get that $Q = (1, -2)$. Therefore, $OP = 2\sqrt{2}$ and $OQ = \sqrt{5}$ where O is the origin. In any triangle, the internal angle bisector of an angle divides the opposite side internally in the ratio of the other two sides. Hence

$$PR{:}RQ = OP{:}OQ = 2\sqrt{2}{:}\sqrt{5}$$

Hence S_1 is true. S_2 is false because in a triangle whose angles are 90°, 60° and 30°, the bisector of 90° cannot divide the triangle into two similar triangles.

Answer: (C)

32. If the lines $2x - 3y + k = 0$, $3x - 4y - 13 = 0$ and $8x - 11y - 33 = 0$ are concurrent, then the value of k is

(A) 7 (B) 6 (C) –7 (D) –6

Solution: By Theorem 2.22, we have

$$\begin{vmatrix} 2 & -3 & k \\ 3 & -4 & -13 \\ 8 & -11 & -33 \end{vmatrix} = 0$$

$$\Rightarrow 2(132 - 143) + 3(-99 + 104) + k(-33 + 32) = 0$$

$$\Rightarrow -22 + 15 - k = 0$$

$$\Rightarrow k = -7$$

Direct Method: Solving $3x - 4y - 13 = 0$ and $8x - 11y - 33 = 0$, we obtain $x = 11$ and $y = 5$ so that $(11, 5)$ is the intersection of these lines. The third line $2x - 3y + k = 0$ also passes through $(11, 5)$. This implies (11.5) satisfying the equation of the third line. So

$$2(11) - 3(5) + k = 0 \Rightarrow k = -7$$

Answer: (C)

33. If the lines $ax + by + c = 0$, $bx + cy + a = 0$ and $cx + ay + b = 0$ are concurrent and $a + b + c \neq 0$, then the quadratic equation $ax^2 + bx + c = 0$ has

(A) equal roots (B) rational roots

(C) irrational roots (D) no real roots

Solution: By Theorem 2.22, we have

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$$

$$\Rightarrow 3abc - a^3 - b^3 - c^3 = 0$$
$$\Rightarrow (a+b+c)(a^2+b^2+c^2-ab-bc-ca) = 0$$

The condition $a+b+c \neq 0$ implies $a^2+b^2+c^2-ab-bc-ca = 0$ which in turn implies that

$$\frac{1}{2}[(a-b)^2+(b-c)^2+(c-a)^2] = 0 \Rightarrow a = b = c$$

Hence, the quadratic equation is $x^2 + x + 1 = 0$ whose roots are

$$\frac{-1 \pm i\sqrt{3}}{2}$$

Answer: (D)

34. The equation of the line passing through the intersection of the lines $2x - 5y + 1 = 0$ and $3x + 2y - 8 = 0$, and having equal non-zero intercepts on the axes, is

(A) $x + y = 3$ (B) $x + y = 2$
(C) $x + y = 1$ (D) $x + y = -3$

Solution: By Theorem 2.20, any line passing through the intersection of the given lines is of the form

$$(2x - 5y + 1) + \lambda(3x + 2y - 8) = 0$$

On simplification we get

$$(2 + 3\lambda)x + (2\lambda - 5)y + 1 - 8\lambda = 0$$

whose intercepts on the axes are

$$\frac{8\lambda - 1}{2 + 3\lambda}, \frac{8\lambda - 1}{2\lambda - 5}$$

Therefore,

$$\frac{8\lambda - 1}{2 + 3\lambda} = \frac{8\lambda - 1}{2\lambda - 5}$$
$$\Rightarrow (8\lambda - 1)(2 + 3\lambda - 2\lambda + 5) = 0$$
$$\Rightarrow \lambda = \frac{1}{8} \text{ or } -7$$

If $\lambda = 1/8$, then the line is $x - 2y = 0$ which is not the case. Hence $x = -7$ and the required line is $-19x - 19y + 57 = 0$ or $x + y - 3 = 0$.

Direct Method: The point of intersection of the given lines is (2, 1). Any line having equal intercepts on the coordinates axes is of the form $x + y = a$. This line passes through the point (2, 1) which implies that $a = 3$. Hence the equation of the required line is $x + y = 3$.

Answer: (A)

35. The straight line $2x + 3y + 1 = 0$ bisects the angle between two straight lines of which one line is $3x + 2y + 4 = 0$. Then, the equation of the other line is

(A) $9x + 26y = 48$ (B) $9x + 46y = 28$
(C) $9x + 26y = 28$ (D) $3x + 26y = 48$

Solution: See Fig. 2.65. Let $L_1 \equiv 3x + 2y + 4 = 0$, $L \equiv 2x + 3y + 1 = 0$ and $L_2 \equiv 0$ be the required lines. Since the line $L_2 = 0$ passes through the intersection of the lines $L_1 = 0$ and $L = 0$, $L_2 = 0$ is of the form (by Theorem 2.20)

$$L_2 \equiv (3x + 2y + 4) + \lambda(2x + 2y + 1) = 0$$

Suppose (α, β) ($\neq$ the intersection of $L_1 = 0$ and $L = 0$) is a point on the line $L = 0$ so that

$$2\alpha + 3\beta + 1 = 0 \tag{2.85}$$

But $L = 0$ is an angular bisection of $L_1 = 0$ and $L_2 = 0$ which implies that

$$\left|\frac{(3\alpha + 2\beta + 4) + \lambda(2\alpha + 3\beta + 1)}{\sqrt{(2\lambda + 3)^2 + (3\lambda + 2)^2}}\right| = \frac{|3\alpha + 2\beta + 1|}{\sqrt{3^2 + 2^2}}$$

$$\Rightarrow 13\lambda^2 + 24\lambda + 13 = 13 \quad \text{[from Eq. (2.85)]}$$

$$\Rightarrow \lambda = 0 \text{ or } -\frac{24}{13}$$

Therefore, $\lambda = 0$ gives the line $L_1 = 0$ and $\lambda = -24/13$ gives the line

$$L_2 \equiv (3x + 2y + 4) - \frac{24}{13}(2x + 3y + 1) = 0$$
$$\equiv -9x - 46y + 28 = 0$$

Therefore, the required line is $L_2 \equiv 9x + 46y - 28 = 0$.

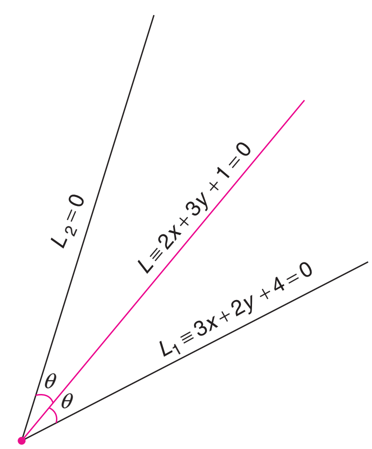

FIGURE 2.65

Answer: (B)

36. A variable straight line passes through the point $P(\alpha, \beta)$ and cuts the axes of coordinates in points

A and B, respectively. If the parallelogram $OACB$ is completed, then the locus of the vertex C (O is the origin) is

(A) $\dfrac{x}{\beta}+\dfrac{y}{\alpha}=1$ (B) $\dfrac{\beta}{x}+\dfrac{\alpha}{y}=1$

(C) $\dfrac{\alpha}{x}+\dfrac{\beta}{y}=1$ (D) $\alpha x+\beta y=(\alpha+\beta)xy$

Solution: See Fig. 2.66. Let the line AB be

$$\frac{x}{a}+\frac{y}{b}=1$$

where $A=(a,0)$ and $B=(0,b)$. This line passes through $P(\alpha,\beta)$ which implies that

$$\frac{\alpha}{a}+\frac{\beta}{b}=1 \tag{2.86}$$

Let $C(h,k)$ be the fourth vertex. Therefore, $h=a$ and $k=b$. Hence, from Eq. (2.86), we have

$$\frac{\alpha}{h}+\frac{\beta}{k}=1$$

Therefore, the locus of C is

$$\frac{\alpha}{x}+\frac{\beta}{y}=1$$

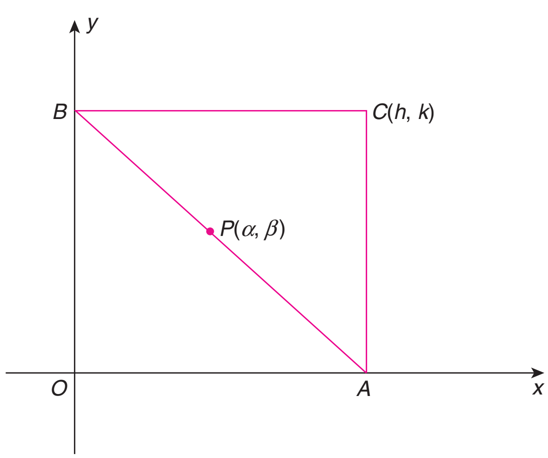

FIGURE 2.66

Answer: (C)

37. If the area of the rhombus enclosed by the lines $lx+my+n=0$, $lx+my-n=0$, $lx-my+n=0$ and $lx-my-n=0$ is 2 sq. unit, then

(A) $m^2=ln$ (B) $n^2=lm$

(C) $m=ln$ (D) $n=lm$

Solution: From Problem 2 of the section 'Subjective Problems', the area of the rhombus is

$$\left|\frac{(n+n)(n+n)}{-lm-lm}\right|=2$$

$$\Rightarrow n^2=lm$$

Answer: (B)

38. Consider the family of lines $2x+y+4+\lambda(x-2y-3)=0$ where λ is a parameter. Then, the number of lines belonging to this family whose distance from the point $(2,-3)$ is $\sqrt{10}$ is

(A) 4 (B) 2 (C) 1 (D) 0

Solution: The given family of lines can be written as

$$(2+\lambda)x+(1-2\lambda)y+(4-3\lambda)=0$$

By hypothesis, the distance of this line from $(2,-3)$ is

$$\left|\frac{(2+\lambda)(2)+(1-2\lambda)(-3)+(4-3\lambda)}{\sqrt{(2+\lambda)^2+(1-2\lambda)^2}}\right|=\sqrt{10}$$

$$\Rightarrow (5\lambda+5)^2=10(5\lambda^2+5)$$
$$\Rightarrow (\lambda+1)^2=2(\lambda^2+1)$$
$$\Rightarrow \lambda^2-2\lambda+1=0$$
$$\Rightarrow \lambda=1$$

Hence, the required line is $3x-y+1=0$.

Answer: (C)

39. The point (2, 1) is shifted through a distance $3\sqrt{2}$ units measured parallel to the line $x+y=1$ in the decreasing direction of ordinates to reach a point B. The image of the point B in the line $x+y=1$ is

(A) (0, 0) (B) (0, –1)

(C) (–3, 2) (D) (3, –4)

Solution: See Fig. 2.67. Let $A=(2,1)$. By hypothesis, A is translated along the line

$$y-1=-1(x-2)$$
$$\Rightarrow x+y=3 \tag{2.87}$$

Therefore abscissa of B is given by the equation

$$(2-x)^2+(2-x)^2=18$$
$$\Rightarrow 2-x=\pm 3$$
$$\Rightarrow x=-1,5$$

Hence, the point $B=(-1,4)$ or $(5,-2)$. However, $A(2,1)$ is translated in the decreasing sense of ordinates. Thus, $B=(5,-2)$. Hence, the image of $B(5,-2)$ in the line $x+5=1$ is $B'(3,-4)$ because the slope of BB' is 1 and the midpoint of BB' is $(4,-3)$ which lies on the line $x+y-1=0$.

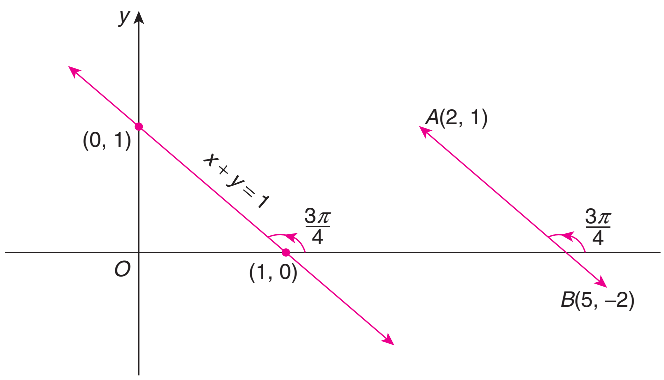

FIGURE 2.67

Answer: (D)

40. If a, b and c are real such that $a^2 + 9b^2 - 4c^2 - 6ab = 0$, then the line $ax + by + c = 0$ always passes through the point with negative ordinate

(A) $\left(-\frac{1}{2}, -\frac{3}{2}\right)$ (B) $\left(\frac{1}{2}, \frac{3}{2}\right)$

(C) $\left(\frac{1}{2}, -\frac{3}{2}\right)$ (D) $\left(-\frac{1}{2}, \frac{3}{2}\right)$

Solution: We have

$$a^2 + 9b^2 - 4c^2 - 6ab = 0$$
$$\Rightarrow (a-3b)^2 - 4c^2 = 0$$
$$\Rightarrow (a-3b+2c)(a-3b-2c) = 0$$
$$\Rightarrow \left(\frac{a}{2} - \frac{3}{2}b + c\right)\left(\frac{-a}{2} + \frac{3b}{2} + c\right) = 0$$

Therefore, the line $ax + by + c = 0$ passes through the points

$$\left(\frac{1}{2}, -\frac{3}{2}\right) \text{ and } \left(-\frac{1}{2}, \frac{3}{2}\right)$$

Since the ordinate is negative, the point is

$$\left(\frac{1}{2}, -\frac{3}{2}\right)$$

Answer: (C)

41. Let $A(5, 12)$, $B(-13\cos\theta, 13\sin\theta)$ and $C(13\sin\theta, -13\cos\theta)$ where θ is real be the vertices of ΔABC. Then, the orthocentre of ΔABC lies on the line

(A) $x - y - 7 = 0$ (B) $x + y - 7 = 0$

(C) $x - y + 7 = 0$ (D) $x + y + 7 = 0$

Solution: Observe that B and C are images of each other on the line $y = x$ and hence the side BC varies and is perpendicular to the line $y = x$. Hence, the orthocentre lies on the altitude through $A(5, 12)$ and parallel to the line $y = x$, whose equation is

$$y - 12 = 1(x - 5)$$
$$\Rightarrow x - y + 7 = 0$$

Answer: (C)

42. Consider the following two families of lines represented by the equations $(x - y - 6) + \lambda(2x + y - 3) = 0$ and $(x + 2y - 4) + \mu(3x - 2y - 4) = 0$. If these families of lines are at right angles to each other, then their points of intersection lie on the curve

(A) $x^2 - y^2 + 3x - 4y - 3 = 0$

(B) $x^2 + y^2 - 5x + 2y + 3 = 0$

(C) $x^2 + y^2 + 3x + 4y - 3 = 0$

(D) $x^2 - y^2 - 3x + 4y - 3 = 0$

Solution: By Theorem 2.20, the equation $(x - y - 6) + \lambda(2x + y - 3) = 0$ represents the family of lines passing through the intersection of the lines $x - y - 6 = 0$ and $2x + y - 6 = 0$ which is $(3, -3)$. Similarly, the second equation represents a family of concurrent lines which is concurrent at the point of intersection of the lines $x + 2y - 4 = 0$ and $3x - 2y - 4 = 0$ which is $(2, 1)$. Suppose the line through the point $(3, -3)$ is

$$y + 3 = m(x - 3) \tag{2.88}$$

Then the line through the point $(2, 1)$ and perpendicular to the line given in Eq. (2.88) is

$$y - 1 = -\frac{1}{m}(x - 2) \tag{2.89}$$

From Eqs. (2.88) and (2.89), we have

$$\frac{y+3}{x-3} = m = -\frac{(x-2)}{y-1}$$
$$\Rightarrow (x-3)(x-2) + (y-1)(y+3) = 0$$
$$\Rightarrow x^2 + y^2 - 5x + 2y + 3 = 0$$

Answer: (B)

Note: In Chapter 3, we will see that the equation $x^2 + y^2 - 3x + 4y - 3 = 0$ represents a circle describable on the segment joining $(3, -3)$ and $(2, 1)$ as ends of a diameter.

43. The line $3x + 2y - 24 = 0$ meets x-axis at A and y-axis at B. The perpendicular bisector of AB meets the line through $(0, -1)$ parallel to x-axis at C. Then, the area of the ΔABC is

(A) 91 sq. unit (B) 12 sq. unit

(C) 36 sq. unit (D) 48 sq. unit

Solution: The given line $3x + 2y = 24$ in the intercept form is

$$\frac{x}{8} + \frac{y}{12} = 1$$

so that $A = (8, 0)$ and $B = (0, 12)$. Equation of the perpendicular bisector of the segments AB is

$$y - 6 = \frac{2}{3}(x - 4)$$

$$\Rightarrow 2x - 3y + 10 = 0$$

This line meets the line $y + 1 = 0$ at the point $C(-13/2, -1)$.

Therefore, the area of the triangle ABC is

$$\frac{1}{2}\begin{vmatrix} 8 & 0 & 1 \\ 0 & 12 & 1 \\ -\frac{13}{2} & -1 & 1 \end{vmatrix} = \frac{1}{4}\begin{vmatrix} 8 & 0 & 1 \\ 0 & 12 & 1 \\ -13 & -2 & 2 \end{vmatrix}$$

$$= \frac{1}{4}\begin{vmatrix} 8 & 0 & 1 \\ 0 & 12 & 1 \\ -13 & -2 & 2 \end{vmatrix}$$

$$= \frac{1}{4}|8(24 + 2) + 1(0 + 156)|$$

$$= \frac{364}{4} = 91$$

Answer: (A)

44. The vertices of a triangle are $A(-8, 5)$, $B(-15, -19)$ and $C(1, -7)$. The internal bisector of the angle A has the equation $px + 2y + r = 0$. Then, $p + r$ is equal to

(A) 78 (B) 88 (C) 98 (D) 89

Solution: See Fig. 2.68. We have

$$AB = \sqrt{7^2 + 24^2} = \sqrt{625} = 25$$

$$BC = \sqrt{16^2 + 12^2} = \sqrt{400} = 20$$

$$CA = \sqrt{9^2 + 12^2} = \sqrt{225} = 15$$

Suppose the internal bisector of the angle A meets the side BC at D so that $BD{:}DC = AB{:}AC = 25{:}15 = 5{:}3$. Therefore

$$D = \left(\frac{-15 \times 3 + 5(1)}{5 + 3}, \frac{-19 \times 3 + 5 \times (-7)}{5 + 3}\right)$$

$$= \left(-5, \frac{-92}{8}\right) = \left(-5, \frac{-23}{2}\right)$$

Therefore, the equation of AD is

$$y - 5 = \frac{5 + (23/2)}{-8 + 5}(x + 8)$$

$$\Rightarrow y - 5 = \frac{33}{-6}(x + 8)$$

$$\Rightarrow y - 5 = -\frac{11}{2}(x + 8)$$

Therefore, $p + r = 11 + 78 = 89$.

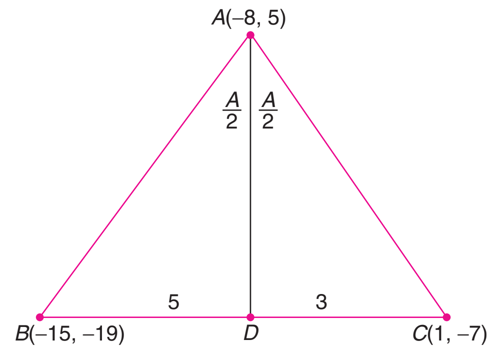

FIGURE 2.68

Answer: (D)

45. The equations of the sides AB, BC and CA of ΔABC are, respectively, $2x + y = 0$, $x + by = c$ and $x - y = 3$. If $G(2, 3)$ is the centroid of ΔABC, then the value of $b + c$ is

(A) 50 (B) 47 (C) 74 (D) 57

Solution: See Fig. 2.69. We have

$$2x + y = 0 \tag{2.90}$$

$$x + by = c \tag{2.91}$$

$$x - y = 3 \tag{2.92}$$

Solving Eqs. (2.90) and (2.92), we get $A = (1, -2)$. Solving Eqs. (2.90) and (2.91), we get

$$B = \left(\frac{c}{1 - 2b}, \frac{-2c}{1 - 2b}\right)$$

Solving Eqs. (2.91) and (2.93), we get

$$C = \left(\frac{3b + c}{1 + b}, \frac{c - 3}{1 + b}\right)$$

Suppose $D(x, y)$ is the midpoint of $\overline{BC}$. Then

$$AG{:}GD = 2{:}1 \Rightarrow \frac{2x + 1}{3} = 2 \text{ and } \frac{2y - 2}{3} = 3$$

$$\Rightarrow D = \left(\frac{5}{2}, \frac{11}{2}\right)$$

D lies on BC. So

$$\frac{5}{2}+b\left(\frac{11}{2}\right)=c$$

$$\Rightarrow 11b-2c=-5 \tag{2.93}$$

Again, if E is the midpoint of AC and $BG:GE$= 2:1, then

$$E=\left(\frac{6-12b-c}{2(1-2b)}, \frac{9-18b+2c}{2(1-2b)}\right)$$

Since E lies on the side CA, we have

$$\frac{6-12b-c}{2(1-2b)}-\frac{9-18b+2c}{2(1-2b)}=3$$

$$\Rightarrow -3+6b-3c=6-12b$$

$$\Rightarrow 6b-c=3 \tag{2.94}$$

Solving Eqs. (2.93) and (2.94), we get $b = 11$ and $c = 63$. Hence, $b + c = 74$.

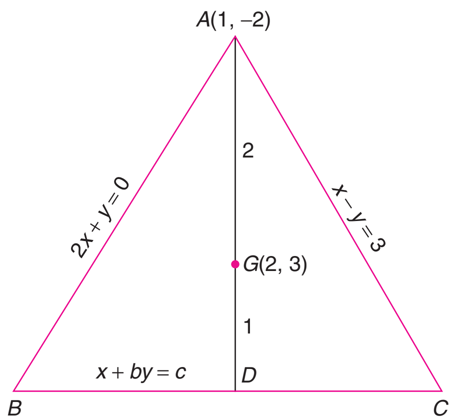

FIGURE 2.69

Answer: (C)

46. In an acute-angled triangle ABC, equation of the side BC is $4x-3y+3=0$, orthocentre H is $(1,2)$ and the circumcentre O is (2, 3). Then the value of the circumradius R is

(A) $\frac{\sqrt{28}}{5}$ (B) $\frac{\sqrt{58}}{5}$ (C) 23 (D) $\frac{\sqrt{23}}{5}$

Solution: See Fig. 2.70. Suppose the image of the orthocentre $H(1,2)$ on the side BC is H'. It is known that the images of the orthocentre of a triangle in the sides lie on the circumcircle. Hence, H' lies on the circumcircle. Suppose $H'=(h,k)$. Therefore, by Theorem 2.13, part (2) we have

$$\frac{h-1}{4}=\frac{k-2}{-3}=-\frac{2(4-6+3)}{4^2+3^2}=\frac{-2}{25}$$

Therefore

$$h=1-\frac{8}{25}=\frac{17}{25}$$

and

$$k=2+\frac{6}{25}=\frac{56}{25}$$

Hence

$$H'=\left(\frac{17}{25}, \frac{56}{25}\right)$$

Therefore, the circumradius is

$$R=OH'$$

$$=\sqrt{\left(2-\frac{17}{25}\right)^2+\left(3-\frac{56}{25}\right)^2}$$

$$=\frac{\sqrt{33^2+19^2}}{25}$$

$$=\frac{\sqrt{1450}}{25}=\frac{\sqrt{\frac{1450}{25}}}{5}=\frac{\sqrt{58}}{5}$$

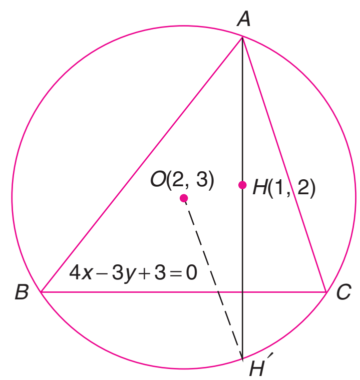

FIGURE 2.70

Answer: (B)

47. A ray of light emanating from the point $(3,4)$ meets the y-axis at $(0, b)$ and reflects towards the x-axis which meets it at $(a, 0)$ and again reflects from x-axis to pass through the point $(8,2)$. Then the value of a is

(A) $\frac{13}{3}$ (B) $\frac{9}{2}$ (C) $\frac{17}{3}$ (D) $\frac{14}{3}$

Solution: Suppose $P = (3, 4)$, $B = (0, b)$, $A(a, 0)$ and $Q = (8, 2)$ (see Fig. 2.71). From the figure, we can see that

$$\frac{2}{8-a}=\frac{4-b}{3-0}=\frac{b}{a}=\frac{4-b+b}{3+a}=\frac{4}{3+a}$$

$$\Rightarrow 2(3+a)=4(8-a)$$

$$\Rightarrow 6a=26$$

$$\Rightarrow a=\frac{13}{3}$$

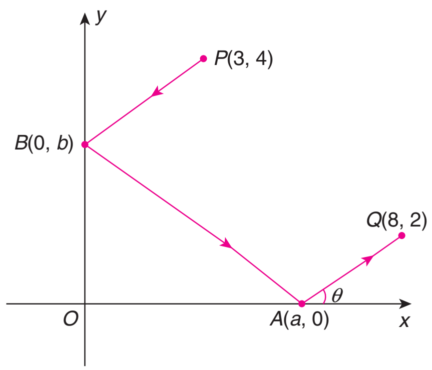

FIGURE 2.71

Answer: (A)

48. Let P be the point $(3, 2)$. Q is the image of P in the x-axis and R is the image of Q in the line $y = -x$. Finally, S is the image of R through the origin and $PQRS$ is a convex quadrilateral. Then, the area of $PQRS$ is

(A) 15 (B) 16 (C) 18 (D) 5

Solution: See Fig. 2.72. By hypothesis, $P = (3, 2)$, $Q = (3, -2)$, $R = (2, -3)$ and $S = (-2, 3)$.

$$\text{Area of } PQRS = \text{Area of } DPQR + \text{Area of } DPRS$$

$$= \frac{1}{2}|3(-2+3) + 3(-3-2) + 2(2+2)| +$$

$$\frac{1}{2}|3(-3-3) + 2(3-2) - 2(2+3)|$$

$$= \frac{1}{2}\left|3 - 15 + 8\right| + \frac{1}{2}\left|-18 + 2 - 10\right|$$

$$= 2 + 13 = 15$$

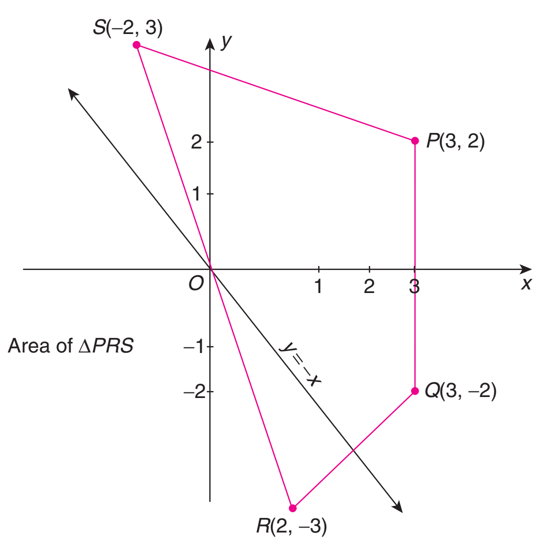

FIGURE 2.72

Answer: (A)

49. In ΔABC, $A = (4, -1)$. If $x - 1 = 0$ and $x - y - 1 = 0$ are the interior angle bisectors of angles B and C, respectively, then the equation of the side BC is

(A) $2x + y - 3 = 0$ (B) $2x - y + 3 = 0$

(C) $x + 2y - 3 = 0$ (D) $x - 2y + 3 = 0$

Solution: Let A' and A'' be the images of $A(4, -1)$ in the bisectors $x - 1 = 0$ and $x - y - 1 = 0$, respectively. Therefore, A' and A'' lie on the line BC (see Problem 27 of the section 'Subjective Problems'). Hence, the equation of BC is the equation of $A'A''$. Now, $A' = (-2, -1)$ and $A'' = (0, 3)$. Therefore, the equation of the side BC is

$$y - 3 = \frac{3+1}{0+2}(x - 0)$$

$$\Rightarrow 2x - y + 3 = 0$$

Answer: (B)

Note: The main tool used here is that if $L = 0$ is the angle bisector of $\underline{|ABC}$, then the image of A on the line $L = 0$ lies on the line BC (see Fig. 2.73).

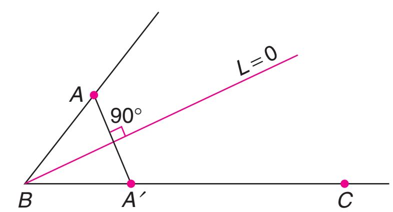

FIGURE 2.73

50. The equations of the sides AB and AC of ΔABC, respectively, are $2x - y = 0$ and $x + y = 3$ and the centroid G is $(2, 3)$. Then, the slope of the side BC is

(A) 4 (B) 3 (C) 2 (D) 5

Solution: See Fig. 2.74. Solving the equations $2x - y = 0$ and $x + y = 3$, we have $A = (1, 2)$. Equation of the median through $A(1, 2)$ is

$$y - 2 = \frac{2-3}{1-2}(x - 1) = x - 1$$

$$\Rightarrow x - y = -1 \tag{2.95}$$

Let $B = (h, 2h)$ and $C = (k, 3 - k)$. Therefore, the midpoint of BC is

$$\left(\frac{h+k}{2}, \frac{2h-k+3}{2}\right) \tag{2.96}$$

Also, let D be the midpoint of BC. Since $AG{:}GD = 2{:}1$, we have

$$D = \left(\frac{5}{2}, \frac{7}{2}\right) \tag{2.97}$$

From Eqs. (2.95) and (2.97), we have

$$\frac{h+k}{2}=\frac{5}{2} \quad \text{and} \quad \frac{2h-k+3}{2}=\frac{7}{2}$$

which implies that

$$h+k=5 \quad \text{and} \quad 2h-k=4$$

Solving these equations, we get $h = 3$ and $k = 2$. Hence, $B = (3, 6)$ and $C = (2, 1)$. Thus, the slope of BC is

$$\frac{6-1}{3-2}=5$$

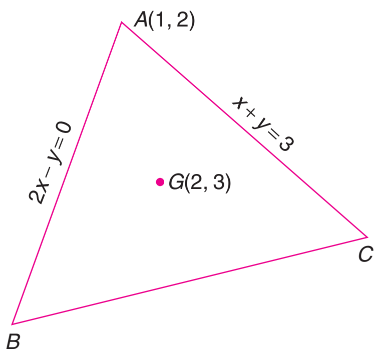

FIGURE 2.74

Answer: (D)

51. If a, b and c are real and $a + b + c = 0$, then the line $3ax + 4by + c = 0$ passes through the point whose coordinates are

(A) $(1, 1)$ (B) $\left(\frac{1}{3}, \frac{1}{4}\right)$

(C) $(3, 4)$ (D) $\left(\frac{1}{4}, \frac{1}{3}\right)$

Solution: We have

$$a+b+c=0 \Rightarrow 3a\left(\frac{1}{3}\right)+4b\left(\frac{1}{4}\right)+c=0$$

which implies that the line $3ax+4by+c=0$ passes through the point

$$\left(\frac{1}{3}, \frac{1}{4}\right)$$

Answer: (B)

52. If the lines $ax + by + c = 0$ (a, b, c are non-zero and real), $x + y - 2 = 0$ and $2x - y + 1 = 0$ are concurrent, then the lines $2ax + 3by + c = 0$ are passing through a fixed point given by

(A) $\left(\frac{1}{2}, \frac{1}{3}\right)$ (B) $(2, 3)$

(C) $\left(\frac{2}{3}, \frac{-7}{5}\right)$ (D) $\left(\frac{1}{6}, \frac{5}{9}\right)$

Solution: Solving $x + y - 2 = 0$ and $2x - y + 1 = 0$, we get that the point of intersection of these two lines as

$$\left(\frac{1}{3}, \frac{5}{3}\right)$$

The line $ax+by+c=0$ also passes through the point $(1/3, 5/3)$ implies that

$$a\left(\frac{1}{3}\right)+b\left(\frac{5}{3}\right)+c=0$$

$$\Rightarrow 2a\left(\frac{1}{6}\right)+3b\left(\frac{5}{9}\right)+c=0$$

Hence, the line $2ax + 3by + c = 0$ passes through the point $\left(\frac{1}{6}, \frac{5}{9}\right)$.

Answer: (D)

53. If a, b and c are real and satisfy the equation $4a^2 + 9b^2 + 12ab - 9c^2 = 0$, then $ax + by + c = 0$ represents two families of concurrent lines. The distance between their points of concurrence is

(A) 1 (B) $3\sqrt{3}$ (C) $\frac{2}{3}\sqrt{13}$ (D) $2\sqrt{13}$

Solution: From the given relation we have

$$(2a+3b)^2 - (3c)^2 = 0$$

$$\Rightarrow (2a+3b+3c)(2a+3b-3c)=0$$

$$\Rightarrow 2a+3b+3c=0 \text{ or } 2a+3b-3c=0$$

$$\Rightarrow a\left(\frac{2}{3}\right)+b(1)+c=0 \text{ or } a\left(\frac{-2}{3}\right)+b(-1)+c=0$$

which implies that $ax+by+c=0$ passes through

$$\left(\frac{2}{3}, 1\right) \text{or} \left(\frac{-2}{3}, -1\right)$$

whose distance between them is

$$\frac{2\sqrt{13}}{3}$$

Answer: (C)

54. Let A be $(1, 1)$. A line through A meets the x-axis in B. A line through A and perpendicular to AB meets the y-axis in C. Then the locus of the midpoint of the segment BC is

(A) $x+y=2$ (B) $x+y=2xy$

(C) $x+y=\frac{1}{2}$ (D) $x+y=1$

Solution: See Fig. 2.75. Let the line passing through the point $A(1, 1)$ be $y - 1 = m(x - 1)$. Therefore

$$B = \left(\frac{m-1}{m}, 0\right)$$

The equation of the line through the point $A(1, 1)$ and perpendicular to AB is

$$y - 1 = \frac{-1}{m}(x-1)$$

Therefore,

$$C = \left(0, \frac{1+m}{m}\right)$$

Let $m(h, k)$ be the midpoint of $\overline{BC}$. Therefore,

$$h = \frac{m-1}{2m} \quad \text{and} \quad k = \frac{m+1}{2m}$$

This gives

$$h - 1 = \frac{-(m+1)}{2m} = -k \Rightarrow h + k = 1$$

Hence, the locus of $M(h, k)$ is the line $x + y = 1$.

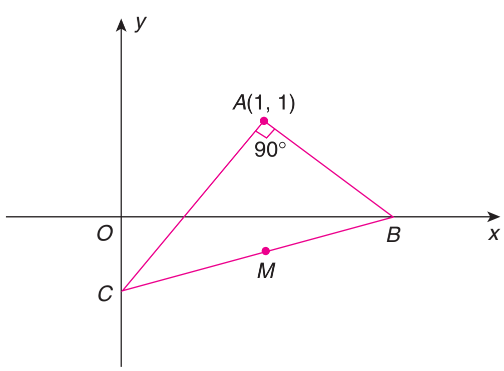

FIGURE 2.75

Answer: (D)

55. Two lines $x + y = 0$ and $x - y = 0$ are fixed sides of a triangle and the third side is a variable line whose equation is $lx + my = 1$ where $l^2 + m^2 = 1$. Then the circumcentre of the triangle lies on the curve

(A) $x^2 - y^2 = (x^2 + y^2)^2$ (B) $x^2 + y^2 = (x^2 - y^2)^2$

(C) $x^2 + y^2 = x^2y^2$ (D) $x^2 + y^2 = 4$

Solution: The vertices of the triangle are

$$O(0,0),\ A\left(\frac{1}{l+m}, \frac{1}{l+m}\right) \text{ and } B\left(\frac{1}{l-m}, \frac{-1}{l-m}\right)$$

Since the lines $y = x$ and $y = -x$ are at right angles, $\underline{|AOB} =$ 90°. Therefore, the circumcentre of ΔOAB is the midpoint of the hypotenuse AB. Let $S(h, k)$ be the circumcentre of ΔOAB. Therefore,

$$h = \frac{1}{2}\left(\frac{1}{l+m} + \frac{1}{l-m}\right) \text{and } k = \frac{1}{2}\left(\frac{1}{l+m} - \frac{1}{l-m}\right)$$

$$\Rightarrow h = \frac{l}{l^2 - m^2} \text{ and } k = \frac{-m}{l^2 - m^2}$$

$$\Rightarrow h^2 + k^2 = \frac{l^2 + m^2}{(l^2 - m^2)^2} = \frac{1}{(l^2 - m^2)^2} \qquad (2.98)$$

Also

$$h^2 - k^2 = \frac{l^2 - m^2}{(l^2 - m^2)^2} = \frac{1}{(l^2 - m^2)} \qquad (2.99)$$

From Eqs. (2.98) and (2.99), we have

$$h^2 + k^2 = (h^2 - k^2)^2$$

Hence (h, k) lies on the curve we get

$$x^2 + y^2 = (x^2 - y^2)^2$$

Answer: (B)

56. The vertices B and C forming the base of an isosceles triangle ABC are (2, 0) and (0, 1), respectively. The vertex A lies on the line $x = 2$. The orthocentre of ΔABC is

(A) $\left(\frac{5}{4}, 1\right)$ (B) $\left(\frac{3}{4}, 1\right)$

(C) $\left(\frac{3}{2}, \frac{3}{2}\right)$ (D) $\left(\frac{4}{3}, \frac{5}{3}\right)$

Solution: See Fig. 2.76. $A = (2, k)$, $B = (2, 0)$, $C = (0, 1)$. Now

$$AB = AC$$
$$\Rightarrow 2^2 + (k-1)^2 = k^2$$
$$\Rightarrow 5 - 2k = 0$$
$$\Rightarrow k = \frac{5}{2}$$

Thus,

$$A = \left(2, \frac{5}{2}\right)$$

Now, the equation of the altitude through $A(2, 5/2)$ is

$$y - \frac{5}{2} = 2(x-2) \quad \left(\because \text{slope of } BC = \frac{-1}{2}\right)$$

$$\Rightarrow 2y - 5 = 4(x-2)$$

$$\Rightarrow 4x - 2y = 3 \qquad (2.100)$$

Altitude through $C(0, 1)$ is

$$y = 1 \qquad (2.101)$$

Therefore, from Eqs. (2.100) and (2.101), the orthocentre of ΔABC is

$$\left(\frac{5}{4}, 1\right)$$

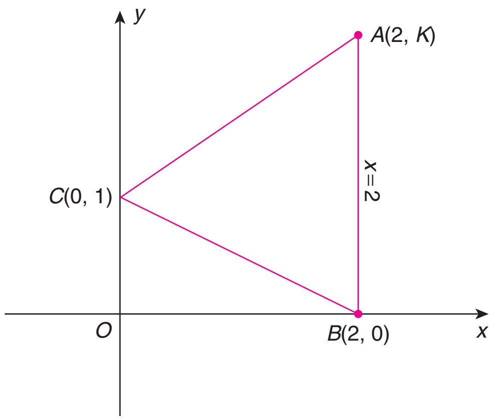

FIGURE 2.76

Answer: (A)

57. Let $A = (3, -4)$, $B = (1, 2)$ and $P = (2\lambda - 1, 2\lambda + 1)$. If the sum $PA + PB$ is minimum, then the value of λ is

(A) $\frac{7}{8}$ (B) 1 (C) $\frac{7}{9}$ (D) $\frac{3}{2}$

Solution: It is known that $PA + PB \geq AB$ and equality occurs if and only if A, P and B are collinear. Hence $PA + PB$ is minimum which implies that

$$\begin{vmatrix} 3 & -4 & 1 \\ 1 & 2 & 1 \\ 2\lambda - 1 & 2\lambda + 1 & 1 \end{vmatrix} = 0$$

$$\Rightarrow 3(2 - 2\lambda - 1) + 4(1 - 2\lambda + 1) + 1(2\lambda + 1 - 4\lambda + 2) = 0$$

$$\Rightarrow -16\lambda + 3 + 8 + 3 = 0$$

$$\Rightarrow \lambda = \frac{7}{8}$$

Answer: (A)

58. The angle between the lines $x\cos\alpha + y\sin\alpha = p$ and $ax + by + p = 0$ is $\pi/4$ where $p > 0$. If these two lines together with the line $x \sin\alpha + y\cos\alpha = 0$ are concurrent, then

(A) $a^2 + b^2 = 2$ (B) $a^2 + b^2 = \frac{1}{2}$

(C) $a^2 + b^2 = 1$ (D) $a^2 - b^2 = 1$

Solution: Since the lines are concurrent, by Theorem 2.22, we have

$$\begin{vmatrix} \cos\alpha & \sin\alpha & -p \\ \sin\alpha & -\cos\alpha & 0 \\ a & b & p \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} a + \cos\alpha & b + \sin\alpha & 0 \\ \sin\alpha & -\cos\alpha & 0 \\ a & b & p \end{vmatrix} = 0$$

$$\Rightarrow p|-\cos\alpha(a + \cos\alpha) - \sin\alpha(b + \sin\alpha)| = 0$$

$$\Rightarrow a\cos\alpha + b\sin\alpha + 1 = 0 \qquad (2.102)$$

Since the angle between $x\cos\alpha + y\sin\alpha = p$ and $x\sin\alpha - y\cos\alpha$ is $\pi/4$, by Theorem 2.12, we have

$$\frac{1}{\sqrt{2}} = \cos\frac{\pi}{4} = \left|\frac{a\cos\alpha + b\sin\alpha}{\sqrt{\cos^2\alpha + \sin^2\alpha}\sqrt{a^2 + b^2}}\right|$$

$$= \frac{1}{\sqrt{a^2 + b^2}} \qquad \text{[by Eq. (2.102)]}$$

$$\Rightarrow a^2 + b^2 = 2$$

Answer: (A)

59. P is a point on the line $2x + y - 1 = 0$. Points Q and R are on the line $2x + y - 2 = 0$ such that ΔPQR is equilateral. Then, the length of the side of the triangle is

(A) $4\sqrt{5}$ (B) $3\sqrt{5}$ (C) $\frac{2}{\sqrt{15}}$ (D) $\sqrt{5}$

Solution: See Fig. 2.77. We have p as the altitude through P. So

$$p = \left|\frac{-1 + 2}{\sqrt{2^2 + 1^2}}\right| = \frac{1}{\sqrt{5}}$$

Suppose a is the length of the side. Therefore

$$\frac{\sqrt{3}}{2} = \sin 60° = \frac{p}{a} = \frac{1}{a\sqrt{5}}$$

Hence

$$a = \frac{2}{\sqrt{15}}$$

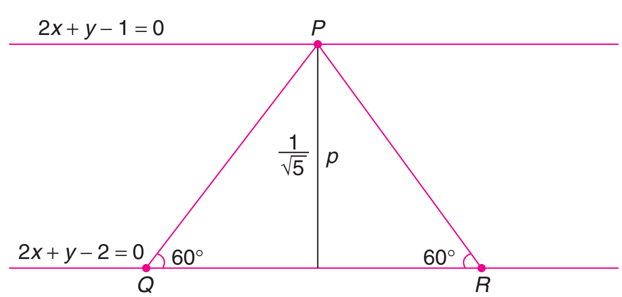

FIGURE 2.77

Answer: (C)

60. If a, b and c are real and are in AP, then the family of lines $ax + by + c = 0$ are concurrent at the point

(A) $(1, -1)$ (B) $(1, 2)$
(C) $(2, -1)$ (D) $(1, -2)$

Solution: Suppose

$$b = \frac{a+c}{2}$$

Then

$$ax + by + c = 0 \Rightarrow 2ax + (a+c)y + 2c = 0$$
$$\Rightarrow (2x + y)a + (y + 2)c = 0$$

which represents family of concurrent lines concurrent at the point $C(1, -2)$ which is the intersection of the lines $2x + y = 0$ and $y + 2 = 0$.

Answer: (D)

61. If $A(1, p^2)$, $B(1, 0)$ and $C(p, 0)$ where $p \neq 0$ are the vertices of a triangle, then the value of p for which the area of ΔABC is minimum is

(A) $\frac{1}{\sqrt{2}}$ (B) $\frac{1}{\sqrt{3}}$
(C) $\frac{-1}{\sqrt{2}}$ (D) None

Solution: Let S be the area of ΔABC so that

$$S = \frac{1}{2}\begin{vmatrix} 1 & p^2 & 1 \\ 0 & 1 & 1 \\ p & 0 & 1 \end{vmatrix} = \frac{1}{2}\left|[1(1-0) - p^2(0-p) + 1(0-p)]\right|$$
$$= \frac{1}{2}\left|p^3 - p + 1\right|$$

Real value of $p = \frac{1}{\sqrt{3}}$ exists such that $3p^2 - 1 = 0$. Hence minimum value of S exists at $p = \frac{1}{\sqrt{3}}$.

Answer: (B)

62. The locus of the point which moves such that its distance from the point (4, 5) is equal to its distance from the line $x - y + 1 = 0$ is

(A) a straight line
(B) a circle with centre at (4, 5)
(C) $(y-5)^2 = 4(x-4)$
(D) $(x-4)^2 = y-5$

Solution: The point (4, 5) lies on the line $x - y + 1 = 0$. Hence, the required locus is the line $y - 5 = -1(x - 4)$ or $x + y - 9 = 0$.

Answer: (A)

63. $A = (-4, 0)$ and $B = (4, 0)$. The points M and N are variable points on the y-axis such that N lies above M and $MN = 4$. The lines AM and BN intersect at P. Then, the locus of P is

(A) $x^2 + 2xy - 16 = 0$ (B) $x^2 = 2xy + 16$
(C) $x^2 + 16 = 2xy$ (D) $x^2 + 2xy + 16 = 0$

Solution: See Fig. 2.78. We have $A = (-4, 0)$ and $B = (4, 0)$ so that y-axis is the perpendicular bisector of $\overline{AB}$. Since $MN = 4$, we can take $N = (0, a + 4)$ and $M = (0, a)$. Equation of the line AM is

$$\frac{x}{-4} + \frac{y}{a} = 1 \tag{2.103}$$

and the equation of BN is

$$\frac{x}{4} + \frac{y}{a+4} = 1 \tag{2.104}$$

From Eq. (2.103), we have

$$a = \frac{4y}{4+x} \tag{2.105}$$

From Eq. (2.104), we get

$$a + 4 = \frac{4y}{4-x} \tag{2.106}$$

Therefore, from Eqs. (2.105) and (2.106), we have

$$\frac{y}{4-x} - \frac{y}{4+x} = 1$$
$$\Rightarrow 2xy = 16 - x^2$$
$$\Rightarrow x^2 + 2xy - 16 = 0$$

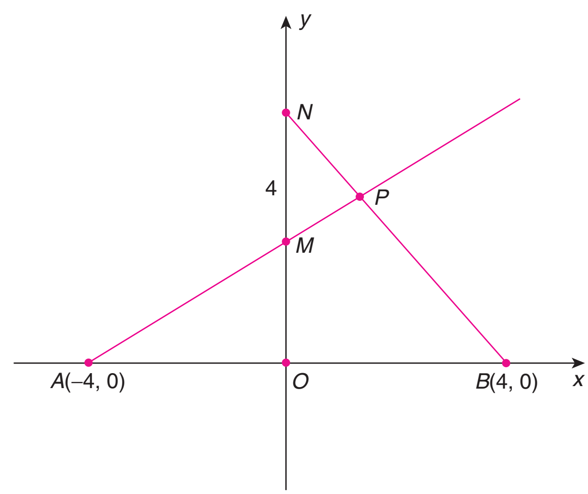

FIGURE 2.78

Answer: (A)

64. The vertices of a parallelogram are described in the order $A(3, 1)$, $B(13, 6)$, $C(13, 21)$ and $D(3, 16)$. If a line through origin divides the parallelogram into two congruent parts, then the slope of the line is

(A) $\dfrac{13}{8}$ (B) $\dfrac{11}{12}$ (C) $\dfrac{8}{11}$ (D) $\dfrac{11}{8}$

Solution: See Fig. 2.79. It can be observed that AD and BC are vertical lines. Suppose the line through the origin meets AD at P and BC at Q in such a way that $CQ = PA$, $DP = BQ$. Also $PQCD$ and $PQBA$ are the congruent parts. Suppose $AP = k$.

Therefore, $P = (3, k+1)$ and $Q = (13, 21-k)$. Hence, the slope of the line is

$$\frac{k+1}{3} = \frac{21-k}{13}$$
$$\Rightarrow 13k + 13 = 63 - 3k$$
$$\Rightarrow 16k = 50$$
$$\Rightarrow k = \frac{25}{8}$$

Thus, the slope of the line is

$$\frac{(25/8)+1}{3} = \frac{33}{3\times 8} = \frac{11}{8}$$

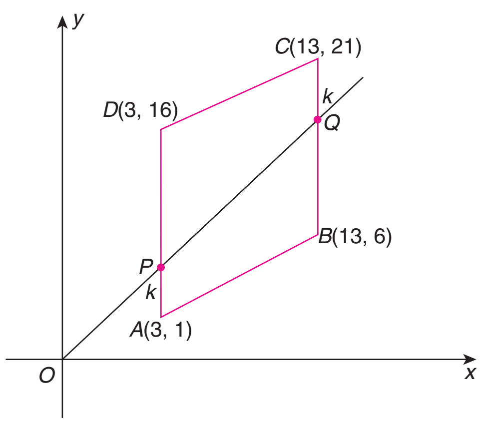

FIGURE 2.79

Answer: (D)

65. The number of straight lines that are equidistant from the vertices of a triangle is

(A) 1 (B) 2
(C) 3 (D) infinite

Solution: See Fig. 2.80. Let ABC be a triangle. Let E and F be the midpoints of AB and AC, respectively. Draw AL, BM and CN perpendicular to the line $\overleftrightarrow{EF}$. It is known that the line $\overleftrightarrow{EF}$ is parallel to $\overleftrightarrow{BC}$ and $EF = (1/2)BC$. It is clear that

$$\Delta BEM \equiv \Delta AEL \text{ and } \Delta CFN \equiv \Delta AFL$$

Therefore, $BM = AL = CN$. Thus the line $\overleftrightarrow{EF}$ is equidistant from A, B and C. Similarly, we have two more lines. Thus, the numbers of lines equidistant from A, B and C is 3.

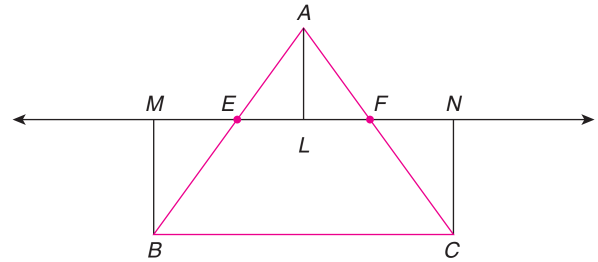

FIGURE 2.80

Answer: (C)

66. The equations of two adjacent sides of a rhombus are $y = x$ and $y = 7x$. The diagonals of the rhombus intersect at the point $(1, 2)$. Thus, the area of the rhombus is

(A) $\dfrac{50}{3}$ (B) $\dfrac{20}{3}$ (C) $\dfrac{10}{3}$ (D) $\dfrac{40}{3}$

Solution: See Fig. 2.81. Let M be the point $(1, 2)$. Since the diagonal AC is perpendicular to the diagonal OB, the equation of the diagonal AC is

$$y - 2 = \frac{-1}{2}(x-1)$$
$$\Rightarrow x + 2y - 5 = 0 \tag{2.107}$$

Solving Eq. (2.107) and the equation $y = x$, we get

$$A = \left(\frac{5}{3}, \frac{5}{3}\right)$$

Solving Eq. (2.107) and the equation $y = 7x$, we get

$$C = \left(\frac{1}{3}, \frac{7}{3}\right)$$

Therefore, area of the rhombus is $2\Delta OAC$ which is given by

$$2\times\frac{1}{2}\left|\frac{5}{3}\times\frac{7}{3} - \frac{5}{3}\times\frac{1}{3}\right| = \frac{30}{9} = \frac{10}{3}$$

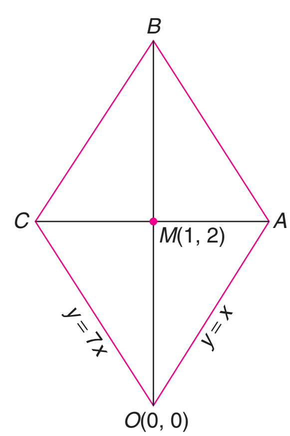

FIGURE 2.81

Answer: (C)

67. A point P on the line $3x + 5y - 15 = 0$ is equidistant from the coordinates axes. P can lie in

(A) first quadrant
(B) first or second quadrant
(C) first or third quadrant
(D) any quadrant

Solution: The point (x, y) is equidistant from both axes $\Leftrightarrow |x| = |y| \Rightarrow y = \pm x$.

1. $y = x$ and (x, y) lies on the line $3x + 5y = 15 \Rightarrow x = \dfrac{15}{8}$, $y = \dfrac{15}{8} \Rightarrow P$ lies in the first quadrant.
2. $y = -x$ and (x, y) lies on $3x + 5y = 15 \Rightarrow x = \dfrac{-15}{2}$, $y = \dfrac{15}{2} \Rightarrow P$ lies in the second quadrant.

Answer: (B)

68. Two equal sides of an isosceles triangle are given by the equations $y = -x$ and $y = 7x$ and its third side passes through the point $(1, -10)$. Then, the equations of the third side is

(A) $x - 3y - 31 = 0$ or $3x - y - 13 = 0$
(B) $x + 3y + 29 = 0$ or $3x - y - 13 = 0$
(C) $3x + y + 7 = 0$ or $x - 3y - 31 = 0$
(D) $3x + y + 7 = 0$ or $x + 3y + 29 = 0$

Solution: See Fig. 2.82. Let

$$y + 10 = m(x - 1) \tag{2.108}$$

be the equation of the third side. Substituting $y = -x$ in Eq. (2.108), we have

$$x = \frac{10 + m}{1 + m} \quad \text{and} \quad y = \frac{-(10 + m)}{1 + m}$$

Let

$$A = \left(\frac{10 + m}{1 + m}, \frac{-(10 + m)}{1 + m}\right)$$

Substituting $y = 7x$ in Eq. (2.108), we have

$$B = \left(\frac{10 + m}{m - 7}, \frac{7(10 + m)}{m - 7}\right)$$

Now,

$$OA = OB$$

$$\Rightarrow \frac{(10 + m)^2 + (10 + m)^2}{(m + 1)^2} = \frac{(10 + m)^2 + 49(10 + m)^2}{(m - 7)^2}$$

$$\Rightarrow \frac{\sqrt{2}(10 + m)}{m + 1} = \pm\frac{5\sqrt{2}(10 + m)}{m - 7}$$

$$\Rightarrow \frac{10 + m}{m + 1} = \pm\frac{5(10 + m)}{m - 7}$$

Case 1: $\dfrac{10 + m}{m + 1} = \dfrac{5(10 + m)}{m - 7}$

$$\Rightarrow (10 + m)[m - 7 - 5(m + 1)] = 0$$

$$\Rightarrow m = -10 \text{ or } m = -3$$

Case 2: $\dfrac{10 + m}{m + 1} = -\dfrac{5(10 + m)}{m - 7}$

$$\Rightarrow (10 + m)[m - 7 + 5(m + 1)] = 0$$

$$\Rightarrow m = -10 \text{ or } m = \frac{2}{6} = \frac{1}{3}$$

If $m = -10$, then the third side passes through the origin $(0, 0)$, which is false in this case. Thus

$$m = -3 \text{ or } \frac{1}{3}$$

Hence, the third side is

$$3x + y + 7 = 0$$

$$\Rightarrow x - 3y - 31 = 0$$

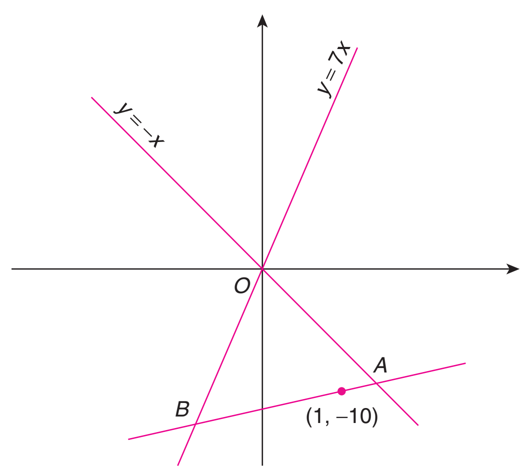

FIGURE 2.82

Answer: (C)

69. The equation of the line which passes through the intersection of the lines $x + 2y + 3 = 0$ and $3x + 4y + 7 = 0$ and is perpendicular the line $x - y + 8 = 0$ is

(A) $x + 2y + 3 = 0$ (B) $x + y - 2 = 0$
(C) $x + y + 2 = 0$ (D) $3x + 3y - 4 = 0$

Solution:

Direct Method: Intersection of the lines $x + 2y + 3 = 0$ and $3x + 4y + 7 = 0$ is $(-1, -1)$. Slope of the line $x - y + 8 = 0$ is 1. Hence, the equation of the required line is

$$y + 1 = -1(x + 1) \text{ or } x + y + 2 = 0$$

Answer: (C)

70. Let A be the point $(t, 2)$ and B be the point on the y-axis such that the slope of AB is $-t$. Then, the locus of the midpoint of AB, as t varies over all real numbers, is

(A) $y = 2 - 2x^2$ (B) $x^2 + y - 1 = 0$

(C) $y = 1 + x^2$ (D) $2x^2 - y + 2 = 0$

Solution: See Fig. 2.83. Equation of the line AB is

$$y - 2 = -t(x - t)$$

Now,

$$x = 0 \Rightarrow y = 2 + t^2$$
$$\Rightarrow B = (0, 2 + t^2)$$

Let $M(x, y)$ be the midpoint of AB. Therefore

$$x = \frac{t+0}{2}, y = \frac{2 + (2 + t^2)}{2}$$
$$\Rightarrow t = 2x, 2y = 4 + t^2 = 4 + 4x^2$$

Thus, the locus of M is $y = 2 + 2x^2$.

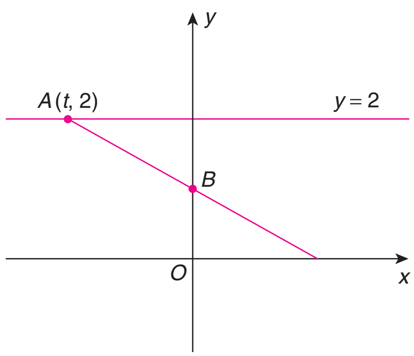

FIGURE 2.83

Answer: (D)

71. If the three families of lines

$$(y - 2x + 1) + \lambda_1(2y - x - 1) = 0$$
$$(3y - x - 6) + \lambda_2(y - 3x + 6) = 0$$

and $$(ax + y - 2) + \lambda_3(6x + ay - a) = 0 \qquad (a > 0)$$

have a common line, then the value of a is

(A) 3 (B) 2 (C) 6 (D) 4

Solution: The first and second families of lines are concurrent at (1, 1) and (3, 3), respectively, while the third family of lines are concurrent at

$$\left(\frac{a}{a^2-6}, \frac{a^2-12}{a^2-6}\right)$$

The three families have a common line if

$$\left(\frac{a}{a^2-6}, \frac{a^2-12}{a^2-6}\right), \ (1, 1) \text{ and } (3, 3)$$

are collinear. Since the equation of the line joining $(1, 1)$ and $(3, 3)$ is $y = x$, we have

$$\frac{a}{a^2-6} = \frac{a^2-12}{a^2-6}$$
$$\Rightarrow a^2 - a - 12 = 0$$
$$\Rightarrow (a-4)(a+3) = 0$$
$$\Rightarrow a = 4, -3$$
$$\Rightarrow a = 4 \qquad (\because a > 0)$$

Answer: (D)

72. If the length of the intercept made on the line $y = ax$ by the lines $y = 2$ and $y = 6$ is less than 5, then

(A) $-\frac{4}{3} < a < \frac{4}{3}$ (B) $a < \frac{-4}{3}$ or $a > \frac{4}{\sqrt{3}}$

(C) $\frac{-3}{4} < a < \frac{4}{3}$ (D) $a \in (-\infty, \infty)$

Solution: The line $y = ax$ intersects the lines $y = 2$ and $y = 6$ at points $A(2/a, 2)$ and $B(6/a, 6)$. Now

$$AB < 5 \Leftrightarrow \left(\frac{2}{a} - \frac{6}{a}\right)^2 + (2-6)^2 < 5^2$$
$$\Leftrightarrow \frac{16}{a^2} < 25 - 16 = 9$$
$$\Leftrightarrow a^2 > \frac{16}{9}$$
$$\Leftrightarrow a < \frac{-4}{3} \text{ or } a > \frac{4}{3}$$

Answer: (B)

73. The equation of the obtuse angle bisector of the angle between the lines $x - 2y + 4 = 0$ and $4x - 3y + 2 = 0$ is

(A) $(4 - \sqrt{5})x + (2\sqrt{5} - 3)y - (4\sqrt{5} - 2) = 0$

(B) $(4 - \sqrt{5})x - (2\sqrt{5} - 3)y - 4\sqrt{5} = 0$

(C) $(4 + \sqrt{5})x + (2\sqrt{5} - 3)y - 4\sqrt{5} + 2 = 0$

(D) $(4 + \sqrt{5})x - (2\sqrt{5} - 3)y + 4\sqrt{5} = 0$

Solution: $c_1 = 4, c_2 = 2$ are positive. Since $a_1a_2 + b_1b_2 = 4 + 6 > 0$, by Theorem 2.26,

$$\frac{x-2y+4}{\sqrt{1^2+2^2}} = +\left(\frac{4x-3y+2}{\sqrt{4^2+3^2}}\right)$$

is the obtuse angle bisector. So we have

$$\sqrt{5}(x-2y+4) = 4x-3y+2$$

$$\Rightarrow (4-\sqrt{5})x+(2\sqrt{5}-3)y-(4\sqrt{5}-2)=0$$

Answer: (A)

74. Two vertices of a triangle are $(5,-1)$ and $(-2,3)$. If origin is the orthocentre of the triangle, then the third vertex is

(A) $(4,7)$ (B) $(-4,7)$
(C) $(4,-7)$ (D) $(-4,-7)$

Solution: See Fig. 2. 84. Let $B=(5,-1)$ and $C=(-2,3)$, and the third vertex $A=(h,k)$. The orthocentre is $O(0,0)$. Since AO is perpendicular to BC, we have

$$\left(\frac{k}{h}\right)\left(\frac{3+1}{-2-5}\right)=-1$$

$$\Rightarrow 4k=7h \tag{2.109}$$

Now BO is perpendicular to AC

$$\Rightarrow \left(\frac{-1}{5}\right)\left(\frac{3-k}{-2-h}\right)=-1$$

$$\Rightarrow 3-k=-5(2+h)$$

$$\Rightarrow 5h-k=-13 \tag{2.110}$$

Solving Eqs. (2.109) and (2.110), we have $h=-4$ and $k=-7$. Hence, $A=(-4,-7)$.

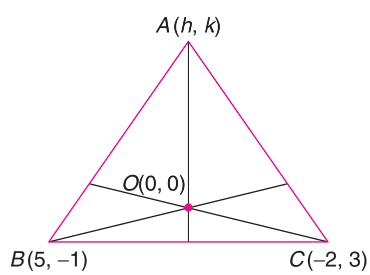

FIGURE 2.84

Answer: (D)

75. Area of the triangle formed by the line $x+y=3$ and the angle bisectors of the pair of lines $x^2-y^2+2y-1=0$ is

(A) 2 (B) 4 (C) 6 (D) 8

(IIT-JEE 2004)

Solution:

$$x^2-y^2+2y-1=0 \Rightarrow x^2-(y-1)^2=0$$

$$\Rightarrow (x+y-1)(x-y+1)=0$$

Therefore, the pair of lines is $x+y-1=0$ and $x-y+1=0$ and their angle bisectors are

$$\frac{x+y-1}{\sqrt{2}} = \pm\left(\frac{x-y+1}{\sqrt{2}}\right)$$

That is $y=1$ and $x=0$. Therefore, the sides of the triangle are $x=0$, $y=1$ and $x+y=3$. The vertices are $(0,1)$, $(0,3)$ and $(2,1)$. Hence the area of the triangle is

$$\frac{1}{2}|0(3-1)+0(1-1)+2(1-3)| = \frac{4}{2}=2$$

Answer: (A)

76. The slope of one of the lines $ax^2+2hxy+by^2=0$ is twice that of the other. Then

(A) $8h^2=9ab$ (B) $4h^2=3ab$
(C) $h^2=4ab$ (D) $9h^2=8ab$

Solution: Let $y=mx$ and $y=2mx$ be the two lines. Therefore

$$ax^2+2hxy+6y^2 \equiv b(y-mx)(y-2mx)$$

Equating the corresponding coefficients, we have

$$-3m=\frac{2h}{b}, 2m^2=\frac{a}{b}$$

$$\Rightarrow \frac{a}{b}=2m^2=2\left(\frac{-2h}{3b}\right)^2=\frac{8h^2}{9b^2}$$

$$\Rightarrow 9ab=8h^2$$

Answer: (A)

77. A is a point on the x-axis. Through the point A, a line is drawn parallel to y-axis so as to meet the lines $ax^2+2hxy+by^2=0$ in B and C. If $AB=BC$, then

(A) $9h^2=8ab$ (B) $8h^2=9ab$
(C) $4h^2=3ab$ (D) $3h^2=8ab$

Solution: See Fig. 2.85. Let $y=m_1x$ and $y=m_2x$ be the lines represented by $ax^2+2hxy+by^2=0$. Therefore

$$m_1+m_2=\frac{-2h}{b}$$

$$m_1m_2=\frac{a}{b}$$

Suppose $A=(\alpha,0)$ and the line $x=\alpha$ meets $y=m_1x$ at B and $y=m_2x$ at C. Thus, $B=(\alpha, m_1\alpha)$ and $C=(\alpha, m_2\alpha)$. Now,

$$AB=BC \Rightarrow \frac{m_2\alpha+0}{2}=m_1\alpha$$

$$\Rightarrow m_2=2m_1$$

Hence, from Worked-Out Problem 76, we get $8h^2 = 9ab$.

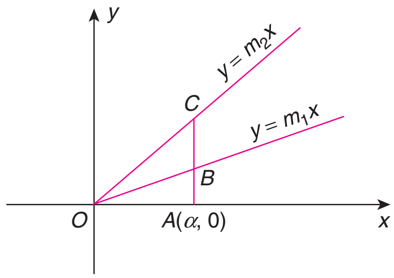

FIGURE 2.85

Answer: (B)

78. If one of the lines represented by $ax^2 + 2hxy + by^2 = 0$ bisects the angle between the axes, then

(A) $4h^2 = ab$ (B) $8h^2 = 9ab$
(C) $(a+b)^2 = 4h^2$ (D) $(a+b)^2 = 2h^2$

Solution: Suppose $y = x$ is one of the lines. That is,

$$ax^2 + 2hx^2 + bx^2 = 0$$
$$\Rightarrow a + b = -2h$$
$$\Rightarrow (a+b)^2 = 4h^2$$

Suppose $y = -x$ is one of the lines. Then

$$a - 2h + b = 0$$
$$\Rightarrow (a+b)^2 = 4h^2$$

Answer: (C)

79. If the equation $12x^2 + 7xy + ky^2 + 13x - y + 3 = 0$ represents a pair of lines, then the value of k is

(A) -5 (B) 5 (C) 10 (D) -10

Solution: Comparing the given equation with the second-degree general equation, we have

$$a = 12, h = \frac{7}{2}, b = k, g = \frac{13}{2}, f = \frac{-1}{2} \text{ and } c = 3$$

Since the equation represents pair of lines, by Theorem 2.31, we get

$$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

That is,

$$(12)(k)(3) + 2\left(\frac{-1}{2}\right)\left(\frac{13}{2}\right)\left(\frac{7}{2}\right) - 12\left(\frac{1}{4}\right) - k\left(\frac{169}{4}\right)$$
$$-3\left(\frac{49}{4}\right) = 0$$
$$\Rightarrow 144k - 91 - 12 - 169k - 147 = 0$$
$$\Rightarrow -25k = 250$$
$$\Rightarrow k = -10$$

Answer: (D)

80. Let PQR be right-angled isosceles triangle right angled at $P(2, 1)$. If the equation of the line QR is $2x + y - 3 = 0$ then the combined equation of the pair of lines PQ and PR is

(A) $3x^2 - 3y^2 + 8xy + 20x + 10y + 25 = 0$
(B) $3x^2 - 3y^2 + 8xy - 20x - 10y + 25 = 0$
(C) $3x^2 - 3y^2 + 8xy + 10x + 15y + 20 = 0$
(D) $3x^2 - 3y^2 - 8xy - 15y - 20 = 0$

(IIT-JEE 1999)

Solution: See Fig. 2.86. Let m be the slope of the line PQ. Since $\underline{|PQR} = 45°$, from the 'Note' of Theorem 2.12, we have

$$1 = \tan 45° = \left|\frac{m+2}{1+m(-2)}\right|$$
$$\Rightarrow m + 2 = \pm(1 - 2m)$$
$$\Rightarrow m = 3, \frac{-1}{3}$$

Therefore, the combined equation of the lines PQ and PR is

$$[y - 1 - 3(x-2)]\left[y - 1 + \frac{1}{3}(x-2)\right]$$
$$\Rightarrow (3x - y - 5)(x + 3y - 5) = 0$$
$$\Rightarrow 3x^2 - 3y^2 + 8xy - 20x - 10y + 25 = 0$$

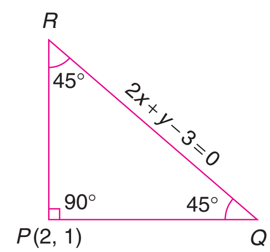

FIGURE 2.86

Answer: (B)

81. Area of the parallelogram formed by the lines $y = mx$, $y = mx + 1$, $y = nx$ and $y = nx + 1$, where $m \neq n$, is equal to

(A) $\dfrac{|m+n|}{(m-n)^2}$ (B) $\dfrac{2}{|m+n|}$
(C) $\dfrac{1}{|m+n|}$ (D) $\dfrac{1}{|m-n|}$

(IIT-JEE 2001)

Solution: From Problem 2 of the section 'Subjective Problems', the area of the parallelogram is

$$\left|\frac{(1-0)(1-0)}{m-n}\right| = \frac{1}{|m-n|}$$

Answer: (B)

82. All chords of the curve $3x^2 - y^2 - 2x + 4y = 0$ which subtend a right angle at the origin pass through a fixed point whose coordinates are

(A) $(0, 0)$ (B) $(1, -2)$

(C) $\left(\frac{1+\sqrt{10}}{3}, -1\right)$ (D) $(2, -1)$

(IIT-JEE 1982)

Solution: See Fig. 2.87. Let $lx + my = 1$ be the chord of the given curve subtending right angle at the origin: Suppose the line meets the curve at A and B. Hence, by Theorem 2.33, the combined equation of the pair of lines OA and OB is

$$3x^2 - y^2 - (2x - 4y)(lx + my) = 0$$

Since $\underline{|AOB} = 90°$, from the above equation and from Theorem 2.28, we have

$$\text{Coefficient of } x^2 + \text{Coefficient of } y^2 = 0$$

$$\Rightarrow (3 - 2l) + (-1 + 4m) = 0$$

$$\Rightarrow l - 2m - 1 = 0$$

$$\Rightarrow l + m(-2) = 1$$

Hence the line $lx + my = 1$ passes through the point $(1, -2)$.

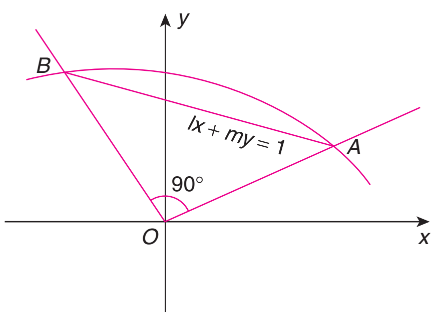

FIGURE 2.87

Answer: (B)

83. The area of the parallelogram formed by the lines $3x^2 + 10xy + 8y^2 + 14x + 22y + 15 = 0$ and $3x^2 + 10xy + 8y^2 = 0$ is

(A) 5 (B) 10 (C) $\frac{15}{2}$ (D) 15

Solution: We have

$$3x^2 + 10xy + 8y^2 \equiv (3x + 4y)(x + 2y)$$

Therefore, for some values of n_1 and n_2

$$3x^2 + 10xy + 8y^2 + 14x + 22y + 15 \equiv$$
$$(3x + 4y + n_1)(x + 2y + n_2) \quad (2.111)$$

Equating the coefficients of x and y on both sides of Eq. (2.111), we have

$$n_1 + 3n_2 = 14 \quad (2.112)$$

$$2n_1 + 4n_2 = 22 \quad (2.113)$$

Solving Eqs. (2.112) and (2.113), we have $n_1 = 5$ and $n_2 = 3$. Therefore, the sides of the parallelogram are

$$3x + 4y = 0$$
$$3x + 4y + 5 = 0$$
$$x + 2y = 0$$

and $$x + 2y + 3 = 0$$

Hence, from Problem 2 of the section 'Subjective Problems', the area of the parallelogram is

$$\left|\frac{(5-0)(3-0)}{3(2)-(4)}\right| = \frac{15}{2} \text{ sq. unit}$$

Answer: (C)

84. If the second-degree general equation $S \equiv ax^2 + 2hmy + by^2 + 2gx + 2fy + c = 0$ represents pair of intersecting lines, then the area of the parallelogram formed by the lines $S = 0$ and $ax^2 + 2hxy + by^2 = 0$ is

(A) $\frac{|C|}{\sqrt{(a-b)^2 + 4h^2}}$ (B) $\frac{|C|}{2\sqrt{(a-b)^2 + 4ab}}$

(C) $\frac{|C|}{2\sqrt{(a-b)^2 + 4h^2}}$ (D) $\frac{|C|}{2\sqrt{h^2 - ab}}$

Solution: Suppose the lines represented by $S = 0$ are $l_1x + m_1y + n_1 = 0$ and $l_2x + m_2y + n_2 = 0$ so that from Theorem 2.32, the equation $ax^2 + 2hxy + by^2 = 0$ represents $l_1x + m_1y = 0$ and $l_2x + m_2y = 0$. Also we have

$$l_1l_2 = a$$
$$l_1m_2 + l_2m_1 = 2h$$
$$m_1m_2 = b$$
$$l_1n_2 + l_2n_1 = 2g$$
$$m_1n_2 + m_2n_1 = 2f$$

and $$n_1n_2 = c$$

Hence the sides of the parallelogram are

$$l_1x + m_1y = 0$$
$$l_1x + m_1y + n_1 = 0$$
$$l_2x + m_2y = 0$$
$$l_2x + n_2y + n_2 = 0$$

Therefore, from Problem 2 of the section 'Subjective Problems', the area of the parallelogram is

$$\left|\frac{(n_1-0)(n_2-0)}{l_1m_2 - l_2m_1}\right| = \left|\frac{n_1n_2}{\sqrt{(l_1m_2 + l_2m_1)^2 - 4l_1l_2m_1m_2}}\right|$$

$$= \frac{|c|}{\sqrt{4h^2 - 4ab}} = \frac{|c|}{2\sqrt{h^2 - ab}}$$

Answer: (D)

Note: By using this formula, for the parallelogram in Worked-Out Problem 83, the area is equal to

$$\frac{|15|}{2\sqrt{25-24}} = \frac{15}{2}$$

85. If $x^2 - 2pxy - y^2 = 0$ and $x^2 - 2qxy - y^2 = 0$ represent a pair of lines ($p \neq q$) such that each pair bisects the angle between the other, then

(A) $pq = 1$ (B) $pq = -1$

(C) $\frac{1}{p} + \frac{1}{q} = 1$ (D) $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$

Solution: By Theorem 2.30, the equation of the pair of angle bisector of the lines $x^2 - 2pxy - y^2 = 0$ is

$$-p(x^2 - y^2) = (1+1)xy$$
$$\Rightarrow px^2 + 2xy - py^2 = 0$$

However, by hypothesis, $x^2 - 2qxy - y^2 = 0$ is the pair of angle bisectors. Therefore,

$$\frac{p}{1} = \frac{2}{-2q} = \frac{-p}{-1} \Rightarrow pq = -1$$

Answer: (B)

86. A straight line is drawn through the point $(1, 2)$ making an angle $\theta (0 < \theta \le \pi/3)$ with the positive direction of the x-axis to intersect the line $x + y - 4 = 0$ at a point P so that the distance between P and $(1, 2)$ is $\sqrt{2/3}$. Then, the value of θ is

(A) $\frac{\pi}{3}$ (B) $\frac{\pi}{10}$ (C) $\frac{\pi}{12}$ (D) $\frac{\pi}{18}$

Solution: Let A be $(1, 2)$ and the line through $A(1, 2)$ be $y - 2 = m(x-1)$. This line meets the line $x + y = 4$ at point P. So

$$4 - x - 2 = m(x-1)$$
$$\Rightarrow x(1+m) = 2+m$$
$$\Rightarrow x = \frac{2+m}{1+m} \text{ and } y = 4 - x = 4 - \frac{2+m}{1+m} = \frac{3m+2}{1+m}$$

Therefore,

$$P = \left(\frac{2+m}{1+m}, \frac{3m+2}{1+m}\right)$$

Now,

$$AP = \sqrt{\frac{2}{3}} \Rightarrow \left(1 - \frac{2+m}{1+m}\right)^2 + \left(2 - \frac{3m+2}{1+m}\right)^2 = \frac{2}{3}$$

$$\Rightarrow \frac{1}{(1+m)^2} + \frac{m^2}{(1+m)^2} = \frac{2}{3}$$
$$\Rightarrow 3m^2 + 3 = 2(1 + 2m + m^2)$$
$$\Rightarrow m^2 - 4m + 1 = 0$$
$$\Rightarrow m = \frac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3}$$

Now

$$0 < \theta \le \frac{\pi}{3} \Rightarrow \tan\theta = m = 2 - \sqrt{3}$$

so that $\theta = 15°$.

Answer: (C)

87. The equation $16x^4 - y^4 = 0$ represents

(A) a single straight line (B) a pair of lines

(C) a single point (D) $xy = 2$

Solution: The given equation is

$$(4x^2 + y^2)(2x + y)(2x - y) = 0$$

The equation $4x^2 + y^2 = 0$ represents the single point (0, 0). Hence, the equation represents the pair of lines $2x + y = 0$ and $2x - y = 0$ which pass through the origin.

Answer: (B)

88. If the lines $x + 2y = 9$, $3x - 5y = 5$ and $ax + by = 1$ are concurrent, then the line $5x + 2y - 1 = 0$ passes through the point

(A) (b, a) (B) (a, b)

(C) $(-a, b)$ (D) $(-a, -b)$

Solution: The point of intersection of the lines $x + 2y - 9 = 0$ and $3x - 5y - 5 = 0$ is $(5, 2)$ which also lies on the line $ax + by = 1$. Hence,

$$a(5) + b(2) = 1$$
$$\Rightarrow 5a + 2b = 1$$

Therefore, the line $5x + 2y = 1$ passes through the point (a, b).

Answer: (B)

89. Straight lines are drawn from the point $A(3, 2)$ to meet the line $6x + 7y - 30 = 0$ at point P. Then, the locus of the midpoints of the segment AP is

(A) $x^2 - y^2 = 30$ (B) $6x + 7y = 31$

(C) $(6x-3)^2 + (7y-2)^2 = 30^2$ (D) $6x + 7y = 32$

Solution: Let the line that passes through $A(3, 2)$ be

$$\frac{x-3}{\cos\theta} = \frac{y-2}{\sin\theta} = r \quad \text{(say)}$$

Every point on this line is of the form $P(3+r\cos\theta, 2+r\sin\theta)$ and this lies on the line $6x + 7y = 30$. It implies that

$$r(6\cos\theta+7\sin\theta) = -2 \qquad (2.114)$$

Suppose $M(h, k)$ be the midpoint of AP. Therefore,

$$2h = 6+r\cos\theta \text{ and } 2k = 4+r\sin\theta$$

$$\Rightarrow 6(2h-6)+7(2k-4) = r(6\cos\theta+7\sin\theta) = -2$$

[from Eq. (2.114)]

$$\Rightarrow 12h+14k-62=0$$
$$\Rightarrow 6h+7k-31=0$$

Hence, the locus of $M(h, k)$ is the line $6x+7y-31=0$.

Answer: (B)

90. If no two lines of the three lines

$$(m-2)x+(2m-5)y=0$$
$$(m-1)x+(m^2-7)y-5=0$$
$$\text{and} \qquad x+y-1=10$$

are parallel, then the three lines are concurrent

(A) for three values of m (B) for two values of m
(C) for one value of m (D) no real value of m

Solution: By Theorem 2.22, the three lines are concurrent if

$$\begin{vmatrix} 1 & 1 & -1 \\ m-2 & 2m-5 & 0 \\ m-1 & m^2-7 & -5 \end{vmatrix} = 0$$

$$\Rightarrow -1[(m-2)(m^2-7)-(m-1)(2m-5)] -5[2m-5-m+2]=0$$

$$\Rightarrow m^3-2m^2-7m+14-2m^2+5m+2m-5 +10m-25-5m+10=0$$

$$\Rightarrow m^3-4m^2+5m-6=0$$

$$\Rightarrow (m-3)(m^2-m+2)=0$$

Since $(m^2-m+2)=0$ has no real solutions, it follows that $m = 3$. However, $m = 3$ implies that the three lines are parallel which is in contradiction of the hypothesis. Hence, if $m \neq 3$, then the lines cannot be concurrent.

Answer: (D)

91. If $25a^2+16b^2-4ab-c^2=0$, then the line $2ax+by+c=0$ passes through a fixed point whose coordinates are

(A) $\left(\frac{5}{2}, 4\right)$ (B) $\left(\frac{5}{2}, -4\right)$
(C) $\left(-\frac{5}{2}, -4\right)$ (D) $(5, -4)$

Solution: The given relation is

$$(5a-4b)^2-c^2=0$$
$$\Rightarrow (5a-4b+c)(5a-4b-c)=0$$
$$\Rightarrow \left[2a\left(\frac{5}{2}\right)+b(-4)+c\right]\left[2a\left(-\frac{5}{2}\right)+b(4)+c\right]=0$$
$$\Rightarrow 2ax+by+c=0$$

It passes through

$$\left(\frac{5}{2}, -4\right) \text{ and } \left(-\frac{5}{2}, 4\right)$$

Answer: (B)

92. The area of the triangle formed by the lines $x + y = 3$ and angle bisectors of the pair of straight lines $x^2 - y^2 + 2y = 1$ is

(A) 2 sq. unit (B) 4 sq. unit
(C) 6 sq. unit (D) 8 sq. unit

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Solution: The given lines are

$$x^2-(y-1)^2=0$$
$$\Rightarrow (x+y-1)(x-y+1)=0$$

Therefore, the given lines are $x-y+1=0$ and $x+y-1=0$ whose angular bisectors are

$$\frac{x+y-1}{\sqrt{2}} = \pm\frac{x-y+1}{\sqrt{2}}$$

That is, $y-1=0$ and $x=0$. Thus, the vertices of the triangle are (0, 1), (0, 3) and (2, 1). Hence, the area is

$$\frac{1}{2}[0(3-1)+0(1-1)+2(1-3)] = 2 \text{ sq. unit}$$

Answer: (A)

93. A straight line is drawn through (1, 0) to the curve $x^2 + y^2 + 6x - 10y + 1 = 0$ such that the intercept made on it by the curve subtends a right angle at the origin. Then, the slope of the line is

(A) 1 or $\frac{1}{9}$ (B) -1 or $-\frac{1}{9}$
(C) -1 or $\frac{1}{9}$ (D) 1 or $-\frac{1}{9}$

Solution: Let $y = m(x - 1)$ be the line meeting the curve in the point A and B. Hence, by Theorem 2.33, the combined equation of the pair of lines OA and OB ('O' is the origin) is

$$x^2+y^2+(6x-10y)\frac{(mx-y)}{m}+1\frac{(mx-y)^2}{m}=0$$

Since $\underline{|AOB} = 90°$, from the above equation, we have

$$\text{Coefficient of } x^2 + \text{Coefficent of } y^2 = 0$$

$$\Rightarrow (1+6+1)+\left(1+\frac{10}{m}+\frac{1}{m^2}\right)=0$$

$$\Rightarrow 9m^2+10m+1=0$$

$$\Rightarrow (9m+1)(m+1)=0$$

$$\Rightarrow m=\frac{-1}{9} \text{ or } m=-1$$

Answer: (B)

94. The area bounded by the curves $y=|x|-1$ and $y=-|x|+1$ is

(A) 1 (B) 2 (C) $2\sqrt{2}$ (D) 4

Solution: The given lines are

$$y=x-1, y=-x-1 \text{ and } y=-x+1, y=-x-1$$

That is, the lines are

$$x-y-1=0, x+y+1=0, x+y-1=0 \text{ and } x-y+1=0$$

Therefore, from Problem 2 of the section 'Subjective Problems', the area of the parallelogram is

$$\left|\frac{(1+1)(1+1)}{1-(-1)}\right|=2$$

Answer: (B)

95. The area of the parallelogram formed by the pairs of lines $x^2 + 4xy + 4y^2 - 5x - 10y + 4 = 0$ and $y^2 - 4y + 3 = 0$ sq. units is

(A) 4 (B) 6 (C) 8 (D) 12

Solution: $y^2-4y+3=0$ represent the lines $y-1=0$ and $y-3=0$.

The lines represented by $x^2 + 4xy + 4y^2 - 5x - 10y + 4 = 0$ are parallel lines and they are $x+2y-4=0$ and $x+2y-1=0$.

Therefore, from Problem 2 of the section 'Subjective Problems', the area is

$$\left|\frac{(-1+3)(-1+4)}{1-0}\right|=6$$

Answer: (B)

96. A straight line L passing through the point $(3, -2)$ is inclined at an angle 60° to the line $y + \sqrt{3}x = 1$. If L intersects the x-axis, then the equation of L is

(A) $y+\sqrt{3}x+2-3\sqrt{3}=0$

(B) $y-\sqrt{3}x+2+3\sqrt{3}=0$

(C) $\sqrt{3}y-x+3+2\sqrt{3}=0$

(D) $\sqrt{3}y+x-3+2\sqrt{3}=0$

(IIT-JEE 2011)

Solution: Let the equation of the line L be $y + 2 = m(x - 3)$. Therefore, by hypothesis

$$\sqrt{3}=\left|\frac{m+\sqrt{3}}{1-m\sqrt{3}}\right|$$

$$\Rightarrow m+\sqrt{3}=\pm\sqrt{3}(1-m\sqrt{3})$$

$$\Rightarrow m=0 \text{ or } m=\sqrt{3}$$

If $m = 0$, then L should be horizontal, which is not true in this case. Hence, $m=\sqrt{3}$ and the equation of L is

$$y+2=\sqrt{3}(x-3)$$

$$\Rightarrow y-\sqrt{3}x+2+3\sqrt{3}=0$$

Answer: (B)

Multiple Correct Choice Type Questions

1. Let L_1 be a straight line passing through the origin and L_2 be the straight line $x + y = 1$. If the intercepts made by the circle $x^2 + y^2 - x + 3y = 0$ on L_1 and L_2 are equal, then L_1 may be represented by the equations

(A) $x+y=0$ (B) $x-y=0$

(C) $x+7y=0$ (D) $x-7y=0$

Solution: The equation of the circle can be written as

$$\left(x-\frac{1}{2}\right)^2+\left(y+\frac{3}{2}\right)^2=\frac{5}{2}$$

so that its centre is at $(1/2, -3/2)$ and radius is $\sqrt{5}/2$. Let $y=mx$ be the equation of L_1. Since the intercepts are made by the circle on L_1 and L_2, it follows that the two chords are equidistant from the centre. Therefore

$$\left|\frac{m\left(\frac{1}{2}\right)+\frac{3}{2}}{\sqrt{1+m^2}}\right|=\left|\frac{\frac{1}{2}-\frac{3}{2}-1}{\sqrt{1^2+1^2}}\right|$$

$$\Rightarrow \frac{(m+3)^2}{4(1+m^2)}=2$$

$$\Rightarrow m^2+6m+9=8+8m^2$$

$$\Rightarrow 7m^2-6m-1=0$$

$$\Rightarrow (7m+1)(m-1)=0$$

$$\Rightarrow m = 1, -\frac{1}{7}$$

Hence, the equations of L_1 are $y = x$ and $y = -x/7$. That is, $x - y = 0$ and $x + 7y = 0$.

Answer: (B), (C)

2. The lines $ax + by + c = 0$, $bx + cy + a = 0$ and $cx + ay + b = 0$ are concurrent if

(A) $a + b + c = 0$
(B) $a^2 + b^2 + c^2 = ab + bc + ca$
(C) $a^3 + b^3 + c^3 = 3abc$
(D) $a^2 + b^2 + c^2 = 2(ab + bc + ca)$

Solution: The three lines are concurrent if

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0 \text{ (By Theorem 2.22)}$$

$$\Rightarrow 3abc - a^3 - b^3 - c^3 = 0$$
$$\Rightarrow a^3 + b^3 + c^3 - 3abc = 0$$

Also

$$a^3 + b^3 + c^3 - 3abc = 0$$
$$\Rightarrow (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) = 0$$
$$\Rightarrow a + b + c = 0 \text{ or } a^2 + b^2 + c^2 = ab + bc + ca$$

Answer: (A), (B), (C)

3. The area of a triangle ABC is 20 sq. unit. The coordinates of A are $(-5, 0)$ and those of B are $(3, 0)$ and the vertex C lies on the line $x - y - 2 = 0$. The coordinates of C are

(A) $(-3, -5)$ (B) $(-5, -7)$
(C) $(5, 3)$ (D) $(7, 5)$

Solution: Let C be $(x, x - 2)$. Therefore

$$20 = \frac{1}{2}|-5(0 - x + 2) + 3(x - 2 - 0) + x(0 - 0)|$$
$$\Rightarrow 40 = |8x - 16|$$
$$\Rightarrow x - 2 = \pm 5$$

Hence, $x = 7, -3$. Therefore, $C = (7, 5), (-3, -5)$.

Answer: (A), (D)

4. If $A(2, -3)$ and $C(-1, 1)$ are the ends of a diagonal of a squares $ABCD$, then the other vertices are

(A) $\left(\frac{1}{2}, \frac{5}{2}\right)$ (B) $\left(\frac{5}{2}, \frac{1}{2}\right)$

(C) $\left(\frac{-3}{2}, \frac{-5}{2}\right)$ (D) $\left(\frac{-1}{2}, \frac{-5}{2}\right)$

Solution: See Fig. 2.88. The point

$$M = \left(\frac{1}{2}, -1\right)$$

is the intersection of the diagonals so that

$$DM = MB = \frac{1}{2}AC = \frac{5}{2}$$

Since the slope of AC is $-4/3$, the equation of the diagonal BD is

$$y + 1 = \frac{3}{4}\left(x - \frac{1}{2}\right)$$
$$\Rightarrow y = \frac{3}{8}(2x - 1) - 1 = \frac{6x - 11}{8}$$

Let

$$B = \left(x, \frac{6x - 11}{8}\right)$$

Now,

$$MB = \frac{5}{2} \Rightarrow \left(x - \frac{1}{2}\right)^2 + \left(\frac{6x - 11}{8} + 1\right)^2 = \frac{25}{4}$$
$$\Rightarrow 16(2x - 1)^2 + (6x - 3)^2 = 25 \times 16$$
$$\Rightarrow 100x^2 - 100x - 375 = 0$$
$$\Rightarrow 4x^2 - 4x - 15 = 0$$
$$\Rightarrow (2x - 5)(2x + 3) = 0$$
$$\Rightarrow x = \frac{5}{2}, \frac{-3}{2}$$

Therefore

$$x = \frac{5}{2} \Rightarrow y = \frac{6x - 11}{8} = \frac{6(5/2) - 11}{8} = \frac{1}{2}$$

and $$x = \frac{-3}{2} \Rightarrow y = \frac{6(-3/2) - 11}{8} = \frac{-5}{2}$$

Therefore, other vertices are

$$\left(\frac{5}{2}, \frac{1}{2}\right) \text{ and } \left(\frac{-3}{2}, \frac{-5}{2}\right)$$

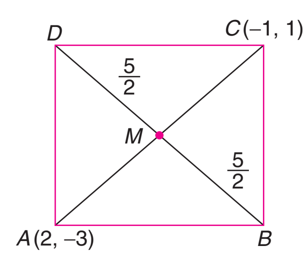


FIGURE 2.88

Aliter (Using Complex Numbers): For complex numbers, see Vol. 1. The points are

$$M=\left(\frac{1}{2},-1\right) \text{ and } C=(-1,1)$$

Replace the point D by Z. Therefore we have

$$\frac{Z-[(1/2)-i]}{(-1+i)-[(1/2)-i]}=i$$

$$\Rightarrow Z-\frac{1}{2}+i=\left(-\frac{3}{2}+2i\right)i=-2-\frac{3i}{2}$$

$$\Rightarrow z=-\frac{3}{2}-\frac{5}{2}i$$

$$\Rightarrow D=\left(-\frac{3}{2},-\frac{5}{2}\right)$$

Hence

$$B=\left(2-1+\frac{3}{2},3-1+\frac{5}{2}\right)=\left(\frac{5}{2},\frac{1}{2}\right)$$

Answer: (B), (C)

5. Equation of a line passing through (1, 1) which is at a distance of 1 unit from the origin is

(A) $x=1$ (B) $y=1$
(C) $x+y=1$ (D) $x+y=2$

Solution: It is clear that the vertical line $x = 1$ passes through (1, 1) and its distance from the origin is 1. Hence, $y=1$ is the other line.

Answer: (A), (B)

6. The equation of the lines passing through the point (1, 1) whose distance from origin is 2 is

(A) $y-1=\dfrac{-1+\sqrt{10}}{3}(x-1)$

(B) $y-1=\dfrac{-1+\sqrt{3}}{3}(x-1)$

(C) $y-1=\dfrac{-1-\sqrt{3}}{3}(x-1)$

(D) $y-1=\dfrac{-1-\sqrt{10}}{3}(x-1)$

Solution: Let $y = m(x - 1)$ be the line so that by hypothesis

$$\left|\frac{m(-1)-0-1}{\sqrt{1+m^2}}\right|=2$$

$$\Rightarrow (m-1)^2=4(1+m^2)$$

$$\Rightarrow 3m^2+2m-3=0$$

$$\Rightarrow m=\frac{-2\pm\sqrt{4+36}}{6}=\frac{-1\pm\sqrt{10}}{3}$$

Answer: (A), (D)

7. A line l is drawn through the point (1, 2) so that it intersects the line $x+y-4=0$ at a point whose distance from the point (1, 2) is $\sqrt{2/3}$. Then the angle made by l with the x-axis is

(A) 15° (B) 75° (C) 105° (D) 60°

Solution: Let $y - 2 = m(x-1)$ be the equation of the line l. That is,

$$y=mx+2-m \tag{2.115}$$

$$y=-x+4 \tag{2.116}$$

Solving Eqs. (2.115) and (2.116), we have

$$x=\frac{m+2}{m+1} \text{ and } y=\frac{3m+2}{m+1}$$

Since the distance between

$$\left(\frac{m+2}{m+1},\frac{3m+2}{m+1}\right) \text{ and } (1,2)$$

is $\sqrt{2/3}$, we have

$$\left(\frac{m+2}{m+1}-1\right)^2+\left(\frac{3m+2}{m+1}-2\right)^2=\frac{2}{3}$$

$$\Rightarrow \left(\frac{1}{m+1}\right)^2+\left(\frac{m^2}{(m+1)^2}\right)=\frac{2}{3}$$

$$\Rightarrow 3+3m^2=2(m+1)^2$$

$$\Rightarrow m^2-4m+1=0$$

$$\Rightarrow m=\frac{4\pm\sqrt{16-4}}{2}=2\pm\sqrt{3}$$

Now, $\tan\theta=2\pm\sqrt{3}$ implies $\theta=75°$ or 15°.

Answer: (A), (B)

8. Let $A(4,3)$, $B(-4,3)$ and $C(0,-5)$ be the vertices of a triangle and $P=(5,0)$. Let L, M and N be the feet of the perpendiculars drawn from P onto the sides BC, CA and AB, respectively. Then

(A) the centroid of ΔLMN does not exist
(B) the orthocentre of ΔLMN is the origin
(C) the area of ΔLMN is 5 sq. unit
(D) L, M and N are collinear

Solution: See Fig. 2.89. The sides BC, CA and AB are represented by the equations $2x+y+5=0$, $2x-y-5=0$ and $y-3=0$, respectively. We can see that $L=(-1,-3)$, $M=(3,1)$ and $N=(5,3)$. Now

$$\text{Slope of } LM=\text{Slope of } MN=1$$

so that L, M and N are collinear.

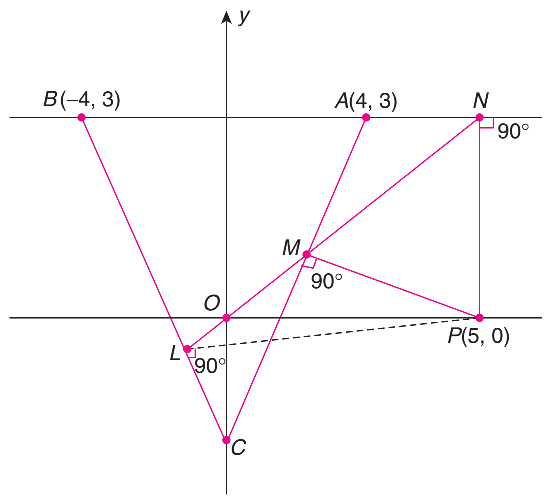


FIGURE 2.89

Note: See Theorem 1.1.

Answer: (A), (D)

9. A line through (2, 2) and the axes form a triangle of area α units. Then, the intercepts on the axes made by the line are roots of the equation

(A) $x^2 - \alpha x + \alpha = 0$ (B) $x^2 + \alpha x - \alpha = 0$
(C) $x^2 - \alpha x + 2\alpha = 0$ (D) $x^2 + \alpha x - 2\alpha$

Solution: Let

$$\frac{x}{a} + \frac{y}{b} = 1$$

be the equation of the line which forms a triangle with the coordinate axes of area α sq. unit. Since the line passes through (2, 2), we have

$$\frac{2}{a} + \frac{2}{b} = 1 \tag{2.117}$$

We have $\frac{1}{2}|ab| = \alpha$ so that

$$ab = \pm 2\alpha \tag{2.118}$$

Now, from Eq. (2.118)

$$2a + 2b = ab = 2\alpha$$

Therefore

$$a + b = \alpha \tag{2.119}$$

Hence, from Eqs. (2.118) and (2.119), a and b are the roots of the equation $x^2 - \alpha x + 2\alpha = 0$. Therefore

$$2a + 2b = -2\alpha \Rightarrow a + b = -\alpha$$

Hence a and b are roots of the equation

$$x^2 + \alpha x - 2\alpha = 0$$

Answer: (C), (D)

10. The points $A(2, 1)$and $B(3, -2)$ are the two vertices of ΔABC and C lies on the line $x - y + 3 = 0$. If the area of ΔABC is 5 sq. units, then the coordinates of the third vertex C are

(A) $\left(\frac{-3}{2}, \frac{3}{2}\right)$ (B) $(0, 0)$
(C) $\left(\frac{5}{2}, \frac{5}{2}\right)$ (D) $\left(\frac{7}{2}, \frac{13}{2}\right)$

Solution: Let C be $(x, x+3)$. By hypothesis, we have

$$5 = \frac{1}{2}|2(-2 - x - 3) + 3(x + 3 - 1) + x(1 + 2)|$$

$$\Rightarrow 10 = |-10 - 2x + 3x + 6 + 3x|$$

$$\Rightarrow 4x - 4 = \pm 10$$

$$\Rightarrow 4x = 14 \text{ or } -6$$

$$\Rightarrow x = \frac{7}{2} \text{ or } \frac{-3}{2}$$

Therefore, the coordinates of the third vertex C are

$$\left(\frac{7}{2}, \frac{13}{2}\right) \text{ or } \left(\frac{-3}{2}, \frac{3}{2}\right)$$

Answer: (A), (D)

Matrix-Match Type Questions

1. $A(-2, 1)$, $B(5, 4)$ and $C(2, -3)$ are the vertices of ΔABC. AD, BE and CF are the altitudes of the triangle and M is the midpoint of BC. Match the items of Column I with those of Column II.

Column I	*Column II*
(A) Equation of AD is	(p) $x - y - 1 = 0$
(B) Equation of BE is	(q) $x + 11y - 9 = 0$
(C) Equation of the median AM is	(r) $7x + 3y - 5 = 0$
(D) Equation of the altitude CF is	(s) $x + 11y - 11 = 0$
	(t) $3x + 7y - 1 = 0$

(*Continued*)

Solution:

(A) Slope of BC is

$$\frac{4+3}{5-2}=\frac{7}{3}$$

Therefore, the equation of the altitude AD is

$$y-1=\frac{-3}{7}(x+2)$$

$$\Rightarrow 3x+7y-1=0$$

Answer: (A) → (t)

(B) Slope of CA is

$$\frac{1+3}{-2-2}=-1$$

Therefore, the equation of the altitude BE is

$$y-4=1(x-5)$$

$$\Rightarrow x-y-1=0$$

Answer: (B) → (p)

(C) The midpoint of BC is

$$\left(\frac{7}{2},\frac{1}{2}\right)$$

and the slope of the median AM is $-1/11$ so that the equation of the median AM is

$$y-1=\frac{-1}{11}(x+2)$$

$$\Rightarrow x+11y-9=0$$

Answer: (C) → (q)

(D) Lastly, the slope of AB is

$$\frac{4-1}{5+2}=\frac{3}{7}$$

and hence the equation of the altitude CF is

$$y+3=\frac{-7}{3}(x-2)$$

$$\Rightarrow 7x+3y-5=0$$

Answer: (D) → (r)

2. Match the items of Column I with those of Column II.

Column I	*Column II*
(A) If the line segment joining the points $P(1, 3)$ and $Q(5, 7)$ subtends a right angle at a point R, such that the area of ΔPQR is 2 sq. unit, then the number of such points R is	(p) 2
(B) If $A(1, 2)$, $B(4, 6)$, $C(5, 7)$ and $S(a, b)$ are the vertices of a parallelogram in the given order, then the value of $a+b$ is	(q) 1
(C) If $\left(\frac{p}{q},\frac{r}{s}\right)$ is the centroid of ΔABC given in (B), then the value of $\frac{p+r}{q+s-1}$ is	(r) 4
(D) Let $p=\lim\limits_{n\to\infty}\lim\limits_{m\to\infty}\cos^{2n}\lfloor m\,\pi x$ where x rational and $q=\lim\limits_{n\to\infty}\lim\limits_{m\to\infty}\cos^{2m}\lfloor n\,x$, where x is irrational. Then the area of the triangle with vertices (p, q), $(2, 1)$ and $(-2, 1)$ is	(s) 3
	(t) 5

Solution:

(A) Since $\lfloor PRQ=90°$, in general, the locus represented by R is a circle with P and Q as ends of the diameter. Because area of ΔPQR is 2 sq. unit, there will be four positions for R (two each in the two semicircles for which PQ is a diameter).

Answer: (A) → (r)

(B) It is known that $a=1+5-4=2$ and $b=2+7-6=3$. Therefore $a+b=5$.

Answer: (B) → (t)

(C) Centroid $\left(\frac{10}{3},\frac{15}{3}\right)\Rightarrow\frac{p+r}{q+s-1}=\frac{25}{5}=5$

Answer: (C) → (t)

(D) We have

$$p=x\ (\because \lfloor m \text{ is even and } \cos\lfloor m\,\pi=1)$$

Similarly, $q=x$. Since $p=x$ is rational and $q=x$ is irrational, we have $p=q=0$. Therefore, $(p,q)=(0,0)$. Hence the area of the triangle is

$$\frac{1}{2}|2(1)-(-2)(1)|=2$$

Answer: (D) → (p)

3. In Column I, a family of concurrent lines is given and their points of concurrence are given in Column II. Match the items of Column I with those of Column II.

Column I	*Column II*
(A) If a, b, c are real as $2a+3b+c=0$, then the lines $ax+by+c=0$ are concurrent at	(p) $\left(\frac{2}{5},\frac{3}{5}\right)$

(*Continued*)

Column I	*Column II*
(B) If a is a parameter, then the family of lines $(1 + a)x + (2 - a)y + 5 = 0$ is concurrent at	(q) $(2, 3)$
(C) The lines $(a - d)x + ay + (a + d) = 0$ for different values of d are concurrent at	(r) $(1, 2)$
(D) For different values m and n, the lines $(m+2n)x+(m-3n)y - m+n=0$ are concurrent at	(s) $\left(\frac{-5}{3}, \frac{-5}{3}\right)$
	(t) $(1, -2)$

Solution:

(A) $2a+3b+c=0 \Rightarrow ax+by+c=0$ passes through $(2, 3)$.

Answer: (A) → (q)

(B) The equation $(1+a)x+(2-a)y+5=0$ is written as $(x+2y+5)+a(x-y)=0$, Hence, by Theorem 2.12, the line passes through the intersection of the lines $x+2y+5=0$ and $x-y=0$ and the point of intersection is

$$\left(\frac{-5}{3}, \frac{-5}{3}\right)$$

Answer: (B) → (s)

(C) The equation $(a-d)x+ay+(a+d)=0$ is written as $a(x+y+1)+d(1-x)=0$ so that this line passes through the intersection of the lines $1-x=0$ and $x+y+1=0$ which is given by $(1, -2)$.

Answer: (C) → (t)

(D) The equation $(m+2n)x+(m-3n)y-m+n=0$ is written as $m(x+y-1)+n(2x-3y+1)=0$. Hence, the line passes through the intersection of the lines $x+y-1=0$ and $2x-3y+1=0$ which is given by

$$\left(\frac{2}{5}, \frac{3}{5}\right)$$

Answer: (C) → (p)

3. Match the items of Column I with those of Column II.

Column I	*Column II*
(A) The equation of the line perpendicular to $4x + y - 1 = 0$ and passing through the intersection of the lines $2x - 5y + 3 = 0$ and $x - 3y - 7 = 0$ is	(p) $22x+25y-69=0$
(B) The equation of the line passing through the intersection of the lines $3x - 2y + 10 = 0$ and $4x + 3y - 7 = 0$ and also passing through the point $(2, 1)$ is	(q) $x-4y-24=0$
(C) Equation of the line which passes through the point $(-2, -4)$ and has sum of its intercepts equal to 3 is	(r) $x-2y-6=0$
(D) $A=(1, 2), B=(3, 4)$ and $C=(2, 7)$. Equation of the line passing through $A=(1, 2)$ and perpendicular to the line BC is	(s) $x-3y+5=0$
	(t) $4x-y+4=0$

(Continued)

Solution:

(A) Any line passing through the intersection of the lines $2x-5y+3=0$ and $x-3y-7=0$ is of the form

$$(2x-5y+3)+\lambda(x-3y-7)=0 \text{ (see Theorem 2.20)}$$
$$\Rightarrow (2+\lambda)x-(5+3\lambda)y+3-7\lambda=0$$

This line is perpendicular to the line

$$4x+y-1=0$$
$$\Rightarrow \left(\frac{2+\lambda}{5+3\lambda}\right)(-4)=-1$$
$$\Rightarrow 8+4\lambda=5+3\lambda$$
$$\Rightarrow \lambda=-3$$

Hence, the required line is

$$-x-(-4y)+3+21=0$$
$$\Rightarrow (x-4y)-24=0$$

Answer: (A) → (q)

(B) The required line equation is of the form $(3x-2y+10)+\lambda(4x+3y-7)=0.$ This passes through the point $(2, 1)$ which implies that

$$[3(2)-2(1)+10]+\lambda(8+3-7)=0$$
$$\Rightarrow 14+4\lambda$$
$$\Rightarrow \lambda=-\frac{7}{2}$$

Hence the required line is

$$(3x-2y+10)-\frac{7}{2}(4x+3y-7)=0$$
$$\Rightarrow -22x-25y+69=0$$
$$\Rightarrow 22x+25y-69=0$$

Answer: (B) → (q)

(C) Let the line be

$$\frac{x}{a}+\frac{y}{b}=1$$

Therefore

$$\frac{-2}{a}-\frac{4}{b}=1$$

or $$4a+2b=-ab \qquad (2.120)$$

and $$a+b=3 \qquad (2.121)$$

Solving Eqs. (2.120) and (2.121), we have $a=-1$, $b=4$ or $a=6, b=-3$.

Case 1: $a=-1$, $b=4$. The line is

$$\frac{x}{-1}+\frac{y}{4}=1$$
$$\Rightarrow 4x-y+4=0$$

Case 2: $a=6, b=-3$. The line is

$$\frac{x}{6}+\frac{y}{-3}=1$$
$$\Rightarrow x-2y-6=0$$

Answer: (C) → (r), (t)

(D) The slope of BC is

$$\frac{7-4}{2-3}=-3$$

Hence, the equation of the line passing through $A(1,2)$ and perpendicular to the line BC is

$$y-2=\frac{1}{3}(x-1)$$
$$\Rightarrow x-3y+5=0$$

Answer: (D) → (s)

4. Match the items of Column I with those of Column II.

Column I	*Column II*
(A) Equation of the line through the point (2, 3), such that its x-intercept is twice its y intercept, is	(p) $2x+33y-46=0$
(B) If $A(-5,6)$, $B(-1,-4)$ and $C(3,2)$ are the vertices of a triangle, then the equation of the line passing through the centroid and the circumcentre of ΔABC is	(q) $9x+y-3=0$ (r) $28x-21y+12=0$
(C) The equation of the line passing through the point $(1,-6)$, whose product of the intercepts on the axes is 1, is	(s) $x+2y-8=0$
(D) The equation of the line whose x-intercept is $-3/7$, and is perpendicular to the line $3x+4y-10=0$, is	(t) $4x+y+2=0$

Solution:

(A) Let the line be

$$\frac{x}{2a}+\frac{y}{a}=1$$

Since it passes through the point $(2,3)$, we have

$$\frac{2}{2a}+\frac{3}{a}=1$$
$$\Rightarrow 2+6=2a$$
$$\Rightarrow a=4$$

Hence the required line equation is

$$\frac{x}{8}+\frac{y}{4}=1$$
$$\Rightarrow x+2y-8=0$$

Answer: (A) → (s)

(B) We have G as the cenrtroid of ΔABC which is given by $(-1,4/3)$. Equation of the perpendicular bisector of the side BC is

$$2x+3y+1=0 \qquad (2.122)$$

Equation of the perpendicular bisector of the side AB is

$$2x-5y+11=0 \qquad (2.123)$$

Solving Eqs. (2.122) and (2.123), the circumcentre of ΔABC is given by

$$\left(\frac{-19}{8},\frac{5}{4}\right)$$

or $$S=\left(\frac{-19}{8},\frac{5}{4}\right) \text{ and } G=\left(-1,\frac{4}{3}\right)$$

Then the slope of the line SG is

$$\frac{(5/4)-(4/3)}{(-19/8)+1}=\frac{-1}{12}\times\frac{8}{-11}=\frac{2}{33}$$

Hence, the equation of the line SG is

$$y-\frac{4}{3}=\frac{2}{33}(x+1)$$

$$\Rightarrow 2x-33y+46=0$$

Answer: (B) → (p)

(C) Let the equation of the line be

$$\frac{x}{a}+ay=1$$

This passes through $(1,-6)$. This implies that

$$\frac{1}{a}-6a=1$$

$$\Rightarrow 6a^2+a-1=0$$

$$\Rightarrow (3a-1)(2a+1)=0$$

$$\Rightarrow a=\frac{-1}{2},\frac{1}{3}$$

Therefore, the required lines are

$$-2x-\frac{1}{2}y=1 \text{ and } 3x+\frac{1}{3}y=1$$

$$\Rightarrow 4x+y+2=0 \text{ or } 9x+y-3=0$$

Answer: (C) → (q), (t)

(D) Let the required line be

$$\frac{x}{(-3/7)}+\frac{y}{b}=1 \quad \Rightarrow \frac{-7x}{3}+\frac{y}{b}=1$$

Since this line is perpendicular to the line $3x+4y-10=0$, we have

$$\left(\frac{+7b}{3}\right)\left(\frac{-3}{4}\right)=-1 \Rightarrow b=\frac{+4}{7}$$

Therefore, the required line is

$$\frac{-7x}{3}+\frac{7y}{4}=1$$

$$\Rightarrow 28x-21y+12=0$$

Answer: (D) → (r)

6. Match the items of Column I with those of Column II.

Column I	*Column II*
(A) Equation of the line with x-intercept 4 and passing through the point $(2,-3)$ is	(p) $x+4y-8=0$
(B) Equation of the line passing through $(4,1)$ and forming a triangle with positive coordinate axes whose area is 8 sq. unit is	(q) $3x-2y=12$
(C) Equation of the line with equal intercepts on the axes and is passing through the point $(2,5)$ is	(r) $x+y-7=0$
(D) Equation of the line which makes an angle of 135° with the positive direction of the axis and makes an intercept of 8 on y-axis is	(s) $2x+y+1=0$
	(t) $x+y-8=0$

Solution:

(A) Equation of the line is

$$\frac{x}{4}+\frac{y}{b}=1$$

It passes through $(2,-3)$. This implies that

$$\frac{2}{4}+\frac{-3}{b}=1\Rightarrow\frac{-3}{b}=\frac{1}{2}\Rightarrow b=-6$$

Hence, the equation of the line is

$$\frac{x}{4}-\frac{y}{6}=1$$

$$\Rightarrow 3x-2y=12=0$$

Answer: (A) → (q)

(B) Let

$$\frac{x}{a}+\frac{y}{b}=1$$

It passes through (4, 1) and forms a triangle with positive axes having area 8. Therefore

$$\frac{4}{a}+\frac{1}{b}=1 \tag{2.124}$$

and $$\frac{1}{2}(ab)=8 \tag{2.125}$$

From Eqs. (2.124) and (2.125), $a=8$ and $b=2$. Hence, the equation of the line is

$$\frac{x}{8}+\frac{y}{2}=1 \text{ or } x+4y-8=0$$

Answer: (B) → (p)

(C) Equation of the line with equal intercepts on the axes is

$$\frac{x}{a}+\frac{y}{a}=1 \text{ or } x+y=a$$

This passes through the point $(2,5)$ implies that $a=7$. Hence, the line is

$$x+y-7=0$$

Answer: (C) → (r)

(D) Let the line be $y=(\tan 135°)x+c$ where $c=8$. That is, $y=-x+8$ or $x+y-8=0$.

Answer: (D) → (t)

7. Match the items of Column I with those of Column II.

Column I	*Column II*
(A) If $6x^2+5xy-6y^2+9x+20y+c=0$ represents a pair of lines, then the value of c is equal to	(p) 8
(B) If $6x^2+2hxy-6y^2+x+5y-1=0$ represent a pair of lines, then $2h$ value $(h \neq 0)$ is	(q) 6
(C) If $12x^2+7xy-ky^2+13x-y+3=0$ represents a pair of lines, then the value of k is	(r) -6
(D) If d is the distance between the parallel lines represented by the equation $9x^2-24xy+16y^2-12x+16y-12=0$, then the value of $5d$ is	(s) -5 (t) -10

Solution:

(A) The given equation represents pair of lines. By Theorem 2.31, we have

$$\begin{vmatrix} 6 & \frac{5}{2} & \frac{9}{2} \\ \frac{5}{2} & -6 & 10 \\ \frac{9}{2} & 10 & c \end{vmatrix}=0$$

$$\Rightarrow 6(-6c-100)-\frac{5}{2}\left(\frac{5c}{2}-45\right)+\frac{9}{2}(25+27)=0$$

$$\Rightarrow 24(-6c-100)-5(5c-90)+18\times 52=0$$

$$\Rightarrow -169c-2400+450+936=0$$

$$\Rightarrow -169c=1014$$

$$\Rightarrow c=-6$$

Answer: (A) → (r)

(B) By Theorem 2.31, we have

$$\begin{vmatrix} 6 & h & \frac{1}{2} \\ h & -6 & \frac{5}{2} \\ \frac{1}{2} & \frac{5}{2} & -1 \end{vmatrix}=0$$

$$\Rightarrow 6\left(6-\frac{25}{4}\right)-h\left(-h-\frac{5}{4}\right)+\frac{1}{2}\left(\frac{5h}{2}+3\right)=0$$

$$\Rightarrow \frac{-6}{4}+h^2+\frac{5h}{4}+\frac{5h}{4}+\frac{3}{2}=0$$

$$\Rightarrow 4h^2+10h=0$$

$$\Rightarrow 2h(2h+5)=0$$

$$\Rightarrow h=0 \text{ or } h=\frac{-5}{2}$$

Since $h \neq 0$, we have $2h=-5$.

Answer: (B) → (s)

(C) We have

$$\begin{vmatrix} 12 & \frac{7}{2} & \frac{13}{2} \\ \frac{7}{2} & k & \frac{-1}{2} \\ \frac{13}{2} & \frac{-1}{2} & 3 \end{vmatrix}=0$$

$$\Rightarrow 12\left(3k-\frac{1}{4}\right)-\frac{7}{2}\left(\frac{21}{2}+\frac{13}{4}\right)+\frac{13}{2}\left(\frac{-7}{4}-\frac{13k}{2}\right)=0$$

$$\Rightarrow \frac{12(12k-1)}{4}-\frac{7\times 55}{8}-\frac{91}{8}-\frac{169k}{4}=0$$

$$\Rightarrow 288k-24-385-91-338k=0$$

$$\Rightarrow -50k=500$$

$$\Rightarrow k=-10$$

Answer: (C) → (t)

(D) If $ax^2+2hxy+by^2+2gx+2xy+c=0$ represents a pair of parallel lines, then the distance between them (see Problem 44 of the section 'Subjective Problems') is

$$2\sqrt{\frac{g^2-ac}{a(a+b)}}$$

We have $a=9$, $b=16$, $g=-6$ and $c=-12$. Therefore

$$d=2\sqrt{\frac{g^2-ac}{a(a+b)}}=2\sqrt{\frac{36+108}{9\times 25}}=\frac{2\times 12}{3\times 5}=\frac{8}{5}$$

$$\Rightarrow 5d=8$$

Answer: (D) → (p)

8. Match the items of Column I with those of Column II.

Column I	*Column II*
(A) The angle between the lines joining the origin to the points of intersection of the line $3x-y=2$ and $7x^2-4xy+8y^2+2x-4y-8=0$ is	(p) $\dfrac{\pi}{2}$
(B) The angle between the lines represented by the equation $12x^2+7xy-10y^2+13x-y+3=0$ is	(q) 1
(C) The angle between the lines joining the origin to the points of intersection of the curve $2x^2+6xy+3y^2+4x+2y-36=0$ and the line $x-2y-6=0$ is	(r) $\dfrac{\pi}{3}$
(D) If the lines joining the origin to the points of intersection of the curve $2x^2-2xy-3y^2+2x-y-1=0$ and the line $x+2y=k$ are at right angles, then the value of k is	(s) $\mathrm{Tan}^{-1}\left(\dfrac{23}{2}\right)$
	(t) -1

Solution:

(A) Suppose the line $3x-y=2$ meets the curve at the points A and B. Then, by Theorem 2.33, the combined equation of the pair of lines OA and OB is

$$7x^2-4xy+8y^2+(2x-4y)\left(\frac{3x-y}{2}\right)-8\left(\frac{3x-y}{2}\right)^2=0$$

In this equation,

Coefficient of x^2 + Coefficient of y^2

$$=(7+3-18)+(8+2-2)$$
$$=-8+8=0$$

Hence

$$\underline{|AOB}=\frac{\pi}{2}$$

Answer: (A) → (p)

(B) If the angle between the lines is θ, then

$$\tan\theta=\frac{2\sqrt{h^2-ab}}{|a+b|}=2\frac{\sqrt{(49/4)+120}}{|12-10|}=\sqrt{\frac{529}{4}}=\frac{23}{2}$$

Hence

$$\theta=\mathrm{Tan}^{-1}\left(\frac{23}{2}\right)$$

Answer: (B) → (s)

(C) Suppose the line $x-2y-6=0$ meets the given curve at points A and B. Hence, the combined equation of the pair of lines OA and OB is

$$2x^2+6xy+3y^2+(4x+2y)\left(\frac{x-2y}{6}\right)-36\left(\frac{x-2y}{6}\right)^2=0$$

In this equation, the coefficient of x^2 + the coefficient of y^2 is

$$\left(2+\frac{2}{3}-1\right)+\left(3-\frac{2}{3}-4\right)=\left(1+\frac{2}{3}\right)+\left(-\frac{2}{3}-1\right)=0$$

Hence

$$\underline{|AOB}=\frac{\pi}{2}$$

Answer: (C) → (p)

(D) The combined equation of the pair of lines is

$$2x^2-2xy+3y^2+(2x-y)\left(\frac{x+2y}{k}\right)-\left(\frac{x+2y}{k}\right)^2=0$$

Since the given two lines are at right angles, in the above equation, the coefficient of x^2 + the coefficient of $y^2=0$. This implies

$$\left(2+\frac{2}{k}-\frac{1}{k^2}\right)+\left(3-\frac{2}{k}-\frac{4}{k^2}\right)=0$$

$$\Rightarrow 5k^2-5=0$$
$$\Rightarrow k=\pm 1$$

Answer: (D) → (q), (t)

Comprehension-Type Questions

1. Passage: Consider the straight line $3x+y+4=0$. Answer the following questions.

(i) The point on the line $3x+y+4=0$ which is equidistant from the points $(-5,6)$ and $(3,2)$ is

(A) $(-1,-1)$ (B) $(-2,2)$

(C) $(-3,5)$ (D) $\left(-\frac{1}{3},-3\right)$

(ii) Equation of the line passing through the point $(1,1)$ and perpendicular to the given line is

(A) $x-3y+4=0$ (B) $x-3y+5=0$

(C) $x-3y-4=0$ (D) $x-3y+2=0$

(iii) If the line $y+5=k(x-3)$ is parallel to the given line then the area of the triangle formed by this line and the coordinate axes (in sq. units) is

(A) $\dfrac{8}{3}$ (B) $\dfrac{16}{3}$ (C) 4 (D) 5

Solution:

(i) Let $A=(-5,6)$ and $B=(3,2)$. The slope of AB is

$$\frac{6-2}{-5-3}=\frac{-1}{2}$$

and the midpoint of $AB=(-1,\ 4)$. Hence, the perpendicular bisector of the segment $\overline{AB}$ is $y-4=2(x+1)$ or $2x-y+6=0$. Solving this equation and the given line equations, we have $x=-2$ and $y=2$. Thus, $(-2,\ 2)$ is the point on the given line which is equidistant from both $A(-5,6)$ and $B(3,2)$.

Answer: (B)

(ii) Line perpendicular to the given line is of the form

$$y=\frac{1}{3}x+c$$

This line passes through $(1,1)$. It implies that

$$1=\frac{1}{3}+c\Rightarrow c=\frac{2}{3}$$

Thus, the required line is

$$y=\frac{x}{3}+\frac{2}{3}\quad\text{or}\quad x-3y+2=0$$

Answer: (D)

(iii) The line $y+5=k(n-3)$ is parallel to the given line $\Rightarrow k=-3$. That is,

$$3x+y=4$$

or

$$\frac{x}{(4/3)}+\frac{y}{4}=1$$

Hence, the area of the triangle is

$$\frac{1}{2}\left(\frac{4}{3}\right)(4)=\frac{8}{3}$$

Answer: (A)

2. Passage: Consider the family of concurrent lines which are concurrent at (1, 2) represented by the equation $(3x-y-1)+\lambda(4x-y-2)=0$, where λ is a parameter. Answer the following questions.

(i) A member of the family with positive slope which makes an angle $45°$ with the line $(3x-4y-2)=0$ is

(A) $7x-y-5=0$ (B) $4x-3y+2=0$

(C) $x+7y-15=0$ (D) $5x-3y-4=0$

(ii) Equation of the line belonging to the given family which is perpendicular to the line $x+y-1=0$ is

(A) $x-y+1=0$ (B) $x+y-3=0$

(C) $2x+y-4=0$ (D) $3x-y-1=0$

(iii) The locus of the feet of the perpendiculars from the origin on each of the lines of the members of the family is

(A) $(2x-1)^2+4(y+1)^2=5$

(B) $(2x-1)^2+(y+1)^2=5$

(C) $(2x+1)^2+4(y-1)^2=5$

(D) $(2x-1)^2+4(y-1)^2=5$

Solution:

(i) The slope of a line belonging to the given family is

$$\frac{3+4\lambda}{1+\lambda}$$

and the slope of the line $3x-4y-2=0$ is 3/4. Therefore, by hypothesis, we have

$$1=\tan 45°=\left|\frac{\dfrac{3+4\lambda}{1+\lambda}-\dfrac{3}{4}}{1+[3(3+4\lambda)/4(1+\lambda)]}\right|=\left|\frac{13\lambda+9}{16\lambda+13}\right|$$

Therefore, $16\lambda+13=\pm(13\lambda+9)$. Hence

$$\lambda=-\frac{4}{3},-\frac{22}{29}$$

Case 1: When we have

$$\lambda=-\frac{4}{3}$$

the slope is

$$\frac{3+4\lambda}{1+\lambda}=\frac{3-(16/3)}{1-(4/3)}=7$$

Hence, the required equation is

$$(3x-y-1)-\frac{4}{3}(4x-y-2)=0$$

$$\Rightarrow -7x+y+5=0$$

$$\Rightarrow 7x-y-5=0$$

Case 2: When we have

$$\lambda=-\frac{22}{29}$$

the slope is

$$\frac{3-(88/29)}{1-(22/29)}=\frac{-1}{7}$$

Thus, the required line is $7x-y-5=0$.

Answer: (A)

(ii) By hypothesis, we have

$$\left(\frac{3+4\lambda}{1+\lambda}\right)(-1) = -1$$
$$\Rightarrow \lambda = -\frac{2}{3}$$

Hence, the required line is

$$(3x-y-1)-\frac{2}{3}(4x-y-2)=0$$
$$\Rightarrow x-y+1=0$$

Answer: (A)

(iii) Let $A=(1,2)$ at which the given family of lines are concurrent and O be the origin. If P is the foot of the perpendicular from origin O onto any line of the family then P lies on the circle drawn on OA as diameter because $\underline{|APO}$ is equal to $90°$. The circle with $(0,0)$ and $(1,2)$ as ends of a diameter is

$$x(x-1)+y(y-2)=0 \quad \text{(see Chapter 3)}$$
$$\Rightarrow x^2+y^2-x-2y=0$$
$$\Rightarrow (2x-1)^2+4(y-1)^2=5$$

Answer: (D)

3. Passage: In ΔABC, $A=(1, 3)$ and $C=(-2/5, -2/5)$ are two vertices and $x+y-2=0$ is the equation of the internal bisector of $\underline{|ABC}$. Answer the following questions.

(i) Equation of the side BC is

(A) $7x-3y-4=0$ (B) $7x+3y+4=0$
(C) $7x+3y-4=0$ (D) $7x-3y+4=0$

(ii) The coordinates of B are

(A) $\left(\frac{17}{10},\frac{3}{10}\right)$ (B) $(1,1)$
(C) $\left(\frac{3}{10},\frac{17}{10}\right)$ (D) $\left(\frac{-5}{2},\frac{9}{2}\right)$

(iii) Equation of AB is

(A) $3x+7y+24=0$
(B) $13x+7y+8=0$
(C) $3x+7y-24=0$
(D) $13x-7y+8=0$

Solution:

(i) The image of $A(1,3)$ on the bisector of $\underline{|ABC}$ lies on the line BC. Therefore, if A' is the image of $A(1,3)$ in the line $x+y-2=0$, then

$$\frac{x-1}{1}=\frac{y-3}{1}=\frac{-2(1+3-2)}{1^2+1^2} \Rightarrow A'=(-1, 1)$$

Since the line BC is same as the line $A'C$, its equation is

$$y-1=\frac{1+(2/5)}{-1+(2/5)}(x+1)$$
$$\Rightarrow y-1=\frac{-7}{3}(x+1)$$
$$\Rightarrow 7x+3y+4=0$$

Answer: (B)

(ii) Solving the equations of the line BC and the angle bisector of $\underline{|ABC}$, we have

$$B=\left(\frac{-5}{2},\frac{9}{2}\right)$$

Answer: (D)

(iii) Since $A=(1,3)$ and $B=(-5/2, 9/2)$, the equation of AB is $3x+7y-24=0$.

Answer: (C)

4. Passage: Suppose a line $lx+my=1$ meets a second-degree curve $ax^2+2hxy+by^2+2gx+2fy+c=0$ at two points A and B. Then, the combined equation of the pair of lines OA and OB is $ax^2+2hxy+by^2+(2gx+2fy)(lx+my)+c(lx+my)^2=0$ which is a second-degree homogeneous equation. Answer the following questions.

(i) All chords of the curve $3x^2-y^2-2x+4y=0$ which subtend right angle at the origin will pass through a fixed point whose coordinates are

(A) $(-1, 2)$ (B) $(1,-2)$
(C) $(1, 2)$ (D) $(-1,-2)$

(IIT-JEE 1991)

(ii) If the intercept of the line $lx+my=1$ made by the curve $x^2+y^2-a^2=0$ subtends right angle at the origin, then l^2+m^2 is equal to

(A) $\frac{2}{a^2}$ (B) $\frac{1}{a^2}$ (C) $2a^2$ (D) $3a^2$

(iii) The line $y=mx+c$ makes an intercept on the curve $y-4ax=0$ which subtends angle at the origin. Then, the line $y=mx+c$ passes through a fixed point whose coordinates are

(A) $(2a, 0)$ (B) $(a, 0)$
(C) $(3a, 0)$ (D) $(4a, 0)$

(IIT-JEE 1994)

Solution:

(i) Suppose $lx+my=1$ is a line meeting the curve $3x^2-y^2-2x+4y=0$ at points A and B. Therefore, the combined equation of the pair of lines OA and OB is

$$3x^2-y^2-(2x-4y)(lx+my)=0$$

Since $\underline{|AOB}=90°$, in the above equation, the coefficient of x^2 + the coefficient of $y^2=0$. Therefore,

$$(3-2l)+(-1+4m)=0$$

$$\Rightarrow l-2m-1=0$$

Hence, the line $lx+my-1=0$ passes through the point $(1,-2)$.

Answer: (B)

(ii) Suppose the line $lx+my=1$ intersects $x^2+y^2=a^2$ at points A and B. Therefore, the combined equation of the pair of lines OA and OB is

$$x^2+y^2=a^2(lx+my)^2=0$$

Since $\underline{|AOB}=90°$, in the above equation, the coefficient of x^2 + the coefficient of $y^2=0$. That is,

$$(1-a^2l^2)+(1-a^2m^2)=0$$

$$\Rightarrow a^2(l^2+m^2)=2$$

$$\Rightarrow l^2+m^2=\frac{2}{a^2}$$

Answer: (A)

(iii) Suppose the line $y=mx+c$, where $c\neq 0$. meets the curve $y^2-4ax=0$ at two points A and B. The combined equation of the pair of lines OA and OB is

$$y^2-4ax\left(\frac{y-mx}{c}\right)=0$$

Now, $\underline{|AOB}=90° \Rightarrow$ the coefficient of x^2 + the coefficient of $y^2=0$. This means

$$\frac{4am}{c}+1=0$$

$$\Rightarrow c=-4am$$

Therefore, the equation of the line is

$$y=mx+c=mx-4am$$

$$\Rightarrow y=m(x-4a)$$

which passes through the fixed point $(4a,0)$.

Answer: (D)

Integer Answer Type Questions

1. The area of the quadrilateral formed by the lines $|x|+|y|=1$ is __________ sq. unit.

Solution: The given quadrilateral is a square with vertices $(1,0)$, $(0,1)$, $(-1,-1)$ and $(0,-1)$, and hence its area is $(\sqrt{2})^2=2$.

Answer: 2

2. Two rays in the first quadrant, $x+y=|a|$ and $ax-y=1$, intersect each other in the interval $a\in(a_0,\infty)$. The value of a_0 is __________. **(IIT-JEE 2006)**

Solution: Solving the given two equations, we have

$$x=\frac{1+|a|}{1+a} \text{ and } y=ax-1=\frac{a(1+|a|)}{1+a}-1=\frac{a|a|-1}{1+a}$$

Since the two rays intersect each other in the first quadrant, we have $x>0$ and $y>0$ which implies that

$$1+a>0 \text{ and } a|a|-1>0$$

Therefore, if $-1<a<0$, then the $a(-a)-1>0$ which is not sensible. Hence, $a\notin(-1,0)$. If $a=0$, then the lines $x+y=0$ and $y=-1$ intersect in fourth quadrant. Thus, $a\neq 0$. Hence, $a>0$ and $a^2-1>0 \Rightarrow a>1$. Therefore, $a_0=1$.

Answer: 1

3. The orthocentre of the triangle formed by the lines $x+y=1$, $2x+3y=6$ and $4x-y+4=0$ lies in the quadrant whose number is __________.

Solution: Solving the above equations taken two by two, the vertices of the triangle are

$$A\left(\frac{-3}{5},\frac{8}{5}\right),\ B(-3,4), \text{ and } C\left(\frac{-3}{7},\frac{16}{7}\right)$$

The equation of the altitude drawn from A to the side BC is

$$y-\frac{8}{5}=\frac{3}{2}\left(x+\frac{3}{5}\right)$$

$$\Rightarrow 3x-2y=-5 \quad (2.126)$$

Again the equation of the altitude from B onto CA is

$$y-4=\frac{-1}{4}(x+3)$$

$$\Rightarrow x+4y=13 \quad (2.127)$$

Solving Eqs. (2.126) and (2.127), the coordinates of the orthocentre are

$$\left(\frac{3}{7}, \frac{22}{7}\right)$$

which lies in quadrant number 1.

Answer: 1

4. If the lines $3x - 5y + 9 = 0$, $4x + ky - 28 = 0$ and $13x - 8y - 1 = 0$ are concurrent, then the value of k is ________.

Solution: Since the lines are concurrent, by Theorem 2.22, we have

$$\begin{vmatrix} 3 & -5 & 9 \\ 4 & k & -28 \\ 13 & -8 & -1 \end{vmatrix} = 0$$

$$\Rightarrow 3(-k-224)+5(-4+364)+9(-32-13k)=0$$
$$\Rightarrow -120k-672+1800-288=0$$
$$\Rightarrow -120k=-840$$
$$\Rightarrow k=7$$

Answer: 7

5. If the slope of the line $ax + (3 - a)y + 7 = 0$ is 7, then the value of integral part of a is ________.

Solution: By hypothesis, we have

$$\frac{-a}{3-a}=7$$
$$\Rightarrow -a=21-7a$$
$$\Rightarrow a=\frac{7}{2}$$
$$\Rightarrow [a]=3$$

Answer: 3

6. If (x_1, y_1) and (x_2, y_2) are two points on the line $5x - 12y + 5 = 0$ that lie at a distance of 3 units from the line $3x + 4y - 12 = 0$, then the value of is $x_1 + x_2$ is ________.

Solution: Let (h, k) be a point on the line $5x - 12y + 15 = 0$ whose distance from the line $3x+4y-12=0$ is 3 units. Then

$$5h-12k=-15 \tag{2.128}$$

and
$$\left|\frac{3h+4k-12}{\sqrt{3^2+4^2}}\right|=3$$

which gives that

$$3h+4k=27 \tag{2.129}$$

or
$$3h+4k=-3 \tag{2.130}$$

Solving Eqs. (2.128) and (2.129), we have

$$h=\frac{33}{7} \text{ and } k=\frac{45}{14}$$

Hence

$$(x_1, y_1)=\left(\frac{33}{7}, \frac{45}{14}\right)$$

Solving Eqs. (2.128) and (2.129), we have

$$h=\frac{-12}{7} \text{ and } k=\frac{15}{28}$$

Thus

$$(x_2, y_2)=\left(\frac{-12}{7}, \frac{15}{28}\right)$$

Therefore

$$x_1+x_2=\frac{33}{7}-\frac{12}{7}=\frac{21}{7}=3$$

Answer: 3

7. The points $A(0, 4)$, $B(5, 1)$ and $C(1, -3)$ are the vertices of a triangle. If h is the altitude from A to BC and Δ is the area of the triangle, then h^2/Δ is equal to ________.

Solution: See Fig. 2.90. We have

$$\Delta=\frac{1}{2}\left|0(1+3)+5(-3-4)+1(4+1)\right|=16$$

Length $BC=\sqrt{(5-1)^2+(1+3)^2}=4\sqrt{2}$. Therefore

$$\frac{1}{2}h(BC)=\Delta \Rightarrow \frac{1}{2}(h)(4\sqrt{2})=16$$

Hence, $h=(4\sqrt{2})$ and $h^2=32$. So

$$\frac{h^2}{\Delta}=\frac{32}{16}=2$$

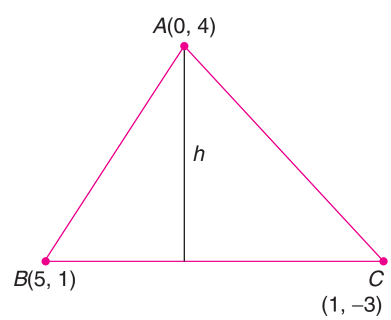


FIGURE 2.90

Answer: 2

8. If one of the lines represented by the equation $mx^2 - (1 - m^2)xy - my^2 = 0$ bisects angles between the coordinate axes, then $|m|$ is equal to ________.

Solution: The given equation is $x^2 + m^2xy - xy - my^2 = 0$. That is, $mx(x+my) - y(x+my) = 0$. Therefore, the lines are $x + my = 0$ and $mx - y = 0$. The angle bisectors of the coordinates are $y = x$ and $y = -x$. Hence, $m = \pm 1$ or $|m| = 1$.

Answer: 1

9. Consider the lines $ax + y + 1 = 0$, $x + by + 1 = 0$ and $x + y + c = 0$, where a, b and c are distinct and different from 1. Then

$$\frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c}$$

is equal to ______.

Solution: Since the lines are concurrent, we have

$$\begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix} = 0$$

Using $C_2 - C_1$ and $C_3 - C_1$ (where C_1, C_2 and C_3 are used to denote the three columns), we have

$$\begin{vmatrix} a & 1-a & 1-a \\ 1 & b-1 & 0 \\ 1 & 0 & c-1 \end{vmatrix} = 0$$

$$\Rightarrow a(b-1)(c-1) - (1-a)[c-1-0] + (1-a)[0-(b-1)] = 0$$

$$\Rightarrow a(1-b)(1-c) + (1-a)(1-c) + (1-a)(1-b) = 0$$

Dividing by $(1-a)(1-b)(1-c)$, we get

$$\frac{a}{1-a}+\frac{1}{1-b}+\frac{1}{1-c} = 0$$

Adding both sides by 1, we get

$$\frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c} = 1$$

Answer: 1

10. In ΔABC, the vertex $A = (1, 2)$, $y = x$ is the perpendicular bisector of the side AB and $x - 2y + 1 = 0$ is the equation of the internal angle bisector of $\underline{|C}$. If the equation of the side BC is $ax + by - 5 = 0$, then the value of $a - b$ is ______.

Solution: Since $y = x$ is the perpendicular bisector of the side AB and $A = (1, 2)$, we have $B = (2, 1)$. Since the image $A'(x, y)$ of A in the angular bisector $x - 2y + 1 = 0$ lies on the line BC, we have

$$\frac{x-1}{1} = \frac{y-2}{-2} = \frac{-2(1-2(2)+1)}{1^2+2^2} = \frac{4}{5}$$

Therefore,

$$A' = \left(\frac{9}{5}, \frac{2}{5}\right)$$

Since equation of BC is the equation of BA', we have the equation of BC as

$$y - 1 = \frac{1-(2/5)}{2-(9/5)}(x-2)$$

$$\Rightarrow y - 1 = 3(x-2)$$

$$\Rightarrow 3x - y - 5 = 0$$

so that $a = 3, b = -1$. Hence, $a - b = 4$.

Answer: 4

11. If the equation of the bisector of the acute angle between the lines $2x - y + 4 = 0$ and $x - 2y - 1 = 0$ is $ax + by + 1 = 0$, then the value of $a - b$ is equal to ______.

Solution: The lines are $2x - y + 4 = 0$ and $x - 2y + 1 = 0$ in which c_1 and c_2 are positive and

$$a_1a_2 + b_1b_2 = 2 + 2 = 4 > 0$$

Hence, by Theorem 2.26, the acute angle bisector is

$$\frac{2x-y+4}{\sqrt{5}} = -\frac{(x-2y-1)}{\sqrt{5}}$$

$$\Rightarrow 3x - 3y + 3 = 0$$

$$\Rightarrow x - y + 1 = 0$$

Hence,

$$a - b = 1 - (-1) = 2$$

Answer: 2

12. The number of possible straight lines passing through (2, 3) and forming a triangle with coordinate axes whose area is 12 sq. unit is ______.

Solution: Let

$$\frac{x}{a}+\frac{y}{b} = 1$$

be the line. Therefore

$$\frac{2}{a}+\frac{3}{b} = 1 \quad (2.131)$$

and

$$\frac{1}{2}|ab| = 12$$

or

$$ab = \pm 24 \quad (2.132)$$

Case 1: When $ab = 24$, from Eq. (2.131), we have $3a + 2b = ab = 24$. Hence

$$3a + 2\left(\frac{24}{a}\right) = 24$$

$$\Rightarrow 3a^2 - 24a + 48 = 0$$

$$\Rightarrow a^2 - 8a + 16 = 0$$

$$\Rightarrow a = 4, b = 6.$$

Case 2: When $ab = -24$, from Eq. (2.131), we have

$$3a + 2b = -24$$
$$\Rightarrow 3a + 2\left(-\frac{24}{a}\right) = -24$$
$$\Rightarrow 3a^2 + 24a - 48 = 0$$
$$\Rightarrow a^2 + 8a - 16 = 0$$
$$\Rightarrow a = \frac{-8 \pm \sqrt{64+64}}{2} = -4 \pm 4\sqrt{2}$$

Therefore, b will have two values corresponding to a. Hence, the number of lines is 3.

Answer: 3

SUMMARY

2.1. ***Slope of line:*** Let l be a non-vertical line (i.e., l is not parallel to y-axis) making an angle θ with the positive direction of x-axis. Then, tan θ is called the slope of the line l. Generally, the slope of a line is denoted by m.

Caution: The concept of slope is followed only for non-vertical lines.

Note: Slope of a horizontal line (which is parallel to x-axis) is always zero.

2.2. If $A(x_1, y_1)$ and $B(x_2, y_2)$ are two points on a non-vertical line, then the slope of the line $\overleftrightarrow{AB}$ is $\dfrac{y_2 - y_1}{x_2 - x_1}$.

2.3. ***Intercepts on the axes:*** If a line l meets x-axis at $(a, 0)$ and y-axis at $(0, b)$, then a is called x-intercept and b is called y-intercept of the line l.

2.4. ***Equations of the axis:*** The equation of x-axis is $y = 0$ and the equation of y-axis is $x = 0$.

2.5. ***Various forms of straight line equations:***

1. ***Two-point form:*** Equation of the line passing through two points (x_1, y_1) and (x_2, y_2) is
$$(x - x_1)(y_1 - y_2) = (y - y_1)(x_1 - x_2)$$
2. ***Point–slope form:*** Equation of the line which is having slope m and passing through the point (x_1, y_1) is
$$y - y_1 = m(x - x_1)$$
3. ***Symmetric form:*** If a non-vertical makes an angle θ with the positive direction of x-axis and passes through a point (x_1, y_1), then its equation is
$$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta}$$
Note: In the above relation, if we consider that each ratio is equal to r (real number), then every point on the line is of the form $(x_1 + r\cos\theta, y_1 + r\sin\theta)$. Also $|r|$ gives the distance of the point (x, y) on the given line from the fixed point (x_1, y_1).
4. ***Intercept form:*** If a and b are x and y intercepts of a line $(ab \neq 0)$, then the line equation is $\dfrac{x}{a} + \dfrac{y}{b} = 1$.
Note: Area of the triangle formed by the coordinate axis and the line $\dfrac{x}{a} + \dfrac{y}{b} = 1$ is $\dfrac{1}{2}|ab|$ sq. unit.
5. ***Slope–intercept form:*** The equation of a non-vertical line which is having slope m and y-intercept c is
$$y = mx + c$$
Note: Equation of any line (except the y-axis) passing through origin is the form $y = mx$.
6. ***Normal form:*** Let l be a line whose distance from the origin is ON $(= p)$ and $\overline{ON}$ make an angle α with the positive direction of the x-axis. Then, the equation of the line l is $x \cos \alpha + y \sin \alpha = p$.

2.6. ***Definition (first-degree equation):*** If a, b and c are real and either a or b is not zero, then $ax + by + c$ is called first-degree expression in x and y and $ax + by + c = 0$ is called first-degree equation in x and y.

2.7. ***Theorem:*** Every first-degree equation in x and y represents a straight line and the equation of any line in the coordinate plane is a first-degree equation in x and y.

2.8. ***General equation of a straight line:*** First-degree equation in x and y is called the general equation of a straight line.

2.9. ***Various forms of $ax + by + c = 0$, where $abc \neq 0$:***

1. ***Slope–intercept form:***
$$y = \left(\frac{-a}{b}\right)x + \left(\frac{-c}{b}\right)$$
2. ***Intercept form:***
$$\frac{x}{(-c/a)} + \frac{y}{(-c/b)} = 1$$

3. ***Normal form:***
   (a) ***When c > 0:***

$$x\left(\frac{-a}{\sqrt{a^2+b^2}}\right)+y\left(\frac{-b}{\sqrt{a^2+b^2}}\right)=\frac{c}{\sqrt{a^2+b^2}}$$

   (b) ***When c < 0:***

$$x\left(\frac{a}{\sqrt{a^2+b^2}}\right)+y\left(\frac{b}{\sqrt{a^2+b^2}}\right)=\frac{-c}{\sqrt{a^2+b^2}}$$

2.10. ***The line $ax + by + c = 0$ in normal form:*** Write $ax + by = -c$. Divide by $\sqrt{a^2+b^2}$ both sides of the equation and then make the right-hand side (RHS) positive.

2.11. ***Distance of a line from a point:*** Suppose (x_1, y_1) is not a point on the line $ax + by + c = 0$. Then, the perpendicular distance drawn from (x_1, y_1) onto the given line is

$$\frac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}$$

In particular, if $c \neq 0$, then the distance of the origin from the line is

$$\frac{|c|}{\sqrt{a^2+b^2}}$$

2.12. ***Distance between two parallel lines:*** The distance between the parallel lines $ax + by + c = 0$ and $ax+by+c'=0 = 0$ is

$$\frac{|c-c'|}{\sqrt{a^2+b^2}}$$

2.13. ***Angle between two lines:*** If θ is an angle between the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, then

$$\cos\theta=\frac{a_1a_2+b_1b_2}{\sqrt{a_1^2+b_1^2}\sqrt{a_2^2+b_2^2}}$$

where θ is acute or obtuse according to the conditions $a_1a_2 + b_2b_2 > 0$ or < 0. To determine the acute angle, we take

$$\cos\theta=\frac{|a_1a_2+b_1b_2|}{\sqrt{a_1^2+b_1^2}\sqrt{a_2^2+b_2^2}}$$

2 14. ***Angle between the lines in terms of their slopes:*** If θ is an angle between the lines whose slopes are m_1 and m_2, then

$$\tan\theta=\frac{m_1-m_2}{1+m_1m_2}$$

and the acute angle is given by

$$\tan\theta=\left|\frac{m_1-m_2}{1+m_1m_2}\right|$$

2.15. ***Condition for parallel and perpendicular:*** Let $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ be two lines. Then the following two conditions are applicable:

1. The lines are parallel $\Leftrightarrow$ $a_1b_2 = a_2b_1$.
2. The lines are perpendicular to each other if and only if $a_1a_2 + b_1b_2 = 0$.

2.16. ***Condition for parallel and perpendicular in terms of slopes:*** Let m_1 and m_2 be the slopes of two lines. Then, the following two conditions are applicable:

1. The lines are parallel $\Leftrightarrow$ $m_1 = m_2$.
2. The lines are at right angles. $\Leftrightarrow$ $m_1\, m_2 = -1$.

2.17. Equation of the line parallel to the line $ax + by + c = 0$ and passing through the point (x_1, y_1) is

$$a(x-x_1)+b(y-y_1)=0$$

2.18. Equation of the line passing through the point (x_1, y_1) and perpendicular to the line $ax + by + c = 0$ is

$$b(x-x_1)-a(y-y_1)=0$$

(2.18)*. The area of the parallelogram formed by the lines $a_1x + b_1y + c_1 = 0$, $a_1x + b_1y + d_1 = 0$, $a_2x + b_2y + c_2 = 0$ and $a_2x + b_2y + d_2 = 0$ is

$$\left|\frac{(c_1-d_1)(c_2-d_2)}{a_1b_2-a_2b_1}\right|$$

2.19. ***Notation:*** $L \equiv ax + by + c$, $L_{11} = ax_1 + by_1 + c$ and $L_{22} = ax_2 + by_2 + \text{c}$.

2.20. ***Theorem:*** Let $L = ax + by + c = 0$ be a straight line and $A(x_1, y_1)$ and $B(x_2, y_2)$ be two points in the plane of $L = 0$. Suppose points A and B are not on the line and the line $\overleftrightarrow{AB}$ is not parallel to the line $L = 0$. Then, the line $L = 0$ divides the segment $\overline{AB}$ in the ratio $-L_{11}:L_{22}$.

1. Points A and B are on the opposite sides of $L = 0$ $\Leftrightarrow$ the division is internal division $\Leftrightarrow$ $-L_{11}:L_{22}$ is positive $\Leftrightarrow$ L_{11} and L_{22} are of opposite sign.
2. Points A and B are on the same side of the line $L = 0$ $\Leftrightarrow$ the division is external

$\Leftrightarrow -L_{11}:L_{22}$ is negative $\Leftrightarrow$ L_{11} and L_{22} are of same sign.

2.21. ***Origin and non-origin sides of a line:*** Suppose l is a straight line which is not passing through the origin. Then, l divides the entire plane into two regions. Origin region means, the region in which the origin lies. The other is called non-origin region.

2.22. ***To determine the position of a point:*** Let $L \equiv ax + by + c$ be a straight line where $c \neq 0$ and $A(x_1, y_1) \neq (0, 0)$ be a point which does not lie on the line $L = 0$. Then

1. $A(x_1, y_1)$ lies on the origin side of $L = 0 \Leftrightarrow c$ and L_{11} has the same sign.
2. $A(x_1, y_1)$ lies on the non-origin side of $L = 0 \Leftrightarrow c$ and L_{11} has the opposite sign.

2.23. Let $L \equiv ax + by + c = 0$ be a line where $c \neq 0$. Then

1. if $c > 0$, then $L_{11} > 0$ for all points on the origin side and $L_{11} < 0$ for all points on non-origin side.
2. if $c < 0$, then $L_{11} < 0$ for all points on the origin side and $L_{11} > 0$ for all points in the non-origin side.

2.24. ***Theorem:*** Suppose $u_1 = 0$ and $u_2 = 0$ are two parallel lines. Then, $\lambda_1 u_1 + \lambda_2 u_2 = 0$ represents lines parallel to the lines $u_1 = 0$ and $u_2 = 0$ for all real values of λ_1 and λ_2 such that $|\lambda_1| + |\lambda_2| \neq 0$.

2.25. ***Theorem:*** If $u_1 = 0$ and $u_2 = 0$ are two intersecting lines, then for all λ_1 and λ_2 $(|\lambda_1| + |\lambda_2| \neq 0)$, the equation $\lambda_1 u_1 + \lambda_2 u_2 = 0$ represents the lines passing through the intersection of $u_1 = 0$ and $u_2 = 0$. Conversely, the equation of any line passing through the intersection of $u_1 = 0$ and $u_2 = 0$ is $\lambda_1 u_1 + \lambda_2 u_2 = 0$ for some λ_1 and λ_2.

Note: Instead of $\lambda_1 u_1 + \lambda_2 u_2 = 0$, we consider the equation $u_1 + \lambda u_2 = 0$ which is practically more useful.

2.26. ***Corollary:*** Suppose $u_1 = 0$ and $u_2 = 0$ are the two intersecting lines. Then, the equation of any line in the plane of $u_1 = 0$ and $u_2 = 0$ is of the form $\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 = 0$.

2.27. ***Theorem:*** When the lines $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$ and $a_3x + b_3y + c_3 = 0$ are concurrent (assuming that no two lines are parallel), then

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

2.28. ***Theorem:*** Suppose $u_1 = 0$, $u_2 = 0$ and $u_3 = 0$ are three lines such that no two are parallel. If there exist non-zero real numbers such that $\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 = 0$, then the lines $u_1 = 0$, $u_2 = 0$ and $u_3 = 0$ are concurrent.

2.29. ***Theorem (Foot of the perpendicular):*** Let (x, y) be the foot of the perpendicular drawn from the point (x_1, y_1) on to the line $ax + by + c = 0$, then

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{-(ax_1 + by_1 + c)}{a^2 + b^2}$$

2.30. ***Theorem (Image):*** If (x, y) is the image of (x_1, y_1) in the mirror line $ax + by + c = 0$, then

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{-2(ax_1 + by_1 + c)}{a^2 + b^2}$$

Note: From 2.29 and 2.30, we can write the foot of the perpendicular and the image of a point with respect to a line.

2.31. ***Angle bisectors:*** If $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are two intersecting lines, then the equations of their angle bisectors are

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm \frac{(a_2x + b_2y + c_2)}{\sqrt{a_2^2 + b_2^2}}$$

2.32. ***Acute angle bisector:*** Suppose $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ where $c_1c_2 \neq 0$ and c_1, c_2 are of the same sign. If $a_1a_2 + b_1b_2 < 0$, then the acute angle bisector is

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = +\left(\frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}\right)$$

2.33. ***Theorem:*** The second-degree homogeneous equation $ax^2 + 2hxy + by^2 = 0$ represents a pair of lines passing through the origin, where $h^2 \geq ab$. If $h^2 = ab$, both lines coincide, otherwise they are distinct lines.

2.34. ***Identities:***

1. Suppose the lines represented by $ax^2 + 2hxy + by^2 = 0$ are $l_1x + m_1y = 0$ and $l_2x + m_2y = 0$. Then

$$l_1l_2 = a,\ l_1m_2 + l_2m_1 = 2h,\ m_1m_2 = b$$

2. If we consider the lines as $y = m_1x$ and $y = m_2x$, then

$$m_1 + m_2 = \frac{-2h}{b} \text{ and } m_1m_2 = \frac{a}{b}$$

2.35. ***Angle between the lines $ax^2 + 2hxy + by^2 = 0$:*** If α is the angle between the lines represented by the equation $ax^2 + 2hxy + by^2 = 0$, then

$$\cos\alpha = \frac{a+b}{\sqrt{(a-b)^2+4h^2}}$$

or $$\tan\alpha = \frac{2\sqrt{h^2-ab}}{a+b}$$

To determine the acute angle, take the absolute value.

2.36. ***Condition for orthogonal lines:*** Suppose $ax^2 + 2hxy + by^2 = 0$ represents pair of straight lines. Then, they are at right angles if and only if $a + b = 0$ (that is, coefficient of x^2 + coefficient of $y^2 = 0$).

2.37. ***Equation of the angle bisectors of the lines $ax^2 + 2hxy + by^2 = 0$:*** If $ax^2 + 2hxy + by^2 = 0$ represents a pair of distinct lines, then the combined equation of the pair of angle bisectors of the lines is $h(x^2 - y^2) = (a - b)xy$.

2.38. ***Theorem:*** The area of the triangle formed by the pair of lines $ax^2 + 2hxy + by^2 = 0$ and the line $lx + my = 1$ is

$$\left|\frac{\sqrt{h^2-ab}}{bl^2-2hlm+am^2}\right|$$

2.39. ***Theorem:*** The second-degree general equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of lines parallel to the lines represented by $ax^2 + 2hxy + by^2 = 0$ if and only if

(i) $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

(ii) $h^2 \ge ab, g^2 \ge ca$ and $f^2 \ge bc$

2.40. ***Formula:*** The area of the parallelogram formed by the pairs of lines $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ and $ax^2 + 2hxy + by^2 = 0$ is

$$\frac{|c|}{2\sqrt{h^2-ab}} \text{ sq. unit}$$

2.41. ***Equally inclined pairs:*** The pair of lines $(\overleftrightarrow{PA}, \overleftrightarrow{PB})$ and $(\overleftrightarrow{PC}, \overleftrightarrow{PD})$ are said to be equally inclined to each other if both pairs have the same pair of angle bisectors.

2.42. ***Theorem:*** Suppose the line $lx + my = 1$ intersects the second-degree curve $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ at points A and B. Then, the combined equation of the pair of lines $\overleftrightarrow{OA}$ and $\overleftrightarrow{OB}$, where O is the origin, is given by the equation

$$ax^2 + 2hxy + by^2 + (2gx + 2fy)(lx + my) + c(lx + my)^2 = 0$$

In particular, OA and OB are at right angles if

Coefficient of x^2 + Coefficient of $y^2 = 0$

in the above equation.

EXERCISES

Single Correct Choice Type Questions

1. Equation of the line through $(0, -3)$ and having slope -2 is

(A) $y - 2x + 3 = 0$ (B) $y + 2x - 3 = 0$
(C) $y + 2x + 3 = 0$ (D) $y - 2x - 3 = 0$

2. Equation of the line passing through $(-5, 2)$ and $(3, 2)$ is

(A) $x - 2 = 0$ (B) $y - 2 = 0$
(C) $x + 2 = 0$ (D) $y + 2 = 0$

3. The points $A(-5, 6)$, $B(-1, -4)$ and $C(3, 2)$ are three non-collinear points. Then, the equation the median through point C of ΔABC is

(A) $7x + 6y - 1 = 0$ (B) $x + 6y + 9 = 0$
(C) $x - 6y + 9 = 0$ (D) $x + 6y - 9 = 0$

4. The circumcentre of the triangle with vertices $A(-5, 6)$, $B(-1, -4)$ and $C(3, 2)$ is

(A) $\left(\frac{-19}{8}, \frac{5}{4}\right)$ (B) $\left(\frac{19}{8}, \frac{5}{4}\right)$
(C) $\left(\frac{-5}{4}, \frac{19}{8}\right)$ (D) $\left(\frac{5}{4}, \frac{-19}{8}\right)$

5. If the area of the triangle formed by the line $2x + 3y + c = 0$ with coordinate axes is 27 sq. units, then c is equal to

(A) ± 16 (B) ± 15 (C) ± 8 (D) ± 18

6. If the line $2x + 3by - 13 = 0$ passes through the point $(-2, 4)$, then the value of b is

(A) $\frac{5}{4}$ (B) $\frac{4}{3}$ (C) $\frac{17}{12}$ (D) $\frac{19}{12}$

7. In the straight line equation $x\cos\alpha + y\sin\alpha = p$, if $p = 6$ or $\alpha = 30°$, then the equation is

(A) $\sqrt{3}x + y - 12 = 0$ (B) $\sqrt{3}x + y + 12 = 0$
(C) $2x + \sqrt{3y} - 12 = 0$ (D) $2x + \sqrt{3}y + 12 = 0$

8. If the line $3x - by - 8 = 0$ makes angle $45°$ with the positive directions of the x-axis, then b is equal to

(A) $7, \frac{9}{7}$ (B) $-7, \frac{9}{7}$

(C) $-7, \frac{-9}{7}$ (D) $7, \frac{-9}{7}$

9. The incentre of the triangle whose sides are $y = 0, 3x - 4y = 0$ and $4x + 3y - 50 = 0$ is

(A) $\left(\frac{15}{2}, \frac{-5}{2}\right)$ (B) $\left(\frac{15}{2}, \frac{5}{2}\right)$

(C) $\left(\frac{-15}{2}, \frac{5}{2}\right)$ (D) $\left(\frac{-15}{2}, \frac{-5}{2}\right)$

10. The incentre of the triangle formed by the lines $15x - 8y + 25 = 0$, $3x - 4y - 10 = 0$ and $5x + 12y - 30 = 0$ is

(A) $\left(\frac{4}{7}, \frac{1}{4}\right)$ (B) $\left(\frac{-4}{7}, \frac{1}{4}\right)$

(C) $\left(\frac{4}{7}, \frac{-1}{4}\right)$ (D) $\left(\frac{-4}{7}, \frac{-1}{4}\right)$

11. The equation of the lines which are parallel to the line $8x - 15y + 34 = 0$ and whose distance from the point $(-2, 3)$ is equal to 3 are $8x - 15y + c_1$ and $8x - 15y + c_2 = 0$. Then $c_1 + c_2$ is equal to

(A) 112 (B) 10 (C) 122 (D) 102

12. The equation of the line which passes through the point of intersection of the lines $3x - 5y + 9 = 0$ and $4x + 7y - 28 = 0$ and the point $(4, 2)$ is

(A) $3x + 2y - 16 = 0$ (B) $38x + 87y - 326 = 0$

(C) $4x + 3y - 22 = 0$ (D) $38x - 87y + 22 = 0$

13. $A(-1,4), B(1,-4)$ and $C(5,4)$ are the vertices of a triangle. Then, the length of the altitude from A onto BC is

(A) $\frac{12}{5}$ (B) $\frac{12}{\sqrt{5}}$ (C) $\frac{12}{5\sqrt{5}}$ (D) 3

14. A point moves such that its distance from the point $(-1, 2)$ is always equal to its distance from the line $3x - 4y - 2 = 0$. Then the locus of the point is

$$(4x + 3)^2 + ax + by + 121 = 0$$

where $a - b$ is equal to

(A) 178 (B) 116 (C) 54 (D) 121

15. The equation of the line through the point of intersection of the lines $x - 3y + 1 = 0$ and $2x + 5y - 9 = 0$, and whose distance from the origin is $\sqrt{5}$ is

(A) $2x + y + 5 = 0$ (B) $2x - y + 5 = 0$

(C) $2x + y - 5 = 0$ (D) $2x - y - 5 = 0$

16. $B(2, 0)$ and $C(0, 1)$ are the ends of the base of an isosceles triangle for which the line $x = 2$ is one side. Then, the orthocentre of the triangle is

(A) $\left(\frac{3}{4}, 1\right)$ (B) $\left(\frac{4}{3}, \frac{7}{12}\right)$

(C) $\left(\frac{3}{2}, \frac{3}{2}\right)$ (D) $\left(\frac{5}{4}, 1\right)$

17. In ΔABC, $A = (1,10)$, circumcentre $= (-1/3, 2/3)$ and orthocentre $= (11/3, 4/3)$. Then, the coordinates of the midpoint of BC are

(A) $(1, 5)$ (B) $\left(1, \frac{-11}{3}\right)$

(C) $(1, 6)$ (D) $(1, -3)$

Hint: The centroid G divides the line joining the circumcentre and orthocentre in the ratio 1:2.

18. A rhombus is situated in the first quadrant with $x - y = 0$ and $7x - y = 0$ as two of its adjust sides. Then the slope of the longer diagonal of the rhombus is

(A) 2 (B) $\frac{1}{2}$ (C) $-\frac{1}{2}$ (D) -2

19. The distance of the line $2x - 3y - 4 = 0$ from the point $(1, 1)$ measured in the direction of the line $x + y - 1 = 0$ is

(A) $\frac{1}{\sqrt{2}}$ (B) $\sqrt{2}$ (C) $5\sqrt{2}$ (D) $2\sqrt{2}$

20. The line parallel to x-axis and passing through the intersection of the lines $ax + 2by + 3b = 0$ and $bx - 2by - 3a = 0$ where $(a, b) \neq (0, 0)$ is

(A) above the x-axis at a distance of 3/2 units from it

(B) above the x-axis at a distance of 2/3 units from it

(C) below the x-axis at a distance of 3/2 units from it

(D) below the x-axis at a distance of 2/3 units from it

21. The $L_1 \equiv 4x + 3y - 12 = 0$ intersect x-axis at A and y-axis at B. A variable line L_2 perpendicular to L_1 intersects x-axis at P as y-axis at Q. Then, the perpendicular circumcentre of ΔABC lies on

(A) $4x + 3y + 7 = 0$ (B) $6x - 8y + 7 = 0$

(C) $3x - 4y + 2 = 0$ (D) $3x + 4y - 2 = 0$

22. If the lines $ax-y+4=0, 3x-y+5=0$ and $x+y+8=0$ are concurrent, then the value of a is

(A) $\frac{35}{13}$ (B) $\frac{-5}{13}$ (C) 35 (D) $\frac{-15}{13}$

23. The lines $x^2+4xy+y^2=0$ and $x-y-4=0$ form a triangle which is

(A) equilateral
(B) right angled
(C) an isosceles
(D) isosceles right angled

24. Equation of the line which is parallel to the common line of the pair of lines $6x^2-xy-12y^2$ and $15x^2+14xy-8y^2=0$ and whose distance from this common line is 7 units is

(A) $3x-4y=\pm 35$ (B) $5x-2y=\pm 7$
(C) $3x+4y=\pm 35$ (D) $2x-3y=\pm 7$

25. The point (4, 1) undergoes the following three transformations successively:

I. Reflection about the line $y=x$.
II. translation through a distance 2 units along the positive direction of the x-axis.
III. Rotation through an angle $\pi/4$ about the origin in the counterclockwise direction.

Then, the final position of the point is given by

(A) $\left(\frac{1}{\sqrt{2}}, \frac{7}{\sqrt{2}}\right)$ (B) $(-\sqrt{2}, 7\sqrt{2})$
(C) $\left(-\frac{1}{\sqrt{2}}, \frac{7}{\sqrt{2}}\right)$ (D) $(\sqrt{2}, 7\sqrt{2})$

26. A straight line is passing through a fixed point (α, β). Then, the foot of the perpendicular drawn the origin onto the line lies on the curve whose equation is given by

(A) $x^2+y^2-\alpha x-\beta y=0$
(B) $x^2+y^2+\alpha x+\beta y=0$
(C) $x^2+y^2-\alpha x+\beta y=0$
(D) $x^2+y^2+\alpha x-\beta y=0$

27. $3x^2-8xy-3y^2+10x+20y-25=0$ are the bisectors of the angles between two lines of which one line is passing through the origin. Then, the equation of the other line is

(A) $x-2y=0$ (B) $2x-y=0$
(C) $x+2y-5=0$ (D) $x-2y-5=0$

28. The straight line $2x+3y+1=0$ bisects the angle between two straight lines of which one line is $3x+2y+4=0$. Then, the equation of the other line is

(A) $3x+23y=28$ (B) $9x+4y=28$
(C) $9x-46y=28$ (D) $9x+46y=28$

29. $A(1, 3)$ and $B(5, 2)$ are two points. If P is a variable points on the line $y=x$, then the minimum value of $|PA-PB|$ is

(A) $2\sqrt{5}$ (B) $3\sqrt{5}$ (C) $4\sqrt{5}$ (D) $\sqrt{5}$

Hint: If A' is the image of A on the line $y=x$, then $A'B$ is the value.

30. The equation of the line passing through the point of intersection of the lines $x-y+1=0$ and $3x+y-5=0$ and is perpendicular to the line $x+3y+1=0$ is

(A) $x-3y+1=0$ (B) $3x-y-1=0$
(C) $x+3y-1=0$ (D) $3x-y+1=0$

Multiple Correct Choice Type Questions

1. If the distance of the line $8x+15y+\lambda=0$ from the point (2, 3) is equal to 5 units, then the value of λ is

(A) 24 (B) –24 (C) 146 (D) –146

2. If the line $\sqrt{3}x+y-9=0$ is reduced to the form $x\cos\alpha + y\sin\alpha = p$, then

(A) $\alpha=60°$ (B) $\alpha=30°$
(C) $p=\frac{9}{2}$ (D) $p=9$

3. If l is the line passing through the point (–2, 3) and perpendicular to the line $2x-3y+6=0$, then

(A) the slope of $l=-\frac{3}{2}$
(B) the line l passes through (0, 0)
(C) the intercept on the axes are 2, 3
(D) the line l forms a triangle of area 5 sq. units with the coordinate axes

4. If l is the line passing through the point (2, –3) and is parallel to the line joining the points (4, 1) and (–2, 3), then

(A) (–10, –1) is a point on l
(B) the slope of l is 6

(C) the area of the triangle formed by the line l and the coordinate axes is 64/3

(D) the orthocentre of the triangle formed by l and the axes is $(1, 1)$

5. If σ is the family of lines passing through the point $(5, 0)$, then

(A) the line belonging to σ and having slope 2 is $2x - y - 5 = 0$

(B) the line belonging to σ and having y-intercept 5 is $x + y - 5 = 0$

(C) the line belonging to σ and perpendicular to the line $x + y + 1 = 0$ is $x + y - 5 = 0$

(D) the line belonging to σ and perpendicular to the line $x - y + 1 = 0$ is $x - y - 5 = 0$

6. A line l is passing through the point $(1, -6)$. If the product of the intercepts of l on the axes is 1, then the equation of l is

(A) $2x + y + 5 = 0$ (B) $2x + y + 4 = 0$
(C) $4x + y + 2 = 0$ (D) $9x + y - 3 = 0$

7. The graph of the function

$$y = \cos x \cos(x + 2) - \cos^2(x + 1) \text{ is}$$

(A) a straight line passing through $(0, -\sin^2 1)$

(B) a straight line passing through $\left(\dfrac{\pi}{2}, -\sin^2 1\right)$

(C) a straight line parallel to x-axis

(D) a straight line parallel to y-axis

8. Two points $A(-1, -1)$ and $B(4, 5)$ and the third vertex C lie on the line $5x - y - 15 = 0$. If the area of the triangle is 19/2, then the coordinates of the vertex C are

(A) $(2, -5)$ (B) $(5, 4)$ (C) $(3, 0)$ (D) $(5, 10)$

9. The line

$$\frac{x}{c} + \frac{y}{d} = 1$$

passes through the intersection of the lines

$$\frac{x}{a} + \frac{y}{b} = 1 \text{ and } \frac{x}{b} + \frac{y}{a} = 1$$

and the lengths of the perpendicular drawn from the origin onto these lines are equal. Then

(A) $\dfrac{1}{a^2} - \dfrac{1}{b^2} = \dfrac{1}{c^2} - \dfrac{1}{d^2}$

(B) $\dfrac{1}{a} + \dfrac{1}{b} = \dfrac{1}{c} + \dfrac{1}{d}$

(C) $\dfrac{1}{a^2} + \dfrac{1}{b^2} = \dfrac{1}{c^2} + \dfrac{1}{d^2}$

(D) $\dfrac{1}{a^2} + \dfrac{1}{b^2} + \dfrac{1}{c^2} + \dfrac{1}{d^2} = 2$

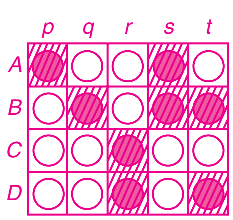
10. The straight lines $3x + y - 4 = 0$, $x + 3y - 4 = 0$ and $x + y = 0$ form a triangle which is

(A) obtuse angled (B) equilateral
(C) isosceles (D) right-angled

Matrix-Match Type Questions

In each of the following questions, statements are given in two columns, which have to be matched. The statements in *column I* are labeled as **(A)**, **(B)**, **(C)** and **(D)**, while those in *column II* are labeled as **(p)**, **(q)**, **(r)**, **(s)** and **(t)**. Any given statement in *column I* can have correct matching with *one or more* statements in *column II*. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following example.

Example: If the correct matches are (A) → (p), (s), (B) → (q), (s), (t), (C) → (r), (D) → (r), (t), that is if the matches are (A) → (p) and (s); (B) → (q), (s) and (t); (C) → (r); and (D) → (r), then the correct darkening of bubbles will look as follows:



1. Let σ be the system of lines passing through the intersection of the lines $x + y - 1 = 0$ and $x - y - 1 = 0$. Match the items of Column I with those of Column II.

Column I	*Column II*
(A) Equation of the line belonging to σ and passing through the point $(2, 3)$ is	(p) $2x - y - 2 = 0$
(B) Equation of the line belonging to σ and parallel to the line $y = 2x + 1$ is	(q) $x + y - 1 = 0$
(C) Equation of the line belonging to σ and having equal intercept (absolutely) is	(r) $x + y + 1 = 0$

(*Continued*)

Column I	Column II
(D) Equation of the line belonging to σ and perpendicular to the line $x + y - 1 = 0$ is	(s) $3x - y - 3 = 0$ (t) $x - y - 1 = 0$

2. Match the items of Column I with those of Column II.

Column I	Column II
(A) If the lines $x + 2ay + a = 0$, $x + 3by + b = 0$ and $x + 4cy + c = 0$ are concurrent, then a, b and c are in	(p) GP
(B) If the lines $ax + by + (ak + b) = 0$, $bx + cy + (bk + c) = 0$ and $(ak + b)x + (bk + c)y = 0$ are concurrent and $b^2 \neq ac$, then	(q) HP
(C) If the lines $ax + 2y + 1 = 0$, $bx + 3y + 1 = 0$ and $cx + 4y + 1 = 0$ pass through a fixed point, then a, b and c form	(r) k is a root of $ax^2 + 2bx + c = 0$
(D) If a, b and c are distinct positive and the lines $a(x + y) + c = 0$, $x + 1 = 0$ and $c(x + y) + b = 0$, then a, c and b are in	(s) AP (t) k is a root of $ax^2 + 2bx + c = 0$

3. Match the items of Column I with those of Column II.

Column I	Column II
(A) Equation of the line whose x-intercept is 4 and passes though the point $(2, -3)$	(p) $x + y - 8 = 0$
(B) Equation of the line having equal intercept on the axes and passing through the point $(2, 5)$ is	(q) $x + y = 7$
(C) Equation of the line which makes an angle of 135° with the positive direction of the x-axis and which cuts the y-axis at a distance of 8 units from the origin is	(r) $3x - 2y = 12$ (s) $x + 4y = 8$
(D) Equation of the line through the point $(4, 1)$ which forms a triangle of 8 sq. unit with positive axes is	(t) $x + y + 8 = 0$

Comprehension Type Questions

1. Passage: Let $u \equiv x + y = 0$, $A = (1, 2)$ and $B = (3, -1)$. Answer the following questions.

(i) If M is a point on the line $u = 0$ such that $AM + BM$ is minimum, then the reflection of M on the line $y = x$ is

(A) $(2, -2)$ (B) $(-2, 2)$
(C) $(1, -1)$ (D) $(-1, 1)$

(ii) If M is a point on $u = 0$ such that $|AM - BM|$ is maximum, then the distance between M and the point $N(1, 1)$ is

(A) $3\sqrt{5}$ (B) $5\sqrt{2}$ (C) 7 (D) 10

(iii) If M is a point on $u = 0$ such that $|AM - BM|$ is minimum, then the area of ΔABM is equal to

(A) $\frac{13}{8}$ (B) $\frac{13}{6}$ (C) $\frac{13}{2}$ (D) $\frac{13}{4}$

2. Passage: ABC is an equilateral triangle with vertex $A = (1, 1)$ and the equation of the side BC is $x + y = 1$. Answer the following questions.

(i) Area of the triangle in square units is

(A) $\frac{1}{\sqrt{3}}$ (B) $\frac{2}{\sqrt{3}}$ (C) $\frac{1}{2\sqrt{3}}$ (D) $\frac{1}{2\sqrt{2}}$

(ii) The gradients of the two sides AB and AC are

(A) $\sqrt{3}, \frac{1}{\sqrt{3}}$ (B) $\sqrt{2}, \frac{1}{\sqrt{2}}$
(C) $\sqrt{2}+1, \sqrt{2}-1$ (D) $2+\sqrt{3}, 2-\sqrt{3}$

(iii) The circumradius of the triangle is

(A) $\frac{1}{3}$ (B) $\frac{\sqrt{2}}{3}$ (C) $\frac{1}{\sqrt{3}}$ (D) $\frac{1}{\sqrt{2}}$

3. Passage: If $u_1 = 0$ and $u_2 = 0$ are two intersecting lines, then for all values of λ and μ, the equation $\lambda u_1 + \mu u_2$ represents straight lines passing through the intersection of the lines $u_1 = 0$ and $u_2 = 0$. In particular, the equation $u_1 + \lambda u_2 = 0$ represents all lines (except $u_2 = 0$) passing through the intersection of $u_1 = 0$ and $u_2 = 0$. The converse of these are also true. Answer the following questions.

(i) The line $(\lambda+1)^2x+\lambda y-2\lambda^2-2=0$ passes through a fixed point. The equation of the line passing through this fixed point and having slope 2 is

(A) $2x-y+8=0$ (B) $2x-y-5=0$
(C) $2x-y-8=0$ (D) $2x-y-4=0$

(ii) Consider the family of lines $p(2x+y+4)+q(x-2y-3)=0$ (p and q are parameters). The number of lines belonging to this family and whose distance from the point $Q(2,-3)$ is $\sqrt{10}$ is

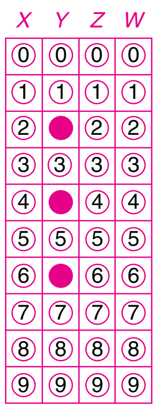
(A) ∞ (B) 1 (C) 2 (D) 0

(iii) In (ii), if L is the required line, then the image of the point $Q(2,-3)$ in the line is

(A) $(4,1)$ (B) $(-4,1)$
(C) $(4,-1)$ (D) $(-4,-1)$

Integer Answer Type Questions

The answer to each of the questions in this section is a non-negative integer. The appropriate bubbles below the respective question numbers have to be darkened. For example, as shown in the figure, if the correct answer to the question number Y is 246, then the bubbles under Y labeled as 2, 4, 6 are to be darkened.



1. If (a, b) is the image of the point $(2,-3)$ on the line $3x-y-1=0$, then $b-a$ is equal to _____.

2. If L is the line belonging to the family of lines represented by the equation $(2x+y+4)+\lambda(x-2y-3)=0$ (where λ is a parameter) whose distance from the point $(2,-3)$ is $\sqrt{10}$ units, then the slope of the line L is ______.

3. If the lines $ax+by-5=0$ and $px+qy+1=0$ are the diagonals of the parallelogram whose sides are $2x-y+7=0$, $3x+2y-5=0$ and $3x+2y+4=0$, then the value of $\frac{1}{8}(a+b+p+q)$ is ________.

4. In ΔABC, the equations of the madians AD and BE, respectively, are $2x+3y-6=0$ and $3x-2y-10=0$. If $AD=6$, $BE=11$, then $\frac{1}{11}$(Area of ΔABC) is ________.

5. $P(1, 2)$, $Q(4, 6)$, $R(5, 7)$ and $S(a, b)$ are the vertices of the parallelogram $PQRS$. Then, $a+b$ is equal to _______.

6. The area of the triangle formed by the line $x+y=3$ and the angle bisectors of the pair of lines $x^2-y^2+2y-1=0$ is ______ sq. unit.

7. A straight line through the origin O meets the parallel lines $4x+2y=9$ and $2x+y=6$ at points P and Q, respectively. If O divides the segment PQ in the ratio $p{:}q$, then the value of $p+q$ is _____.

8. If a, b and c are real such that $3a+2b+4c=0$, then the line $ax+by+c=0$ passes through a fixed point (h, k) where $[h+k]$ ($[\cdot]$ is the usual symbol) is _______.

9. The number of integral values of m, for which the x-coordinate of the point of intersection of the lines $3x+4y-9=0$ and $y=mx+1$ is also an integer is _______.

10. $P(m, n)$ is an interior point (where m and n are positive integers) of a quadrilateral formed by the lines $y=0$, $x=0$, $2x+y-2=0$ and $4x+5y-20=0$. The possible number of positions of P is _____.

ANSWERS

Single Correct Choice Type Questions

1. (C)

2. (B)

3. (C)

4. (A)

5. (D)
6. (C)
7. (A)
8. (D)
9. (B)
10. (A)
11. (C)
12. (B)
13. (B)
14. (A)
15. (C)
16. (D)
17. (B)
18. (A)
19. (B)
20. (C)
21. (B)
22. (A)
23. (A)
24. (C)
25. (C)
26. (A)
27. (C)
28. (D)
29. (D)
30. (B)

Multiple Correct Choice Type Questions

1. (A), (D)
2. (B), (C)
3. (A), (B)
4. (A), (C)
5. (B), (D)
6. (C), (D)
7. (A), (B), (C)
8. (C), (D)
9. (A), (B)
10. (A), (C)

Matrix-Match Type Questions

1. (A) $\rightarrow$ (s); (B) $\rightarrow$ (p); (C) $\rightarrow$ (q), (t); (D) $\rightarrow$ (t)
2. (A) $\rightarrow$ (q); (B) $\rightarrow$ (r); (C) $\rightarrow$ (s), (t); (D) $\rightarrow$ (p)
3. (A) $\rightarrow$ (r); (B) $\rightarrow$ (q); (C) $\rightarrow$ (p), (t); (D) $\rightarrow$ (s)

Comprehension Type Questions

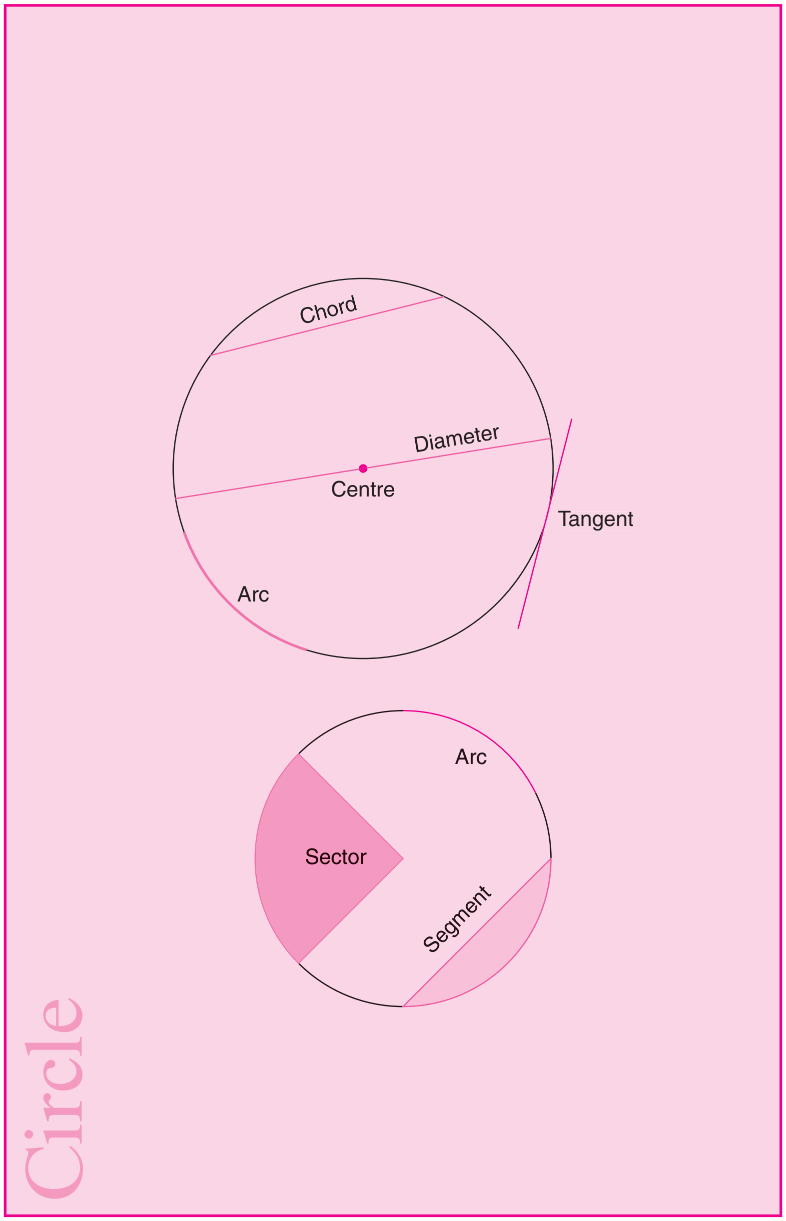
1. (i) (D); (ii) (D); (iii) (D)
2. (i) (C); (ii) (D); (iii) (B)
3. (i) (C); (ii) (B); (iii) (D)

Integer Answer Type Questions

1. 3
2. 3
3. 5
4. 4
5. 5
6. 2
7. 7
8. 2
9. 2
10. 5

Circle

3



Contents

A **circle** is a simple shape of Euclidean geometry consisting of those points in a plane that are equidistant from a given point, the centre. A circle can be defined as the curve traced out by a point that moves so that its distance from a given point is constant.

In the previous chapter, we discussed about the straight line and pair of lines. It is known that a straight line is represented by first-degree equation in x and y and hence it is called first-degree curve. Curves represented by second-degree equation in x and y are called second-degree curves. Some of the second-degree curves are pair of lines (studied in the previous chapter), circle and conics. Among these second-degree curves, the circle has been known since ancient times and has some special properties. In this chapter, we study the general equation of a circle, equation of the tangent at a point, chord equation interval of its midpoint, chord of contact, orthogonal circles, etc. Subjective Problems have been provided for the preceding sections. Students are advised to solve each and every problem to grasp the topics.

3.1 | Introduction

We begin with the following definition.

DEFINITION 3.1 **Circle** Let A be a fixed point in a plane and $r > 0$ a given real number. Then the locus of the point P such that the distance AP is equal to r is called a circle with centre A and radius r (Fig. 3.1).

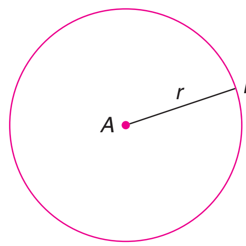


FIGURE 3.1 A circle.

We translate the definition of the circle, and obtain the equations of a circle.

THEOREM 3.1 The equation of the circle with centre at the point $A(h, k)$ and radius r is

$$(x-h)^2 + (y-k)^2 = r^2$$

PROOF $P(x, y)$ is a point on the given circle $\Leftrightarrow (AP)^2 = r^2 \Leftrightarrow (x-h)^2 + (y-k)^2 = r^2$, because the distance

$$AP = \sqrt{(x-h)^2 + (y-k)^2}$$

■

Note: If $h = 0, k = 0$ (i.e., origin is the centre), then the equation of the circle is $x^2 + y^2 = r^2$.

QUICK LOOK 1

1. Equation of the circle with centre at origin and radius r is $x^2 + y^2 = r^2$.
2. Equation of the circle with centre at (h, k) and radius r is $x^2 + y^2 - 2hx - 2ky + h^2 + k^2 - r^2 = 0$, which is in the form $x^2 + y^2 + 2gx + 2fy + c = 0$, where g, f and c are real numbers.

THEOREM 3.2 If g, f, c are real numbers and $g^2 + f^2 - c > 0$, then the equation $x^2 + y^2 + 2gx + 2fy + c = 0$ represents circle with centre at $(-g, -f)$ and radius $\sqrt{g^2 + f^2 - c}$.

PROOF $x^2 + y^2 + 2gx + 2fy + c = 0$ can be written as $(x+g)^2 + (y+f)^2 = \left(\sqrt{g^2+f^2-c}\right)^2$ which represents circle with centre $(-g, -f)$ and radius $\sqrt{g^2 + f^2 - c}$ according to Theorem 3.1. ■

Note:

1. The locus represented by the equation $(x-h)^2+(y-k)^2=0$ is the single point (h, k) which is called point circle. In fact, if the radius of a circle is zero, then it is called *point circle*.
2. If $g^2+f^2-c<0$, then the locus represented by $x^2+y^2+2gx+2fy+c=0$ is the empty set and it represents point circle if $g^2+f^2-c=0$. Further, the equation of the circle with centre (h, k) and radius r is of the form $x^2+y^2+2gx+2fy+c=0$ (see Quick Look 1). With this understanding we refer the equation $x^2+y^2+2gx+2fy+c=0$ as the *general equation of a circle*. Usually, when $g^2+f^2-c<0$, then we call the circle as *imaginary circle*.
3. If $a \neq 0$, then the equation $ax^2+ay^2+2gx+2fy+c=0$ can be written as $x^2+y^2+2g'x+2f'y+c'=0$, where $g'=g/a, f'=f/a, c'=c/a$ which represents circle in the broad perspective as per point (2). The equation $ax^2+ay^2+2gx+2fy+c=0$ is called *universal equation of the circle*.
4. The second degree general equation $S \equiv ax^2+2hxy+by^2+2gx+2fy+c=0$ represents a circle with positive radius if and only if $a=b=0$ and $g^2+f^2>ac$. The proof is not necessary.

THEOREM 3.3 If $A(x_1, y_1)$ and $B(x_2, y_2)$ are extremities of a diameter of a circle, then the equation of the circle is $(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$.

PROOF Consider Fig. 3.2. Let P be any point on the circle whose coordinates are (x, y). It is clear that both the points A and B satisfy the equation

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0 \tag{3.1}$$

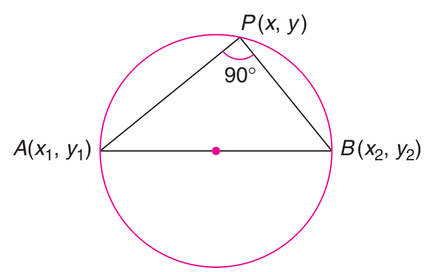


FIGURE 3.2

Hence, we may assume P is not both A and B. From the elementary plane geometry, it is known that angle in a semicircle is a right angle. Therefore, $\underline{|APB}$ is a right angle. That is, the segments $\overline{AP}$ and $\overline{BP}$ are at right angle to each other so that the product of their slopes is equal to -1. Therefore

$$\left(\frac{y-y_1}{x-x_1}\right)\left(\frac{y-y_2}{x-x_2}\right)=-1$$

Hence

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$$

That is, every point on the given circle satisfies Eq. (3.1) and conversely, if any point $Q(x, y)$ satisfies Eq. (3.1), then we know that $\overline{AQ}$ and $\overline{BQ}$ are at right angles and hence Q must lie on the circle. Hence, Eq. (3.1) represents the circle for which $A(x_1, y_1)$ and $B(x_2, y_2)$ are ends of a diameter. ■

THEOREM 3.4 If $P(x, y)$ is a point on the circle $x^2+y^2=r^2$, then there exists θ such that $x=r\cos\theta$ and $y=r\sin\theta$ and conversely the point $(r\cos\theta, r\sin\theta)$ lies on the circle $x^2+y^2=r^2$ for all θ. The equations $x=r\cos\theta$, $y=r\sin\theta$ are called *parametric equations* of the circle $x^2+y^2=r^2$.

PROOF Let $P(x, y)$ be a point on the circle $x^2+y^2=r^2$ (see Fig. 3.3). Draw PM perpendicular to the x-axis and let $\underline{|MOP}=\theta$. From ΔMOP, we have that

$$\cos\theta=\frac{x}{r} \quad \text{and} \quad \sin\theta=\frac{y}{r}$$

Therefore, $x = r\cos\theta$, $y = r\sin\theta$. Also

$$r^2\cos^2\theta + r^2\sin^2\theta = r^2$$

shows that $(r\cos\theta, r\sin\theta)$ lies on the circle $x^2 + y^2 = r^2$.

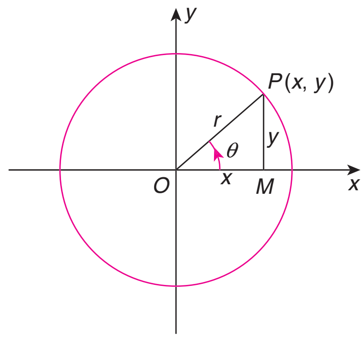


FIGURE 3.3

Note:

1. By shifting the origin to the centre (h, k) of the circle $(x - h)^2 + (y - k)^2 = r^2$ and using Theorem 3.4 we can see that

$$x = h + r\cos\theta$$

and $$y = k + r\sin\theta$$

are the parametric equations of the circle $(x - h)^2 + (y - k)^2 = r^2$.

2. Since $(-g, -f)$ and $r = \sqrt{g^2 + f^2 - c}$ are, respectively, the centre and radius of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$, it follows that

$$x = -g + r\cos\theta \quad \text{and} \quad y = -f + r\sin\theta$$

are its parametric equations.

3.2 | Relation Between a Circle and a Line in its Plane

Let $\mathcal{C}$ be a circle with centre A and radius r and l be a straight line in the plane of the circle. Draw AM perpendicular to the line l. Then

1. $AM > r \Leftrightarrow$ the line l and the circle $\mathcal{C}$ have no common points [see Fig. 3.4(a)].
2. $AM = r \Leftrightarrow$ the line touches the circle [see Fig. 3.4(b)].
3. $AM < r \Leftrightarrow$ the line l intersects the circle $\mathcal{C}$ in two distinct points [see Fig. 3.4(c)].

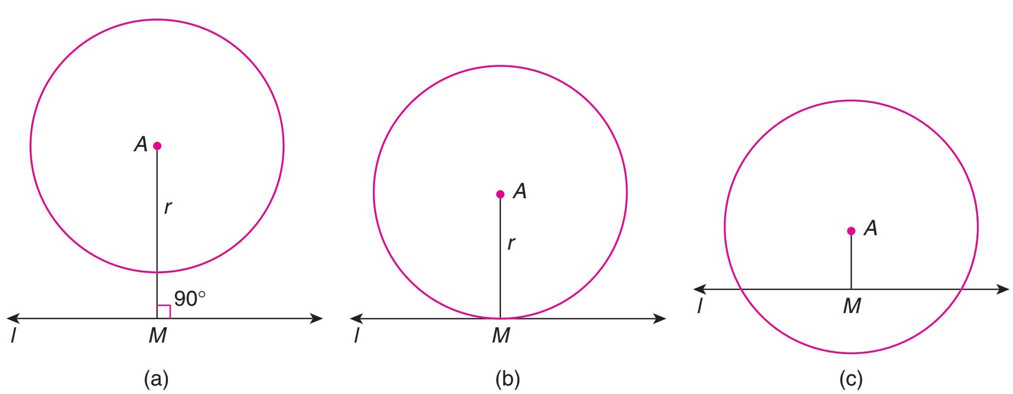


FIGURE 3.4

General Note: In the proofs of theorems, we consider the circle $x^2 + y^2 = r^2$ only to avoid tediousness of the proofs.

THEOREM 3.5 The perpendicular bisector of a chord of a circle passes through the centre of the circle.

PROOF Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be ends of a chord of the circle $x^2 + y^2 = r^2$ whose centre is $O(0, 0)$ (see Fig. 3.5). Let M be the midpoint of the chord $\overline{AB}$ so that

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Now, A and B are points on the circle implies

$$x_1^2 + y_1^2 = r^2$$

and
$$x_2^2 + y_2^2 = r^2$$

Therefore

$$(x_1^2 - x_2^2) + (y_1^2 - y_2^2) = 0$$
$$\Rightarrow (x_1 + x_2)(x_1 - x_2) + (y_1 + y_2)(y_1 - y_2) = 0$$
$$\Rightarrow \frac{y_1 - y_2}{x_1 - x_2} = \frac{-(x_1 + x_2)}{y_1 + y_2}$$

Now,

$$\text{Slope of line } OM = \frac{\dfrac{y_1 + y_2}{2} - 0}{\dfrac{x_1 + x_2}{2} - 0} = \frac{y_1 + y_2}{x_1 + x_2} = \frac{-(x_1 - x_2)}{y_1 - y_2}$$

Therefore

$$(\text{Slope of chord } AB)(\text{Slope of } \overline{OM}\,) = \left(\frac{y_1 - y_2}{x_1 - x_2}\right)\left(\frac{y_1 + y_2}{x_1 + x_2}\right) = -1$$

and hence $\overline{OM}$ is perpendicular to $\overline{AB}$.

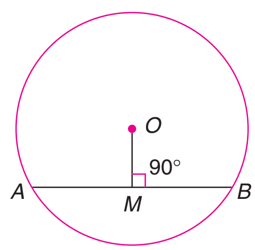


FIGURE 3.5

■

Notation: Here onwards, we use the following notation.

$$S \equiv x^2 + y^2 + 2gx + 2fy + c$$
$$S_1 \equiv xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$$
$$S_2 \equiv xx_2 + yy_2 + g(x + x_2) + f(y + y_2) + c$$
$$S_{21} = S_{12} = x_1x_2 + y_1y_2 + g(x_1 + x_2) + f(y_1 + y_2) + c$$
$$S_{11} = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$$

In general

$$S_{ii} = x_i^2 + y_i^2 + 2gx_i + 2fy_i + c$$

In particular if $S \equiv x^2 + y^2 - a^2$, then $S_1 \equiv xx_1 + yy_1 - a^2$, $S_{11} = x_1^2 + y_1^2 - a^2$, etc.

THEOREM 3.6 If $S = 0$ is a circle and (x_1, y_1) is a point in the plane of the circle (not the centre), then the equation $S_1 = 0$ represents a straight line which is perpendicular to the line joining the centre and the point (x_1, y_1).

PROOF Let

$$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$$

Therefore

$$\begin{aligned} S_1 &\equiv xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c \\ &\equiv (g + x_1)x + (f + y_1)y + gx_1 + fy_1 + c \end{aligned}$$

Since $(x_1, y_1) \neq (-g, -f)$, it follows that

$$S_1 \equiv (g + x_1)x + (f + y_1)y + gx_1 + fy_1 + c = 0$$

is a first-degree equation in x and y and hence it represents a straight line. Also, since the slope of the line $S_1 = 0$ is $-(g + x_1)/(f + y_1)$, it follows that it is perpendicular to the line joining $(-g, -f)$ and (x_1, y_1). ■

THEOREM 3.7 The equation of the chord joining two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ on a circle $S = 0$ is $S_1 + S_2 = S_{12}$ and hence the equation of the tangent at (x_1, y_1) is $S_1 = 0$.

PROOF Let $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$. Since $P(x_1, y_1)$ and $Q(x_2, y_2)$ lie on the circle (see Fig. 3.6), we have

$$\begin{aligned} S_{11} &= x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0 \\ S_{22} &= x_2^2 + y_2^2 + 2gx_2 + 2fy_2 + c = 0 \end{aligned}$$

Let $C = (-g, -f)$ (centre) and $M = \left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$ (the midpoint of $\overline{AB}$). From Theorem 3.5, $\overline{AB}$ is perpendicular to $\overline{CM}$ so that the equation of the chord $\overline{AB}$ is

$$\begin{aligned} &(x - x_1)\left(\frac{x_1 + x_2}{2} + g\right) + (y - y_1)\left(\frac{y_1 + y_2}{2} + f\right) = 0 \\ &\Rightarrow (x - x_1)(x_1 + x_2 + 2g) + (y - y_1)(y_1 + y_2 + 2f) = 0 \\ &\Rightarrow x(x_1 + x_2) + y(y_1 + y_2) + 2gx + 2fy = x_1^2 + y_1^2 + x_1x_2 + y_1y_2 + 2gx_1 + 2fy_1 \\ &\Rightarrow x(x_1 + x_2) + y(y_1 + y_2) + 2gx + 2fy = x_1x_2 + y_1y_2 - c \quad (\because S_{11} = 0) \\ &\Rightarrow [xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c] + [xx_2 + yy_2 + g(x + x_2) + f(y + y_2) + c] \\ &\quad = x_1x_2 + y_1y_2 + gx_1 + fy_1 + gx_2 + fy_2 + 2c - c \end{aligned}$$

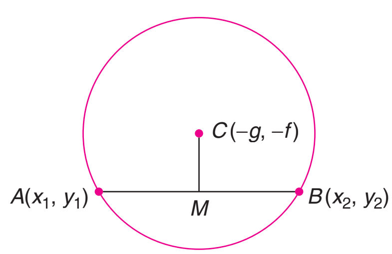


FIGURE 3.6

Therefore

$$S_1 + S_2 = x_1x_2 + y_1y_2 + g(x_1 + x_2) + f(y_1 + y_2) + c = S_{12}$$

That is, equation of the chord AB is $S_1 + S_2 = S_{12}$.

Since the tangent at $P(x_1, y_1)$ to the circle is the limiting position of the chord $\overline{PQ}$ as Q approaches P along the circle (see Chapter 3, Vol. 3), the equation of the tangent is

$$S_1 + S_1 = S_{11} = 0 \quad [\because P(x_1, y_1) \text{ lies on the circle}]$$

Hence, $S_1 \equiv xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$ is the tangent at (x_1, y_1). ■

QUICK LOOK 2

If $S = x^2 + y^2 - a^2 = 0$ is the circle, then the tangent at (x_1, y_1) is $S_1 \equiv xx_1 + yy_1 - a^2 = 0$.

Example 3.1

Find the equation of the tangent to the circle $x^2 + y^2 - 2x - 4y + 3 = 0$ at the point $(2, 3)$.

Solution: We have

$$2^2 + 3^2 - 2(2) - 4(3) + 3 = 16 - 16 = 0$$

which implies that $(2, 3)$ lies on the circle $S = x^2 + y^2 - 2x - 4y + 3 = 0$. Here, $(x_1, y_1) = (2, 3)$ so that the equation of the tangent at $(2, 3)$ is

$$S_1 = x(2) + y(3) - (x + 2) - 2(y + 3) + 3 = 0$$

that is, $S_1 = x + y - 5 = 0$.

Example 3.2

Find the equation of the tangent to the circle $3x^2 + 3y^2 - 4x - 6y = 0$ at $(0, 0)$.

Solution: The equation of the circle is in the universal form so that its general form is

$$S \equiv x^2 + y^2 - \left(\frac{4}{3}\right)x - 2y = 0$$

Here, $g = -2/3, f = -1, c = 0$ and $(x_1, y_1) = (0, 0)$. The equation of the tangent at $(0, 0)$ is

$$S_1 \equiv x(0) + y(0) - \frac{2}{3}(x + 0) - 1(y + 0) = 0$$

That is, $S_1 \equiv 2x + 3y = 0$.

Example 3.3

Find the equation of the tangent to the circle

$$x^2 + y^2 + 2ay \cot \alpha - a^2 = 0$$

at $(a, 0)$.

Solution: Clearly, $(a, 0)$ lies on the circle. Equation of the tangent at $(a, 0)$ is

$$x(a) + y(0) + a \cot \alpha\,(y + 0) - a^2 = 0$$

$$\Rightarrow ax + (c \cot \alpha)y - a^2 = 0$$

$$\Rightarrow x + y \cot \alpha - a = 0$$

Example 3.4

Find the equation of the tangent to the circle $x^2 + y^2 = a^2$ at $(a \cos \theta, a \sin \theta)$.

Solution: We have

$$S \equiv x^2 + y^2 - a^2 = 0$$

and $\quad (x_1, y_1) = (a \cos \theta, a \sin \theta)$

The equation of the tangent at $(a \cos \theta, a \sin \theta)$ is

$$x(a \cos \theta) + y(a \sin \theta) - a^2 = 0$$

$$\Rightarrow x \cos \theta + y \sin \theta - a = 0$$

Example 3.5

Find the equation of the tangent to the circle $S \equiv x^2 + y^2 + 2x - 2y - 3 = 0$ at the point $(1, 2)$ and also find the tangent to the circle parallel to this tangent.

Solution: The equation of the tangent at $A(1, 2)$ (see Fig. 3.7) is

$$S_1 \equiv x(1) + y(2) + 1(x+1) - 1(y+2) - 3 = 0$$

$$\Rightarrow S_1 \equiv 2x + y - 4 = 0 \qquad (3.2)$$

The tangent to the circle parallel to the tangent given by Eq. (3.2) must be at the other end $B(-3, 0)$ of the diameter through the point $A(1, 2)$. Therefore, the tangent at $B(-3, 0)$ is

$$x(-3) + y(0) + 1(x-3) - 1(y+0) - 3 = 0$$

$$\Rightarrow -2x - y - 6 = 0$$

$$\Rightarrow 2x + y + 6 = 0$$

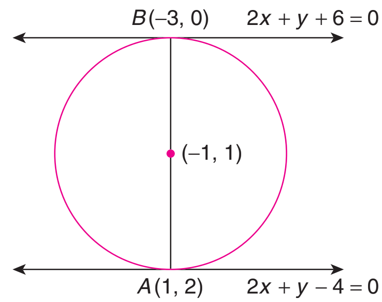


FIGURE 3.7

Theorem 3.8 The condition for the line $y = mx + c$, $c \neq 0$ to touch the circle $x^2 + y^2 = a^2$ is that $c^2 = a^2(1 + m^2)$ and in such a case the point of contact is $(-a^2m/c, a^2/c)$.

Proof Suppose the line $y = mx + c$ touches the circle at the point $P(x_1, y_1)$. Hence, by Theorem 3.7, the equation of the tangent at $P(x_1, y_1)$ is

$$S_1 \equiv xx_1 + yy_1 - a^2 = 0$$

That is, $S_1 = 0$ and $y = mx + c$ represent the same line. Therefore

$$\frac{x_1}{m} = \frac{y_1}{-1} = \frac{-a^2}{c}$$

Therefore

$$x_1 = \frac{-a^2m}{c} \text{ and } y_1 = \frac{a^2}{c}$$

Since (x_1, y_1) lies on the circle, we have

$$\frac{a^4m}{c^2} + \frac{a^4}{c^2} = a^2$$

$$\Rightarrow c^2 = a^2(1 + m^2)$$

Conversely, suppose $c^2 = a^2(1 + m^2)$. Therefore

$$\left|\frac{m(0) - 0 + c}{\sqrt{1+m^2}}\right| = a$$

That is, the length of the perpendicular drawn from $(0, 0)$ (i.e. the centre of the circle) onto the line $y = mx + c$ is equal to the radius a. Hence, by point (2) in Section 3.2, the line $y = mx + c$ touches the circle. ■

Note:

1. For any $m \neq 0$, the two lines $y = mx \pm a\sqrt{1+m^2}$ are parallel tangents to the circle $x^2 + y^2 = a^2$.

2. In general, to show that a line touches a circle, it is enough if we show that the length of the perpendicular drawn from the centre onto the line is equal to the radius of the circle.

3.3 | Classification of Points in a Plane w.r.t. a Circle in the Same Plane

Let $\mathcal{C}$ be a circle with centre at the point A and radius r and P be any point in the plane (Fig. 3.8). Then

1. P lies outside the circle $\Leftrightarrow AP > r$.

2. P lies on the circle (on the circumference) $\Leftrightarrow AP = r$.

3. P lies inside the circle $\Leftrightarrow AP < r$.

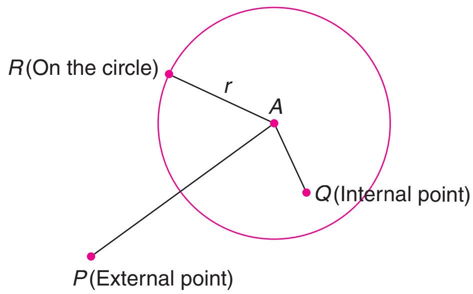


FIGURE 3.8

THEOREM 3.9 Let $S = 0$ be a circle and $P(x_1, y_1)$ be a point in the plane of the circle. Then $P(x_1, y_1)$ lies outside or inside or on the circle according as $S_{11} > = < 0$.

PROOF Let $S = x^2 + y^2 + 2gx + 2fy + c = 0$, $A = (-g, -f)$ and $r = \sqrt{g^2 + f^2 - c}$. Now

$$\begin{aligned} P(x_1, y_1) \text{ lies outside } S = 0 &\Leftrightarrow AP > r \\ &\Leftrightarrow (AP)^2 > r^2 \\ &\Leftrightarrow (x_1 + g)^2 + (y_1 + f)^2 > g^2 + f^2 - c \\ &\Leftrightarrow x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c > 0 \\ &\Leftrightarrow S_{11} > 0 \end{aligned}$$

Similarly, $P(x_1, y_1)$ lies inside $S = 0 \Leftrightarrow S_{11} < 0$. ■

THEOREM 3.10 Through every external point, two distinct tangents can be drawn to a circle.

PROOF Let $S \equiv x^2 + y^2 - a^2 = 0$ be a circle and $P(x_1, y_1)$ be an external point to the circle, so that

$$S_{11} = x_1^2 + y_1^2 - a^2 > 0$$

By Note (1) under Theorem 3.8, we know that $y = mx + a\sqrt{1+m^2}$ touches the circle $S \equiv x^2 + y^2 - a^2 = 0$. This line passes through $P(x_1, y_1)$

$$\begin{aligned} &\Leftrightarrow y_1 = mx_1 + a\sqrt{1+m^2} \\ &\Leftrightarrow (y_1 - mx_1)^2 = a^2(1+m^2) \\ &\Leftrightarrow (x_1^2 - a^2)m^2 - 2x_1y_1m + y_1^2 - a^2 = 0 \end{aligned} \tag{3.4}$$

Equation (3.4) is a quadratic equation in m whose discriminant is

$$4x_1^2y_1^2 - 4(x_1^2 - a^2)(y_1^2 - a^2) = 4a^2(x_1^2 + y_1^2 - a^2)$$
$$= 4a^2S_{11} > 0 \qquad (\because S_{11} > 0)$$

Therefore, the quadratic equation [Eq. (3.4)] in m has two distinct roots, say m_1 and m_2, so that there are two tangents through (x_1, y_1) with slopes m_1 and m_2 and

$$m_1m_2 = \frac{y_1^2 - a^2}{x_1^2 - a^2}$$

QUICK LOOK 3

The two tangents through (x_1, y_1) to the circle $S \equiv x^2 + y^2 - a^2 = 0$ are at right angles implies and is implied by ($\Leftrightarrow$)

$$m_1m_2 = -1 \Leftrightarrow \frac{y_1^2 - a^2}{x_1^2 - a^2} = -1$$
$$\Leftrightarrow x_1^2 + y_1^2 = 2a^2$$

Therefore, the locus of (x_1, y_1) is the circle $x^2 + y^2 = 2a^2$, which is a circle concentric with $S = 0$ and having radius $\sqrt{2}$ times the radius of $S = 0$. That is, the locus of the point through perpendicular tangents drawn to a circle $S = 0$ is also a circle concentric with $S = 0$ and radius equal to $\sqrt{2}$ times the radius of $S = 0$.

DEFINITION 3.2 **Director Circle** The locus of the point through which perpendicular tangents are drawn to a given circle $S = 0$ is a circle called the *director circle of* $S = 0$.

QUICK LOOK 4

If the centre and radius of a circle are A and r, respectively, then the centre and radius of its director circle are A and $r\sqrt{2}$, respectively.

Example 3.6

Find the locus of the point of intersection of perpendicular tangents to the circle $S \equiv x^2 + y^2 - 2x + 2y - 2 = 0$. That is, find the director circle of $S = 0$.

Solution: The centre and radius of the circle $S = 0$ are $(1, -1)$ and 2, respectively. Hence, the equation of the director circle of $S = 0$ is

$$(x-1)^2 + (y+1)^2 = (2\sqrt{2})^2 = 8$$

That is,

$$x^2 + y^2 - 2x + 2y - 6 = 0$$

THEOREM 3.11 The length of the tangent drawn from an external point $P(x_1, y_1)$ to the circle $S = 0$ is $\sqrt{S_{11}}$.

PROOF Let $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$, centre $A = (-g, -f)$ and radius $r = \sqrt{g^2 + f^2 - c}$. Let T be the point of contact of the tangent from P to the circle. See Fig. 3.9. From Pythogoras theorem, we have

$$(AP)^2 = (PT)^2 + (AT)^2$$
$$\Rightarrow (x_1 + g)^2 + (y_1 + f)^2 = (PT)^2 + (g^2 + f^2 - c)$$
$$\Rightarrow (PT)^2 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = S_{11} > 0$$

because $P(x_1, y_1)$ is an external point (see Theorem 3.9). Therefore

$$PT = \sqrt{S_{11}}$$

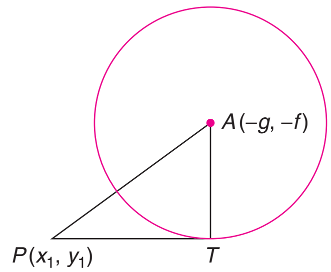


FIGURE 3.9

DEFINITION 3.3 **Chord of Contact** Let $\mathcal{C}$ be a circle and P be an external point to $\mathcal{C}$. Let A and B be the points of contact of the tangents drawn from P to $\mathcal{C}$. Then the chord $\overline{AB}$ is called the chord of contact of the point P with respect to the circle $\mathcal{C}$.

THEOREM 3.12 The equation of the chord of contact of a point $P(x_1, y_1)$ with respect to the circle $S = 0$ is $S_1 = 0$.

PROOF Let $\overline{AB}$ be the chord of contact of P (see Fig. 3.10). Suppose $A = (x_2, y_2)$ and $B = (x_3, y_3)$. The equation of the tangent at $A(x_2, y_2)$ is $S_2 = 0$.

This tangent passes through $P(x_1, y_1) \Rightarrow S_{21} = 0 \Rightarrow S_{12} = 0$. Therefore, the point $A(x_2, y_2)$ satisfies the first-degree equation $S_1 = 0$. Similarly, $S_{13} = 0$ implies that the point $B(x_3, y_3)$ satisfies the first-degree equation $S_1 = 0$. Hence, the equation of the chord $\overline{AB}$ is $S_1 = 0$.

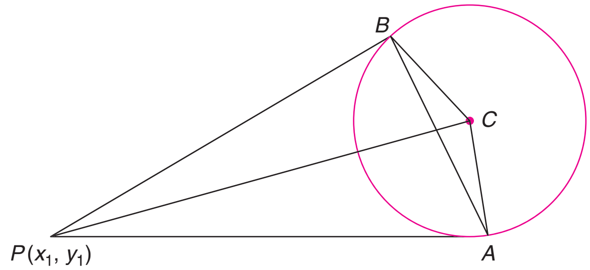


FIGURE 3.10

QUICK LOOK 5

The chord of contact $\overline{AB}$ of P is perpendicular to the line joining P with the centre of the circle (see Theorem 3.6).

THEOREM 3.13 The equation of the chord of the circle $S = 0$ whose midpoint is $M(x_1, y_1)$ is $S_1 = S_{11}$ (see Fig. 3.11).

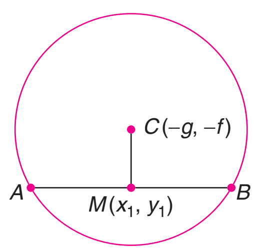


FIGURE 3.11

Proof Let $C(-g, -f)$ be the centre of the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$. Let $M(x_1, y_1)$ be the midpoint of the chord $\overline{AB}$. Since $\overline{AB}$ is a chord perpendicular to $\overline{CM}$, the equation of $\overline{AB}$ is

$$(x - x_1)(x_1 + g) + (y - y_1)(y_1 + f) = 0$$
$$\Rightarrow xx_1 + yy_1 + gx + fy = x_1^2 + y_1^2 + gx_1 + fy_1$$
$$\Rightarrow xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$$
$$\Rightarrow S_1 = S_{11}$$

■

Subjective Problems (Sections 3.1 till 3.3)

1. If the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ intersects the x-axis in two points, then show that the length of the intercept is $2\sqrt{g^2 - c}$.

Solution: Let the circle meet the x-axis in $A(x_1, 0)$ and $B(x_2, 0)$ (see Fig. 3.12) so that x_1, x_2 are the roots of the equation $x^2 + 2gx + c = 0$ (since the x-axis equation is $y = 0$). Therefore

$$x_1 + x_2 = -2g \quad \text{and} \quad x_1x_2 = c$$

Hence

$$(AB)^2 = (x_1 - x_2)^2 = (x_1 + x_2)^2 - 4x_1x_2 = 4g^2 - 4c = 4(g^2 - c)$$
$$\Rightarrow AB = 2\sqrt{g^2 - c}$$

Similarly, the length of the intercept made by the circle $S = 0$ on y-axis is $2\sqrt{f^2 - c}$.

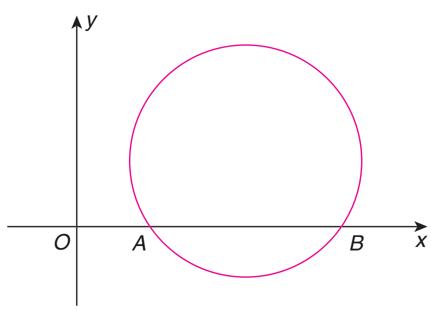


FIGURE 3.12

QUICK LOOK 6

1. x-axis touches the circle $S = 0 \Leftrightarrow g^2 = c$.
   y-axis touches the circle $S = 0 \Leftrightarrow f^2 = c$.
2. The circle $S = 0$ touches both the axes $\Leftrightarrow g^2 = c = f^2 \Leftrightarrow |g| = \sqrt{c} = |f|$.
3. The equation of the circle, with radius a, which touches both axes, is given by $x^2 + y^2 \pm 2ax \pm 2ay + a^2 = 0$ (one in each of the quadrants).

2. Find the length of the intercept on a straight line by a circle with radius r and the length of the perpendicular from the centre of the circle onto the line being p.

Solution: Let the circle intersect the line in A and B and M be the midpoint of AB. If C is the centre of the circle (see Fig. 3.13) then $p = CM$. Using Pythagoras theorem, we have

$$AB = 2AM = 2\sqrt{r^2 - p^2}$$

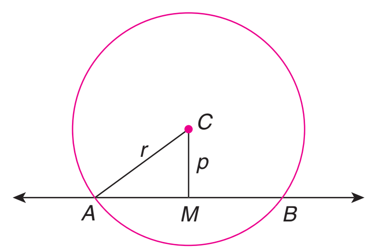


FIGURE 3.13

3. Two rods of lengths $2a$ and $2b$ slide along the coordinate axes such that their ends are always concyclic. Find the locus of the centre of the circle.

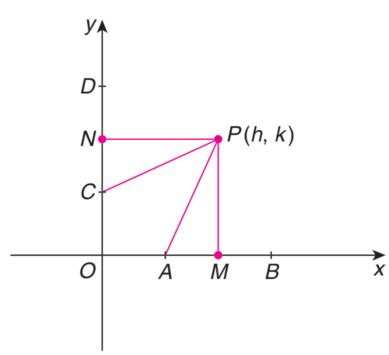


FIGURE 3.14

Solution: $P(h, k)$ is the centre of the circle passing through points A, B, C and D (Fig. 3.14) where $AB = 2a$ and $DC = 2b$. This implies and is implied by ($\Leftrightarrow$)

$$PA = PC = \text{radius of the circle}$$
$$\Leftrightarrow (PA)^2 = (PC)^2$$
$$\Leftrightarrow (AM)^2 + (PM)^2 = (PA)^2 = (PC)^2 = (PN)^2 + (CN)^2$$

where M and N are the midpoints of AB and CD, respectively. From the above, we have

$$a^2 + k^2 = h^2 + b^2$$

$$\Leftrightarrow k^2 - h^2 = b^2 - a^2$$

$$\Leftrightarrow \text{Locus of } (h,k) \text{ is } y^2 - x^2 = b^2 - a^2$$

4. Find the equation of the circle passing through the points $A(0, 1)$, $B(2, 3)$ and $C(-2, 5)$.

Solution: Let $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ be the circle passing through points A, B and C. Therefore

$$\begin{aligned} 2f + c &= -1 \\ 4g + 6f + c &= -13 \\ -4g + 10f + c &= -29 \end{aligned} \tag{3.5}$$

Solving the system of equations provided in Eq. (3.5), we get $g = 1/3$, $f = -10/3$ and $c = 17/3$ so that the equation of the circle is

$$x^2 + y^2 + \frac{2}{3}x - \frac{20}{3}y + \frac{17}{3} = 0$$

$$3x^2 + 3y^2 + 2x - 20y + 17 = 0$$

Note: Under the given hypothesis, to find the equation of the circle, it is sufficient if we find its centre and radius or assume the circle as $x^2 + y^2 + 2gx + 2fy + c = 0$ and find the values of g, f and c.

5. Find the equation of a circle with the centre at point $(6, 1)$ and touching the line $5x + 12y - 3 = 0$.

Solution: Let A be $(6, 1)$. Since the line $5x + 12y - 3 = 0$ touches the circle (say, at M as shown in Fig. 3.15), the distance of the line from the centre is equal to the radius. Therefore, the radius is given by

$$r = \left| \frac{5(6) + 12(1) - 3}{\sqrt{5^2 + 12^2}} \right| = \frac{39}{12} = 3$$

Therefore, the equation of the circle is $(x - 6)^2 + (y - 1)^2 = 9$. That is, $x^2 + y^2 - 12x - 2y + 28 = 0$.

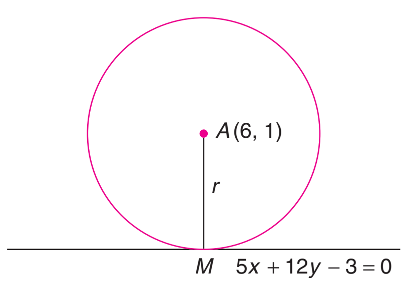


FIGURE 3.15

6. Find the equation of the circle passing through the points $P(-1, 2)$, $Q(3, -2)$ and whose centre lies on the line $x = 2y$.

Solution: Let $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ be the required circle. Since its centre $(-g, -f)$ lies on the line $x = 2y$, we have

$$g - 2f = 0 \tag{3.6}$$

The circle passes through the points $P(-1, 2)$ and $Q(3, -2)$. Hence

$$-2g + 4f + c = -5 \tag{3.7}$$

$$6g - 4f + c = -13 \tag{3.8}$$

Solving Eqs. (3.6), (3.7) and (3.8), we obtain $g = -2$, $f = -1$ and $c = -5$. Therefore, the equation of the circle is given by $S \equiv x^2 + y^2 - 4x - 2y - 5 = 0$.

7. The line $x = y$ is tangent at $(0, 0)$ to a circle of radius 1. Find the centre of the circle.

Solution: Let $C(x_1, y_1)$ be the centre of the circle (see Fig. 3.16). The line joining $C(x_1, y_1)$ and $O(0, 0)$ is perpendicular to the line $y = x$. Therefore

$$\frac{y_1}{x_1} = -1 \Rightarrow y_1 = -x_1$$

Hence

$$1 = x_1^2 + y_1^2 = 2x_1^2 \Rightarrow x_1 = \pm\frac{1}{\sqrt{2}}$$

and
$$y_1 = \pm\frac{1}{\sqrt{2}}$$

Therefore

$$(x_1, y_1) = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) \text{ or } \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

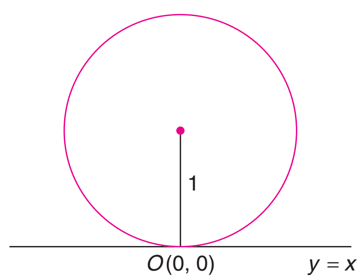


FIGURE 3.16

8. Determine the position of the point $(-1, -2)$ relative to the circle $S \equiv x^2 + y^2 + 4x + 6y + 9 = 0$.

Solution: We have

$$S_{11} = (-1)^2 + (-2)^2 + 4(-1) + 6(-2) + 9$$
$$= 14 - 16 = -2 < 0$$

Therefore, by Theorem 3.9, the point $(-1, -2)$ lies inside the circle.

9. From each point on the line $2x + y - 4 = 0$, a pair of tangents are drawn to the circle $x^2 + y^2 = 1$. Prove that the chords of contact pass through a fixed point.

Solution: Let $P(x_1, y_1)$ be a point on the line $2x + y - 4 = 0$. Therefore

$$2x_1 + y_1 - 4 = 0 \tag{3.9}$$

By Theorem 3.12, the equation of the chord of contact of (x_1, y_1) with respect to the circle

$$S \equiv x^2 + y^2 - 1 = 0$$

is given by

$$S_1 \equiv xx_1 + yy_1 - 1 = 0 \tag{3.10}$$

From Eq. (3.9), we get $y_1 = 4 - 2x_1$. Substituting the value of y_1 in Eq. (3.10), we have

$$(x - 2y)x_1 + (4y - 1) = 0$$

so that the line passes through the point of intersection of the lines $x - 2y = 0$ and $4y - 1 = 0$ which is given by

$$\left(\frac{1}{2}, \frac{1}{4}\right)$$

10. Find the equations of the tangent to the circle $x^2 + y^2 = 9$ which is perpendicular to the line $2x + 3y + 7 = 0$.

Solution: Any line perpendicular to the line $2x + 3y + 7 = 0$ is of the form $3x - 2y + c = 0$. This line touches the circle $x^2 + y^2 = 9$ if and only if the perpendicular drawn onto the line $3x - 2y + c = 0$ from the centre $(0, 0)$ is equal to the radius 3. That is,

$$\left|\frac{c}{\sqrt{3^2 + 2^2}}\right| = 3$$

Therefore

$$c = \pm 3\sqrt{13}$$

Hence, the equations of the required tangent are

$$3x - y + 3\sqrt{13} = 0$$

and $$3x - y - 3\sqrt{13} = 0$$

11. If the line $lx + my = 1$ touches the circle $x^2 + y^2 = a^2$, then show that the point (l, m) lies on the circle $x^2 + y^2 = 1/a^2$.

Solution: The line $lx + my = 1$ touches the circle $x^2 + y^2 = a^2$. This implies that the length of the perpendicular drawn from the centre $(0, 0)$ onto the line is equal to the radius a. Therefore,

$$\left|\frac{-1}{\sqrt{l^2 + m^2}}\right| = a$$

$$\Rightarrow l^2 + m^2 = \frac{1}{a^2}$$

Therefore, the point (l, m) lies on the circle $x^2 + y^2 = 1/a^2$.

12. Show that the line $3x - 4y - 1 = 0$ touches the circle $x^2 + y^2 - 2x + 4y + 1 = 0$ and find the coordinates of the point of contact.

Solution: The centre of the circle is $(1, -2)$ and its radius is $\sqrt{1^2 + 2^2 - 1} = 2$. The distance of the line from the centre $(1, -2)$ is given by

$$\left|\frac{3(1) - 4(-2) - 1}{\sqrt{3^2 + 4^2}}\right| = \frac{10}{5} = 2$$

which is equal to the radius of the circle. Therefore, the line touches the circle. Let (x_1, y_1) be the point of contact. That is, at (x_1, y_1), the line

$$3x - 4y - 1 = 0 \tag{3.11}$$

is the tangent. But, by Theorem 3.7, the equation of the tangent at (x_1, y_1) is

$$S_1 \equiv xx_1 + yy_1 - (x + x_1) + 2(y + y_1) + 1 = 0$$

That is,

$$S_1 \equiv (x_1 - 1)x + (y_1 + 2)y - x_1 + 2y_1 + 1 = 0 \tag{3.12}$$

Equations (3.11) and (3.12) represent the same straight line. Therefore,

$$\frac{x_1 - 1}{3} = \frac{y_1 + 2}{-4} = \frac{-x_1 + 2y_1 + 1}{-1} = t \text{ (say)}$$

Hence

$$x_1 = 3t + 1, \quad y_1 = -4t - 2$$

and $$-t = -x_1 + 2y_1 + 1 = (-3t - 1) - (8t + 4) + 1 = -11t - 4$$

$$\Rightarrow t = \frac{-2}{5}$$

Therefore,

$$x_1 = 3t + 1 = \frac{-6}{5} + 1 = -\frac{1}{5}$$

and $$y_1 = -4t - 2 = \frac{8}{5} - 2 = -\frac{2}{5}$$

Hence

$$(x_1, y_1) = \left(\frac{-1}{5}, \frac{-2}{5}\right)$$

13. Find the length of the chord of the circle $x^2 + y^2 - 10x - 20y - 44 = 0$ on the line $3x - 4y = 0$.

Solution: $C = (5, 10)$ is the centre and $r = 13$ is the radius of the given circle. Suppose the line $3x - 4y = 0$ cuts the circle at points A and B and M is the midpoint of AB (see Fig. 3.17). Therefore, CM is the perpendicular drawn to AB (see Theorem 3.5) which is given by

$$CM = \left|\frac{3(5) - 4(10)}{\sqrt{3^2 + 4^2}}\right| = 5$$

Therefore,

$$\begin{aligned} AB &= 2AM \\ &= 2\sqrt{CA^2 - CM^2} \\ &= 2\sqrt{13^2 - 5^2} = 2 \times 12 = 24 \end{aligned}$$

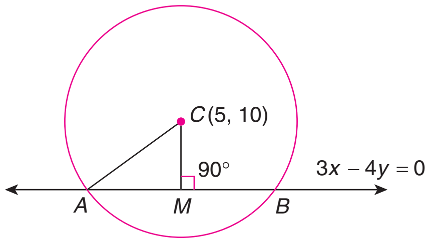


FIGURE 3.17

Aliter: Put $y = 3x/4$ in the equation of the circle so that we have

$$x^2 + \frac{9x^2}{16} - 10x - 20\left(\frac{3x}{4}\right) - 44 = 0$$

That is,

$$\begin{aligned} &25x^2 - 400x - 704 = 0 \\ \Rightarrow &(5x + 8)(5x - 88) = 0 \end{aligned}$$

Therefore,

$$x = \frac{88}{5}, \frac{-8}{5}$$

Hence, the points are $A = (88/5, 66/5)$ and $B = (-8/5, -6/5)$ and the length is given by

$$AB = \sqrt{\left(\frac{88}{5} + \frac{8}{5}\right)^2 + \left(\frac{66}{5} + \frac{6}{5}\right)^2} = \sqrt{576} = 24$$

14. Show that the equation of the locus of the foot of the perpendicular drawn from the origin upon any line passing through the point $(2, 3)$ is the circle $x^2 + y^2 - 2x - 3y = 0$.

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Solution: Let $M(h, k)$ be the foot of the perpendicular drawn from the origin $O(0, 0)$ to a line passing through the point $(2, 3)$ as shown in Fig. 3.18. Therefore, OM is the perpendicular drawn to the line so that

$$\begin{aligned} &(\text{Slope of } OM) \times (\text{Slope of the line}) = -1 \\ \Rightarrow &\left(\frac{k}{h}\right)\left(\frac{k-3}{h-2}\right) = -1 \\ \Rightarrow &h(h-2) + k(k-3) = 0 \end{aligned}$$

Therefore, the locus of the point (h, k) is $x^2 + y^2 - 2x - 3y = 0$ which is equivalent to the circle described on the line joining the points $(0, 0)$ and $(2, 3)$ as diameter.

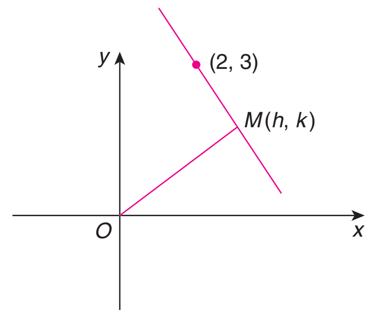

FIGURE 3.18

15. Find the equation of the circle passing through the origin which is cutting the chord of equal length $\sqrt{2}$ on the lines $y = x$ and $y = -x$.

Solution: Let $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ be the required circle. Since it passes through the origin $(0, 0)$, $c = 0$. Put $y = x$ in $S = 0$. Then $x^2 + (g + f)x = 0$. Therefore, $x = 0$, $x = -(g + f)$ so that the points of intersection are $A(0, 0)$ and $B[-(f + g), -(f + g)]$. Now

$$\begin{aligned} &AB = \sqrt{2} \\ \Rightarrow &f + g = \pm 1 \end{aligned}$$

Similarly,

$$g - f = \pm 1$$

Therefore, the centres of circles are given by $(1, 0)$, $(-1, 0)$, $(0, 1)$ and $(0, -1)$ and the equations of the circle are given by $x^2 + y^2 \pm 2x = 0$ and $x^2 + y^2 \pm 2y = 0$.

16. The angle between the pair of tangents from a point P to the circle $S \equiv x^2 + y^2 + 4x - 6y + 9 + 4\cos^2\alpha = 0$ is 2α. Show that the point P lies on the circle $x^2 + y^2 + 4x - 6y + 9 = 0$ and hence find the equation of the director circle of $S = 0$.

Solution: The centre of the circle $S = 0$ is $(-2, 3)$ and its radius is $2\sin\alpha$ (note that 2α being the angle between the tangents, we have $0 < \alpha < \pi/2$). Let $P = (h, k)$ as shown in Fig. 3.19. Then

$$\sin\alpha = \frac{CT}{CP} = \frac{2\sin\alpha}{\sqrt{(h+k)^2 + (k-3)^2}}$$

Therefore

$$(h+2)^2 + (k-3)^2 = 4$$

Hence, (h, k) lies on the circle

$$(x+2)^2 + (y-3)^2 = 4 \text{ or } x^2 + y^2 + 4x - 6y + 9 = 0$$

If $\alpha = \pi/4$, then $2\alpha = \pi/2$ and therefore the equation of the director circle of $S \equiv x^2 + y^2 + 4x - 6y + 11 = 0$ is given by $x^2 + y^2 + 4x - 6y + 9 = 0$.

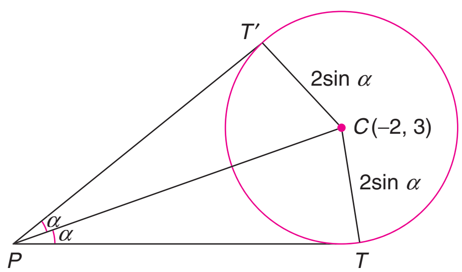

FIGURE 3.19

17. Prove that the locus of the point whose chord of contact with respect to a circle subtends a right angle at the centre of the circle is its director circle.

Solution: P is a point and AB its chord of contact with respect to a circle with centre C (see Fig. 3.20) such that $\underline{|BCA} = 90°$. Since CA and CB are at right angles to the tangents PA and PB, respectively. It follows that $\underline{|APB}$ is a right-angled triangle. Hence, point P lies on the director circle of the given circle.

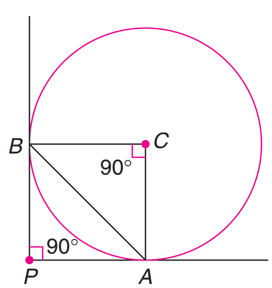

FIGURE 3.20

18. Point P is on the circle $x^2 + y^2 - 2ax = 0$. A circle is drawn on OP as diameter where O is the origin. As P moves on the circle, find the locus of the centre of the circle.

Solution: Let $S \equiv x^2 + y^2 - 2ax = 0$ and $P = (h, k)$ be a point on $S = 0$. Therefore

$$h^2 + k^2 - 2ah = 0 \tag{3.13}$$

Now, $Q\,(h/2, k/2)$ is the centre of the circle drawn on OP as the diameter. From Eq. (3.13), we have

$$\left(\frac{h}{2}\right)^2 + \left(\frac{k}{2}\right)^2 - a\left(\frac{h}{2}\right) = 0$$

Therefore, the locus of Q is $x^2 + y^2 - ax = 0$.

19. Find the point for which the line $9x + y - 28 = 0$ is the chord of contact with respect to the circle $2x^2 + 2y^2 - 3x + 5y - 7 = 0$.

Solution: The given circle is

$$S \equiv x^2 + y^2 - \frac{3}{2}x + \frac{5}{2}y - \frac{7}{2} = 0$$

Let $P(x_1, y_1)$ be the point whose chord of contact with respect to $S = 0$ is

$$9x + y - 28 = 0 \tag{3.14}$$

But, in fact, the chord of contact of $P(x_1, y_1)$ with respect to $S = 0$ is given by

$$xx_1 + yy_1 - \frac{3}{4}(x + x_1) + \frac{5}{4}(y + y_1) - \frac{7}{2} = 0$$

That is,

$$(4x_1 - 3)x + (4y_1 + 5)y - 3x_1 + 5y_1 - 14 = 0 \tag{3.15}$$

Equations (3.14) and (3.15) represent the same line. Therefore,

$$\frac{4x_1 - 3}{9} = \frac{4y_1 + 5}{1} = \frac{-3x_1 + 5y_1 - 14}{-28} = t \quad \text{(say)}$$

Hence

$$x_1 = \frac{3 + 9t}{4}$$

and

$$y_1 = \frac{t - 5}{4}$$

so that

$$-28t = -3\left(\frac{3 + 9t}{4}\right) + 5\left(\frac{t - 5}{4}\right) - 14$$

$$\Rightarrow -112t = -9 - 27t + 5t - 25 - 56$$

$$\Rightarrow -90t = -90$$

$$\Rightarrow t = 1$$

Therefore

$$x_1 = \frac{3+9t}{4} = \frac{12}{4} = 3$$

and $$y_1 = \frac{t-5}{4} = \frac{-4}{4} = -1$$

Hence, $P = (3, -1)$.

20. Show that the circle $x^2 + y^2 + 4x - 4y + 4 = 0$ touches both axes and find the points of contact.

Solution: The centre is $(-2, 2)$ and the radius is 2. The distance of the centre $(-2, 2)$ from the coordinate axes is equal to 2. Hence, the circle touches both coordinate axes, and the points of contact are given by $(-2, 0)$ and $(0, 2)$.

21. Find the equations of the circle touching both axes and passing through the point $(2, 1)$.

Solution: Since the circle touches both axes and passes through the point $(2, 1)$, the centre of the circle must be in the first quadrant. Hence, its equation should be of the form $x^2 + y^2 - 2ax - 2ay + a^2 = 0$. It passes through the point $(2, 1)$ which implies that

$$5 - 4a - 2a + a^2 = 0$$
$$\Rightarrow (a-1)(a-5) = 0$$
$$\Rightarrow a = 1 \text{ or } 5$$

The equations of the circle are $x^2 + y^2 - 2x - 2y + 1 = 0$ and $x^2 + y^2 - 10x - 10y + 25 = 0$.

22. The chords of the circle $x^2 + y^2 + 2gx + 2fy + 2c = 0$ subtend right angle at the origin. Show that the locus of the foot of the perpendiculars from the origin to these chords is the circle $x^2 + y^2 + gx + fy + c = 0$.

Solution: Let AB be a chord of the circle $S \equiv x^2 + y^2 + 2gx + 2fy + 2c = 0$ subtending right angle at the origin ($\angle AOB = 90°$) and $N(h, k)$ be the foot of the perpendicular drawn from the origin O to the chord AB (see Fig. 3.21). Since the slope of ON is k/h, the equation of the chord AB is

$$y - k = -\frac{h}{k}(x - h)$$
$$\Rightarrow hx + ky = h^2 + k^2$$

Therefore, the combined equation of the pair of lines $\overleftrightarrow{OA}$ and $\overleftrightarrow{OB}$ is given by

$$x^2 + y^2 + (2gx + 2fy)\left(\frac{hx+ky}{h^2+k^2}\right) + 2c\left(\frac{hx+ky}{h^2+k^2}\right)^2 = 0$$

(by Theorem 2.33)

Since $\angle AOB = 90°$, from the above equation,

Coefficient of x^2 + Coefficient of $y^2 = 0$

$$\Rightarrow 1 + \frac{2gh}{(h^2+k^2)} + \frac{2ch^2}{(h^2+k^2)^2} + 1 + \frac{2fk}{(h^2+k^2)} + \frac{2ck^2}{(h^2+k^2)^2} = 0$$

$$\Rightarrow 2(h^2+k^2)^2 + 2(gh+fk)(h^2+k^2) + 2c(h^2+k^2) = 0$$

$$\Rightarrow h^2 + k^2 + gh + kf + c = 0$$

Hence, the locus of (h, k) is given by

$$x^2 + y^2 + gx + fy + c = 0$$

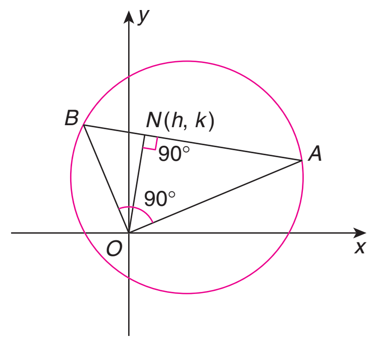

FIGURE 3.21

23. Find the equation of the circle which touches the x-axis at $(a, 0)$ and cuts of chord of length l on the positive y-axis. Determine the equation of the circle when $a = 12$ and $l = 10$.

Solution: Let $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$. Since the circle touches x-axis (see Fig. 3.22), we have

$$g^2 = c \quad \text{(Quick Look 6)}$$

Therefore

$$a^2 = c \quad \text{(Problem 1)}$$

Also l is the y-intercept. This implies that

$$2\sqrt{f^2 - c} = l \quad \text{(Problem 1)}$$

$$\Rightarrow f^2 - c = \frac{l^2}{4}$$

$$\Rightarrow f^2 = \frac{l^2}{4} + c = \frac{l^2}{4} + a^2$$

$$\Rightarrow f = \pm\sqrt{\frac{l^2 + 4a^2}{2}}$$

Since the intercept is on the positive y-axis, we have

$$f = \pm\sqrt{\frac{l^2 + 4a^2}{2}}$$

Hence, the required equation of the circle is

$$x^2 + y^2 - 2ax + (\sqrt{l^2 + 4a^2})y + a^2 = 0$$

$$\Rightarrow (x-a)^2 + \left(y + \frac{\sqrt{l^2+4a^2}}{2}\right)^2 = \frac{l^2+4a^2}{4}$$

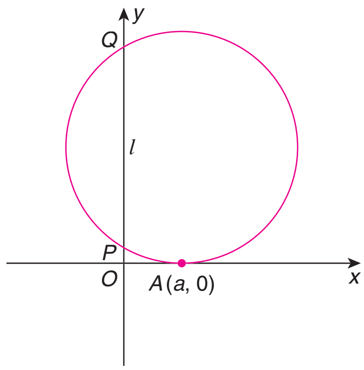

FIGURE 3.22

24. A point moves such that the length of the tangent from it to the circle $x^2 + y^2 + 4x - 5y + 6 = 0$ is double the length of the tangent to the circle $x^2 + y^2 = 4$. Show that the locus is a circle. Find its centre and radius.

Solution: Let $S \equiv x^2 + y^2 + 4x - 5y + 6 = 0$ and $S' \equiv x^2 + y^2 - 4 = 0$. Let $P(x, y)$ be a point from which the tangent to $S = 0$ is double the tangent to $S' = 0$. By hypothesis,

$$\sqrt{S_{11}} = 2\sqrt{S'_{11}} \quad \text{(see Theorem 3.11)}$$

Therefore

$$S_{11} = 4S'_{11}$$

$$\Rightarrow x_1^2 + y_1^2 + 4x_1 - 5y_1 + 6 = 4(x_1^2 + y_1^2 - 4)$$

$$\Rightarrow 3x_1^2 + 3y_1^2 - 4x_1 + 5y_1 - 22 = 0$$

Hence, the locus is a circle whose equation is given by $3x^2 + 3y^2 - 4x + 5y - 22 = 0$.

25. If A and B are two fixed points and P is a variable point such that $PA:PB = n:1$, then show that the locus of P is a circle if $n \neq 1$.

Solution: Without loss of generality, we take $A = (a, 0)$ and $B(-a, 0)$. Let P be (x_1, y_1). So

$$PA = nPB$$

$$\Leftrightarrow (PA)^2 = n^2(PB)^2$$

$$\Leftrightarrow (x_1 - a)^2 + y_1^2 = n^2[(x_1+a)^2 + y_1^2]$$

$$\Leftrightarrow (n^2-1)(x_1^2+y_1^2) + 2a(n^2+1)x_1 + (n^2-1)a^2 = 0$$

Since $n \neq 1$, the locus is the circle $(n^2 - 1)(x^2 + y^2) + 2a(n^2 + 1)x + (n^2 - 1)a^2 = 0$.

Note: In the above problem, if $n = 1$, then the locus of P is the perpendicular bisector of the segment $\overline{AB}$.

26. Prove that from a point (a, b) of the circle $x(x - a) + y(y - b) = 0$, two chords, each bisected by the x-axis, can be drawn if and only if $a^2 > 8b^2$.

Solution: If $P = (a, b)$, then the equation $x(x - a) + y(y - b) = 0$ represents the circle with O and P as extremities of a diameter (see Fig. 3.23). Let $M(h, 0)$ be the midpoint of a chord of the circle. Equation of this chord is

$$xh + y(0) - \frac{a}{2}(x+h) - \frac{b}{2}(y+0) = h^2 - ah$$

(see Theorem 3.13)

That is,

$$\left(h - \frac{a}{2}\right)x - \frac{b}{2}y = h^2 - \frac{ah}{2}$$

This chord is passing through (a, b). This implies

$$\left(h - \frac{a}{2}\right)a - \frac{b^2}{2} = h^2 - \frac{ah}{2}$$

$$\Leftrightarrow 2ah - a^2 - b^2 = 2h^2 - ah$$

$$\Leftrightarrow 2h^2 - 3ah + a^2 + b^2 = 0 \quad \text{(which has two distinct roots)}$$

$$\Leftrightarrow 9a^2 > 8(a^2 + b^2)$$

$$\Leftrightarrow a^2 > 8b^2$$

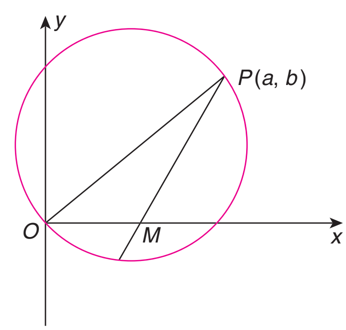

FIGURE 3.23

27. Find the points on the line $x - y + 1 = 0$, the tangents from which to the circle $x^2 + y^2 - 3x = 0$ are of length 2.

Solution: Let $P(x_1, y_1)$ be a point on the line $x - y + 1 = 0$ from which the length of the tangents to the circle $S \equiv x^2 + y^2 - 3x = 0$ is of length 2. Therefore

$$x_1 - y_1 + 1 = 0 \quad \text{and} \quad \sqrt{S_{11}} = 2$$

$$x_1 - y_1 + 1 = 0 \quad \text{and} \quad x_1^2 + y_1^2 - 3x_1 = 4$$

Therefore

$$x_1^2 + (x_1+1)^2 - 3x_1 = 4$$
$$\Rightarrow 2x_1^2 - x_1 - 3 = 0$$
$$\Rightarrow 2x_1^2 + 2x_1 - 3x_1 - 3 = 0$$
$$\Rightarrow (2x_1 - 3)(x_1 + 1) = 0$$

Hence

$$x_1 = -1, \frac{3}{2}$$

x_1	$y_1 = x_1 + 1$
-1	0
$\frac{3}{2}$	$\frac{5}{2}$

Therefore, the points on the line are given by $(-1, 0)$ and $(3/2, 5/2)$.

28. Show that the area of the triangle formed by the two tangents from (x_1, y_1) to the circle $x^2 + y^2 = a^2$ and their chord of contact is

$$\frac{a(x_1^2 + y_1^2 - a^2)^{3/2}}{x_1^2 + y_1^2}$$

Solution: Let $P = (x_1, y_1)$, $O = (0, 0)$ and AB be the chord of contact (see Fig. 3.24). Suppose OP meets the chord AB at point M so that $AM = MD$ and OM is perpendicular to AB. The equation of chord AB is given by

$$S_1 \equiv xx_1 + yy_1 - a^2 = 0 \quad \text{(see Theorem 3.12)}$$

Therefore

$$OM = \left| \frac{-a^2}{\sqrt{x_1^2 + y_1^2}} \right|$$

and

$$AM = \sqrt{OA^2 - OM^2} = \sqrt{a^2 - \frac{a^4}{x_1^2 + y_1^2}} = a\frac{\sqrt{x_1^2 + y_1^2 - a^2}}{\sqrt{x_1^2 + y_1^2}}$$

Also

$$PM = \frac{|x_1^2 + y_1^2 - a^2|}{\sqrt{x_1^2 + y_1^2}}$$
$$= \frac{x_1^2 + y_1^2 - a^2}{\sqrt{x_1^2 + y_1^2}} \quad (\because P \text{ lies outside the circle})$$

Now, the area of ΔPAB is given by

$$\frac{1}{2} AB \cdot PM = AM \cdot PM = \frac{a\sqrt{x_1^2 + y_1^2 - a^2}}{\sqrt{x_1^2 + y_1^2}} \times \frac{(x_1^2 + y_1^2 - a^2)}{\sqrt{x_1^2 + y_1^2}}$$
$$= \frac{a(x_1^2 + y_1^2 - a^2)^{3/2}}{x_1^2 + y_1^2} = \frac{aS_{11}^{3/2}}{(OP)^2}$$

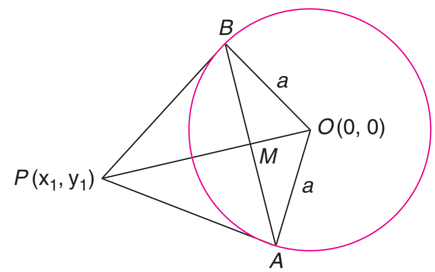

FIGURE 3.24

29. Tangents are drawn to the circles $x^2 + y^2 = a^2$ and $x^2 + y^2 = b^2$ at right angles to one another. Find the locus of their point of intersection.

Solution: Both circles are concentric at the origin (see Fig. 3.25). From the Note given under Theorem 3.8, the lines

$$y = mx + a\sqrt{1+m^2}$$
$$y = \frac{-1}{m}x + b\sqrt{a + \frac{1}{m^2}}$$

are tangents to the given circles and are at right angles. Therefore,

$$(y - mx)^2 + (x + my)^2 = (a^2 + b^2)(1 + m^2)$$
$$\Rightarrow (x^2 + y^2)(1 + m^2) = (a^2 + b^2)(1 + m^2)$$

That is, $x^2 + y^2 = a^2 + b^2$ is the required locus.

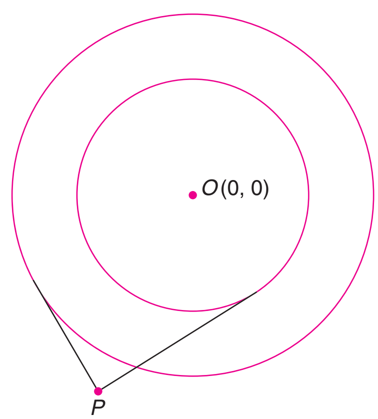

FIGURE 3.25

Note: In the problem, if $b = a$, then the equation of the *director circle* of $x^2 + y^2 = a^2$ is $x^2 + y^2 = 2a^2$.

30. Find the equations of the tangents to the circle $x^2 + y^2 - 6x - 4y + 5 = 0$ which make an angle of 45° with positive direction of the x-axis.

Solution: In the circle $S \equiv x^2 + y^2 - 6x - 4y + 5 = 0$, the centre is (3, 2) and radius is $2\sqrt{2}$. The given line is

$$y = x + c \ (\because \tan 45° = 1)$$

The line touches the given circle if and only if the distance of the line from the centre is equal to the radius. Therefore

$$\frac{|3-2+c|}{\sqrt{1^2+1^2}} = 2\sqrt{2}$$
$$\Rightarrow (c+1)^2 = 16$$
$$\Rightarrow c+1 = \pm 4$$
$$\Rightarrow c = 3 \text{ or } -5$$

Hence, the required equations of the tangents are $x - y + 3 = 0$ and $x - y - 5 = 0$.

31. Prove that the tangent to the circle $x^2 + y^2 = 5$ at the point (1, –2) also touches the circle $x^2 + y^2 - 8x - 6y - 20 = 0$ and find the coordinates of the point of contact.

Solution: Tangent to the circle $S \equiv x^2 + y^2 - 5 = 0$ at (–1, –2) is given by

$$S_1 \equiv x(-1) + y(-2) - 5 = 0$$
$$S_1 \equiv -x - 2y - 5 = 0$$
$$S_1 \equiv x + 2y + 5 = 0 \quad (3.16)$$

Now, this line [Eq. (3.16)] also touches the circle $x^2 + y^2 - 8x - 6y - 20 = 0$. This implies that the distance of its centre (4, 3) from the line [Eq. (3.16)] is equal to the radius $3\sqrt{5}$. So

$$\frac{|4+6+5|}{\sqrt{1^2+2^2}} = 3\sqrt{5}$$

Hence, the tangent at (–1, –2) to $x^2 + y^2 = 5$ also touches the circle $x^2 + y^2 - 8x - 6y - 20 = 0$ at say (x_1, y_1). But, the tangent at (x_1, y_1) is given by

$$xx_1 + yy_1 - 4(x + x_1) - 3(y + y_1) - 20 = 0$$
$$\Rightarrow (x_1 - 4)x + (y_1 - 3)y - (4x_1 + 3y_1 + 20) = 0$$
$$\Rightarrow \frac{x_1 - 4}{1} = \frac{y_1 - 3}{2} = \frac{-(4x_1 + 3y_1 + 20)}{5} = t \ \text{(say)}$$
$$\Rightarrow x_1 = t + 4,\ y_1 = 2t + 3$$
$$\text{and } 5t = -(4x_1 + 3y_1 + 20) = -[4(t+4) + 3(2t+3) + 20]$$
$$\Rightarrow 5t = -10t - 45$$
$$\Rightarrow t = -3$$

Therefore,

$$(x_1, y_1) = (-3 + 4, 2(-3) + 3) = (1, -3)$$

Note: When a line touches a circle, then the point of contact is the foot of the perpendicular drawn from the centre of the circle onto the line.

32. Find the equations of the tangents to the circle $S \equiv x^2 + y^2 + 8x + 4y - 5 = 0$ from the point (3, –3).

Solution: The centre and the radius of the circle $S = 0$ are (–4, –2) and 5, respectively. We have $S_{11} = 3^2 + (-3)^2 + 8(3) + 4(-3) - 5 = 25 > 0$. Hence, (3, –3) is external to $S = 0$. Let $y + 3 = m(x - 3)$ be a line passing through (3, –3). This line touches the circle $S = 0$. This implies that

$$\frac{|m(-4-3)+2-3|}{\sqrt{m^2+1}} = 5$$
$$\Rightarrow (7m-1)^2 = 25(m^2+1)$$
$$\Rightarrow 24m^2 - 14m - 24 = 0$$
$$\Rightarrow 12m^2 - 7m - 12 = 0$$
$$\Rightarrow 12m^2 - 16m + 9m - 12 = 0$$
$$\Rightarrow 4m(3m-4) + 3(3m-4) = 0$$
$$\Rightarrow m = \frac{4}{3}, \frac{-3}{4}$$

Therefore, the tangents from (3, –3) to the circle $S = 0$ are

$$y + 3 = \frac{4}{3}(x-3)$$

and

$$y + 3 = -\frac{3}{4}(x-3)$$

That is,

$$4x - 3y - 21 = 0$$
$$3x + 4y + 3 = 0$$

33. A circle passes through the points (–1, 1), (0, 6) and (5, 5). On this circle, find the points at which the tangents are parallel to the line joining origin to the centre of the circle.

Solution: Let $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ be the circle passing through the points (–1, 1), (0, 6) and (5, 5). Therefore,

$$\left.\begin{aligned} -2g + 2f + c &= -2 \\ 12f + c &= -36 \\ 10g + 10f + c &= -50 \end{aligned}\right\} \quad (3.17)$$

Solving the set of equations given in Eq. (3.17), we obtain $g = -2$, $f = -3$ and $c = 0$. Therefore, $S \equiv x^2 + y^2 - 4x - 6y = 0$ is the required circle. Let $P(x_1, y_1)$ be a point on the circle $S = 0$ at which the tangent to the circle is parallel to the line joining origin to the centre (2, 3). Therefore, the slope of the tangent $S_1 \equiv xx_1 + yy_1 - 2(x + x_1) - 3(y + y_1) = 0$ is equal to 3/2 and the slope of the line $S_1 \equiv (x_1 - 2)x + (y_1 - 3)y - 2x_1 - 3y_1 = 0$ is 3/2. Hence

$$-\left(\frac{x_1-2}{y_1-3}\right)=\frac{3}{2}$$

$$\Rightarrow \frac{2-x_1}{3}=\frac{y_1-3}{2}=t \quad \text{(say)}$$

Therefore, $x_1 = 2 - 3t$ and $y_1 = 3 + 2t$. Sine (x_1, y_1) lies on the circle $S = 0$, we have

$$(2 - 3t)^2 + (3 + 2t)^2 - 4(2 - 3t) - 6(3 + 2t) = 0$$

Therefore,

$$13t^2 - 13 = 0 \Rightarrow t = \pm 1$$

Hence, the point $(x_1, y_1) = (-1, 5), (5, 1)$.

34. Find the locus of the midpoints of the portions of the tangents to the circle $x^2 + y^2 = a^2$ terminated by the coordinate axes.

Solution: Tangent at $P(x_1, y_1)$ to the circle $S \equiv x^2 + y^2 - a^2 = 0$ is $S_1 \equiv xx_1 + yy_1 - a^2 = 0$. Therefore, the intercepts of the tangent $S_1 = 0$ on the axes are $A(a^2/x_1, 0)$ and $B(0, a^2/y_1)$. $M(h, k)$ is the midpoint of $\overline{AB}$. This means there

$$h=\frac{a^2}{2x_1}, k=\frac{a^2}{2y_1}$$

$$\Leftrightarrow x_1=\frac{a^2}{2h}, y_1=\frac{a^2}{2k}$$

$$\Leftrightarrow a^2 = x_1^2+y_1^2=\frac{a^4}{4}\left(\frac{1}{h^2}+\frac{1}{k^2}\right)$$

$$\Leftrightarrow \frac{4}{a^2}=\frac{1}{h^2}+\frac{1}{k^2}$$

Therefore, the locus of $M(h, k)$ is given by

$$\frac{1}{x^2}+\frac{1}{y^2}=\frac{4}{a^2}$$

35. $ABCD$ is a square of side $2a$ units. Taking AB and AD as coordinate axes, find the equation of the circle which touches all the four sides of the square. If E is a point on DC such that $3DE = DC$ and F is a point on BA produced such that $FA = AB$, prove that EF touches the circle and also find the coordinates of the point of contact.

Solution: By hypothesis, A is $(0, 0)$, $B = (2a, 0)$, $C = (2a, 2a)$ and $D = (0, 2a)$. Therefore, $O = (a, a)$ is the centre of the circle so that a is its radius (see Fig. 3.26). Hence, the equation of the circle is $(x - a)^2 + (y - a)^2 = a^2$. Further, E divides the segment DC in the ratio 1:2, that is, DE:EC = 1:2. Therefore,

$$E=\left(\frac{2a}{3}, 2a\right)$$

Also

$$F = (-2a, 0)$$

Now, the equation of EF is

$$y=\frac{3}{4}(x+2a)$$

$$\Rightarrow 3x-4y+6a=0$$

Therefore, the distance of the centre (a, a) from the line EF is equal to

$$\frac{|3a-4a+6a|}{\sqrt{3^2+4^2}}=a$$

Hence, the line EF touches the circle. In Chapter 2, we discussed the following result: If (h, k) is the foot of the perpendicular from (x_1, y_1) onto a line $lx + my + n = 0$, then

$$\frac{h-x_1}{l}=\frac{k-y_1}{m}=-\frac{(lx_1+my_1+n)}{l^2+m^2}$$

[see Theorem 2.13(1)]

Now, equation for the line EF is $3x - 4y + 6a = 0$ and $(x_1, y_1) = (a, a)$. Therefore,

$$\frac{h-a}{3}=\frac{k-a}{-4}=-\left(\frac{3-4a+6a}{3^2+4^2}\right)=\frac{-a}{5}$$

This gives

$$h=a-\frac{3a}{5}=\frac{2a}{5} \quad \text{and} \quad k=a+\frac{4a}{5}=\frac{9a}{5}$$

Therefore, the point of contact is given by

$$\left(\frac{2a}{5}, \frac{9a}{5}\right)$$

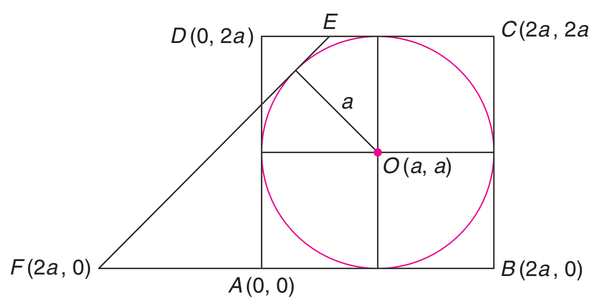

FIGURE 3.26

36. Find the point of intersection of the tangents to the circle passing through the points (4, 7), (5, 6) and (1, 8) at the point where it is cut by the line $5x + y + 17 = 0$.

Solution: Let $A = (4, 7)$, $B = (5, 6)$ and $C = (1, 8)$. Let $S(x_1, y_1)$ be the circumcentre of ΔABC. Therefore, $SA = SB = SC = R$ (circumradius). Hence

$$SA = SB \Rightarrow (x_1-4)^2 + (y_1-7)^2 = (x_1-5)^2 + (y_1-6)^2$$
$$\Rightarrow 2x_1 - 2y_1 = -4$$
$$\Rightarrow x_1 - y_1 = -2 \qquad (3.18)$$

$$SB = SC \Rightarrow (x_1-5)^2 + (y_1-6)^2 = (x_1-1)^2 + (y_1-8)^2$$
$$\Rightarrow 8x_1 - 4y_1 = -4$$
$$\Rightarrow 2x_1 - y_1 = -1 \qquad (3.19)$$

Solving Eqs. (3.18) and (3.19), we have $x_1 = 1$ and $y_1 = 3$. Therefore, the centre of the circle is (1, 3) and the radius of the circle is $R = SA = 5$. Therefore, the equation of the circle is $(x-1)^2 + (y-3)^2 = 5^2$. That is,

$$x^2 + y^2 - 2x - 6y - 15 = 0$$

Let us consider that the line

$$5x + y + 17 = 0 \qquad (3.20)$$

meets the circle at points P and Q and the tangents at points P and Q meet at $T(x_1, y_1)$ as shown in Fig. 3.27. Then, Eq. (3.20) is the equation of the chord of contact of $T(x_1, y_1)$. However, the chord of contact of $T(x_1, y_1)$ is given by

$$S_1 \equiv xx_1 + yy_1 - (x + x_1) - 3(y + y_1) - 15 = 0$$
$$\Rightarrow S_1 \equiv (x_1 - 1)x + (y_1 - 3)y - (x_1 + 3y_1 + 15) = 0 \qquad (3.21)$$

That is, Eqs. (3.20) and (3.21) represent the same line. Therefore, by Theorem 2.10, we have

$$\frac{x_1 - 1}{5} = \frac{y_1 - 3}{1} = -\left(\frac{x_1 + 3y_1 + 15}{17}\right) = t \quad \text{(say)}$$

This gives

$$x_1 = 1 + 5t;\ y_1 = 3 + t$$

and
$$17t = (1 + 5t) + 3(3 + t) + 15$$

Solving, we get

$$-25t = 25 \Rightarrow t = -1$$

Therefore, $(x_1, y_1) = (-4, 2)$.

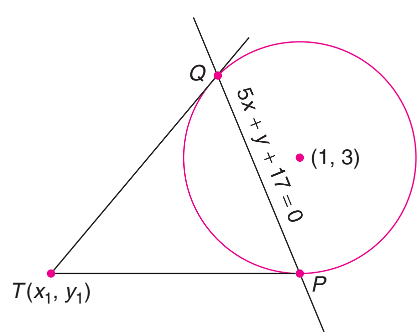

FIGURE 3.27

37. If the tangents drawn from the origin to the circle $x^2 + y^2 - 2px - 2qy + q^2 = 0$ $(q \neq 0)$ are at right angles, then show that $p^2 = q^2$.

Solution: We know that y-axis touches the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ if $f^2 = c$. For the circle mentioned in this problem also, we have $q^2 = f^2 = c = q^2$. Therefore, y-axis touches the circle. Since the tangents drawn from the origin are at right angles, the other tangent touches the x-axis. Therefore,

$$p^2 = q^2 \quad \text{(see Quick Look 6 and Problem 1)}$$

Aliter: Since the tangents drawn from (0, 0) are at right angles, origin must lie on the director circle of the given circle (see Definition 3.2 and Quick Look 4). Therefore, (0, 0) lies on the circle $(x-p)^2 + (y-q)^2 = 2p^2$.

38. If the points

$$\left(m_i, \frac{1}{m_i}\right) \qquad (i = 1, 2, 3, 4)$$

are concyclic, then show that $m_1 m_2 m_3 m_4 = 1$.

(IIT-JEE 1989)

Solution: Let $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ be a circle passing through the four points

$$\left(m_i, \frac{1}{m_i}\right)$$

Therefore,

$$m_i^2 + \frac{1}{m_i^2} + 2g(m_i) + \frac{2f}{m_i} + c = 0$$

Hence, m_i, where $i = 1, 2, 3$ and 4, are the roots of the equation

$$m^4 + 2gm^3 + cm^2 + 2fm + 1 = 0$$

Therefore, the product of the roots is 1. That is, $m_1 m_2 m_3 m_4 = 1$.

39. Find the values of λ for which the point $(\lambda - 1, \lambda + 1)$ lies in the larger segment of the circle $x^2 + y^2 - x - y - 6 = 0$ made by the chord whose equation is $x + y - 2 = 0$.

(IIT-JEE 1989)

Solution: Let $S \equiv x^2 + y^2 - x - y - 6 = 0$. Let C be the centre of the circle which is given by (1/2, 1/2) and its radius is given by

$$\sqrt{\frac{1}{4} + \frac{1}{4} + 6} = \sqrt{\frac{13}{2}}$$

Draw CM perpendicular to the line $x + y - 2 = 0$ (see Fig. 3.28). Therefore,

$$CM = \frac{|(1/2) + (1/2) - 2|}{\sqrt{1^2 + 1^2}} = \frac{1}{\sqrt{2}} < \sqrt{\frac{13}{2}} \quad \text{(radius)}$$

Therefore, $x + y - 2 = 0$ intersects the circle $S = 0$ in two distinct points, say, points A and B. Now,

$P(\lambda - 1, \lambda + 1)$ lies inside the circle $S = 0$

$$\Leftrightarrow (\lambda - 1)^2 + (\lambda + 1)^2 - (\lambda - 1) - (\lambda + 1) - 6 < 0$$
$$\Leftrightarrow 2\lambda^2 - 2\lambda - 4 < 0$$
$$\Leftrightarrow (\lambda + 1)(\lambda - 2) < 0$$
$$\Leftrightarrow -1 < \lambda < 2 \tag{3.22}$$

Let $L = x + y - 2 = 0$. Now, $P(\lambda - 1, \lambda + 1)$ lies inside the circle, in the larger segment of the circle if $-1 < \lambda < 2$ and P and C lie on the same side of the chord AB. $P(\lambda - 1, \lambda + 1)$ and $C(1/2, 1/2)$ lie on the same side of $L = 0$, if $L_{11} = (\lambda - 1) + (\lambda + 1) - 2$ and $L_{22} = (1/2) + (1/2) - 2$ have the same sign. But $L_{22} = -1 < 0$. Therefore,

$$L_{11} = (\lambda - 1) + (\lambda + 1) - 2 < 0$$
$$\Rightarrow 2\lambda - 2 < 0$$
$$\Rightarrow \lambda < 1 \tag{3.23}$$

From Eqs. (3.22) and (3.23), we have $-1 < \lambda < 1$.

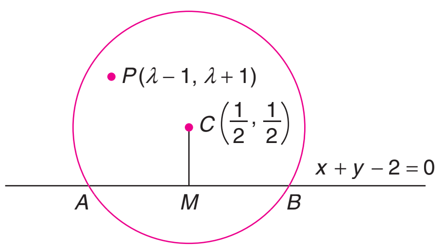

FIGURE 3.28

40. Find the range of λ for which the line $3x + 4y - \lambda = 0$ lies in between the circles $S \equiv x^2 + y^2 - 2x - 2y + 1 = 0$ and $S' \equiv x^2 + y^2 - 18x - 2y + 78 = 0$ without intersecting and without touching the circles.

Solution: Let $L \equiv 3x + 4y - \lambda = 0$.

$$S \equiv x^2 + y - 2x - 2y + 1 = 0$$
$$S' \equiv x^2 + y^2 - 18x - 2y + 78 = 0$$

$C_1 = (1, 1)$ and $C_2 = (9, 1)$ are the centres and $r_1 = 1$ and $r_2 = 2$ are the radii of $S = 0$ and $S' = 0$, respectively. First, the line $L = 0$ lies in between the circles. The centres C_1 and C_2 lie on the opposite sides of $L = 0$ (see Fig. 3.29). Now

$$L_{11} = 3 + 4 - \lambda$$

and

$$L_{22} = 3(9) + 4(1) - \lambda$$

have opposite signs

$$\Leftrightarrow (7 - \lambda)(31 - \lambda) < 0$$
$$\Leftrightarrow (\lambda - 7)(\lambda - 31) < 0$$
$$\Leftrightarrow 7 < \lambda < 31 \tag{3.24}$$

Second, $L = 0$ does not have common point with $S = 0 \Leftrightarrow C_1M_1 > r_1 = 1$, where C_1M_1 is the perpendicular drawn from C_1 onto the line. This implies

$$\frac{|3(1) + 4(1) - \lambda|}{\sqrt{3^2 + 4^2}} > 1$$
$$\Leftrightarrow (\lambda - 7)^2 > 25$$
$$\Leftrightarrow \lambda^2 - 14\lambda + 24 > 0$$
$$\Leftrightarrow (\lambda - 2)(\lambda - 12) > 0$$
$$\Leftrightarrow \lambda < 2 \text{ or } \lambda > 12 \tag{3.25}$$

Similarly

$$C_2M_2 > 2 \Leftrightarrow \lambda^2 - 62\lambda + 861 > 0$$
$$\Leftrightarrow (\lambda - 21)(\lambda - 41) > 0$$
$$\Leftrightarrow \lambda < 21 \text{ or } \lambda > 41 \tag{3.26}$$

The given line $L = 0$ satisfies the specified condition if all the three conditions given in Eqs. (3.24)–(3.26) simultaneously hold. Therefore, $12 < \lambda < 21$.

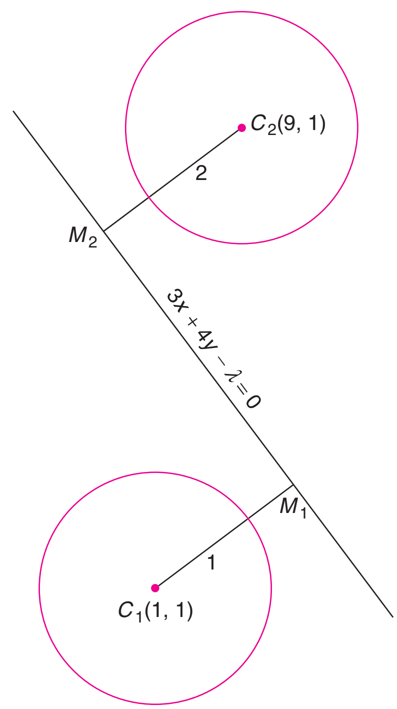

FIGURE 3.29

41. $ABCD$ is a rectangle. A circle passing through the vertex C touches the sides AB and AD at M and N, respectively. If the distance of the line MN from the vertex C is p units, then show that the area of the rectangle $ABCD$ is p^2.

Solution: Take A as origin. Let $AB = a$ and $AD = b$ so that $C = (a, b)$. Let r be the radius of the circle so that its centre $O = (r, r)$ (see Fig. 3.29). Equation of the chord

MN is $x/r + y/r = 1$ or $x + y = r$. Now, $C(a, b)$ is a point on the circle $(x - r)^2 + (y - r)^2 = r^2$, which implies

$$a^2 + b^2 - 2r(a+b) + r^2 = 0 \qquad (3.27)$$

The distance of $C(a, b)$ from the chord MN is given by

$$\frac{|a+b-r|}{\sqrt{2}} = p$$
$$\Rightarrow (a+b-r)^2 = 2p^2$$
$$\Rightarrow (a+b)^2 - 2r(a+b) + r^2 = 2p^2$$
$$\Rightarrow a^2 + b^2 - 2r(a+b) + r^2 + 2ab = 2p^2$$
$$\Rightarrow 2ab = 2p^2$$
$$\Rightarrow ab = p^2 \qquad \text{[by Eq.(3.27)]}$$

Therefore, the area of $ABCD$ is given by p^2.

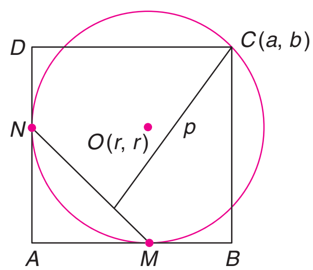

FIGURE 3.30

42. A is a point on the circle $x^2 + y^2 = 2a^2$. From A, two tangents are drawn to the circle $x^2 + y^2 = a^2$ whose points of contact are B and C. As A moves on the circle $x^2 + y^2 = 2a^2$, show that the locus of the circumcentre of ΔABC is the circle $x^2 + y^2 = a^2/2$.

Solution: According to Quick Look 4, A lies on the director circle of $x^2 + y^2 = a^2$. Hence, $\underline{|BAC} = 90°$ (see Fig. 3.31). Therefore, the circumcentre of ΔABC = midpoint of BC = midpoint of OA because $ABOC$ is a square, where O is (0, 0). If $A = (h, k)$, the circumcentre is $(h/2, k/2)$ and hence $h^2 + k^2 = 2a^2$. This implies $(h/2)^2 + (k/2)^2 = a^2/2$. Therefore, the locus is given by

$$x^2 + y^2 = \frac{a^2}{2}$$

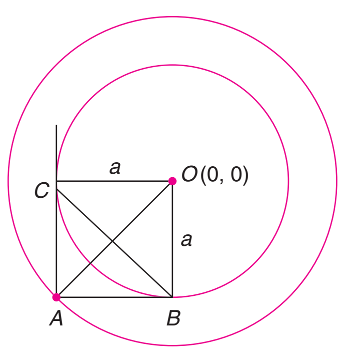

FIGURE 3.31

43. A circle $S \equiv x^2 + y^2 + 4x - 2\sqrt{2} + c = 0$ is the director circle of S_1, S_1 is the director circle of S_2, S_2 is the director circle of S_3 and so on. If the sum of all the radii of these circles is 2, then find C.

Solution: According to Quick Look 4, the radius of the director circle of a given circle is $\sqrt{2}$ times the radius of the given circle. Therefore, the radius of $S = 0$ is given by

$$\sqrt{2^2 + (\sqrt{2})^2 - c} = \sqrt{6-c} = a \qquad \text{(say)}$$

Therefore, the radius of S_1 is $a/\sqrt{2}$, the radius of S_2 is $a/2$ and so on. Therefore,

$$2 = a + \frac{a}{\sqrt{2}} + \frac{a}{2} + \frac{a}{2\sqrt{2}} + \cdots\infty$$
$$= a\left(1 + \frac{1}{\sqrt{2}} + \frac{1}{2} + \frac{1}{2\sqrt{2}} + \cdots\infty\right)$$
$$= \frac{a}{1 - 1/\sqrt{2}} = \frac{\sqrt{2}a}{\sqrt{2} - 1}$$

This gives

$$4(\sqrt{2} - 1)^2 = 2a^2 = 2(6 - c)$$
$$\Rightarrow 2(2 + 1 - 2\sqrt{2}) = 6 - c$$
$$\Rightarrow c = 4\sqrt{2}$$

44. Find the point of intersection of tangents to the circle $x^2 + y^2 = a^2$ which are inclined at angles α and β with the positive direction of the x-axis such that $\cot\alpha + \cot\beta = 0$.

Solution: Suppose $P(h, k)$ is the point of intersection of the tangents which are inclined at angles α and β with x-axis. $y = mx + a\sqrt{1+m^2}$ always touches $x^2 + y^2 = a^2$ $(k - mh)^2 = a^2(1 + m^2)$ which has two distinct roots, say, m_1 and m_2. Let $m_1 = \tan\alpha$, $m_2 = \tan\beta$. Therefore,

$$\cot\alpha + \cot\beta = 0$$
$$\Rightarrow \frac{1}{\tan\alpha} + \frac{1}{\tan\beta} = 0$$
$$\Rightarrow \frac{m_1 + m_2}{m_1 m_2} = 0$$
$$\Rightarrow 2hk = 0$$

Therefore, locus of (h, k) is $xy = 0$.

45. Find the midpoint of the chord of the circle $x^2 + y^2 = 25$ intercepted on the line $x - 2y - 2 = 0$.

Solution: Let $M(x_1, y_1)$ be the midpoint of the chord. Therefore, equation of the chord is $S_1 = S_{11}$. That is,

$$xx_1 + yy_1 - 25 = x_1^2 + y_1^2 - 25$$
$$xx_1 + yy_1 = x_1^2 + y_1^2 \quad (3.28)$$

However, by hypothesis,

$$x - 2y - 2 = 0 \quad (3.29)$$

is the chord. From Eqs. (3.28) and (3.29), we have

$$\frac{x_1}{1} = \frac{y_1}{-2} = \frac{x_1^2 + y_1^2}{2} = t \quad \text{(say)}$$
$$x_1 = t, y_1 = -2t$$

and $$2t = x_1^2 + y_1^2 = 5t^2 \Rightarrow t = 0 \text{ or } \frac{2}{5}$$

Here, $t = 0$ implies that $(x_1, y_1) = (0, 0)$ is not on the line $x - 2y - 2 = 0$. Therefore,

$$(x_1, y_1) = \left(\frac{2}{5}, \frac{-4}{5}\right)$$

46. Tangent at any point on the circle $x^2 + y^2 = a^2$ meets the circle $x^2 + y^2 = b^2$ at points P and Q. If the tangents drawn at points P and Q of this circle intersect at right angles, then show that $b^2 = 2a^2$.

Solution: Let $S \equiv x^2 + y^2 - a^2 = 0$ and $S' \equiv x^2 + y^2 - b^2 = 0$. By hypothesis, $b > a$. Suppose that the tangents at points P and Q to $S' = 0$ meet in $T(h, k)$. Since $\underline{|PTQ} = 90°$, T must lie on the director circle of $S' = 0$. Therefore,

$$h^2 + k^2 = 2b^2 \quad (3.30)$$

Now, equation of PQ is $hx + ky - b^2 = 0$ and it touches the circle $S = 0$ so that

$$\frac{|-b^2|}{\sqrt{h^2 + k^2}} = a$$

and hence $b^4 = a^2(h^2 + k^2)$. Therefore, from Eq. (3.30), we get $b^2 = 2a^2$.

47. Let us consider that the line $lx + my + n = 0$ does not pass through the origin O and P is a point on the line. On the segment OP, let Q be a point such that $OP \cdot OQ = k^2$, where k is a fixed number. Then show that Q lies on the curve $n^2(x^2 + y^2) = k^2(lx + my)$.

Solution: Let $P = (\alpha, \beta)$. Since P lies on the line $lx + my + n = 0$, we have

$$l\alpha + m\beta + n = 0 \quad (3.31)$$

Let $Q = (p, q)$. Since the equation of OP is $\beta x - \alpha y = 0$ and $Q(p, q)$ lies on OP (Fig. 3.32), we have

$$p\beta - \alpha q = 0 \quad (3.32)$$

From Eqs. (3.31) and (3.32), we get

$$\alpha = \frac{-pn}{lp + mq}$$

and $$\beta = \frac{-qn}{lp + mq}$$

Therefore

$$OP \cdot OQ = k^2 \Rightarrow \sqrt{\alpha^2 + \beta^2} \cdot \sqrt{p^2 + q^2} = k^2$$
$$\Rightarrow \sqrt{\frac{n^2(p^2 + q^2)}{(lp + mq)^2}} \cdot \sqrt{p^2 + q^2} = k^2$$
$$\Rightarrow n(p^2 + q^2) = k^2(lp + mq)$$

Hence, $Q(p, q)$ *lies on the curve* $n(x^2 + y^2) = k^2(lx + my)$.

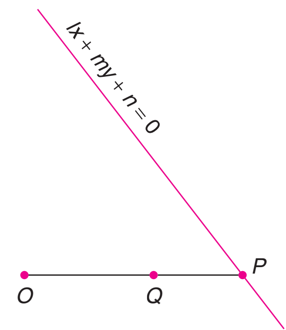

FIGURE 3.32

48. For all values of the parameter α, show that the locus of the point of intersection of the lines $x \cos \alpha + y \sin \alpha = p$ and $x \sin \alpha - y \cos \alpha = q$ is the circle $x^2 + y^2 = p^2 + q^2$.

Solution: Squaring and adding the given equations, we have $x^2 + y^2 = p^2 + q^2$ which represents circle with centre as origin and radius $\sqrt{p^2 + q^2}$.

49. A circle touches the line $y = x$ at a point P whose distance from the origin is $4\sqrt{2}$. The point $(-10, 2)$ is an interior point of the circle. The length of the chord on the line $x + y = 0$ is $6\sqrt{2}$. Find the equation, of the circle. **(IIT-JEE 1990)**

Solution: Let $C(h, k)$ be the centre of the circle. Now, $P = (x, x)$ and $OP = 4\sqrt{2} \Rightarrow 2x^2 = 32 \Rightarrow x = \pm 4$. Let M be the midpoint of AB (see Fig. 3.33) where $AB = 6\sqrt{2}$. Therefore

$$AM = MB = 3\sqrt{2}$$

Also $OPCM$ is a rectangle $\Leftrightarrow CM = OP = 4\sqrt{2}$. Therefore

$$AC^2 = (AM)^2 + (CM)^2 = (3\sqrt{2})^2 + (4\sqrt{2})^2 = 50$$

Hence, the radius (r) of the circle is $AC = 5\sqrt{2}$. Now

$$4\sqrt{2} = CM = \left|\frac{h+k}{\sqrt{2}}\right| \Rightarrow h+k = \pm 8 \qquad (3.33)$$

$$5\sqrt{2} = PC = \left|\frac{h-k}{\sqrt{2}}\right| \Rightarrow h-k = \pm 10 \qquad (3.34)$$

From Eqs. (3.33) and (3.34), we have $(h, k) = (9, -1), (1, -9)$, $(-1, 9)$ or $(-9, 1)$. Therefore, equation of the circle is given by

$$S \equiv (x-9)^2 + (y-1)^2 = 50$$

or $$S \equiv (x-1)^2 + (y+9)^2 = 50$$

or $$S \equiv (x+1)^2 + (y-9)^2 = 50$$

or $$S \equiv (x+9)^2 + (y-1)^2 = 50$$

However, $(-10, 2)$ is an interior point of the circle for which $S_{11} < 0$ which is satisfied by the equation

$$S \equiv (x+9)^2 + (y-1)^2 = 50$$

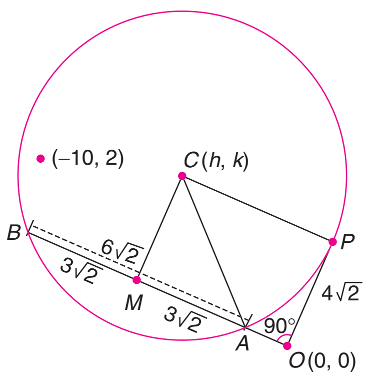

FIGURE 3.33

50. Two vertices of an equilateral triangle are $(-1, 0)$ and $(1, 0)$ and its third vertex lies above the x-axis. Find the equation of the circumcircle of the triangle.

Solution: Let $B = (-1, 0)$ and $C = (1, 0)$ be two vertices of an equilateral triangle ABC (see Fig. 3.34). Now, $BC = AC = AB = 2$ and $OC = OB = 1$. Since the third vertex lies above the x-axis on the perpendicular bisector of the side BC follows $A = (O, k)$, where $k > 0$, we have

$$\tan 60^\circ = \frac{OA}{OC} = \frac{k}{1}$$

$$\Rightarrow k = \sqrt{3}$$

Therefore, $A = (0, \sqrt{3})$. The equation of the perpendicular bisector of AB is given by

$$y - \frac{\sqrt{3}}{2} = \frac{-1}{\sqrt{3}}\left(x + \frac{1}{2}\right)$$

which meets y-axis at

$$\left(0, \frac{\sqrt{3}}{2} - \frac{1}{2\sqrt{3}}\right) = \left(0, \frac{1}{\sqrt{3}}\right)$$

Therefore, centre is given by

$$\left(0, \frac{1}{\sqrt{3}}\right)$$

Also

$$2R = \frac{BC}{\sin A} = \frac{2}{\sin 60^\circ} = \frac{4}{\sqrt{3}} \Rightarrow R = \frac{2}{\sqrt{3}}$$

Therefore, circumradius is given by $2/\sqrt{3}$. Hence, the equation of the circumcircle is given by

$$x^2 + \left(y - \frac{1}{\sqrt{3}}\right)^2 = \left(\frac{2}{\sqrt{3}}\right)^2$$

$$\Rightarrow x^2 + y^2 - \frac{2}{\sqrt{3}}y + \frac{1}{3} - \frac{4}{3} = 0$$

$$\Rightarrow \sqrt{3}(x^2 + y^2) - 2y - \sqrt{3} = 0$$

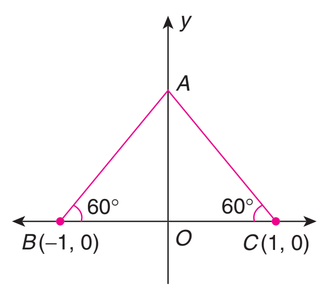

FIGURE 3.34

3.4 | Relation Between Two Circles

Let C_1 and C_2 be two circles with centres A and B and radii r_1 and r_2, respectively.

1. C_1 and C_2 do not have any common point $\Leftrightarrow AB > r_1 + r_2$ [see Fig. 3.35(a)].

2. C_1 and C_2 touch each other externally $\Leftrightarrow AB = r_1 + r_2$ [see Fig. 3.35(b)].
3. C_1 and C_2 intersect in two distinct points $\Leftrightarrow |r_1 - r_2| < AB < r_1 + r_2$ [see Fig. 3.35(c)].
4. C_1 and C_2 touch each other internally $\Leftrightarrow AB = |r_1 - r_2|$ [see Fig. 3.35(d)].
5. One circle lies completely within the other without having common point $\Leftrightarrow AB < |r_1 - r_2|$ [see Fig. 3.35(e)].

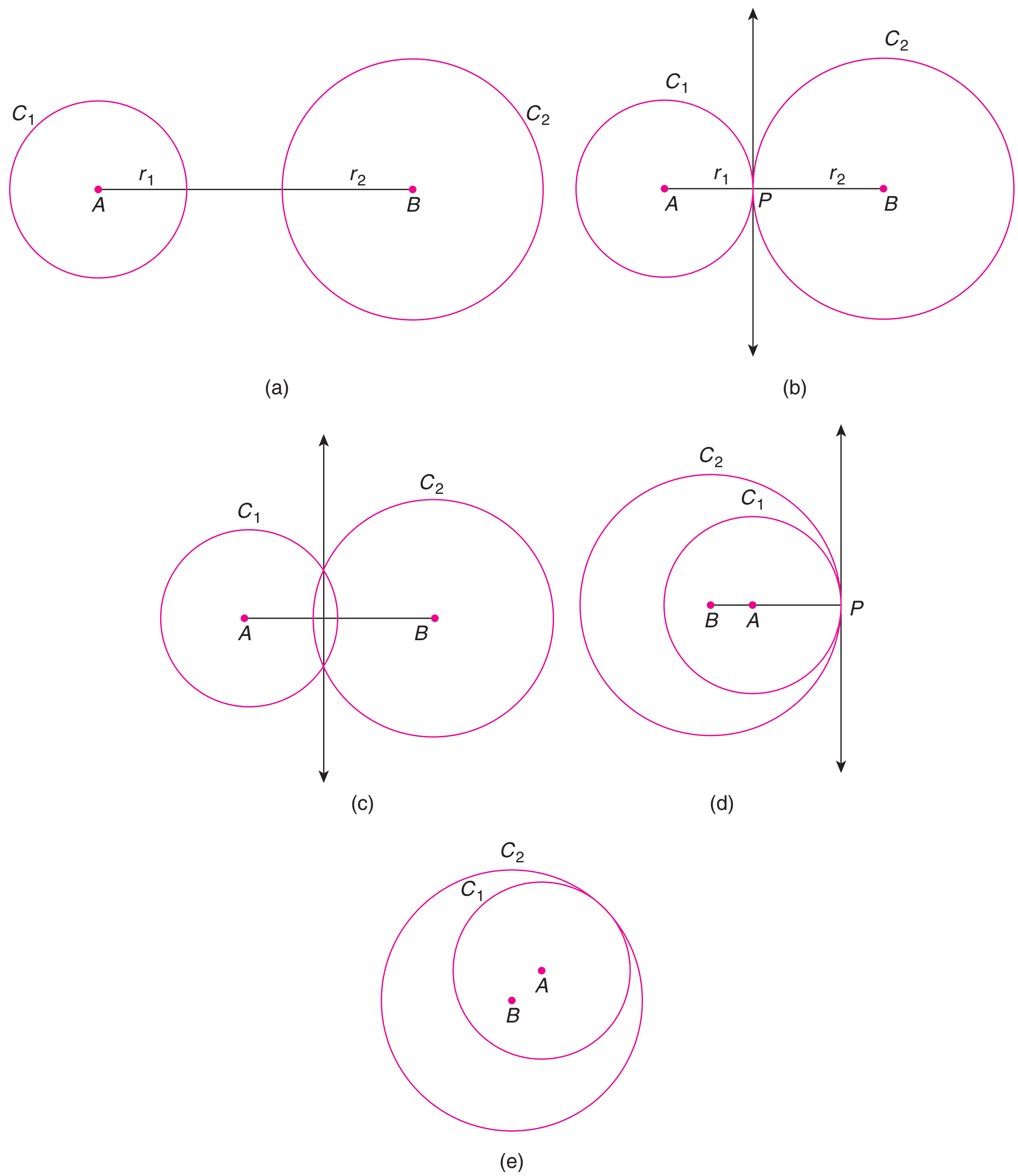

FIGURE 3.35

QUICK LOOK 7

1. If the two circles touch externally at a point P, then A, P and B are collinear and P divides $\overline{AB}$ internally in the ratio $r_1 : r_2$.
2. If the two circles touch internally, then P divides $\overline{AB}$ externally in the ratio $r_1 : r_2$.
3. Whether the two circles touch internally or externally, the point of contact is the foot of the perpendicular drawn from either of the centres onto the common tangent.

Theorem 3.14 If $S = 0$ and $S' = 0$ are non-concentric circles in the standard form, then the equation $S - S' = 0$ represents a straight line perpendicular to the line joining the centres.

Proof Let

$$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$$
$$S' \equiv x^2 + y^2 + 2g'x + 2f'y + c' = 0$$

and $(-g, -f) \neq (-g', -f')$. Therefore, $S - S' \equiv 2(g - g')x + 2(f - f')y + c - c' = 0$ is a first-degree equation and hence it represents a straight line. Since the slope of the line joining the centres is $f - f'/g - g'$ and the slope of $S - S' = 0$ is $-(g - g')/f - f'$, $S - S' = 0$ is perpendicular to the line joining the centres. ■

QUICK LOOK 8

1. If the circles $S = 0$ and $S' = 0$ cut in points P and Q, then $S - S' = 0$ passes through both P and Q and hence $S - S' = 0$ is the common chord PQ of the circles.
2. If the circles $S = 0$ and $S' = 0$ touch each other at point P, then $S - S' = 0$ is the common tangent of the two circles at point P and P divides the line joining the centres in the ratio of their radii.

Theorem 3.15 Let $S \equiv x^2 + y^2 + 2gx + 2fy + c' = 0$ and $L \equiv ax + by + c = 0$ be a line. Then for all values of λ, the equation $S + \lambda L \equiv x^2 + y^2 + 2gx + 2fy + c' + \lambda(ax + by + c) = 0$ represents a family of circles.

Proof $S + \lambda L \equiv x^2 + y^2 + (2g + \lambda a)x + (2f + \lambda b)y + \lambda c + c' = 0$ which represents circle in the sense of the Note given under Theorem 3.2. ■

QUICK LOOK 9

1. If the line $L = 0$ intersects the circle $S = 0$ in two different points P and Q, then $S + \lambda L = 0$ represents a circle passing through points P and Q.
2. If $L = 0$ touches $S = 0$ at point P, then $S + \lambda L = 0$ represents a circle touching $S = 0$ at point P and $L = 0$ is the common tangent at point P.
3. If $S = 0$ and $S' = 0$ are two non-concentric circles in the standard form and $L \equiv S - S' = 0$, then $S + \lambda L = 0$ where $L \equiv S - S' = 0$ represents a circle. In particular, if $S = 0$ and $S' = 0$ intersect, then $S + \lambda L = 0$ represents a circle passing through their points of intersection. Further, if $S = 0$ and $S' = 0$ touch each other at point P then $S + \lambda L = 0$ where $L \equiv S - S' = 0$ represents circle touching both $S = 0$ and $S' = 0$ at point P.

Example 3.7

Show that the two circles $x^2 + y^2 - 4x + 6y + 8 = 0$ and $x^2 + y^2 - 10x - 6y + 14 = 0$ touch each other and find the coordinates of the point of contact.

Solution: Let us consider that

$$S \equiv x^2 + y^2 - 4x + 6y + 8 = 0$$
$$S' \equiv x^2 + y^2 - 10x - 6y + 14 = 0$$

Centre and radius of $S = 0$ are $A(2, -3)$ and $r_1 = \sqrt{5}$; centre and radius of $S' = 0$ are $B(5, 3)$ and $r_2 = 2\sqrt{5}$. The distance between the centres (see Fig. 3.36) is given by

$$AB = \sqrt{(5-2)^2 + (3+3)^2} = \sqrt{45} = 3\sqrt{5}$$

Therefore

$$3\sqrt{5} = AB = r_1 + r_2$$

So, the two circles touch each other externally. Suppose $P(x, y)$ is the point of contact. Then

$$AP : PB = \sqrt{5} : 2\sqrt{5} = 1 : 2$$

Hence

$$x = \frac{1(5)+2(2)}{1+2} = 3$$

and
$$y = \frac{1(3)+2(-3)}{1+2} = -1$$

Therefore, $P = (3, -1)$.

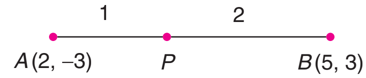

FIGURE 3.36

Example 3.8

Show that the two circles $S \equiv x^2 + y^2 + 4y - 1 = 0$ and $S' \equiv x^2 + y^2 + 6x + y + 8 = 0$ touch each other. Find the common tangent at the point of contact and the point of contact.

Solution: $A = (0, -2)$ and $r_1 = \sqrt{5}$, respectively, are the centre and the radius of $S = 0$. Similarly $B = (-3, -1/2)$ and $r_2 = \sqrt{5}/2$, respectively, are the centre and the radius of $S' = 0$. The distance between the centres (see Fig. 3.37) is given by

$$AB = \sqrt{(-3-0)^2 + \left(\frac{-1}{2} - (-2)\right)^2} = \sqrt{9 + \frac{9}{4}} = \frac{3\sqrt{5}}{2} = r_1 + r_2$$

Therefore, $S = 0$, $S' = 0$ touch each other externally. The common tangent is $S - S' \equiv 2x - y + 3 = 0$. Suppose $P(x, y)$ is the point of contact. Therefore

$$AP : PB = r_1 : r_2 = 2 : 1$$

So

$$x = \frac{1(0)+2(-3)}{2+1} = -2$$

and
$$y = \frac{1(-2)+2\left(-\frac{1}{2}\right)}{2+1}$$

Therefore, $P = (-2, -1)$.

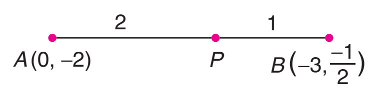

FIGURE 3.37

Example 3.9

Find the equation of the circle whose diameter is the common chord of the circles $(x - a)^2 + y^2 = a^2$ and $x^2 + (y - b)^2 = b^2$.

Solution: Let us consider that

$$S \equiv (x-a)^2 + y^2 - a^2 = x^2 + y^2 - 2ax = 0$$

$$S' \equiv x^2 + (y-b)^2 - b^2 = x^2 + y^2 - 2by = 0$$

The common chord equation is $L \equiv ax - by = 0$ which is $S - S' = 0$. Equation of the circle described on $L = 0$ as diameter is of the form

$$S = \lambda L = x^2 + y^2 - 2ax + \lambda(ax - by) = 0$$

whose centre is given by

$$\left(\frac{2a - \lambda a}{2}, \frac{\lambda b}{2}\right)$$

which lies on the common chord $L \equiv S - S' = ax - by = 0$. Therefore

$$a\left(\frac{2a - \lambda a}{2}\right) - b\left(\frac{\lambda b}{2}\right) = 0$$

$$\Rightarrow 2a^2 - \lambda a^2 - \lambda b^2 = 0$$

$$\Rightarrow \lambda = \frac{2a^2}{a^2 + b^2}$$

Hence, equation of the given circle is given by

$$S + \lambda L \equiv S + \frac{2a^2}{a^2 + b^2} L = 0$$

$$\Rightarrow x^2 + y^2 - 2ax + \frac{2a^2}{a^2 + b^2}(ax - by) = 0$$

$$\Rightarrow (a^2 + b^2)(x^2 + y^2) - 2ab^2 x - 2a^2 by = 0$$

Example 3.10

Find the equation of a circle passing through the point $(2, 1)$ and the points of intersection of the circles $S \equiv x^2 + y^2 - 2x + 3y - 1 = 0$ and $S' \equiv x^2 + y^2 + 3x - 2y - 1 = 0$.

Solution: $L \equiv S - S' \equiv -5x + 5y = 0$ is the common chord of $S = 0$ and $S' = 0$. Any circle passing through the intersection of $S = 0$ and $S' = 0$ is of the form

$$S + \lambda L = x^2 + y^2 - 2x + 3y - 1 + \lambda(x - y) = 0$$

The given circle passes through the point (2, 1). This implies that

$$2^2 + 1^2 - 2(2) + 3(1) - 1 + \lambda(2-1) = 0 \Rightarrow \lambda = -3$$

Therefore, the required equation is $x^2 + y^2 - 2x + 3y - 1 - 3(x - y) = 0$. That is,

$$x^2 + y^2 - 5x + 6y - 1 = 0$$

Example 3.11

The line $2x + 3y = 1$ cuts the circle $x^2 + y^2 = 4$ at points A and B. Show that the equation of the circle described on AB as diameter is $13(x^2 + y^2) - 4x - 6y - 50 = 0$.

Solution: Let $S \equiv x^2 + y^2 - 4 = 0$ and $L \equiv 2x + 3y - 1 = 0$. The required circle equation is

$$S + \lambda L = x^2 + y^2 - 4 + \lambda(2x + 3y - 1) = 0$$

The centre of the circle is

$$\left(-\lambda, \frac{-3\lambda}{2}\right)$$

Since AB is the diameter of $S + \lambda L = 0$ and $(-\lambda, -3\lambda/2)$ lies on $L = 0$, we have

$$2(-\lambda) + 3\left(\frac{-3\lambda}{2}\right) = 1$$

$$\Rightarrow \lambda = \frac{-2}{13}$$

Therefore

$$S + \lambda L \equiv x^2 + y^2 - 4 - \frac{2}{13}(2x + 3y - 1) = 0$$

$$\Rightarrow 13(x^2 + y^2) - 4x - 6y - 50 = 0$$

Example 3.12

The condition for the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ which bisects the circumference of the circle $S' \equiv x^2 + y^2 + 2g'x + 2f'y + c' = 0$ is that $2(g - g')g' + 2(f - f') = c - c'$.

Solution: We have

$S = 0$ bisects the circumference of $S' = 0$

$\Leftrightarrow$ Common chord $L \equiv S - S' = 0$ is diameter of the circle $S' = 0$

$\Leftrightarrow$ The centre of $S' = 0$ lies on $L = 0$

$\Leftrightarrow 2(g - g')(-g') + 2(f - f')(-f') + c - c' = 0$

$\Leftrightarrow 2(g - g')g' + 2(f - f')f' = c - c'$.

DEFINITION 3.4 **Angle of Intersection** Suppose point P is a common point of two circles C_1 and C_2. Then the angle between the tangents drawn to the circles at point P is called the angle of intersection of C_1 and C_2 (see Chapter 3, Vol. 3).

DEFINITION 3.5 **Orthogonal Circles** If the angle of intersection of two circles is a right angle, then the two circles are said to intersect orthogonally.

QUICK LOOK 10

Two circles C_1 and C_2 with centres A and B, respectively, intersect each other orthogonally at P if and only if AP is tangent to C_2 and BP is tangent to C_1 (see Fig. 3.38). That is, if and only if ΔAPB is right-angled with right angle at P (see Fig. 3.38)

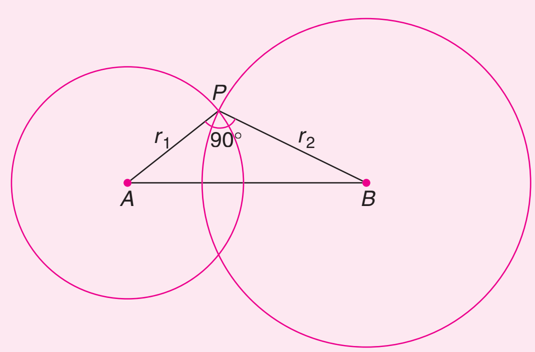

FIGURE 3.38

THEOREM 3.16 Let C_1 and C_2 be two circles with centres at points A and B, respectively, which intersect at point P. If θ is the angle of intersection of the circles, then

$$\cos\theta = \frac{(AB)^2 - (AP)^2 - (BP)^2}{2(AP)(BP)} = \frac{(AB)^2 - r_1^2 - r_2^2}{2r_1r_2}$$

where $r_1 = AP$ and $r_2 = BP$.

PROOF Suppose the tangent to circle C_2 at point P meets the line AB at point T_1 and the tangent to circle C_1 at point P meets AB at point T_2 (see Fig. 3.39). Therefore, PT_2 is at right angles at AP and PT_1 is at right angles to PB. It is given that $\angle T_1PT_2 = \theta$. Now,

$$\angle APT_1 = 90° - \theta \text{ and } \angle BPT_2 = 90° - \theta \Rightarrow \angle APB = 90° + (90° - \theta) = 180° - \theta$$

Therefore, using cosine rule to ΔPAB, we get

$$\begin{aligned}(AB)^2 &= (AP)^2 + (BP)^2 - 2(AP)(BP)\cos(180° - \theta)\\ &= (AP)^2 + (BP)^2 + 2(AP)(BP)\cos\theta\\ \Rightarrow \cos\theta &= \frac{(AB)^2 - (AP)^2 - (BP)^2}{2(AP)(BP)} = \frac{(AB)^2 - r_1^2 - r_2^2}{2r_1r_2}\end{aligned}$$

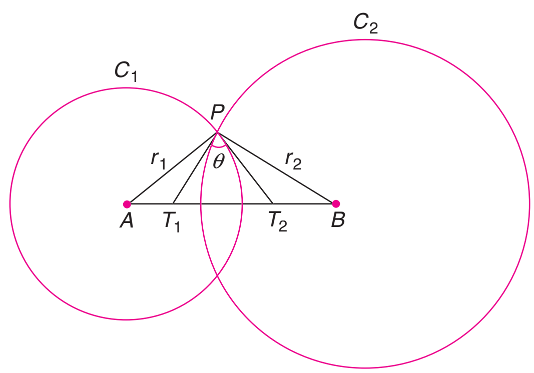

FIGURE 3.39 ■

QUICK LOOK 11

The circles intersect orthogonally if and only if $\theta = 90°$. That is, if and only if $(AB)^2 = r_1{}^2 + r_2{}^2$.

Theorem 3.17 gives algebraic condition for two circles to cut orthogonally.

THEOREM 3.17 The condition for two circles $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ and $S' \equiv x^2 + y^2 + 2g'(x) + 2f'y + c' = 0$ to cut orthogonally is that $2gg' + 2ff' = c + c'$.

PROOF Let $A = (-g, -f)$, $r_1 = \sqrt{g^2 + f^2 - c}$, $B = (-g', -f')$ and $r_2 = \sqrt{g'^2 + f'^2 - c'}$. The two circles cut each other orthogonally

$$\begin{aligned}&\Leftrightarrow (AB)^2 = r_1{}^2 + r_2{}^2 \quad \text{(Quick Look 11)}\\ &\Leftrightarrow (g - g')^2 + (f - f')^2 = (g^2 + f^2 - c) + (g'^2 + f'^2 - c')\\ &\Leftrightarrow 2gg' + 2ff' = c + c'\end{aligned}$$

■

Example 3.13

Show that the circles $S \equiv x^2 + y^2 - 8x - 6y + 21 = 0$ and $S' \equiv x^2 + y^2 - 2y - 15 = 0$ cut orthogonally.

Solution: For $S \equiv 0$, the centre and radius are

$$A = (4, 3),\ r_1 = \sqrt{4^2 + 3^2 - 21} = 2$$

For $S' \equiv 0$, the centre and radius are

$$B = (0, 1),\ r_2 = \sqrt{1 + 15} = 4$$

Now,

$$(AB)^2 = 4^2 + 2^2 = 20 = r_1^2 + r_2^2$$

According to Quick Look 11, the two circles cut orthogonally. Also, we have $g = -4, f = -3, c = 21$ and $g' = 0, f' = -1, c' = -15$. Therefore,

$$2gg' + 2ff' = 2(-4)(0) + 2(-3)(-1) = 6 = c + c'$$

Hence, by Theorem 3.17, the two circles cut orthogonally.

Example 3.14

Find the angle of intersection of the circles $S \equiv x^2 + y^2 - 16 = 0$ and $S' \equiv x^2 + y^2 - 4x - 2y - 4 = 0$.

Solution: Let θ be the angle of intersection of the circles. For $S \equiv 0$, centre and radius are $A = (0, 0)$, $r_1 = 4$. For $S' \equiv 0$, centre and radius are $B = (2, 1)$ and $r_2 = 3$. By Theorem 3.16, we have

$$\cos\theta = \frac{(AB)^2 - r_1^2 - r_2^2}{2r_1r_2} = \frac{5 - (16) - (9)}{2(4)(3)} = \frac{-5}{6}$$

$$\Rightarrow \theta = \mathrm{Cos}^{-1}\left(\frac{-5}{6}\right) = \pi - \mathrm{Cos}^{-1}\left(\frac{5}{6}\right)$$

Note: One may think that we can find the angle of intersection by using calculus method. The difficulty in using the calculus method is that we should find the coordinates of the point of intersection which is a cumbersome process. This is not necessary in geometrical method.

THEOREM 3.18 Let $S = 0$ be a circle and $A(x_1, y_1)$ be a point in the plane of the circle. Let L is a line through A meeting the circle $S = 0$ in P and Q. Then $AP \cdot AQ$ is constant. That is, $AP \cdot AQ$ is independent of the line L.

PROOF Let $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$. Suppose the parametric equations of the line $L = 0$ be

$$x = x_1 + r\cos\theta \text{ and } y = y_1 + r\sin\theta$$

Put $x = x_1 + r\cos\theta$ and $y = y_1 + r\sin\theta$ in $S = 0$. Therefore,

$$(x_1 + r\cos\theta)^2 + (y_1 + r\sin\theta)^2 + 2g(x_1 + r\cos\theta) + 2f(y_1 + r\sin\theta) + c = 0$$

That is,

$$r^2 + 2[(x_1 + g)\cos\theta + (y_1 + f)\sin\theta]r + S_{11} = 0 \qquad (3.35)$$

Since Eq. (3.35) is a quadratic equation in r, it will have two roots, say, r_1 and r_2. $L = 0$ meeting the circle implies that r_1 and r_2 are real and AP, AQ are equal to r_1 and r_2. Therefore, $AP \cdot AQ = r_1r_2 = S_{11}$, which is constant. S_{11} value depends on $A(x_1, y_1)$, but not on $L = 0$. ■

QUICK LOOK 12

If $A(x_1, y_1)$ is an external point, then we know that $S_{11} > 0$ and $\sqrt{S_{11}}$ is the length of the tangent drawn from $A(x_1, y_1)$ (see Theorem 3.11). Hence, $A(x_1, y_1)$ is external point. This implies $AP \cdot AQ = S_{11}$ = Square of the tangent from point A.

DEFINITION 3.6 **Power of a Point** Let $S = 0$ be a circle and $A(x_1, y_1)$ a point in the plane of the circle. If a line L through point A meets the circle at points P and Q, then the power of A with respect to the circle is defined as

1. $AP \cdot AQ$ if A is external point.
2. $-(AP \cdot AQ)$ if A is internal point.
3. 0 if A lies on the circle.

QUICK LOOK 13

According to Theorem 3.18, the power of a point $A(x_1, y_1)$ with respect to the circle $S = 0$ is S_{11}. The power is positive or negative or zero when the point lies outside or inside or on the circle, respectively.

Examples

1. The power of the point $A(2, 3)$ with respect to the circle $S \equiv x^2 + y^2 - 4x - 2y - 6 = 0$ is $S_{11} = 2^2 + 3^2 - 4(2) - 2(3) - 6 = -7$.
2. The power of the point $(1, 1)$ with respect to the circle $S \equiv x^2 + y^2 + 4x + 2y - 1 = 0$ is $S_{11} = 1^2 + 1^2 + 4(1) - 2(1) - 1 = 7$.

THEOREM 3.19 Let $S = 0$ and $S' = 0$ be two non-concentric circles. Then, the locus of the point whose powers with respect to the two circles are equal is a straight line perpendicular to the line joining the centres.

PROOF Let $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ and $S' \equiv x^2 + y^2 + 2g'x + 2f'y + c' = 0$ be two non-concentric circles [i.e. $(-g, -f) \neq (-g', -f')$]. $A(x_1, y_1)$ is a point on the locus $\Leftrightarrow S_{11} = S'_{11}$. This implies

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = x_1^2 + y_1^2 + 2g'x_1 + 2f'y_1 + c'$$
$$\Leftrightarrow 2(g - g')x_1 + 2(f - f')y_1 + c - c' = 0$$

Therefore, the locus of $A(x_1, y_1)$ is the line

$$S - S' \equiv 2(g - g')x + 2(f - f')y + c - c' = 0$$

■

Note: Since the circles are non-concentric, $S - S' = 0$ is a first degree equation and hence it is a straight line. Also, one can see that the line $S - S' = 0$ is perpendicular to the line joining the centres.

DEFINITION 3.7 **Radical Axis** The locus of the point whose powers with respect to two non-concentric circles are equal is a straight line called the *radical axis* of the two circles.

QUICK LOOK 14

1. The radical axis of the circles $S = 0, S' = 0$ is $S - S' = 0$.
2. $S - S' = 0$ is the common chord if they intersect. That is, if the two circles intersect, then their radical axis is their common chord.
3. If the two circles touch, then the common target at the point of contact is the radical axis.

THEOREM 3.20 The radical axes of three circles (see Fig. 3.40) with non-collinear centres taken two by two are concurrent and this point is called the *radical centre* of the three circles.

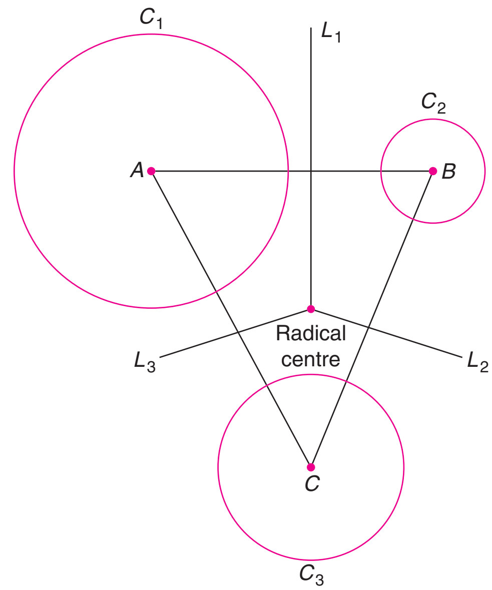

FIGURE 3.40

PROOF Let $S = 0$, $S' = 0$ and $S'' = 0$ be those circles (in the standard form) with non-collinear centres. Then by Theorem 3.19, $L_1 = S - S' = 0$, $L_2 = S' - S'' = 0$ and $L_3 = S'' - S = 0$ are the radical axes of the pairs (C_1, C_2), (C_2, C_3) and (C_3, C_1), respectively. Now $L_1 + L_2 + L_3 = 0$ implies that the three lines $L_1 = 0$, $L_2 = 0$, and $L_3 = 0$ are concurrent (see Theorem 2.14) where C_1, C_2 and C_3 are $S = 0$, $S' = 0$ and $S'' = 0$, respectively. ■

Since the radical centre lies on all the three radical axes, its powers with respect to all the three circles are equal. Hence the radical centre lies outside the three circles or lies inside the three circles.

If the radical centre lies outside the circles, then the length of the tangent drawn from it to the circles is same and hence if a circle is drawn with the centre at the radical centre and length of the tangent drawn from it to the circles as radius, then this circle cuts all the three circles orthogonally.

Example 3.15

Find the radical centre of the circles

$$S \equiv x^2 + y^2 - 16x + 60 = 0$$
$$S' \equiv 3x^2 + 3y^2 - 36x + 81 = 0$$
$$S'' \equiv x^2 + y^2 - 16x - 12y + 88 = 0$$

Solution: We have

$$L_1 = S - S' \equiv 4x - 33 = 0$$
$$L_2 \equiv S' - S'' \equiv 4x + 12y - 61 = 0$$
$$L_3 \equiv S'' - S \equiv -12y + 28 = 0$$

From $L_1 = 0$ and $L_3 = 0$, we have $x = 33/4$, $y = 7/3$. Also $(33/4, 7/3)$ lies on $L_2 = 0$. Hence, the radical centre of the circles is $(33/4, 7/3)$ with respect to $S = 0$. The power of $(33/4, 7/3)$ is equal to

$$\left(\frac{33}{4}\right)^2 + \left(\frac{7}{3}\right)^2 - 16\left(\frac{33}{4}\right) + 60 = \left(\frac{1089}{16}\right) + \left(\frac{49}{9}\right) - 72$$
$$= \frac{(10585 - 10368)}{144}$$
$$= \frac{217}{144}$$

Therefore, the required circle is

$$\left(x - \frac{33}{4}\right)^2 + \left(y - \frac{7}{3}\right)^2 = \frac{217}{144}$$

Example 3.16

Find the equation of the circle which cuts orthogonally all the three circles $S \equiv x^2 + y^2 + x + 2y + 3 = 0$, $S' \equiv x^2 + y^2 + 2x + 4y - 4 = 0$ and $S'' \equiv x^2 + y^2 - 7x - 8y - 9 = 0$.

Solution: We have

$$L_1 \equiv S - S' \equiv -x - 2y + 7 = 0$$
$$L_2 \equiv S' - S'' \equiv 9x + 12y + 5 = 0$$
$$L_3 \equiv S'' - S \equiv -8x - 10y - 12 \equiv 4x + 5y + 6 = 0$$

Solving $L_1 = 0$ and $L_3 = 0$, we have $x = (-47)/3$ and $y = 34/3$. Therefore, the radical centre is given by $(-47/3, 34/3)$. Now

$$S_{11} = \left(-\frac{47}{3}\right)^2 + \left(\frac{34}{3}\right)^2 - \frac{47}{3} + 2\left(\frac{34}{3}\right) + 3 = \frac{3455}{9}$$

So, the required circle is

$$\left(x + \frac{47}{3}\right)^2 + \left(y - \frac{34}{3}\right)^2 = \frac{3455}{9}$$

3.5 | Common Tangents of Two Circles

Let C_1 and C_2 be two non-concentric circles. Suppose A and B are their centres and r_1 and r_2 are their radii, respectively.

Case 1: Suppose circles C_1 and C_2 are such that one circle completely lies outside the other. That is, $AB > r_1 + r_2$ [see Fig. 3.41(a)]. In this case, there are *four common tangents*. Two are called direct tangents and the other two are called transverse tangents. A tangent is called direct common tangent if the centres of the circles lie on the same side of the target. Transverse common tangent means that the centres lie on the opposite sides. If the radii are different, then the direct common tangents intersect in a point which is called the *external centre of similitude* denoted by S_2. S_2 divides externally the line joining the centres in the ratio of their radii. The two transverse common tangents intersect in a point is called the *internal centre of similitude* denoted by S_1. S_1 divides the line joining the centres internally in the ratio of their radii [see Fig. 3.41(b)].

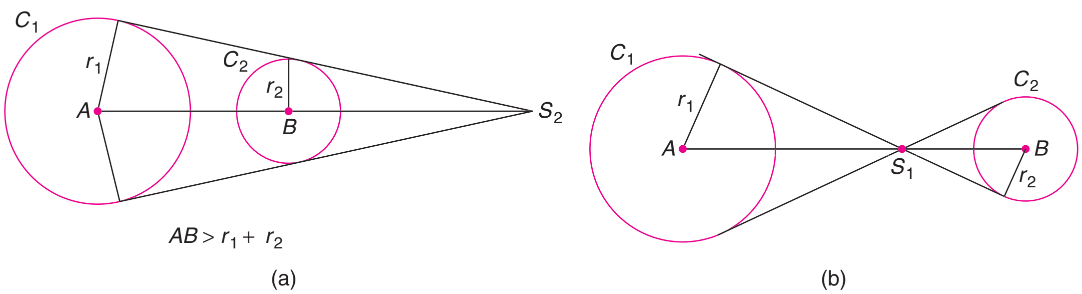

FIGURE 3.41 (a) Direct common tangents. (b) Transverse common tangents.

Case 2: Suppose circles C_1 and C_2 touch each other externally. Then the number of common tangents is three. Two tangents are direct and one tangent is transverse (see Fig. 3.42).

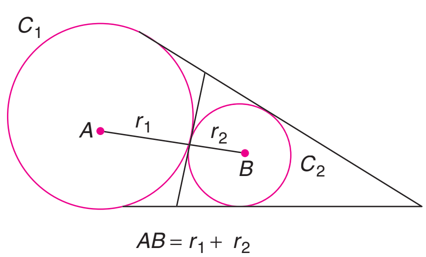

FIGURE 3.42

Case 3: Suppose circles C_1 and C_2 intersect in two distinct points. Then the number of common tangents is two which are direct. In this case, $|r_1 - r_2| < AB < r_1 + r_2$.

Case 4: Suppose circles C_1 and C_2 touch each other internally. That is, $AB = |r_1 - r_2|$. In this case, there is only one common tangent.

Case 5: If one of the two circles C_1 and C_2 completely lies within the other, there is no common tangent.

Example 3.17

Consider $S \equiv x^2 + y^2 - 14x + 6y + 33 = 0$ and $S' \equiv x^2 + y^2 + 30x - 2y + 1 = 0$. Find the number of common tangents and their equations.

Solution: The centre A and radius r_1 of $S = 0$ are $(7, -3)$ and 5. The centre B and radius r_2 of $S' = 0$ are $(-15, 1)$ and 15. The distance between the centres is given by

$$AB = \sqrt{(7+15)^2 + (-3-1)^2} = \sqrt{22^2 + 4^2} = \sqrt{500}$$
$$= 10\sqrt{5} > 5 + 15$$

Therefore, the two circles do not have any common point. Suppose S_2 is the external centre of similitude. Then S_2 divides BA in the ratio $15:5 = 3:1$ externally. Therefore

$$S_2 = \left(\frac{-1(-15)+3(7)}{2}, \frac{-1(1)+3(-3)}{2}\right) = (18, -5)$$

Let $y + 5 = m(x - 18)$ touch $S = 0$. Therefore

$$\left|\frac{m(7-18)+3-5}{\sqrt{1+m^2}}\right| = 5$$

$$\Rightarrow (11m+2)^2 = 25(1+m^2)$$
$$\Rightarrow 96m^2 + 44m - 21 = 0$$
$$\Rightarrow 96m^2 + 72m - 28m - 21 = 0$$
$$\Rightarrow 24m(4m+3) - 7(4m+3) = 0$$
$$\Rightarrow (24m-7)(4m+3) = 0$$
$$\Rightarrow m = \frac{-3}{4}, \frac{7}{24}$$

Hence, the direct common tangents are

$$y + 5 = \left(\frac{-3}{4}\right)(x - 18)$$

and
$$y + 5 = \left(\frac{7}{24}\right)(x - 18)$$

Suppose S_1 is the internal centre of similitude so that S_1 divides BA internally in the ratio $15:5 = 3:1$. Therefore,

$$S_1 = \left(\frac{3}{2}, -2\right)$$

Now, $y + 2 = m[x - (3/2)]$ touches $S = 0$ which implies that

$$\left|\frac{m(7-\frac{3}{2})+3-2}{\sqrt{1+m^2}}\right| = 5$$

$$\Rightarrow (11m+2)^2 = 100m^2 + 100$$
$$\Rightarrow 21m^2 + 44m - 96 = 0$$
$$\Rightarrow 21m^2 + 72m - 28m - 96 = 0$$
$$\Rightarrow 3m(7m+24) - 4(7m+24) = 0$$
$$\Rightarrow m = \frac{4}{3}, \frac{-24}{7}$$

Therefore, the transverse common tangents are

$$y + 2 = \frac{4}{3}\left(x - \frac{3}{2}\right)$$

and
$$y + 2 = \frac{-24}{7}\left(x - \frac{3}{2}\right)$$

Though the following theorem is not of much practical usage, we state it without proof.

THEOREM 3.21 (PAIR OF TANGENTS) If (x_1, y_1) is an external point to a circle $S = 0$, then the combined equation of the pair of tangents drawn from (x_1, y_1) to the circle is

$$S_1^2 = SS_{11}$$ ■

Note: If $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ is a circle and (x_1, y_1) lies outside the circle, then the combined equation of the pair of tangents is

$$[xx_1 + yy_1 + g(x+x_1) + f(y+y_1) + c]^2 = (x^2 + y^2 + 2gx + 2fy + c)(x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c)$$

Subjective Problems (Sections 3.4 and 3.5)

1. Find the equations of the tangents to the circle $S \equiv x^2 + y^2 + 8x + 4y - 5 = 0$ from the point $(3, -3)$ and also write their combined equation.

Solution: Let $y + 3 = m(x - 3)$ be a tangent to $S = 0$. The centre of the circle is $(-4, -2)$ and its radius is 5. Therefore

$$\left|\frac{m(-4-3)+2-3}{\sqrt{1+m^2}}\right| = 5$$

$$\Rightarrow (7m+1)^2 = 25(1+m^2)$$

$$\Rightarrow 24m^2 + 14m - 24 = 0$$

$$\Rightarrow 12m^2 + 7m - 12 = 0$$

$$\Rightarrow 12m^2 + 16m - 9m - 12 = 0$$

$$\Rightarrow 4m(3m+4) - 3(3m+4) = 0$$

$$\Rightarrow m = \frac{3}{4}, \frac{-4}{3}$$

Therefore, the tangents are given by $y + 3 = (3/4)(x - 3)$ or $3x - 4y - 21 = 0$ and $y + 3 = (-4/3)(x - 3)$ or $4x + 3y - 3 = 0$. Combined equation is $(3x - 4y - 21)(4x + 3y - 3) = 0$.

Note: The two tangents from $(3, -3)$ are at right angles and hence $(3, -3)$ lies on the director circle of $S = 0$. One can check this fact.

2. Find the equations of the tangents drawn from origin to the circle $S \equiv x^2 + y^2 - 6x - 2y + 8 = 0$.

Solution: $S_{11} = 8 > 0 \Rightarrow (0, 0)$ is external point to $S = 0$. The centre of the circle is $(3, 1)$ and its radius is $\sqrt{2}$. We know that $y = mx$ touches the circle. So

$$\frac{|m(3)-1|}{\sqrt{1+m^2}} = \sqrt{2}$$

$$\Leftrightarrow (3m-1)^2 = 2(1+m^2)$$

$$\Leftrightarrow 7m^2 - 6m - 1 = 0$$

$$\Leftrightarrow (m-1)(7m+1) = 0$$

Therefore

$$m = 1, \frac{-1}{7}$$

Hence, the tangents are $y = x$ and $x + 7y = 0$.

3. Let the circles $S \equiv x^2 + y^2 - 2cy - a^2 = 0$ and $S' \equiv x^2 + y^2 - 2bx + a^2 = 0$ whose centres are A and B, respectively, intersect at points P and Q. Show that the points P, Q, A, B and origin are concyclic.

Solution: Let $L \equiv S - S' \equiv bx - cy - a^2 = 0$. The required circle equation is of the form

$$S + \lambda L \equiv x^2 + y^2 - 2cy - a^2 + \lambda(bx - cy - a^2) = 0$$

(See point 3 of Quick Look 9). This circle passes through origin $\Leftrightarrow -a^2 - \lambda a^2 = 0 \Leftrightarrow \lambda = -1$. The required circle is

$$x^2 + y^2 - 2cy - a^2 - (bx - cy - a^2) = 0$$

$$\Rightarrow x^2 + y^2 - bx - cy = 0$$

which also passes through the centres $(0, c)$ and $(b, 0)$ of the circles $S = 0$ and $S' = 0$, respectively.

4. Suppose the lengths of the tangents drawn from $A(x_1, y_1)$ and $B(x_2, y_2)$ to the circle $x^2 + y^2 = a^2$ are of lengths l_1 and l_2, respectively, and $x_1x_2 + y_1y_2 = a^2$. Then show that $(AB)^2$ is equal to $l_1^2 + l_2^2$.

Solution: We have

$$l_1^2 = S_{11} = x_1^2 + y_1^2 - a^2 \text{ and } l_2^2 = x_2^2 + y_2^2 - a^2$$

Therefore

$$\begin{aligned}(AB)^2 &= (x_1 - x_2)^2 + (y_1 - y_2)^2 \\ &= (x_1^2 + y_1^2) + (x_2^2 + y_2^2) - 2(x_1x_2 + y_1y_2) \\ &= (x_1^2 + y_1^2) + (x_2^2 + y_2^2) - 2a^2 \qquad \text{(by hypothesis)} \\ &= (x_1^2 + y_1^2 - a^2) + (x_2^2 + y_2^2 - a^2) \\ &= l_1^2 + l_2^2\end{aligned}$$

5. Find the equation of a circle which bisects the circumferences of the circles $x^2 + y^2 = 1$, $x^2 + y^2 + 2x = 3$ and $x^2 + y^2 + 2y = 3$.

Solution: Let the given circles be C_1, C_2 and C_3, respectively. Let C be the required circle and its equation be $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$. Let $S' \equiv x^2 + y^2 - 1 = 0$. Since C bisects the circumference of C_1, the line $S - S' \equiv 2gx + 2fy + c + 1 = 0$ passes through the centre of $C_1 = (0, 0)$. Therefore,

$$c = -1$$

Now, C bisects the circumference of $C_2 \Rightarrow$ the line $S - S'' \equiv 2(g - 1)x + 2fy + c + 3 = 0$ passes through the centre $(-1, 0)$ of C_2. Therefore

$$2(g-1)(-1) - 1 + 3 = 0$$

$$\Rightarrow -2g + 2 - 1 + 3 = 0$$

$$\Rightarrow g = 2$$

Similarly, since the circle C bisects the circumference of circle C_3, we have $f = 2$. Therefore, equation of the required circle C is

$$S \equiv x^2 + y^2 + 4x + 4y - 1 = 0$$

6. Find the equation of a circle that passes through the point (1, 2) which bisects the circumference of the circle $x^2 + y^2 = 9$ and is orthogonal to the circle $x^2 + y^2 - 2x + 8y - 7 = 0$.

Solution: Let

$$S' \equiv x^2 + y^2 - 9 = 0$$
$$S'' \equiv x^2 + y^2 - 2x + 8y - 7 = 0$$

Let $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ be the required circle. Since $S = 0$ passes through $(1, 2)$, we have

$$2g + 4f + c = -5 \tag{3.36}$$

$S = 0$ bisects the circumference of $S' = 0 \Rightarrow S - S' = 0$ passes through $(0, 0)$. This gives

$$c + 9 = 0$$
$$\Rightarrow c = -9 \tag{3.37}$$

$S = 0$ and $S'' = 0$ cut orthogonally which implies that

$$2g(-1) + 2f(4) = c - 7$$
$$\Rightarrow -2g + 8f = c - 7 \tag{3.38}$$

From Eqs. (3.36)–(3.38), we have

$$2g + 4f = 9 - 5 = 4$$
and $$-2g + 8f = -9 - 7 = -16$$

Therefore, $g = 4, f = -1$, and $c = -9$. Hence the required circle is

$$S \equiv x^2 + y^2 + 8x - 2y - 9 = 0$$

7. Find the equation of the circle which passes through origin, has its centre on the line $x + y = 4$ and cuts orthogonally the circle $x^2 + y^2 - 4x + 2y + 4 = 0$.

Solution: Let $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ be the required circle. $S = 0$ passes through $(0,0) \Rightarrow c = 0$. The centre $(-g, -f)$ lies on the line $x + y = 4$ implies that

$$-g - f = 4 \tag{3.39}$$

$S = 0$ cuts the circle $S' \equiv x^2 + y^2 - 4x + 2y + 4 = 0$ implies that

$$2g(-2) + 2f(1) = c + 4$$
$$\Rightarrow -4g + 2f = c + 4$$
$$\Rightarrow -4g + 2f = 4 \qquad (3.40)\ (\because c = 0)$$

Solving Eqs. (3.39) and (3.40), we have $g = -2$ and $f = -2$. Therefore

$$S \equiv x^2 + y^2 - 4x - 4y = 0$$

8. Find the equation of the locus of the centres of all circles which touch the line $x = 2a$ and cut orthogonally the circle $x^2 + y^2 = a^2$.

Solution: Let $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ be the required circle. Now $S = 0$ touches the line $x = 2a$ implies

$$|-g - 2a| = \sqrt{g^2 + f^2 - c} \tag{3.41}$$

$S = 0$ cuts orthogonally $x^2 + y^2 - a^2 = 0$ implies

$$2g(0) + 2f(0) = c - a^2 \Rightarrow c = a^2 \tag{3.42}$$

From Eqs. (3.41) and (3.42),

$$(g + 2a)^2 = g^2 + f^2 - a^2$$
$$\Rightarrow y^2 - g^2 + 4a^2 + 4ag = g^2 + f^2 - a^2$$
$$f^2 + 4a(-g) - 5a^2 = 0$$

Therefore, the locus of the centre $(-g, -f)$ is given by

$$y^2 + 4ax - 5a^2 = 0$$
$$\Rightarrow y^2 = -4a\left(x - \frac{5a}{4}\right)$$

9. Show that the equation of a straight line meeting the circle $x^2 + y^2 = a^2$ in two points at equal distance d from a point (x_1, y_1) on the circumference is $2(xx_1 + yy_1 - a^2) + d^2 = 0$.

Solution: Let $S \equiv x^2 + y^2 - a^2 = 0$. See Fig. 3.43. The required line is the common chord of the circle $S = 0$ and $S' \equiv (x - x_1)^2 + (y - y_1)^2 = d^2$. Therefore, the line equation is given by

$$S - S' \equiv -a^2 + 2xx_1 + 2yy_1 - (x_1^2 + y_1^2) + d^2 = 0$$
$$\Rightarrow 2xx_1 + 2yy_1 - 2a^2 + d^2 = 0 \qquad (\because x_1^2 + y_1^2 = a^2)$$

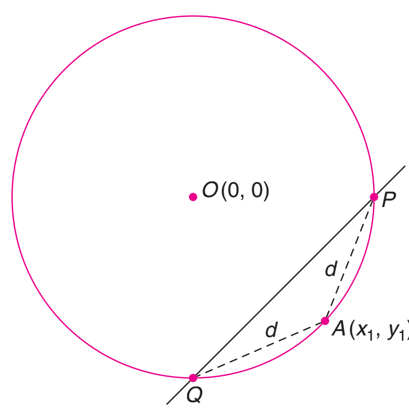

FIGURE 3.43

10. Prove that the two circles $x^2 + y^2 + 2ax + c = 0$ and $x^2 + y^2 + 2by + c = 0$ touch each other if and only if

$$\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c}$$

Solution: Let

$$S \equiv x^2 + y^2 + 2ax + c = 0$$
$$S' \equiv x^2 + y^2 + 2by + c = 0$$

The two circles touch each other $\Leftrightarrow S - S' = 0$ is the common tangent at the point of contact. This implies that the distance of the centre of one of the circles from the line $S - S' = 0$ is equal to the radius of the circle. Therefore

$$\frac{|a(-a)-b(0)|}{\sqrt{a^2+b^2}} = \sqrt{a^2-c} \qquad (\because S - S' \equiv ax - by = 0)$$

$$\Rightarrow a^4 = (a^2+b^2)(a^2-c)$$

$$\Rightarrow a^4 = a^4 - c(a^2+b^2) + a^2b^2$$

$$\Rightarrow c(a^2+b^2) = a^2b^2$$

$$\Rightarrow \frac{a^2+b^2}{a^2b^2} = \frac{1}{c}$$

$$\Rightarrow \frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c}$$

11. Find the equation of the circle which cuts orthogonally each of the circles

$$S' \equiv x^2 + y^2 + 2x + 17y + 4 = 0$$

$$S'' \equiv x^2 + y^2 + 7x + 6y + 11 = 0$$

$$S''' \equiv x^2 + y^2 - x + 22y + 3 = 0$$

Solution: Radical axis of $S' = 0$ and $S'' = 0$ is $S' - S'' \equiv 5x - 11y + 7 = 0$ and the radical axis of $S'' = 0$ and $S''' = 0$ is $S'' - S''' \equiv 8x - 16y + 8 = 0$. That is,

$$5x - 11y = -7$$

and

$$x - 2y = -1$$

Solving these equations, we obtain the radical centre as (3, 2). If t is the length of the tangent from (3, 2) to the circle $S' = 0$, we have

$$t = \sqrt{S'_{11}} = \sqrt{9+4+6+34+4} = \sqrt{57}$$

Therefore, the required circle is $(x-3)^2 + (y-2)^2 = 57$.

12. Prove that the locus of the midpoint of a system of parallel chords of a circle is a diameter of the circle.

Solution: Let $S \equiv x^2 + y^2 - a^2 = 0$. Let $M(x_1, y_1)$ be the midpoint of a chord parallel to a line $lx + my + n = 0$. Equation of the chord in terms of (x_1, y_1) as its midpoint is

$$xx_1 + yy_1 = x_1^2 + y_1^2$$

But this is parallel to $lx + my + n = 0$. This implies $lx_1 - my_1 = 0$. Therefore, the locus of $M(x_1, y_1)$ is $lx - my = 0$ which passes through the centre (0, 0).

13. Prove that any two circles with different centres can be in the form $x^2 + y^2 + 2ax + c = 0$ and $x^2 + y^2 + 2bx + c = 0$, c is same for both the equations.

Solution: Let C_1 and C_2 be two circles with A and B as their centres. Take the line AB as x-axis, their radical axis as y-axis. Their centres lie on the x-axis. Therefore, we can write their equations as

$$S \equiv x^2 + y^2 + 2ax + c = 0$$

and

$$S' \equiv x^2 + y^2 + 2bx + c' = 0$$

Since the origin lies on the radical axis of the circles C_1 and C_2, its powers with respect to the circles are equal. Hence, $C = C_1$. Therefore, the equations of C_1 and C_2 are $S \equiv x^2 + y^2 + 2ax + c = 0$ and $S' \equiv x^2 + y^2 + 2bx + c = 0$.

14. Find the equation of the circle whose centre lies on the line $x + y - 11 = 0$ and which passes through the intersection of the circle $x^2 + y^2 - 3x + 2y - 4 = 0$ and the line $2x + 5y + 2 = 0$.

Solution: Let $S \equiv x^2 + y^2 - 3x + 2y - 4 = 0$ and $L \equiv 2x + 5y + 2 = 0$. The required circle equation is of the form

$$S + \lambda L \equiv x^2 + y^2 - 3x + 2y - 4 + \lambda(2x + 5y + 2) = 0$$

$$\equiv x^2 + y^2 - (3 - 2\lambda)x + (2 + 5\lambda)y - 4 + 2\lambda = 0$$

The centre $[(3 - 2\lambda)/2, -(2 + 5\lambda)/2]$ lies on the line

$$x + y - 11 = 0$$

$$\Rightarrow \frac{3-2\lambda}{2} - \frac{2+5\lambda}{2} - 11 = 0$$

$$\Rightarrow -7\lambda - 21 = 0$$

$$\Rightarrow \lambda = -3$$

Therefore, the required circle is

$$x^2 + y^2 - 3x + 2y - 4 - 3(2x + 5y + 2) = 0$$

$$\Rightarrow x^2 + y^2 - 9x - 13y - 10 = 0$$

15. A circle touches both axes and also touches the line $4x + 3y - 6 = 0$ and lies in the first quadrant. Find the equation of the circle.

Solution: $(x-h)^2 + (y-h)^2 = h^2$ $(h > 0)$ touches both positive axes. This circle also touches the line $4x + 3y - 6 = 0$. Therefore

$$\frac{|4(h) + 3h - 6|}{\sqrt{4^2+3^2}} = h$$

$$\Rightarrow (7h-6)^2 = 25h^2$$

$$\Rightarrow (7h - 6 + 5h)(7h - 6 - 5h) = 0$$

$$\Rightarrow (12h - 6)(2h - 6) = 0$$

$$\Rightarrow h = \frac{1}{2}, \frac{3}{2}$$

Therefore, the equations of the circles are given by

$$\left(x-\frac{1}{2}\right)^2+\left(y-\frac{1}{2}\right)^2=\frac{1}{4} \tag{3.43}$$

and

$$\left(x-\frac{3}{2}\right)^2+\left(y-\frac{3}{2}\right)^2=\frac{9}{4} \tag{3.44}$$

Note: If the line $4x + 3y - 6 = 0$ cuts the axes in A and B, then Eq. (3.43) represents the incircle of $\Delta\, OAB$ and Eq. (3.44) represents the excircle opposite to the vertex O.

16. Determine the equation of the circle which touches the line $y = x$ at the origin and bisects the circumference of the circle $x^2 + y^2 + 2y - 3 = 0$.

Solution: Let $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ be the required circle. It passes through $(0, 0)$. This implies $c = 0$. Now $S = 0$ touches the line $x - y = 0$. Therefore

$$\left|\frac{-g+f}{\sqrt{2}}\right| = \sqrt{g^2+j}$$

$$\Rightarrow (g+f)^2 = 0$$

$$\Rightarrow g = -f$$

If $S = 0$ bisects the circumference of $S' \equiv x^2 + y^2 + 2y - 3 = 0$, then $S - S' = 0$ passes through the centre $(0, -1)$ of $S' = 0$. This implies that

$$2gx - 2(g+1)y + 3 = 0$$

$$\text{passes through } (0,-1) \qquad (\because f = -g)$$

$$\Rightarrow 0 - 2(g+1)(-1) + 3 = 0$$

$$\Rightarrow 2g = -5 = -2f$$

Therefore, $S \equiv x^2 + y^2 - 5x + 5y = 0$.

17. Find the equation of the described on the common chord of the circles $(x - a)^2 + y^2 = a^2$ and $x^2 + (y - b)^2 = b^2$ as diameter.

Solution: Let $S \equiv x^2 + y^2 - 2ax = 0$ and $S' \equiv x^2 + y^2 - 2by = 0$. Common chord equation is

$$L \equiv S - S' \equiv ax - by = 0$$

Equation of the circle passing through the intersection of $S = 0$ and $S' = 0$ is of the form

$$S + \lambda L \equiv x^2 + y^2 - 2ax + \lambda(ax - by) = 0$$

whose centre is $(a - \lambda a/2, \lambda b/2)$. Since the common chord is a diameter, we have $a(a - \lambda a/2) - b(\lambda b/2) = 0$. Therefore

$$\lambda(a^2 + b^2) = 2a^2$$

$$\Rightarrow \lambda = \frac{2a^2}{a^2+b^2}$$

Hence, the required circle is

$$(a^2 + b^2)(x^2 + y^2) = 2ab(bx + ay)$$

18. On the line $x + 5y - 22 - \lambda(x - 8y + 30) = 0$, the circle $S \equiv x^2 + y^2 - 2x + 2y - 14 = 0$ makes an interrupt of length $2\sqrt{3}$. Find the value(s) of λ and the line(s) equations.

Solution: Centre and radius of the circle are $(1, -1)$ and 4 (see Fig. 3.44). The given line equation is

$$(1-\lambda)x + (5+8\lambda)y - 22 - 30\lambda = 0$$

$$\Rightarrow \frac{|(1-\lambda)(1) + (5+8\lambda)(-1) - 22 - 30\lambda|}{\sqrt{(1-\lambda)^2 + (5+8\lambda)^2}} = \sqrt{16-3}$$

(by hypothesis)

$$\Rightarrow \frac{|-39\lambda - 26|}{\sqrt{65\lambda^2 + 78\lambda + 26}} = \sqrt{13}$$

$$\Rightarrow (3\lambda+2)^2 = 5\lambda^2 + 6\lambda + 2$$

$$\Rightarrow 4\lambda^2 + 6\lambda + 2 = 0$$

$$\Rightarrow 2\lambda^2 + 3\lambda + 1 = 0$$

$$\Rightarrow (2\lambda+1)(\lambda+1) = 0$$

$$\Rightarrow \lambda = -1, \frac{-1}{2}$$

When $\lambda = -1$, the equation of the line is $2x - 3y + 8 = 0$ and when $\lambda = -1/2$, the equation of the line is $3x + 2y - 14 = 0$.

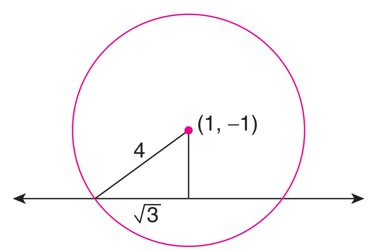

FIGURE 3.44

19. Find the equation of the circle which passes through the point $(1, 1)$ and which touches the circle

$$x^2 + y^2 + 4x - 6y - 3 = 0$$

at the point $(2, 3)$ on it.

Solution: Let $S \equiv x^2 + y^2 + 4x - 6y - 3 = 0$. The equation of the tangent to $S = 0$ at $(2, 3)$ is

$$S_1 \equiv x(2) + y(3) + 2(x+2) - 3(y+3) - 3 = 0$$

$$\Rightarrow S_1 \equiv 4x - 8 = 0 \Rightarrow x - 2 = 0$$

Any circle touching the circle $S = 0$ and having $x - 2 = 0$ as a common tangent is of the form $S + \lambda(x - 2) = 0$ is

$$x^2 + y^2 + 4x - 6y - 3 + \lambda(x - 2) = 0$$

This circle passes through (1, 1) implies that

$$1+1+4-6-3+\lambda(1-2)=0$$
$$\Rightarrow -3-\lambda=0$$
$$\Rightarrow \lambda=-3$$

Therefore, the required circle is

$$x^2 + y^2 + 4x - 6y - 3 - 3(x - 2) = 0$$
$$\Rightarrow x^2+y^2+x-6y+3=0$$

20. Find the equation of the circle which touches the line $3x - y - 6 = 0$ at the point (1, −3) and having radius $2\sqrt{10}$.

Solution: $S \equiv (x - 1)^2 + (y + 3)^2 = 0$ is a point circle and $L \equiv 3x - y - 6 = 0$. Therefore, required circle is of the form

$$S+\lambda L \equiv x^2+y^2-2x+6y+10+\lambda(3x-y-6)=0$$
$$\equiv x^2+y^2-(2-3\lambda)x-(\lambda-6)y+10-6\lambda=0$$

Radius of the circle $= 2\sqrt{10}$ implies that

$$\left(\frac{2-3\lambda}{2}\right)^2+\left(\frac{\lambda-6}{2}\right)^2-10+6\lambda=40$$
$$\Rightarrow (2-3\lambda)^2+(\lambda-6)^2-40+24\lambda=-160$$
$$\Rightarrow 10\lambda^2=160$$
$$\Rightarrow \lambda=\pm 4$$

Hence, the equations of the circle are given by

$$x^2+y^2+10x+2y-14=0 \quad (\text{when } \lambda=4)$$
$$x^2+y^2-14x+10y+34=0 \quad (\text{when } \lambda=-4)$$

21. Let T_1 and T_2 be two tangents drawn from (−2, 0) to the circle $C: x^2 + y^2 = 1$. Determine the circles touching C and having T_1 and T_2 as their pair of tangents. Also find the equations of all possible common tangents to those circles when taken two by two.

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Solution: Any line through (−2, 0) is of the form $y = m(x + 2) = mx + 2m$. This touches the circle C. Now

$$x^2+y^2=1$$
$$\Rightarrow (2m)^2=1(1+m^2) \qquad (\text{by Theorem 3.8})$$
$$\Rightarrow m=\pm\frac{1}{\sqrt{3}}$$

The tangents are obtained as

$$y=\frac{1}{\sqrt{3}}(x+2) \Rightarrow x-\sqrt{3}y+2=0$$

and $$y=\frac{-1}{\sqrt{3}}(x+2) \Rightarrow x+\sqrt{3}y+2=0$$

Therefore, T_1 is given by

$$x-\sqrt{3}y+2=0$$

and T_2 is given by

$$x+\sqrt{3}y+2=0$$

Any circle touching C and the tangents T_1 and T_2 must have its centre on the x-axis and also touch C at (1, 0) and (−1, 0) only (see Fig. 3.45). Equation of the tangent to C at (1, 0) is $x - 1 = 0$. Equation of the circle touching C at (1, 0) is of the form $x^2 + y^2 - 1 + \lambda(x - 1) = 0$ whose centre is $(-\lambda/2, 0)$ and radius $\sqrt{(\lambda^2/4)+\lambda+1}$. If this circle also touches $x - \sqrt{3}y + 2 = 0$, we have

$$\frac{\left|\frac{-\lambda}{2}-\sqrt{3}(0)+2\right|}{\sqrt{1+3}}=\sqrt{\frac{\lambda^2}{4}+\lambda+1}$$
$$\Rightarrow \frac{|4-\lambda|}{4}=\frac{1}{2}\sqrt{(\lambda+2)^2}$$
$$\Rightarrow 4-\lambda=\pm 2(\lambda+2)$$
$$\Rightarrow 4-\lambda=2\lambda+4 \Rightarrow \lambda=0$$

which corresponds to C. Therefore,

$$4-\lambda=-2(\lambda+2) \Rightarrow \lambda=-8$$

and the circle is

$$x^2+y^2-1-8(x-1)=0$$
$$\Rightarrow x^2+y^2-8x+7=0$$

Similarly, any circle touching C at (−1, 0) is of the form $x^2 + y^2 - 1 + \lambda(x - 1) = 0$. The circle also touches the tangent $T_1 \Rightarrow \lambda = 8/3$ and the circle is $[x + (4/3)]^2 + y^2 = 1/9$.

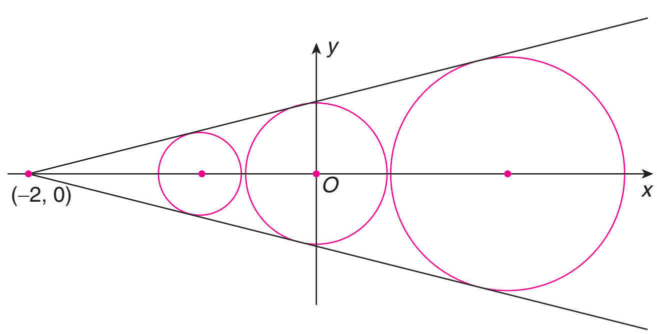

FIGURE 3.45

The remaining part is left as an exercise for the students.

22. C_1 and C_2 are two concentric circles. The radius of C_2 is twice that of C_1. From a point P on C_2, tangents PA and PB are drawn to C_1. Prove that the centroid of

$\Delta\, PAB$ lies on C_1. **(IIT-JEE 1998)**

Solution: Let the circle C_1 be $x^2 + y^2 = a^2$. So the equation of C_2 must be $x^2 + y^2 = 4a^2$. Let $P(h, k)$ be on C_2 so that

$$h^2 + k^2 = 4a^2 \tag{3.45}$$

Equation of AB is

$$hx + ky = a^2 \tag{3.46}$$

Substituting $y = (a^2 - hx)/k$ in $x^2 + y^2 = a^2$, we get

$$x^2 + \left(\frac{a^2 - hx}{k}\right)^2 = a^2$$

$$\Rightarrow (h^2 + k^2)x^2 - 2a^2hx + (a^2 - k^2)a^2 = 0$$

$$\Rightarrow 4a^2x^2 - 2a^2hx + (a^2 - k^2)a^2 = 0 \quad \text{[from Eq. (3.45)]}$$

Therefore, $4x^2 - 2hx + a^2 - k^2 = 0$ has two distinct roots, say x_1 and x_2. Hence

$$x_1 + x_2 = \frac{2h}{4} = \frac{h}{2}$$

$$x_1x_2 = \frac{a^2 - k^2}{4}$$

Therefore

$$\begin{aligned} y_1 + y_2 &= \frac{a^2 - hx_1}{k} + \frac{a^2 - hx_2}{k} \\ &= \frac{2a^2 - h(x_1 + x_2)}{k} \\ &= \frac{2a^2 - (h^2/2)}{k} \\ &= \frac{4a^2 - h^2}{2k} \\ &= \frac{h^2 + k^2 - h^2}{2k} = \frac{k}{2} \quad \text{[from Eq. (3.45)]} \end{aligned}$$

Suppose $G(x, y)$ is the centroid of $\Delta\, PAB$ (see Fig. 3.46). In such case

$$x = \frac{x_1 + x_2 + h}{3} = \frac{(h/2) + h}{3} = \frac{h}{2}$$

and $$y = \frac{y_1 + y_2 + k}{3} = \frac{k}{2}$$

Therefore

$$x^2 + y^2 = \frac{1}{4}(h^2 + k^2) = \frac{1}{4}(4a^2) \quad \text{[from Eq. (3.45)]}$$

Hence, the centroid $G(x, y)$ lies on C_1.

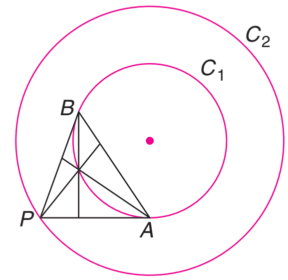

FIGURE 3.46

23. The chord of contact of a point on the circle $x^2 + y^2 = a^2$ with respect to the circle $x^2 + y^2 = b^2$ touches the circle $x^2 + y^2 = c^2$. Show that $b^2 = ac$ (i.e. a, b, c are in GP).

Solution: By hypothesis, we have $a > b > c$. Let $P(x_1, y_1)$ be a point on $x^2 + y^2 = a^2$ so that

$$x_1^2 + y_1^2 = a^2 \tag{3.47}$$

The equation of the chord of contact of $P(x_1, y_1)$ with respect to the circle $x^2 + y^2 = b^2$ is

$$xx_1 + yy_1 - b^2 = 0 \tag{3.48}$$

The line given in Eq. (3.48) touches the circle $x^2 + y^2 = c^2$. This gives

$$\frac{|0x_1 + 0y_1 - b^2|}{\sqrt{x_1^2 + y_1^2}} = c$$

$$\Rightarrow b^2 = \sqrt{x_1^2 + y_1^2} \cdot c = ac \quad \text{[from Eq. (3.47)]}$$

24. Prove that the equation $x^2 + y^2 - 2x - 2ay - 8 = 0$ represents a family of circles intersecting in two points P and Q. Determine a member of this family of circles such that tangents drawn to this circle at P and Q intersect on the line $x + 2y + 5 = 0$.

Solution: The given equation is $(x^2 + y^2 - 2x - 8) - (2a)y = 0$ which is of the form $S + \lambda L = 0$, where $S = 0$ is the circle $x^2 + y^2 - 2x - 8 = 0$ and $L = 0$ is the line $y = 0$ (x-axis). The line $y = 0$ intersects the circle $x^2 + y^2 - 2x - 8 = 0$ at two points $P(-2, 0)$ and $Q(4, 0)$. Therefore, the given equation represents a family of circles all passing through $P(2, 0)$ and $Q(4, 0)$.

Since the tangents at points P and Q intersect at (x_1, y_1) which lies on the line $x + 2y + 5 = 0$, we get

$$x_1 + 2y_1 + 5 = 0 \tag{3.49}$$

Also the equation of the line PQ is given by

$$xx_1 + yy_1 - (x + x_1) - a(y + y_1) - 8 = 0$$

$$\Rightarrow (x_1 - 1)x + (y_1 - a)y - x_1 - ay_1 - 8 = 0 \tag{3.50}$$

However, actually the equation of PQ is

$$y = 0 \tag{3.51}$$

From Eqs. (3.50) and (3.51), we get $x_1 - 1 = 0$ and $x_1 + ay_1 + 8 = 0$. Therefore $x_1 = 1$ and since $x_1 + 2y_1 + 5 = 0$ [from Eq. (3.49)], we have $y_1 = -3$. Substituting $x_1 = 1$ and $y_1 = -3$ in $x_1 + ay_1 + 8 = 0$, we have

$$1 - 3a + 8 = 0 \Rightarrow a = 3$$

Hence, the required member of the family is

$$x^2 + y^2 - 2x - 6y - 8 = 0$$

25. Prove that the centre of a circle which cuts orthogonally two given circles lies on the radical axes of the two circles.

Solution: Let

$$S' \equiv x^2 + y^2 + 2g'x + 2f'y + c' = 0$$

$$S'' \equiv x^2 + y^2 + 2g''x + 2f''y + c'' = 0$$

be the two given circles. Suppose a circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ cuts orthogonally $S' = 0$ and $S'' = 0$. Therefore, by Theorem 3.17, we have

$$2gg' + 2ff' = c + c'$$

and

$$2gg'' + 2ff'' = c + c''$$

Therefore, $2(g' - g'')g + 2(f' - f'')f = c' - c''$. That is, the radical axis $S' - S'' = 0$ of $S' = 0$ and $S'' = 0$ passes through $(-g, -f)$.

26. The common chord of two intersecting circles subtends angles 90° and 60°, respectively, at the centres of the circles and the distance between the centres is $\sqrt{3} + 1$. Find the radii of the circles.

Solution: Let the common chord AB meet the line joining the centres O_1 and O_2 of the circles at point M (see Fig. 3.47). Then

$$\angle AO_1B = 90°, \quad \angle AO_2B = 60°$$

$$\Rightarrow \angle AO_1O_2 = 45°, \quad \angle AO_2O_1 = 30°$$

Therefore, $\angle O_1AO_2 = 105°$. Using sine value for ΔO_1AO_2, we get

$$\frac{O_1O_2}{\sin 105°} = \frac{O_1A}{\sin 30°} = \frac{O_2A}{\sin 45°}$$

$$O_1A = \frac{(\sqrt{3}+1)\sin 30°}{\sin 105°}$$

$$= \frac{(\sqrt{3}+1)(1/2)}{\cos 15°} = \frac{(\sqrt{3}+1)2\sqrt{2}}{2(\sqrt{3}+1)} = \sqrt{2} \Rightarrow r_1 = \sqrt{2}$$

$$O_2A = \frac{(\sqrt{3}+1)\sin 45°}{\sin 105°} = \frac{(\sqrt{3}+1)2\sqrt{2}}{\sqrt{2}(\sqrt{3}+1)} = 2 \Rightarrow r_2 = 2$$

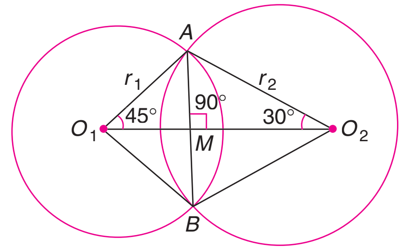

FIGURE 3.47

27. Find the centre of the smallest circle which cuts orthogonally the two circles $x^2 + y^2 = 1$ and $x^2 + y^2 + 8x + 8y - 33 = 0$.

Solution: Suppose $x^2 + y^2 + 2gx + 2fy + c = 0$ is the required circle. Since this circle cuts orthogonally the given two circles, we have $c = 11$ and $g + f = -4$. The radius of the circle is

$$\sqrt{g^2 + f^2 - c} = \sqrt{g + (g+4)^2 - 1}$$

$$= \sqrt{2g^2 + 8g + 15} = \sqrt{2(g+2)^2 + 7}$$

which is minimum if and only if $g = -2$. Hence, the centre is (2, 2).

28. The tangents to a circle at two points P and Q meet at point T. The lines joining points P and Q to one extremity of the diameter parallel to PQ meet the diameter perpendicular to PQ at points R and S. Prove that $RT = ST$.

Solution: Without loss of generality, let us consider that the circle as $x^2 + y^2 = a^2$ and let $P(h, k)$ and $Q(h, -k)$ be the ends of the chord PQ (see Fig. 3.48). Equation of the tangent at $P(h, k)$ is $hx + ky - a^2 = 0$ so that $T = (a^2/h, 0)$. The equation of the line joining $P(h, k)$ and $B(0, a)$ is

$$y - a = \frac{a-k}{-h}(x - 0) \tag{3.52}$$

Substituting $y = 0$ in Eq. (3.52) we get $R = (ah/a - k, 0)$. Similarly, the line QB meets the x-axis at $S(ah/a + k, 0)$. Let $M = (x, 0)$ be the midpoint of RS. Then

$$x = \frac{[ah/(a-k)] + [ah/(a+k)]}{2}$$

$$= \frac{a^2h}{a^2 - k^2} = \frac{a^2h}{h^2} \qquad (\because h^2 + k^2 = a^2)$$

$$= \frac{a^2}{h}$$

Therefore, $M = T$ and hence $RT = ST$.

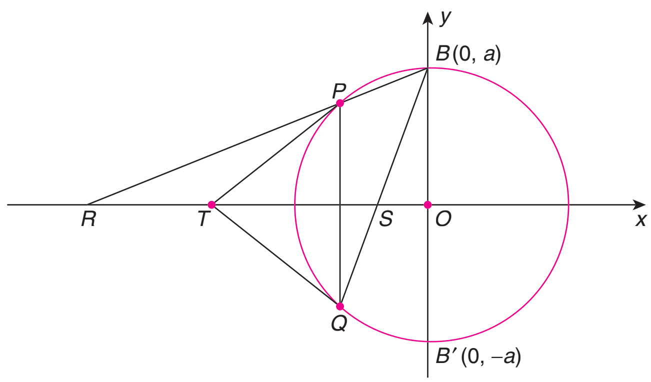

FIGURE 3.48

29. Suppose two circles pass through the points $(0, a)$ and $(0, -a)$ and touch the line $y = mx + c$. If the two circles cut orthogonally, then show that $c^2 = a^2(2 + m^2)$.

Solution: The centre of a circle passing through the points $(0, -a)$ and $(0, a)$ must lie on x-axis. Let the equation of any such circle be

$$S \equiv x^2 + y^2 + 2gx - a^2 = 0$$

$S = 0$ touches the line $y = mx + c$ implies that

$$\frac{|m(-g) - 0 + c|}{\sqrt{1+m^2}} = \sqrt{g^2 + a^2}$$
$$\Rightarrow m^2g^2 - 2cmg + c^2 = (g^2 + a^2)(1 + m^2)$$
$$\Rightarrow g^2 + 2cmg + a^2(1 + m^2) - c^2 = 0$$

Let g_1 and g_2 be two the roots of this quadratic equation. Hence

$$g_1 + g_2 = -2cm$$
$$g_1g_2 = a^2(a + m^2) - c^2$$

Now, the circles $x^2 + y^2 + 2g_1x - a^2$ and $x^2 + y^2 + 2g_2x - a^2 = 0$ cut orthogonally implies that

$$2g_1g_2 + 2f_1f_2 = c_1 + c_2$$
$$\Rightarrow 2g_1g_2 = -a^2 - a^2$$
$$\Rightarrow g_1g_2 = -a^2$$
$$\Rightarrow a^2(1 + m^2) - c^2 = -a^2$$
$$\Rightarrow a^2(2 + m^2) = c^2$$

30. Prove that the radical centre of the circles described on the sides of a triangle as diameters is the *orthocentre* of the triangle.

Solution: The circles described on AB and AC as diameters (see Fig. 3.49) will intersect at a point D on the side BC and because $\underline{|ADB} = \underline{|ADC} = 90°$. We have that AD is an altitude of ΔABC which is also the radical axis of the two circles with AB and AC as diameters. Similarly, the other altitudes BE and CF are the radical axes of the pairs described on AB, BC and AC, BC. Thus, the altitudes of ΔABC are radical axes. Hence, the orthocentre is their radical centre.

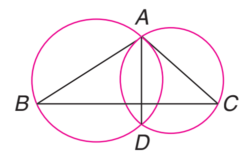

FIGURE 3.49

31. Show that two direct common tangents and the transverse common tangent to the two circles $S \equiv x^2 + y^2 - 6x = 0$ and $S' \equiv x^2 + y^2 + 2x = 0$ form an equilateral triangle.

Solution: $A(3, 0)$ and 3 are the centre and radius of $S = 0$ (see Fig. 3.50). $B(-1, 0)$ and 1 are the centre and radius of $S' = 0$. Now $AB = 4 = 3 + 1 \Rightarrow S = 0$ and S' touch each other externally at $(0, 0)$ and $x = 0$ (i.e., y-axis) is the common tangent at the origin. Let T be the external centre of similitude. Therefore, T divides the line joining the centres A and B externally in the ratio 3:1. That is,

$$AT : TB = 3 : -1$$

or
$$T = \left(\frac{3(-1) + (-1)3}{3 - 1}, 0\right) = (-3, 0)$$

Let $y = m(x + 3)$ be a tangent from $T(-3, 0)$ to $S' = 0$. Therefore

$$\frac{|m(-1+3) - 0|}{\sqrt{1+m^2}} = 1$$
$$\Rightarrow 4m^2 = 1 + m^2$$
$$\Rightarrow m = \pm\frac{1}{\sqrt{3}}$$

Hence, the tangents from $T(-3, 0)$ are $x - \sqrt{3}\,y + 3 = 0$ and $x + \sqrt{3}\,y + 3 = 0$. Therefore, the vertices of the triangle are $P(0, \sqrt{3})$, $T(-3, 0)$ and $Q(0, -\sqrt{3})$, which form an equilateral triangle.

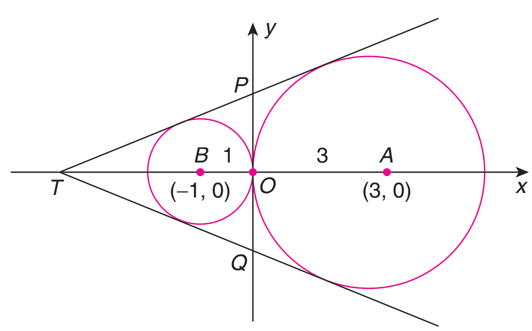

FIGURE 3.50

32. Tangents PQ and PR are drawn to the circle $S \equiv x^2 + y^2 + - a^2 = 0$ from the point $P(x_1, y_1)$ to touch the circle at Q and R. Determine the equation of the circumcircle of ΔPQR.

Solution: QR being the chord of contact of $P(x_1, y_1)$ with respect to the circle $S = 0$, its equation is $S_1 \equiv xx_1 + yy_1 - a^2 = 0$ Any circle passing through Q and R is of the form $S + \lambda L = 0$ where $S \equiv x^2 + y^2 + - a^2 = 0$ and $L \equiv xx_1 + yy_1 - a^2 = 0$. Therefore,

$$S + \lambda L \equiv x^2 + y^2 + - a^2 + \lambda(xx_1 + yy_1 - a^2) = 0$$

which is the circumcircle of ΔPQR, with the condition that it passes through $P(x_1, y_1)$. Therefore

$$x_1^2 + y_1^2 - a^2 + \lambda(x_1^2 + y_1^2 - a^2) = 0$$
$$\lambda = -1$$

Therefore, the equation of the circumcircle of ΔPQR is $x^2 + y^2 - xx_1 - yy_1 = 0$.

33. If $4l^2 - 5m^2 + 6l + 1 = 0$, then show that the line $lx + my + 1 = 0$ touches a fixed circle. Find the equation of the circle.

Solution: Suppose the line $lx + my + 1 = 0$ touches circle $(x - h)^2 + (y - k)^2 = a^2$. So

$$\frac{|lh + mk + 1|}{\sqrt{l^2 + m^2}} = a$$
$$\Rightarrow (hl + km + 1)^2 = a^2(l^2 + m^2)$$

Therefore

$$l^2(h^2 - a^2) + m^2(k^2 - a^2) + 2hklm + 2hl + 2km + 1 = 0 \tag{3.53}$$

However, by hypothesis,

$$4l^2 - 5m^2 + 6l + 1 = 0 \tag{3.54}$$

From Eqs. (3.53) and (3.54), we have

$$\frac{h^2 - a^2}{4} = \frac{k^2 - a^2}{-5} = \frac{2hk}{0} = \frac{2h}{6} = \frac{2k}{0} = \frac{1}{1}$$

Therefore, $k = 0, h = 3$ and $a = \sqrt{5}$. The circle equation is obtained as $(x - 3)^2 + y^2 = 5$.

34. Find the equation of the circle having the pair of lines $x^2 + 2xy + 3x + 6y = 0$ as its normals and having the size just sufficient to contain the circle $x(x - 4) + y(y - 3) = 0$.

Solution: We have

$$x^2 + 2xy + 3x + 6y \equiv x(x + 2y) + 3(x + 2y) \equiv (x + 2y)(x + 3)$$

The point of intersection of the lines = $(-3, 3/2)$, which is the centre of the required circle. Also the centre and radius of the circle $x(x - 4) + y(y - 3) = 1$ are $(2, 3/2)$ and $5/2$, respectively.

If the required circle is to just contain the circle with centre $(2, 3/2)$ and radius $5/2$, then this circle should touch the required circle internally. If r is the radius of the required circle the distance between their centres $(-3, 3/2)$ and $(2, 3/2)$ must be equal to $r - 5/2$ (since $r > 5/2$). Therefore

$$r - \frac{5}{2} = \sqrt{(2+3)^2 + 0} = 5$$
$$\Rightarrow r = 5 + \frac{5}{2} = \frac{15}{2}$$

Hence, the equation of the required circle is

$$(x+3)^2 + \left(y - \frac{3}{2}\right)^2 = \left(\frac{15}{2}\right)^2$$
$$\Rightarrow x^2 + y^2 + 6x - 3y = \frac{225}{4} - \frac{9}{4} - 9$$
$$\Rightarrow x^2 + y^2 + 6x - 3y = \frac{180}{4} = 45$$
$$\Rightarrow x^2 + y^2 + 6x - 3y - 45 = 0$$

35. From each point on the line $2x + y = 4$, tangents are drawn to the circle $x^2 + y^2 = 1$. Find the point through which those chords of contact pass. **(IIT-JEE 1997)**

Solution: Let $P(h, k)$ be a point on the line $2x + y = 4$. Therefore,

$$2h + k = 4 \tag{3.55}$$

The chord of contact of $P(h, k)$ with respect to the circle $x^2 + y^2 = 1$ is $hx + ky - 1 = 0$. That is,

$$hx + (4 - 2h)y - 1 = 0$$

From Eq. (3.55), we have $k = 4 - 2h$. Therefore

$$h(x - 2y) + 4y - 1 = 0$$

Hence these lines are concurrent at the point of intersection of the lines $x - 2y = 0$ and $4y - 1 = 0$ which is $(1/2, 1/4)$.

36. Consider a curve $ax^2 + 2hxy + by^2 = 1$ and a point P not on it. A line drawn from the point P intersects the curve at points Q and R. If the product $PQ \cdot PR$ is independent of the slope of the line, then show that the curve is a circle. **(IIT-JEE 1997)**

Solution: Let P be (x_1, y_1) and line through $P(x_1, y_1)$ (Fig. 3.51) be

$$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r \quad \text{(say)}$$

Substitute $x = x_1 + r\cos\theta$ and $y = y_1 + r\sin\theta$ in the equation $ax^2 + 2hxy + by^2 = 0$, so that we have

$$a(x_1 + r\cos\theta)^2 + 2h(x_1 + r\cos\theta)(y_1 + r\sin\theta)$$
$$+ b(y_1 + r\sin\theta)^2 - 1 = 0$$

That is,

$$r^2(a\cos^2\theta + 2h\cos\theta\sin\theta + b\sin^2\theta)$$
$$+2[(ax_1 + hy_1)\cos\theta + (hx_1 + by_1)\sin\theta]r + ax_1^2$$
$$+2hx_1y_1 + by_1^2 - 1 = 0 \qquad (3.56)$$

If $PQ = r_1$ and $PR = r_2$, then r_1 and r_2 are the roots of Eq. (3.56). Hence

$$PQ \cdot PR = r_1 r_2 = \frac{ax_1^2 + 2hx_1y_1 + by_1^2 - 1}{a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta}$$

Hence $r_1 r_2$ is independent of the slope $\tan\theta$ if $a = b$ and $h = 0$. In such a case, the given curve becomes $ax^2 + ay^2 = 1$, which is a circle,

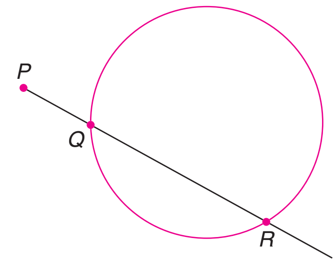

FIGURE 3.51

WORKED-OUT PROBLEMS

Single Correct Choice Type Questions

1. The angle between the tangents drawn from the origin to the circle $x^2 + y^2 - 14x + 2y + 25 = 0$ is

(A) $\frac{\pi}{3}$ (B) $\frac{\pi}{4}$ (C) $\frac{5\pi}{12}$ (D) $\frac{\pi}{2}$

Solution: Centre and radius of the circle are (7, –1) and 5, respectively. Now $y = mx$ touches the circle

$$\Leftrightarrow \frac{|m(7)+1|}{\sqrt{1+m^2}} = 5$$

$$\Leftrightarrow (7m+1)^2 = 25(1+m^2)$$

$$\Leftrightarrow 24m^2 + 14m - 24 = 0$$

$$\Leftrightarrow 12m^2 + 7m - 12 = 0$$

$$\Leftrightarrow 12m^2 + 16m - 9m - 12 = 0$$

$$\Leftrightarrow 4m(3m+4) - 3(3m+4) = 0$$

$$\Leftrightarrow m = \frac{3}{4}, \frac{-4}{3} \qquad (3.57)$$

Since the product of the slopes is –1, the tangents from (0, 0) are at right angles.

Note: From Eq. (3.57), the product of the roots is –1 which shows that the product of the slopes is –1.

Answer: (D)

2. The line $3x + y = 0$ is a tangent to the circle which has its centre at the point (2, –1). Then, the radius is

(A) $\frac{5}{2}$ (B) $\sqrt{\frac{5}{2}}$ (C) $\frac{\sqrt{5}}{2}$ (D) $5\sqrt{2}$

Solution: The radius (r) is equal to the distance of the centre (2,–1) from the line $3x + y = 0$, which is given by

$$\frac{|3(2)-1|}{\sqrt{3^2+1}} = \sqrt{\frac{5}{2}}$$

Answer: (B)

3. The angle at which the circle $x^2 + y^2 = 16$ can be seen from the point (8, 0) is

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{6}$

Solution: The required angle is the angle between the tangents drawn from the point (8, 0) to the circle $x^2 + y^2 = 16$. Let $y = m(x - 8)$ be a line through (8, 0). This line touches the circle. So

$$\Leftrightarrow \frac{|m(0-8)-0|}{\sqrt{1+m^2}} = 4$$

$$\Leftrightarrow 64m^2 = 16(1+m^2)$$

$$\Leftrightarrow 3m^2 = 1$$

$$\Leftrightarrow m = \pm\frac{1}{\sqrt{3}}$$

Therefore, the tangents are inclined at angles of 30° and 150° with the positive direction of the x-axis. Hence, the acute angle between tangents to the circle can be seen is 60°.

Answer: (C)

4. The radius of the circle passing through the points $(-1, 1)$, $(0, 6)$ and $(5, 5)$ is

(A) $2\sqrt{3}$ (B) 2 (C) $2\sqrt{2}$ (D) $\sqrt{13}$

Solution: Let $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ be the required circle. Therefore

$$-2g + 2f + c = -2 \tag{3.58}$$

$$12f + c = -36 \tag{3.59}$$

$$10g + 10f + c = -50 \tag{3.60}$$

Solving Eqs. (3.58)–(3.60), we have $g = -2$, $f = -3$ and $c = 0$. Therefore, the radius of the circle is

$$\sqrt{(-2)^2 + (-3)^2 + c} = \sqrt{13}.$$

Answer: (D)

5. The radius of the circle which passes through the point $(2, 3)$ and touches the line $2x - 3y - 13 = 0$ at the point $(2, -3)$ is

(A) $2\sqrt{2}$ (B) $\sqrt{8}$ (C) $\sqrt{13}$ (D) $2\sqrt{3}$

Solution: Let $A = (2, 3)$ and $B = (2, -3)$. Let M be the midpoint of $AB = (2, 0)$ and C be the centre of the circle (see Fig. 3.54). Therefore, C is the intersection of the lines CB and CM. Equation of the line CM is

$$y = 0 \tag{3.61}$$

Equation of the line CB is

$$y + 3 = \frac{-3}{2}(x - 2)$$

That is,

$$3x + 2y = 0 \tag{3.62}$$

Equations (3.61) and (3.62) imply that the centre of the circle is $(0, 0)$. Therefore, the radius of the circle is

$$CB = \sqrt{(2-0)^2 + (-3-0)^2} = \sqrt{13}$$

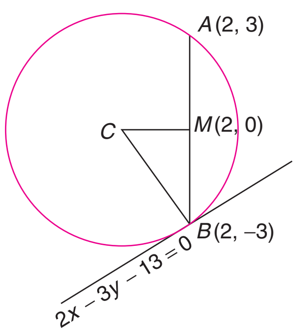

FIGURE 3.52

Answer: (C)

6. The angle between the tangents drawn from the point $(13, 0)$ to the circle $x^2 + y^2 = 25$ is

(A) $2\,\mathrm{Tan}^{-1}\left(\frac{5}{12}\right)$ (B) $\mathrm{Tan}^{-1}\left(\frac{7}{12}\right)$

(C) $\frac{\pi}{3}$ (D) $2\,\mathrm{Tan}^{-1}\left(\frac{7}{12}\right)$

Solution: Let $y = m(x - 13)$ be a line through $(13, 0)$. This line touches the circle

$$\Leftrightarrow \frac{|m(0-13) - 0|}{\sqrt{1+m^2}} = 5$$

$$\Leftrightarrow 169m^2 = 25(1 + m^2)$$

$$\Leftrightarrow 144m^2 = 25$$

$$\Leftrightarrow m = \pm\frac{5}{12}$$

Therefore, the angle between the tangents is

$$2\,\mathrm{Tan}^{-1}\left(\frac{5}{12}\right)$$

Answer: (A)

7. The tangent to the circle $x^2 + y^2 = 5$ at the point $(1, -2)$ also touches the circle $x^2 + y^2 - 8x + 6y + 20 = 0$ at the point

(A) $(3, -1)$ (B) $(2, -2)$

(C) $(2, -4)$ (D) $(4, 2)$

Solution: Tangent to the circle $x^2 + y^2 = 5$ at the point $(1, -2)$ is

$$x(1) + y(-2) - 5 = 0$$

$$\Rightarrow x - 2y - 5 = 0 \tag{3.63}$$

Point of contact of the tangent given in Eq. (3.63) with the circle $x^2 + y^2 - 8x + 6y + 20 = 0$ is the foot of the perpendicular drawn from the centre $(4, -3)$ onto the tangent given in Eq. (3.63). Suppose (x_1, y_1) is the foot of the perpendicular from $(4, -3)$ onto the line $x - 2y - 5 = 0$ (see Fig. 3.53). Therefore, by Theorem 2.13(1), we have

$$x_1 = 4 - \frac{[4 - 2(-3) - 5]}{1^2 + 2^2} = 4 - 1 = 3$$

and
$$y_1 = -3 - [(-2)(1)] = -1$$

Therefore, the point of contact is $(3, -1)$.

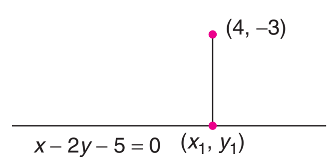

FIGURE 3.53

Answer: (A)

8. The number of common tangents to the circles $x^2 + y^2 - 2x - 6y + 9 = 0$ and $x^2 + y^2 + 6x - 2y + 1 = 0$ is

(A) 4 (B) 3 (C) 2 (D) 1

Solution: $A = (1, 3)$ and $r_1 = 1$, respectively, are the centre and the radius of the first circle. $B = (-3, 1)$ and $r_2 = 3$, respectively, are the centre and radius of the second circle. So

$$AB = \sqrt{(1+3)^2 + (3-1)^2} = \sqrt{20} = 2\sqrt{5} > r_1 + r_2$$

Therefore, the two circles do not have a common point. Hence, the number of common tangents is 4.

Answer: (A)

9. Any tangent to the circle $x^2 + y^2 = a^2$ meets the axes at points A and B, respectively. The rectangle $OACB$ is completed. The locus of the vertex C of the rectangle is

(A) $x^2 + y^2 = \dfrac{1}{a^2}$ (B) $x^{-2} + y^{-2} = a^{-2}$

(C) $x^2 + y^2 = \dfrac{a^2}{2}$ (D) $x^2 + y^2 = 2a^2$

Solution: Tangent at (x_1, y_1) to the circle (see Fig. 3.54) is $xx_1 + yy_1 = a^2$. Therefore

$$A = \left(\frac{a^2}{x_1}, 0\right)$$

$$B = \left(0, \frac{a^2}{y_1}\right)$$

Hence

$$C = \left(\frac{a^2}{x_1}, \frac{a^2}{y_1}\right)$$

If $C = (x, y)$, then

$$\frac{1}{x^2} + \frac{1}{y^2} = \frac{x_1^2}{a^4} + \frac{y_1^2}{a^4} = \frac{x_1^2 + y_1^2}{a^4} = \frac{a^2}{a^4} = \frac{1}{a^2}$$

Therefore, the locus of C is $x^{-2} + y^{-2} = a^{-2}$.

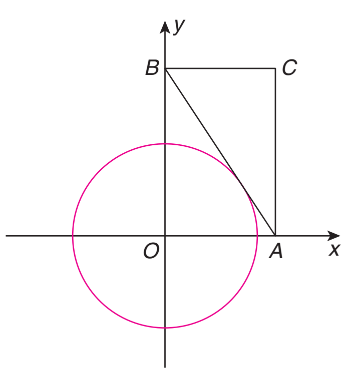

FIGURE 3.54

Answer: (B)

10. Let $C_1: x^2 + y^2 = 5$ and $C_2: x^2 + y^2 = 9$. The tangent at (1, 2) to C_1 meets the circle C_2 at A and B. The tangents drawn at A and B to C_2 meets in T. Then the coordinates of T are

(A) $(4, -5)$ (B) $\left(\dfrac{9}{15}, \dfrac{18}{15}\right)$

(C) $(4, 5)$ (D) $\left(\dfrac{9}{5}, \dfrac{18}{5}\right)$

Solution: Equation of the tangent at (1, 2) to C_1 is

$$x + 2y - 5 = 0 \qquad (3.64)$$

Suppose $T(x_1, y_1)$ is the point where the tangents to C_2 at points A and B meet (see Fig. 3.55). Hence, the equation of the chord AB is

$$xx_1 + yy_1 - 9 = 0 \qquad (3.65)$$

Equations (3.64) and (3.65) represent the same line AB. Therefore

$$\frac{x_1}{1} = \frac{y_1}{2} = \frac{-9}{-5}$$

$$\Rightarrow x_1 = \frac{9}{5}, y_1 = \frac{18}{5}$$

So

$$T = \left(\frac{9}{5}, \frac{18}{5}\right)$$

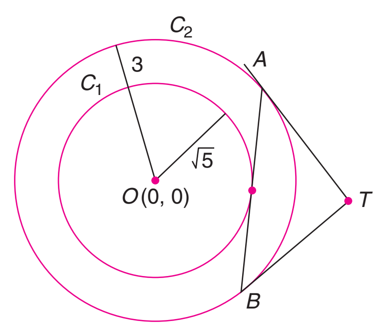

FIGURE 3.55

Answer: (D)

11. Tangents are drawn from any point on the circle $x^2 + y^2 = R^2$ to the circle $x^2 + y^2 = r^2$. If the line joining the points of intersection of these tangents with the first circle also touches the second, then $R{:}r$ is

(A) 2:1 (B) 1:2 (C) 3:1 (D) $\sqrt{2}$:1

Solution: Let $P(x_1, y_1)$ be a point on $x^2 + y^2 = R^2$. Suppose the tangents from point P to the circle $x^2 + y^2 = r^2$ meet the circle with radius R in A and B such that AB touches the circle with centre r. Thus, for ΔPAB, $x^2 + y^2 = R^2$ is the circumcircle and $x^2 + y^2 = r^2$ is the incircle and hence the circumcentre and incentre are the same. Therefore, the triangle is equilateral so that $R = 2r$.

Answer: (A)

12. If the circles $(x-1)^2+(y-3)^2=r^2$ and $x^2+y^2-8x+2y+8=0$ intersect at two distinct points, then

(A) $2<r<8$ (B) $r<2$
(C) $r=2$ (D) $r>2$

Solution: For the first circle $A=(1,3)$ and radius $=r$, and for second circle $B=(4,-1)$ and radius $r=3$. Since the two circles intersect in two distinct points, we have

$$|r'-r|<AB<r+r'$$
$$\Rightarrow |3-r|<\sqrt{3^2+4^2}<3+r$$

Therefore, $2<r<8$.

Answer: (A)

13. If the line $x\cos\theta+y\sin\theta=2$ is the equation of a transverse common tangent to the circles $x^2+y^2=4$ and $x^2+y^2-6\sqrt{3}x-6y+20=0$, then the value of θ is

(A) $\frac{\pi}{3}$ (B) $\frac{\pi}{6}$ (C) $\frac{2\pi}{3}$ (D) $\frac{5\pi}{6}$

Solution: $A=(0,0)$ and $r_1=2$, respectively, are the centre and radius of the first circle. $B=(3\sqrt{3},3)$ and $r_2=4$, respectively, are the centre and the radius of the second circle. So

$$AB=\sqrt{27+9}=6=r_1+r_2$$

Therefore, the two circles touches each other externally, and the transverse common tangent is

$$(x^2+y^2-6\sqrt{3}x-6y+20)-(x^2+y^2-4)=0$$
$$\Rightarrow -6\sqrt{3}x-6y+24=0$$
$$\Rightarrow \sqrt{3}x+y-4=0$$

However, $x\cos\theta+y\sin\theta-2=0$ is the transverse common tangent. Therefore

$$\frac{\cos\theta}{\sqrt{3}}=\frac{\sin\theta}{1}=\frac{-2}{-4}=\frac{1}{2}$$

Hence

$$\cos\theta=\frac{\sqrt{3}}{2} \quad \text{and} \quad \sin\theta=\frac{1}{2}$$

Therefore

$$\theta=\frac{\pi}{6}$$

Answer: (B)

14. The equation of the circle passing through the point $(2,0)$ and touching the line $y=x$ at the origin is

(A) $x^2+y^2-2x+2y=0$ (B) $x^2+y^2+2x+2y=0$
(C) $x^2+y^2-2x-2y=0$ (D) $x^2+y^2+2x-2y=0$

Solution: Let $S\equiv x^2+y^2+2gx+2fy+c=0$ be the required circle which passes through origin. This implies that

$$c=0 \tag{3.66}$$

It passes through $(2,0)$ which implies that

$$4g+c=-4 \tag{3.67}$$

It touches the line $y=x$ which implies that

$$\frac{|-g+f|}{\sqrt{2}}=\sqrt{g^2+f^2-c}$$
$$\Rightarrow (g-f)^2=2(g^2+f^2-c)$$
$$\Rightarrow -2gf=g^2+f^2-c$$
$$\Rightarrow (g+f)^2=0 \qquad \text{[from Eq. (3.66)]}$$
$$\Rightarrow g=-f$$

From Eq. (3.67), we get $g=-1$ and $f=1$. Therefore, equation of the circle is $S\equiv x^2+y^2-2x+2y=0$.

Answer: (A)

15. C_1 is a circle with centre at A and radius 2. C_2 is a circle with centre at B and radius 3. The distance AB is 7. If P is the point of intersection of a transverse common tangent with line AB, then the distance AP is

(A) $\frac{14}{5}$ (B) $\frac{7}{3}$ (C) $\frac{9}{4}$ (D) $\frac{8}{3}$

Solution: It is known that point P is the internal centre of similitude and $AP:PB=2:3$ (see Section 3.5). Therefore,

$$3AP=2PB=2(AB-AP)=2(7-AP)$$
$$\Rightarrow 5AP=14$$
$$\Rightarrow AP=\frac{14}{5}$$

Answer: (A)

16. Two circles with centres at $A(2,3)$ and $B(5,6)$ and having equal radii are intersecting orthogonally. Then the radius of the circles is

(A) $3\sqrt{2}$ (B) $2\sqrt{2}$ (C) 3 (D) 2

Soution: Two circles intersect orthogonally. Therefore, the square of the distance between the two circles is equal to the sum of the squares of the radii (see Quick Look 11). Therefore, if r is the radius of the two equal circles, then

$$(AB)^2=2r^2$$
$$\Rightarrow \left(\sqrt{(5-2)^2+(6-3)^2}\right)^2=2r^2$$

$$\Rightarrow 3^2 + 3^2 = 2r^2$$
$$\Rightarrow r = 3$$

Answer: (C)

17. A square $ABCD$ has area 1 sq. unit. A circle touches the sides AB and AD and passes through the vertex C. Then, the radius of the circle is

(A) $\dfrac{1}{\sqrt{2}}$ (B) $\dfrac{2}{\sqrt{2}}$

(C) $\sqrt{2} - 1$ (D) $2 - \sqrt{2}$

Solution: Let us consider A as origin, $\overrightarrow{AB}$ as positive x-axis and $\overrightarrow{AD}$ as positive y-axis (see Fig. 3.56). Suppose S is the centre of the circle and r its radius. We have $S = (r, r)$. Therefore

$$r = SC = \sqrt{(r-1)^2 + (r-1)^2}$$
$$\Rightarrow r^2 = 2(r-1)^2$$
$$\Rightarrow r^2 - 4r + 2 = 0$$
$$\Rightarrow r = \frac{4 \pm \sqrt{16-8}}{2} = 2 - \sqrt{2} \qquad (\because\ r < 1)$$

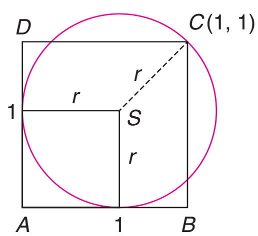

FIGURE 3.56

Answer: (D)

18. In $\Delta\, ABC$, $B = (3, 0)$ and $C = (-3, 0)$ and A is a variable vertex such that $\underline{|BAC} = 90°$. Then the locus of the centroid of $\Delta\, ABC$ is the circle

(A) $x^2 + y^2 = 2$ (B) $x^2 + y^2 = \dfrac{4}{9}$

(C) $x^2 + y^2 = \dfrac{1}{9}$ (D) $x^2 + y^2 = 1$

Solution: It is clear that A moves on the circle with BC as diameter and its equation is

$$(x+3)(x-3) + y^2 = 0$$
$$x^2 + y^2 = 9 \qquad (3.68)$$

Let $G(h, k)$ be the centroid of $\Delta\, ABC$. Therefore

$$h = \frac{x+3-3}{3},\ k = \frac{y+0+0}{3}$$
$$\Rightarrow 3h = x, 3k = y$$

From Eq. (3.68), we have

$$x^2 + y^2 = 9$$
$$\Rightarrow h^2 + k^2 = 1$$

Therefore, the locus of the centroid G is $x^2 + y^2 = 1$.

Answer: (D)

19. A circle is given by $x^2 + (y - 1)^2 = 1$. Another circle C touches it externally and also the x-axis. Then the locus of its centre is

(A) $\{(x, y): x^2 = 4y\} \cup \{(x, y): y \le 0\}$
(B) $\{(x, y): x^2 + (y - 1)^2 = 4\} \cup \{(x, y): y \le 0\}$
(C) $\{(x, y): x^2 = y\} \cup \{(0, y): y \le 0\}$
(D) $\{(x, y): x^2 = 4y\} \cup \{(0, y): y \le 0\}$

(IIT-JEE 2005)

Solution: Let $P(h, k)$ be the centre of C so that $|k|$ is the radius of C because it touches x-axis (see Fig. 3.57). Also, since C touches the circle $x^2 + (y - 1)^2 = 1$ externally, we have

$$\sqrt{(h-0)^2 + (k-1)^2} = 1 + |k|$$
$$\Rightarrow h^2 + (k-1)^2 = 1 + 2|k| + k^2$$
$$\Rightarrow h^2 - 2k = 2|k|$$
$$\Rightarrow h^2 = 4k$$

where $k > 0$ or $h_2 = 0$ when $k \le 0$. Therefore, the locus of the centre $P(h, k)$ is $\{(x, y): x^2 = 4y\} \cup \{(0, y): y \le 0\}$.

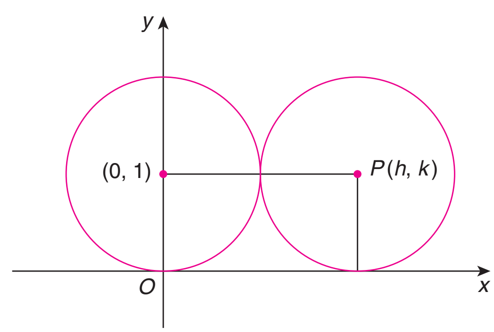

FIGURE 3.57

Answer: (D)

20. The equation of the circle touching the line $2x + 3y + 1 = 0$ at the point $(1, -1)$ and cutting orthogonally the circle $x^2 + y^2 + 2x - 2y - 3 = 0$ is

(A) $x^2 + y^2 - 5x - \dfrac{5}{2}y + \dfrac{1}{2} = 0$
(B) $x^2 + y^2 - 10x - 5y + 3 = 0$
(C) $x^2 + y^2 - 20x - 10y + 8 = 0$
(D) $x^2 + y^2 + 5x + 5y - 2 = 0$

Solution: The required circle is of the form

$$(x-1)^2+(y+1)^2+\lambda(2x+3y+1)=0$$

That is,

$$x^2+y^2+2(\lambda-1)x+(3\lambda+2)y+2+\lambda=0 \quad (3.69)$$

This circle cuts the circle $x^2+y^2+2x-2y-3=0$ orthogonally. From this and Theorem 3.17 we get

$$2(\lambda-1)(1)+2\left[\frac{3\lambda+2}{2}\right](-1)=2+\lambda-3$$

Therefore

$$\begin{aligned}&2\lambda-2-3\lambda-2=2+\lambda-3\\ &\Rightarrow 2\lambda=-3\\ &\Rightarrow \lambda=-\frac{3}{2}\end{aligned}$$

Substituting the value of λ in Eq. (3.69), we get the required circle as

$$x^2+y^2-5x-\frac{5}{2}y+\frac{1}{2}=0$$

Answer: (A)

21. The centre of the circle inscribed in a square formed by the lines $x^2-8x+12=0$ and $y^2-14y+45=0$ is

(A) $(4,7)$ (B) $(7,4)$ (C) $(9,4)$ (D) $(4,9)$

(IIT-JEE 2003)

Solution: The lines are $x=2$, $x=6$ and $y=5$, $y=9$. Therefore, the vertices of the square are $(2, 5)$, $(2, 9)$, $(6, 9)$ and $(6, 5)$.

$$\begin{aligned}\text{Centre of the circle} &= \text{Centre of the square}\\ &=\left(\frac{2+6}{2},\frac{9+5}{2}\right)\\ &=(4,7)\end{aligned}$$

Answer: (A)

22. If one of the diameters of the circle $x^2+y^2-2x-6y+6=0$ is a chord to the circle with centre $(2, 1)$, then the radius of the circle is

(A) $\sqrt{3}$ (B) $\sqrt{2}$ (C) 3 (D) $\sqrt{2}$

(IIT-JEE 2004)

Solution: Let AB be a diameter of the circle $x^2+y^2-2x-6y+6=0$ (see Fig. 3.58). Centre of this circle is $M=(1,3)$ and radius $\sqrt{1^2+3^2-6}=2$. $C(1,2)$ is the centre of the required circle. Therefore

$$\begin{aligned}(CA)^2&=(AM)^2+(CM)^2\\ &=4+(2-1)^2+(3-1)^2\\ &=4+1+4=9\end{aligned}$$

This implies $CA=3$

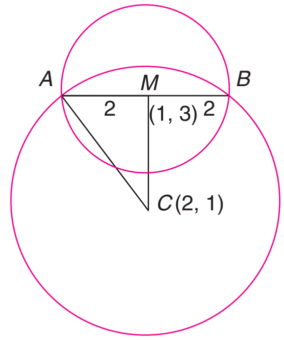

FIGURE 3.58

Answer: (C)

23. If the tangent at the point P on the circle $x^2+y^2+6x+6y-2=0$ meets the straight line $5x-2y+6=0$ at a point Q on the y-axis, then the length of PQ is

(A) 4 (B) $2\sqrt{5}$ (C) 5 (D) $3\sqrt{5}$

Solution: The line $5x-2y+6=0$ meets y-axis in $(0,3)$ so that $Q=(0,3)$. Hence, QP is the length of the tangent drawn from Q. It is given by

$$\sqrt{S_{11}}=\sqrt{0+9+6(0)+6(3)-2}=\sqrt{25}=5$$

Therefore, $PQ=5$.

Answer: (C)

24. Let AB be a chord of the circle $x^2+y^2=r^2$ subtending a right angle at the centre. Then, the locus of the centroid of $\Delta\,PAB$ as P moves on the circle is

(A) a parabola (B) a circle

(C) an ellipse (D) a pair of lines

Solution: Let $A=(r\cos\alpha, r\sin\alpha)$ and

$$\begin{aligned}B&=\left(r\cos\left(\frac{\pi}{2}+\alpha\right),\ r\sin\left(\alpha+\frac{\pi}{2}\right)\right)\\ &=(-r\sin\alpha, r\cos\alpha)\end{aligned}$$

Let $P(h,k)$ be a point on the circle (see Fig. 3.59). Therefore

$$h^2+k^2=r^2 \quad (3.70)$$

Let $G(x,y)$ be the centroid of $\Delta\,PAB$. Therefore

$$x=\frac{r\cos\alpha-r\sin\alpha+h}{3}$$

and

$$y=\frac{r\sin\alpha+r\cos\alpha+k}{3}$$

Therefore

$$\begin{aligned}r^2&=h^2+k^2\\ &=[3x-r(\cos\alpha-\sin\alpha)]^2+[3y-r(\sin\alpha+\cos\alpha)]^2\end{aligned}$$

That is,

$$\left[x-\frac{r}{3}(\cos\alpha-\sin\alpha)\right]^2+\left[y-\frac{r}{3}(\sin\alpha+\cos\alpha)\right]^2=\left(\frac{r}{3}\right)^2$$

which is a circle.

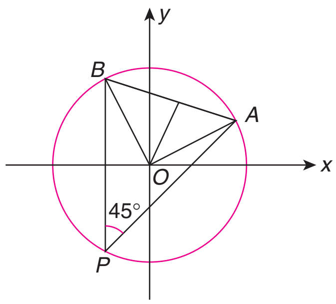

FIGURE 3.59

Answer: (B)

25. Tangents drawn from the point $P(1, 8)$ to the circle $x^2 + y^2 - 6x - 4y - 11 = 0$ to touch it at the points A and B. Then, the equation of the circumcircle of ΔPAB is

(A) $x^2 + y^2 - 4x - 6y - 19 = 0$
(B) $x^2 + y^2 - 4x - 10y + 19 = 0$
(C) $x^2 + y^2 - 2x + 6y - 29 = 0$
(D) $x^2 + y^2 - 6x - 4y + 19 = 0$

Solution: Equation of AB is

$$S_1 \equiv x(1) + y(8) - 3(x + 1) - 2(y + 8) - 11 = 0$$

because AB is the chord of contact. That is,

$$S_1 \equiv -2x+6y-30=0$$
$$x-3y+15=0$$

Therefore, any circle passing through A and B is of the form

$$x^2+y^2-6x-4y-11+\lambda(x-3y+15)=0$$

This represents circumcircle of $\Delta PAB \Leftrightarrow$ it passes through $P(1, 8)$. Therefore

$$1+64-6-32-11+\lambda(1-24+15)=0$$
$$-8\lambda=-16$$
$$\lambda=2$$

Therefore, the circumcircle of ΔPAB is

$$x^2 + y^2 - 6x - 4y - 11 + 2(x - 3y + 15) = 0$$

That is,

$$x^2+y^2-4x-10y+19=0$$

Aliter: Suppose $C(3, 2)$ is the centre of the circle. Since $\underline{|PAC} = 90° = \underline{|PBC}$, the circumcircle of the quadrilateral $PACB$ is also the circumcircle of ΔPAB. Therefore, PC is a diameter of the circumcircle of ΔPAB. Hence, the equation of the circumcircle is $(x - 1)(x - 3) + (y - 8)(y - 2) = 0$. That is,

$$x^2+y^2-4x-10y+19=0$$

Answer: (B)

26. A line is touching two circles at T_1 and T_2, respectively, such that the length T_1T_2 is 36. The minimum distance between the circles is 14. If the radius of the larger circle is four times the radius of the smaller circle, then the radius of the smaller circle is

(A) 5 (B) 4 (C) 3 (D) 10

Solution: Points A and B are the centres of the two circles. $PQ = 14$ and $T_1T_2 = 36$. Draw T_1M parallel to AB to BT_2 (see Fig. 3.60). Let r and $4r$ be the radii. From $\Delta\, T_1T_2M$, we have $(T_1M)^2 = (T_1T_2)^2 + (T_2M)^2$. Therefore,

$$(14+5r)^2=36^2+9r^2$$
$$\Rightarrow 16r^2+140r-50\times 22=0$$
$$\Rightarrow 4r^2+35r-275=0$$
$$\Rightarrow 4r^2-20r+55r-275=0$$
$$\Rightarrow 4r(r-5)+55(r-5)=0$$
$$\Rightarrow r=5$$

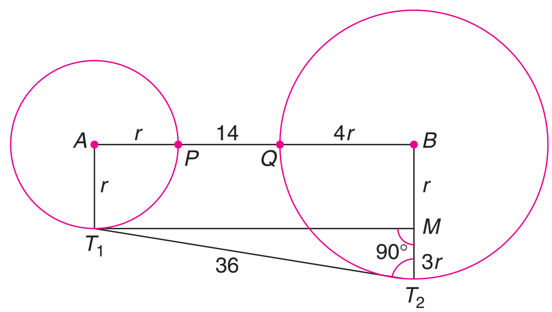

FIGURE 3.60

Answer: (A)

27. Let PQ and RS be tangents at the extremities of the diameter PR of a circle of radius r. If PS and RQ intersect at point X on the circumference of the circle, then $2r$ is equal to

(A) $\sqrt{PQ\cdot RS}$ (B) $(PQ+RS)/2$
(C) $[2(PQ)(RS)]/PR+RS$ (D) $\sqrt{(PQ^2+RS^2)/2}$

Solution: In ΔPQR, $\underline{|QPR} = 90°$ (see Fig. 3.61). The angle in the semicircle is given by $\underline{|PXR} = 90°$. Therefore, $\Delta\, PQX$ and $\Delta\, PXR$ are similar. Hence

$$\frac{PQ}{PR}=\frac{PX}{RX} \tag{3.71}$$

Similarly, $\Delta\, PRX$ and $\Delta\, RXS$ are similar. Therefore

$$\frac{PR}{RS}=\frac{PX}{RX} \tag{3.72}$$

From Eqs. (3.71) and (3.72), we have

$$\frac{PQ}{PR} = \frac{PR}{RS}$$

Therefore

$$(PR)^2 = PQ \cdot RS$$
$$2r = PR = \sqrt{PQ \cdot RS}$$

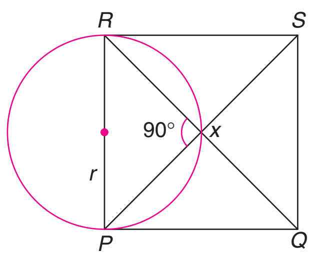

FIGURE 3.61

Answer: (A)

28. ΔPQR is inscribed in the $x^2 + y^2 = 25$. If Q and R have coordinates (3, 4) and (–4, 3), respectively, then $\angle PQR$ is equal to

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{3}$ (C) $\frac{\pi}{4}$ (D) $\frac{\pi}{6}$

Solution: From Fig. 3.62, we have the following:

$$\text{Slope of } OQ \times \text{Slope of } OR = \frac{4}{3} \times \frac{3}{-4} = -1$$

where O is the origin. Therefore $\angle QOR = 90°$. Hence

$$\angle QPR = \frac{\pi}{4}$$

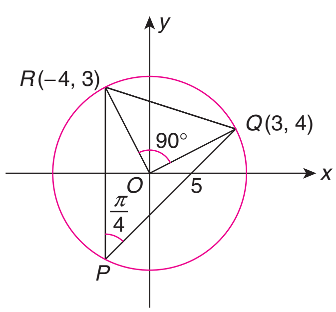

FIGURE 3.62

Answer: (C)

29. The equation of the circle passing through the point of contact of the direct common tangents of the circles $x^2 + y^2 = 16$ and $x^2 + y^2 - 12x + 32 = 0$ is

(A) $x^2 + y^2 + 6x - 8 = 0$ (B) $x^2 + y^2 - 6x - 8 = 0$
(C) $x^2 + y^2 - 4x - 6 = 0$ (D) $x^2 + y^2 + 4x - 6 = 0$

Solution: Let S_2 be the external centre of similitude (see Fig. 3.63). Therefore, S_2 divides the line joining $O(0, 0)$ and $A(6, 0)$ externally in the ratio 2:1. Hence $S_2 = (12, 0)$. Therefore, equation of the chord of contact of $S_2(12, 0)$ with respect to $x^2 + y^2 = 16$ is

$$L \equiv 3x - 4 = 0 \tag{3.73}$$

Equation of the chord of contact of $S_2(12, 0)$ with respect to $x^2 + y^2 - 12x + 32 = 0$ is

$$L' \equiv 3x - 20 = 0 \tag{3.74}$$

Now, equation of the circle passing through the points of intersection of $x^2 + y^2 - 16 = 0$ and $L = 0$ is

$$x^2 + y^2 - 16 + \lambda(3x - 4) = 0 \tag{3.75}$$

Equation of the circle passing through the intersection of $x^2 + y^2 - 12x + 32 = 0$ and $L' = 0$ is

$$x^2 + y^2 - 12x + 32 + \mu(3x - 20) = 0 \tag{3.76}$$

Equations (3.75) and (3.76) represent the same circle

$$\Leftrightarrow 3\lambda = 3\mu - 12 \text{ and } -4\lambda - 16 = -20\mu + 32$$
$$\Leftrightarrow \lambda - \mu = -4 \text{ and } \lambda - 5\mu = -12$$

Therefore, $\lambda = -2$ and $\mu = 2$, so that the equation of the required circle is $x^2 + y^2 - 6x - 8 = 0$.

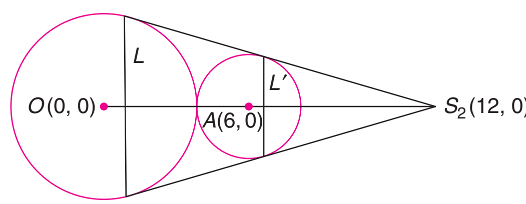

FIGURE 3.63

Answer: (B)

30. In ΔABC, if $\sin^2 A + \sin^2 B + \sin^2 C = 1$, then the angle of intersection of the circumcircle of ΔABC and the circumcircle of its pedal triangle is

(A) $\frac{2\pi}{3}$ (B) $\frac{\pi}{2}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{4}$

Solution: For pedal triangle and its properties, see Chapter 4, Vol. 2. Let O and H be the circumcentre and orthocentre, respectively, of ΔABC. Take O as origin of reference and $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$ and $\overrightarrow{OC} = \vec{C}$. Hence $|\vec{a}| = |\vec{b}| = |\vec{c}| = R$ (which is the circumradius). According to Example 5.2, part (2), Chapter 5, Vol. 2, we have $\overrightarrow{OH} = \vec{a} + \vec{b} + \vec{c}$. Therefore

$$\begin{aligned} |\overrightarrow{OH}|^2 &= 3R^2 + 2R^2(\cos 2A + \cos 2B + \cos 2C) \\ &= 3R^2 + 2R^2[3 - 2(\sin^2 A + \sin^2 B + \sin^2 C)] \\ &= 3R^2 + 2R^2(3 - 2) = 5R^2 \end{aligned} \tag{3.77}$$

If N is the centre of the circumcircle of the pedal triangle, then N is the midpoint of OH and the circumradius of the pedal triangle is $R/2$. Therefore

$$(ON)^2 = \left(\frac{\vec{a}+\vec{b}+\vec{c}}{2}\right)^2 = \frac{5R^2}{4} = R^2 + \left(\frac{R}{2}\right)^2$$

That is, square of the distance between the centres of the circles is equal to the sum of the squares of their radii. Therefore, the circumcircle of ΔABC and the circumcircle of its pedal triangle intersect orthogonally (see Quick Look 11).

Answer: (B)

31. Two equal circles of radius $\sqrt{2}+1$ touch each other externally. Another circle of radius 1 touches both the circles externally. Then, the perimeter of the region bounded between the three circles is

(A) $\frac{\pi}{2}(2+\sqrt{2})$ (B) $\frac{\pi}{4}(2+\sqrt{2})$

(C) $\frac{\pi}{4}(2-\sqrt{2})$ (D) $\frac{\pi}{4}(2\sqrt{2}-1)$

Solution: Let A and B be the centres of the circles of equal radius $\sqrt{2}+1$ and C be the circle with unit radius (see Fig. 3.64). Therefore, $AC = BC = 2+\sqrt{2}$ and $AB = 2(\sqrt{2}+1)$. Now

$$\begin{aligned}(AC)^2 + (BC)^2 &= (2+\sqrt{2})^2 + (2+\sqrt{2})^2 \\ &= 4(\sqrt{2}+1)^2 = (AB)^2\end{aligned}$$

Therefore, $\underline{|C} = 90°$. The perimeter of the shaded region is given by

$$2(\sqrt{2}+1)\frac{\pi}{4} + \frac{\pi}{2} = \frac{\pi}{2}(2+\sqrt{2})$$

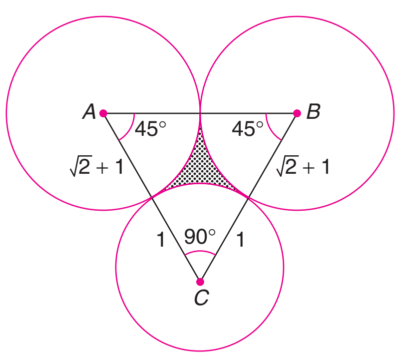

FIGURE 3.64

Answer: (A)

32. If a circle having centre at (a, b) and radius r completely lies within the two lines $x+y+2=0$ and $x+y-2=0$, then the minimum value of $\{|a+b+2|, |a+b-2|\}$ is

(A) greater than $r\sqrt{2}$ (B) less than $r\sqrt{2}$

(C) greater than $2r$ (D) less than $2r$

Solution: The two given lines are parallel (see Fig. 3.65). Therefore, the circle completely lies within the lines. This implies that

$$\frac{|a+b\pm 2|}{\sqrt{1^2+2^2}} > r$$
$$\Rightarrow |a+b\pm 2| > r\sqrt{2}$$

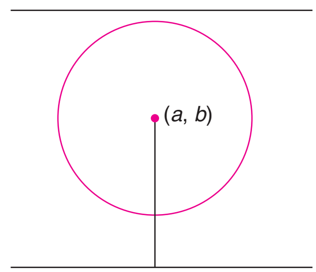

FIGURE 3.65

Answer: (A)

33. If A is a point on the circle $x^2+y^2=1$ and B is a point on the circle $x^2+y^2-10x+21=0$, then the minimum value of AB is

(A) 1 (B) 2 (C) 4 (D) 8

Solution: Suppose the centres of the two circles are $O(0, 0)$ and $C(5, 0)$ (Fig. 3.66). The radii of the circles are 1 and 2. Suppose the line OC meets the circles at points P and Q, respectively, so that $P = (1, 0)$ and $Q = (3, 0)$. Minimum value of AB occurs when $A = P$ and $B = Q$. Therefore, the minimum value of $AB = 5 - (1+2) = 2$.

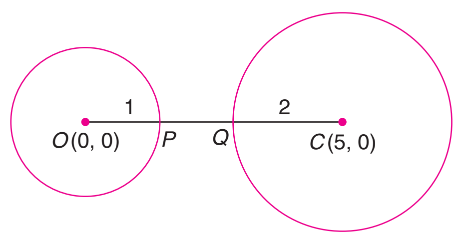

FIGURE 3.66

Answer: (B)

34. Three circles with radii a, b and c touch each other externally. The tangents drawn at their points of contact meet at a point P whose distance from the points of contact is 4. Then $(abc)/(a+b+c)$ is equal to

(A) 8 (B) 16 (C) 24 (D) 32

Solution: P is equidistant from the sides of ΔABC and that distance is 4 (Fig. 3.67). Therefore, 4 is the inradius of ΔABC. The semiperimeter s of ΔABC is given by

$$\frac{(a+b)+(b+c)+(c+a)}{2} = a+b+c$$

Now Δ is given by

$$\sqrt{(a+b+c)(a+b+c-b-c)(a+b+c-c-a)(a+b+c-a-b)}$$
$$=\sqrt{abc(a+b+c)}$$

Therefore

$$4 = r = \frac{\Delta}{s} = \sqrt{\frac{abc}{a+b+c}} \text{ or } \frac{abc}{a+b+c} = 16$$

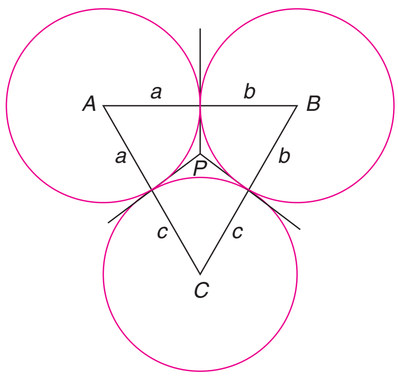

FIGURE 3.67

Answer: (B)

35. Three circles of equal radius r touch each other externally. Then the radius of the circle having internal contact with all the three circles is

(A) $(2+\sqrt{3})r$ (B) $(2-\sqrt{3})r$

(C) $(2+\sqrt{2})r$ (D) $\left(\frac{2+\sqrt{3}}{\sqrt{3}}\right)r$

Solution: Let A, B and C be the centres of three circles. Therefore, ΔABC is an equilateral triangle with side length $2r$. If R is the circumradius of ΔABC, then

$$R = \frac{2r}{2\sin 60°} = \frac{2r}{\sqrt{3}}$$

Therefore, the radius of the required circle is obtained as

$$R + r = \frac{2r}{\sqrt{3}} + r = \left(\frac{2+\sqrt{3}}{\sqrt{3}}\right)r$$

Answer: (D)

36. If $y = 2x$ is a chord of the circle $S \equiv x^2 + y^2 - 10x = 0$, then the equation of the circle described on this chord as diameter is

(A) $x^2 + y^2 - 2x - 4y = 0$ (B) $x^2 + y^2 - 14x + 2y = 0$

(C) $x^2 + y^2 - 6x - 2y = 0$ (D) $x^2 + y^2 - 18x + 14y = 0$

Solution: The required circle of the form $S + \lambda(2x - y) = 0$ is

$$x^2 + y^2 - 10x + \lambda(2x - y) = 0$$

For this circle, the chord $y = 2x$ is a diameter and if the center of this circle lies on $y = 2x$, then

$$2(5-\lambda) + \left(\frac{-\lambda}{2}\right) = 0$$
$$\Rightarrow 10 - 2\lambda - \frac{\lambda}{2} = 0$$
$$\Rightarrow \lambda = 4$$

Therefore, the required circle is $x^2 + y^2 - 2x - 4y = 0$.

Answer: (A)

37. The centre of the circle passing through the points $(0, 0)$ and $(1, 0)$ and touching the circle $x^2 + y^2 = 9$ in the first quadrant is

(A) $\left(\frac{3}{2}, \frac{1}{2}\right)$ (B) $\left(\frac{1}{2}, \frac{3}{2}\right)$

(C) $\left(\frac{1}{2}, \frac{1}{2}\right)$ (D) $\left(\frac{1}{2}, \sqrt{2}\right)$

Solution: Let $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ be the required circle. If it passes through $(0, 0)$, we have $c = 0$. If it passes through $(1, 0)$, we have $2g = -1$. Touching internally the circle $x^2 + y^2 = 9$ implies that

$$\sqrt{g^2+f^2} = 3 - \sqrt{g^2+f^2-c}$$
$$\Rightarrow 4(g^2+f^2) = 9$$
$$\Rightarrow 4\left(\frac{1}{4}+f^2\right) = 9$$
$$\Rightarrow 4f^2 = 8$$
$$\Rightarrow f = \pm\sqrt{2}$$

But the centre lies in the first quadrant. Therefore

$$g = \frac{-1}{2}, f = -\sqrt{2}$$

so that the centre is

$$\left(\frac{1}{2}, \sqrt{2}\right)$$

Answer: (D)

38. From the point $A(0, 3)$ on the circle $x^2 + 4x + (y-3)^2 = 0$ a chord AB is drawn and extended to a point M such that $AM = 2AB$. Then, the equation of the locus of M is

(A) $x^2 + y^2 + 8x - 4y + 3 = 0$

(B) $x^2 + y^2 - 8x + 6y - 27 = 0$

(C) $x^2 + y^2 + 8x - 6y + 9 = 0$

(D) $x^2 + y^2 - 8x - 9y + 18 = 0$

Solution: The given circle is

$$(x+2)^2 + (y-3)^2 = 4$$

which touches the y-axis at $A(0, 3)$. Let $M = (h, k)$ so that $B = [h/2, (k + 3)/2]$ is the midpoint of AM. Since B lies on the circle

$$(h/2 + 2)^2 + (k + 3/2 - 3)^2 = 4$$

we have

$$(h+4)^2 + (k-3)^2 = 16$$

Therefore, locus of M is $x^2 + y^2 + 8x - 6y + 9 = 0$.

Answer: (C)

39. If the chord of contact of tangents from a point P to a given circle passes through Q, then the circle described on PQ as diameter

(A) touches the given circle externally.
(B) touches the given circle internally.
(C) cuts the given circle orthogonally.
(D) does not have common points with the given circle.

Solution: Let $S \equiv x^2 + y^2 - a^2 = 0$ and $P(x_1, y_1)$ be a point whose chord of contact is $S_1 \equiv xx_1 + yy_1 - a^2 = 0$ which passes through $Q(x_2, y_2)$. Therefore

$$x_2x_1 + y_2y_1 = a^2 \tag{3.78}$$

Equation of the circle with PQ as diameter is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$$

$$x^2 + y^2 - (x_1 + x_2)x - (y_1 + y_2)y + x_1x_2 + y_1y_2 = 0 \tag{3.79}$$

The original circle is

$$x^2 + y^2 - a^2 = 0 \tag{3.80}$$

From Eqs. (3.79) and (3.80), we get

$$2gg' + 2ff' = 2(0)\left[\frac{-(x_1 + x_2)}{2}\right] + 2(0)\left[\frac{-(y_1 + y_2)}{2}\right] = 0$$

and from Eq. (3.78), we have

$$c + c' = (x_1x_2 + y_1y_2) - a^2 = 0$$

Hence, the circle described on PQ as diameter cuts the original circle orthogonally.

Answer: (C)

40. The condition that the chord $x \cos\alpha + y \cos\alpha = p$ subtends a right angle at the centre of the circle $x^2 + y^2 = a^2$ is

(A) $a^2 = 2p^2$ (B) $p^2 = 2a^2$
(C) $a = 2p$ (D) $p = 2a$

Solution: Suppose the chord $x \cos\alpha + y \cos\alpha = p$ meets the circle at points A and B (see Fig. 3.68). Therefore, the combined equation of the pair of lines OA and OB is

$$x^2 + y^2 - a^2\left(\frac{x\cos\alpha + y\sin\alpha}{p}\right)^2 = 0 \quad \text{(see Theorem 2.33)}$$

Since $\underline{|AOB} = 90°$, in the above equation, the sum of the coefficient of x^2 and the coefficient of y^2 is 0. That is,

$$\left(1 - \frac{a^2\cos^2\alpha}{p^2}\right) + \left(1 - \frac{a^2\sin^2\alpha}{p^2}\right) = 0$$

$$2p^2 = a^2$$

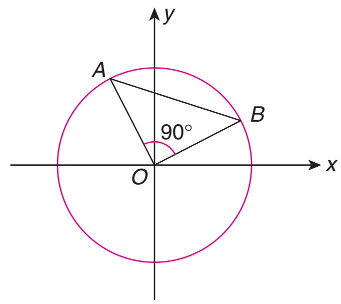

FIGURE 3.68

Answer: (A)

41. If the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is touched by the line $y = x$ at P such that $OP = 6\sqrt{2}$, then the value of c is

(A) 144 (B) 72 (C) 36 (D) 26

Solution: By Pythagoras theorem we have $(OC)^2 = (OP)^2 + (CP)^2$ where $C = (-g, -f)$ (Fig. 3.69). Therefore

$$g^2 + f^2 = 72 + (g^2 + f^2 - c)$$
$$\Rightarrow c = 72$$

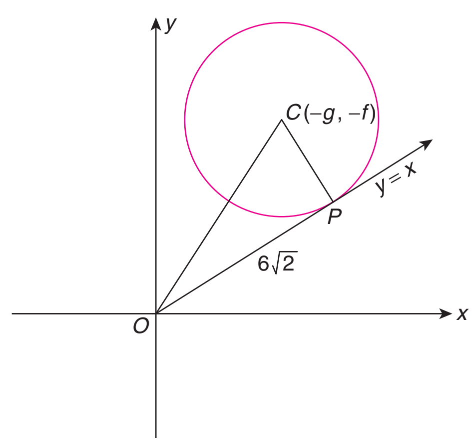

FIGURE 3.69

Answer: (B)

42. The equation of the circle passing through the points of intersection of the circle $x^2 + y^2 = 4$ and the line $2x + y = 1$ and having minimum radius is

(A) $5(x^2+y^2)+18x+6y-5=0$
(B) $5(x^2+y^2)+9x+8y-15=0$
(C) $5(x^2+y^2)+4x+9y-5=0$
(D) $5(x^2+y^2)-4x-2y-18=0$

Solution: Equation of the required circle is of the form

$$x^2+y^2-4+\lambda(2x+y-1)=0 \tag{3.81}$$

Radius of this circle is

$$\sqrt{\lambda^2+\frac{\lambda^2}{4}+\lambda+4}=\frac{1}{2}\sqrt{5\lambda^2+4\lambda+16}$$
$$=\frac{\sqrt{5}}{2}\sqrt{\lambda^2+\frac{4\lambda}{5}+\frac{16}{5}}$$
$$=\frac{\sqrt{5}}{2}\sqrt{\left(\lambda+\frac{2}{5}\right)^2+\frac{16}{5}-\frac{4}{25}}$$
$$=\frac{\sqrt{5}}{2}\sqrt{\left(\lambda+\frac{2}{5}\right)^2+\frac{76}{5}}$$

This value is minimum when

$$\lambda=\frac{-2}{5}$$

Substituting the value of λ in Eq. (3.81), we have the required circle equation as

$$x^2+y^2-4-\frac{2}{5}(2x+y-1)=0$$
$$\Rightarrow 5(x^2+y^2)-4x-2y-18=0$$

Answer: (D)

43. If the angle of intersection of the circles $x^2+y^2+x+y=0$ and $x^2+y^2+x-y=0$ is α, then the equation of the line passing through (1, 2) and inclined at an angle α with the positive direction of the x-axis is

(A) $x-y-3=0$ (B) $y=2$
(C) $x+y-3=0$ (D) $x=1$

Solution: $S \equiv x^2+y^2+x+y=0$ so that $g=1/2, f=1/2$ and $c=0$.

$S' \equiv x^2+y^2+x-y=0$ so that $g'=1/2, f'=-1/2$ and $c'=0$.

Now

$$2gg'+2ff'=2\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)+2\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)=0=c+c'$$

Therefore, $S=0$ and $S'=0$ intersect orthogonally so that

$$\alpha=\frac{\pi}{2}$$

Therefore, equation of the line is $x=1$.

Answer: (D)

44. From point P on the line $x+y-25=0$, tangents PA and PB are drawn to the circle $x^2+y^2=9$. As point P moves on the line, the locus of the midpoint of the chord AB is

(A) $5(x^2+y^2)=9(x+y)$ (B) $5(x^2+y^2)=3(x+y)$
(C) $25(x^2+y^2)=9(x+y)$ (D) $25(x^2+y^2)=3(x+y)$

Solution: Let $P(h,k)$ be a point on the line $x+y-25=0$. Therefore

$$h+k=25 \tag{3.82}$$

Since AB being the chord of contact of P with respect to the circle $x^2+y^2=9$, its equation is

$$hx+ky=9 \tag{3.83}$$

If $M(x_1,y_1)$ is the midpoint of the chord AB, its equation is

$$xx_1+yy_1={x_1}^2+{y_1}^2 \tag{3.84}$$

Both Eqs. (3.83) and (3.84) represent the chord AB. Therefore

$$\frac{h}{x_1}=\frac{k}{y_1}=\frac{9}{x_1^2+y_1^2}$$

Therefore

$$h=\frac{9x_1}{x_1^2+y_1^2}$$
$$k=\frac{9y_1}{x_1^2+y_1^2}$$

Hence, from Eq. (3.82), we have

$$25=h+k=\frac{9(x_1+y_1)}{x_1^2+y_1^2}$$
$$\Rightarrow 25(x_1^2+y_1^2)=9(x_1+y_1)$$

Therefore, the locus of $M(x_1,y_1)$ is

$$25(x^2+y^2)=9(x+y)$$

Answer: (C)

45. If the circles $x^2+y^2+2x+2ky+6=0$ and $x^2+y^2+2ky+k=0$ intersect orthogonally, then k is

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(A) 2 or $-\frac{3}{2}$ (B) -2 or $-\frac{3}{2}$
(C) 2 or $\frac{3}{2}$ (D) -2 or $\frac{3}{2}$

Solution: We have $g=1, f=k, c=6$ and $g'=0, f'=k$, $c'=k$. Therefore,

$$2gg'+2ff'=c+c'$$

$$\Rightarrow 2(1)(0)+2(k)(k)=6+k$$
$$\Rightarrow 2k^2-k-6=0$$
$$\Rightarrow 2k^2+3k-4k-6=0$$
$$\Rightarrow k(2k+3)-2(2k+3)=0$$
$$\Rightarrow (k-2)(2k+3)=0$$

Therefore

$$k=2 \text{ or } \frac{-3}{2}$$

Answer: (A)

46. A circle passes through origin and has its centre on the line $y = x$. If the circle cuts the circle $x^2 + y^2 - 4x - 6y + 10 = 0$ orthogonally, then its equation is

(A) $x^2 + y^2 + 2x + 2y = 0$ (B) $x^2 + y^2 + 2x - 2y = 0$
(C) $x^2 + y^2 - 2x + 2y = 0$ (D) $x^2 + y^2 - 2x - 2y = 0$

Solution: Let $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ be the required circle which passes through $(0, 0)$. This implies that

$$c=0 \tag{3.85}$$

The circle has the centre on the line $y = x$ which implies that

$$\begin{aligned}-g&=-f\\ \Rightarrow g&=f\end{aligned} \tag{3.86}$$

The circle cuts the circle $x^2 + y^2 - 4x - 6y + 10 = 0$ orthogonally implies that

$$2(g)(-2)+2f(-3)=c+10$$
$$-4g-6f=c+10 \tag{3.87}$$

From Eqs. (3.85)–(3.87), we have $g = f = -1$ and $c = 0$. Therefore

$$S \equiv x^2+y^2-2x-2y=0$$

Answer: (D)

47. The number of common tangents of the circles $S \equiv x^2 + y^2 - 4 = 0$ and $S' \equiv x^2 + y^2 - 6x - 8y - 24 = 0$ is

(A) 0 (B) 1 (C) 3 (D) 4

Solution: $O = (0, 0)$ and $r_1 = 2$, respectively, are the centre and the radius of $S = 0$. $A = (3, 4)$ and $r_2 = 7$, respectively, are the centre and the radius of $S' = 0$. Now

$$OA=\sqrt{3^2+4^2}=5=7-2=r_2-r_1$$

Thus, $S = 0$ and $S' = 0$ touch each other internally. Therefore, there is only one common tangent for the given circles.

Answer: (B)

48. Let the circles $S \equiv x^2 + y^2 - 12 = 0$ and $S' \equiv x^2 + y^2 - 5x + 3y - 2 = 0$ intersect at points P and Q. Tangents are drawn to the circle $S \equiv x^2 + y^2 - 12 = 0$ at points P and Q. The point of intersection of these tangents is

(A) $\left(6,-\frac{18}{5}\right)$ (B) $\left(6,\frac{18}{5}\right)$

(C) $\left(-6,\frac{18}{5}\right)$ (D) $\left(-6,-\frac{18}{5}\right)$

Solution: The common chord of the circles is $L \equiv S - S' \equiv 5x - 3y - 10 = 0$. Note that PQ is $L = 0$. Suppose the tangents at points P and Q meet in $T(h, k)$. Therefore, the equation of the chord PQ is

$$hx+ky=12 \tag{3.88}$$

However,

$$L \equiv 5x-3y-10=0 \tag{3.89}$$

is PQ. Therefore, from Eqs. (3.88) and (3.89), we get $h/5 = k/-3 = 12/10 = 6/5$. Hence

$$h=6, k=\frac{-18}{5}$$

Therefore

$$T(h,k)=\left(6,\frac{-18}{5}\right)$$

Answer: (A)

49. If $a > 2b > 0$, then the positive value of m for which $y = mx - b\sqrt{1+m^2}$ is a common tangent to the circles $x^2 + y^2 = b^2$ and $(x - a)^2 + y^2 = b^2$ is

(A) $\frac{2b}{\sqrt{a^2-4b^2}}$ (B) $\frac{\sqrt{a^2-4b^2}}{2b}$

(C) $\frac{2b}{a-2b}$ (D) $\frac{b}{a-2b}$

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Solution: We have $y = mx - b\sqrt{1+m^2}$ which is a common tangent. This implies that

$$\frac{\left|m(0)-0-b\sqrt{1+m^2}\right|}{\sqrt{1+m^2}}=b=\frac{\left|ma-0-b\sqrt{1+m^2}\right|}{\sqrt{1+m^2}}$$
$$\Rightarrow b^2(1+m^2)=(ma-b\sqrt{1+m^2})^2=m^2a^2-2abm\sqrt{1+m^2}+b^2(1+m^2)$$
$$\Rightarrow m^2a^2-2abm\sqrt{1+m^2}=0$$
$$\Rightarrow ma=2b\sqrt{1+m^2}$$
$$\Rightarrow m^2a^2=4b^2(1+m^2)$$

$$\Rightarrow m^2(a^2-4b^2)=4b^2$$

$$\Rightarrow m=\frac{2b}{\sqrt{a^2-4b^2}} \qquad (\because m>0)$$

Answer: (A)

50. One of the diameters of the circle circumscribing the rectangle $ABCD$ is $x-4y+7=0$. If $A=(-3,4)$ and $B=(5,4)$, then the area of $ABCD$ is

(A) 72 (B) 64 (C) 32 (D) 36

Solution: Let $P(h,k)$ be the centre and $Q(1,4)$ be the midpoint of AB (see Fig. 3.70) Since AB is a horizontal segment, it follows that PQ is vertical. Therefore, $h=1$ and $h-4k+7=0 \Rightarrow k=2$. Hence, the centre of the circle is $(1,2)$. It is clear that $BC=2PQ$ which implies that

$$BC=2(2)=4$$

Now $AB=8$ and $BC=4$ which implies that

$$\text{Area of } ABCD = 8\times 4=32$$

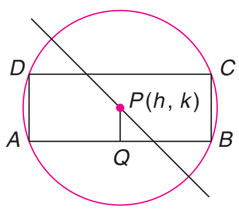

FIGURE 3.70

Answer: (C)

51. Let L_1 be a straight line passing through the origin and L_2 be the line $x+y=1$. If the intercepts made by the circle $x^2+y^2-x+3y=0$ on L_1 and L_2 are equal, then which of the following equations can represent by L_1.

(A) $x+y=0$ (B) $x-y=0$

(C) $2x+7y=0$ (D) $x-7y=0$

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Solution: Centre of the circle is (1/2, –3/2) and the radius of the circle is $\sqrt{5/2}$. Let P be the length of the perpendicular from the centre (1/2, –3/2) onto the line L_2. Therefore

$$P=\frac{|(1/2)-(3/2)-1|}{\sqrt{2}}=\frac{2}{\sqrt{2}}$$

So, the intercept is

$$2\sqrt{r^2-p^2}=2\sqrt{\frac{5}{2}-\frac{4}{2}}=\frac{2}{\sqrt{2}}=\sqrt{2}$$

Let $y=mx$ be the equation of L_1. Hence, we have

$$4=\frac{(m+3)^2}{2(1+m^2)}$$

so that $m=1$ or –1/7. Here $m=1$ implies that the equations of L_1 is $y=x$ or $x-y=0$.

Answer: (B)

52. The equation of the circle passing through the intersection of the circles $x^2+y^2=4$ and $x^2+y^2-2x-4y+4=0$ and touching line $x+2y=0$ is

(A) $x^2+y^2-x-2y=0$ (B) $x^2+y^2+x+2y=0$

(C) $x^2+y^2-x+2y=0$ (D) $x^2+y^2+x-2y=0$

Solution: The circle passing through the intersection of the given circles is

$$x^2+y^2-4+\lambda(x+2y-4)=0$$

where $x+2y-4=0$ is the common chord. The centre of this circle is

$$\left(\frac{-\lambda}{2},-\lambda\right)$$

and the radius is

$$\sqrt{\frac{\lambda^2}{4}+\lambda^2+4\lambda+4}$$

This circle touches the line $x+2y=0$. This implies that

$$\frac{|-(\lambda/2)+2(-\lambda)|}{\sqrt{1^2+2^2}}=\sqrt{\frac{\lambda^2}{4}+\lambda^2+4\lambda+4}$$

$$\Rightarrow \left|\frac{5\lambda}{2\sqrt{5}}\right|=\frac{1}{2}\sqrt{5\lambda^2+16\lambda+16}$$

$$\Rightarrow 5\lambda^2=5\lambda^2+16\lambda+16$$

$$\Rightarrow \lambda=-1$$

Hence, the required circle is $x^2+y^2-x-2y=0$.

Answer: (A)

53. If the line $3x-4y-k=0$ meets the circle $x^2+y^2-4x-8y-5=0$ in two distinct points, then

(A) $-35<k<15$

(B) $k<-35$

(C) $k>15$

(D) $k\in(-\infty,-35)\cup(15,\infty)$

Solution: Centre of the circle is (2, 4) and the radius is 5. The line meets the circle in two distinct points. This implies that

$$\frac{|3(2)-4(4)-k|}{\sqrt{3^2+4^2}}<5$$

$$\Leftrightarrow |10+k|<25$$

$$\Leftrightarrow -25<10+k<25$$

$$\Leftrightarrow -35<k<15$$

Answer: (A)

54. The circumference of the circle $x^2 + y^2 - 2x + 8y - a = 0$ is bisected by the circle $x^2 + y^2 + 4x + 22y + b = 0$. Then $a + b$ equals

(A) 25 (B) 35 (C) 45 (D) 50

Solution: $S \equiv x^2 + y^2 - 2x + 8y - a = 0$ and $S' \equiv x^2 + y^2 + 4x + 22y + b = 0$. Since $S' = 0$ bisects the circumference of $S = 0$, the centre of $S = 0$ lies on the common chord $S - S' = 0$. Therefore

$$S - S' \equiv -6x - 14y - a - b = 0$$

$$\Rightarrow S - S' \equiv 3x + 7y + \frac{a+b}{2} = 0$$

The centre $(1, -4)$ of $S = 0$ lies on $S - S' = 0$. This implies that

$$3(1) + 7(-4) + \frac{a+b}{2} = 0$$

$$\Rightarrow a + b = 50$$

Answer: (D)

55. The equation of the circle which touches the line $x = y$ at the origin and bisects the circumference of the circle $x^2 + y^2 + 2y - 3 = 0$ is

(A) $x^2 + y^2 - 5x - 5y = 0$
(B) $2x^2 + 2y^2 + 5x - 5y = 0$
(C) $2x^2 + 2y^2 - 5x + 5y = 0$
(D) $x^2 + y^2 - 5x + 5y = 0$

Solution: Let $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ be the required circle. It passes through the origin. This implies that

$$c = 0 \qquad (3.90)$$

It touches the line $x - y = 0$. This implies that

$$\frac{|-g + f|}{\sqrt{2}} = \sqrt{g^2 + f^2 - c}$$

$$\Rightarrow (g - f)^2 = 2(g^2 + f^2) \qquad (\because c = 0)$$

$$\Rightarrow g^2 + f^2 + 2gf = 0$$

$$\Rightarrow (g + f)^2 = 0$$

$$\Rightarrow g + f = 0 \qquad (3.91)$$

It bisects the circumference of the circle $x^2 + y^2 + 2y - 3 = 0$. The centre $(0, -1)$ lies on the common chord $2gx + 2(f - 1)y + 3 = 0$. Therefore

$$2(f - 1)(-1) + 3 = 0$$

$$\Rightarrow -2f = -5$$

$$\Rightarrow f = \frac{5}{2}$$

Hence, $S \equiv x^2 + y^2 - 5x + 5y = 0$.

Answer: (D)

56. A circle of radius 4 is drawn passing through origin and whose diameter is along the x-axis. The line $y = 2x$ is a chord of this circle. Then the equation of the circle whose diameter is this chord is

(A) $5(x^2 + y^2) + 8x + 16y = 0$
(B) $5(x^2 + y^2) - 8x + 16y = 0$
(C) $5(x^2 + y^2) + 8x - 16y = 0$
(D) $5(x^2 + y^2) - 8x - 16y = 0$

Solution: The given circle equation is

$$S \equiv (x - 4)^2 + y^2 - 16 = 0$$

$$\Rightarrow S \equiv x^2 + y^2 - 8x = 0$$

The required circle is of the form

$$S + \lambda(2x - y) = 0$$

$$\Rightarrow x^2 + y^2 - 8x + \lambda(2x - y) = 0$$

The centre of this circle (see Fig. 3.71) lies on the line $2x - y = 0$. This means

$$2(4 - \lambda) - \frac{\lambda}{2} = 0$$

$$\Rightarrow 16 - 5\lambda = 0$$

$$\Rightarrow \lambda = \frac{16}{5}$$

Therefore, the required circle is

$$x^2 + y^2 - 8x + 16/5\ (2x - y) = 0$$

$$\Rightarrow 5(x^2 + y^2) - 8x - 16y = 0$$

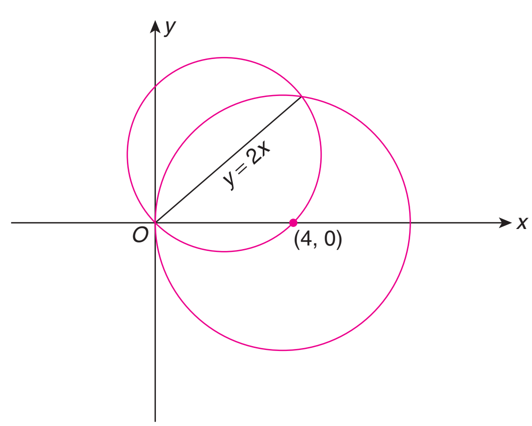

FIGURE 3.71

Answer: (D)

57. Two circles $x^2 + y^2 + 2ax - c^2 = 0$ and $x^2 + y^2 + 2bx - c^2 = 0$ meet at points P and Q. If R is a point on the first circle and S is a point on the second circle such that PR and QS are parallel, then the locus of the midpoint of RS is

(A) $x^2 + y^2 - (a + b)x = 0$
(B) $x^2 + y^2 + (a + b)x = 0$
(C) $x^2 + y^2 - (a + b)x + c^2 = 0$
(D) $x^2 + y^2 + (a + b)x + c^2 = 0$

Solution: The two circles intersect in $P(0, c)$ and $Q(0, -c)$. The lines through P and Q which are parallel to each other are

$$y = mx + c \text{ and } y = mx - c$$

Now $y = mx + c$ meets the first circle in R whose coordinates are

$$\left(\frac{-2(a+mc)}{1+m^2}, \frac{-2m(a+mc)}{1+m^2} + c\right)$$

and the line $y = mx - c$ meets the second circle in S whose coordinates are

$$\left(\frac{-2(b-mc)}{1+m^2}, \frac{-2m(b-mc)}{1+m^2} - c\right)$$

Let $M(x, y)$ be the midpoint of RS. Therefore

$$x = \frac{-(a+b)}{1+m^2} \text{ and } y = \frac{-m(a+b)}{1+m^2}$$

Now $y/x = m$. Substituting in

$$x(1+m^2) = -(a+b)$$

we get

$$\left(1+\frac{y^2}{x^2}\right) = -(a+b)x$$

$$\frac{x(x^2+y^2)}{x^2} = -(a+b)$$

$$x^2 + y^2 + (a+b)x = 0$$

Answer: (B)

58. The extremities of a diagonal of a rectangle are (–4, 4) and (6, –1). A circle circumscribes the rectangle and cuts an intercept AB on the y-axis. Then the area of the triangle formed by AB and the two tangents at A and B is

(A) $\left(\frac{11}{2}\right)^2$ (B) $\frac{(11)^2}{2}$

(C) $\left(\frac{11}{2}\right)^3$ (D) $\left(\frac{11}{2}\right)^4$

Solution: The circumcircle of the rectangle is

$$(x+4)(x-6) + (y-4)(y+1) = 0$$

$$\Rightarrow x^2 + y^2 - 2x - 3y - 28 = 0 \tag{3.92}$$

From Problem 1 of the section 'Subjective Problems (Sections 3.1 till 3.3)', the length of the y-intercept of the circle provided in Eq. (3.92) is

$$2\sqrt{f^2 - c} = 2\sqrt{\frac{9}{4} + 28} = 11$$

Therefore, $AB = 11$. The ordinates of A and B are given by

$$y^2 - 3y - 28 = 0$$
$$\Rightarrow (y-7)(y+4) = 0$$
$$\Rightarrow y = 7, -4$$

Therefore, $A = (0, 7)$ and $B = (0, -4)$. Equation of the tangent at $A(0, 7)$ is

$$x(0) + y(7) - (x+0) - \frac{3}{2}(y+7) - 28 = 0$$

$$\Rightarrow -2x + 11y - 77 = 0$$

$$\Rightarrow 2x - 11y + 77 = 0 \tag{3.93}$$

Equation of the tangent at $B(0, -4)$ is

$$x(0) + y(-4) - (x+0) - \frac{3}{2}(y-4) - 28 = 0$$

$$\Rightarrow -2x - 11y - 44 = 0$$

$$\Rightarrow 2x + 11y + 44 = 0 \tag{3.94}$$

Solving Eqs. (3.93) and (3.94), we have $C = (-121/4, 3/2)$ which is the intersection of the tangents at A and B. Therefore, area of ΔABC is

$$\frac{1}{2}|AB \times \text{Height}| = \frac{1}{2} \times 11 \times \frac{121}{4} = \left(\frac{11}{2}\right)^3$$

Answer: (C)

59. The locus of the centre of the circle whose intercept on x-axis is of constant length $2a$ and which passes through a fixed point $(0, b)$ is

(A) $x^2 - 2by + b^2 - a^2 = 0$
(B) $x^2 + 2by + b^2 - a^2 = 0$
(C) $x^2 - 2by + (a + b) = 0$
(D) $x^2 + 2by + a^2 + b^2 = 0$

Solution: See Fig. 3.72.

$$k^2 + a^2 = (\text{Radius})^2 = h^2 + (k-b)^2$$

where $C(h, k)$ is the centre. Therefore, $h^2 - 2bk + b^2 - a^2 = 0$. Hence, the locus of $C(h, k)$ is

$$x^2 - 2by + b^2 - a^2 = 0$$

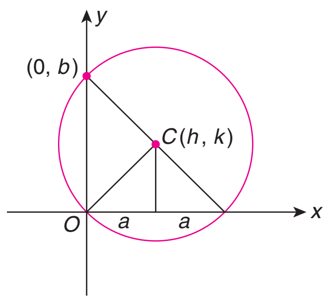

FIGURE 3.72

Answer: (A)

60. Two circles of radii a and b touch externally. If x is the radius of a third circle which is between them and touches them externally and also touching their direct common tangent, then $1/x$ is equal to

(A) $\dfrac{1}{a}+\dfrac{1}{b}$ (B) $\dfrac{1}{a}+\dfrac{1}{b}-\dfrac{1}{ab}$

(C) $\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{2}{\sqrt{ab}}$ (D) $\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{\sqrt{ab}}$

Solution: See Fig. 3.73. Points A and B are the centres with radii a and b, respectively. LM is a direct common tangent. AP is drawn perpendicular to BM so that $APML$ is a rectangle. Through the centre C of the circle with radius x, draw a line parallel to AP meeting the line AL at R and the line BM at Q. Now, $\Delta\, BCQ$ is right-angled triangle in which BC is the hypotenuse. We have $BC = b + x$ and $BQ = b - x$. By Pythagoras theorem, we have

$$(BC)^2 = (BQ)^2 + (CQ)^2 \Rightarrow (b+x)^2 = (b-x)^2 + (CQ)^2$$

$$CQ = 2\sqrt{bx} \tag{3.95}$$

$$CR = 2\sqrt{ax} \tag{3.96}$$

Also $AB = b + a$, $BP = b - a$ and $\angle APB = 90°$ which implies that

$$(AB)^2 = (AP)^2 + (BP)^2$$

$$\Rightarrow (b+a)^2 = (AP)^2 + (b-a)^2$$

$$\Rightarrow AP = 2\sqrt{ab} \tag{3.97}$$

Further $AP = RQ = LM$. Therefore, from Eqs. (3.95)–(3.97), we get

$$AP = RQ = CR + CQ$$

$$\Rightarrow 2\sqrt{ab} = 2\sqrt{ax} + 2\sqrt{bx}$$

$$\Rightarrow ab = ax + bx + 2\sqrt{ab}(x)$$

$$= x(a + b + 2\sqrt{ab})$$

Therefore

$$\frac{1}{x} = \frac{a+b+2\sqrt{ab}}{ab} = \frac{1}{b}+\frac{1}{a}+\frac{2}{\sqrt{ab}}$$

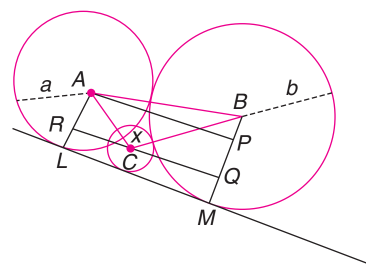

FIGURE 3.73

Answer: (C)

Multiple Correct Choice Type Questions

1. The centres of the circles passing through (0,0) and (1,0) and touching the circle $x^2 + y^2 = 9$ are

(A) $\left(\dfrac{1}{2}, \sqrt{2}\right)$ (B) $\left(-\dfrac{1}{2}, \sqrt{2}\right)$

(C) $\left(\dfrac{1}{2}, -\sqrt{2}\right)$ (D) $\left(-\dfrac{1}{2}, -\sqrt{2}\right)$

Solution: Suppose $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ is the required circle. It passes through (0, 0) $\Rightarrow$ $c = 0$. It passes through (1, 0) $\Rightarrow$ $g = -1/2$. It is clear that $S = 0$ touches internally the circle $x^2 + y^2 = 9$. Therefore

$$\sqrt{g^2+f^2} = 3 - \sqrt{g^2+f^2-c} = 3 - \sqrt{g^2+f^2} \qquad (\because c = 0)$$

$$\Rightarrow 4(g^2 + f^2) = 9$$

$$\Rightarrow 4\left(\frac{1}{4} + f^2\right) = 9 \qquad \left(\because g = -\frac{1}{2}\right)$$

$$\Rightarrow f = \pm\sqrt{2}$$

Therefore, the centres of the circles are given by

$$(-g, -f) = \left(\frac{1}{2}, \pm\sqrt{2}\right)$$

Answers: (A), (C)

2. The equations of the chords of length 5 and passing through the point (3, 4) on the circle $4x^2 + 4y^2 - 24x - 7y = 0$ are

(A) $4x + 3y = 0$ (B) $4x - 3y = 0$

(C) $4x + 3y - 24 = 0$ (D) $4x + 3y - 12 = 0$

Solution: We have that the centre is $C = (3, 7/8)$ and the radius $= 25/8$ (see Fig. 3.74). Any line through (3, 4) of the form $x - 3/\cos\theta = y - 4/\sin\theta = r$ (say) so that the other point of the extremity of the chord is $(3 + r\cos\theta, 4 + r\sin\theta)$. Since the length of the chord is 5, we have $r = 5$ and

$$(3+5\cos\theta-3)^2+\left(4+5\sin\theta-\frac{7}{8}\right)^2=\left(\frac{25}{8}\right)^2$$

Therefore

$$25\cos^2\theta+\left(5\sin\theta+\frac{25}{8}\right)^2=\left(\frac{25}{8}\right)^2$$

$$\Rightarrow 25+\frac{125}{4}\sin\theta=0$$

Hence

$$\sin\theta=\frac{-4}{5}$$

$$\cos\theta=\pm\frac{3}{5}$$

$$\tan\theta=\pm\frac{4}{3}$$

Therefore, the chord equation is given by

$$y-4=\pm\frac{4}{3}(x-3)$$

Hence

$$4x-3y=0 \text{ and } 4x+3y-24=0$$

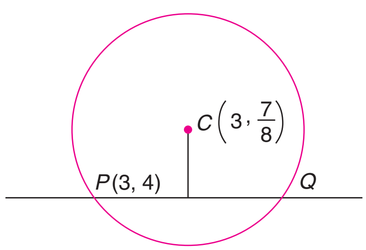

FIGURE 3.74

Answers: (B), (C)

3. $A(2, 0)$ is a point on the circle $(x+2)^2+(y-3)^2=25$. A line through $A(2, 0)$ making an angle of 45° with the tangent to the circle at A is drawn. Then the equations of the circles with the centres on these lines which are at a distance of $5\sqrt{2}$ units from point A and of radius 3 are

(A) $(x-1)^2+(y-7)^2=9$
(B) $(x-3)^2+(y+7)^2=9$
(C) $(x-9)^2+(y-1)^2=9$
(D) $(x+9)^2+(y+1)^2=9$

Solution: The given circle is $x^2+y^2+4x-6y-12=0$. Therefore, the equation of the tangent to this circle at $A(2, 0)$ is

$$x(2)+y(0)+2(x+2)-3(y+0)-12=0$$

$$\Rightarrow 4x-3y-8=0 \quad (3.98)$$

Let $y=m(x-2)$ be a line through $A(2, 0)$. This line makes an angle of 45° with the line [given in Eq. (3.98)]. So we have

$$\left|\frac{m-4/3}{1+4m/3}\right|=1$$

$$\Rightarrow |3m-4|=|3+4m|$$

$$\Rightarrow 3m-4=\pm|3+4m|$$

Therefore

$$m=-7 \text{ or } m=\frac{1}{7}$$

Hence, the lines are $x-7y-2=0$ and $7x+y-14=0$. Since the centres of the circles are at a distance of $5\sqrt{2}$ from $A(2, 0)$, we have

$$\frac{x-2}{(-1/\sqrt{50})}=\frac{y}{(7/\sqrt{50})}=\pm5\sqrt{2}$$

and

$$\frac{x-2}{(7/\sqrt{50})}=\frac{y}{(1/\sqrt{50})}=\pm5\sqrt{2}$$

Therefore, the centres of the circles are (1, 7), (3, –7), (9, 1) and (–5, –1).

Answers: (A), (B), (C), (D)

4. The point(s) on the line $x=2$ from which the tangents drawn to the circle $x^2+y^2=16$ are at right angles is (are)

(A) $(2, -2\sqrt{5})$ (B) $(2, 2\sqrt{5})$
(C) $(2, 2\sqrt{7})$ (D) $(2, -2\sqrt{7})$

Solution: If the tangents drawn are at right angles, the points must be the intersection of the line $x=2$ with the director circle $x^2+y^2=32$ of the circle $x^2+y^2=16$. Substituting $x=2$ in $x^2+y^2=32$, we have

$$y^2=28 \text{ or } y=\pm2\sqrt{7}$$

Therefore, the points are $(2, 2\sqrt{7})$ and $(2, -2\sqrt{7})$.

Answers: (C), (D)

5. If the circles $S\equiv x^2+y^2+2x+2\lambda y+6=0$ and $S'\equiv x^2+y^2+2\lambda y+\lambda=0$ cut each other orthogonally, then λ is equal to

(A) $-\frac{3}{2}$ (B) -2 (C) 2 (D) $\frac{3}{2}$

Solution: $S=0$ and $S'=0$ cut orthogonally. This implies that

$$2(1)(0)+2(\lambda)(\lambda)=6+\lambda$$

$$\Rightarrow 2\lambda^2-\lambda-6=0$$

$$\Rightarrow 2\lambda^2-4\lambda+3\lambda-6=0$$

$$\Rightarrow 2\lambda(\lambda-2)+3(\lambda-2)=0$$

Therefore

$$\lambda = 2, \frac{-3}{2}$$

Answers: (A), (C)

6. The equations of the circles of radius 1 unit and touching the circles $x^2 + y^2 + 2x = 0$ and $x^2 + y^2 - 2x = 0$ are

(A) $x^2 + y^2 + 2\sqrt{3}\, y + 2 = 0$

(B) $x^2 + y^2 - 2\sqrt{2}\, x = 0$

(C) $x^2 + y^2 + 2\sqrt{2}\, x + 1 = 0$

(D) $x^2 + y^2 - 2\sqrt{3}\, y + 2 = 0$

Solution: The two given circles are of unit radius and having centres at (–1, 0) and (1, 0), respectively. The required circle is also of unit radius. Hence, the required circle must touch these two given circles externally. Also, the given circles $x^2 + y^2 + 2x = 0$ and $x^2 + y^2 - 2x = 0$ touch each other externally at origin. Hence, the centre of the required circle from the origin is $\sqrt{3}$. Therefore, the required circles are

$$x^2 + (y - \sqrt{3})^2 = 1$$

and

$$x^2 + (y + \sqrt{3})^2 = 1$$

Answers: (A), (D)

7. On which of the following lines, the circle $S \equiv x^2 + y^2 - 2x + 4y = 0$ makes equal lengths of intercepts?

(A) $3x - y - 10 = 0$ (B) $x + 3y = 0$

(C) $3x - y = 0$ (D) $x + 3y + 10 = 0$

Solution: $A(1, -2)$ and $\sqrt{5}$ are centre and radius of the circle $S = 0$.

(i) Let p be the length of the perpendicular from the centre onto the line $L_1 \equiv 3x - y - 10 = 0$. Therefore

$$\left|\frac{3(1) - (-2)10}{\sqrt{3^2 + 1^2}}\right| = \sqrt{\frac{5}{2}}$$

Hence, the length of the intercept is

$$2\sqrt{5 - \frac{5}{2}} = \sqrt{10}$$

(ii) Let p be the perpendicular from (1, –2) onto the line $L_2 \equiv x + 3y = 0$. Therefore

$$\frac{|1 + 3(-2)|}{\sqrt{10}} = \sqrt{\frac{5}{2}}$$

Hence, the intercept is

$$2\sqrt{5 - \frac{5}{2}} = \sqrt{10}$$

(iii) Let p be the perpendicular from (1, –2) onto the line $L_3 \equiv 3x - y = 0$. Therefore

$$\frac{|3(1) - (-1)|}{\sqrt{10}} = \sqrt{\frac{5}{2}}$$

Hence the intercept is

$$2\sqrt{5 - \frac{5}{2}} = \sqrt{10}$$

(iv) Let p be the perpendicular from (1, –2) onto $L_4 \equiv x + 3y + 10 = 0$. Therefore

$$\frac{|1 + 3(-2) + 10|}{\sqrt{10}} = \sqrt{\frac{5}{2}}$$

Hence the intercept is

$$2\sqrt{5 - \frac{5}{2}} = \sqrt{10}$$

Note: You can guess that the chords are of equal length if they are located at equal distance from the centre.

Answers: (A), (B), (C), (D)

8. The equations of the tangents drawn from the origin to the circle $S \equiv x^2 + y^2 - 2rx - 2hy + h^2 = 0$ are

(A) $x = 0$ (B) $y = 0$

(C) $(h^2 - r^2)x - 2rhy = 0$ (D) $(h^2 - r^2)x + 2rhy = 0$

Solution:

$$x = 0$$
$$\Rightarrow y^2 - 2hy + h^2 = 0$$
$$\Rightarrow (y - h)^2 = 0$$

Therefore, y-axis touches the circle at $(0, h)$. That is, $x = 0$ is a tangent. Suppose $y = mx$ is a tangent to $S = 0$. Then

$$\frac{|m(r) - h|}{\sqrt{1 + m^2}} = \sqrt{r^2 + h^2 - h^2}$$
$$\Rightarrow (h - mr)^2 = r^2(1 + m^2)$$
$$\Rightarrow h^2 - 2mrh - r^2 = 0$$
$$\Rightarrow m = \frac{h^2 - r^2}{2hr}$$

Therefore

$$y = \left(\frac{h^2 - r^2}{2hr}\right)x \text{ or } (h^2 - r^2)x - 2rhy = 0$$

Answers: (A), (C)

9. A straight line through the vertex P of a triangle PQR, intersects the side QR at the point S and the circumcircle of the triangle PQR at the point T. If S is not the circumcentre of the triangle, then

(A) $\dfrac{1}{PS}+\dfrac{1}{ST}<\dfrac{2}{\sqrt{QS\times SR}}$

(B) $\dfrac{1}{PS}+\dfrac{1}{ST}>\dfrac{2}{\sqrt{QS\times SR}}$

(C) $\dfrac{1}{PS}+\dfrac{1}{ST}<\dfrac{4}{QR}$

(D) $\dfrac{1}{PS}+\dfrac{1}{ST}>\dfrac{4}{QR}$

(IIT-JEE 2008)

Solution: Since S is not the circumcentre, the two chords $\overline{PT}$ and $\overline{QR}$ cannot bisect each other (see Fig. 3.75), but $\overline{QR}$ may bisect $\overline{PT}$ (see the Note at the end of the solution). Using $AM-GM$ inequality, the equality

$$\frac{1}{PS}+\frac{1}{ST}\geq 2\sqrt{\frac{1}{PS}\cdot\frac{1}{ST}}$$

occurs if $PS = ST$. However,

$$PS\cdot ST = QS\cdot SR \qquad \text{(by similar triangle)}$$

Therefore

$$\frac{1}{PS}+\frac{1}{ST}\geq\frac{2}{\sqrt{QS\cdot SR}} \tag{3.99}$$

Also the equality

$$\frac{QR}{QS\cdot SR}=\frac{QS+SR}{QS\cdot SR}=\frac{1}{SR}+\frac{1}{QS}\geq 2\sqrt{\frac{1}{QS}\cdot\frac{1}{SR}}$$

The equality or occurs if $QS = SR$. Therefore

$$\frac{QR}{2}\geq\sqrt{QS\cdot SR}$$
$$\frac{4}{QR}\leq\frac{1}{QS\cdot SR} \tag{3.100}$$

From Eqs. (3.99) and (3.100), we have

$$\frac{1}{PS}+\frac{1}{ST}>\frac{4}{QR}$$

because $PS = ST$ and $QS = SR$ cannot hold simultaneously (by the hypothesis, S is not the circumcentre).

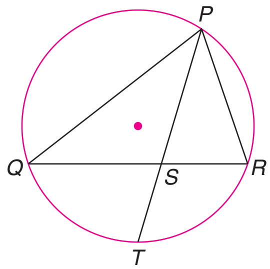

FIGURE 3.75

Note: In choice (B), equality may occur when S is the midpoint of PT. This can happen if we consider a right-angled triangle PQR, where $P = (a, b)(a > 0, b > 0)$, $Q = (0, 0)$ and $R = (a, 0)$ such that $a^2 > 8b^2$ [see Problem 26 in the section 'Subjective Problems (Sections 3.1 till 3.3)']. In particular, take $P = (3, 1)$, $Q = (0, 0)$, $R = (3, 0)$ which is right-angled triangle right angled at R and take $T = (1, -1)$ so that the midpoint of $\overline{PT}$ lies on QR.

Answers: (B), (D)

10. The circles $x^2 + y^2 = 400$ and $x^2 + y^2 - 10x - 24y + 120 = 0$

(A) touch each other externally.

(B) touch each other internally.

(C) point of contact is (100/13, 240/13).

(D) point of contact is (100/13, –240/13).

Solution: The centres of the circles are $O(0, 0)$ and $A(5, 12)$, respectively. The radii are $r_1 = 20$ and $r_2 = 7$, respectively. $OA = 13 = r_1 - r_2$ implies that the two circles touch internally. Let P be the point of contact so that $OP{:}PA = 20{:}-7$. Therefore

$$P=\left(\frac{100}{13},\frac{240}{13}\right)$$

Answers: (B), (C)

11. If the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ cuts the curve $xy = 1$ in four points (x_i, y_i) $(i = 1, 2, 3, 4)$, then

(A) $x_1 x_2 x_3 x_4 = 1$

(B) $y_1 y_2 y_3 y_4 = 1$

(C) $x_1 + x_2 + x_3 + x_4 = -2g$

(D) $y_1 + y_2 + y_3 + y_4 = -2f$

Solution: Substituting $y = 1/x$ in the circle equation, we get

$$x^2+\frac{1}{x^2}+2gx+\frac{2f}{x}+c=0$$

Therefore, $x^4 + 2gx^3 + cx^2 + 2fx + 1 = 0$ whose roots are x_1, x_2, x_3 and x_4. Hence,

$$x_1+x_2+x_3+x_4=2g,$$
$$\sum x_1x_2x_3=-2f$$

and $$x_1x_2x_3x_4=1$$

Now

$$y=\frac{1}{x}=y_1y_2y_3y_4=1$$

and

$$y_1+y_2+y_3+y_4=\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_4}=\frac{\sum x_1x_2x_3}{x_1x_2x_3x_4}=-2f$$

Answers: (A), (B), (C), (D)

12. Let C_1: $x^2 + y^2 = 4$ and C_2: $x^2 + y^2 - 6x - 8y - 24 = 0$. Then

(A) C_1 and C_2 touch each other internally
(B) C_1 and C_2 do not intersect
(C) the number of common tangents is 2
(D) the number of common tangents is 1

Solution: We have

$$A = (0, 0) \text{ and } r_1 = 2$$
$$B = (3, 4) \text{ and } r_2 = 7$$

Now, $AB = 5 = 7 - 2 = r_2 - r_1$. Therefore, the two circles C_1 and C_2 touch each other internally and hence they have only one common tangent.

Answers: (A), (D)

13. The equation of the circle with centre (4, 3) and touching circle $x^2 + y^2 = 1$ is

(A) $x^2 + y^2 - 8x - 6y + 11 = 0$
(B) $x^2 + y^2 - 8x - 6y + 9 = 0$
(C) $x^2 + y^2 - 8x - 6y - 9 = 0$
(D) $x^2 + y^2 - 8x - 6y - 11 = 0$

Solution: Since (3, 4) lies outside the circle $x^2 + y^2 = 1$, one circle has the external contact with $x^2 + y^2 = 1$ and the other circle has the internal contact. $O = (0, 0)$ and $r_1 = 1$. Let $A = (4, 3)$ and r_2 be the radius of the required circle $OA = 5$.

Case 1: $r_2 = 5 - 1 = 4$ (i.e. external contact). Hence, the required circle is

$$(x - 4)^2 + (y - 3)^2 = 16$$
$$\Rightarrow x^2 + y^2 - 8x - 6y + 9 = 0$$

Case 2: $r_2 = 5 + 1 = 6$ (i.e. internal contact). Hence, the required circle is

$$(x - 4)^2 + (y - 3)^2 = 36$$
$$\Rightarrow x^2 + y^2 - 8x - 6y - 11 = 0$$

Answers: (B), (D)

14. The equation of the tangents to the circle $x^2 + y^2 - 6x - 4y + 5 = 0$ which is inclined at an angle of 45° with the x-axis is

(A) $x - y + 3 = 0$ (B) $x - y - 3 = 0$
(C) $x - y - 5 = 0$ (D) $x - y + 5 = 0$

Solution: Required tangent equation is of the form $y = x + c$. $A = (3, 2)$ and $r = 2\sqrt{2}$, respectively, are the centre and the radius of the given circle. Therefore, the line $y = x + c$ touches the given circle implies that

$$\frac{|3 - 2 + c|}{\sqrt{1^2 + 1^2}} = 2\sqrt{2}$$

$$\Rightarrow c + 1 = \pm 4$$

Therefore, $c = 3, -5$. Hence, the required tangents are

$$y = x + 3 \quad \text{and} \quad y = x - 5$$

Answers: (A), (C)

15. The centre of a circle C lies on the line $2x - 2y + 9 = 0$ and the circle cuts orthogonally the circle C_1: $x^2 + y^2 = 4$. Then C passes through

(A) $(-3, 3)$ (B) $(-1/2, 1/2)$
(C) $(-4, 4)$ (D) $(-2, 2)$

Solution: Let the equation of C be $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$. The centre $(-g, -f)$ lies on $2x - 2y + 9 = 0$ implies that

$$-2g + 2f = -9 \tag{3.101}$$

The circle C cuts the circle C_1 orthogonally. This implies that

$$2g(0) + 2f(0) = c - 4$$
$$\Rightarrow c = 4 \tag{3.102}$$

From Eqs. (3.101) and (3.102), the equation of the circle C is given by

$$S \equiv x^2 + y^2 + 2gx + (2g - 9)y + 4 = 0$$
$$\equiv x^2 + y^2 - 9y + 4 + 2g(x + y) = 0$$

The circle C passes through the intersection of the circle $x^2 + y^2 - 9y + 4 = 0$ and the line $x + y = 0$ and these points are $(-1/2, 1/2)$ and $(-4, 4)$.

Answers: (B), (C)

16. The equation of the circle passing through origin and touching the lines $x + 2 = 0$ and $3x + 4y - 50 = 0$ is

(A) $x^2 + y^2 - 6x - 8y = 0$
(B) $x^2 + y^2 + 6x - 8y = 0$
(C) $x^2 + y^2 - 16x + 12y = 0$
(D) $x^2 + y^2 + 8x - 12y = 0$

Solution: The centre of the required circle lies on the line $2x + y - 10 = 0$ which is an angular bisector of the lines $x + 2 = 0$ and $3x + 4y - 50 = 0$. Let $C(\alpha, 10 - 2\alpha)$ be the centre of the required circle. Hence

$$\sqrt{\alpha^2 + (10 - 2\alpha)^2} = |\alpha + 2|$$
$$\Rightarrow 5\alpha^2 - 40\alpha + 100 = \alpha^2 + 4\alpha + 4$$
$$\Rightarrow 4\alpha^2 - 44\alpha + 96 = 0$$
$$\Rightarrow \alpha^2 - 11\alpha + 24 = 0$$
$$\Rightarrow (\alpha - 3)(\alpha - 8) = 0$$
$$\Rightarrow \alpha = 3, 8$$

Therefore, the required centre of the circles are $(3, 4)$ and $(8, -6)$, respectively, and their radii are the distances of their centre from the origin which are 5 and 10, respectively. Hence, the circles are

$$(x-3)^2+(y-4)^2=25 \text{ and } (x-8)^2+(y+6)^2=100$$

That is,

$$x^2+y^2-6x-8y=0 \text{ and } x^2+y^2-16x+12y=0$$

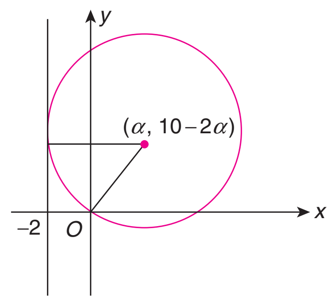

FIGURE 3.77

Answers: (A), (C)

Matrix-Match Type Questions

1. Match items of Column I with those of Column II.

Column I	Column II
(A) If x-axis bisected each of two chords drawn from the point $(a, b/2)$ on the circle $2x(x-a)+y(2y-b)=0$ $(ab \neq 0)$, then a/b belongs to	(p) $(-\infty,-2)\cup(2,\infty)$
	(q) $(-2, 2)$
(B) If the circles $x^2+y^2-10x+16=0$ and $x^2+y^2=r^2$ intersect in two distinct points, then r lies in the interval	(r) $(-\infty,-\sqrt{2})\cup(\sqrt{2},\infty)$
(C) If the line $y+x=0$ bisects chords drawn from the point $(1+a\sqrt{2}/2,\ 1-a\sqrt{2}/2)$ to the circle $2x^2+2y^2-(1+a\sqrt{2})x-(1-a\sqrt{2})y=0$, then a belongs to	(s) $(-3, 3)$
(D) Point $(2, \lambda)$ lies inside the circle $x^2+y^2=13$ if and only if λ belongs to	(t) $(2, 8)$

Solution:

(A) The given circle equation is

$$S \equiv 2x^2+2y^2-2ax-by=0$$

$$S \equiv x^2+y^2-ax-\frac{b}{2}y=0$$

Let $(x_1, 0)$ be the midpoint of a chord of the circle. Therefore, the equation of the chord is

$$xx_1+y(0)-\frac{a}{2}(x+x_1)-\frac{b}{4}(y+0)=x_1^2-ax_1$$

$$\Rightarrow 4xx_1-2ax-2ax_1-by=4x_1^2-4ax_1$$

$$\Rightarrow 2(2x_1-a)x-by+2ax_1-4x_1^2=0$$

This passes through the point $(a, b/2)$. This implies that

$$2(2x_1-a)a-b\left(\frac{b}{2}\right)+2ax_1-4x_1^2=0$$

$$\Rightarrow -4x_1^2+6ax_1-\left(2a^2+\frac{b^2}{2}\right)=0$$

(which has two distinct real roots)

$$\Rightarrow 4x_1^2-6ax_1+\left(2a^2+\frac{b^2}{2}\right)=0$$

(which has two distinct real roots)

$$\Rightarrow (6a)^2>4(4)\left(2a^2+\frac{b^2}{2}\right)$$

$$\Rightarrow 9a^2>2(4a^2+b^2)$$

$$\Rightarrow a^2>2b^2$$

$$\left|\frac{a}{b}\right|>\sqrt{2}$$

Therefore

$$\frac{a}{b}<-\sqrt{2} \quad \text{or} \quad \frac{a}{b}>\sqrt{2}$$

Answer: (A) → (r)

(B) $O=(0,0)$ and $A=(5,0)$ are the centres and r, 3 are the radii of the circles. The two circles intersect in two distinct points. So

$$|r-3|<OA<r+3$$

$$\Leftrightarrow |r-3|<5<r+3$$

$$5 < r+3 \Rightarrow 2 < r \text{ and } r-3 < 5 \Rightarrow r < 8$$
$$\Leftrightarrow 2 < r < 8$$

Answer: (B) → (p), (r), (t)

(C) Let

$$\alpha = \frac{1+\sqrt{2}a}{2}$$

so that

$$\bar{\alpha} = \frac{1-\sqrt{2}a}{2}$$

Therefore, the given circle equation is $S \equiv x^2 + y^2 - \alpha x - \bar{\alpha} y = 0$ which passes through origin. Let $(x_1, -x_1)$ be the midpoint of the chord. Therefore, the chord equation is $S_1 = S_{11}$. That is,

$$xx_1 + y(-x_1) - \frac{\alpha}{2}(x+x_1) - \frac{\bar{\alpha}}{2}(y-x_1) = x_1^2 + x_1^2 - \alpha x_1 + \bar{\alpha} x_1$$

$$\Rightarrow 2xx_1 - 2x_1 y - \alpha x - \alpha x_1 - \bar{\alpha} y + \bar{\alpha} x_1 = 4x_1^2 - 2\alpha x_1 + 2\bar{\alpha} x_1$$

$$\Rightarrow (2x_1 - \alpha)x - (2x_1 + \bar{\alpha})y + \alpha x_1 - \bar{\alpha} x_1 - 4x_1 = 0$$

This passes through $(\alpha, -\bar{\alpha})$. This implies that

$$(2x_1 - \alpha)\alpha - (2x_1 + \bar{\alpha})\bar{\alpha} + (\alpha - \bar{\alpha})x_1 - 4x_1^2 = 0$$

$$\Rightarrow -4x_1^2 + 3(\alpha - \bar{\alpha})x_1 - (\alpha^2 + \alpha^{-2}) = 0$$

$$\Rightarrow 4x_1^2 - 3(\alpha - \bar{\alpha})x_1 + (\alpha^2 + \alpha^{-2}) = 0$$

(has two distinct real roots in x_1)

$$\Rightarrow 9(\alpha - \bar{\alpha})^2 > 16(\alpha^2 + \bar{\alpha}^2)$$

$$\Rightarrow 9(2a^2) > 16\left(\frac{1+2a^2}{2}\right)$$

$$\left[\because \alpha = \frac{1+a\sqrt{2}}{2} \text{ and } \bar{\alpha} = \frac{1-a\sqrt{2}}{2}\right]$$

$$\Rightarrow 9a^2 > 4(1+2a^2)$$

$$\Rightarrow a^2 > 4$$

$$\Rightarrow a \in (-\infty, -2) \cup (2, \infty)$$

Answer: (C) → (p), (r)

(D) $O = (0, 0)$ and $r = \sqrt{13}$. Let $P = (2, \lambda)$. P lies inside the circle

$$\Leftrightarrow OP < r$$

$$\Leftrightarrow \sqrt{\lambda^2 + 2^2} < \sqrt{13}$$

$$\Leftrightarrow |\lambda| < 3$$

$$\Leftrightarrow -3 < \lambda < 3$$

Answer: (D) → (s)

2. Match the items of Column I with those of Column II.

Column I	*Column II*
(A) The intercept on the line $y = x$ by the circle $x^2 + y^2 - 2x = 0$ is AB. Equation of the circle described on AB as diameter is	(p) $x^2 + y^2 - 18x - 6y + 120 = 0$
(B) The equation of the circle with radius 5 and touching the circle $x^2 + y^2 - 2x - 4y - 20 = 0$ at the point (5, 5) is	(q) $25(x^2 + y^2) - 20x + 2y - 60 = 0$
(C) The equation of a circle which passes through the point (2, 0) whose centre is the limit of the point of intersection of the lines $3x + 5y = 1$ and $(2 + c)x + 5c^2y = 1$ as c tends to 1 is	(r) $x^2 + y^2 - 3x - 3y - 8 = 0$
(D) The equation of the circle passing through the points of intersections of the circles $x^2 + y^2 - 4x - 2y - 8 = 0$ and $x^2 + y^2 - 2x - 4y - 8 = 0$ at the point $(-1, 4)$ is	(s) $x^2 + y^2 - x - y = 0$

Solution:

(A) Substitute $y = x$ in the circle equation. Therefore

$$2x^2 - 2x = 0$$
$$\Rightarrow x = 0, 1$$

Hence, $A(0, 0)$ and $B = (1, 1)$. Therefore, the equation of the circle with AB as diameter is given by

$$x(x-1) + y(y-1) = 0$$
$$\Rightarrow x^2 + y^2 - x - y = 0$$

Answer: (A) → (s)

(B) The equation of the tangent at (5, 5) to the given circle is

$$x(5) + y(5) - (x+5) - 2(y+5) - 20 = 0$$

That is, $4x + 3y - 35 = 0$. The required circle is of the form $x^2 + y^2 - 2x - 4y - 20 + \lambda(4x + 3y - 35) = 0$. The radius of the circle is 5. This implies that

$$(1-2\lambda)^2+\left(2-\frac{3\lambda}{2}\right)^2+35\lambda+20=25$$

$$\Rightarrow 4(1-2\lambda)^2+(4-3\lambda)^2+140\lambda-20=0$$

$$\Rightarrow 25\lambda^2+100\lambda=0$$

$$\Rightarrow 5\lambda^2+20\lambda=0$$

$$\Rightarrow \lambda=0 \text{ or } -4$$

$\lambda = 0$ gives the circle $x^2 + y^2 - 2x - 4y + 20 = 0$ and $\lambda = -4$ gives the required circle which is $x^2 + y^2 - 18x - 16y + 120 = 0$.

Aliter: Since the radius of the circle $x^2 + y^2 - 2x - 4y - 20 = 0$ is 5 and the radius of the required circle is also 5, it follows that the contact must be external contact and the point $(5, 5)$ must be the midpoint of the segment joining the centres. Hence, if (h, k) is the centre of the required circle, we have

$$\frac{1+h}{2}=5 \text{ and } \frac{2+k}{2}=5$$

$$\Rightarrow h=9, k=8$$

Hence, the equation of the required circle is

$$(x-9)^2+(y-8)^2=25$$

$$\Rightarrow x^2+y^2-18x-16y+120=0$$

Answer: (B) → (p)

(C) We have the equation of the two lines as

$$3x+5y=1$$

and

$$(2+c)x+5c^2y=1$$

On solving these equations, we get

$$x=\frac{1-c^2}{2+c-3c^2}$$

$$y=\frac{c-1}{5(2+c-3c^2)}$$

Therefore

$$\lim_{c\to 1} x=\lim_{c\to 1}\frac{1+c}{2+3c}=\frac{2}{5}$$

and

$$\lim_{c\to 1} y=\lim_{c\to 1}\frac{-1}{5(2+3c)}=\frac{-1}{25}$$

Hence, the centre of the circle = $(2/5, -1/25)$. Since the circle passes through $(2, 0)$, its radius is

$$\sqrt{\left(2-\frac{2}{5}\right)^2+\left(\frac{-1}{25}\right)}=\frac{\sqrt{1601}}{25}$$

Therefore, equation of the required circle is

$$\left(x-\frac{2}{5}\right)^2+\left(y+\frac{1}{25}\right)^2=\frac{1601}{625}$$

That is,

$$25(x^2+y^2)-20x+2y-60=0$$

Answer: (C) → (q)

(D) Let

$$S\equiv x^2+y^2-4x-2y-8=0$$

and

$$S'\equiv x^2+y^2-2x-4y-8=0$$

Let

$$L\equiv S-S'\equiv -2x+2y=0$$

Hence, the required circle is of the form

$$S+\lambda L\equiv x^2+y^2-4x-2y-8+\lambda(x-y)=0$$

This passes through $(-1, 4)$. This implies

$$1+16+4-8-8+\lambda(-1-4)=0 \Rightarrow \lambda=1$$

Therefore, the required circle is

$$x^2+y^2-3x-3y-8=0$$

Answer: (D) → (r)

3. Match the items of Column I with those of Column II.

Column I	*Column II*
(A) The equation of the circle passing through the points (1, 1), (2, 2) and having radius 1 is	(p) $x^2 + y^2 - 4x - 2y + 4 = 0$
(B) If the two lines $3x - 2y - 8 = 0$ and $2x - y - 5 = 0$ lie along two diameters of a circle which touches the x-axis, then the equation of the circle is	(q) $x^2 + y^2 - 2x + 4y - 11 = 0$
	(r) $x^2 + y^2 - 4x + 2y + 4 = 0$
(C) The equation of the circle which touches the lines $4x - 3y + 10 = 0$ and $4x - 3y - 30 = 0$ and whose centre lies on the line $2x + y = 0$ is	(s) $3x^2 + 3y^2 - 10x - 6y - 45 = 0$
(D) Equation of the circle which passes through the points (0, 5) (6, 1) whose centre lies on the line $12x + 5y = 25$ is	(t) $x^2 + y^2 - 2x - 4y + 4 = 0$

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Solution:

(A) Let $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ be the required circle. Therefore

$$2g + 2f + c = -2 \tag{3.103}$$

and

$$4g + 4f + c = -8 \tag{3.104}$$

The radius is 1. This implies that

$$g^2 + f^2 + c = 1 \tag{3.105}$$

Solving Eqs. (3.103)–(3.105), we have $g = -1, f = -2$ and $c = 4$ or $g = -2, f = -1$ and $c = 4$. Therefore, there are two circles

$$x^2 + y^2 - 2x - 4y + 4 = 0$$

and

$$x^2 + y^2 - 4x - 2y + 4 = 0$$

which satisfy the given conditions.

Answer: (A) → (p), (t)

(B) The point of intersection of the lines $3x - 2y - 8 = 0$ and $2x - y - 5 = 0$ is $(2, -1)$ which is the centre of the required circle. Since the circle touches the x-axis, its radius must be the absolute value of the ordinate of the centre. Hence, the radius is 1. Therefore, the required circle is $x^2 + y^2 - 4x + 2y + 4 = 0$.

Answer: (B) → (r)

(C) The two lines $4x - 3y + 10 = 0$, $4x - 3y - 30 = 0$ are parallel tangents to the required circle. Therefore, the length of the diameter of the required circle is the distance between the parallel tangents which is equal to

$$\frac{|-30 - 10|}{\sqrt{4^2 + 3^2}} = 8$$

Hence, the radius of the required circle is 4. Also the line $2x + y = 0$ intersects the two parallel tangents in the points $A(-1, 2)$ and $B(3, 6)$ (these points A and B are obtained on substituting $y = -2x$ in the tangents equations). Therefore, the centre of the required circle is $(1, -2)$ and the circle equation is given by

$$(x-1)^2 + (y+2)^2 = 4^2$$

$$\Rightarrow x^2 + y^2 - 2x + 4y - 11 = 0$$

Answer: (C) → (q)

(D) Let $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ be the required circle. It passes through $(0, 5)$. This implies that

$$10f + c = -25 \tag{3.106}$$

It passes through $(6, 1)$. This implies that

$$12g + 2f + c = -37 \tag{3.107}$$

The centre lies on the line $12x + 5y = 25$. This implies that

$$-12g - 5f = 25 \tag{3.108}$$

Solving Eqs. (3.106)–(3.108), we get $g = -5/3, f = -1$ and $c = -15$. Therefore, the required circle is

$$S \equiv x^2 + y^2 - \frac{10x}{3} - 2y - 15 = 0$$

$$\Rightarrow 3x^2 + 3y^2 - 10x - 6y - 45 = 0$$

Answer: (D) → (s)

4. Match the items of Column I with those of Column II.

Column I	*Column II*
(A) Equations of the circle circumscribing the rectangle whose sides are $x - 3y - 4 = 0$, $3x + y - 22 = 0$, $x - 3y - 14 = 0$ and $3x + y - 62 = 0$ is	(p) $x^2 + y^2 - 9x - 9y + 36 = 0$
(B) Two vertices of an equilateral triangle are $(-1, 0)$ and $(1, 0)$ and its third vertex lies above the x-axis. The equation of its circumcircle is	(q) $\sqrt{3}(x^2 + y^2) - 2y - \sqrt{3} = 0$
(C) The equations of a circle with origin at centre and passing through the vertices of an equilateral triangle whose median is of length 6 units is	(r) $x^2 + y^2 - 9 = 0$
(D) The vertices of a triangle are (6, 0), (0, 6) and (7, 7). The equation of the incircle of the triangle is	(s) $x^2 + y^2 - 27x - 3y + 142 = 0$
	(t) $x^2 + y^2 - 16 = 0$

Solution:

(A) Solving the given equations, the two opposite vertices of the rectangle are obtained as $(7, 1)$ and $(20, 2)$ and hence the equation of the circle is

$$(x - 7)(x - 20) + (y - 1)(y - 2) = 0$$

$$\Rightarrow x^2 + y^2 - 27x - 3y + 142 = 0$$

Answer: (A) → (s)

(B) Suppose $A = (-1, 0)$ and $B = (1, 0)$ which lies on x-axis such that the origin is the midpoint of the segment $\overline{AB}$ (see Fig. 3.77). Hence, the third vertex lies on the positive y-axis. Suppose the third vertex is $C(0, k)$. Since CO is the median and CO is perpendicular to the side AB,

$$\frac{\sqrt{3}}{2} = \sin 60° = \frac{CO}{2} \qquad (\because AB = BC = CA = 2)$$

Therefore

$$CO = \sqrt{3}$$
$$\Rightarrow k = \sqrt{3}$$

So $C = (0, \sqrt{3})$. In an equilateral triangle, the circumcentre and the centroid are the same. If G is the centroid of $\Delta\, ABC$, then

$$CG : GO = 2:1 \Rightarrow G = \left(0, \frac{k}{3}\right) = \left(0, \frac{1}{\sqrt{3}}\right)$$

Therefore, the radius of the circumcircle is $1/\sqrt{3}$. Hence, the circumcircle equation is given by

$$(x-0)^2 + \left(y - \frac{1}{\sqrt{3}}\right)^2 = \frac{4}{3}$$
$$\Rightarrow x^2 + y^2 - \frac{2y}{\sqrt{3}} + \frac{1}{3} = \frac{4}{3}$$
$$\Rightarrow \sqrt{3}(x^2+y^2) - 2y - \sqrt{3} = 0$$

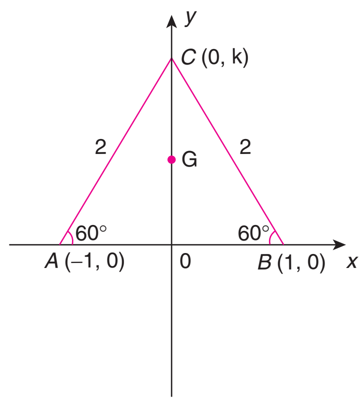


FIGURE 3.77

Answer: (B) → (q)

(C) In an equilateral triangle, the circumcentre and the centroid are the same and hence the circumcentre is (0, 0). Since the centroid divides the median in the ratio 2:1 (from vertex to the base) it follows that the radius is $(2/3)6 = 4$. Hence, the circle equation is $x^2 + y^2 = 16$.

Answer: (C) → (t)

(D) Perimeter of the triangle is $16\sqrt{2}$. Using the incentre formula, the incentre is obtained as

$$\left(\frac{72\sqrt{2}}{16\sqrt{2}}, \frac{72\sqrt{2}}{16\sqrt{2}}\right) = \left(\frac{9}{2}, \frac{9}{2}\right)$$

We know that if r is the radius of the incircle, then it is the distance of the centre from any side of the triangle (see Fig. 3.78). One side of the triangle is $x + y = 6$. Therefore

$$r = \frac{\left|\frac{9}{2} + \frac{9}{2} - 6\right|}{\sqrt{1^2+1^2}} = \frac{3}{\sqrt{2}}$$

Therefore, the incircle equation is

$$\left(x - \frac{9}{2}\right)^2 + \left(y - \frac{9}{2}\right)^2 = \frac{9}{2}$$
$$\Rightarrow x^2 + y^2 - 9x - 9y + 36 = 0$$

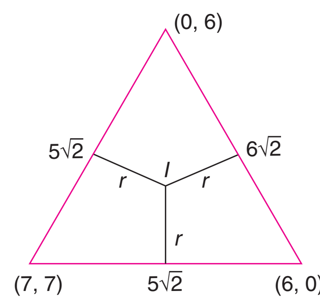


FIGURE 3.78

Answer: (D) → (p)

5. Match the items of Column I with those of Column II.

Column I	*Column II*
(A) The equation of the locus of the midpoints of the chords of the circle $4x^2+4y^2-12x+4y+1=0$ that subtend an angle $2\pi/3$ at its centre is	(p) $16(x^2+y^2) - 48x + 16y + 31 = 0$
(B) The locus of the midpoint of chords of the circle $x^2+y^2-2x-2y-2=0$ which subtend an angle $2\pi/3$ at the centre is	(q) $x^2+y^2-2x-2y+1=0$
(C) The locus of the midpoint of the chords of the circle $x^2+y^2=4$ such that the segment intercepted by the chord on the curve $x^2-2x-2y=0$ subtends a right angle at the origin is	(r) $x^2+y^2-2x-2y=2$

(*Continued*)

Column I	*Column II*
(D) Through the point (2, 3), secants are drawn to the circle $x^2 + y^2 = 4$. Then, the locus of the midpoints of these secants intercepted by the circle is	(s) $x^2 + y^2 - 2x - 3y = 0$ (t) $x^2 + y^2 - 3x - 2y + 4 = 0$

Solution:

(A) Given circle is $S \equiv x^2 + y^2 - 3x + y + (1/4) = 0$. $A = (3/2, -1/2)$ is the centre and $r = 3/2$ is the radius. Let $M(x_1, y_1)$ be the midpoint of a chord BC. Therefore, AM is perpendicular to BC and from Fig. 3.79, we have

$$\underline{|BAM} = \underline{|CAM} = 60°$$

Now

$$\cos 60° = \frac{AM}{AB} \Rightarrow \frac{1}{2} = \frac{AM}{AB}$$

$$\Rightarrow \frac{3}{2} = 2AM$$

$$\Rightarrow 9 = 16\left[\left(x_1 - \frac{3}{2}\right)^2 + \left(y_1 + \frac{1}{2}\right)^2\right]$$

$$\Rightarrow 9 = 16\left[x_1^2 + y_1^2 - 3x_1 + y_1 + \frac{9}{4} + \frac{1}{4}\right]$$

Therefore, the locus is

$$16(x^2 + y^2) - 48x + 16y + 31 = 0$$

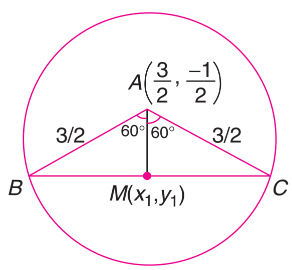


FIGURE 3.79

Answer: (A) → (p)

(B) $A = (1, 1)$ is the centre and $r = 2$ is the radius. Proceeding as above, we get $1/2 = AM/2$. Therefore

$$(AM)^2 = 1$$

$$\Rightarrow (x_1 - 1)^2 + (y_1 - 1)^2 = 1$$

Therefore, the locus is $x^2 + y^2 - 2x - 2y + 1 = 0$.

Answer: (B) → (q)

(C) Let $M(x_1, y_1)$ be the midpoint of a chord so that its equation is $xx_1 + yy_1 = x_1{}^2 + y_1{}^2$. Suppose the chord intersects the curve $x^2 = 2(x + y)$ in two points P and Q. Therefore, the combined equation of the pair of lines OP and OQ is obtained as

$$x^2 = 2(x+y)\left(\frac{xx_1 + yy_1}{x_1^2 + y_1^2}\right)$$

$$\Rightarrow (x_1^2 + y_1^2)x^2 - 2(x+y)(xx_1 + yy_1) = 0 \quad (3.109)$$

Since $\underline{|POQ} = 90°$, from Eq. (3.109), we have

Coefficient of x^2 + Coefficient of $y^2 = 0$

Therefore

$$(x_1^2 + y_1^2 - 2x_1) - 2y_1 = 0$$

So, the locus of (x_1, y_1) is

$$x^2 + y^2 - 2x - 2y = 0$$

Answer: (C) → (r)

(D) Let $M(x_1, y_1)$ be the midpoint of a secant chord through (2, 3) of $x^2 + y^2 = 4$ so that its equation is

$$xx_1 + yy_1 = x_1^2 + y_1^2$$

This passes through the point (2, 3). This implies

$$2x_1 + 3y_1 = x_1^2 + y_1^2$$

Therefore, the locus of (x_1, y_1) is

$$x^2 + y^2 - 2x - 3y = 0$$

Answer: (D) → (s)

Comprehension Type Questions

1. Passage: If $S = 0$ is a circle in standard form and $L = 0$ is a straight line in the plane of the circle, then, in general, the equation $S + \lambda L = 0$ represents a circle. If $L = 0$ cuts $S = 0$ in two points P and Q, then $S + \lambda L = 0$ represents circle passing through P and Q. In particular, if $L = 0$ touches the circle $S = 0$, then $S + \lambda L = 0$ also touches $L = 0$. Answer the following questions.

(i) The equation of the circle whose centre lies on the line $x + y - 11 = 0$ and which passes through the intersection of the circle $x^2 + y^2 - 3x + 2y - 4 = 0$ and the line $2x + 5y + 2 = 0$ is

(A) $x^2 + y^2 + 9x + 13y + 10 = 0$

(B) $x^2 + y^2 - 9x + 13y - 10 = 0$

(C) $x^2 + y^2 - 9x - 13y - 10 = 0$

(D) $x^2 + y^2 + 9x - 13y + 10 = 0$

(ii) The equation of the circle touching the line $2x+3y+1=0$ at $(1,-1)$ and cutting orthogonally the circle having the line segment joining (0, 3) and $(-2,-1)$ as diameter is

(A) $2x^2+2y^2-10x+5y-1=0$

(B) $x^2+y^2-10x-5y+1=0$

(C) $2(x^2+y^2)+10x+5y-1=0$

(D) $2(x^2+y^2)-10x-5y+1=0$

(iii) The equation of the circle passing through the intersection of circles $x^2+y^2=4$ and $x^2+y^2-2x-4y+4=0$ and touching the line $x+2y-5=0$ is

(A) $x^2+y^2-x-2y=0$

(B) $x^2+y^2-x+2y=0$

(C) $x^2+y^2+x+2y=0$

(D) $x^2+y^2+x-2y=0$

(IIT-JEE 2004)

Solution:

(i) Let $S\equiv x^2+y^2-3x+2y-4=0$ and $L\equiv 2x+5y+2=0$. The required circle is of the form

$$S+\lambda L\equiv x^2+y^2-3x+2y-4+\lambda(2x+5y+2)=0$$

whose centre is

$$\left(\frac{3-2\lambda}{2},\frac{-(5\lambda+2)}{2}\right)$$

which lies on the line $x+y-11=0$. That is,

$$3-2\lambda-5\lambda-2-22=0$$
$$\Rightarrow -7\lambda-21=0$$
$$\Rightarrow \lambda=-3$$

Therefore, the required circle is $x^2+y^2-9x-13y-10=0$.

Answer: (C)

(ii) The circle described on the line joining the points (0, 3) and $(-2,-1)$ as diameter is

$$x(x+2)+(y-3)(y+1)=0$$

$$\Rightarrow x^2+y^2+2x-2y-3=0 \qquad (3.110)$$

The required circle is of the form

$$(x-1)^2+(y+1)^2+\lambda(2x+3y+1)=0$$

$$\Rightarrow x^2+y^2+2(\lambda-1)x+2\left(\frac{3\lambda}{2}+1\right)y+2+\lambda=0 \qquad (3.111)$$

The circle given in Eq. (3.111) cuts orthogonally the circle given in Eq. (3.110). This implies that

$$2(\lambda-1)(1)+2\left(\frac{3\lambda}{2+1}\right)(-1)=2+\lambda-3$$

$$\Rightarrow -\lambda-4=\lambda-1$$

$$\Rightarrow \lambda=\frac{-3}{2}$$

Therefore, the required circle is

$$x^2+y^2-5x-\frac{5}{2}y+2-\frac{3}{2}=0$$

$$\Rightarrow 2x^2+2y^2-10x-5y+1=0$$

Answer: (D)

(iii) Let $S\equiv x^2+y^2-4=0$ and $S'\equiv x^2+y^2-2x-4y+4=0$. Let $L\equiv S-S'\equiv 2x+4y-8=0$. That is,

$$L\equiv x+2y-4=0$$

Now, the required circle is

$$S+\lambda L\equiv x^2+y^2-4+\lambda(x+2y-4)=0$$

$$\equiv x^2+y^2+2\left(\frac{\lambda}{2}\right)x+2\lambda y-4-4\lambda=0$$

Therefore, the centre is

$$\left(\frac{-\lambda}{2},-\lambda\right)$$

and the radius is

$$\sqrt{\frac{\lambda^2}{4}+\lambda^2+4+4\lambda}$$

$S+\lambda L=0$ touches the line $x+2y-5=0$. This implies that

$$\frac{\left|\frac{-\lambda}{2}+2(-\lambda)-5\right|}{\sqrt{1^2+2^2}}=\sqrt{\frac{\lambda^2}{4}+\lambda^2+4+4\lambda}$$

$$\Rightarrow \frac{|5\lambda+10|}{2\sqrt{5}}=\frac{1}{2}\sqrt{5\lambda^2+16\lambda+16}$$

$$\Rightarrow 5(\lambda+2)^2=5\lambda^2+16\lambda+16$$

$$\Rightarrow 4\lambda+4=0$$

$$\Rightarrow \lambda=-1$$

$\lambda=-1$ gives the circle $x^2+y^2-x-2y=0$.

Answer: (A)

2. Passage: $ABCD$ is a square of side length 2 units. C_1 is the circle inscribed in $ABCD$ and C_2 is the circumcircle of $ABCD$. L is a fixed line in the same plane through the point A. Answer the following questions:

(i) If P is any point on C_1 and Q is any point on C_2, then

$$\frac{(PA)^2+(PB)^2+(PC)^2+(PD)^2}{(QA)^2+(QB)^2+(QC)^2+(QD)^2}=$$

(A) 0.75 (B) 1.25 (C) 1 (D) 0.5

(ii) A circle touches the line L and C_1 externally such that both circles are on the same side of L. Then the locus of the centre of C is

(A) an ellipse (B) a parabola
(C) hyperbola (D) a pair of lines

(iii) A line M through A is drawn parallel to BD. Point S moves such that its distance from the line BD and vertex are equal. If locus of S cuts M at T_2 and T_3 and AC at T_1. Then the area of $\Delta\, T_1T_2T_3$ is

(A) $\frac{1}{2}$ sq. unit (B) $\frac{2}{3}$ sq. unit
(C) 1 sq. unit (D) 2 sq. unit

Solution:

(i) Let O be the centre of the square so that O is the centre of both C_1 and C_2 (see Fig. 3.80). Take O as origin. Let $\overrightarrow{OA} = \vec{\alpha}$ and $\overrightarrow{OB} = \vec{\beta}$ so that $\overrightarrow{OC} = -\vec{\alpha}$, $\overrightarrow{OD} = -\vec{\beta}$ and $|\overrightarrow{OA}| = |\vec{\alpha}| = \sqrt{2}$ and $|\vec{\beta}| = \sqrt{2}$. Let $\overrightarrow{OP} = \vec{\beta}$ and $|\overrightarrow{OC}| = \vec{q}$ so that $|\vec{\beta}| = 1$ and $|\vec{q}| = \sqrt{2}$. Now

$$\begin{aligned}
&(PA)^2+(PB)^2+(PC)^2+(PD)^2\\
&=(\vec{\alpha}+\vec{p})+(\vec{\beta}+\vec{q})+(\vec{\alpha}+\vec{p})+(\vec{\beta}+\vec{p})^2\\
&=2(\alpha^2+p^2)+2(\beta^2+p^2)\\
&=2(2+1)+2(2+1)=12
\end{aligned}$$

Also

$$(QA)^2+(QB)^2+(QC)^2+(QD)^2$$
$$=2(2+2)+2(2+2)=16$$

Therefore

$$\frac{\sum (PA)^2}{\sum (QA)^2}=\frac{12}{16}=\frac{3}{4}=0.75$$

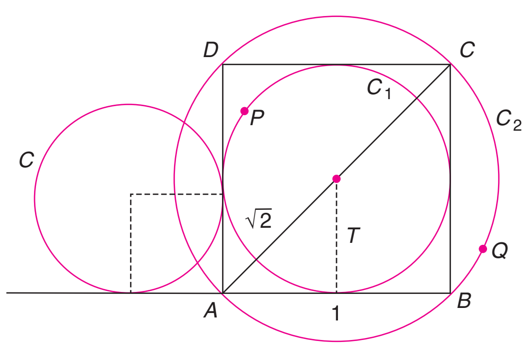


FIGURE 3.80

Answer: (A)

(ii) Suppose L_1 is a line parallel to L at a unit distance from L such that L_1 and the circle C_1 are on opposite sides to the line L. Therefore, the centre of the circle C is equidistant from the line L_1 and the centre of the circle C_2. Hence, the centre of C is a parabola with focus at the centre of C_2 and directrix L_1.

Answer: (B)

(iii) Clearly, the locus S is a parabola with A as a focus and BD as directrix (see Fig. 3.81).

$$AT_1 = OT_1 = \frac{1}{2}OA = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

$T_2\,T_3$ is the latus rectum which is given by

$$4\times\frac{1}{\sqrt{2}} = 2\sqrt{2}$$

Therefore, the area of $\Delta T_1T_2T_3$ is

$$\frac{1}{2}T_2T_3\cdot AT_1 = \frac{1}{2}\times 2\sqrt{2}\times\frac{1}{\sqrt{2}} = 1$$

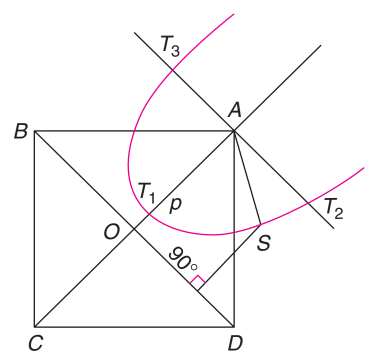


FIGURE 3.81

Answer: (C)

3. Passage: A circle C of radius 1 is inscribed in a equilateral PQR. The points of control of C with the side PQ, QR and RP are D, E and F, respectively. The line PQ is given by the equation $\sqrt{3}x+y-6=0$ and the point D is $(3\sqrt{3}/2, 3/2)$. Further, it is given that the origin and the centre of C are on the same side of the side PQ. Answer the following questions.

(i) The equation of the circle C is

(A) $(x-2\sqrt{3})^2+(y-1)^2=1$

(B) $(x-2\sqrt{3})^2+\left(y+\frac{1}{2}\right)^2=1$

(C) $(x-\sqrt{3})^2+(y+1)^2=1$

(D) $(x-\sqrt{3})^2+(y-1)^2=1$

(ii) Points E and F are given by

(A) $\left(\frac{\sqrt{3}}{2},\frac{3}{2}\right),(\sqrt{3},0)$

(B) $\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right),(\sqrt{3},0)$

(C) $\left(\frac{\sqrt{3}}{2},\frac{3}{2}\right),\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)$

(D) $\left(\frac{3}{2},\frac{\sqrt{3}}{2}\right),\left(\frac{3}{2},\frac{1}{2}\right)$

(iii) Equation of the sides QR and RP are

(A) $y=\frac{2}{\sqrt{3}}x+1, y=\frac{-2}{\sqrt{3}}x+1$

(B) $y=\frac{x}{\sqrt{3}}, y=0$

(C) $y=\frac{\sqrt{3}}{2}x+1, y=\frac{-\sqrt{3}}{2}x-1$

(D) $y=\sqrt{3}x, y=0$

Solution:

(i) C is the incircle of ΔPQR (see Fig. 3.82) where its incentre is I and its radius is 1 (by hypothesis). In an equilateral triangle, the altitudes are the angle bisector. The incentre I lies on the altitude DR whose equation is

$$y-\frac{3}{2}=\frac{1}{\sqrt{3}}\left(x-\frac{3\sqrt{3}}{2}\right)$$

$$\Rightarrow x=\sqrt{3}y$$

Let I be (x_1, y_1) so that $x_1=\sqrt{3}y_1$. Therefore,

$$\left(x_1-\frac{3\sqrt{3}}{2}\right)^2+\left(y_1-\frac{3}{2}\right)^2=1 \qquad (\because ID=1)$$

$$\left(\sqrt{3}y_1-\frac{3\sqrt{3}}{2}\right)^2+\left(y_1-\frac{3}{2}\right)^2=1 \qquad (\because x_1=\sqrt{3}y_1)$$

Therefore

$$(2\sqrt{3}y_1-3\sqrt{3})^2+(2y_1-3)^2=4$$

$$\Rightarrow 16y_1^2-48y_1+32=0$$

$$\Rightarrow y_1^2-3y_1+2=0$$

$$\Rightarrow y_1=1,2$$

Hence, $y_1=2 \Rightarrow x_1=2\sqrt{3}$ and $y_1=1 \Rightarrow x_1=\sqrt{3}$. Incentre I and the origin lie on the same side of PQ $\Rightarrow I=(\sqrt{3},1)$. Therefore, equation of the circle C is

$$(x-\sqrt{3})^2+(y-1)^2=1$$

Answer: (D)

(ii) Since PQ makes angle 120° with the positive direction of the x-axis and $\angle QPR=60°$, it is clear that PR is parallel to the x-axis (see Fig. 3.82). Also the distance of I from the side PR is 1 implies that PR is the x-axis and R is the origin. Now, the equation of PR is $y=0$ and that of QR is $y=\sqrt{3}x$. Therefore

$$F=(\sqrt{3},0)$$

and

$$E=(\sqrt{3}\cos 60°, \sqrt{3}\sin 60°)=\left(\frac{\sqrt{3}}{2},\frac{3}{2}\right)$$

Answer: (A)

(iii) See Fig. 3.83. $R=(0,0)$ and $F=(\sqrt{3},0)$. This implies that the equation PR is $y=0$. Again $R=(0,0)$ and $E=\left(\frac{\sqrt{3}}{2},\frac{3}{2}\right)$. Therefore, the equation of QR is $y=\sqrt{3}x$.

Answer: (D)

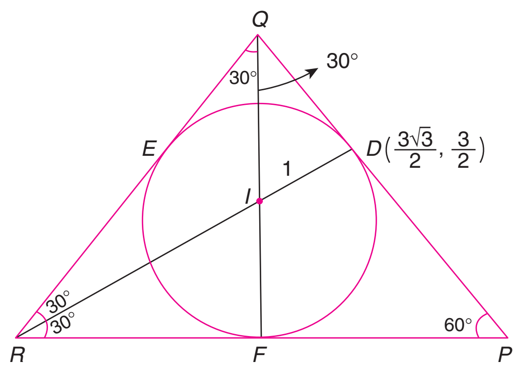


FIGURE 3.82

4. Passage: Let Σ be a family of circles passing through the two points $P(3,7)$ and $Q(6,5)$. Answer the following questions.

(i) The number of circles belonging to Σ and touching the x-axis is

(A) 1 (B) 2
(C) infinite (D) 0

(ii) If each of the circles of Σ cuts the circle $x^2+y^2-4x-6y-3=0$, then all these chords pass through a fixed point whose coordinates are

(A) $\left(2+\frac{23}{3}\right)$ (B) $\left(-3+\frac{23}{3}\right)$
(C) $(1,23)$ (D) $(23,1)$

(iii) The centre of the circle belonging to Σ and cutting orthogonally the circle $x^2+y^2=29$ is

(A) $\left(\frac{1}{2},\frac{3}{2}\right)$ (B) $\left(\frac{7}{2},\frac{9}{2}\right)$
(C) $\left(3,-\frac{7}{9}\right)$ (D) $\left(\frac{65}{18},\frac{14}{3}\right)$

(i) Equation of PQ is $2x + 3y - 27 = 0$. Also the circle described on PQ is a diameter is $S \equiv x^2 + y^2 - 9x - 12y + 53 = 0$. Any circle passing through P and Q is of the form

$$x^2 + y^2 - 9x - 12y + 53 + \lambda(2x + 3y - 27) = 0$$
$$\Rightarrow x^2 + y^2 - (9 - 2\lambda)x - (12 - 3\lambda)y + 53 - 27\lambda = 0 \quad (3.112)$$

This touches the x-axis. That is

$$\frac{|12 - 3\lambda|^2}{4} = 53 - 27\lambda$$
$$\Rightarrow 9\lambda^2 - 72\lambda + 144 = 212 - 108\lambda$$
$$\Rightarrow 9\lambda^2 + 36\lambda - 68 = 0$$

The discriminant of the quadratic is positive so that it has two distant roots. Hence, there are two circles belonging to Σ which touch the x-axis.

Answer: (B)

(ii) Let

$$S' \equiv S + \lambda L$$
$$= x^2 + y^2 - 9x - 12y + 53 + \lambda(2x + 3y - 27) = 0$$
$$S'' \equiv x^2 + y^2 - 4x - 6y - 3 = 0$$

Common chord of $S' = 0$ and $S'' = 0$ is $S' - S'' = -5x - 6y + 50 + \lambda(2x + 3y - 27) = 0$.

This chord passes through the intersection of $-5x - 6y + 56 = 0$ and $2x + 3y - 27 = 0$ which is $(2, 23/3)$.

Answer: (A)

(iii) The circle given in Eq. (3.112) cuts orthogonally the circle

$$x^2 + y^2 = 9$$
$$\Rightarrow \frac{2(2\lambda - 9)}{2}(0) + \frac{2(2\lambda - 12)}{2}(0)$$
$$= 53 - 27\lambda - 29$$
$$\Rightarrow 27\lambda = 24$$
$$\Rightarrow \lambda = \frac{8}{9}$$

Therefore, the circle is

$$x^2 + y^2 - 9x - 12y + 53 + \frac{8}{9}(2x + 3y - 27) = 0$$
$$\Rightarrow x^2 + y^2 + \left(\frac{16}{9} - 9\right)x + \left(\frac{24}{9} - 12\right)y + 53 - 24 = 0$$
$$\Rightarrow x^2 + y^2 - \frac{65}{9}x - \frac{28}{3}y + 29 = 0$$

Therefore, the centre is

$$\left(\frac{65}{18}, \frac{14}{3}\right)$$

Answer: (D)

Integer Answer Type Questions

1. If $x^2 + y^2 - 4x - 4y - k = 0$ is the equation of the locus of the point from which perpendicular tangents can be drawn to the circle $x^2 + y^2 - 4x - 4y = 0$, then the value k is _____.

Solution: The equation

$$x^2 + y^2 - 4x - 4y = 0 \quad (3.113)$$

represents a circle with centre (2, 2) and radius $\sqrt{8} = 2\sqrt{2}$. Then the locus of point from which perpendicular tangents can be drawn to the circle given in Eq. (3.113) is a concentric circle whose radius is $\sqrt{2}$ times the radius of the circle [this circle is called the director circle of the circle given in Eq. (3.113)]. Hence, the radius of the director circle is $\sqrt{2}(2\sqrt{2}) = 4$. Thus $k = 8$.

Answer: 8

2. If the equation of the chord of contact of (h, k) with respect to the circle $x^2 + y^2 - 4x - 2y - 11 = 0$ is $x + 2y - 12 = 0$, then $h + k$ is equal to _____.

Solution: The equation

$$x + 2y - 12 = 0 \quad (3.114)$$

is chord of contract of (h, k) with respect to the circle $S \equiv x^2 + y^2 - 4x - 2y - 11 = 0$. But the chord of contact is

$$xh + yk - 2(x + h) - (y + k) - 11 = 0$$
$$\Rightarrow (h - 2)x + (k - 1)y - (2h + k + 11) = 0 \quad (3.115)$$

Equations (3.114) and (3.115) represent the same straight line. Therefore

$$\frac{h - 2}{1} = \frac{k - 1}{2} = \frac{-(2h + k + 11)}{-12} = \lambda \quad \text{(say)}$$

Hence

$$h = \lambda + 2$$
$$k = 2\lambda + 1$$

Now

$$12\lambda = 2(\lambda + 2) + (1 + 2\lambda) + 11$$

so that

$$8\lambda = 16 \Rightarrow \lambda = 2$$

Therefore

$$h = \lambda + 2 = 4$$
$$k = 1 + 2\lambda = 5$$

so that

$$h + k = 9$$

Answer: 9

3. The line $\lambda x - y + 1 = 0$ cuts the coordinate axes at points P and Q. The line $x - 2y + 3 = 0$ intersects the coordinate axes at points R and S. If P, Q, R and S are concyclic, then the value of λ is _____.

Solution: Equation of the circle passing through points P, Q, R and S is of the form

$$(\lambda x - y + 1)(x - 2y + 3) + \mu(xy) = 0$$

This equation represents a circle if the coefficient of x^2 is equal to the coefficient of y^2 and coefficient of $xy = 0$. Therefore,

$$\lambda = 2 \text{ and } \mu = 5$$

Hence, the value of $\lambda = 2$.

Answer: 2

4. A rational point in an analytical plane means that both the coordinates of the point are rational numbers. Then the maximum number of rational points on a circle C with centre $(0, \sqrt{2})$ is _____.

Solution: Suppose, there are three rational points on circle C. Now consider that the triangle with those rational point vertices whose circumcenter is $(0, \sqrt{2})$. Since the vertices of this triangle are rational points, its equations of the perpendicular bisectors are first-degree equations in x and y with rational coefficients and hence the circumcentre must be a rational point, but here it is not a rational point because $(0, \sqrt{2})$ is the circumcentre. Hence, maximum number of rational points on the circle is 2.

Answer: 2

5. Three circles of radii 3, 4 and 5 units touch each other externally. The tangents drawn at the points of contact are concurrent at point P. If k is the distance of P from the points of contact, then $[k]$ (which is the integer part of k) is _____.

Solution: Let A, B and C be the centres of the circles with radii 3, 4 and 5, respectively. Hence, the sides of the $\Delta\, ABC$ are $AB = 7$, $BC = 9$ and $CA = 8$. Therefore, $s = 12$ and $\Delta = \sqrt{12\times3\times4\times5} = 12\sqrt{5}$ k is the inradius of $\Delta\, ABC$ which is given by

$$\frac{\Delta}{s} = \sqrt{5}$$

Therefore, $[k] = 2$.

Answer: 2

6. A and B are two fixed points in a plane and $k > 0$, Then the locus of P such that $PA:PB = k:1$ is a circle, provided k is *not equal to* _____.

Solution: If $k = 1$, then $PA = PB$ so that the locus of P is the perpendicular bisector of $\overline{AB}$.

Answer: 1

7. Two parallel chords of a circle of radius 2 are at a distance $\sqrt{3}+1$ apart. If the chords subtend at the center, angles π/k and $2\pi/k$ where $k > 0$, then $[k]$ (which is the integer part of k) is _____.

(IIT-JEE 2010)

Solution: Since the distance between the chords is $\sqrt{3} + 1 > 2$, the chords must be on the opposite sides of the origin (see Fig. 3.83). $\sqrt{3} + 1$ is the distance between PQ and RS which is given by

$$2\cos\frac{\pi}{k} + 2\cos\frac{\pi}{2k}$$

This holds when

$$\frac{\pi}{k} = \frac{\pi}{3}$$

so that $k = 3$.

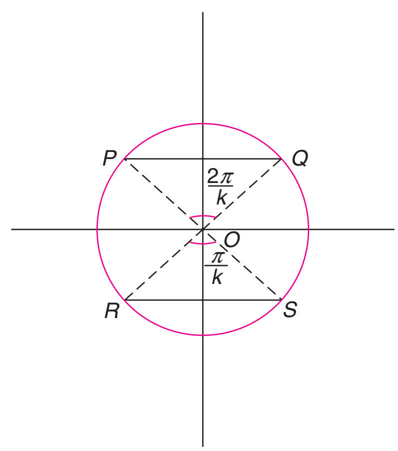


FIGURE 3.83

Answer: 3

8. The centres of two circles C_1 and C_2 each of unit radius are at a distance of 6 units from each other. Let P be the midpoint of the line segment joining the centres of C_1 and C_2 and C be a circle touching the circles C_1 and C_2 externally. If a common tangent to C_1, which is passing through P, is also a common tangent to C_1 and C_2, then the radius of C is _____.

(IIT-JEE 2009)

Solution: We have

$$MN = MP + PN = \sqrt{3^2 - 1^2} + \sqrt{3^2 - 1^2} = 4\sqrt{2}$$

However, $MN = BC = \sqrt{OB^2 - OC^2}$, where BC is drawn parallel to MN meeting OA in C (see Fig. 3.84). Therefore

$$\sqrt{(R+1)^2 - (R-1)^2}$$

where R is the radius of the circle. Hence

$$16 \times 2 = 4R$$
$$\Rightarrow R = 8$$

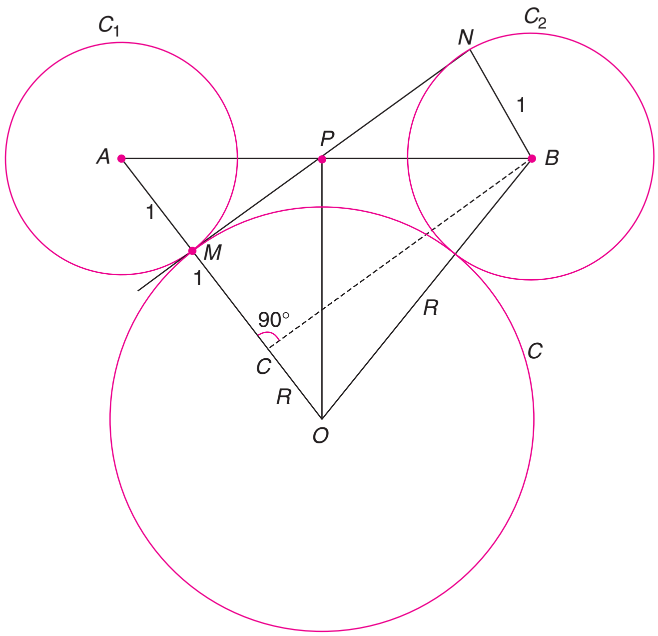


FIGURE 3.84

Answer: 8

9. If Δ is the area of the triangle formed by the positive x-axis and the normal and the tangent to the circle $x^2 + y^2 = 4$ at $(1, \sqrt{3})$, then $[\Delta]$ (which is integer part of Δ is _____.

Solution: Tangent at $(1, \sqrt{3})$ is $x(1) + y(\sqrt{3}) = 4$ (see Fig. 3.85). $y = 0 \Rightarrow x = 4$. The tangent meets the x-axis at $T(4, 0)$. Therefore

$$PT = \sqrt{(4-1)^2 + 3} = 2\sqrt{3}$$

Hence

$$\Delta = \text{Area of } \Delta OTP = \frac{1}{2} \times PT \times OP = \frac{1}{2}(2\sqrt{3}) \times 2 = 2\sqrt{3}$$

Therefore

$$[\Delta] = [2\sqrt{3}] = 3$$

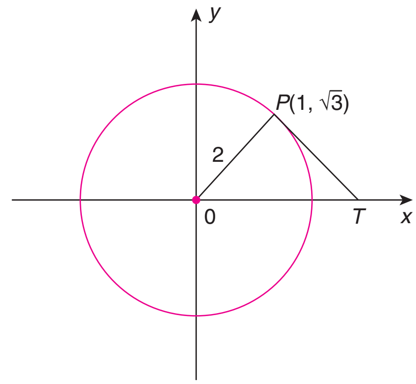


FIGURE 3.85

Answer: 3

10. Let $x^2 + y^2 - 4x - 2y - 11 = 0$ be a circle. A pair of tangents from (4, 5) with a pair of radii form a quadrilateral of area _____.

Solution: See Fig. 3.86. Let A be (4, 5) and AP be a tangent given by

$$\sqrt{S_{11}} = \sqrt{4^2 + 5^2 - 4(4) - 2(5) - 1} = 2$$

Area of the quadrilateral is given by

$$2(\text{Area of } \Delta APC) = 2\left(\frac{1}{2} \times 2 \times 2\right) = 4$$

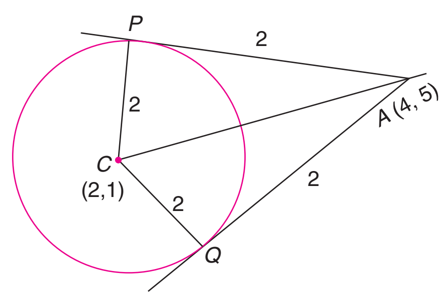


FIGURE 3.86

Answer: 4

11. If the points of intersection of the line $4x - 3y - 10 = 0$ and the circle $x^2 + y^2 - 2x + 4y - 20 = 0$ are (a, b) and (c, d) where a and b are positive and c and d are negative, then $a - c + b + d$ is _____.

Solution: Substituting $y = (4x - 10)/3$ in the circle equation, we get

$$x^2 + \left(\frac{4x-10}{3}\right)^2 - 2x + 4\frac{(4x-10)}{3} - 20 = 0$$

$$\Rightarrow 9x^2+16x^2-80x+100-18x+48x-120-180=0$$
$$\Rightarrow 25x^2-50x-200=0$$
$$\Rightarrow x^2-2x-8=0$$
$$\Rightarrow (x-4)(x+2)=0$$
$$\Rightarrow x=4,-2$$

Calculation

x	$y=\dfrac{4x-10}{3}$
4	2
–2	–6

Hence

$$(a,b)=(4,2) \text{ and } (c,d)=(-2,-6)$$

Therefore

$$a-c+b+d=4+2+2-6=2$$

Answer: 2

12. The line joining the points $A(3, 4)$ and $B(1, 0)$ cuts the circle $x^2+y^2=4$ at points P and Q. If $AP/PQ=\lambda$ and $AQ/QB=\mu$, then the value of $|\lambda\mu|$ is ________.

Solution: The point dividing the segment $\overline{AB}$ in the ratio k:1 is

$$\left(\frac{k+3}{k+1}, \frac{4}{k+1}\right)$$

This lies on the circle

$$x^2+y^2=4$$
$$\Rightarrow (k+3)^2+16=4(k+1)^2$$
$$\Rightarrow 3k^2+2k-21=0$$

for which λ and μ are the roots. Therefore

$$|\lambda\mu|=\left|\frac{-21}{3}\right|=7$$

Answer: 7

13. If $ax+by+c=0$ is the locus of the centre of the circle which passes through the point (1, 2) and cuts orthogonally the circle $x^2+y^2=4$ then $-c-(a-b)$ value is _____.

Solution: Let $S\equiv x^2+y^2+2gx+2fy+c=0$ be the required circle. $S=0$ passes through (1, 2). This implies that

$$2g+4f+c=-5 \tag{3.116}$$

$S=0$ cuts orthogonally $x^2+y^2=4$ implies that

$$2g(0)+2f(0)=c-4$$

Therefore

$$c=4 \tag{3.117}$$

From Eqs. (3.116) and (3.117), we get

$$2g+4f=-9$$

Therefore

$$2(-g)+4(-f)=9$$

Hence, the locus of $(-g,-f)$ is $2x+4y-9=0$. Therefore, $a=2$, $b=4$ and $c=-9$ so that

$$-c-(a+b)=9-6=3$$

Answer: 3

14. If $ax+by+c=0$ is the equation of the common chord of the circles $3x^2+3y^2-2x+12y-9=0$ and $3x^2+3y^2-2x+12y-9=0$ then $b-a-c$ is equal to __________

Solution: The given circles are

$$S\equiv x^2+y^2-(2/3)x+4y-3=0$$

and $$S'\equiv x^2+y^2+6x+2y-15=0$$

Therefore, the common chord is

$$S-S'\equiv\left(\frac{-2x}{3}-6x\right)+(4y-2y)-3+15=0$$
$$\Rightarrow -20x+6y+36=0$$
$$\Rightarrow 10x-3y-18=0$$

Therefore

$$a=10, b=-3 \text{ and } c=-18$$

and hence

$$b-a-c=-13+18=5$$

Answer: 5

15. The lines $3x-4y+4=0$ and $6x-8y-22=0$ are the tangents to the same circle whose diameters is _____.

Solution: Since the lines $3x-4y+4=0$ and $6x-8y-22=0$ are parallel lines, the distance between them is the diameter of the circle. Therefore the diameter of the circle is

$$\frac{|4+11|}{\sqrt{3^2+4^2}}=\frac{15}{5}=3$$

Answer: 3

16. The value of r such that the area of the triangle formed by the pair of tangents drawn to the circle $x^2+y^2=r^2$ from the point $P(6, 8)$ and its chord of contact is maximum is ____. **(IIT-JEE 2003)**

Solution: $PA = PB$ which is equal to the length of the tangent (see Fig. 3.87). This is given by

$$\sqrt{6^2+8^2-r^2} = \sqrt{100-r^2}$$

In Problem 28 in the section 'Subjective Problems (Sections 3.1 till 3.3)', it is worked out that the area of ΔPAB is equal to

$$\frac{aS_{11}{}^{3/2}}{x_1{}^2+y_1{}^2}$$

Therefore, the area of ΔPAB is given by

$$\frac{r(100-r^2)^{3/2}}{6^2+8^2}$$

Let $\Delta = \dfrac{r(100-r^2)^{3/2}}{100}$

$$\frac{d\Delta}{dr} = \frac{1}{100}\left[(100-r^2)^{3/2} - \frac{3}{2}r(100-r^2)^{1/2}(2r)\right]$$

$$= \frac{1}{100}[100-r^2]^{1/2}(100-r^2-3r^2)$$

$$\frac{d\Delta}{dr} = 0 \Rightarrow 100-4r^2 = 0 \qquad (\because r \neq 10)$$

$$\Rightarrow r = 5$$

Also $d\Delta/dr > 0$ for $r < 5$ and is negative for $r > 5$. Therefore, Δ is maximum when $r = 5$.

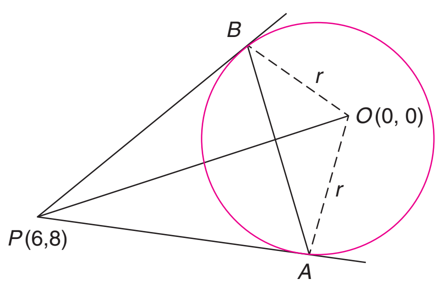


FIGURE 3.87

Answer: 5

17. If (h, k) is the center of smallest circle cutting orthogonally the circles $x^2 + y^2 = 1$ and $x^2 + y^2 + 8x + 8y - 33 = 0$, then $h + k$ is equal to ______.

Solution: Let $S \equiv x^2+y^2+2gx+2fy+c=0$ be the required circle. Therefore, $2g(0)+2f(0)=c-1$. This implies that

$$c = 1 \qquad (3.118)$$

Also

$$2g(4)+2f(4) = c-33 = 1-33 = -32$$

$$\Rightarrow g+f = -4$$

$$\Rightarrow f = -4-g \qquad (3.119)$$

Therefore, from Eqs. (3.118) and (3.119), the radius is

$$\sqrt{g^2+f^2-c} = \sqrt{g^2+(4+g)^2-1}$$

$$= \sqrt{2g^2+8g+15}$$

$$= \sqrt{2[(g+2)^2-4]+15}$$

$$= \sqrt{2(g+2)^2+7}$$

Therefore, the radius is minimum if $g = -2$. In such a case, $f = -2$. Hence

$$(h, k) = \text{Centre} = (-g, -f) = (2, 2)$$

Therefore, $h + k = 4$.

Answer: 4

18. The number of integer values of k for which the chord of the circle $x^2 + y^2 = 125$ passing through the point $P(8, k)$ and is bisected at the point $P(8, k)$ with integer slope is ______.

Solution: Since $P(8, k)$ is the midpoint of the chord, the chord is perpendicular to OP. Slope of OP $= k/8$. Therefore, the slope of the chord is $-8/k$ which is an integer if $k = \pm1, \pm2, \pm4$ and ± 8. If $k = \pm 8$, then the point P lies outside the circle so that it cannot be the midpoint of any chord. Therefore

$$k \neq \pm 8$$

$$k = \pm1, \pm2, \pm4$$

so that the integer values of k is 6.

Answer: 6

19. A light ray gets reflected from the line $x = -2$. If the reflected ray touches the circle $x^2 + y^2 = 4$ at the point of incidence on line is $(-2, -4)$, then the equation of the incident ray is $3x + 4y + k = 0$ where $[k/7]$ (which is the integer part of $k/7$) is _____.

Solution: Let $y = mx + 2\sqrt{1+m^2}$ be a tangent to the circle $x^2 + y^2 = 4$ [see Note (1) of Theorem 3.8] (See Fig. 3.88). This passes through $(-2, -4)$. This implies

$$(2m-4)^2 = 4(1+m^2)$$

$$\Rightarrow -16m = -12$$

$$\Rightarrow m = \frac{3}{4}$$

Hence, the slope of the reflected ray is 3/4. Thus, the slope of the incident ray is $-3/4$. Therefore, the equation of the incident ray is $3x + 4y + 22 = 0$.

Answer: 3

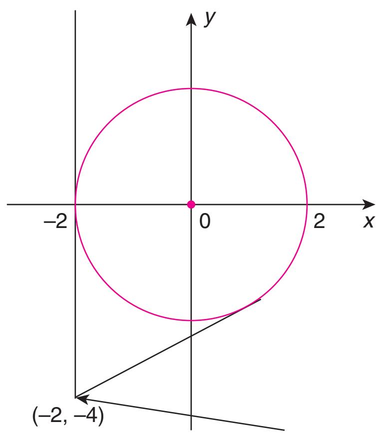


FIGURE 3.88

20. The distance of the common chord of the circles $S \equiv x^2 + y^2 + 5x - 8y + 1 = 0$ and $S'' \equiv x^2 + y^2 - 3x + 7y - 25 = 0$ from the centre of the circle $x^2 + y^2 - 2x = 0$ is ______.

Solution: Clearly, the circles $S = 0$ and $S' = 0$ intersect and hence the equation of the common chord is

$$S - S' \equiv 8x - 15y + 26 = 0 \qquad (3.120)$$

Centre of the circle $x^2 + y^2 - 2x$ is (1, 0). Therefore, the distance of (1, 0) from the line $S - S' = 0$ [i.e. Eq. (3.120)] is

$$\left|\frac{8(1) - 15(0) + 26}{\sqrt{8^2 + 15^2}}\right| = \frac{34}{17} = 2$$

Answer: 2

SUMMARY

3.1 ***Theorem:*** Equation of the circle with centre at the point (x_1, y_1) and radius $r > 0$ is

$$(x - x_1)^2 + (y - y_1)^2 = r^2$$

If $x_1 = y_1 = 0$, then the equation of the circle with centre at the point origin with radius r is $x^2 + y^2 = r^2$.

3.2 ***Theorem:*** Equation of the circle with (x_1, y_1) and (x_2, y_2) as extremities of a diameter is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$$

3.3 ***Parametric equations of the circle:*** Suppose $P(x, y)$ is a point on the circle with $A(x_1, y_1)$ as its centre and r as radius. Suppose the line $\overline{AP}$ makes an angle θ with positive direction of the x-axis. Then $x = x_1 + r\cos\theta$ and $y = y_1 + r\sin\theta$ are the coordinates of the point P. Conversely, $(x_1 + r\cos\theta, y_1 + r\sin\theta)$ lies on the circle with centre at $A(x_1, y_1)$ and radius r for all real values of θ. Hence, $x = x_1 + r\cos\theta$ and $y = y_1 + r\sin\theta$ (θ being parameter) are called the parametric equations of the circle $(x - x_1)^2 + (y - y_1)^2 = r^2$. In particular, the parametric equations of the circle $x^2 + y^2 = r^2$ are $x = r\cos\theta$, $y = r\sin\theta$.

3.4 ***Point circle:*** Circle having radius zero is called point circle. That is, $(x - x_1)^2 + (y - y_1)^2 = 0$ is the point circle with centre at (x_1, y_1).

3.5 ***General equation of the circle:*** If g, f and c are real numbers, then the equation $x^2 + y^2 + 2gx + 2fy + c = 0$ represents a circle with centre at $(-g, -f)$ and radius $\sqrt{g^2 + f^2 - c}$ with an understanding of the following conditions:

(i) $g^2 + f^2 - c > 0 \Rightarrow$ the circle is a real circle.
(ii) $g^2 + f^2 - c = 0 \Rightarrow$ the circle is a point circle.
(iii) $g^2 + f^2 - c < 0 \Rightarrow$ the circle is imaginary.

3.6 ***Universal equation of a circle:*** The equation $ax^2 + ay^2 + 2gx + 2fy + c = 0$ $(a \neq 0)$ is called universal equation of the circle with centre at the point $(-g/a, -f/a)$ and radius $\sqrt{\dfrac{g^2}{a^2} + \dfrac{f^2}{a^2} - \dfrac{c}{a}}$.

3.7 The second-degree general equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a circle if and only if $a = b$, $h = 0$ and $g^2 + f^2 > ac$.

3.8 ***Theorem:*** The perpendicular bisector of a chord of a circle passes through the centre of the circle.

3.9 ***Relation between a circle and a line in the same plane:*** Let C be a circle with centre at A and radius r. Let l be a line in the plane of the circle C. Then

(i) The circle C and the line l have no common points $\Leftrightarrow$ The perpendicular distance of the centre A from the line l is greater than the radius r.
(ii) The line touches the circle $\Leftrightarrow$ The perpendicular distance drawn from A onto the line is equal to the radius r.
(iii) The line l intersects the circle in two distinct points $\Leftrightarrow$ The perpendicular distance of the line from the centre is less than the radius r.

3.10 ***Notation:*** S is denoted by $x^2 + y^2 + 2gx + 2fy + c$. That is, $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ is a circle. Then

(i) S_1 means that $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$
That is, $S_1 \equiv xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$
(ii) $S_2 \equiv xx_2 + yy_2 + g(x + x_2) + f(y + y_2) + c$

(iii) $S_{12} = x_1x_2 + y_1y_2 + g(x_1 + x_2) + f(y_1 + y_2) + c = S_{21}$

(iv) $S_{11} = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$

3.11 ***Property of $S_1 = 0$:*** If (x_1, y_1) is not the centre of the circle $S = 0$, then

$$S_1 \equiv (g + x_1)x + (f + y_1)y + gx_1 + fy_1 + c = 0$$

is a first-degree equation in x and y and hence it represents a straight line and is perpendicular to the line joining the centre of the circle and the point (x_1, y_1).

3.12 ***Theorem (Equation of a chord):*** If $P(x_1, y_1)$ and $Q(x_2, y_2)$ are two points on a circle $S = 0$, then the equation of the chord joining points P and Q is $S_1 + S_2 = S_{12}$. In particular, equation of the tangent at (x_1, y_1) is $S_1 = 0$.

Note: If $S \equiv x^2 + y^2 - a^2 = 0$, then the equation of the chord joining $P(x_1, y_1)$ and $Q(x_2, y_2)$ is

$$(xx_1 + yy_1 - a^2) + (xx_2 + yy_2 - a^2) = x_1x_2 + y_1y_2 - a^2$$

In particular, the equation of the chord joining $(a \cos \alpha, a \sin \alpha)$ and $(a \cos \beta, a \cos \beta)$ is

$$x \cos \frac{(\alpha + \beta)}{2} + y \sin \frac{(\alpha + \beta)}{2} = a \cos \frac{(\alpha - \beta)}{2}$$

3.13 Equation of a tangent:

(i) ***Cartesian form:*** The equation of the tangent to the circle $x^2 + y^2 = a^2$ at the point (x_1, y_1) is

$$xx_1 + yy_1 - a^2 = 0$$

(ii) ***Parametric form:*** Equation of the tangent to the circle $x^2 + y^2 = a^2$ at the point $(a \cos \alpha, a \sin \alpha)$ is

$$x \cos \alpha + y \sin \alpha = a$$

3.14 ***Position of a point with respect a circle:*** Let C be a circle with centre at point A and radius r. Let P be a point in the plane of the circle, then

(i) P lies outside the circle if and only if $AP > r$.
(ii) P lies inside the circle if and only if $AP < r$.
(iii) P lies on the circle if and only if $AP = r$.

3.15 Condition for a point to lie outside or inside on the circle. Let $S = 0$ be a circle and $P(x_1, y_1)$ a point in the plane of the circle, then

(i) $P(x_1, y_1)$ lies outside the circle $\Leftrightarrow S_{11} > 0$.
(ii) $P(x_1, y_1)$ lies inside the circle $\Leftrightarrow S_{11} < 0$.
(iii) $P(x_1, y_1)$ lies on the circle $\Leftrightarrow S_{11} = 0$.

3.16 ***Theorem:*** From any external point of a circle, two tangents can be drawn to the circle.

3.17 ***Condition for line to be tangent:*** The line $y = mx + c$, where $c \neq 0$, touches the circle $x^2 + y^2 = a^2$ if and only if $c^2 = a^2(1 + m^2)$ and in such a case, the point of contact is

$$\left(\frac{-a^2m}{c}, \frac{a^2}{c}\right)$$

3.18 ***Director circle:*** The locus of the point through which perpendicular tangents can be drawn to a circle is again concentric circle whose radius equals $\sqrt{2}$ times the radius of the given circle and is called the director circle of the given circle. The director circle of the circle $(x - h)^2 + (y - k)^2 = r^2$ is $(x - h)^2 + (y - k)^2 = 2r^2$.

3.19 ***Equation of the chord in terms of its midpoint:*** Equation of the chord of a circle $S = 0$ whose midpoint is (x_1, y_1) is $S_1 = S_{11}$.

3.20 ***Definition (chord of contact):*** Let C be a circle and P be an external point to C. Suppose the two tangents drawn from P to C touch the circle at A and B. Then the chord AB is called chord of contact of P with respect to the circle C.

3.21 ***Equation of the chord of contact:*** Equation of the chord of contact of a point (x_1, y_1) with respect to a circle $S = 0$ is $S_1 = 0$.

3.22 ***Length of the chord:*** Suppose PQ is a chord of a circle with centre A and radius r. AM is the line drawn perpendicular to the chord PQ so that M is the midpoint of PQ. Then the length of the chord PQ is $2\sqrt{r^2 - (AM)^2}$.

3.23 ***Length of the tangent:*** The length of the tangent drawn from an external point (x_1, y_1) to a circle $S = 0$ is $\sqrt{S_{11}}$.

3.24 ***Relation between two circles:*** Let C_1 and C_2 be two circles with centres A_1 and A_2, respectively, and radii r_1 and r_2, respectively. Then

(i) C_1 and C_2 do not have a common point and neither is being completely within the other $\Leftrightarrow A_1A_2 > r_1 + r_2$.
(ii) C_1 and C_2 touch each other externally $\Leftrightarrow A_1A_2 = r_1 + r_2$.
(iii) C_1 and C_2 intersect in two distinct points $\Leftrightarrow |r_1 - r_2| < A_1A_2 < r_1 + r_2$.
(iv) C_1 and C_2 touch each other internally $\Leftrightarrow A_1A_2 = |r_1 - r_2|$.
(v) One lies completely within the other without touching $\Leftrightarrow A_1A_2 < |r_1 - r_2|$.

3.25 ***Angle of intersection of circles:*** Let C_1 and C_2 be two circles intersecting at points A and B. Then the angle between the tangents drawn to the circles at either of the points A and B is same and this angle is called the angle of intersection of the two circles C_1 and C_2.

3.26 ***Definition (Orthogonal circles):*** If the angle of intersection of two circles is a right angle, then the two circles are said to intersect each other orthogonally.

3.27 ***Angle of intersection formula:*** Let C_1 and C_2 be two circles with centres A_1 and A_2, respectively, and radii r_1 and r_2, respectively. If α is the angle of intersection of the circles, then

$$\cos\alpha = \frac{(A_1A_2)^2 - (A_1P)^2 - (A_2P)^2}{2(A_1P)(A_2P)}$$

where P is one of the points of intersection of C_1 and C_2.

3.28 ***Theorem (Two circles intersecting orthogonally):*** Two circles $x^2 + y^2 + 2gx + 2fy + c = 0$ and $x^2 + y^2 + 2g'x + 2f'y + c' = 0$ cut each other orthogonally if and only if $2gg' + 2ff' = c + c'$.

3.29 Suppose $S = 0$ and $S' = 0$ are two non-concentric circles in the standard form (i.e., general form). Then, the equation $S - S' = 0$ is a first-degree equation in x and y. Hence it represents a line which is also perpendicular to the line joining the centres of the circles.

3.30 ***About the line $S - S' = 0$:*** Let $S = 0$ and $S' = 0$ be two circles with different centres in the standard form. Then

(i) $S - S' = 0$ represents the common chord of the circles if the circles intersect.

(ii) If the circles touch each other at a point P, then $S - S' = 0$ represents the common tangent to the circles at P.

3.31 ***Definition (Power of a point):*** Let C be a circle and P be a point in the plane of C. Suppose a line through P meets the circle C at two points A and B. Then the value $PA \cdot PB$ is independent of the line through P. The power of P with respect to the circle C is defined as follows:

(i) The value $PA \cdot PB$ if P is an external point to C.

(ii) The value $-(PA \cdot PB)$ if P lies inside C.

(iii) The value zero if P lies on the circle.

3.32 ***Power of a point formula:*** The power of $P(x_1, y_1)$ with respect to a circle $S = 0$ is S_{11}.

3.33 ***Theorem (radical axis):*** The locus of the point whose powers with respect to two non-concentric circles are equal is a straight line and is perpendicular to the line joining the centres of the circles. This line is called the radical axis of the two circles.

3.34 ***Equation of the radical axis:*** If $S = 0$ and $S' = 0$ are two non-concentric circles, then their radical axis equation is $S - S' = 0$

3.35 ***About the radical axis:***

(i) If the circles $S = 0$ and $S' = 0$ intersect, then their radical axis $S - S' = 0$ is also a common chord.

(ii) If the two circles touch each other, then their radical axis $S - S' = 0$ is a common tangent to the circles at their point of contact.

3.36 ***Theorem:*** Let C_1 and C_2 be two non-concentric circles. If a circle C cuts both C_1 and C_2 orthogonally, then the centre of C lies on the radical axis of C_1 and C_2.

3.37 ***Radical centre:*** If $S = 0$, $S' = 0$ and $S'' = 0$ are three circles with non-collinear centres, then their radical axes taken two by two are concurrent and this point of concurrence is called the radical centre of the three circles.

Note: (1) The tangents drawn from the radical centre to each of the three circles are of equal length. (2) Taking the radical centre as the centre and the length of the tangent drawn from the radical centre to the circle as radius, if a circle is drawn, then this circle cuts all the circles orthogonally.

3.38 ***Common tangents to two circles:*** Let C_1 and C_2 be two circles with centres A_1 and A_2, respectively, and radii r_1 and r_2, respectively. The following table shows the number of common tangent(s) to the two circles:

Nature of the circles	*Number of common tangents*	*Number of direct common tangents*	*Number of transverse common tangents*
Circles do not have common points ($A_1A_2 > r_1 + r_2$)	4	2	2
Circles touch externally ($A_1A_2 = r_1 + r_2$)	3	2	1
Circles intersect in two district points ($\lvert r_1 - r_2\rvert < A_1A_2 < r_1 + r_2$)	2	2	0
Circles touch internally ($A_1A_2 = \lvert r_1 - r_2\rvert$)	1	—	—
One circle completely within the other ($A_1A_2 < \lvert r_1 - r_2\rvert$)	0	0	0

3.39 ***Centres of similitude:***

(i) Suppose that the two circles C_1 and C_2 (discussed in 3.38) do not intersect and the radius r_1 is not equal to r_2. Then the two direct common tangents intersect on the line of centres and this point is called external centre of similitude. This external centre of similitude divides the segment $\overline{A_1A_2}$ in the ratio $r_1{:}r_2$ externally.

(ii) The two transverse common tangents also intersect the line of centres in between the centres. This point is called the internal centre of similitude and this point divides $\overline{A_1A_2}$ internally in the ratio $r_1{:}r_2$.

(iii) If the circles touch each other externally, then the internal centre of similitude is their point of contact and the external centre of similitude exits as usual.

(iv) If the two circles intersect, there is no internal centre of similitude, but external centre of similitude exits.

3.40 ***Theorem:*** Combined equation of the pair of tangents drawn from an external point (x_1, y_1) to a circle $S = 0$ is $S_1^2 = S_{11}$.

3.41 **Finding the common tangents to two circles**:
Step1: Determine the nature of the circles.
Step2: If external centre of similitude exists, then find its coordinates [using 3.39(i)].
Step3: Consider a line through the external centre of similitude and impose the condition that it touches one of the circles. Then that line automatically touches the other circle also. Thus, we get direct common tangents. Perform same with regards to transverse common tangents by finding the internal centre of similitude.

3.42 ***Second method to find the common tangents:*** Suppose the circles are $(x - x_1)^2 + (y - y_1)^2 = r_1^2$ and $(x - x_2)^2 + (y - y_2)^2 = r_2^2$. We know that the line $(x - x_1)\cos\theta + (y - y_1)\sin\theta = r_1$ is a tangent to the first circle. Now, impose the condition that this line also touches the second circle and find the values of $\sin\theta$ and $\cos\theta$.

Note: In this second method, we get the common targets, but we do not know their nature unless we use further test.

3.43 **$S + \lambda L = 0$:** Let $S \equiv x^2 + y^2 = 2gn + 2fy + c = 0$ be a circle and $L \equiv ax + by + c = 0$ a straight line. Then, the equation $S + \lambda L = 0$ represents a family of circles for all real values of λ.

Note:

(i) If the line $L = 0$ intersects the circle $S = 0$ at two points P and Q, then $S + \lambda L = 0$ represents circles, all passing through points P and Q.

(ii) If $L = 0$ touches the circle $S = 0$ at the point P, then $S + \lambda L = 0$ represents circles, all touching at point P. In such cases, $L = 0$ is a common tangent to all these circles at point P.

(iii) If $S = 0$ and $S = 0$ are two non-concentric circles, then $S + \lambda L = 0$, where $S - S = 0$ represents a family of circles.

EXERCISES

1. Find the equation of the circumcircle of the triangle whose vertices are $(0, 0)$, $(a, 0)$ and $(0, b)$.

2. Find the equation of the circumcircle of the quadrilateral whose sides are $5x + 3y - 9 = 0$, $y = x/3$, $2x = y$ and $x + 4y = 2 = 0$ in the given order.

3. If x_1 and x_2 are the roots of $x^2 + 2ax - b^2 = 0$ and y_1 and y_2 are the roots of $x^2 + 2px - q^2 = 0$, then find the equation of the circle described on the segment joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ as diameter.

4. If (h, k) is the centre of the circle whose parametric equations are $x = -1 + 2\cos\theta$ and $y = 3 + 2\sin\theta$, then find (h, k).

5. Find the length of the intercept made by the circle $x^2 + y^2 + 10x - 6y + 9 = 0$ on x-axis.

6. Show that $x^2 + y^2 + 10x - 6y + 9 = 0$ touches y-axis.

7. Find the equation of the circle inscribed in the triangle whose sides are x-axis, y-axis and the line $x + y = 1$.

8. The line $x\cos\alpha + y\sin\alpha - \rho = 0$ cuts the circle $x^2 + y^2 = a^2$ in A and B. Find the equation of the circle described on $\overline{AB}$ as diameter.

9. Show that the condition that the circle circumscribing the triangle formed by the three lines $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$ and $a_3x + b_3y + c_3 = 0$ should have its centre lying on the x-axis so that

$$\left|\begin{pmatrix} a_2a_3 - b_2b_3 & a_3a_1 - b_3b_1 & a_1a_2 - b_1b_2 \\ a_3b_2 + a_2b_3 & a_1b_3 + b_3b_1 & a_2b_1 + a_1b_2 \\ b_3c_2 + b_2c_3 & b_1c_3 + b_3c_1 & b_2c_1 + b_1c_2 \end{pmatrix}\right| = 0$$

10. Prove that the circles $x^2 + y^2 + 2gx + 2fy + c = 0$ and $x^2 + y^2 + 2g'x + 2f'y + c' = 0$ should touch if

$$(2gg' + 2ff' - c - c')^2 = 4(g^2 + f^2 - c)(g'^2 + f'^2 - c')$$

11. Find the centre and radius of the circle which is inscribed in the triangle formed by the lines $y = 0$, $12x - 5y = 0$ and $3x + 4y - 7 = 0$.

12. Find the equation of the circle passing through the points $(0, a)$ and (b, h) and having its centre on the axis of x.

13. Show that the equation of the circle passing through the points (a, b), $(a, -b)$ and $(a + b, a - b)$ is $b(x^2 + y^2) - (a^2 + b^2)x + (a - b)(a^2 + b^2) = 0$.

14. Show that the equation of the circle passing through origin and having intercepts of lengths 3 and 4 on positive x-and y-axis, respectively, is $x^2 + y^2 - 3x - 4y = 0$.

15. For all values of λ and μ, prove that the circles $x^2 + y^2 + 2\lambda x + c = 0$ and $x^2 + y^2 + 2\mu y - c = 0$ cut each other orthogonally.

16. For two distinct values of λ if the circle $x^2 + y^2 + 2\lambda x + c = 0$ cuts the circle $x^2 + y^2 + 2gx + 2fy + c' = 0$ orthogonally, then show that $g = 0$ and $c' = -c$.

17. Prove that the radical axis of two circles bisects their common tangent.

18. If $c^2 > 1/2(a - b)^2$, then show that the length of the common chord of the circles and $(x - a)^2 + (y - b)^2 = c^2$ is $\sqrt{4c^2 - 2(a-b)^2}$ and $(x - b)^2 + (y - a)^2 = c^2$.

19. Find the equations to the circles which intersect the circles $x^2 + y^2 - 6y + 1 = 0$ and $x^2 + y^2 - 4y + 1 = 0$ orthogonally and touch the line $3x + 4y + 5 = 0$.

20. Find the equation to the circle which passes through the origin and cuts orthogonally each of the circles $x^2 + y^2 - 6x + 8 = 0$ and $x^2 + y^2 - 2x - 2y - 7 = 0$.

21. Find the equation to the circle cutting orthogonally the three circles $x^2 + y^2 + 2x + 17y + 4 = 0$, $x^2 + y^2 + 7x + 6y + 11 = 0$ and $x^2 + y^2 - x - 22y + 3 = 0$.

22. If the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ cut the coordinate axes in concyclic points, then show that $a_1^2a_2^2 = b_1^2 b_2^2$.

23. If the line $y = mx + 2$ cuts the circle $x^2 + y^2 = 1$ in two distinct points, then show that either $m < -\sqrt{3}$ or $m > \sqrt{3}$.

24. Suppose λ_1, λ_2 and λ_3 are positive real numbers that are in GP. If t_1, t_2 and t_3 are the tangents drawn from any point on the circle $x^2 + y^2 = a^2$ to the circles, $x^2 + y^2 + 2\lambda_i x - a^2 = 0$ $(i = 1, 2, 3)$, then prove that t_1, t_2 and t_3 are in GP.

25. The straight line $x - 2y + 1 = 0$ intersects the circle $x^2 + y^2 = 25$ at points A and B. Find the coordinates of the point of intersection of the tangents drawn to the circle at points A and B.

26. Find the area of the triangle formed by the chord of contact of the point $P(4, 3)$ with respect to the circle $x^2 + y^2 = 9$ and the two tangents drawn from $P(4, 3)$ to the circle.

[***Hint:*** use the formula that the area is equal to $(rS_{11}^{3/2})/(x_1^2 + y_1^2)$.]

27. A circle passes through origin and has its centre on the line $y = x$. If it cuts $x^2 + y^2 - 4x - 6y + 10 = 0$ orthogonally, then find its equation.

28. Let C be the circle $x^2 + y^2 - 2x - 4y - 2\sqrt{3} + 1 = 0$. Let C' be a variable circle touching internally C and the tangents drawn from the point $(1, 2)$ to C' include an angle of $60°$. Show that the center of C' lies on the circle $(x - 1)^2 + (y - 2)^2 = 3$.

29. A variable circle passes through the point $A(p, q)$ and touches the x-axis. Show that the other end of the diameter through A lies on the curve $(x - p)^2 = 4qy$.

30. If the tangents drawn from the origin to the circle $x^2 + y^2 - 2ax - 2by + 2 = 0$ are at right angles, then show that $a^2 + b^2 = 4$.

31. Let $1 - 2r$, $1 - r$ and 1 be the radii of three concentric circles with centres at origin $(r > 0)$. If the line $y = x + 1$ cuts the circle $x^2 + y^2 = (1 - 2r)^2$ in two distinct points, then show that $0 < r < \sqrt{2} - 1/2\sqrt{2}$.

32. Consider a family of circles passing through two fixed points $A(3, 7)$ and $B(6, 5)$. Show that the chords in which the circle $x^2 + y^2 - 4x - 6y - 3 = 0$ cuts the members of the family are concurrent at a point. Find the coordinates of this point.

(***Hint:*** Equation of the given family by of circles $S + \lambda L = 0$ where $L = 0$ is the equation of AB and $S = 0$ in the equation of the circle described on AB as diameter.)

33. The equation $(x + 5y - 22) + \lambda(-x + 8y - 30) = 0$ (λ is a parameter) represents a family of lines. Find the lines belonging to the family on which the circle $x^2 + y^2 - 2x + 2y - 14 = 0$ cuts chords of length $2\sqrt{3}$.

34. Find the centres of the circles passing through the points $(0, 0)$ and $(1, 0)$ and touching the circle $x^2 + y^2 = 9$.

35. A circle of radius 2 units rolls on the outside of the circle $x^2 + y^2 + 4x = 0$, touching it externally. Find the locus of the centre of this outer circle.

ANSWERS

1. $x^2 + y^2 - ax - by = 0$

2. $9x^2 + 9y^2 - 20x + 15y = 0$

3. $x^2 + y^2 + 2ax + 2py - b^2 - q^2 = 0$

4. $(-1, 3)$

5. 8

7. $\left(x-1+\frac{1}{\sqrt{2}}\right)^2+\left(y-1+\frac{1}{\sqrt{2}}\right)^2=\left(1-\frac{1}{\sqrt{2}}\right)^2$

8. $x^2 + y^2 - a^2 - 2p(x\cos\alpha + y\sin\alpha - p) = 0$

11. Centre is $\left(\frac{7}{9},\frac{14}{27}\right)$ and radius is $\frac{14}{27}$

12. $b(x^2 + y^2 - a^2) = x(h^2 + b^2 - a^2)$

19. $x^2 + y^2 = 1$ and $4(x^2 + y^2) - 15x - 4 = 0$

20. $3x^2 + 3y^2 - 8x - 29y = 0$

21. $x^2 + y^2 - 4x - 6y - 44 = 0$

25. $(-25, 50)$

26. $\frac{192}{25}$ sq. unit

27. $x^2 + y^2 - x - y = 0$

32. $\left(2,\frac{23}{3}\right)$

33. $2x - 3y + 8 = 0$ and $3x + 2y - 14 = 0$

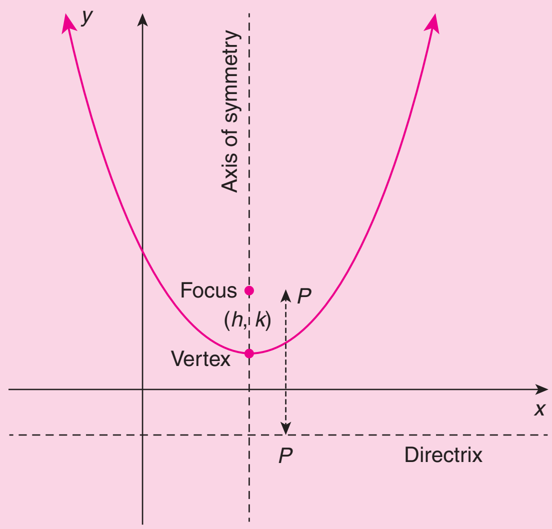
34. $\left(\frac{1}{2},\pm\sqrt{2}\right)$

35. $x^2 + y^2 + 4x - 12 = 0$

Parabola

4

Contents



Parabola

The locus of the point which moves in such a way that it is equidistant from a fixed point and a fixed line is known as a **parabola**.

In the previous chapter, we discussed about circle, general equation of a circle, equation of the tangent at a point, chord equation in terms of its midpoint, chord of contact, orthogonal circles, etc. In this chapter, we will discuss broadly about one of the conics, namely, parabola. The chapter discusses about cone, base curve or guiding curve, circular cone and right circular cone, general conic, classification of conics, double ordinate and latus rectum, parametric equations of $y^2 = 4ax$, normal, normal chord, chord of contact, diameter of a parabola, focal chord and focal radius and conormal points. 'Subjective Problems' section provides subjective worked-out problems for the preceding sections. Students are advised to solve each and every problem to grasp the topics.

4.1 | Conic Section

Before we discuss the properties of the conics, namely, parabola, ellipse and hyperbola, let us understand the term *conic* or *conic section*. Conic sections are the curves obtained by projecting a circle through a fixed point V not in the plane of the circle onto another plane. In fact, conics are sections of a cone by a plane cutting the cone in various ways (of course, the base of the cone is a circle). In this context, we used the word *cone* and *base of the cone*. While studying the elementary geometry and elementary solid geometry, a student visualises the shape of the cone as given in his or her respective textbooks. In this section, we introduce the concept of a cone in a more general way.

DEFINITION 4.1 **Cone** Let S be a non-empty set of points in the space. Then, S is called a cone if there exists a point $V \in S$ such that the line $\overleftrightarrow{VP}$ is contained in S for all points P in S. This point V is called the *vertex* of the cone and the line $\overleftrightarrow{VP}$ where $P \in S$ is called *generator* of the cone S.

Examples

(1) Every line is a cone with every point on the line as vertex and the line is the only generator.

(2) Every plane is a cone with all of its points as vertices.

(3) Two intersecting planes form a cone with every point on their line of intersection as vertex.

DEFINITION 4.2 **Degenerate and Non-degenerate Cones** The cones described in the examples of Definition 4.1 are called *degenerate cones*. Generally, cones that are having more than one vertex are called *degenerate cones*. Cones which do not degenerate are called *non-degenerate cones*. Using the three-dimensional analytic geometry (Chapter 6), we can verify that the locus represented by the equation $x^2 - y^2 + z^2 = 0$ is a cone with origin as the vertex.

DEFINITION 4.3 **Base Curve or Guiding Curve** If a plane is not passing through the vertex and intersects all the generators of a cone, then the intersection of the plane and the cone are called *base curve* or *guiding curve*.

DEFINITION 4.4 **Circular Cone and Right Circular Cone** If the base curve is a circle, then it is called a *circular* cone (see Fig. 4.1). If the base curve is a circle and the line connecting the centre of the base and the vertex of the cone is perpendicular to the plane of the circle, then the cone is called *right circular cone*.

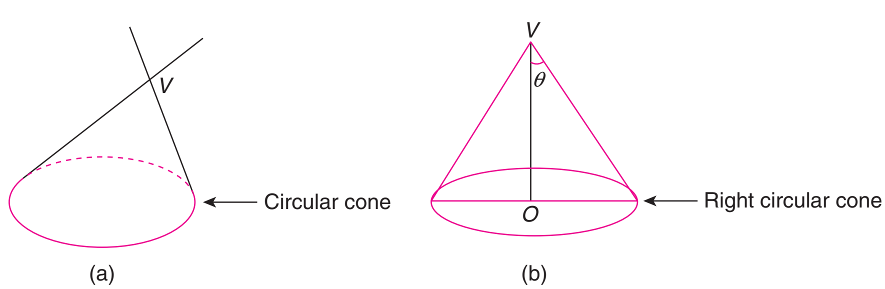


FIGURE 4.1

Note: In a right circular cone, the line passing through the vertex and the centre of the base circle is called the axis of the cone and it makes a constant angle with each of the generators and this constant angle is called *semi-vertical angle* of the cone [θ in Fig. 4.1(b)]. In a right circular cone, a plane which is not passing through the vertex and perpendicular to the axis cuts the cone in a circle. Definition 4.5 explains the geometrical concept of a parabola.

DEFINITION 4.5 Let V be the vertex of a right circular cone with semi-vertical angle θ whose axis is l. Suppose a plane which is not passing through the vertex makes an angle θ with l. Then, the cone section is a plane curve called *parabola* (see Fig. 4.2).

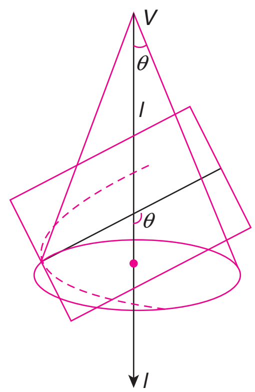


FIGURE 4.2

Now, let us define a general conic using two-dimensional geometry.

DEFINITION 4.6 **General Conic** Let l be a fixed line, S be a fixed point which does not lie on l and e be a positive real number. Then, the locus of the point P, such that the distance SP is equal to e times the perpendicular distance of l from P, is called a conic. For this conic, the fixed line l is called *directrix*, the fixed point S is called *focus* and the number e is called the *eccentricity*.

DEFINITION 4.7 **Classification of Conics** Conics are classified according to their eccentricity e.

1. If $e = 1$, then the conic is called *parabola*.
2. If $0 < e < 1$, then the conic is called *ellipse*.
3. If $e > 1$, then the conic is called *hyperbola*.

In this chapter, we will discuss about parabola.

4.2 | Parabola

THEOREM 4.1 Standard equation of a parabola is $y^2 = 4ax$ where $a > 0$.

PROOF Let the line l be the directrix and point S be the focus. From point S, draw SZ perpendicular to the directrix l and let O be the midpoint of SZ. Let $ZO = OS = a$ (see Fig. 4.3). Consider $\overrightarrow{Ox}$ as positive x-axis and $\overrightarrow{Oy}$ perpendicular to $\overrightarrow{Ox}$ as y-axis so that $Z = (-a, 0)$ and $S = (a, 0)$.

$P(x, y)$ is a point on the parabola

$\Leftrightarrow SP = PM$ where PM is the perpendicular distance of the directrix $l(x + a = 0)$ from P

$$\Leftrightarrow (x-a)^2 + y^2 = |x+a|^2$$

$$\Leftrightarrow y^2 = 4ax$$

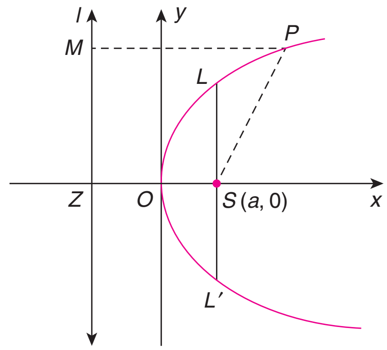


FIGURE 4.3

QUICK LOOK 1

1. The focus and the directrix of $y^2 = 4ax$ are $(a, 0)$ and $x + a = 0$, respectively, and $O(0, 0)$ is called the *vertex of the parabola*.
2. The curve $y^2 = 4ax$ is symmetrix about the x-axis (also called the *axis of the parabola*).
3. Other forms of parabola:
   a. If the focus is to the left of the directrix, then its equation is $y^2 = -4ax$.
   b. If the focus is on the y-axis, then the equation of the parabola is $x^2 = \pm 4ay$.

THEOREM 4.2 Equation of the parabola with the vertex at (h, k), the axis parallel to the x-axis, the focus at a distance a to the right of the vertex and the directrix parallel to the y-axis which is at a distance of $2a$ to the left focus is $(y-k)^2 = 4a(x-h)$ (see Fig. 4.4).

PROOF Since the directrix equation is $x = h - a$ and the focus is $(h + a, k)$, we have

$$SP = PM$$
$$\Leftrightarrow (x-h-a)^2 + (y-k)^2 = |x-h+a|^2$$
$$\Leftrightarrow (y-k)^2 = 4a(x-h)$$

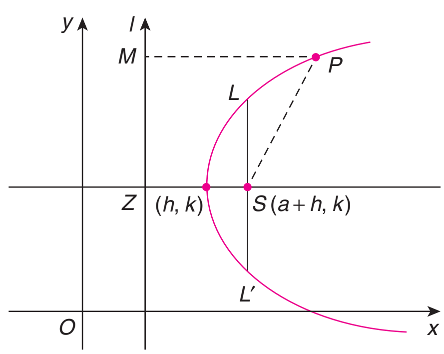


FIGURE 4.4

QUICK LOOK 2

Other forms of the equations of parabola are as follows:

1. $(y-k)^2 = -4a(x-h)$
2. $(x-h)^2 = 4a(y-k)$
3. $(x-h)^2 = -4a(y-k)$

DEFINITION 4.8 **Double Ordinate and Latus Rectum** Suppose a line perpendicular to the axis of parabola meets the curve at P and Q. Then, PQ is called double ordinate of the curve. In particular, *double ordinate* passing through the focus is called *latus rectum*. Usually, the ends of the latus rectum are denoted by L and L' (see Fig. 4.5).

THEOREM 4.3 The length of the latus rectum of the parabola $y^2 = 4ax$ is $4a$.

PROOF Let L and L' be the ends of the latus rectum. Since $S = (a, 0)$, let L be (a, y).

$$L(a, y) \text{ lies on the curve } \Rightarrow y^2 = 4a(a)$$
$$\Rightarrow y = \pm 2a$$

Therefore, $L = (2a, 0)$ and $L' = (-2a, 0)$ so that $LL' = 4a$.

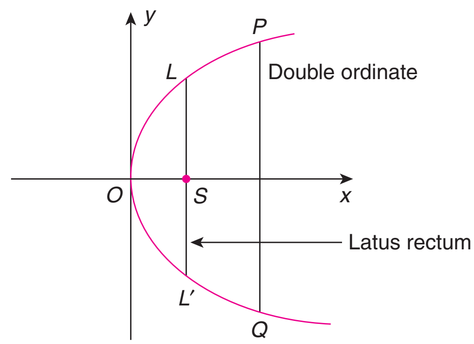


FIGURE 4.5

■

QUICK LOOK 3 (SOME PROPERTIES OF $y^2 = 4ax$)

1. For any point (x, y) on the curve, $x \geq 0$ and $y = \pm\sqrt{4ax}$. That is, to each value of $x \geq 0$, there are two corresponding values for y. In this case we say that the curve is symmetric about x-axis.
2. $x \to +\infty \Rightarrow y \to \pm\infty$.
3. The point on the curve where the axis meets the curve is called the vertex.
4. For $y^2 = 4ax$, $(0, 0)$ is the vertex and y-axis is the tangent at the vertex.

DEFINITION 4.9 **Parametric Equations of $y^2 = 4ax$** For all real values t, the point $(at^2, 2at)$ lies on the parabola $y^2 = 4ax$. Conversely, if (x_1, y_1) is a point on the curve $y^2 = 4ax$, then take $t = y_1/2a$ so that $x_1 = at^2$. That is, if (x_1, y_1) is a point on the curve $y^2 = 4ax$, there corresponds a real t such that $x_1 = at^2$, $y_1 = 2at$. Hence, the *equations $x = at^2$, $y = 2at$, $t \in R$ are called the parametric equations of the parabola $y^2 = 4ax$.*

Notation: S denotes the expression $y^2 - 4ax$. That is,

$$S \equiv y^2 - 4ax$$
$$S_1 \equiv yy_1 - 2a(x + x_1)$$
$$S_2 \equiv yy_2 - 2a(x + x_2)$$
$$S_{12} = S_{21} = y_1 y_2 - 2a(x_1 + x_2)$$
$$S_{11} = y_1^2 - 4ax_1$$

4.2.1 Classification of Points

Let $P(x, y)$ be a point in the plane of the parabola. If $x < 0$, then P is called external point to the curve. Suppose $x > 0$. Then, draw PM perpendicular to the axis (i.e., x-axis) meeting the curve at point Q. Point P is called external or internal point to the curve according as $PM > QM$ or $PM < QM$ (see Fig. 4.6).

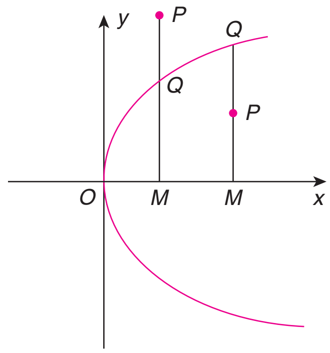


FIGURE 4.6

THEOREM 4.4 Let $P(x_1, y_1)$ be a point in the plane of the parabola $S \equiv y^2 - 4ax = 0$. Then

1. P lies outside the parabola $\Leftrightarrow S_{11} > 0$.
2. P lies inside the parabola $\Leftrightarrow S_{11} < 0$.

PROOF Draw PM perpendicular to the axis of the parabola meeting the curve at $Q(x_1, y_2)$ (see Fig. 4.6). Point $P(x_1, y_1)$ lies outside the parabola

$$\Leftrightarrow PM > QM$$
$$\Leftrightarrow |y_1| > |y_2|$$
$$\Leftrightarrow y_1^2 > y_2^2 = 4ax_1 \qquad (\because Q \text{ lies on the curve})$$
$$\Leftrightarrow y_1^2 - 4ax_1 > 0$$
$$\Leftrightarrow S_{11} > 0$$

In the above argument, when the point lies inside the curve, change 'greater than' symbol to 'less than' symbol. ■

Example 4.1

Find the equation of the parabola whose focus is (1, – 1) and directive is the line $x + y + 7 = 0$.

Solution: We have

$P(x, y)$ is a point on the curve

$$\Leftrightarrow SP = PM$$

where $S = (1, -1)$ and PM is the distance of P from the directrix.

$$\Leftrightarrow (SP)^2 = (PM)^2$$
$$\Leftrightarrow (x-1)^2 + (y+1)^2 = \frac{|x+y+7|^2}{2}$$
$$\Leftrightarrow 2x^2 + 2y^2 - 4x + 4y + 4 = x^2 + y^2 + 2xy + 14x + 14y + 1 + 9$$
$$\Leftrightarrow x^2 - 2xy + y^2 - 18x - 10y - 45 = 0$$

Note: In general, when the focus, directrix and the eccentricity of the conic are given, its equation is a second-degree general equation in x and y. The converse can be proved under certain conditions. Discussion of second-degree general equation is out of scope of this book and generally it is not taught in Class 12 level.

Example 4.2

Determine the coordinates of the focus, the equation of the directrix and the length of the latus rectum of the parabola

$$y^2 = \left(\frac{8}{3}\right)x.$$

Solution: The equation of the curve is

$$y^2 = \left(\frac{8}{3}\right)x = 4\left(\frac{2}{3}\right)x$$

so that $a = 2/3$. Hence, $S = (a,0) = (2/3,0)$ is the focus,

$$x + \frac{2}{3} = 0 \quad \text{or} \quad 3x + 2 = 0$$

is the equation of the directrix and $4a = 8/3$ is the length of the latus rectum.

Example 4.3

Find the equation of the parabola with its vertex at (3, 2) and focus at (5, 2).

Solution: The points (3, 2) and (5, 2) lie on the horizontal line $y = 2$. Therefore, the axis of the parabola is the horizontal line $y - 2 = 0$. Distance between the vertex and the focus is 2 so that $a = 2$. Hence, the equation of the parabola is

$$(y-k)^2 = 4a(x-h)$$

$$\Rightarrow (y-2)^2 = 4(2)(x-3)$$

$$\Rightarrow (y-2)^2 = 8(x-3)$$

Example 4.4

Show that $y = ax^2 + bx + c$, $a \neq 0$ represents a parabola and find its vertex, focus, directrix and latus rectum.

Solution: We have

$$y = ax^2 + bx + c$$

$$\Rightarrow \frac{y}{a} = x^2 + \frac{b}{a}x + \frac{c}{a}$$

$$\Rightarrow \frac{y}{a} = \left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}$$

$$\Rightarrow \left(x + \frac{b}{2a}\right)^2 = \frac{1}{a}\left(y - \frac{4ac - b^2}{4a}\right)$$

$$\Rightarrow X^2 = \frac{1}{a}Y \qquad \left(\text{where } X = x + \frac{b}{2a} \text{ and } Y = y - \frac{4ac - b^2}{4a}\right)$$

Therefore, the vertex $(X = 0, Y = 0)$ is given by

$$\left(\frac{-b}{2a}, \frac{4ac - b^2}{4a}\right)$$

The focus $(X = 0, Y = 1/4a)$ is given by

$$\left(\frac{-b}{2a}, y - \frac{4ac - b^2}{4a} = \frac{1}{4a}\right) = \left(\frac{-b}{2a}, \frac{1}{4a} + \frac{4ac - b^2}{4a}\right)$$

The directrix equation is given by

$$Y = -\frac{1}{4a} \text{ or } y = \frac{4ac - b^2}{4a} - \frac{1}{4a}$$

The latus rectum is given by

$$4\left(\frac{1}{4a}\right) = \frac{1}{a}$$

Example 4.5

Find the vertex, focus, directrix and latus rectum of the parabola $y^2 + 4x - 2y + 3 = 0$.

Solution: The given equation is written as

$$(y-1)^2 = -4\left(x + \frac{1}{2}\right)$$

Let

$$X = x + \frac{1}{2} \text{ and } Y = y - 1$$

Then we have

$$Y^2 = -4X$$

Therefore, the vertex is given by

$$(X = 0, Y = 0) = \left(\frac{-1}{2}, 1\right)$$

Since $a = 1$, the focus $(X = -1, Y = 0)$ is

$$\left(\frac{-3}{2}, 1\right)$$

The directrix is given by

$$X = a \text{ or } x + \frac{1}{2} = 1 \text{ or } x - \frac{1}{2} = 0$$

The latus rectum is given by $4a = 4$.

Example 4.6

Find the vertex, focus, directrix and latus rectum of the parabola $x^2 - 4x - 5y - 1 = 0$.

Solution: The given equation is written as

$$(x-2)^2 = 5(y+1) = 4\left(\frac{5}{4}\right)(y+1)$$

Let $X = x - 2$ and $Y = y + 1$ so that

$$X^2 = 4aY$$

where $a = 5/4$. Therefore

1. The vertex $(X = 0, Y = 0)$ is $(2, -1)$
2. The focus $(X=0, Y=a)$ is $\left(x=2,\ y+1=\frac{5}{4}\right)=\left(2,\ \frac{1}{4}\right)$
3. The directrix equation is
$$Y = -a \text{ or } y+1 = \frac{-5}{4} \text{ or } 4y+9=0$$
4. The latus rectum is $4a = 4\left(\frac{5}{4}\right) = 5.$

THEOREM 4.5 (EQUATION OF THE CHORD)

The equation of the chord joining two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ on the parabola $S \equiv y^2 - 4ax = 0$ is $S_1 + S_2 = S_{12}$ and in particular, equation of the tangent at (x_1, y_1) is $S_1 \equiv yy_1 - 2a\,(x + x_1) = 0$.

PROOF Since $P(x_1, y_1)$ and $Q(x_2, y_2)$ are the points on the curve, we have $y_1^2 = 4ax_1$ and $y_2^2 = 4ax_2$. Therefore

$$y_1^2 - y_2^2 = 4a(x_1 - x_2)$$

$$\Rightarrow \frac{4a}{y_1 + y_2} = \frac{y_1 - y_2}{x_1 - x_2} = \text{Slope of the chord } PQ$$

Hence, the equation of the chord PQ is

$$y - y_1 = \frac{4a}{y_1 + y_2}(x - x_1)$$

$$\Rightarrow (y - y_1)\,(y_1 + y_2) = 4a(x - x_1)$$

$$\Rightarrow yy_1 + yy_2 - y_1^2 - y_1y_2 = 4ax - 4ax_1$$

$$\Rightarrow yy_1 + yy_2 - 4ax = y_1y_2 \qquad (\because y_1^2 = 4ax_1)$$

$$\Rightarrow [yy_1 - 2a(x+x_1)] + [yy_2 - 2a(x+x_2)] = y_1y_2 - 2ax_1 - 2ax_2$$

$$\Rightarrow S_1 + S_2 = S_{12}$$

Now, as the point Q approaches point P along the curve, in the limiting case, the position of the chord PQ takes the position of the tangent at P so that the equation of the tangent at point P is

$$S_1 + S_1 = S_{11} = 0 \qquad [\because (x_1, y_1) \text{ lies on the curve}]$$

Thus, $S_1 \equiv yy_1 - 2a\,(x + x_1) = 0$ is the equation of the tangent at (x_1, y_1). ■

QUICK LOOK 4

In the equation $S_1 \equiv yy_1 - 2a\,(x + x_1) = 0$, if we replace x_1 with at^2 and y_1 with $2at$, then the equation of the tangent at $(at^2, 2at)$ is $ty = x + at^2$. Also note that the slope of the tangent at $(at^2, 2at)$ is $1/t$.

THEOREM 4.6 (INTERSECTION OF THE TANGENTS AT t_1 AND t_2)

The point of intersection of the tangents to the parabola $y^2 = 4ax$ at the points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ is $[at_1t_2, a(t_1 + t_2)]$.

PROOF By Quick Look 4, the tangents at t_1 and t_2, respectively, are given by

$$t_1 y = x + at_1^2 \tag{4.1}$$

and

$$t_2 y = x + at_2^2 \tag{4.2}$$

Solving Eqs. (4.1) and (4.2), we get $x = at_1t_2$ and $y = a(t_1 + t_2)$. ■

QUICK LOOK 5

If the tangents are at right angles, the product of their slopes is equal to –1. This implies

$$\frac{1}{t_1} \times \frac{1}{t_2} = -1 \Rightarrow t_1t_2 = -1$$

so that their point of intersection lies on the directrix $x + a = 0$

THEOREM 4.7 Condition for the line $y = mx + c$ to touch the parabola is

$$c = \frac{a}{m}$$

and in such case, the point of contact is

$$\left(\frac{a}{m^2}, \frac{2a}{m}\right)$$

PROOF 1 Suppose the line $y = mx + c$ touches the parabola $S \equiv y^2 - 4ax = 0$ at (x_1, y_1). By Theorem 4.5, the equation of the tangent at point (x_1, y_1) is $S_1 \equiv yy_1 - 2a\,(x + x_1) = 0$. That is, the equations $y = mx + c$ and $S_1 \equiv yy_1 - 2a\,(x + x_1) = 0$ represent the same straight line. Hence,

$$\frac{-2a}{m} = \frac{y_1}{-1} = \frac{-2ax_1}{c}$$

Therefore

$$x_1 = \frac{c}{m} \quad \text{and} \quad y_1 = \frac{2a}{m}$$

Now, (x_1, y_1) lies on the parabola $y^2 - 4ax$. So we have

$$y_1^2 = 4ax_1$$

$$\Rightarrow \frac{4a^2}{m^2} = 4a\left(\frac{c}{m}\right)$$

$$\Rightarrow c = \frac{a}{m}$$

This gives

$$x_1 = \frac{a}{m^2},\ y_1 = \frac{2a}{m}$$

That is, if the line $y = mx + c$ touches the parabola $y^2 = 4ax$, then

$$c = \frac{a}{m}$$

and the point of contact is

$$\left(\frac{a}{m^2}, \frac{2a}{m}\right)$$

Conversely, suppose

$$c = \frac{a}{m}$$

then the point of intersection of the line

$$y = mx + \frac{a}{m} \text{ and } y^2 = 4ax$$

given by the equations

$$\left(mx + \frac{a}{m}\right)^2 = 4ax \text{ and } \left(mx - \frac{a}{m}\right)^2 = 0$$

$$\Rightarrow x = \frac{a}{m^2}$$

is a double root. Hence, the line

$$y = mx + \frac{a}{m}$$

intersects the parabola $y^2 = 4ax$ at the coincident point

$$\left(\frac{a}{m^2}, \frac{2a}{m}\right)$$

Thus, $y = mx + \dfrac{a}{m}$ touches the parabola.

Proof 2

Line $y = mx + c$ touches the parabola $y^2 = 4ax$

$\Leftrightarrow$ The quadratic equation $(mx + c)^2 - 4ax = 0$ has equal roots

$\Leftrightarrow m^2x^2 + 2(cm - 2a)x + c^2 = 0$ has equal roots

$\Leftrightarrow 4(cm - 2a)^2 - 4c^2m^2 = 0$

$\Leftrightarrow -4cam + 4a^2 = 0$

$\Leftrightarrow c = \dfrac{a}{m}$

Note: Difference between Proof 1 and Proof 2: In Proof 1, we obtain both the condition and the coordinates of the point of contact, but in Proof 2 we obtain only the condition

$$c = \frac{a}{m}$$

QUICK LOOK 6

For all real values of $m \neq 0$, the line

$$y = mx + \frac{a}{m}$$

touches the parabola $y^2 = 4ax$ at the point

$$\left(\frac{a}{m^2}, \frac{2a}{m}\right)$$

Theorem 4.8 From any external point in the plane of a parabola, two tangents can be drawn to the parabola.

Proof Let $S \equiv y^2 - 4ax = 0$ be a parabola and $P(x_1, y_1)$ be a point located in the plane of the parabola and external to it. Therefore, by Theorem 4.4, we get

$$S_{11} = y_1^2 - 4ax_1 > 0$$

Also, by Theorem 4.7, the line

$$y = mx + \frac{a}{m}$$

touches the parabola. Now, this line passes through the point

$$\begin{aligned} P(x_1, y_1) &\Leftrightarrow y_1 = mx_1 + \frac{a}{m} \\ &\Leftrightarrow x_1m^2 - y_1m + a = 0 \text{ has distinct real roots} \\ &\Leftrightarrow y_1^2 - 4ax_1 > 0 \text{ which is true because } (x_1, y_1) \text{ is an external point.} \end{aligned}$$

Hence, through (x_1, y_1), there are two tangents whose slopes, say m_1 and m_2, are the roots of the quadratic equation $x_1m^2 - y_1m + a = 0$. ■

In the quadratic equation obtained in Theorem 4.8, the sum of the roots is

$$m_1 + m_2 = \frac{y_1}{x_1}$$

and the product of the roots is

$$m_1m_2 = \frac{a}{x_1}$$

Now, the tangents through (x_1, y_1) are at right angles

$$\begin{aligned} &\Leftrightarrow -1 = m_1m_2 = \frac{a}{x_1} \\ &\Leftrightarrow x_1 + a = 0 \\ &\Leftrightarrow \text{Locus of } (x_1, y_1) \text{ is the directrix } x + a = 0 \end{aligned}$$

Theorem 4.9 provides the clear interpretation of the intersection of perpendicular tangents.

Theorem 4.9 (Director Circle) The locus of the point of intersection of perpendicular tangents to a parabola is the directrix of the parabola. This locus is called director circle even though it is a line.

Proof According to Quick Look 6, the lines

$$y = mx + \frac{a}{m} \text{ and } y = \frac{-x}{m} - am$$

are tangents to the parabola and they intersect at right angles. Their point of intersection satisfies the equation

$$x\left(m + \frac{1}{m}\right) = -a\left(m + \frac{1}{m}\right)$$

Hence, the locus of the point of intersection is the line $x = -a$ which is the directrix. ■

Example 4.7

Find the locus of the point of intersection of perpendicular tangents to the parabola $y^2+4x-2y+3=0$.

Solution: The given equation is written as

$$(y-1)^2=-4\left(x+\frac{1}{2}\right)$$

$$\Rightarrow Y^2=-4aX$$

where

$$a=1,\ X=x+\frac{1}{2} \text{ and } Y=y-1$$

Directrix equation is

$$X=a \quad \text{or} \quad x+\frac{1}{2}=1$$

Thus, the required locus is

$$2x-1=0$$

Example 4.8

Find the director circle of the parabola $x^2+2y=4x-3$.

Solution: The given equation is written as

$$(x-2)^2=1-2y=-2\left(y-\frac{1}{2}\right)$$

Let $X = x - 2$ and $Y = y - (1/2)$. Substituting in above equation we get

$$X^2=-4aY$$

where $a = 1/2$.

The directrix equation $Y = a$ or $y - 1 = 0$.

Therefore, the director circle is the directrix $y - 1 = 0$.

DEFINITION 4.10 **Normal** As defined in Vol. 3, the *normal to a parabola* at point P is defined to be the line perpendicular to the tangent to the parabola at point P and passing through point P (see Fig. 4.7).

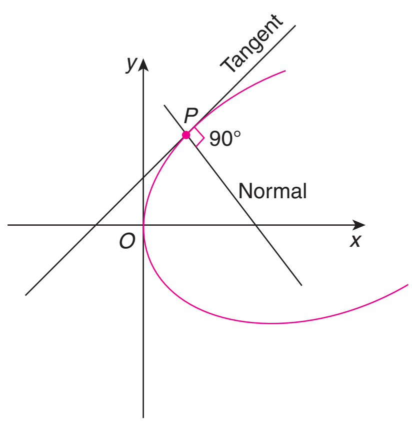


FIGURE 4.7

THEOREM 4.10 [NORMAL EQUATION AT $(at^2, 2at)$] Equation of the normal at t is

$$tx+y=2at+at^3$$

PROOF Tangent at $(at^2, 2at)$ is $ty = x + at^2$ (see Quick Look 4). Thus the slope of the tangent at t is $1/t$. Hence, the equation of the normal at $(at^2, 2at)$ is

$$y-2at=-t(x-at^2)$$

$$\Rightarrow tx+y=2at+at^3$$

■

QUICK LOOK 7

Since the slope of the normal at $(at^2, 2at)$ is $-t$, replacing $-t$ by m, the equation of the normal in terms of its slope m at the point $(am^2, -2am)$ is

$$y = mx - 2am - am^3$$

THEOREM 4.11 (POINT OF INTERSECTION OF NORMALS AT t_1 AND t_2) The point of intersection of the normals at t_1 and t_2 is

$$[2a + a(t_1^2 + t_1 t_2 + t_2^2), -at_1 t_2(t_1 + t_2)]$$

PROOF According to Theorem 4.10, the two equations

$$t_1 x + y = 2at_1 + at_1^3 \tag{4.3}$$

$$t_2 x + y = 2at_2 + at_2^3 \tag{4.4}$$

are normals at t_1 and t_2. Solving Eqs. (4.3) and (4.4), we get

$$x = 2a + a(t_1^2 + t_1 t_2 + t_2^2) \text{ and } y = -at_1 t_2 (t_1 + t_2)$$

■

THEOREM 4.12 If the normal to the parabola at t_1 meets the curve again at t_2, then

$$t_2 = -t_1 - \frac{2}{t_1}$$

PROOF Equation of the normal at $(at_1^2, 2at_2)$ is

$$t_1 x + y = 2at_1 + at_1^3$$

This line meets the curve again at $(at_2^2, 2at_2)$. So we have

$$t_1(at_2^2) + 2at_2 = 2at_1 + at_1^3$$

$$\Rightarrow t_1(t_2^2 - t_1^2) = 2(t_1 - t_2)$$

$$\Rightarrow t_1(t_2 + t_1) = -2 \qquad (\because t_1 \neq t_2)$$

$$\Rightarrow t_2 = -t_1 - \frac{2}{t_1}$$

■

THEOREM 4.13 If the normals at $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ intersect at a point R on the curve again, then

1. $t_1 t_2 = 2$
2. The product of the abscissa of P and Q is $4a^2$
3. The product of the ordinates of P and Q is $8a^2$

PROOF 1. Let R be $(at_3^2, 2at_3)$. Then, by Theorem 4.12, we have

$$-t_1 - \frac{2}{t_1} = t_3 = -t_2 - \frac{2}{t_2}$$

$$\Rightarrow t_2 - t_1 = 2\left(\frac{1}{t_1} - \frac{1}{t_2}\right)$$

$$\Rightarrow t_1 t_2 = 2$$

2. The product of the abscissa $= (at_1^2)(at_2^2) = a^2(t_1t_2)^2 = a^2(2^2) = 4a^2$.
3. The product of the ordinates $= (2at_1)(2at_2) = 4a^2(t_1t_2) = 8a^2$. ■

DEFINITION 4.11 **Normal Chord** A chord of a parabola is called a *normal chord* if it is normal at one of its extremities.

THEOREM 4.14 If the line $lx + my + n = 0$ is a normal to the parabola $y^2 = 4ax$, then $al^3 + 2alm^2 + m^2n = 0$.

PROOF Suppose the line

$$lx + my + n = 0 \tag{4.5}$$

is normal at $(at^2, 2at)$. However, at $(at^2, 2at)$, the equation of the normal is

$$tx + y = 2at + at^3 \tag{4.6}$$

Equations (4.5) and (4.6) represent the same straight line. Therefore

$$\frac{t}{l} = \frac{1}{m} = \frac{-(2at + at^3)}{n}$$

$$\Rightarrow t = \frac{l}{m} \text{ and } \frac{-n}{m} = 2at + at^3 = 2a\left(\frac{l}{m}\right) + a\left(\frac{l^3}{m^3}\right)$$

$$\Rightarrow -m^2n = 2alm^2 + al^3$$

$$\Rightarrow al^3 + 2alm^2 + m^2n = 0$$ ■

DEFINITION 4.12 **Chord of Contact** Let P be an external point to a parabola and the tangents drawn from P to the parabola touch the curve at A and B. Then, the chord AB is called the *chord of contact* of P with respect to the given parabola.

THEOREM 4.15 Equation of the chord of contact of the point $P(x_1, y_1)$ with respect to the parabola $S \equiv y^2 - 4ax = 0$ is $S_1 = 0$.

PROOF See Fig. 4.8. Suppose $A(x_2, y_2)$ and $B(x_3, y_3)$ are the points of contact of the tangents drawn from $P(x_1, y_1)$ to the parabola. Hence, by Theorem 4.5, the equation of the tangent at $A(x_2, y_2)$ is

$$S_2 \equiv yy_2 - 2a(x + x_2) = 0$$

This tangent passes through the point $P(x_1, y_1)$. Therefore,

$$y_1y_2 - 2a(x_1 + x_2) = 0$$

Thus, $A(x_2, y_2)$ satisfies the equation

$$S_1 \equiv yy_1 - 2a(x + x_1) = 0$$

Similarly, $B(x_3, y_3)$ satisfies the equation $S_1 = 0$. Hence, the equation of the line AB is $S_1 = 0$.

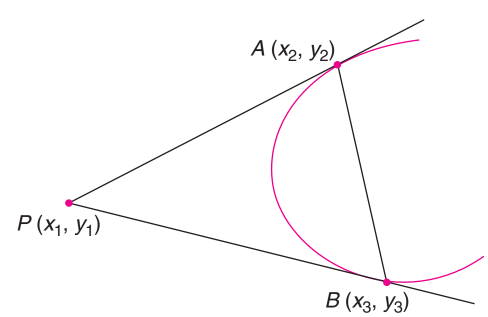


FIGURE 4.8 ■

THEOREM 4.16 (PAIR OF TANGENTS) The combined equation of the pair of tangents drawn from an external point $P(x_1, y_1)$ to the parabola $S \equiv y^2 - 4ax = 0$ is $S_1^2 = SS_{11}$.

PROOF See Fig. 4.9. Let $Q(x_2, y_2)$ $(\neq P)$ be a point on the pair of tangents drawn through P. According to Theorem 4.8, two tangents can be drawn from P. Suppose point R divides the segment $\overline{PQ}$ in the ratio l:λ $(\lambda \neq -1)$. Hence, the coordinates of R are

$$\left(\frac{\lambda x_1 + x_2}{\lambda + 1}, \frac{\lambda y_1 + y_2}{\lambda + 1}\right)$$

R lies on the curve. So we have

$$\left(\frac{\lambda y_1 + y_2}{\lambda + 1}\right)^2 - 4a\left(\frac{\lambda x_1 + x_2}{\lambda + 1}\right) = 0$$

$$\Leftrightarrow \lambda^2 S_{11} + 2\lambda S_{12} + S_{22} = 0 \tag{4.7}$$

Now

$$\begin{aligned} &\overleftrightarrow{PQ} \text{ touches the parabola} \\ &\Leftrightarrow \text{Equation (4.7) has equal roots in } \lambda \\ &\Leftrightarrow 4S_{12}^2 = 4S_{11}S_{22} \\ &\Leftrightarrow S_{12}^2 = S_{11}S_{22} \end{aligned}$$

Therefore, the locus of $Q(x_2, y_2)$ is $S_1^2 = S_{11}S$ or $S_1^2 = SS_{11}$.

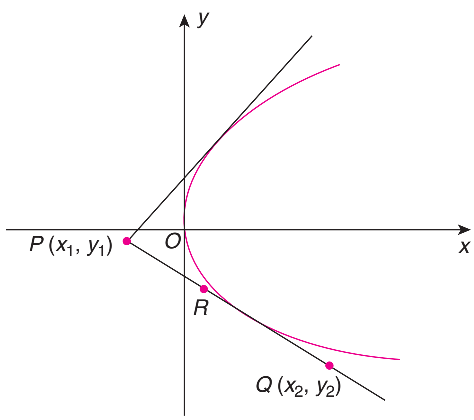


FIGURE 4.9

QUICK LOOK 8

Note

$$\begin{aligned} &S_1^2 = SS_{11} \\ &\Rightarrow [yy_1 - 2a(x + x_1)]^2 = (y^2 - 4ax)(y_1^2 - 4ax_1) \end{aligned}$$

Note: The equation of the pair of tangents in Quick Look 8 is generally discussed only in theory and it is not useful in solving problems. In most of the problems, we use that

$$y = mx + \frac{a}{m}$$

is a tangent to the parabola $y^2 = 4ax$.

THEOREM 4.17 (EQUATION OF THE CHORD USING ITS MIDPOINT) The equation of the chord of $S \equiv y^2 - 4ax = 0$ whose midpoint is (x_1, y_1) is

$$S_1 = S_{11}$$

$$\Rightarrow yy_1 - 2a(x + x_1) = y_1^2 - 4ax_1$$

PROOF Suppose $A(x_2, y_2)$ and $B(x_3, y_3)$ are the ends of the chord whose midpoint is $M(x_1, y_1)$. By Theorem 4.5, the equation of the line $\overleftrightarrow{AB}$ is

$$S_2 + S_3 = S_{23}.$$

That is,

$$[yy_2 - 2a(x + x_2)] + [yy_3 - 2a(x + x_3)] = y_2y_3 - 2a(x_2 + x_3).$$

$$y(y_2 + y_3) - 2a(x_2 + x_3) - 4ax = y_2y_3 - 2a(x_2 + x_3).$$

Since $x_2 + x_3 = 2x_1$ and $y_2 + y_3 = 2y_1$, we have

$$2yy_1 - 4ax_1 - 4ax = y_2y_3 - 4ax_1$$

That is,

$$yy_1 - 2ax = \frac{y_2y_3}{2} \tag{4.8}$$

Since the chord AB is passing through (x_1, y_1), from Eq. (4.8), we have

$$\frac{y_2y_3}{2} = y_1^2 - 2ax_1 \tag{4.9}$$

Therefore, from Eqs. (4.8) and (4.9), the chord equation is

$$yy_1 - 2ax = y_1^2 - 2ax_1$$

$$\Rightarrow yy_1 - 2a(x + x_1) = y_1^2 - 4ax_1$$

Hence, $S_1 = S_{11}$ is the equation of the chord. ■

DEFINITION 4.14 **Diameter of a Parabola** A line intersecting the parabola and parallel to the axis of the parabola is called a *diameter of the parabola*.

QUICK LOOK 9

The line $y = k$ (i.e., a diameter) meets the parabola at the point

$$\left(\frac{k^2}{4a}, k\right)$$

THEOREM 4.18 In a parabola, the midpoints of parallel chords lie on a line which is a diameter of the parabola.

PROOF See Fig. 4.10. Suppose $M(x_1, y_1)$ be the midpoint of a chord which is parallel to the line $lx + my + n = 0$. By Theorem 4.17, equation of the chord whose midpoint is (x_1, y_1) is

$$S_1 = S_{11}$$

$$\Rightarrow yy_1 - 2a(x + x_1) = y_1^2 - 4ax_1$$

$$\Rightarrow 2ax - y_1y + y_1^2 - 2ax_1 = 0$$

This chord is parallel to the line

$$lx + my + n = 0 \Rightarrow 2am + ly_1 = 0$$

This implies that (x_1, y_1) lies on the line $2am + ly = 0$, which is horizontal, and hence it is a diameter.

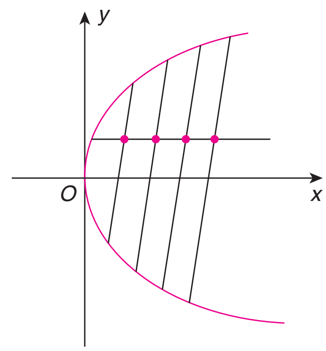


FIGURE 4.10

Note: If the chords are vertical, then their midpoints lie on the axis of the parabola.

Example 4.9

Find the coordinates of the point whose chord of contact with respect to the parabola $y^2 = 4x$ is $2x - 7y + 2 = 0$.

Solution: Suppose (x_1, y_1) is the point whose chord of contact with respect to $y^2 = 4x$ is

$$2x - 7y + 2 = 0 \tag{4.10}$$

However, the chord of contact of (x_1, y_1) is

$$S_1 \equiv yy_1 - 2(x + x_1) = 0 \tag{4.11}$$

Therefore, from Eqs. (4.10) and (4.11), we get

$$\frac{-2}{2} = \frac{y_1}{-7} = \frac{-2x_1}{2}$$

Hence, $x_1 = 1$ and $y_1 = 7$. Thus, the required point is $(1, 7)$.

Example 4.10

Show that the chords of contacts of points on the line $2x - 3y + 4 = 0$ with respect to the parabola $y^2 = 4ax$ pass through a fixed point.

Solution: Let $P(x_1, y_1)$ be a point on the line $2x - 3y + 4 = 0$. Therefore

$$2x_1 - 3y_1 + 4 = 0 \tag{4.12}$$

Now, the chord of contact of (x_1, y_1) with respect to $y^2 = 4ax$ is

$$yy_1 - 2a(x + x_1) = 0 \tag{4.13}$$

From Eqs. (4.12) and (4.13), we get

$$yy_1 - 2ax - a(3y_1 - 4) = 0$$

$$\Rightarrow y_1(y - 3a) - 2a(x - 2) = 0 \tag{4.14}$$

By Theorem 2.20, Eq. (4.14) represents the lines passing through the fixed point which is the intersection of the lines $x = 2$ and $y = 3a$. Hence, the fixed point is $(2, 3a)$.

Example 4.11

The normals at P and Q of the parabola $y^2 = 4ax$ meet at a point (x_1, y_1) on the curve. Show that

$$(PQ)^2 = (x_1 + 4a)(x_1 - 8a)$$

Solution: Let $P = (at_1^2, 2at_1)$ and $Q = (at_2^2, 2at_2)$. Suppose $(x_1, y_1) = (at_3^2, 2at_3)$. Therefore, by Theorems 4.12 and 4.13, we have

$$-t_1 - \frac{2}{t_1} = t_3 = -t_2 - \frac{2}{t_2}$$

and $$t_1 t_2 = 2 \tag{4.15}$$

Now,

$$\begin{aligned}(PQ)^2 &= a^2(t_1^2 - t_2^2)^2 + 4a^2(t_1 - t_2)^2 \\ &= a^2(t_1 - t_2)^2\left[(t_1 + t_2)^2 + 4\right] \\ &= a^2[t_1^2 + t_2^2 - 2t_1t_2][t_1^2 + t_2^2 + 2t_1t_2 + 4] \\ &= [a(t_1^2 + t_2^2) - 2at_1t_2][a(t_1^2 + t_2^2) + 2at_1t_2 + 4a]\end{aligned} \tag{4.16}$$

Also by Theorem 4.11, we have

$$x_1 = 2a + a(t_1^2 + t_1t_2 + t_2^2) \text{ and } y_1 = -at_1t_2(t_1 + t_2)$$

Therefore

$$a(t_1^2 + t_2^2) = x_1 - 2a - at_1t_2 \tag{4.17}$$

Using Eqs. (4.15) and (4.17) in Eq. (4.16), we have

$$\begin{aligned}(PQ)^2 &= (x_1 - 2a - 2a - 4a)(x_1 - 2a - 2a + 8a) \\ &= (x_1 - 8a)(x_1 + 4a)\end{aligned}$$

DEFINITION 4.15 **Focal Chord and Focal Radius** The chord passing through the focus of a parabola is called the *focal chord*. If P is a point on a parabola with focus S, then SP is called the *focal radius* of P.

THEOREM 4.19 (PROPERTIES OF FOCAL CHORDS) Let $y^2 = 4ax$ be a parabola with focus $S(a, 0)$ and directrix $x + a = 0$. Then, the following properties hold good.

1. If $P(at_1^2, 2at_1)$ is one end of the focal chord, then the other end is
$$\left(\frac{a}{t_1^2}, \frac{-2a}{t_1}\right)$$
That is, if t is the parameter of one end of the focal chord, then the parameter of the other end is $-1/t$.
2. Length of the focal chord whose one end is $(at^2, 2at)$ is
$$a\left(t + \frac{1}{t}\right)^2$$
3. If PSQ is a focal chord, then
$$\frac{1}{SP} + \frac{1}{SQ} = \frac{1}{a} = \frac{2}{2a}$$
In other words, the semi-latus rectum is the harmonic mean between the focal radii of the ends of a focal chord.
4. The circle described on a focal chord touches the directrix.
5. The circle described on a focal radius of a point on the parabola touches the tangent at the vertex.
6. The circle described on a focal radius of a point $P(at^2, 2at)$ cuts an intercept of length $a\sqrt{1 + t^2}$ on the normal at P.
7. The tangents drawn at the extremities of a focal chord are at right angles and hence intersect on the directrix of the parabola.

PROOF See Fig. 4.11.

1. Let $Q(at_2^2, 2at_2)$ be the other end of the focal chord for which $P(at_1^2, 2at_1)$ is one end. Equation of the focal chord PQ is
$$y - 2at_1 = \frac{2}{t_1 + t_2}(x - at_1^2)$$
This chord passes through $S(a, 0)$. So we have
$$0 - 2at_1 = \frac{2}{t_1 + t_2}(a - at_1^2)$$

$$\Rightarrow -at_1^2 - at_1t_2 = a - at_1^2$$
$$\Rightarrow t_1t_2 = -1$$
$$\Rightarrow t_2 = \frac{-1}{t_1}$$

2. Let $P(at^2, 2at)$ be one end of the focal chord so that by point (1), the other end is

$$Q\left(\frac{a}{t^2}, \frac{-2a}{t}\right)$$

Therefore,

$$(PQ)^2 = \left(at^2 - \frac{a}{t^2}\right)^2 + \left(2at + \frac{2a}{t}\right)^2$$
$$= a^2\left(t + \frac{1}{t}\right)^2\left[\left(t - \frac{1}{t}\right)^2 + 4\right]$$
$$= a^2\left(t + \frac{1}{t}\right)^2\left(t + \frac{1}{t}\right)^2$$
$$= a^2\left(t + \frac{1}{t}\right)^4$$

Hence,

$$PQ = a\left(t + \frac{1}{t}\right)^2$$

3. Let $P = (at^2, 2at)$ and $Q = \left(\frac{a}{t^2}, \frac{-2a}{t}\right)$. Then, we know that

$$SP = at^2 + a, \quad SQ = \frac{a}{t^2} + a$$

Therefore,

$$\frac{1}{SP} + \frac{1}{SQ} = \frac{1}{a(t^2+1)} + \frac{t^2}{a(1+t^2)} = \frac{1}{a} = \frac{2}{2a}$$

and $2a$ is the semi-latus rectum. Hence, SP, semi-latus rectum, and SQ are in HP.

4. $P(at^2, 2at)$ and $Q(a/t^2, -2a/t)$ are the ends of a focal chord. It is known that the centre of the circle described on PQ as diameter is

$$\left(\frac{1}{2}\left(at^2 + \frac{a}{t^2}\right), a\left(t - \frac{1}{t}\right)\right)$$

and radius [by point (2)] is

$$\frac{PQ}{2} = \frac{a}{2}\left(t + \frac{1}{t}\right)^2$$

Now, the distance of the centre of the circle from the directrix is

$$\frac{1}{2}\left(at^2 + \frac{a}{t^2}\right) + a$$

$$= \frac{1}{2}a\left(t^2 + \frac{1}{t^2} + 2\right)$$

$$= \frac{a}{2}\left(t + \frac{1}{t}\right)^2 = \frac{PQ}{2}$$

Hence, the circle described on PQ as diameter touches the directrix.

5. The circle with SP as diameter is $(x-a)(x-at^2)+y(y-2at)=0$. The centre is

$$\left(\frac{a+at^2}{2}, \frac{2at}{2}\right) = \left(\frac{a}{2}(1+t^2), at\right)$$

and the radius is

$$\frac{1}{2}SP = \frac{1}{2}\sqrt{(at^2-a)^2+4a^2t^2}$$

$$= \frac{a}{2}\sqrt{(t^2-1)^2+4t^2} = \frac{a}{2}(t^2+1)$$

Distance of the centre from the y-axis is the radius, which is given by

$$\frac{a}{2}(1+t^2)$$

Thus, the circle described on SP as diameter touches the tangent at the vertex.

6. Let the circle described on SP meet the normal at P in N. Draw SM perpendicular to the tangent at P. Since $SNPM$ is a rectangle, we have (see Fig. 4.11)

$$PN = SM = \frac{|a - t(0) + at^2|}{\sqrt{1+t^2}} \quad (\because \text{ equation of the tangent at } P \text{ is } ty = x + at^2)$$

$$= a\sqrt{1+t^2}$$

7. Let $P(at^2, 2at)$ and $Q(a/t^2, -2a/t)$. The tangent at P is

$$ty = x + at^2$$

whose slope is $1/t$. The tangent at Q is

$$\frac{-y}{t} = x + \frac{a}{t^2}$$

whose slope is $-t$. Therefore, the tangents at P and Q intersect at right angles.

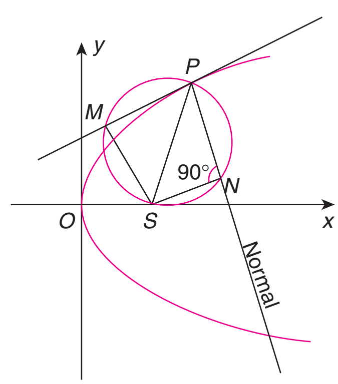


FIGURE 4.11

QUICK LOOK 10

Extremities of focal chords are the only points where the tangents drawn to the parabola intersect at right angles on the direction and conversely.

THEOREM 4.20 (NUMBER OF NORMALS FROM A POINT) In general, three normals can be drawn through a point in the plane of a parabola to the curve.

PROOF Let $P(h, k)$ be a point in the plane of $y^2 = 4ax$. It is known that the normal to parabola at $(at^2, 2at)$ (see Theorem 4.10) is $tx + y = 2at + at^3$. This normal passes through $P(h,k)$

$$\Leftrightarrow th + k = 2at + at^3$$

$$\Leftrightarrow at^3 + (2a - h)t - k = 0 \tag{4.18}$$

Equation (4.18) is a cubic equation in t and hence, in general, it has three roots and hence, there are three points on the curve at which normals drawn to the parabola are concurrent at $P(h, k)$. ■

QUICK LOOK 11

If t_1, t_2 and t_3 are the roots of Eq. (4.18), then we have

1. $t_1 + t_2 + t_3 = 0$
2. $t_1t_2 + t_2t_3 + t_3t_1 = \dfrac{2a - h}{a}$
3. $t_1t_2t_3 = \dfrac{k}{a}$

Also note that all the three roots t_1, t_2 and t_3 need not necessarily be real and distinct.

DEFINITION 4.16 **Conormal Points** Points on a parabola are called *conormal points* if the normals drawn at them are concurrent.

Section 4.2.2 provides a procedure to determine the number of normals from a given point to a parabola. Since the proof involves *Cardon's method* of solving a cubic equation, generally, the proof is avoided. Students are advised to adopt only the procedure.

4.2.2 Procedure to Determine Number of Normals

Let $P(x_1, y_1)$ be a point in the plane of the parabola. From Theorem 4.20, the normal at $(at^2, 2at)$ passes through $P(x_1, y_1)$ if and only if

$$tx_1 + y_1 = 2at + at^3 \tag{4.19}$$

so that the number of normals through is equal to the number of real roots of Eq. (4.19). Therefore, Eq. (4.19) is written as

$$t^3 + \frac{(2a - x_1)}{a}t - \frac{y_1}{a} = 0$$

Let

$$H = \frac{2a - x_1}{3a}, G = \frac{-y_1}{a} \text{ and } \Delta = G^2 + 4H^3$$

Then we have

1. Only one normal if $\Delta > 0$.
2. Two normals if $\Delta = 0$.
3. Three normals if $\Delta < 0$.

Example 4.12

Find the number of normals to the parabola $y^2 = 4ax$ from the point $(6a, 0)$ and also find the feet of the normals.

Solution: Normal $tx + y = 2at + at^3$ passes through $(6a, 0)$. So we have

$$6at = 2at + at^3$$
$$\Rightarrow t^3 - 4t = 0$$
$$\Rightarrow t(t^2 - 4) = 0$$
$$\Rightarrow t = 0, \pm 2$$

Hence, there are three normals to the parabola from the point $(6a, 0)$ and the feet of the normals are $(0, 0)$, $(4a, 4a)$ and $(4a, -4a)$. Also, according to the procedure given in Section 4.2.2, we have

$$x_1 = 6a,\ y_1 = 0,\ H = \frac{2a - x_1}{3a} = \frac{2a - 6a}{3a} = \frac{-4}{3}$$

and

$$G = \frac{-y_1}{a} = 0 \text{ and } \Delta = G^2 + 4H^3 = 0 + 4\left(\frac{-4}{3}\right)^3 < 0$$

Hence, there are three normals to the given parabola.

Example 4.13

Find the number of normals to $y^2 = 4ax$ from the point (a, a).

Solution: We have

$$x_1 = a,\ y_1 = a,\ H = \frac{2a - x_1}{3a} = \frac{2a - a}{3a} = \frac{1}{3}$$

and
$$G = \frac{-y_1}{a} = \frac{-a}{a} = -1$$

Now

$$\Delta = G^2 + 4H^3 = (-1)^2 + 4\left(\frac{1}{3}\right)^3 > 0$$

Hence, from (a, a), there is only one normal. Also it is clear that the cubic equation in t is

$$P(t) \equiv t^3 + t - 1 = 0$$

which has only one real root because if it has two real roots, then by Rolle's theorem (Theorem 3.4, Vol. 3)

$$P'(t) = 3t^2 + 1 = 0$$

has a root in between them which is absurd. Thus, the cubic equation

$$P(t) \equiv t^3 + t - 1 = 0$$

has only one real root and hence it has only one normal through (a, a).

Example 4.14

Let $(at_r^2, 2at_r)$ for $r = 1, 2$ and 3 be conormal points on the parabola $y^2 = 4ax$. If the normals at those points to the curve meet the axis of the parabola in points whose distances from the vertex are in AP, then prove that the point of concurrence of the three normals lies on the curve. $27ay^2 = 2(x - 2a)^3$.

Solution: Let $P(x_1, y_1)$ be the point where the normals at t_1, t_2 and t_3 are concurrent. Hence, by Theorem 4.20, t_1, t_2 and t_3 are roots of the equation

$$at^3 + (2a - x_1)t - y_1 = 0 \tag{4.20}$$

Therefore, by Quick Look 11, we have

$$t_1 + t_2 + t_3 = 0 \tag{4.20a}$$

$$t_1t_2 + t_2t_3 + t_3t_1 = \frac{2a - x_1}{a} \tag{4.20b}$$

$$t_1t_2t_3 = \frac{y_1}{a} \tag{4.20c}$$

The normal at t_1, namely, $t_1x + y = 2at_1 + at_1^3$ meets the axis at the point $(2a + at_1^2, 0)$ and similarly the normals at t_2 and t_3 meet the axis at $(2a + at_2^2, 0)$ and $(2a + at_3^2, 0)$. By hypothesis,

$$(2a + at_1^2) + (2a + at_3^2) = 4a + 2at_2^2$$

$$\Rightarrow t_1^2 + t_3^2 = 2t_2^2 \tag{4.20d}$$

Also, from Eq. (4.20a), we have

$$(t_1 + t_3)^2 = t_2^2 \Rightarrow t_1^2 + 2t_1t_3 + t_3^2 = t_2^2$$

Therefore, from Eq. (4.20d), we get

$$2t_2^2 + 2t_1t_3 = t_2^2$$

and from Eq. (4.20c), we get

$$t_2^2 = -2t_1t_3 = -2\left(\frac{y_1}{at_2}\right)$$

$$\Rightarrow t_2^3 = \frac{-2y_1}{a}$$

Since t_2 is a root of Eq. (4.20), we have

$$-2y_1 + (2a - x_1)\left(\frac{-2y_1}{a}\right)^{1/3} - y_1 = 0$$

$$\Rightarrow 3y_1 = -(2a - x_1)\left(\frac{2y_1}{a}\right)^{1/3} \qquad (4.21)$$

Taking cube on both sides of Eq. (4.21), we get

$$27y_1^3 = -(2a - x_1)^3 \frac{2y_1}{a}$$

$$\Rightarrow 27ay_1^2 = 2(x_1 - 2a)^3$$

Hence, the point (x_1, y_1) lies on the curve

$$27ay^2 = 2(x - 2a)^3$$

THEOREM 4.21 In general, a circle and parabola will intersect in four points such that the algebraic sum of the ordinates of the common points is zero.

PROOF Suppose $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ is a circle and $y^2 = 4ax$ is a parabola. Every point on the parabola is of the form $(at^2, 2at)$, $t \in \mathbb{R}$. Substituting $x = at^2$ and $y = 2at$ in the circle equation $S = 0$, we have

$$a^2t^4 + 2a(g + 2a)t^2 + 4aft + c = 0 \qquad (4.22)$$

Equation (4.22), being a fourth-degree equation in t, has four real roots, at the most. If t_1, t_2, t_3 and t_4 are the roots of Eq. (4.22), then

$$t_1 + t_2 + t_3 + t_4 = 0$$

so that

$$2at_1 + 2at_2 + 2at_3 + 2at_4 = 0$$

or

$$y_1 + y_2 + y_3 + y_4 = 0$$

where $y_r = 2at_r$ $(r = 1, 2, 3, 4)$. Thus, the algebraic sum of the ordinates of the common points is zero. ∎

QUICK LOOK 12

If t_1, t_2, t_3 and t_4 are the parameters of the common points of the parabola $y^2 = 4ax$ and the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$, then t_r, where $r = 1, 2, 3$ and 4, are roots of the equation

$$a^2t^4 + 2a(g + 2a)t^2 + 4aft + c = 0$$

and hence by the relation between the roots and the coefficients, we have

$$t_1 + t_2 + t_3 + t_4 = 0 \qquad (4.23a)$$

$$\sum t_1t_2 = \frac{2(g + 2a)}{a} \qquad (4.23b)$$

$$\sum t_1t_2t_3 = \frac{-4f}{a} \qquad (4.23c)$$

$$t_1t_2t_3t_4 = \frac{c}{a^2} \qquad (4.23d)$$

Subjective Problems

1. Prove that the foot of the perpendicular drawn from the focus onto any tangent to a parabola falls on the tangent at the vertex (see Fig. 4.12).

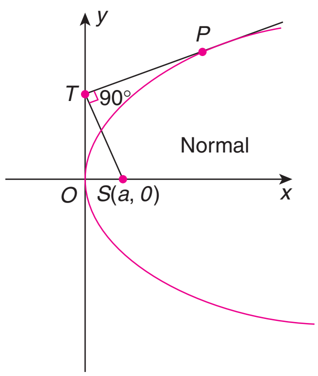


FIGURE 4.12

Solution: Let $y^2 = 4ax$ be a parabola and $P(at^2, 2at)$ be a point on the curve. The tangent at $P(at^2, 2at)$ is given by

$$ty = x + at^2$$

This meets the tangent at the vertex (i.e., y-axis) in the point $T(0, at)$. Since the focus is $S(a, 0)$, the slope of ST is given by

$$\frac{at-0}{0-a} = -t$$

However, the slope of the tangent at $P = 1/t$. So

$$\text{Slope of } ST \times \text{Slope of the tangent at } P = -t\left(\frac{1}{t}\right) = -1$$

Therefore, ST is perpendicular to the tangent at P.

2. Prove that the portion of the tangent to a parabola intercepted between the point of contact and the directrix subtends a right angle at the focus of the parabola (see Fig. 4.13).

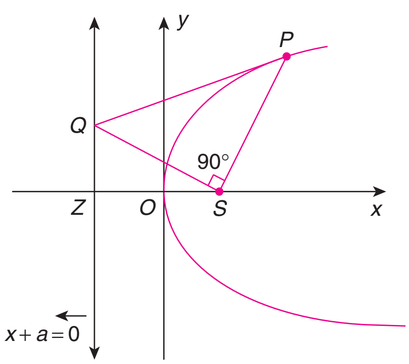


FIGURE 4.13

Solution: The tangent at $P(at^2, 2at)$ is $ty = x + at^2$. This meets the directrix at the point

$$Q\left(-a, \frac{a(t^2-1)}{t}\right)$$

Now,

$$\text{Slope of } SQ \times \text{Slope of } SP = \left(\frac{[a(t^2-1)/t]-0}{-a-a}\right)\left(\frac{2at-0}{at^2-a}\right)$$

$$= \frac{-(t^2-1)}{2t} \times \frac{2t}{t^2-1} = -1$$

Hence, $\angle PSQ = 90°$.

3. Prove that the circumcircle of a triangle formed by three tangents to a parabola passes through the focus of the parabola.

Solution: Let the tangents be

$$t_r y = x + at_r^2$$

where $r = 1, 2$ and 3. By Problem 1, the feet of the perpendiculars drawn from the focus of the parabola onto the three tangents (which are the sides of a triangle) are collinear on the tangent at the vertex. Hence, from the section 'Pedal Line (or Simson's Line)', Chapter 1, the circumcircle of the triangle passes through the focus.

4. Prove that the orthocentre of a triangle formed by three tangents to a parabola lies on the directrix.

Solution: Let $t_r y = x + at_r^2$, $r = 1, 2, 3$ be three tangents to a parabola so that, from Theorem 4.6, the vertices of the triangle formed by the three are given by

$$A[at_1t_2,\ a(t_1+t_2)]$$

$$B[at_2t_3,\ a(t_2+t_3)]$$

$$C[at_3t_1,\ a(t_3+t_1)]$$

So

$$\text{Slope of } BC = \frac{a(t_1-t_2)}{at_3(t_1-t_2)} = \frac{1}{t_3}$$

Therefore, the equation of the altitude through A of ΔABC is

$$y - a(t_1+t_2) = -t_3(x - at_1t_2) \tag{4.24}$$

Similarly, the equation of the altitude through B is

$$y - a(t_2+t_3) = -t_1(x - at_2t_3) \tag{4.25}$$

Subtracting Eq. (4.25) from Eq. (4.24), we get

$$a(t_3-t_1) = (t_1-t_3)x$$

$$\Rightarrow x = -a$$

That is, the abscissa of the orthocentre is $-a$ and hence the orthocentre lies on the directrix.

5. If $P(x_1, y_1), Q(x_2, y_2)$ and $R(x_3, y_3)$ are the three points on the parabola $y^2 = 4ax$, then show that the area of the triangle PQR is the absolute value of

$$\frac{1}{8a}(y_1-y_2)(y_2-y_3)(y_3-y_1)$$

Solution: The area of ΔPQR is

$$\frac{1}{2}\left\|\begin{matrix} x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1\end{matrix}\right\| = \frac{1}{8a}\left(\text{absolute value of} \begin{vmatrix} 4ax_1 & y_1 & 1\\ 4ax_2 & y_2 & 1\\ 4ax_3 & y_3 & 1\end{vmatrix}\right)$$

$$= \text{Absolute value of } \frac{1}{8a}\begin{vmatrix} y_1^2 & y_1 & 1\\ y_2^2 & y_2 & 1\\ y_3^2 & y_3 & 1\end{vmatrix}$$

$$= \frac{1}{8a}|(y_1-y_2)(y_2-y_3)(y_3-y_1)|$$

QUICK LOOK 13

If we replace y_1, y_2 and y_3, respectively, by $2at_1$, $2at_2$ and $2at_3$, then the area of the triangle is equal to

$$a^2|(t_1-t_2)(t_2-t_3)(t_3-t_1)|$$

6. Prove that the area of the triangle formed by three tangents to a parabola is half of the area of the triangle whose vertices are the points of contact of the tangents. **(IIT-JEE 1996)**

Solution: Let $P(at_1^2, 2at_1), Q(at_2^2, 2at_2)$ and $R(at_3^2, 2at_3)$ be the three points on $y^2 = 4ax$ and the three points A, B, and C be the points of intersection of the tangents drawn at points P, Q and R of the triangle. Therefore, by Theorem 4.6, we have

$$A[at_1t_2,\ a(t_1+t_2)]$$

$$B[at_2t_3,\ a(t_2+t_3)]$$

$$C[at_3t_1,\ a(t_3+t_1)]$$

Therefore

Area of ΔABC

$$= \text{Absolute value of the determinant } \frac{1}{2}\begin{vmatrix} at_1t_2 & a(t_1+t_2) & 1\\ at_2t_3 & a(t_2+t_3) & 1\\ at_3t_1 & a(t_3+t_1) & 1\end{vmatrix}$$

$$= \text{Absolute value of } \frac{a^2}{2}(t_3-t_1)(t_1-t_2)\begin{vmatrix} at_1t_2 & a(t_1+t_2) & 1\\ t_2 & 1 & 0\\ t_3 & 1 & 0\end{vmatrix}$$

(By $R_3 - R_2$ and then $R_2 - R_1$)

$$= \frac{a^2}{2}|(t_1-t_2)(t_2-t_3)(t_3-t_1)|$$

$$= \frac{1}{2}(\text{Area of } \Delta PQR) \qquad \text{(By Quick Look 13)}$$

7. Prove that the area of the triangle formed by the three normals to the parabola $y^2 = 4ax$ at the points t_1, t_2 and t_3 is

$$\frac{1}{2}a^2|(t_1-t_2)(t_2-t_3)(t_3-t_1)|(t_1+t_2+t_3)^2$$

Solution: By Theorem 4.11, the vertices of the triangle are

$$[2a+a(t_1^2+t_1t_2+t_2^2), -at_1t_2(t_1+t_2)]$$

$$[2a+a(t_2^2+t_2t_3+t_3^2), -at_2t_3(t_2+t_3)]$$

and $$[2a+a(t_3^2+t_3t_1+t_1^2), -at_3t_1(t_3+t_1)]$$

Let these points be A, B and C, respectively. Therefore

Area of the ΔABC

= Absolute value of

$$\frac{1}{2}\begin{vmatrix} 2a+a(t_1^2+t_1t_2+t_2^2) & -at_1t_2(t_1+t_2) & 1\\ 2a+a(t_2^2+t_2t_3+t_3^2) & -at_2t_3(t_2+t_3) & 1\\ 2a+a(t_3^2+t_3t_1+t_1^2) & -at_3t_1(t_3+t_1) & 1\end{vmatrix}$$

= Absolute value of

$$\frac{-a^2}{2}\begin{vmatrix} 2+t_1^2+t_1t_2+t_2^2 & t_1t_2(t_1+t_2) & 1\\ (t_3-t_1)(t_2+t_1+t_3) & t_2(t_3-t_1)(t_2+t_1+t_3) & 0\\ (t_3-t_2)(t_1+t_2+t_3) & t_1(t_3-t_2)(t_1+t_2+t_3) & 0\end{vmatrix}$$

(By $R_2 - R_1$ and $R_3 - R_1$)

= Absolute value of

$$\left[\frac{a^2}{2}(t_3-t_1)(t_3-t_1)(t_1+t_2+t_3)^2\right]$$

$$\times\begin{vmatrix} 2+t_1^2+t_1t_2+t_2^2 & t_1t_2(t_1+t_2) & 1\\ 1 & t_2 & 0\\ 1 & t_1 & 0\end{vmatrix}$$

$$= \frac{a^2}{2}|(t_1-t_2)(t_2-t_3)(t_3-t_1)|(t_1+t_2+t_3)^2$$

8. Show that the length of the side of an equilateral triangle inscribed in the parabola $y^2 = 4ax$ with one vertex at the vertex of the parabola is $8a\sqrt{3}$.

Solution: Let OPQ be the equilateral triangle inscribed in the parabola (see Fig. 4.14), where $O(0, 0)$ is the vertex so that $\angle POM = \angle QOM = 30°$. Let $OP = OQ = r$. Therefore

$$P = (r\cos 30°,\ r\sin 30°) = \left(\frac{r\sqrt{3}}{2}, \frac{r}{2}\right)$$

Now, P lies on the parabola. So

$$\frac{r^2}{4} = 4a\left(\frac{r\sqrt{3}}{2}\right)$$

$$\Rightarrow r = 8a\sqrt{3}$$

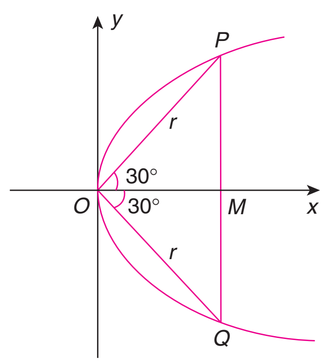


FIGURE 4.14

9. Show that the locus of the midpoints of chords of a parabola passing through the vertex is in turn a parabola whose latus rectum is half of the latus rectum of the original.

Solution: By Theorem 4.17, the equation of the chord in terms of its midpoint $M(x_1, y_1)$ is

$$S_1 = S_{11}$$

$$\Rightarrow yy_1 - 2a(x + x_1) = y_1^2 - 4ax_1$$

$$\Rightarrow yy_1 - 2ax = y_1^2 - 2ax_1 \quad \text{which passes through } (0,0)$$

$$\Rightarrow y_1^2 - 2ax_1 = 0$$

Hence, the locus of $M(x_1, y_1)$ is the parabola $y^2 = 2ax$.

10. Prove that the chords of a parabola subtending right angle at the vertex pass through a fixed on the axis.

(IIT-JEE 1994)

Solution: See Fig. 4. 15. Let $y = mx + c$ be a chord (say, PQ) subtending a right angle at the vertex. Therefore, by Theorem 2.33, the combined equation of the pair of lines $\overleftrightarrow{OP}$ and $\overleftrightarrow{OQ}$ is

$$y^2 - 4ax\left(\frac{y - mx}{c}\right) = 0$$

Since $\angle POQ = 90°$, in the above equation,

$$\text{Coefficient of } x^2 + \text{ Coefficient of } y^2 = 0$$

$$\Rightarrow 1 + \frac{4am}{c} = 0$$

$$\Rightarrow c = -4am$$

Therefore, the equation of the chord PQ is

$$y = mx - 4am$$

$$\Rightarrow m(x - 4a) - y = 0$$

Certainly, this line passes through the fixed point $(4a, 0)$.

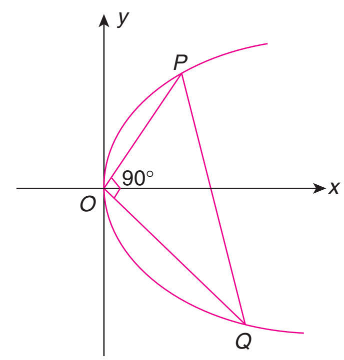


FIGURE 4.15

11. Show that the intersection of two perpendicular normals to the parabola lies on the curve $y^2 = a(x - 3a)$.

Solution: The normals at $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ are

$$t_1x + y = 2at_1 + at_1^3$$

and $$t_2x + y = 2at_2 + at_2^3$$

respectively. Therefore, their point of intersection is

$$[2a + a(t_1^2 + t_1t_2 + t_2), -at_1t_2(t_1 + t_2)]$$

Suppose this point of intersection is R. Since the normals are at right angles, we have $(-t_1)(-t_2) = -1$ or $t_1t_2 = -1$. Let

$$x_1 = 2a + a(t_1^2 + t_1t_2 + t_2^2) = 2a + a\left[(t_1 + t_2)^2 - t_1t_2\right] \quad (4.26)$$

and $$y_1 = -at_1t_2(t_1 + t_2) = -a(-1)(t_1 + t_2) \text{ or } t_1 + t_2 = \frac{y_1}{a}$$

Therefore, from Eq. (4.26), we have

$$x_1 = 2a + a\left[(t_1 + t_2)^2 - t_1t_2\right]$$

$$= 2a + a\left[\frac{y_1^2}{a^2} + 1\right] = 3a + \frac{y_1^2}{a}$$

or $$y_1^2 = a(x_1 - 3a)$$

Hence, $R(x_1, y_1)$ lies on $y^2 = a(x - 3a)$.

12. Let P be a point on the parabola $y^2 = 4ax$ and N be the foot of the perpendicular drawn from point P onto the axis. If T is the intersection of the tangent at P with the axis of the parabola, then show that vertex is the midpoint of TN and $SP = ST$, where S is the focus.

Solution: Let P be $(at^2, 2at)$ so that the equation of the tangent at P is $ty = x + at^2$ and it meets the axis $(y = 0)$ at point $T(-at^2, 0)$. However, $N = (at^2, 0)$. Hence, the vertex is the midpoint of TN. Also,

$$ST = SO + OT = a(1+t^2) = SP$$

13. Prove that the locus of the midpoints of chords of a parabola which subtend right angle at the vertex is another parabola whose latus rectum is half that of the original parabola. Further, all such chords pass through a fixed point on the axis of the parabola.

Solution: See Fig. 4. 16. Let $M(x_1, y_1)$ be the midpoint of a chord of $S \equiv y^2 - 4ax = 0$ which subtends a right angle at the vertex. Hence, by Theorem 4.17, equation of the chord is

$$S_1 = S_{11}$$
$$\Rightarrow yy_1 - 2a(x + x_1) = y_1^2 - 4ax_1$$
$$\Rightarrow yy_1 - 2ax = y_1^2 - 2ax_1$$
$$\Rightarrow \frac{yy_1 - 2ax}{y_1^2 - 2ax_1} = 1 \tag{4.27}$$

Suppose the chord provided in Eq. (4.27) cuts the parabola at points P and Q. Therefore, by Theorem 2.33, the combined equation of the pair of lines $\overleftrightarrow{OP}$ and $\overleftrightarrow{OQ}$ is

$$y^2 - 4a\left(\frac{yy_1 - 2ax}{y_1^2 - 2ax_1}\right) = 0 \tag{4.28}$$

Since $\angle POQ = 90°$, in Eq. (4.28)

$$\text{Coefficient of } x^2 + \text{Coefficient of } y^2 = 0$$
$$\Rightarrow \frac{8a^2}{y_1^2 - 2ax_1} + 1 = 0$$
$$\Rightarrow y_1^2 = 2a(x_1 - 4a)$$

Hence, the locus of (x_1, y_1) is the parabola $y^2 = 2a(x - 4a)$. If we put $y_1^2 = 2ax_1 - 8a^2$ in Eq. (4.27), we get the equation of the chord as $yy_1 - 2ax = -8a^2$ which passes through $(4a, 0)$.

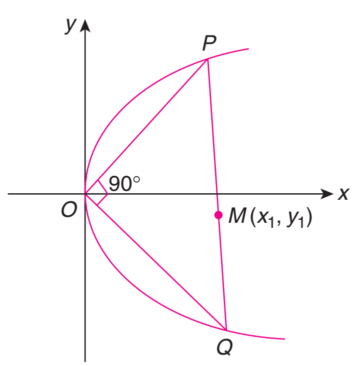


FIGURE 4.16

14. Prove that the area of the triangle formed by pair of tangents drawn from (x_1, y_1) to the parabola $S \equiv y^2 - 4ax = 0$ and the chord of contact of (x_1, y_1) is

$$\frac{S^{3/2}}{2a}$$

Solution: See Fig. 4.17. The equation of the chord of contact of point $P(x_1, y_1)$ is

$$S_1 = 0$$
$$\Rightarrow yy_1 - 2a(x + x_1) = 0 \tag{4.29}$$

Suppose $Q(h, k)$ and $R(l, m)$ are the ends of the chord of contact of P. Hence, k and m are the roots of the equation. By Eq. (4.29), and $y^2 = 4ax$, we have

$$yy_1 = 2a\left(\frac{y^2}{4a} + x_1\right)$$

so that

$$k + m = 2y_1 \tag{4.30}$$

and

$$km = 4ax_1 \tag{4.31}$$

Also, Q and R lie on the line provided in Eq. (4.29). So

$$ky_1 = 2ah + 2ax_1 \text{ and } my_1 = 2al + 2ax_1$$
$$\Rightarrow (k - m)y_1 = 2a(h - l)$$
$$\Rightarrow \frac{h - l}{k - m} = \frac{y_1}{2a} \tag{4.32}$$

Therefore

$$RQ = \sqrt{(h-l)^2 + (k-m)^2} = \sqrt{\frac{y_1^2}{4a^2}(k-m)^2 + (k-m)^2}$$
$$= \frac{|k-m|\sqrt{y_1^2 + 4a^2}}{2a} = \frac{\sqrt{(k+m)^2 - 4km}\sqrt{y_1^2 + 4a^2}}{2a}$$

Therefore, from Eqs. (4.30) and (4.31), we get

$$RQ = \frac{\sqrt{4y_1^2 - 16ax_1}\sqrt{y_1^2 + 4a^2}}{2a}$$
$$= \frac{1}{a}\sqrt{y_1^2 - 4ax_1}\sqrt{y_1^2 + 4a^2} \tag{4.33}$$

Now,

PM = Perpendicular distance from P onto the chord QR

$$= \frac{|y_1^2 - 2a(x_1 + x_1)|}{\sqrt{y_1^2 + 4a^2}} = \frac{y_1^2}{\sqrt{y_1^2 + 4a^2}}$$

$$[\because (x_1, y_1) \text{ lies outside}] \tag{4.34}$$

Hence, from Eqs. (4.33) and (4.34), the area of ΔPQR is

$$\frac{1}{2}(QR)(PM) = \left(\frac{(y_1^2 - 4ax_1)^{3/2}}{2a}\right) = \frac{S_{11}^{3/2}}{2a}$$

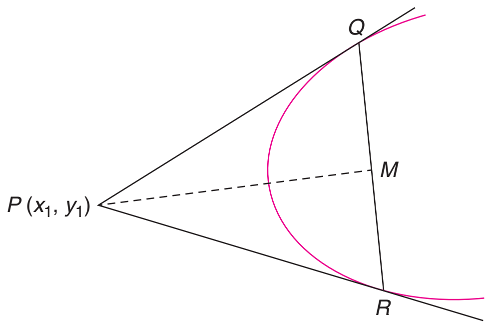


FIGURE 4.17

15. Show that the locus of the midpoint of chords of a parabola which pass through a fixed point is in turn a parabola.

Solution: Let $M(x_1, y_1)$ be the midpoint of a chord of $S \equiv y^2 - 4ax = 0$ which passes through a fixed point, say, (h, k). So

$$yy_1 - 2a(x + x_1) - (y_1^2 - 4ax_1) = 0$$

$(S_1 = S_{11})$ passes through (h, k). Therefore we have

$$ky_1 - 2a(h + x_1) - y_1^2 + 4ax_1 = 0$$
$$\Rightarrow ky_1 - 2ah - y_1^2 + 2ax_1 = 0$$
$$\Rightarrow y_1^2 - ky_1 = 2a(x_1 - h)$$
$$\Rightarrow \left(y_1 - \frac{k}{2}\right)^2 = 2a(x_1 - h) + \frac{k^2}{4}$$
$$\Rightarrow \left(y_1 - \frac{k}{2}\right)^2 = 2a\left[x_1 - h + \frac{k^2}{8a}\right]$$

Therefore, the locus of (x_1, y_1) is the parabola

$$\left(y - \frac{k}{2}\right)^2 = 2a\left(x - h + \frac{k^2}{8a}\right)$$

16. Show that the locus of the midpoints of chords of the parabola $y^2 = 4ax$ which are of constant length $2l$ is $(4ax - y^2)(y^2 + 4a^2) = 4a^2l^2$.

Solution: Let $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ be the extremities of a chord of length $2l$ and $M(x_1, y_1)$ be the midpoint of PQ so that

$$x_1 = \frac{a(t_1^2 + t_2^2)}{2} \qquad (4.35)$$

and

$$y_1 = \frac{2a(t_1 + t_2)}{2} = a(t_1 + t_2) \qquad (4.36)$$

By hypothesis, the length of the segment PQ is

$$a|t_1 - t_2|\sqrt{(t_1 + t_2)^2 + 4} = 2l$$

Thus,

$$(PQ)^2 = a^2(t_1 - t_2)^2[(t_1 + t_2)^2 + 4] = 4l^2$$
$$\Rightarrow a^2[(t_1 + t_2)^2 - 4t_1t_2][(t_1 + t_2)^2 + 4] = 4l^2 \qquad (4.37)$$

Now, from Eqs. (4.35) and (4.36), we get

$$t_1 + t_2 = \frac{y_1}{a}$$

and

$$t_1^2 + t_2^2 = \frac{2x_1}{a}$$

which imply that

$$\frac{y_1^2}{a^2} = (t_1 + t_2)^2 = t_1^2 + t_2^2 + 2t_1t_2 = \frac{2x_1}{a} + 2t_1t_2$$

Thus,

$$t_1t_2 = \frac{y_1^2 - 2ax_1}{2a^2} \qquad (4.38)$$

Substituting the value of t_1t_2 from Eq. (4.38) and $t_1 + t_2 = y_1/a$ in Eq. (4.37), we have

$$a^2\left[\frac{y_1^2}{a^2} - \frac{4(y_1^2 - 2ax_1)}{2a^2}\right]\left[\frac{y_1^2}{a^2} + 4\right] = 4l^2$$
$$\Rightarrow (4ax_1 - y_1^2)(y_1^2 + 4a^2) = 4a^2l^2$$

Hence, the locus of (x_1, t_1) is

$$(4ax - y^2)(y^2 + 4a^2) = 4a^2l^2$$

17. Find the locus of the midpoints of focal chords of $S = y^2 - 4ax = 0$.

Solution: Let $M(x_1, y_1)$ be the midpoint of a focal chord. Then

$$yy_1 - 2a(x + x_1) = y_1^2 - 4ax_1$$

It passes through $(a, 0)$. So

$$0 - 2a(a + x_1) = y_1^2 - 4ax_1$$
$$\Rightarrow y_1^2 = 2ax_1 - 2a^2$$
$$\Rightarrow y_1^2 = 2a(x_1 - a)$$

Thus, the locus of $M(x_1, y_1)$ is the parabola $y^2 = 2a(x - a)$ whose vertex is $S\ (a, 0)$ and latus rectum is $2a$.

18. Show that the midpoints of normal chords of the parabola $S \equiv y^2 - 4ax = 0$ lies on the curve

$$\frac{y^2}{2a} + \frac{4a^3}{y^2} = x - 2a$$

Solution: Let $M(x_1, y_1)$ be the midpoint of a normal chord, which is normal at the point $(at^2, 2at)$. Hence, the two equations $yy_1 - 2ax = y_1^2 - 2ax_1$ and $tx + y = 2at + at^3$ represent the same normal. Therefore,

$$\frac{t}{-2a} = \frac{1}{y_1} = \frac{2at + at^3}{y_1^2 - 2ax_1}$$

$$\Rightarrow t = \frac{-2a}{y_1} \quad \text{and} \quad 2at + at^3 = \frac{y_1^2 - 2ax_1}{y_1}$$

$$\Rightarrow 2a\left(\frac{-2a}{y_1}\right) + a\left(\frac{-2a}{y_1}\right)^3 = \frac{y_1^2 - 2ax_1}{y_1}$$

$$\Rightarrow \frac{-4a^2}{y_1} - \frac{8a^4}{y_1^3} = \frac{y_1^2 - 2ax_1}{y_1}$$

$$\Rightarrow -4a^2 - \frac{8a^4}{y_1^2} = y_1^2 - 2ax_1$$

$$\Rightarrow -2a - \frac{4a^3}{y_1^2} = \frac{y_1^2}{2a} - x_1$$

$$\Rightarrow \frac{y_1^2}{2a} + \frac{4a^3}{y_1^2} = x_1 - 2a$$

Hence, (x_1, y_1) lies on the curve

$$\frac{y^2}{2a} + \frac{4a^3}{y^2} = x - 2a$$

19. Prove that tangent to a parabola bisects the angle between its focal radius and the line drawn parallel to the axis through the point of contact.

Solution: See Fig. 4.18. Suppose the tangent at point P meets the axis at point T. Tangent at $P(at^2, 2at)$ is

$$ty = x + at^2$$

so that

$$T = (-at^2, 0)$$

Hence, $ST = a + at^2 = SP$. Therefore, ΔSPT is isosceles and hence

$$\angle SPT = \angle STP = \angle TPT'$$

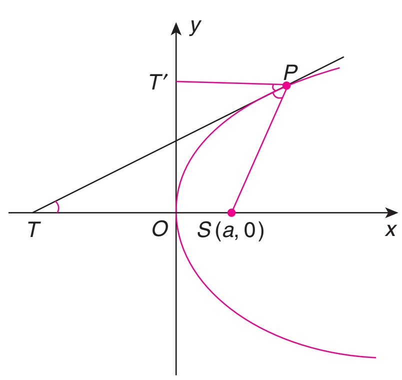


FIGURE 4.18

20. $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ are two points on the parabola $S \equiv y^2 - 4ax = 0$ such that $|t_1 - t_2| = 4$. Show that the circle described on PQ as diameter touches the parabola.

Solution: The equation of the circle described as PQ as diameter is

$$(x - at_1^2)(x - at_2^2) + (y - 2at_1)(y - 2at_2) = 0 \quad (4.39)$$

and the equation of the parabola $y^2 = 4ax$ in parametric form is

$$x = at^2 \text{ and } y = 2at$$

The circle provided in Eq. (4.39) touches the parabola if the equation obtained by substituting $x = at^2$ and $y = 2at$ in the circle has equal roots. The common points of circle and the parabola are given by

$$(at^2 - at_1^2)(at^2 - at_2^2) + (2at - 2at_1)(2at - 2at_2) = 0$$

$$\Rightarrow (t + t_1)(t + t_2) + 4 = 0$$

$$\Rightarrow t^2 + (t_1 + t_2)t + t_1t_2 + 4 = 0$$

whose discriminant is

$$(t_1 + t_1^2)^2 - 4(t_1t_2 + 4) = (t_1 - t_2)^2 - 16$$

$$= 4^2 - 16 \qquad (\text{by hypothesis } |t_1 - t_2| = 4)$$

$$= 0$$

Hence, the circle touches the parabola.

QUICK LOOK 14

Problem 20 is also stated as follows: Prove that the circle described on a chord of a parabola as diameter, such that the difference of the ordinates of the ends of the chord is equal to twice the latus rectum, touches the parabola.

21. Prove that the locus of the point, through which only two normals can be drawn to the parabola $y^2 = 4ax$, is the curve $27ay^2 = 4(x - 2a)^3$.

Solution: Let $P(x_1, y_1)$ be a point through which the two normals are drawn to the parabola. Therefore, if t_1, t_2 and t_3 are the parameters of the feet of the normals, then by Theorem 4.20, t_1, t_2 and t_3 are the roots of the cubic equation

$$at^3 + (2a - x_1)t - y_1 = 0$$

and hence we have

$$t_1 + t_2 + t_3 = 0 \quad (4.40)$$

$$t_1t_2 + t_2t_3 + t_3t_1 = \frac{2a - x_1}{a} \quad (4.41)$$

$$t_1t_2t_3 = \frac{y_1}{a} \tag{4.42}$$

From point (x_1, y_1), only two normals exist $\Leftrightarrow$ two of the parameters t_1, t_2 and t_3 are equal. Suppose $t_1 = t_2$. Therefore, from Eq. (4.40),

$$2t_1 = -t_3 \tag{4.43}$$

From Eqs. (4.42) and (4.43), we get

$$-2t_1^3 = \frac{y_1}{a} \tag{4.44}$$

Also from Eq. (4.41),

$$t_1^2 + 2t_1t_3 = \frac{2a - x_1}{a}$$

$$\Rightarrow t_1^2 + 2t_1(-2t_1) = \frac{2a - x_1}{a} \qquad (\because 2t_1 = -t_3)$$

$$\Rightarrow 3t_1^2 = \frac{x_1 - 2a}{a} \tag{4.45}$$

From Eqs. (4.44) and (4.45), we get

$$\left(\frac{-y_1}{2a}\right)^2 = (t_1^3)^2 = (t_1^2)^3 = \left(\frac{x_1 - 2a}{3a}\right)^3$$

$$\Rightarrow \frac{y_1^2}{4a^2} = \frac{(x_1 - 2a)^3}{27a^3}$$

$$\Rightarrow 27ay_1^2 = 4(x_1 - 2a)^3$$

Hence, the locus of the point is

$$27ay^2 = 4(x - 2a)^3$$

22. Show that the locus of the point through which two of the three normals drawn to the parabola $y^2 = 4ax$ are at right angles to the curve is $y^2 = a(x - 3a)$.

Solution: Suppose $t_2t_3 = -1$ so that

$$t_1t_2t_3 = \frac{y_1}{a} \Rightarrow t_1 = \frac{-y_1}{a}$$

Since t_1 is a root of the cubic equation $at^3 + (2a - x_1)t + y_1 = 0$, we have

$$a\left(\frac{-y_1}{a}\right)^3 + (2a - x_1)\left(\frac{-y_1}{a}\right) - y_1 = 0$$

$$\Rightarrow \frac{-y_1^2}{a^2} - \frac{2a - x_1}{a} - 1 = 0$$

$$\Rightarrow y_1^2 + a(3a - x_1) = 0$$

Hence, the locus of (x_1, y_1) is

$$y^2 = a(x - 3a)$$

23. There are three points A, B and C on a parabola at which the normals drawn are concurrent at a point (h, k). Show that the circumcircle of the triangle ABC passes through the vertex of the parabola and also find its equation.

Solution: Suppose $A = (a t_1^2, 2at_1)$, $B = (at_2^2, 2at_2)$ and $C = (at_3^2, 2at_3)$. Therefore, by Theorem 4.20, t_1, t_2 and t_3 are the roots of the cubic equation

$$at^3 + (2a - h)t - k = 0 \tag{4.46}$$

where

$$t_1 + t_2 + t_3 = 0 \tag{4.46a}$$

$$t_1t_2 + t_2t_3 + t_3t_1 = \frac{2a - h}{a} \tag{4.46b}$$

$$t_1t_2t_3 = \frac{k}{a} \tag{4.46c}$$

Suppose the equation of the circumcircle of ΔABC is

$$x^2 + y^2 + 2gx + 2fy + c = 0 \tag{4.47}$$

Since A, B and C are the common points of the circle provided in Eq. (4.46) and the parabola, let $D(at_4^2, 2at_4)$ be the fourth point (by Theorem 4.21). Further, t_1, t_2, t_3 and t_4 are roots of the equation

$$a^2t^4 + 2a(g + 2a)t^2 + 4aft + c = 0$$

So we have

$$t_1 + t_2 + t_3 + t_4 = 0$$

$$\sum t_1t_2 = \frac{2(g + 2a)}{a}$$

$$\sum t_1t_2t_3 = -\frac{4f}{a}$$

$$t_1t_2t_3t_4 = \frac{c}{a^2}$$

Now, $t_1 + t_2 + t_3 + t_4 = 0$ and $t_1 + t_2 + t_3 = 0$ [from Eq. (4.46a)] imply that $t_4 = 0$. Hence, the fourth point D is the vertex. Also

$$\frac{2(g + 2a)}{a} = \sum t_1t_2 = t_1t_2 + t_1t_3 + t_1t_4 + t_2t_3 + t_2t_4 + t_3t_4$$

$$= t_1t_2 + t_2t_3 + t_3t_1 \qquad (\because t_4 = 0)$$

Therefore, from Eq. (4.46b), we get

$$\frac{2(g + 2a)}{a} = \frac{2a - h}{a}$$

$$\Rightarrow 2g = -2a - h \tag{4.48}$$

Further,

$$\frac{-4f}{a} = \sum t_1t_2t_3 = t_1t_2t_3 \qquad (\because t_4 = 0)$$

Therefore, from Eq. (4.46c), we have

$$\frac{-4f}{a} = \sum t_1 t_2 t_3 = t_1 t_2 t_3 = \frac{k}{a}$$

$$\Rightarrow 2f = \frac{-k}{2} \tag{4.49}$$

Moreover,

$$0 = t_1 t_2 t_3 t_4 = \frac{c}{a^2} \Rightarrow c = 0 \tag{4.50}$$

Substituting the values of $2g$, $2f$ and c from Eqs. (4.48), (4.49) and (4.50), respectively, in Eq. (4.47), the equation of the circumcircle of ΔABC is obtained as

$$x^2 + y^2 - (2a+h)x - \left(\frac{k}{2}\right)y = 0$$

$$\Rightarrow 2x^2 + 2y^2 - 2(2a+h)x - ky = 0$$

24. Suppose QR is the chord of contact of a point P with respect to the parabola $S \equiv y^2 - 4ax = 0$. If QR is a normal chord, normal at Q, then show that the directrix bisects PQ.

Solution: Let $Q = (at_1^2, 2at_1)$ and $R = (at_2^2, 2at_2)$ so that $P = [at_1 t_2, a(t_1 + t_2)]$. Since QR is normal at Q, from Theorem 4.8, we have

$$t_2 = -t_1 - \frac{2}{t_1}$$

or

$$t_1 + t_2 = \frac{-2}{t_1} \tag{4.51}$$

Let (x, y) be the midpoint of PQ so that, from Eq. (4.51), we get

$$x = \frac{at_1^2 + at_1 t_2}{2} = \frac{at_1(t_1 + t_2)}{2} = \frac{at_1}{2}\left(\frac{-2}{t_1}\right)$$

That is, $x = -a$ and hence the directrix bisects PQ.

25. A diameter, which is through the point P on a parabola, meets a chord QR of the parabola in point A. The tangents at Q and R meet the diameter through P in B and C. Show that PA is geometric mean between PB and PC.

Solution: See Fig. 4.19. Let $P = (at^2, 2at)$, $Q = (at_1^2, 2at_1)$, $R = (at_2^2, 2at_2)$. The diameter through P is

$$y = 2at \tag{4.52}$$

Tangents at points Q and R, respectively, are

$$t_1 y = x + at_1^2 \tag{4.53}$$

and

$$t_2 y = x + at_2^2 \tag{4.54}$$

The equation of the chord QR is

$$y - 2at_1 = \frac{2}{t_1 + t_2}(x - at_1^2) \tag{4.55}$$

Substituting $y = 2at$ in Eq. (4.55), we get

$$A = (att_1 + att_2 - at_1 t_2, 2at)$$

Substituting $y = 2at$ in Eqs. (4.53) and (4.54), we get

$$B = (2att_1 - at_1^2, 2at)$$

and

$$C = (2att_2 - at_2^2, 2at)$$

Therefore

$$PB = \left|at^2 - 2att_1 + at_1^2\right| = a(t - t_1)^2 \tag{4.56}$$

and

$$PC = a(t - t_2)^2 \tag{4.57}$$

Also

$$(PA)^2 = (att_1 + att_2 - at_1 t_2 - at^2)^2$$
$$= a^2[t(t_1 - t) - t_2(t_1 - t)]^2$$
$$= a^2(t_1 - t)^2(t_2 - t)^2$$

Therefore, from Eqs. (4.56) and (4.57), we have

$$(PA)^2 = PB \cdot PC$$

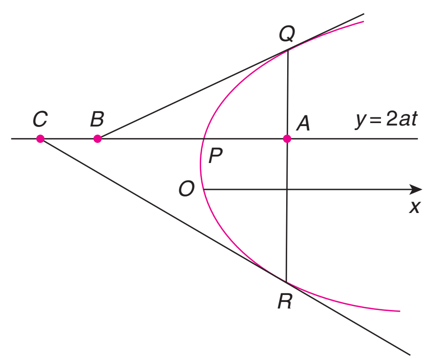


FIGURE 4.19

26. The normal at point P of a parabola meets the same curve at Q. Show that the tangents at P and Q intersect on the diameter passing through the other end of the focal chord passing through P.

Solution: See Fig. 4.20. PQ is a normal chord normal at P. Let $P = (at_1^2, 2at_1)$ and $Q = (at_2^2, 2at_2)$ so that by Theorem 4.12, we get

$$t_2 = -t_1 - \frac{2}{t_1} \tag{4.58}$$

Let R be the point of intersection of the tangents at P and Q so that by Theorem 4.6, we have

$$R = [(at_1t_2, a(t_1+t_2)]$$

Therefore, from Eq. (4.58), we get

$$R = \left(at_1t_2,\ a\left(\frac{-2}{t_1}\right)\right)$$

$$\Rightarrow \left(at_1t_2, \frac{-2a}{t_1}\right) \tag{4.59}$$

Now, if P' is the other end of the focal chord through P, then by Theorem 4.19, part (1), we have

$$P' = \left(\frac{a}{t_1^2}, \frac{-2a}{t_1}\right)$$

Therefore, from Eq. (4.59), it is clear that the point R lies on the diameter passing through P'.

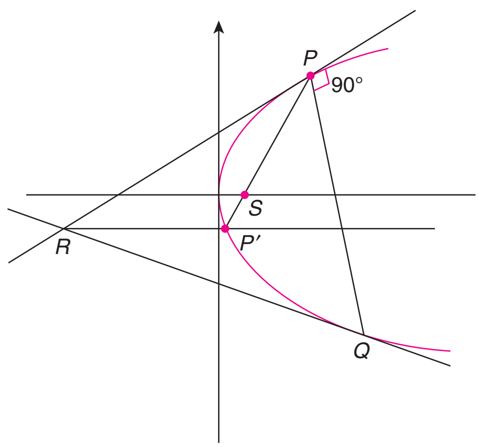


FIGURE 4.20

27. If the normals at three points P, Q and R on the parabola $y^2 = 4ax$ meet at point A, then show that

$$SP \cdot SQ \cdot SR = a(AS)^2$$

where S is the focus.

Solution: Let $A = (h, k)$ and $P=(at_1^2, 2at_1)$, $Q=(at_2^2, 2at_2)$ and $R=(at_3^2, 2at_3)$ so that t_1, t_2 and t_3 are roots of the equation

$$at^3 + (2a-h)t - k = 0$$

Since t_1, t_2 and t_3 are the roots, we have

$$t_1 + t_2 + t_3 = 0$$

$$t_1t_2 + t_2t_3 + t_3t_1 = \frac{2a-h}{a}$$

and
$$t_1t_2t_3 = \frac{k}{a}$$

Now,

$$\begin{aligned}
SP\cdot SQ\cdot SR &= (a+at_1^2)(a+at_2^2)(a+at_3^2)\\
&= a^3(1+t_1^2)(1+t_2^2)(1+t_3^2)\\
&= a^3\left[1+(t_1^2+t_2^2+t_3^2)+(t_1^2t_2^2+t_2^2t_3^2+t_3^2t_1^2)\right.\\
&\qquad\left.+t_1^2t_2^2t_3^2\right]\\
&= a^3\left[1+(t_1+t_2+t_3)^2-2\sum t_1t_2+\left(\sum t_1t_2\right)^2\right.\\
&\qquad\left.-2t_1t_2t_3(t_1+t_2+t_3)+t_1^2t_2^2t_3^2\right]\\
&= a^3\left[1+0-2\left(\frac{2a-h}{a}\right)+\left(\frac{2a-h}{a}\right)^2-0+\frac{k^2}{a^2}\right]\\
&= a[a^2-4a^2+2ah+4a^2-4ah+h^2+k^2]\\
&= a[a^2-2ah+h^2+k^2]\\
&= a[(a-h)^2+k^2]\\
&= a(AS)^2
\end{aligned}$$

28. Show that the locus of the point from which the pair of tangents drawn to the parabola $y^2 = 4ax$ including constant angle α is $\cot^2\alpha(y^2-4ax) = (x+a)^2$.

Solution: Let $P(x_1, y_1)$ be the point of intersection of tangents drawn at $Q(at_1^2, 2at_1)$ and $R(at_2^2, 2at_2)$ so that

$$x_1 = at_1t_2 \tag{4.60a}$$

and
$$y_1 = a(t_1+t_2) \tag{4.60b}$$

It is known that $1/t_1$ and $1/t_2$ are the slopes of the tangent at Q and R so that by hypothesis, we have

$$\tan^2\alpha = \left(\frac{(1/t_1)-(1/t_2)}{1+(1/t_1t_2)}\right)^2 = \left(\frac{t_2-t_1}{t_1t_2+1}\right)^2 = \frac{(t_1+t_2)^2-4t_1t_2}{(1+t_1t_2)^2}$$

Therefore, from Eqs. (4.60a) and (4.60b), we get

$$\tan^2\alpha = \frac{(y_1^2/a^2)-(4x_1/a)}{1+(x_1/a)^2}$$

$$= \frac{y_1^2-4ax_1}{(x_1+a)^2}$$

Therefore, the locus of (x_1, y_1) is

$$(x+a)^2 = \cot^2\alpha\,(y^2-4ax)$$

QUICK LOOK 15

If $\alpha = 90°$, then the tangents from (x_1, y_1) are at right angles so that $x_1 + a = 0$ and hence (x_1, y_1) lies on the directrix.

29. A straight line touches both $x^2 + y^2 = 2a^2$ and $y^2 = 8ax$. Show that its equation is $y = \pm(x + 2a)$.

Solution: We have

$$y = mx + \frac{2a}{m}$$

is a tangent to $y^2 = 8ax$ for all $m \neq 0$. This line also touches the circle $x^2 + y^2 = 2a^2$. So

$$\frac{4a^2}{m^2} = (\sqrt{2a})^2(1+m^2)$$
$$\Rightarrow \frac{4}{m^2} = 2(1+m^2)$$
$$\Rightarrow m^4 + m^2 - 2 = 0$$
$$\Rightarrow (m^2+2)(m^2-1) = 0$$
$$\Rightarrow m^2 = 1 \text{ or } m = \pm 1$$

Therefore, the tangent equation is $y = \pm(x + 2a)$.

30. If two tangents to a parabola make complementary angles with the axis, then show that their point of intersection lies on the line $x = a$.

Solution: Let the tangents at t_1 and t_2 make complementary angles, say α and β, with the axis of the parabola. Therefore

$$\tan\alpha = \frac{1}{t_1}$$

and $$\tan\beta = \frac{1}{t_2}$$

and suppose $\alpha + \beta = 90° \Rightarrow t_1 t_2 = 1$. Since the abscissa of the point of intersection of the tangent at t_1 and t_2 is $at_1t_2 = a$, the point of intersection lies on the line $x = a$.

31. A pair of tangents is drawn to a parabola such that the sum of the angles made by them with the x-axis is constant. Prove that the point lies on a line through the focus.

Solution: Let the tangents at t_1 and t_2 intersect at (x_1, y_1) so that

$$x_1 = at_1t_2 \tag{4.61a}$$

and $$y_1 = a(t_1 + t_2) \tag{4.61b}$$

Let $\tan\alpha = \dfrac{1}{t_1}$ at $\tan\beta = \dfrac{1}{t_2}$ and suppose $\alpha + \beta = \theta$, where θ is constant, we have

$$\tan\theta = \tan(\alpha+\beta) = \frac{(1/t_1)+(1/t_2)}{1-[(1/t_1)\times(1/t_2)]} = \frac{t_1+t_2}{(t_1t_2)-1} = \frac{y_1/a}{(x_1/a)-1}$$

Therefore, from Eqs. (4.61a) and (4.61b), we get

$$\tan\theta = \frac{y_1}{x_1 - a} \Rightarrow (x_1 - a)\tan\theta - y_1 = 0$$

This implies that (x_1, y_1) lies on the line $(x - a)\tan\theta - y = 0$ which passes through the focus $(a, 0)$.

32. Show that the point from which a tangent to $y^2 = 4a(x + a)$ and a tangent to $y^2 = 4b(x + a)$ are at right angles lies on the line $x + a + b = 0$.

Solution: The parabola $y^2 = 4a(x + a)$ is written as $Y^2 = 4aX$ where $Y = y$ and $X = x + a$. For $Y^2 = 4aX$, a tangent is of the form

$$Y = mX + \frac{a}{m}$$
$$\Rightarrow y = m(x+a) + \frac{a}{m}$$

Therefore

$$y = m_1(x+a) + \frac{a}{m_1}$$

is a tangent to $y^2 = 4a(x + a)$ and

$$y = m_2(x+b) + \frac{b}{m_2}$$

is a tangent to $y^2 = 4b(x + b)$. These tangents are perpendicular to each other, which implies that $m_1m_2 = -1$. Now, the abscissa of the point of intersection satisfies the equation

$$m_1(x+a) + \frac{a}{m_1} - m_2(x+b) - \frac{b}{m_2} = 0$$
$$\Rightarrow (m_1 - m_2)x + am_1 + \frac{a}{m_1} - bm_2 - \frac{b}{m_2} = 0$$
$$\Rightarrow (m_1 - m_2)x + am_1 - am_2 - bm_2 + bm_1 \qquad (\because m_1m_2 = -1)$$
$$\Rightarrow (m_1 - m_2)x + a(m_1 - m_2) + b(m_1 - m_2) = 0$$

Therefore, the point of intersection of the tangent lies on the line $x + a + b = 0$.

33. In a parabola, prove that the tangents drawn at the extremities of a chord intersect on the diameter bisecting the chord.

Solution: Let $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ be two points on the parabola $y^2 = 4ax$. Therefore, the point of the intersection of these tangents is $[at_1t_2, a(t_1 + t_2)]$ and the midpoint of the chord PQ is

$$\left(\frac{a}{2}(t_1^2, + t_2^2), \frac{2a(t_1+t_2)}{2}\right) = \left(\frac{a}{2}(t_1^2 + t_2^2), a(t_1+t_2)\right)$$

Hence, the point of intersection of the tangents at P and Q lies on the diameter $y = a(t_1 + t_2)$ which bisects the chord PQ.

34. If r_1 and r_2 are the lengths of two perpendicular chords passing through the vertex of the parabola $y^2 = 4ax$, then show that

$$16a^2(r_1^{2/3} + r_2^{2/3}) = (r_1r_2)^{4/3}$$

Solution: See Fig. 4.21. Let OP and OQ be perpendicular chords such that $OP = r_1$ and $OQ = r_2$. Suppose OP makes angle θ with the axis so that OQ makes $90° - \theta$ with the axis. Therefore, $P = (r_1\cos\theta, r_1\sin\theta)$ and $Q = (r_2\sin\theta, r_2\cos\theta)$. Therefore,

$$r_1^2\sin^2\theta = 4a(r_1\cos\theta)$$

$$\Rightarrow r_1 = \frac{4a\cos\theta}{\sin^2\theta}$$

Similarly,

$$r_2 = \frac{4a\sin\theta}{\cos^2\theta}$$

$$\Rightarrow r_1^{2/3} + r_2^{2/3} = (4a)^{2/3}\left[\frac{\cos^{2/3}\theta}{\sin^{4/3}\theta} + \frac{\sin^{2/3}\theta}{\cos^{4/3}\theta}\right] = \frac{(4a)^{2/3}}{(\sin\theta\cos\theta)^{4/3}}$$

$$\Rightarrow 16a^2\left(r_1^{2/3} + r_2^{2/3}\right) = \frac{(16a^2)(16a^2)^{1/3}}{(\sin\theta\cos\theta)^{4/3}} = \frac{(16a^2)^{4/3}}{(\sin\theta\cos\theta)^{4/3}}$$

Also

$$(r_1r_2)^{4/3} = \left(\frac{4a\cos\theta}{\sin\theta}\times\frac{4a\sin\theta}{\cos^2\theta}\right)^{4/3} = \left(\frac{16a^2}{\sin\theta\cos\theta}\right)^{4/3}$$

$$\Rightarrow 16a^2(r_1^{2/3} + r_2^{2/3}) = (r_1r_2)^{4/3}$$

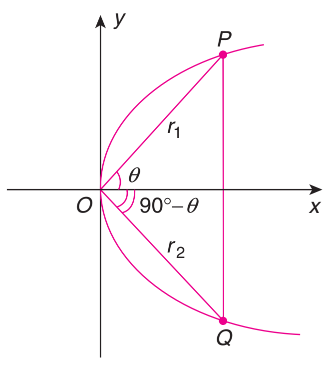


FIGURE 4.21

35. The tangent at a point P to the parabola $y^2 - 2y - 4x + 5 = 0$ intersects the directrix at Q. Find the locus of the point R which divides the chord PQ externally in the ratio $1/2 : 1$. **(IIT-JEE 2004)**

Solution: The given parabola equation can be written as

$$(y-1)^2 = 4(x-1)$$

which is of the form $Y^2 = 4X$ where $X = x - 1$ and $Y = y - 1$ and $a = 1$. Therefore, the vertex is (1, 1), focus at $(X = 1, Y = 0)$ is

$$(x-1=1, y-1=0) = (2,0)$$

The directrix equation is $X + 1 = 0$ or $x = 0$. Every point on the given parabola is of the form

$$(X = t^2, Y = 2t) = (1+t^2, 1+2t)$$

Hence, the equation of the tangent at $P(1+t^2, 1+2t)$ is

$$t(y-1) = (x-1) + t^2 \tag{4.62}$$

Since the directrix is $x = 0$, by substituting $x = 0$ in Eq. (4.62), we have

$$Q = \left(0, \frac{t^2+t-1}{t}\right)$$

Therefore, we have

$$P = (1+t^2, 1+2t) \quad \text{and} \quad Q = \left(0, \frac{t^2+t-1}{t}\right)$$

Let R be (x_1, y_1) and $PR:RQ = 1:-2$ (see Fig. 4.22.) Therefore

$$x_1 = \frac{-2(1+t^2)+1(0)}{-2+1} = 2(1+t^2) \tag{4.63}$$

and

$$y_1 = \frac{-2(1+2t)+[(t^2+t-1)/t]}{-2+1}$$

$$= \frac{3t^2+t+1}{t} \tag{4.64}$$

From Eq. (4.64), we have

$$ty_1 = 3t^2+t+1 = 3(1+t^2)+t-2$$

From Eq. (4.63), we get

$$ty_1 = \frac{3x_1}{2} + t - 2$$

$$\Rightarrow t(y_1, -1) = \frac{3x_1-4}{2}$$

$$\Rightarrow t = \frac{3x_1-4}{2(y_1-1)}$$

Substituting the value of t in Eq. (4.63), we have

$$x_1 = 2[1+t^2] = 2\left[1+\left(\frac{3x_1-4}{2(y_1-1)}\right)^2\right]$$

$$\Rightarrow 4x_1(y_1-1)^2 = 2\left[4(y_1-1)^2 + (3x_1-4)^2\right]$$

$$\Rightarrow 2x_1(y_1-1)^2 = 4(y_1-1)^2 + (3x_1-4)^2$$

$$\Rightarrow 2(x_1-2)(y_1-1)^2 = (3x_1-4)^2$$

Therefore, the locus of $R(x_1, x_2)$ is

$$2(x-2)(y-1)^2 = (3x-4)^2$$

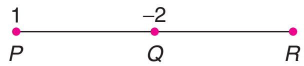


FIGURE 4.22

36. Show that the locus of the point that divides a chord, whose slope is 2, of the parabola $y^2 = 4x$ internally in the ratio 1:2 is a parabola. Find the vertex of this parabola. **(IIT-JEE 1995)**

Solution: Let $P(t_1^2, 2t_1)$ and $Q(t_2^2, 2t_2)$ be the ends of a chord whose slope is 2. Therefore

$$\frac{2}{t_1+t_2} = 2$$

$$\Rightarrow t_1 + t_2 = 1 \qquad (4.65)$$

Let $R\ (x_1, y_1)$ divide $\overline{PQ}$ internally in the ratio 1:2 (see Fig. 4.23). Hence

$$3x_1 = 2t_1^2 + t_2^2 \qquad (4.66)$$

and $$3y_1 = 4t_1 + 2t_2 = 2(t_1+t_2) + 2t_1$$

Therefore, from Eq. (4.65), we get

$$3y_1 = 2 + 2t_1$$

Therefore,

$$t_1 = \frac{3y_1 - 2}{2}$$

and $$t_2 = 1 - t_1 = 1 - \frac{3y_1-2}{2} = \frac{4-3y_1}{2}$$

Substituting the values of t_1 and t_2 in Eq. (4.66), we have

$$3x_1 = 2\left(\frac{3y_1-2}{2}\right)^2 + \left(\frac{4-3y_1}{2}\right)^2$$

$$= \frac{9y_1^2 - 12y_1 + 4}{2} + \frac{16 - 24y_1 + 9y_1^2}{4}$$

$$= \frac{27y_1^2 - 48y_1 + 24}{4}$$

$$\Rightarrow 12x_1 = 27y_1^2 - 48y_1 + 24$$

$$\Rightarrow 4x_1 = 9y_1^2 - 16y_1 + 8$$

$$\Rightarrow 4x_1 = 9\left(y_1^2 - \frac{16}{9}y_1 + \frac{8}{9}\right)$$

$$\Rightarrow 4x_1 = 9\left[\left(y_1 - \frac{8}{9}\right)^2 + \frac{8}{9} - \frac{64}{81}\right]$$

$$\Rightarrow \frac{4x_1}{9} = \left(y_1 - \frac{8}{9}\right)^2 + 8 - \frac{64}{9}$$

$$\Rightarrow \frac{4x_1}{9} = \left(y_1 - \frac{8}{9}\right)^2 + \frac{8}{9}$$

$$\Rightarrow \left(y_1 - \frac{8}{9}\right)^2 = \frac{4}{9}(x_1 - 2)$$

Hence, the locus of (x_1, y_1) is the parabola

$$\left(y - \frac{8}{9}\right)^2 = \frac{4}{9}(x-2)$$

whose vertex is $(2, 8/9)$.

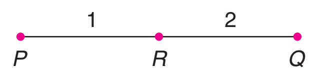


FIGURE 4.23

37. The angle between a pair of tangents drawn from point P to the parabola $y^2 = 4ax$ is 45°. Show that the locus of point P is a hyperbola. **(IIT-JEE 1998)**

Solution: From Problem 28, the locus of the point P is

$$\cot^2 45°(y^2 - 4ax) = (x+a)^2$$

$$\Rightarrow (y^2 - 4ax) = (x+a)^2$$

$$\Rightarrow x^2 + 6ax + a^2 - y^2 = 0$$

$$\Rightarrow (x+3a)^2 - 8a^2 - y^2 = 0$$

$$\Rightarrow \frac{(x+3a)^2}{8a^2} - \frac{y^2}{8a^2} = 1$$

which is a hyperbola, in fact, a rectangular hyperbola.

38. Suppose the normals at three different points on the parabola $y^2 = 4x$ pass through the point $(h, 0)$. Show that $h > 2$. **(IIT-JEE 1984)**

Solution: Let t_1, t_2 and t_3 be the parameters of the feet of the normals drawn from $(h, 0)$. Hence, t_1, t_2 and t_3 are the roots of the cubic equation $t^3 + (2-h)t = 0$ (see Theorem 4.20). That is, t_1, t_2 and t_3 are the roots of $t(t^2 + 2 - h) = 0$. Since the equation has three roots, h must be greater than 2.

39. Three normals with slopes m_1, m_2 and m_3 are drawn from point P, which is not located on the axis of the parabola $y^2 = 4x$. If $m_1 m_2 = \alpha$ results in the locus of P being part of the parabola, then find the value of α. **(IIT-JEE 2003)**

Solution: Let $P = (h, k)$, where $(k \neq 0)$, and $(t_r^2, 2t_r)$, where $r = 1$, 2 and 3, be the points on the parabola at which the normals pass through $P(h, k)$. Since the equation of normal at $(at^2, 2at)$ to the parabola $y^2 = 4ax$ is $tx + y = 2at + at^3$, the slope of the normal is $-t$. We have $m_1 = -t_1, m_2 = -t_2$ and $m_3 = -t_3$, so by hypothesis we get

$$m_1 m_2 = \alpha$$

$$\Rightarrow t_1 t_2 = \alpha \tag{4.67}$$

Since t_1, t_2 and t_3 are the roots of

$$t^3 + (2 - h)t - k = 0 \tag{4.68}$$

we have

$$t_1 + t_2 + t_3 = 0$$

$$t_1t_2 + t_2t_3 + t_3t_1 = 2 - h$$

and $$t_1 t_2 t_3 = k$$

Since $t_1 t_2 = \alpha$ [from Eq. (4.67)], we have

$$t_3 = \frac{k}{\alpha} \qquad (\because t_1\, t_2\, t_3 = k)$$

Since t_3 is a root of Eq. (4.68), we have

$$\left(\frac{k}{\alpha}\right)^3 + (2-h)\frac{k}{\alpha} - k = 0$$

$$\Rightarrow k^2 + \alpha^2(2-h) - \alpha^3 = 0$$

$$\Rightarrow k^2 = \alpha^2(\alpha - 2 + h)$$

Hence, the locus of (h, k) is the curve $y^2 = \alpha^2(x + \alpha - 2)$ and this is identical with $y^2 = 4x$ only when $\alpha = 2$.

40. Show that the locus of the point of intersection of tangents drawn at the ends of normal chords to the parabola $y^2 = 8(x - 1)$ is $y^2(x + 3) + 32 = 0$.

Solution: The given parabola equation is $Y^2 = 8X$ where $Y = y$ and $X = x - 1$. Every point on this parabola is

$$(x = 2t^2, y = 4t) = (2t^2 + 1, 4t)$$

Normal at $(2t^2 + 1, 4t)$ is

$$tX + Y = 2(2t) + 2t^3$$

$$\Rightarrow t(x-1) + y = 4t + 2t^3 \tag{4.69}$$

Suppose the tangents at the ends of normal chord intersect at $P(x_1, y_1)$. Then, the normal chord is the chord of contact of $P(x_1, y_1)$ and hence its equation is

$$Yy_1 - 4(X + x_1) = 0$$

$$\Rightarrow yy_1 - 4(x - 1 + x_1) = 0$$

$$\Rightarrow yy_1 - 4x + 4 - 4x_1 = 0 \tag{4.70}$$

Equations (4.69) and (4.70) represent the same straight line. Therefore

$$\frac{t}{-4} = \frac{1}{y_1} = \frac{4t + 2t^3}{4x_1 - 4}$$

$$\Rightarrow t = \frac{-4}{y_1} \text{ and } -1 = \frac{4 + 2t^2}{x_1 - 1}$$

$$\Rightarrow -(x_1 - 1) = 4 + 2\left(\frac{16}{y_1^2}\right)$$

$$\Rightarrow -x_1 - 3 = \frac{32}{y_1^2}$$

$$\Rightarrow y_1^2(x_1 + 3) + 32 = 0$$

Hence, the locus of (x_1, y_1) is

$$y^2(x + 3) + 32 = 0$$

41. Find the locus of the point of intersection of perpendicular normals drawn to the parabola $x^2 = 8y$.

Solution: Let $P(4t_1, 2t_1^2)$ and $Q(4t_2, 2t_2^2)$ be the points on $x^2 = 8y$. Equation of the normals at P and Q are

$$y - 2t_1^2 = \frac{-1}{t_1}(x - 4t_1^2) \tag{4.71}$$

and $$y - 2t_2^2 = \frac{-1}{t_2}(x - 4t_2^2) \tag{4.72}$$

Since the normals are at right angles, we have

$$\left(\frac{-1}{t_1}\right)\left(\frac{-1}{t_2}\right) = -1$$

$$\Rightarrow t_1 t_2 = -1 \tag{4.73}$$

Solving Eqs. (4.71) and (4.72) and from Eq. (4.73), we have

$$x = -2t_1t_2(t_1 + t_2) = 2(t_1 + t_2)$$

and $$y = 2(t_1^2 + t_1t_2 + t_2^2 + 2) = 2(t_1^2 + t_2^2 + 1) \quad (\because t_1t_2 = -1)$$

$$= 2[(t_1 + t_2)^2 - 2t_1t_2 + 1] = 2\left[\frac{x^2}{4} + 3\right]$$

$$\left(\because t_1t_2 = -1 \text{ and } t_1 + t_2 = \frac{x}{2}\right)$$

$$\Rightarrow 2y = x^2 + 12$$

which is the required locus.

42. Two tangents of the parabola $y^2 = 8x$ meet the tangent at the vertex in P and Q. If $PQ = 4$, prove that the locus of the point of intersections of the tangents is $y^2 = 8(x + 2)$.

Solution: Let the two tangents be

$$t_1 y = x + 2t_1^2$$

and $$t_2 y = x + 2t_2^2$$

so that their point of intersection is $[2t_1t_2, 2(t_1+t_2)]$. Now, $P = (0, 2t_1)$ and $Q = (0, 2t_2)$. Since $PQ = 4$, we have

$$|t_1 - t_2| = 2$$

Suppose $x_1 = 2t_1t_2$ and $y_1 = 2(t_1 + t_2)$. Therefore

$$\frac{y_1^2}{4} = (t_1+t_2)^2 = (t_1-t_2)^2 + 4t_1t_2 = 4 + 2x_1$$

Thus, the locus of (x_1, y_1) is

$$y^2 = 8(x+2)$$

43. A chord of the parabola $y^2 = 4ax$ subtends right angle at the vertex. Show that the locus of the point of intersection of the normals drawn at the extremities of the chords is $y^2 = 16a(x - 6a)$.

Solution: Suppose $Q(at_1^2, 2at_1)$ at $R(at_2^2, 2at_2)$ are the extremities of a chord subtending right angle at the vertex. Therefore

Slope of $QO \times$ Slope of $RO = -1$ (where O is the vertex)

$$\Rightarrow \left(\frac{2at_1}{at_1^2}\right)\left(\frac{2at_2}{at_2^2}\right) = -1$$

$$\Rightarrow t_1t_2 = -4 \tag{4.74}$$

Suppose (x_1, y_1) is the point of intersection of the normals drawn at points Q and R so that

$$x_1 = 2a + a(t_1^2 + t_1t_2 + t_2^2)$$

and
$$y_1 = -at_1t_2(t_1+t_2)$$

Therefore, from Eq. (4.74), we get

$$x_1 = 2a + a(t_1^2 + t_2^2 - 4) \tag{4.75a}$$

and
$$y_1 = -a(-4)(t_1+t_2) \tag{4.75b}$$

which implies that

$$\frac{x_1 + 2a}{a} = t_1^2 + t_2^2 \quad \text{and} \quad \frac{y_1}{4a} = t_1 + t_2$$

Therefore

$$\frac{y_1^2}{16a^2} = (t_1+t_2)^2 = t_1^2 + t_2^2 + 2t_1t_2 = \frac{x_1+2a}{a} - 8 \qquad (\because t_1t_2 = -4)$$

$$\Rightarrow y_1^2 = 16a(x_1 - 6a)$$

Hence, the locus of the point of intersection of the normals drawn at the extremities of the chords subtending right angle at the vertex is

$$y^2 = 16a(x - 6a)$$

44. From the vertex of a parabola if a pair of chords is drawn at right angles to one another and with these chords as adjacent sides a rectangle be formed, prove that the locus of the fourth vertex is $y^2 = 4a(x - 8a)$.

Solution: Let $y^2 = 4ax$ be the parabola, $P = (at_1^2, 2at_1)$ and $Q = (at_2^2, 2at_2)$ such that $\angle POQ = 90°$ (see Fig. 4.24). Now

$$\angle POQ = 90° \Rightarrow \left(\frac{2at_1}{at_1^2}\right)\left(\frac{2at_2}{at_2^2}\right) = -1$$

$$\Rightarrow t_1t_2 = -4 \tag{4.76}$$

Let $R = (h, k)$ be the fourth vertex of the rectangle $POQR$. Therefore (by Worked-Out Problem 10, Chapter 1), we have

$$h = at_1^2 + at_2^2 - 0$$

and
$$k = 2at_1 + 2at_2 - 0$$

Therefore

$$\left(\frac{k}{2a}\right)^2 = (t_1+t_2)^2 = t_1^2 + t_2^2 + 2t_1t_2 = \frac{h}{a} - 8 \qquad (\because t_1t_2 = -4)$$

$$\Rightarrow k^2 = 4a(h - 8a)$$

Hence, the locus of (h, k) is

$$y^2 = 4a(x - 8a)$$

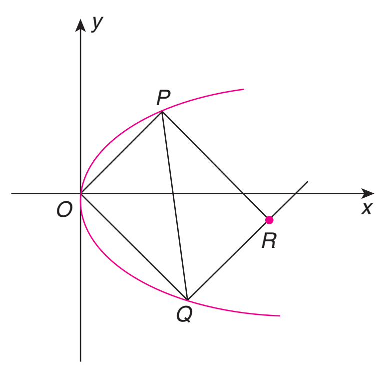


FIGURE 4.24

45. Let PSP' and QSQ' be two local chords of the parabola $y^2 = 4ax$ where S is the focus. If the line PQ passes through a fixed point (α, β), then show that $P'Q'$ also passes through the fixed point $(a^2/\alpha, -a\beta/\alpha)$

Solution: Let $P = (at_1^2, 2at_1)$ and $Q = (at_2^2, 2at_2)$ so that by Theorem 4.19, part (1), we have

$$P' = \left(\frac{a}{t_1^2}, \frac{-2a}{t_1}\right)$$

and $$Q' = \left(\frac{a}{t_2^2}, \frac{-2a}{t_2}\right)$$

Equation of PQ is

$$y - 2at_1 = \frac{2}{t_1 + t_2}(x - at_1^2)$$

$$\Rightarrow y(t_1 + t_2) = 2x + 2at_1t_2$$

Since this chord passes through (α, β), we have

$$\beta(t_1 + t_2) = 2\alpha + 2at_1t_2 \tag{4.77}$$

The equation of line $P'Q'$ is

$$y\left(\frac{-1}{t_1} - \frac{1}{t_2}\right) = 2x + 2a\left(-\frac{1}{t_1}\right)\left(-\frac{1}{t_2}\right)$$

$$\Rightarrow -(t_1 + t_2)y = (2t_1t_2)x + 2a$$

Now, $P'Q'$ passes through $(a^2/\alpha, -a\beta/\alpha)$

$$\Leftrightarrow -(t_1 + t_2)\left(\frac{-a\beta}{\alpha}\right) = (2t_1t_2)\left(\frac{a^2}{\alpha}\right) + 2a$$

$$\Leftrightarrow (t_1 + t_2)a\beta = 2a^2t_1t_2 + 2a\alpha$$

$$\Leftrightarrow \beta(t_1 + t_2) = 2at_1t_2 + 2\alpha$$

which is true according to Eq. (4.77). Thus $P'Q'$ passes through $(a^2/\alpha, -a\beta/\alpha)$.

46. Circle with centre at the focus of the parabola $y^2 = 2px$ touches the directrix of the parabola. Show that the points of intersection of the circle and the parabola are $(p/2, p)$ and $(p/2, -p)$. **(IIT-JEE 1995)**

Solution: Focus $S = (p/2, 0)$. Let the radius be r so that the equation of the circle is

$$\left(x - \frac{p}{2}\right)^2 + y^2 = r^2$$

This touches the directrix. So

$$\left|\frac{p}{2} + \frac{p}{2}\right| = r$$

$$\Leftrightarrow r = p$$

Hence, the equation of the circle is

$$\left(x - \frac{p}{2}\right)^2 + y^2 = p^2 \tag{4.78}$$

The abscissa of the point of intersection of the circle Q with the parabola are the roots of

$$\left(x - \frac{p}{2}\right)^2 + 2px = p^2$$

$$\Rightarrow x^2 + px - \frac{3p^2}{4} = 0$$

$$\Rightarrow 4x^2 + 4px - 3p^2 = 0$$

$$\Rightarrow 4x^2 + 6px - 2px - 3p^2 = 0$$

$$\Rightarrow 2x(2x + 3p) - p(2x + 3p) = 0$$

$$\Rightarrow x = \frac{p}{2}, \frac{-3p}{2}$$

Since $p > 0$ and (x, y) a point on the parabola, x cannot be negative. Hence

$$x = \frac{p}{2} \text{ and } y = \pm p$$

Thus, the points of intersection are $(p/2, p)$ and $(p/2, -p)$.

47. The normals at P and Q on a parabola $y^2 = 4ax$ intersect at a point R on the curve. If M is the midpoint of PQ and N is the midpoint of MR, then show that the locus of the point N is a parabola.

Solution: Let $P = (at_1^2, 2at_1)$, $Q = (at_2^2, 2at_2)$ and $R = (at_3^2, 2at_3)$. Hence, by Theorem 4.12, we have

$$-t_1 - \frac{2}{t_1} = t_3 = -t_2 - \frac{2}{t_2}$$

$$\Rightarrow t_1t_2 = 2 \tag{4.79}$$

Let $M(x_1, y_1)$ be the midpoint of PQ. Therefore

$$2x_1 = a(t_1^2 + t_2^2) \tag{4.80a}$$

and $$y_1 = a(t_1 + t_2) \tag{4.80b}$$

By Theorem 4.11, if $R = (h, k)$, then

$$h = 2a + a(t_1^2 + t_1t_2 + t_2^2)$$

and $$k = -at_1t_2(t_1 + t_2)$$

Therefore, from Eq. (4.79), we get

$$h = 4a + a(t_1^2 + t_2^2) \tag{4.81a}$$

and $$k = -2a(t_1 + t_2) \tag{4.81b}$$

Let (α, β) be the midpoint of MR. Therefore,

$$2\alpha = x_1 + h \text{ and } 2\beta = y_1 + k$$

which are written [from Eqs. (4.80a), (4.80b), (4.81a) and (4.81c)] as follows:

$$2\alpha = \frac{a(t_1^2+t_2^2)}{2} + 4a + a(t_1^2+t_2^2)$$

$$\Rightarrow \alpha = \frac{3a}{4}(t_1^2+t_2^2) + 2a$$

and $$2\beta = a(t_1+t_2) - 2a(t_1+t_2)$$

$$\beta = -\frac{a}{2}(t_1+t_2)$$

Now $$\beta^2 = \frac{a^2}{4}(t_1+t_2)^2 = \frac{a^2}{4}\left[\frac{4\alpha-8a}{3a}+4\right]$$

$$= \frac{a}{3}[\alpha + a] \qquad (\because t_1t_2 = 2)$$

$$= \frac{a}{3}(x+a)$$

Therefore, the locus of the midpoint (α, β) of MR is $y^2 = \frac{a}{4}(\alpha + 2a)$.

48. Prove that the locus of the midpoint of the portion of a normal to $y^2 = 4ax$ intercepted between the curve and the axis is another parabola.

Solution: Normal at $P(at^2, 2at)$ is

$$tx + y = 2at + at^3$$

Substituting $y = 0$ in the above equation, we have

$$x = 2a + at^2$$

Hence, the normal meets the axis at $N(2a + at^2, 0)$. Let (α, β) be the midpoint of PN so that $\alpha = a + at^2$ and $\beta = at$. Therefore

$$\frac{\alpha - a}{a} = t^2 = \left(\frac{\beta}{a}\right)^2$$

$$\Rightarrow \beta^2 = a(\alpha - a)$$

Hence, the locus of the midpoint of PN is

$$y^2 = a(x-a)$$

which is a parabola.

49. If $P(x_1, y_1)$, $Q(x_2, y_2)$ and $R(x_3, y_3)$ are three points on $y^2 = 4ax$ and the normals at P, Q and R meet at a point, then prove that

$$\frac{x_1-x_2}{y_3} + \frac{x_2-x_3}{y_1} + \frac{x_3-x_1}{y_2} = 0$$

Solution: Suppose $x_r = at_r^2$ and $y_r = 2at_r$, for $r = 1, 2$ and 3. The normals at P, Q and R meet at point $A(h, k)$. Hence, by Theorem 4.20 and Quick Look 11, t_1, t_2 and t_3 are the roots of the equation

$$at^3 + (2a-h)t - k = 0$$

Also $$t_1 + t_2 + t_3 = 0$$

$$t_1t_2 + t_2t_3 + t_3t_1 = \frac{2a-h}{a}$$

$$t_1t_2t_3 = \frac{k}{a}$$

Now,

$$\frac{x_1-x_2}{y_3} = \frac{a(t_1^2-t_2^2)}{2at_3} = \frac{(t_1+t_2)(t_1-t_2)}{2t_3} = \frac{-t_3(t_1-t_2)}{2}$$
$$(\because t_1+t_2+t_3 = 0)$$

Similarly,

$$\frac{x_2-x_3}{y_1} = \frac{-t_1(t_2-t_2)}{2}$$

and $$\frac{x_3-x_1}{y_2} = \frac{-t_2(t_3-t_1)}{2}$$

Hence

$$\frac{x_1-x_2}{y_3} + \frac{x_2-x_3}{y_1} + \frac{x_3-x_1}{y_2} = \frac{-1}{2}[t_3(t_1-t_2) + t_1(t_2-t_3) + t_2(t_3-t_1)]$$

$$= \frac{-1}{2}(0) = 0$$

WORKED-OUT PROBLEMS

Single Correct Choice Type Questions

1. L is the normal to the parabola $y^2 = 4x$ and passes through the point (9, 6). If the slope of the normal is positive, then its equation is

(A) $y - x + 3 = 0$ (B) $y + 3x - 33 = 0$

(C) $y - 2x + 12 = 0$ (D) $y + x - 15 = 0$

(IIT-JEE 2011)

Solution: Observe that (9, 6) lies on the parabola $y^2 = 4x$. Normal at $(t^2, 2t)$ is $tx + y = 2t + t^3$ which passes through $(9, 6)(t = 3)$; therefore, we have

$$3x + y = 6 + 27$$

$$\Rightarrow 3x + y - 33 = 0$$

Aliter (Using Calculus): Differentiating $y^2 = 4x$ with respect to x, we get

$$\frac{dy}{dx} = \frac{2}{y}$$

Therefore

$$\left(\frac{dy}{dx}\right)_{(9,6)} = \frac{2}{6} = \frac{1}{3}$$

Hence, equation of the normal at (9, 6) is

$$y - 6 = -3(x - 9)$$
$$\Rightarrow y + 3x - 33 = 0$$

Answer: (B)

2. Let S be the focus of the parabola $y^2 = 8x$ and PQ be the common chord of the circle $x^2 + y^2 - 2x - 4y = 0$ and the given parabola. Then the area of ΔPQS is

(A) 8 (B) 5 (C) 4 (D) 10

(IIT-JEE 2012)

Solution: Every point on $y^2 = 8x$ is of the form $(2t^2, 4t)$. Substituting $x = 2t^2$ and $y = 4t$ in the equation of the circle, we have

$$4t^4 + 16t^2 - 4t^2 - 16t = 0$$
$$\Rightarrow t(t^3 + 3t - 4) = 0$$
$$\Rightarrow t(t-1)(t^2 + t + 4) = 0$$

Since $t^2 + t + 4 = 0$ has normal roots, the values of t are 0 and 1. Hence, the common chord end points are $P(0, 0)$ and $Q(2, 4)$ and the focus is $S(2, 0)$. Hence, the area of ΔPQS is

$$\frac{1}{2}|4 \times 2 - 0| = 4$$

Answer: (C)

3. The orthocentre of a triangle formed by three tangents to a parabola $y^2 = 8x$ lies on

(A) $x + 1 = 0$ (B) $x + 2 = 0$

(C) $x + 4 = 0$ (D) $x - 1 = 0$

Solution: Orthocentre of a triangle formed by three tangents to the parabola $y^2 = 4ax$ lies on the directrix $x + a = 0$ (see Problem 4 in the section 'Subjective Problems'). Here, $a = 2$ gives that the orthocentre lies on the line $x + 2 = 0$.

Answer: (B)

4. The midpoints of the chords of the parabola $y^2 = 8x$ which subtend right angle at the vertex lie on the curve $y^2 = kx$, where k is equal to

(A) 1 (B) 2 (C) $\frac{1}{2}$ (D) 4

Solution: According to Problem 18 of the section 'Subjective Problems', the midpoints lie on a parabola whose latus rectum is half of the latus rectum of the original parabola. Here, the latus rectum of the original parabola is 8 and hence the latus rectum of the required parabola is 4.

Answer: (D)

5. If $x + y = k$ is a normal to the parabola $y^2 = 12x$, then the value of k is

(A) 3 (B) (C) 9 (D) -3

(IIT-JEE 2000)

Solution: We have $y^2 = 12x = 4(3)$, $x = 4ax$ where $a = 3$. Every point on the parabola is of the form $(3t^2, 6t)$. Suppose $x + y = k$ is normal at $(3t^2, 6t)$. Equation of the normal at $(3t^2, 6t)$ is $tx + y = 6t + 3t^3$. Therefore

$$\frac{t}{1} = \frac{1}{1} = \frac{6t + 3t^3}{k}$$
$$\Rightarrow t = 1 \text{ and } k = 6 + 3 = 9$$

Answer: (C)

6. If the line $x - 1 = 0$ is the directrix of the parabola $y^2 - kx + 8 = 0$, then one of the values of k is

(A) $\frac{1}{8}$ (B) 8 (C) 4 (D) $\frac{1}{4}$

(IIT-JEE 2000)

Solution: The given parabola equation can be written as

$$y^2 + k\left(x - \frac{8}{k}\right), Y^2 = k(X)$$

where

$$X = x - \frac{8}{k} \text{ and } Y = y$$

Therefore

$$Y^2 = kX = 4\left(\frac{k}{4}\right)X$$

so that $a = k/4$. The equation of the directrix is

$$X + a = 0 \Rightarrow x - \left(\frac{8}{k} - \frac{k}{4}\right) = 0$$

Now $x - 1 = 0$ is the directrix. This implies

$$\frac{8}{k} - \frac{k}{4} = 1$$

$$\Rightarrow 32-k^2 = 4k$$
$$\Rightarrow k^2+4k-32=0$$
$$\Rightarrow (k+8)(k-4)=0$$
$$\Rightarrow k = 4 \text{ or } -8$$

Therefore, one of the values of k is 4.

Answer: (C)

7. A normal at $P(at_1^2, 2at_1)$ meets the curve again at $Q(at_2^2, 2at_2)$. If PQ subtends a right angle at the vertex, then

(A) $t_1^2 = 2$ (B) $t_2^2 = 2$
(C) $t_1 = 2t_2$ (D) $t_2 = 2t_1$

Solution: From Theorem 4.12, we have

$$t_2 = -t_1 - \frac{2}{t_1}$$

or
$$t_1 + t_2 = \frac{-2}{t_1} \tag{4.82}$$

Also $\angle POQ = 90°$. This implies

$$\left(\frac{2at_1}{at_1^2}\right)\left(\frac{2at_2}{at_2^2}\right) = -1$$
$$\Rightarrow t_1t_2 = -4 \tag{4.83}$$

From Eq. (4.82), we have

$$t_1^2 + t_1t_2 = -2$$

which can written as [from Eq. (4.83)]

$$t_1^2 - 4 = -2$$
$$\Rightarrow t_1^2 = 2$$

Answer: (A)

8. The line $x - y - 1 = 0$ meets the parabola $y^2 = 4x$ at A and B. Normals at A and B meet at C. If D is a point on the curve such that CD is normal at D, then the coordinates of D are

(A) $(4, 4)$ (B) $(4, -4)$
(C) $(2, -4)$ (D) $(2, 4)$

Solution: Substituting $y = x - 1$ in $y^2 = 4x$ we get

$$x^2 - 6x + 1 = 0$$

whose roots are

$$\frac{6 \pm \sqrt{36-4}}{2} = 3 \pm 2\sqrt{2}$$

Now

$$x = 3+2\sqrt{2} \Rightarrow y = x-1 = 2+2\sqrt{2}$$

$$x = 3-2\sqrt{2} \Rightarrow y = 2-2\sqrt{2}$$

Let $A=(3+2\sqrt{2}, 2+2\sqrt{2})$ and $B=(3-2\sqrt{2}, 2-2\sqrt{2})$. By hypothesis, the normals at A, B and D meet at C. Hence, by Quick Look 11, the algebraic sum of the ordinates of A, B and D is zero. This implies

$$2+2\sqrt{2}+2-2\sqrt{2}+y_3 = 0$$

Here $D = (x_3, y_3)$.

Now

$$4x = 16 \Rightarrow x = 4$$

Hence, the required point is (4,–4).

Answer: (B)

9. Let θ be the angle of intersection of the parabolas $y^2 = 8ax$ and $x^2 = 27ay$ at the point other than the origin. Then, $\tan\theta$ is equal to

(A) $\frac{8}{13}$ (B) $\frac{13}{8}$ (C) $\frac{13}{9}$ (D) $\frac{9}{13}$

Solution:

$$y^4 = 64a^2x^2 = 64a^2(27ay) = 64\times 27a^3y$$
$$\Rightarrow y(y^3 - 64\times 27\times a^3) = 0$$
$$\Rightarrow y = 0, y = 12a$$

The point of intersection of the parabolas other than the origin is $(18a, 12a)$. The equation of the tangent to $y^2 = 8ax$ at $(18a, 12a)$ is

$$y(12a) - 4a(x+18a) = 0$$
$$\Rightarrow 3y - x - 18a = 0$$

The slope is 1/3. The equation of the tangent to $x^2 - 27ay$ at $(18a, 12a)$ is

$$x(18a) - \frac{27}{2}a(y+12a) = 0$$
$$\Rightarrow 4x - 3y - 12a = 0$$

whose slope is 4/3. Hence

$$\tan\theta = \left|\frac{4/3-1/3}{1+4/9}\right| = \frac{9}{13}$$

Aliter (Using Calculus): Differentiating $y^2 = 8ax$ on both sides with respect to x, we get

$$\frac{dy}{dx} = \frac{4a}{y}$$

so that

$$\left(\frac{dy}{dx}\right)_{(18a,12a)} = \frac{4a}{12a} = \frac{1}{3} = m_1 \quad \text{(say)}$$

When $x^2 = 27ay$, we have

$$\left(\frac{dy}{dx}\right)_{(18a,12a)} = \frac{2\times 18a}{27a} = \frac{4}{3} = m_2$$

Hence

$$\tan\theta = \left|\frac{m_1 - m_2}{1+m_1m_2}\right| = \left|\frac{1/3-4/3}{1+4/4}\right| = \frac{9}{13}$$

Answer: (D)

10. From the point (15, 12), three normals are drawn to the parabola $y^2 = 4x$. The centroid of the triangle formed by the feet of the normals is

(A) $\left(\frac{26}{3}, 0\right)$ (B) $(4, -0)$

(C) $\left(\frac{16}{3}, 0\right)$ (D) $(16, 0)$

Solution: Let $P(t_1^2, 2t_1), Q(t_2^2, 2t_2)$ and $R(t_3^2, 2t_3)$ be the feet of the normals drawn from (15,12). Hence, by Theorem 4.2 and Quick Look 11, t_1, t_2, t_3 are the roots of

$$t^3 - 13t - 12 = 0 \tag{4.84}$$

so that $t_1+t_2+t_3 = 0$, $t_1t_2+t_2t_3+t_3t_1 = -13$, and $t_1t_2t_3 = 12$. Let $G(x_1, y_1)$ be the centroid of ΔPQR. Therefore

$$x_1 = \frac{t_1^2+t_2^2+t_3^2}{3}$$

and

$$y_1 = \frac{2t_1+2t_2+2t_3}{3}$$

Clearly,

$$y_1 = 0 \qquad (\because t_1+t_2+t_3 = 0)$$

Now,

$$x_1 = \frac{1}{3}[(t_1+t_2+t_3)^2 - 2\sum t_1t_2]$$

$$= \frac{1}{3}[0-2(-13)] = \frac{26}{3}$$

Thus G is

$$(x_1, y_1) = \left(\frac{26}{3}, 0\right)$$

Answer: (A)

Note:

1. It can be observed that $t = -1, -3$ and 4 are the roots of Eq. (4.84) so that the feet of the normals are $(1, -2)$, $(9, -6)$, $(16, 8)$ and hence the centroid is $(26/3, 0)$.
2. In general, the cubic equation in t similar to the one provided in Eq. (4.84) is not so easy to solve. Relying upon Quick Look 11 is always useful in this case.

11. The length of the normal chord of the parabola $y^2 = 4x$ which makes an angle $\pi/4$ with the positive direction of the axis is

(A) $4\sqrt{2}$ (B) $8\sqrt{2}$ (C) 8 (D) 4

Solution: Slope m of the chord is $\tan\pi/4 = 1$. By Quick Look 7, the normal chord equation is $y = x - 2 - 1$. That is, $y = x - 3$. Substituting this value in $y^2 = 4x$ we get

$$(x-3)^2 = 4x$$

$$\Rightarrow x^2 - 10x + 9 = 0$$

Solving this, we get $x = 1$ and 9 and so $y = -2$ and 6. That is, the ends of the normal chord are $(1, -2)$ and $(9, 6)$. Hence, the length of the normal chord is

$$\sqrt{(9-1)^2+(6+2)^2} = 8\sqrt{2}$$

Answer: (B)

12. If $2x + y + k = 0$ is a normal to the parabola $y^2 = -8x$, then the value of k is

(A) 12 (B) 24 (C) -12 (D) -24

Solution: Suppose $2x + y + k = 0$ is normal at $(-2t^2, -4t)$. However, the normal at $(-2t^2, -4t)$ is

$$tx + y = -4t - 2t^3$$

$$\Rightarrow tx + y + 4t + 2t^3 = 0$$

Therefore

$$\frac{t}{2} = \frac{1}{1} = \frac{4t+2t^3}{k}$$

$$\Rightarrow t = 2 \text{ and } k = 24$$

Answer: (B)

13. The centre of the circle which cuts the parabola $y^2 = 4x$ orthogonally at the point $(1, 2)$ passes through the point

(A) $(5, 3)$, (B) $(3, 4)$, (C) $(2, 4)$ (D) $(4, 3)$

Solution: Since the circle cuts the parabola $y^2 = 4x$ orthogonally at (1, 2), the tangent to the parabola at (1, 2) should pass through the centre of the circle. That is, $y(2) - 2(x+1) = 0$ passes through the centre. That is, $y = x + 1$ passes through $(3, 4)$.

Answer: (B)

14. If a normal chord of $y^2 = 4ax$ subtends a right angle at the vertex, then its slope is

(A) $\sqrt{2}$ (B) $\frac{1}{\sqrt{2}}$ (C) $\frac{1}{2}$ (D) 2

Solution: The equation of the normal with slope m is

$$y - mx = -2am - am^3 \tag{4.85}$$

Suppose this chord meets the parabola at P and Q. Therefore, by Theorem 2.33, the combined equation of the pair of lines OP and OQ is

$$y^2 - 4ax\left(\frac{mx-y}{2am+am^3}\right) = 0$$

Now

$$\angle POQ = 90°$$

$\Rightarrow$ Coefficient of x^2 + Coefficient y^2 (in the above equation) = 0

$$\Rightarrow 1 - \frac{4ma}{2am+am^3} = 0$$

$$\Rightarrow m^2 + 2 - 4 = 0$$

$$\Rightarrow m = \pm\sqrt{2}$$

Answer: (A)

15. If the line $lx + my + n = 0$ touches the parabola $y^2 = 4ax$, then

(A) $ln = am$ (B) $ln^2 = am$

(C) $l^2 n = am^2$ (D) $ln = am^2$

Solution: Suppose $lx + my + n = 0$ touches the parabola $y^2 = 4ax$ at $(at^2, 2at)$. However, $ty = x + at^2$ is the equation of the tangent at $(at^2, 2at)$. Therefore

$$\frac{l}{1} = \frac{m}{-t} = \frac{n}{at^2}$$

$$\Rightarrow -\frac{m}{l} = t = \frac{-n}{ma}$$

$$\Rightarrow am^2 = ln$$

Answer: (D)

16. If (x_1, y_1) is the point of contact of the tangent parallel to the line $4y - x + 3 = 0$ and touching the parabola $y^2 = 7x$, then $x_1 + y_1$ is equal to

(A) 40 (B) 28 (C) 42 (D) 32

Solution: The line

$$y = mx + \frac{a}{m}$$

touches the parabola $y^2 = 4ax$ at $(a/m^2, 2a/m)$ (see Theorem 4.7). Here

$$m = \frac{1}{4}, a = \frac{7}{4}$$

Therefore

$$\frac{a}{m^2} = \frac{7}{4} \times 16 = 28; \quad \frac{2a}{m} = \frac{14}{4} \times 4 = 14$$

Hence, $(x_1, y_1) = (28, 14)$ so that $x_1 + y_1 = 42$.

Answer: (C)

17. If $L \equiv x + y - 1 = 0$ is a line and $S \equiv y - x + x^2 = 0$ is a parabola, then which of the following is true?

(A) $L = 0$ and $S = 0$ do not have common points

(B) $L = 0$ cuts $S = 0$ in two distinct points

(C) $L = 0$ touches the parabola $S = 0$

(D) $L = 0$ is the directrix of the parabola $S = 0$

Solution: Substituting $y = 1 - x$ in the equation of the parabola, we get

$$1 - x - x + x^2 = 0$$

$$\Rightarrow x^2 - 2x + 1 = 0$$

$$\Rightarrow (x-1)^2 = 0$$

Hence, $L = 0$ touches the parabola at $(1, 1)$.

Answer: (C)

18. If $c > 1/2$, then three normals can be drawn from the point $(c, 0)$ to the parabola $y^2 = x$ out of which one is the x-axis. If the other two normals are at right angles, then the value of c is

(A) $\frac{4}{3}$ (B) $\frac{3}{4}$ (C) $\frac{2}{3}$ (D) $\frac{3}{2}$

Solution: The normal at $(at^2, 2at)$, where $a = 1/4$, which is drawn to $y^2 = x$ is

$$tx + y = 2\left(\frac{1}{4}\right)t + \frac{1}{4}t^3$$

$$\Rightarrow 4tx + 4y = 2t + t^3$$

This passes through $(c, 0)$. So we get

$$4tc = 2t + t^3$$

$$\Rightarrow t(t^2 + 2 - 4c) = 0$$

$$\Rightarrow t = 0 \text{ or } t = \pm\sqrt{4c-2} \quad \left(\because c > \frac{1}{2} \Rightarrow 4c - 2 > 0\right)$$

It is clear that $t = 0$ gives that x-axis is the normal at $(0, 0)$. The other two normals are at right angles $\Leftrightarrow$ product of their slopes $= -1$.

That is

$$\sqrt{4c-2}(-\sqrt{4c-2}) = -1$$

$$\Rightarrow 4c - 2 = -1$$

$$\Rightarrow c = \frac{3}{4}$$

Answer: (B)

19. If P is a variable point on the parabola $y^2 = 4ax$ whose focus is S, then the locus of the midpoint of the segment $\overline{SP}$ is in turn a parabola whose directrix is

(A) $x = -\dfrac{a}{2}$ (B) $x = -a$

(C) $x = 0$ (D) $x = a$

Solution: $P = (at^2, 2at)$, $t \in \mathbb{R}$ and $S = (a, 0)$. Let $M(x_1, y_1)$ be the midpoint of SP. Then and

$$x_1 = \frac{a(1+t^2)}{2}$$

and $$y_1 = \frac{2at}{2} = at$$

Therefore

$$\frac{2x_1}{a} - 1 = t^2 = \left(\frac{y_1}{a}\right)^2$$

$$\Rightarrow y_1^2 = 2ax_1 - a^2$$

$$\Rightarrow y_1^2 = 2a\left(x_1 - \frac{a}{2}\right)$$

Hence, the locus of (x_1, y_1) is the parabola

$$y^2 = 2a\left(x - \frac{a}{2}\right)$$

Substituting $Y = y$ and $X = x - (a/2)$ so that

$$y^2 = 2a\left(x - \frac{a}{2}\right)$$

is given by

$$Y^2 = 4\left(\frac{a}{2}\right)X$$

whose directrix is

$$X = \frac{-a}{2}$$

$$\Rightarrow x - \frac{a}{2} = \frac{-a}{2}$$

$$\Rightarrow x = 0$$

Answer: (C)

20. A focal chord of the parabola $y^2 = 16x$ touches the circle $(x-6)^2 + y^2 = 2$. Then the possible values of the slope of this chord are

(A) $\{-1, 1\}$ (B) $\{-2, 2\}$

(C) $\left\{-2, \dfrac{1}{2}\right\}$ (D) $\left\{2, -\dfrac{1}{2}\right\}$

Solution: Let $P = (at^2, 2at)$, where $a = 4$, be a point on the parabola $y^2 = 16x$ so that $S = (a, 0) = (4, 0)$ is the focus. Equation of SP is

$$y = \frac{8t}{4(t^2-1)}(x-4)$$

$$\Rightarrow 2tx - (t^2-1)y - 8t = 0 \tag{4.86}$$

It touches the circle $(x - 6)^2 + y^2 = 2$ which implies that

$$\frac{|2t(6) - (8t)|}{\sqrt{4t^2 + (t^2-1)^2}} = \sqrt{2}$$

$$\Rightarrow |4t| = \sqrt{2}(t^2+1)$$

$$\Rightarrow t^2 - 2\sqrt{2}t + 1 = 0$$

$$\Rightarrow t^2 + 2\sqrt{2}t + 1 = 0$$

$$\Rightarrow t = \frac{2\sqrt{2} \pm \sqrt{8-4}}{2} = \sqrt{2} \pm 1$$

Hence the slope of SP is

$$\frac{2t}{t^2-1} = \frac{2(\sqrt{2}\pm 1)}{(3 \pm 2\sqrt{2}) - 1} = \frac{2(\sqrt{2}\pm 1)}{2 \pm 2\sqrt{2}} = \pm 1$$

That is, the chord $y = \pm(x - 4)$ touches the circle $(x - 6)^2 + y^2 = 2$.

Aliter: Let $y = m(x - 4)$ be a focal chord of $y^2 = 16x$. This chord touches the circle $(x - 6)^2 + y^2 = 2$. This implies

$$\frac{|6m - 4m|}{\sqrt{m^2+1}} = \sqrt{2}$$

$$\Rightarrow 4m^2 = 2(m^2+1)$$

$$\Rightarrow m^2 = 1$$

$$\Rightarrow m = \pm 1$$

Answer: (A)

21. The equation of the common tangent touching the circle $(x-3)^2 + y^2 = 9$ and the parabola $y^2 = 4x$ above the x-axis is

(A) $\sqrt{3}y = 3x + 1$ (B) $\sqrt{3}y = -(x+3)$

(C) $\sqrt{3}y = x + 3$ (D) $\sqrt{3}y = -(3x+1)$

Solution: The line

$$y = mx + \frac{1}{m}$$

touches $y^2 = 4x$ for all values of $m \neq 0$. This line also touches the circle $(x - 3)^2 + y^2 = 9$. So we have

$$\frac{\left|m(3) + \dfrac{1}{m}\right|}{\sqrt{m^2+1}} = 3$$

$$\Rightarrow \left(3m+\frac{1}{m}\right)^2 = 9(m^2+1)$$

$$\Rightarrow 9m^2+6+\frac{1}{m^2} = 9m^2+9$$

$$\Rightarrow \frac{1}{m^2} = 3$$

$$\Rightarrow m = \pm\frac{1}{\sqrt{3}}$$

Since the tangent is above x-axis, we have

$$m = \frac{1}{\sqrt{3}}$$

Therefore, the tangent equation is given by

$$y = \frac{x}{\sqrt{3}}+\sqrt{3}$$

$$\Rightarrow \sqrt{3}y = x+3$$

Answer: (C)

22. The equation of a line inclined at an angle of 45° to the line $y = 3x + 5$ and touching the parabola $y^2 = 8x$ is

(A) $y = 2x+1$ (B) $2x+y+1=0$

(C) $x+2y+1=0$ (D) $x-2y+1=0$

Solution: Given

$$y^2 = 8x = 4(2)x$$

so that $a = 2$. Therefore, the line

$$y^2 = mx+\frac{2}{m}$$

touches the parabola for all values of $m \neq 0$. The angle between this tangent and the line $y = 3x + 5$ is 45°. So

$$\left|\frac{m-3}{1+3m}\right| = \tan 45° = 1$$

$$\Rightarrow m-3 = \pm(1+3m)$$

$$\Rightarrow m = -2, \frac{1}{2}$$

Therefore, $m = -2$. So the tangent is

$$y = -2x-1$$

$$\Rightarrow 2x+y+1=0$$

Answer: (B)

23. Two equal parabolas have the same focus and their axes are at right angles. A normal to one parabola is perpendicular to a normal to the other parabola. Then the locus of the point of intersection of those normals is

(A) $(x-y)^2 = 2a(x+y)$ (B) $(x+y)^2 = 2a(x+y)$

(C) $(x+y)^2 = 4a(x-y)$ (D) $(x-y)^2 = 4a(x+y)$

Solution: Consider the common focus as the origin and their axes as coordinate axes. We can suppose that the parabolas are $y^2 = 4a(x+a)$ and $x^2 = 4a(y+a)$. Therefore, by Quick Look 7, their normals in terms of the slopes are

$$y = m(x+a)-2am-am^3 \qquad (4.87)$$

and

$$x = m'(y+a)-2am'-am'^3 \qquad (4.88)$$

Since the normals are at right angles, we have

$$m\left(\frac{1}{m'}\right) = -1 \Rightarrow m' = -m$$

Substituting the value of m' in Eq. (4.88), we have

$$x = -m(y+a)+2am+am^3 \qquad (4.89)$$

Therefore, from Eqs. (4.87) and (4.89), we get

$$m = \frac{x+y}{x-y}$$

Substituting the value of m in Eq. (4.87), we obtain the locus as

$$y = \left(\frac{x+y}{x-y}\right)(x+a)-2a\left(\frac{x+y}{x-y}\right)-a\left(\frac{x+y}{x-y}\right)^3$$

$$\begin{aligned}\Rightarrow y(x-y)^3 &= (x+y)[(x+a)(x-y)^2-2a(x-y)^2 \\ &\quad -a(x+y)^2] \\ &= (x+y)[(x-y)^2(x-a)-a(x+y)^2] \\ &= (x+y)[x(x-y)^2-a[(x-y)^2+(x+y)^2]] \\ &= (x+y)[x(x-y)^2-2a(x^2+y^2)]\end{aligned}$$

$$\begin{aligned}\Rightarrow 2a(x+y)(x^2+y^2) &= x(x+y)(x-y)^2-y(x-y)^3 \\ &= (x-y)^2[x^2+xy-xy+y^2] \\ &= (x-y)^2(x^2+y^2)\end{aligned}$$

$$\Rightarrow (x-y)^2 = 2a(x+y)$$

Answer: (A)

24. If P is a point on the line $x + 4a = 0$ and QR is the chord of contact of P with respect to $y^2 = 4ax$, then $\angle QOR$ (where O is the vertex) is equal to

(A) 45° (B) 60° (C) 30° (D) 90°

Solution: Let P be (x_1, y_1) so that

$$x_1 + 4a = 0 \qquad (4.90)$$

Chord of contact of $P(x_1, y_1)$ is

$$yy_1 - 2a(x + x_1) = 0$$

$$\Rightarrow yy_1 - 2ax + 8a^2 = 0 \qquad [\because x_1 = -4a \text{ from Eq. (4.90)}]$$

$$\Rightarrow \frac{2ax - y_1 y}{8a^2} = 1$$

Hence, the combined equation of the pair of lines OQ and OR is

$$y^2 - 4ax\left(\frac{2ax - y_1 y}{8a^2}\right)$$

In this equation, the coefficient of x^2 + the coefficient of $y^2 = -1 + 1 = 0$. Hence, $\angle QOR = 90°$.

Answer: (D)

25. Locus of the point of intersection of the tangents to the parabola $y^2 = 4ax$ which include 60°angle is

(A) $y^2 = 4ax + 3(x + a)^2$ (B) $y^2 - 4ax = 2(x + a)^2$

(C) $y^2 - 4ax = (x + a)^2$ (D) $y^2 - 4ax = 3(x + a)^3$

Solution: According to Problem 28 of the section 'Subjective Problems', the locus of the point of intersection of the tangents to the parabola $y^2 = 4ax$ which include angle α is

$$\cot^2 \alpha (y^2 - 4ax) = (x + a)^2$$

Now, substituting $\alpha = 60°$ the required locus is

$$\frac{1}{3}(y - 4ax) = (x + a)^2$$

$$\Rightarrow y^2 - 4ax = 3(x + a)^2$$

Answer: (A)

26. The equation of the circle with centre at the focus of the parabola $8y = (x - 1)^2$ and touching the parabola at the vertex is

(A) $x^2 + y^2 - 4y = 0$

(B) $x^2 + y^2 - 4y + 1 = 0$

(C) $x^2 + y^2 - 2x - 4y = 0$

(D) $x^2 + y^2 - 2x - 4y + 1 = 0$

Solution: The parabola is $X^2 = 4(2)Y$ where $X = x - 1$, $Y = y$, and $a = 2$. Therefore, the vertex = (1, 0) and the focus = (1, 2). Also the radius of the circle is equal to 2. Hence, the circle equation is

$$(x - 1)^2 + (y - 2)^2 = 2^2$$

$$\Rightarrow x^2 + y^2 - 2x - 4y + 1 = 0$$

Answer: (D)

27. The curve described parametrically by $x = t^2 + t + 1$ and $y = t^2 - t + 1$ represents

(A) a pair of straight lines (B) an ellipse

(C) a parabola (D) a hyperbola

Solution: By the two parametric equations we have

$$x - y = 2t$$

$$\Rightarrow t = \frac{x - y}{2}$$

Substituting the value of t in $x = t^2 + t + 1$, we have

$$x = \left(\frac{x - y}{2}\right)^2 + \left(\frac{x - y}{2}\right) + 1$$

$$\Rightarrow 4x = (x - y)^2 + 2(x - y) + 4$$

$$\Rightarrow (x - y)^2 = 2(x + y - 2)$$

which is a parabola.

Answer: (C)

28. Consider the following statements:
I: A parabola is symmetric about its axis.

II: The curve

$$y = \frac{-x^2}{2} + x + 1$$

is symmetric with respect to the line $x = 1$.
Then which one of the following is true?

(A) Both I and II are true

(B) Both I and II are false

(C) I is true and II is false

(D) I is false and II is true

Solution: $y^2 = 4ax$ is symmetric about the x-axis which is its axis. Therefore, I is true. Now,

$$y = \frac{-x^2}{2} + x + 1$$

$$\Rightarrow 2y = -x^2 + 2x + 1$$

$$\Rightarrow 2y = -(x - 1)^2 + 2(x - 1)^2 = -2(y - 1)$$

$$\Rightarrow X^2 = -2Y$$

where $X = x - 1$ and $Y = y - 1$. This implies that the curve is symmetric about $X = 0$ or $x = 1$. Therefore, II is also true.

Answer: (A)

29. The point of intersection of the tangents drawn at the ends of the latus rectum of the parabola $y^2 = 4x$ is

(A) $(-1, 0)$ (B) $(-1, 1)$

(C) $(-1, 2)$ (D) $(-1, -2)$

Solution: The ends of the latus rectum are $(1, 2)$ and $(1, -2)$. Therefore, the tangents at these points are

$$2y - 2(x+1) = 0 \Rightarrow y = x+1$$
$$-2y - 2(x+1) = 0 \Rightarrow y = -x-1$$

Hence, the point of intersection is $(-1, 0)$.

Answer: (A)

30. If P, Q and R are three points on the parabola $y^2 = 4ax$ at which the normals intersect at the point (h, k), then the centroid of $\angle PQR$ is

(A) $\frac{1}{3}(h-2a,0)$ (B) $\frac{4}{3}(h-2a,0)$

(C) $\frac{2}{3}(h-2a,0)$ (D) $\left(\frac{2a-h}{3},0\right)$

Solution: Let $P = (at_1^2, 2at_1)$, $Q = (at_2^2, 2at_2)$ and $R = (at_3^2, 2at_3)$. Therefore, from Theorem 4.20, t_1, t_2 and t_3 are the roots of the equation

$$at^3 + (2a-h)t - k = 0$$

so that

$$t_1 + t_2 + t_3 = 0 \tag{4.90a}$$

$$t_1t_2 + t_2t_3 + t_3t_1 = \frac{2a-h}{a} \tag{4.90b}$$

$$t_1t_2t_3 = \frac{k}{a} \tag{4.90c}$$

Let $G = (x_1, y_1)$ be the centroid of $\angle PQR$. Therefore

$$x_1 = \frac{at_1^2 + at_2^2 + at_3^2}{3} = \frac{a}{3}[(t_1+t_2+t_3)^2 - 2\sum t_1t_2]$$

Therefore, from Eqs. (4.90a) and (4.90b), we get

$$x_1 = \frac{a}{3}\left[0 - \frac{2(2a-h)}{a}\right] = \frac{2}{3}(h-2a)$$

and
$$y_1 = \frac{2a}{3}(t_1+t_2+t_3) = 0$$

Hence

$$G = \left[\frac{2}{3}(h-2a), 0\right]$$

Answer: (C)

31. Let $P(at^2, 2at)$ be a point on the parabola $y^2 = 4ax$ and Q and R be the feet of the normals drawn from P. Then $(PQ)^2\,(PR)^2$ is equal to

(A) $16a^4(1+t^2)^2$ (B) $16a^2(1+t^2)^3$

(C) $64a^4(1+t^2)^3$ (D) $16a^4(1+t^2)^3$

Solution: Let $Q = (at_1^2, 2at_1)$ and $R = (at_2^2, 2at_2)$. Since the normals at Q and R meet the curve again at P, we have

$$-t_1 - \frac{2}{t_1} = t = -t_2 - \frac{2}{t_2} \quad \text{and} \quad t_1t_2 = 2 \tag{4.91}$$

Now,

$$\begin{aligned}(PQ)^2 &= a^2(t^2 - t_1^2)^2 + 4a^2(t-t_1)^2 \\ &= a^2(t-t_1)^2[(t+t_1)^2 + 4] \\ &= a^2[(t+t_1)^2 - 4tt_1][(t+t_1)^2 + 4] \\ &= a^2\left[\frac{4}{t_1^2} - 4tt_1\right]\left[\frac{4}{t_1^2} + 4\right] \\ &= 16a^2\left(\frac{1}{t_1^2} - tt_1\right)\left(\frac{1}{t_1^2} + 1\right)\end{aligned}$$

Therefore,

$$\begin{aligned}(PQ)^2 &= 16a^2\left(\frac{1}{t_1^2} + t_1^2 + 2\right)\left(\frac{1}{t_1^2} + 1\right) \\ &\qquad \left[\because -t_1 - \frac{2}{t_1} = t \text{ from Eq. (4.91)}\right] \\ &= \frac{16a^2(1+t_1^2)^3}{t_1^4}\end{aligned}$$

Similarly,

$$(PR)^2 = \frac{16a^2(1+t_2^2)^3}{t_2^4}$$

Therefore,

$$\begin{aligned}(PQ)^2(PR)^2 &= \frac{16a^2(1+t_1^2)^3(1+t_2^2)^3}{(t_1t_2)^4} \\ &= \frac{16a^2(1+t_1^2+t_2^2+t_1^2t_2^2)^3}{(t_1t_2)^4} \\ &= \frac{16^2a^4[1+(t_1+t_2)^2 - 2t_1t_2 + t_1^2t_2^2]^3}{(t_1t_2)^4} \\ &= \frac{16^2a^4[1+t^2-4+4]^3}{2^4} \\ &\qquad [\text{from Eq. (4.91) and } t+t_1+t_2=0] \\ &= 16a^4(1+t^2)^3\end{aligned}$$

Answer: (D)

32. The chords of a parabola $y^2 = 6x$ are passing through the point $(9, 5)$. Then the middle points of these chords lie on the curve represented by the equation

(A) $y^2 - 5y - 3x + 27 = 0$

(B) $y^2 + 5y - 3x - 27 = 0$

(C) $y^2 - 5y - 3x - 27 = 0$

(D) $y^2 + 5y - 3x + 27 = 0$

Solution:

$$y^2 = 6x = 4\left(\frac{3}{2}\right)x$$

so that $a = 3/2$. The equation of the chord with midpoint (x_1, y_1) is

$$S_1 = S_n$$

$$\Rightarrow yy_1 - 2\left(\frac{3}{2}\right)(x + x_1) = y_1^2 - 6x_1$$

$$\Rightarrow yy_1 - 3(x + x_1) = y_1^2 - 6x_1$$

This chord passes through the point (9, 5). So

$$\Rightarrow 5y_1 - 3(9 + x_1) = y_1^2 - 6x_1$$

$$\Rightarrow y_1^2 - 5y_1 - 3x_1 + 27 = 0$$

Therefore, (x_1, y_1) lies on the curve $y^2 - 5y - 3x + 27 = 0$.

Answer: (A)

33. PQ is a chord of the parabola $y^2 = 4ax$ such that the normals at P and Q intersect on the parabola. Then the midpoint of the chord PQ lies on the curve

(A) $y^2 = 2a(x - 2a)$ (B) $y^2 = 2a(x + a)$

(C) $y^2 = 2a(x + 2a)$ (D) $y^2 = a(x + 2a)$

Solution: Let $P = (at_1^2, 2at_1)$ and $Q = (at_2^2, 2at_2)$ and let $M(x_1, y_1)$ be the midpoint of PQ so that

$$2x_1 = a(t_1^2 + t_2^2) \text{ and } y_1 = a(t_1 + t_2) \tag{4.92}$$

Since the normals at P and Q meet on the parabola, by Theorem 4.13, we have

$$t_1 t_2 = 2 \tag{4.93}$$

From Eq. (4.92),

$$2x_1 = a\,[(t_1 + t_2)^2 - 2t_1t_2] = a\left[\frac{y_1^2}{a^2} - 4\right] = \frac{y_1^2 - 4a^2}{a}$$

$$\Rightarrow y_1^2 - 4a^2 = 2ax_1$$

$$\Rightarrow y_1^2 = 2a(x_1 + 2a)$$

Therefore, (x_1, y_1) lies on the curve $y^2 = 2a(x + 2a)$.

Answer: (C)

34. Tangent to the parabola $y^2 = -4ax$ meets the parabola $y^2 = 4ax$ in P and Q. Then the midpoint of PQ lies on the curve

(A) $y^2 = \dfrac{4a^2x}{a+b}$ (B) $y^2 = \dfrac{4a^2x}{2a+b}$

(C) $y^2 = \dfrac{4ax}{a+b}$ (D) $y^2 = \dfrac{4ax}{2a+b}$

Solution: Equation of the chord of $y^2 = 4ax$, whose midpoint is (x_1, y_1), is

$$S_1 = S_{11}$$

$$\Rightarrow yy_1 - 2a(x + x_1) = y_1^2 - 4ax_1$$

$$\Rightarrow y = \frac{2ax}{y_1} + \frac{y_1^2 - 2ax_1}{y_1}$$

This chord touches the parabola $y^2 = -4ax$. Therefore

$$\frac{y_1^2 - 2ax_1}{y_1} = \frac{-b}{2a / y_1} \qquad \left(\because c = -\frac{b}{m} \text{ is the condition}\right)$$

For the line $y = mx + c$ to touch $y^2 = -4bx$ we have

$$2a(y_1^2 - 2ax_1) = -by_1^2$$

$$\Rightarrow y_1^2(2a + b) = 4a^2x_1$$

Therefore, the point (x_1, y_1) lies on the curve

$$y^2 = \frac{4a^2x}{2a+b}$$

Answer: (B)

35. P is a point on the directrix of the parabola $y^2 = 4ax$ and Q is the point of contact of a tangent drawn from P to the parabola. Then the midpoint of PQ lies on the curve

(A) $y^2(2x + a) = a(3x + a)$

(B) $y^2(3x + a) = a(2x + a)^2$

(C) $y^2(2x + a) = a(3x + 2a)^2$

(D) $y^2(2x + a) = a(3x + a)^2$

Solution: Let $P = (-a, y)$ be a point on the directrix and $Q = (at^2, 2at)$ be the point of contact of the tangent from P. The tangent at Q is $ty = x + at^2$. This cuts the directrix at P. So

$$y = \frac{a(t^2 - 1)}{t}$$

Therefore

$$P = \left(-a, \frac{a(t^2 - 1)}{t}\right) \tag{5.94a}$$

Suppose $M(x_1, y_1)$ is the midpoint of PQ. Therefore

$$2x_1 = a(t^2 - 1) \Rightarrow 2x_1 + a = at^2$$

and

$$2y_1 = 2at + \frac{a(t^2 - 1)}{t}$$

$$\Rightarrow 2y_1 t = \frac{3at^2 - a}{t}$$

$$\Rightarrow 2y_1 t = 3at^2 - a \qquad (5.94b)$$

Now from Eq. (5.94a)

$$t = \sqrt{\frac{2x_1 + a}{a}}$$

Substituting this is Eq. (5.94b) we get

$$2y_1\sqrt{\frac{2x_1+a}{a}} = 6x_1 + 3a - a$$

$$\Rightarrow \frac{4y_1^2(2x_1+a)}{a} = 4(3x_1+a)^2$$

$$\Rightarrow y_1^2(2x_1+a) = a(3x_1+a)^2$$

Therefore, (x_1, y_1) lies on the curve $y^2(2x+a) = a(3x+a)^2$.

Answer: (D)

36. If PNP' is a double ordinate of the parabola $y^2 = 4x$, then the locus of intersection of the normal at P and a line parallel to the axis through P' is

(A) $y^2 = 4(x-2)$ (B) $y^2 = 4(x-3)$

(C) $y^2 = 4(x-4)$ (D) $y^2 = 4(x-1)$

Solution: Let $P = (t^2, 2t)$ and $P' = (t^2, -2t)$. The normal at P is

$$tx + y = 2t + t^3 \qquad (4.95a)$$

The line passing through P' and parallel to the axis is

$$y = -2t \qquad (4.95b)$$

Intersection of the line in Eq. (4.95a) and the line in (4.95b) is

$$\left(\frac{-y}{2}\right)x + y = -y - \frac{y^3}{8}$$

$$\Rightarrow x - 2 = +2 + \frac{y^2}{4}$$

$$\Rightarrow 4x - 16 = y^2$$

$$\Rightarrow 4(x-4) = y^2$$

Answer: (C)

37. From point A, common tangents are drawn to the circle

$$x^2 + y^2 = \frac{a^2}{2}$$

and parabola $y^2 = 4ax$. The area of the quadrilateral formed by the common tangent, the chords of contact of point A with respect to the circle and the parabola is

(A) $\dfrac{15a^2}{2}$ (B) $\dfrac{15a^2}{4}$ (c) $4a^2$ (D) $5a^2$

Solution: See Fig. 4.25. The line

$$y = mx + \frac{a}{m}$$

is always a tangent to $y^2 = 4ax$, which is also the tangent to the circle

$$x^2 + y^2 = \frac{a^2}{2}$$

This implies

$$\Rightarrow \left|\frac{0 - 0 + \frac{a}{m}}{\sqrt{1+m^2}}\right| = \frac{a}{\sqrt{2}}$$

$$\Rightarrow 2 = m^2(1+m^2)$$

$$\Rightarrow m^4 + m^2 - 2 = 0$$

$$\Rightarrow (m^2+2)(m^2-1) = 0$$

$$\Rightarrow m = \pm 1$$

Therefore, the tangents are $y = x + a$ and $y = -x - a$ which intersect at $A(-a, 0)$ which lies on the directrix. The chord of contact of $A(-a, 0)$ with respect to the circle $x^2 + y^2 = a^2/2$ is

$$x = \frac{-a}{2}$$

and with respect to the $y^2 = 4ax$ is $x = a$. Let PQ be $x = -a/2$ and RS be $x = a$ so that $PQRS$ is a trapezium whose area is

$$\frac{1}{2}(PQ + RS) \times (\text{Distance between } PQ \text{ and } RS)$$

$$= \frac{1}{2}(a + 4a)\left(a + \frac{a}{2}\right) = \frac{15a^2}{4}$$

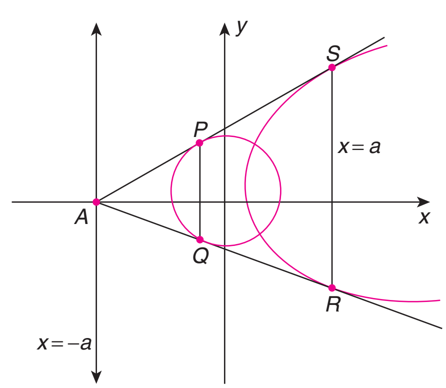


FIGURE 4.25

Answer: (B)

38. *PSQ* is a focal chord of a parabola where *S* is the focus. If $SP = 3$ and $SQ = 2$, then the length of the latus rectum is

(A) $\frac{12}{5}$ (B) $\frac{6}{5}$ (C) $\frac{8}{5}$ (D) $\frac{24}{5}$

Solution: By Theorem 4.19, part (3), we know that semi-latus rectum is the harmonic mean (HM) between *SP* and *SQ*. If *l* is the length of the semi-latus rectum, then

$$l = \frac{2(SP)(SQ)}{(SP)+(SQ)} = \frac{2(3)(2)}{3+2} = \frac{12}{5}$$

Hence,

$$2l = \frac{24}{5}$$

Answer: (D)

39. The ordinates of the points *P* and *Q* of a parabola $y^2 = 4x$ are in the ratio 1:2. Then the locus of the point of intersection of the normals at *P* and *Q* is

$$y^2 = \frac{k}{343}(x-2)^3$$

where the value of *k* is

(A) 18 (B) 36 (C) 54 (D) 12

Solution: Let $P = (t_1^2, 2t_1)$ and $Q = (t_2^2, 2t_2)$ so that, by hypothesis, we have

$$(2t_1):(2t_2) = 1:2$$

$$\Rightarrow t_2 = 2t_1 \quad (4.96)$$

Let (x_1, y_1) be the intersection of the normals at *P* and *Q* so that, by Theorem 4.11 and Eq. (4.96), we have

$$x_1 = 2 + t_1^2 + t_1t_2 + t_2^2 = 2 + 7t_1^2 \quad (4.97)$$

and

$$y_1 = -t_1t_2(t_1+t_2) = -2t_1^2(3t_1) = -6t_1^3 \quad (4.98)$$

From Eqs. (4.97) and (4.98), we have

$$\left(\frac{x_1-2}{7}\right)^3 = t_1^6 = \left(\frac{-y_1}{6}\right)^2 = \frac{y_1^2}{36}$$

Hence, the locus of (x_1, y_1) is

$$\frac{36}{343}(x-2)^3 = y^2$$

Answer: (B)

40. *PQ* is a chord of the parabola $y^2 = 4x$ whose perpendicular bisector meets the axis at *M* and the ordinate of the midpoint *PQ* meets the axis at *N*. Then the length *MN* is equal to

(A) 2 (B) 2.5 (C) 3 (D) 4

Solution: See Fig. 4.26. Let $P = (t_1^2, 2t_1)$ and $Q = (t_2^2, 2t_2)$. Let *L* be the midpoint of

$$PQ = \left(\frac{1}{2}(t_1^2+t_2^2), t_1+t_2\right)$$

Slope of the chord *PQ* is

$$\frac{2}{t_1+t_2}$$

Hence, the equation of the perpendicular bisector of the chord *PQ* is

$$y-(t_1+t_2) = \frac{-(t_1+t_2)}{2}\left(x - \frac{t_1^2+t_2^2}{2}\right)$$

Substituting $y = 0$ in this equation, we have

$$M = \left(2 + \frac{t_1^2+t_2^2}{2}, 0\right) \quad (4.99)$$

Also

$$N = \left(\frac{t_1^2+t_2^2}{2}, 0\right) \quad (4.100)$$

From Eqs. (4.99) and (4.100), we get

$$OM = 2 + \frac{t_1^2+t_2^2}{2}$$

and

$$ON = \frac{t_1^2+t_2^2}{2}$$

so that $MN = OM - ON = 2$ which is also the semi-latus rectum.

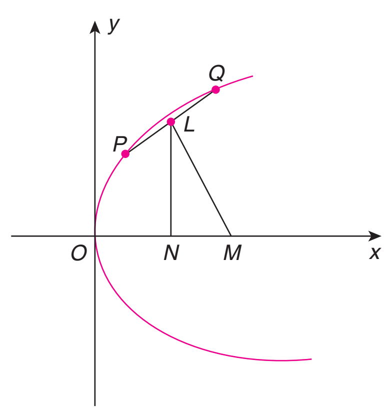


FIGURE 4.26

Answer: (A)

41. If the tangent at two points of the parabola $y^2 = 4ax$ intersect at (h, k), then the normals at these two points intersect in

(A) $\left(2a-h+\frac{k^2}{a}, \frac{-hk}{a}\right)$ (B) $\left(h-2a+\frac{k^2}{a}, \frac{hk}{a}\right)$

(C) $\left(2a-h+\frac{k^2}{a}, \frac{hk}{a}\right)$ (D) $\left(h-2a+\frac{k^2}{a}, \frac{-hk}{a}\right)$

Solution: Let $P=(at_1^2, 2at_1)$ and $Q=(at_2^2, 2at_2)$. Therefore the point of intersection of the tangent is

$$T=(at_1t_2, a(t_1+t_2))=(h,k)$$

so that

$$h=at_1t_2$$

and
$$k=a(t_1+t_2) \tag{4.101}$$

Let N be the intersection of the normals at P and Q so that, say,

$$N=(2a+a(t_1^2+t_1t_2+t_2^2), -at_1t_2(t_1+t_2))=(x,y)$$

Then

$$x=2a+a(t_1^2+t_1t_2+t_2^2)=2a+a\left[(t_1+t_2)^2-t_1t_2\right]$$
$$=2a+a\left[\frac{k^2}{a^2}-\frac{h}{a}\right]=2a-h+\frac{k^2}{a}$$

and
$$y=-at_1t_2(t_1+t_2)=-h\left(\frac{k}{a}\right)=\frac{-hk}{a}$$

Answer: (A)

42. The point of intersection of the normals at A and B of a parabola $y^2=4ax$ meet on the line $y=-a$. Then the point of intersection of the tangents at A and B lies on the curve

(A) $xy=a$ (B) $x^2=4ay$

(C) $xy=a^2$ (D) $x^2=2ay$

Solution: Let $A=(at_1^2, 2at_1)$ and $B=(at_2^2, 2at_2)$ so that the point of intersection N of normals drawn at A and B is

$$N=[2a+a(t_1^2+t_1t_2+t_2^2), -at_1t_2(t_1+t_2)]$$

and lies on $y=-a$. This implies

$$-at_1t_2(t_1+t_2)=-a$$
$$\Rightarrow t_1t_2(t_1+t_2)=1 \tag{4.102}$$

Let $T\,(x,y)$ be the intersection of the tangents at A and B. Hence, $x=at_1t_2$ and $y=a(t_1+t_2)$. Therefore, from Eq. (4.102), we get

$$xy=(at_1t_2)\times(a)\times(t_1+t_2)=a^2$$

Answer: (C)

43. A point P of the parabola $y^2=4ax$ lies on the line $y=x$. The normal chord PQ, normal at P, subtends an angle at the focus of the parabola which equals

(A) $60°$ (B) $45°$ (C) $30°$ (D) $90°$

Solution: Let P be $(at^2, 2at)$ so that $t=2$ because it lies on the line $y=x$. Suppose normal at P meets the curve at $Q=(at'^2, 2at')$. So

$$t'^2=-t-\frac{2}{t}=-2-1 \qquad (\because t=2)$$

Therefore, $Q=(9a,-6a)$ and $P=(4a,4a)$. Now

$$\text{Slope of } SP\times\text{Slope of } SQ=\left(\frac{4a}{4a-a}\right)\left(\frac{6a}{a-9a}\right)$$
$$=\frac{4}{3}\left(\frac{-6}{8}\right)=-1$$

Hence, $\angle PSQ=90°$.

Answer: (D)

44. The angle between the tangents drawn from the point $(1,4)$ to the parabola $y^2=4x$ is

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{2}$

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Solution: The line

$$y=mx+\frac{1}{m}$$

touches $y^2=4x$ for all $m\neq 0$. This passes through $(1,4)$. So

$$4=m+\frac{1}{m}$$
$$\Rightarrow m^2-4m+1=0 \tag{4.103}$$

Let m_1 and m_2 be the roots of Eq. (4.103) so that $m_1+m_2=4$ and $m_1m_2=1$. If θ is the acute angle between those tangents, then

$$\tan\theta=\left|\frac{m_1-m_2}{1+m_1m_2}\right|$$
$$=\left|\frac{\sqrt{(m_1+m_2)^2-4m_1m_2}}{1+m_1m_2}\right|$$
$$=\left|\frac{\sqrt{16-4}}{2}\right|=\sqrt{3}$$

Hence, $\theta=\pi/3$.

Answer: (C)

45. If P is a point on the parabola $y^2=8x$ above the axis and S is the focus with $SP=4$, then the ordinate of P is

(A) 4 (B) 2 (C) 8 (D) $2\sqrt{2}$

Solution: Let $P = (x, y)$ so that

$$4 = SP = 2 + x \qquad (\because SP = a + x)$$

$$\Rightarrow x = 2 \text{ and } y^2 = 16$$

$$\Rightarrow x = 2 \text{ and } y = \pm 4$$

Since P lies on the curve above the axis, we have $y = 4$.

Answer: (A)

46. A circle is drawn with centre at the focus S of the parabola $y^2 = 4x$ so that a common chord of the parabola and the circle is equidistant from the focus and the vertex. Then the equation of the circle is

(A) $(x-1)^2 + y^2 = \dfrac{9}{4}$ (B) $(x-1)^2 = \dfrac{9}{16} - y^2$

(C) $(y-1)^2 + x^2 = \dfrac{9}{4}$ (D) $(y-1)^2 + x^2 = \dfrac{9}{16}$

Solution: Since $S = (a, 0) = (1, 0)$, the circle is of the form

$$(x-1)^2 + y^2 = r^2 \qquad (4.104)$$

Suppose AB is a common chord. Since this is equidistant from the focus and the vertex $M(1/2, 0)$ lies on AB and AB is double ordinate of the parabola, let $A = (1/2, y)$ so that

$$y^2 = 4\left(\frac{1}{2}\right)$$

$$\Rightarrow y = \pm\sqrt{2}$$

$$\Rightarrow A = \left(\frac{1}{2}, \sqrt{2}\right) \text{and } B = \left(\frac{1}{2}, -\sqrt{2}\right)$$

Since $\Delta\, AMS$ is right-angled triangle, we have

$$SA^2 = SM^2 + MA^2 = \frac{1}{4} + 2$$

$$= \frac{9}{4} = (\text{Radius})^2$$

Hence, the equation of the circle is

$$(x-1)^2 + y^2 = \frac{9}{4}$$

Answer: (A)

47. A circle touches the parabola $y^2 = 4x$ at the point $(1, 2)$ and also the directrix. The y-coordinate of the point of contact of the circle and the directrix is

(A) $\sqrt{2}$ (B) 2 (C) $2\sqrt{2}$ (D) 4

Solution: See Fig. 4.27. Clearly, the point $(1, 2)$ is the upper end of the latus rectum of the parabola

$$y^2 = 4x$$

The tangent at the point $(1, 2)$ to the parabola is

$$y(2) - 2(x+1) = 0$$

$$\Rightarrow y = x + 1 \qquad (4.105)$$

Let (h, k) be the centre of the circle that lies on the normal to the parabola at $(1, 2)$ whose equation is $x + y = 3$. Hence

$$h + k = 3 \qquad (4.106)$$

Also, since the circle touches the directrix $x + 1 = 0$, we have

$$(h+1)^2 = (h-1)^2 + (k-2)^2 = (\text{Radius})^2$$

Therefore, from Eq. (4.106), we get

$$4h = (k-2)^2 = (3-h-2)^2$$

$$\Rightarrow h^2 - 6h + 1 = 0$$

$$\Rightarrow h = 3 \pm 2\sqrt{2}$$

Therefore

$$k = 3 - (3 \pm 2\sqrt{2}) = \pm 2\sqrt{2}$$

Since the circle is above the axis, we have $k = 2\sqrt{2}$ which is the y-coordinate of the point of contact of the circle with the directrix.

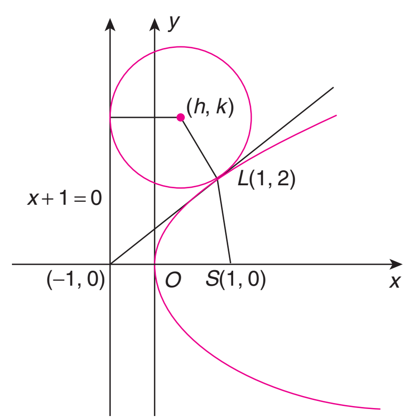


FIGURE 4.27

Answer: (A)

48. The directrix of the parabola traced out by the centre of a moving circle which touches both the line $y = -x$ and the circle $(x-3)^2 + (y-4)^2 = 9$ is

(A) $x + y - 3 = 0$ (B) $x + y + 3 = 0$

(C) $x + y - 3\sqrt{2} = 0$ (D) $x - y - \sqrt{2} = 0$

Solution: See Fig. 4.28. The directrix is the line parallel to $x + y = 0$ at a distance of 3 units from $x + y = 0$ to the left of $x + y = 0$. Therefore, the required line is $x + y + 3 = 0$.

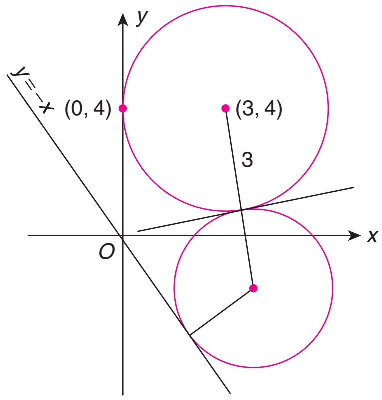


FIGURE 4.28

Answer: (B)

49. Let L be the point $(t, 2)$ and M be a point on the y-axis such that the slope of LM is $-t$. Then, the locus of the midpoint of LM is a parabola whose latus rectum is

(A) 2 (B) $\frac{1}{2}$ (C) 4 (D) $\frac{1}{4}$

Solution: Let $M = (0, k)$ so that the slope of LM is

$$\frac{2-k}{t-0} = -t$$

$$\Rightarrow 2-k = -t^2 \tag{4.107}$$

Let (x, y) be the midpoint of LM. Therefore

$$x = \frac{t}{2} \text{ and } y = \frac{2+k}{2}$$

Hence, from Eq. (4.107), we have

$$2y = 2+k = 2+(2+t^2)$$
$$= 4+t^2$$
$$= 4+4x^2$$
$$\Rightarrow y = 2+2x^2$$
$$\Rightarrow x^2 = \frac{1}{2}(y-2)$$

Hence, the latus rectum is 1/2.

Answer: (B)

50. Let P be a point on the parabola $y = 4 - x^2$ lying in the first quadrant (i.e., $x > 0$, $y > 0$). Tangent to the parabola at P meets the x- and y-axis at A and B, respectively. The minimum possible area of ΔOAB (where O is the origin) is

(A) $\frac{64}{3\sqrt{3}}$ (B) $\frac{32}{3\sqrt{3}}$

(C) $96\sqrt{3}$ (D) $192\sqrt{3}$

Solution: The given equation is

$$x^2 = -(y-4)$$

Now

$$\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = -2x_1$$

Hence, the equation of the tangent at $P(x_1, y_1)$ is

$$y - y_1 = -2x_1(x - x_1)$$
$$\Rightarrow 2x_1x + y = y_1 + 2x_1^2$$

Hence

$$A = \left(\frac{y_1 + 2x_1^2}{2x_1}, 0\right)$$

and $$B = (0, y_1 + 2x_1^2)$$

Thus, the area of ΔOAB is given by

$$\Delta OAB = \frac{1}{2}(OA)(OB) = \frac{(y_1 + 2x_1^2)^2}{4x_1}$$
$$= \frac{(4 - x_1^2 + 2x_1^2)^2}{4x_1} = \frac{(x_1^2 + 4)^2}{4x_1}$$

So

$$\frac{d\Delta}{dx_1} = \frac{1}{4}\left[\frac{4x_1^2(x_1^2+4) - (x_1^2+4)^2}{x_1^2}\right] = \frac{x_1^2+4}{4x_1^2}(3x_1^2 - 4)$$

Therefore

$$\frac{d\Delta}{dx_1} = 0$$
$$\Rightarrow 3x_1^2 = 4 \text{ or } x_1 = \frac{2}{\sqrt{3}}$$

We can see easily that $d\Delta/dx_1$ changes sign from – to + when $x_1 = 2/\sqrt{3}$. Hence Δ is minimum at $x_1 = 2/\sqrt{3}$ and the minimum value of Δ is given by

$$\Delta = \frac{[(2/3)+4]^2}{4(2/\sqrt{3})} = \frac{256}{9} \times \frac{\sqrt{3}}{8} = \frac{32}{3\sqrt{3}}$$

Answer: (B)

Try it out Solve Worked-Out Problem 50 using parametric equation of the tangent.

51. Let P be the point $(-3, 0)$ and Q be a moving point $(0, 3t)$. Suppose PQ is trisected at R so that R is nearer to Q. The line RN which is drawn perpendicular to PQ meets the x-axis at N. The locus of the midpoint of PN is a parabola whose directrix is

(A) $x = 2$ (B) $y = 2$
(C) $x = -2$ (D) $y = -2$

Solution: Point R is nearer to Q implies that $PR:RQ = 2:1$ so that $R = (-1, 2t)$. Equation of the line through R and perpendicular to PQ is

$$y - 2t = -\frac{1}{t}(x+1) \quad \text{or} \quad x + ty = 2t^2 - 1$$

and hence $N = (2t^2 - 1, 0)$. Let $M\,(x, y)$ be the midpoint of RN so that

$$x = \frac{2t^2 - 1 - 1}{2}, y = \frac{2t}{2} = t$$

Therefore,

$$2x = 2t^2 - 2 = 2y^2 - 2$$
$$\Rightarrow y^2 = x + 1$$

Hence, the directrix is $x + 2 = 0$.

Answer: (C)

52. A circle of radius r touches the parabola $y^2 = 4ax$ ($a > 0$) at the vertex and the centre lies on the axis of the parabola. Further, the circle completely lies within the parabola. Then the largest possible value of r is

(A) $2a$ (B) $3a$ (C) $4a$ (D) a

Solution: Equation of the circle is $(x - y)^2 + y^2 = r^2$. See Fig. 4.29. If $P(at^2, 2at)$ is any point on the parabola, then

$$(r - at^2)^2 + 4a^2t^2 \geq r^2$$

(equality holds when $t = 0$). This implies

$$a^2t^4 - 2art^2 + 4a^2t^2 \geq 0$$
$$\Rightarrow at^2 - 2r + 4a \geq 0$$
$$\Rightarrow r \leq \frac{a}{2}(t^2 + 4) \leq 2a$$

where the equality holds when $t = 0$. Hence the maximum value of r is $2a$.

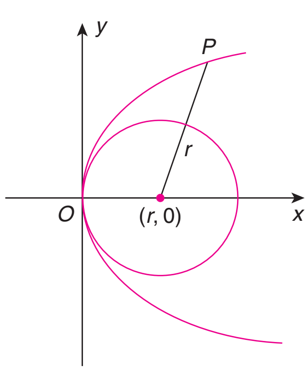


FIGURE 4.29

Answer: (A)

53. If the line $y = x$ touches the parabola $y = x^2 + bx + c$ at the point $(1, 1)$, then

(A) $b = 0, c = -1$ (B) $b = -1, c \in \mathbb{R}$
(C) $b = -1, c = -1$ (D) $b = -1, c = 1$

Solution: The point (1, 1) lies on the parabola, which implies that

$$b + c = 0 \qquad (4.108)$$

Also, $y = x$ touches the parabola $\Leftrightarrow$ the quadratic $x^2 + (b-1)x + c = 0$ has equal roots $\Leftrightarrow (b-1)^2 - 4c = 0$. Therefore, from Eq. (4.108), we get

$$(b-1)^2 + 4b = 0$$
$$\Rightarrow (b+1)^2 = 0$$
$$\Rightarrow b = -1 \text{ and } c = 1$$

Answer: (D)

54. If $P(x_1, y_1)$ and $Q(x_2, y_2)$ are two points on the parabola $y^2 = 8ax$, at which the normal meets in $(18, 12)$, then the length of the chord PQ is

(A) $2\sqrt{15}$ (B) $4\sqrt{15}$
(C) $2\sqrt{13}$ (D) $4\sqrt{13}$

Solution: Observe that (18,12) lies on the parabola. Let

$$P = (at^2, 2at^2) = (2t^2, 4t) \quad (\because a = 2)$$

If the normal at P meets the circle at $(18, 12) = (2t_1^2, 4t_1)$, where $t_1 = 3$, by Theorem 4.12, we have

$$-t - \frac{2}{t} = 3$$

so that

$$t^2 + 3t + 2 = 0$$

Hence, $t = -1, -2$. Therefore, $P = (2, -4)$ and $Q = (8, -8)$. Thus

$$PQ = \sqrt{(8-2)^2 + (-8+4)^2} = \sqrt{52} = 2\sqrt{13}$$

Answer: (C)

55. P is one end of the latus rectum of the parabola $y^2 = 4ax$. If the normal at P to the parabola meets the circle again in Q then the length of the chord PQ is

(A) $8a\sqrt{2}$ (B) $4a\sqrt{2}$
(C) $2a\sqrt{2}$ (D) $8a$

Solution: Let $P = (a, 2a)$ so that the value of the parameter t is 1. Suppose normal at P to the parabola meets the curve again at $(at_1^2, 2at_1)$. Therefore,

$$t_1 = -1 - \frac{2}{1} = -3$$

Hence, $Q = (9a - 6a)$and

$$(PQ)^2 = (9a - a)^2 + (-6a - 2a)^2 = 2 \times 64a^2$$

$$\Rightarrow PQ = 8a\sqrt{2}$$

Answer: (A)

56. In the parabola $y^2 = 4x$, the tangent at the point P, whose abscissa is equal to the latus rectum, meets the axis at T and the normal at P cuts the curve again at Q. Then the ratio PT:PQ is

(A) 5:4 (B) 4:5 (C) 2:3 (D) 3:2

Solution: We have

$$y^2 = 4ax (a = 1)$$

By hypothesis, at $P = (4, 4)$ so that the tangent equation at P (4, 4) is $2y = x + 4$ which implies that

$$T = (-4, 0) \qquad (4.109)$$

Also the normal at $P(4, 4)$ meets the curve again at Q $(t^2, 2t)$. So

$$t = -2 - \frac{2}{2} = -3$$

so that $Q = (9, -6)$. Now

$$PT = \sqrt{8^2 + 4^2} = 4\sqrt{5}$$

and

$$PQ = \sqrt{(9-4)^2 + (-6-4)^2}$$

$$= \sqrt{125} = 5\sqrt{5}$$

Hence

$$PT : PQ = 4\sqrt{5} : 5\sqrt{5} = 4 : 5$$

Answer: (B)

57. For the parabola $y^2 = 4ax$, let $T = (-a, 0)$. If PP' is a double ordinate of the parabola at PT and meets the curve again at Q, then $P'Q$ passes through the point

(A) $(4a, 0)$ (B) $(3a, 0)$ (C) $(2a, 0)$ (D) $(a, 0)$

Solution: Let $P = (at^2, 2at)$. so that $P' = (at^2, -2at)$. Let $Q = (at_1^2, 2ct_1)$ so that the equation of the line PQ is

$$y - 2at = \frac{2}{t + t_1}(x - at^2)$$

The line PQ passes through T $(-a, 0)$. So

$$t = \frac{1}{t + t_1}(1 + t^2)$$

$$\Rightarrow tt_1 = 1 \qquad (4.110)$$

Now, the equation of $P'Q$ is

$$y + 2at = \frac{2}{t_1 - t}(x - at^2)$$

Now

$P'Q$ passes through $S(a, 0)$

$$\Leftrightarrow t = \frac{1}{t_1 - t}(1 - t^2)$$

$$\Leftrightarrow tt_1 = 1$$

which is true according to Eq. (4.110). Hence, $P'Q$ passes through $S(a, 0)$.

Answer: (D)

58. P is a point on the parabola $y^2 = 4x$ and M is the foot of the perpendicular drawn from P onto the directrix. S is the focus. If ΔPSM is an equilateral triangle, then the area of the triangle is

(A) $3\sqrt{3}$ (B) $4\sqrt{3}$

(C) $2\sqrt{3}$ (D) $8\sqrt{3}$

Solution: It is clear that the focus S is $(1, 0)$ and $M = (-1, 2t)$ where $p = (t^2, 2t)$. Since ΔPSM is equilateral, we have

$$SM = SP$$

$$\Rightarrow 4 + 4t^2 = (t^2 + 1)^2$$

$$\Rightarrow t^2 + 1 = 4 \text{ or } t = \pm\sqrt{3}$$

Now $P = (3, 2\sqrt{3})$ implies

$$\text{Area of } \Delta PSM = \frac{\sqrt{3}}{4}(16) = 4\sqrt{3}$$

Answer: (B)

59. The tangents at P and Q on a parabola $y^2 = 4ax$ meet at point T. If the chord PQ passes through the fixed point $(-a, b)$ then the point T lies on the curve

(A) $bx = 2a(y - a)$ (B) $ay = 2b(x - b)$

(C) $ax = 2b(y - b)$ (D) $by = 2a(x - a)$

Solution: Let $P = (at_1^2, 2at_1)$ and $Q = (at_2^2, 2at_2)$ so that

$$T = (at_1t_2, a(t_1 + t_2))$$

The equation of the chord PQ is

$$y - 2at_1 = \frac{2}{t_1 + t_2}(x - at_1^2)$$

The chord PQ passes through the point $(-a, b)$. So

$$b - 2at_1 = \frac{2}{t_1 + t_2}(-a - at_1^2)$$

$$\Rightarrow b(t_1 + t_2) - 2at_1t_2 + 2a = 0 \qquad (4.111)$$

Suppose $(x, y) = T = [at_1t_2, a(t_1 + t_2)]$. Then

$$x = at_1t_2 \text{ and } y = a(t_1 + t_2)$$

Substituting the values of x and y in Eq. (4.111), we have

$$b\left(\frac{y}{a}\right)-2x+2a=0$$
$$\Rightarrow by=2a(x-a)$$

Answer: (D)

60. The distance between a tangent to the parabola $y^2 = 8x$ and a parallel normal which is inclined at an angle of 30° with the axis is

(A) $\frac{16}{\sqrt{3}}$ (B) $\frac{2}{3}$ (C) $\frac{16}{3}$ (D) $\frac{16}{3\sqrt{3}}$

Solution: The slope of the normal is

$$\tan 30° = \frac{1}{\sqrt{3}}$$

The line with slope $1/\sqrt{3}$ is

$$y=\frac{1}{\sqrt{3}}x+c$$

This line touches the parabola. So

$$c=\frac{2}{(1/\sqrt{3})}=2\sqrt{3}$$

Therefore, the tangent equation is

$$\sqrt{3}y=x+2$$
$$\Rightarrow x-\sqrt{3}y+6=0 \qquad (4.112)$$

Equation of the normal with slope $1/\sqrt{3}$ (see Quick Look 7) is

$$y=\frac{1}{\sqrt{3}}x-\frac{4}{\sqrt{3}}-\frac{2}{3\sqrt{3}}$$
$$\Rightarrow x-\sqrt{3}y-\frac{14}{3}=0 \qquad (4.113)$$

Hence, the distance between the lines provided in Eq. (4.112) and (4.113) is

$$\left|\frac{6+14/3}{\sqrt{1^2+3}}\right|=\frac{32}{3(2)}=\frac{16}{3}$$

Answer: (C)

Multiple Correct Choice Type Questions

1. If C is a circle described on the focal chord of the parabola $y^2 = 4x$ as diameter which is inclined at an angle of 45° with the axis, then the

(A) radius of the circle is 2.

(B) the centre of the circle is (3, 2).

(C) the line $x + 1 = 0$ touches the circle.

(D) the circle $x^2+y^2+2x-6y+3=0$ is orthogonal to C.

Solution: Let

$$P=(t_1^2, 2t_1) \quad \text{and} \quad Q=\left(\frac{1}{t_1^2}, \frac{-2}{t_1}\right)$$

be the extremities of the focal chord. By Theorem 4.19, part (4), the circle touches the directrix $x + 1 = 0$. The equation of the circle is

$$(x-t_1^2)\left(x-\frac{1}{t_1^2}\right)+(y-2t_1)\left(y+\frac{2}{t_1}\right)=0$$
$$\Rightarrow x^2-\left(t_1^2+\frac{1}{t_1^2}\right)x+1+y^2-2\left(t_1-\frac{1}{t_1}\right)y-4=0 \qquad (4.114)$$

The slope of the focal chord PQ is equal to 1. So we have

$$\frac{2(t_1+1/t_1)}{t_1^2-1/t_1^2}=1$$
$$\Rightarrow \frac{2}{t_1-1/t_1}=1$$
$$\Rightarrow t_1-\frac{1}{t_1}=2$$
$$\Rightarrow t_1^2-2t_1+1=0$$
$$\Rightarrow t_1=1$$

Therefore, $P = (1, 2)$ and $Q = (1, -2)$. From Eq. (4.114), the equation of the circle is $x^2+y^2-2x-3=0$. Hence, the radius is 2 and the centre is (1, 0).

Answer: (A), (C)

2. For the parabola $y^2-2x-6y+5=0$, which of the following are true?

(A) Vertex is $(-2, 3)$

(B) Focus is $\left(\frac{-3}{2}, 3\right)$

(C) Directrix is $2x+5=0$

(D) Latus rectum is 2

Solution: Given equation is $(y-3)^2=2(x+2)$ which represents parabola whose

Vertex = (–2, 3)

$$\text{Focus} = \left(x+2=\frac{1}{2}, y-3=0\right) = \left(\frac{-3}{2}, 3\right)$$

$$\text{Directrix} = x+2+\frac{1}{2}=0 \quad \text{or} \quad 2x+5=0$$

Latus rectum = 4

Answer: (A), (B), (C)

3. If (x_1, y_1) and (x_2, y_2) are two points on the parabola $y^2 = 8x$, at which the normals to the curve intersect on the curve, then

(A) $x_1 x_2 = 8$ (B) $y_1 y_2 = 16$

(C) $y_1 y_2 = 32$ (D) $x_1 x_2 = 16$

Answer: (C), (D)

Solution: If the normals at $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ meet again on the curve, then by Theorem 4.13, $x_1x_2 = 4a^2$ and $y_1y_2 = 8a^2$. Here, $a = 2$.

4. PQ is a double ordinate of the parabola $y^2 = 4ax$. If the normal at P intersects the line passing through Q and is parallel to the axis at G, then the locus of G is a parabola with

(A) vertex at $(4a, 0)$

(B) directrix $x = 3a$

(C) focus $(5a, 0)$

(D) latus rectum $4a$

Solution: Let $P = (at^2, 2at)$ and $Q = (at^2, -2at)$. The normal at P is

$$tx + y = 2at + at^3 \tag{4.115}$$

The line passing through Q and parallel to the axis (i.e., x-axis) is

$$y = -2at \tag{4.116}$$

From Eqs. (4.115) and (4.116), we have

$$G = (4a + at^2, -2at) = (h, k) \quad \text{(say)}$$

Therefore

$$h = 4a + at^2 \text{ and } k = -2at$$

$$\Rightarrow h - 4a = a\left(\frac{k^2}{4a^2}\right) = \frac{k^2}{4a}$$

$$\Rightarrow k^2 = 4a(h-4a)$$

$$\Rightarrow y^2 = 4a(x-4a)$$

Hence, the vertex is $(4a, 0)$, latus rectum is $4a$, the directrix is $x - 4a = -a$ or $x - 3a = 0$ and the focus $= (x - 4a = a, 0) = (5a, 0)$.

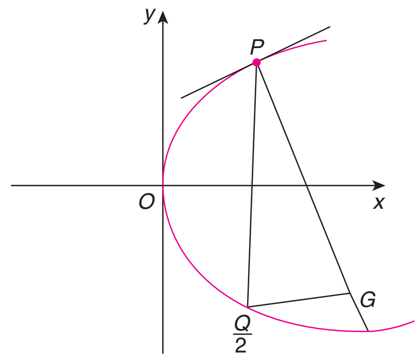


FIGURE 4.30

Answer: (A), (B), (C), (D)

5. Let A and B be two distinct points on the parabola $y^2 = 4x$. If the axis of the parabola touches a circle of radius r and AB is its diameter, then the slope of the line AB is

(A) $\frac{-1}{r}$ (B) $\frac{1}{r}$ (C) $\frac{2}{r}$ (D) $\frac{-2}{r}$

(IIT-JEE 2010)

Solution: Let $A = (t_1^2, 2t_1)$ and $B = (t_2^2, 2t_2)$ $(t_1 \neq t_2)$. Therefore, the slope of AB is

$$\frac{2}{t_1 + t_2} \tag{4.117}$$

Now, C is the centre of the circle described on AB as diameter (see Fig. 4.30), which is given by

$$\left(\frac{t_1^2 + t_2^2}{2}, t_1 + t_2\right)$$

Therefore, the radius is

$$|t_1 + t_2| = r \qquad (\because \text{ the circle touches the axis})$$

So, the slope of $AB = \pm 2/r$.

Answer: (C), (D)

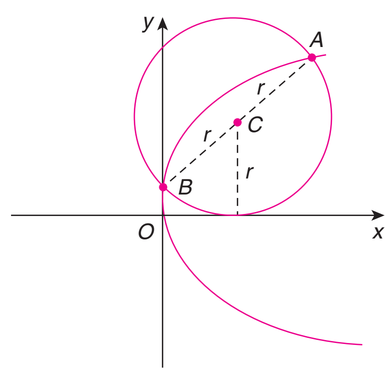


FIGURE 4.31

6. The tangent PT and the normal PN to the parabola $y^2 = 4ax$ at a point P on it meet its axis at points T

and N, respectively. The locus of the centroid of the ΔPTN is a parabola whose

(A) vertex is $\left(\frac{2a}{3}, 0\right)$ (B) direction is $x = 0$

(C) latus rectum is $\frac{2a}{3}$ (D) focus is $(a, 0)$

(IIT-JEE 2009)

Solution: See Fig. 4.32. $P = (at^2, 2at)$. Tangent at P is

$$ty = x + at^2$$

so that $T = (-at^2, 0)$. Normal at P is

$$tx + y = 2at + at^3$$

and hence $N = (2a + at^2, 0)$.

Now $P = (at^2, 2at)$, $T = (-at^2, 0)$ and $N = (2a + at^2, 0)$. Let $G(x, y)$ be the centroid of ΔPTN. Therefore

$$x = \frac{2a + at^2}{3} \quad \text{and} \quad y = \frac{2at}{3}$$

Hence

$$\frac{3x - 2a}{a} = t^2 = \left(\frac{3y}{2a}\right)^2$$

$$\Rightarrow 3x - 2a = \frac{9}{4a} y^2$$

$$\Rightarrow y^2 = \frac{4a}{9}(3x - 2a)$$

$$\Rightarrow y^2 = \frac{4a}{3}\left(x - \frac{2a}{3}\right)$$

Therefore,

Vertex = $(2a/3, 0)$

Focus = $\left(x - \frac{2a}{3} = \frac{a}{3}, y = 0\right) = (a, 0)$

Directrix = $x - \frac{2a}{3} = \frac{-a}{3} \Rightarrow x = \frac{a}{3}$

Latus rectum = $4a/3$

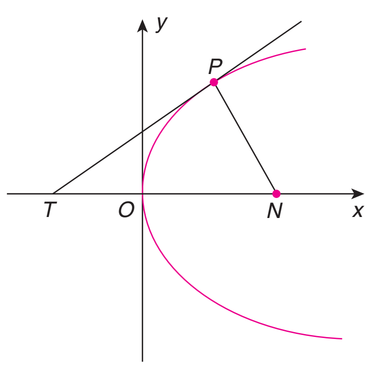


FIGURE 4.32

Answer: (A), (D)

7. The equation of the common tangents to the parabolas $x^2 = y$ and $(x - 2)^2 = -y$ are

(A) $y = 4(x - 1)$ (B) $y = 0$

(C) $y = -4(x - 1)$ (D) $y = -30x - 50$

(IIT-JEE 2006)

Solution: We have

$$x^2 = y = 4\left(\frac{1}{4}\right)y\left(a = \frac{1}{4}\right)$$

Hence, any tangents to $x^2 = y$ are of the form

$$x = my + \frac{1}{4m} \quad \forall m \neq 0$$

This line meets the parabola $y = -(x - 2)^2$ at the point whose ordinates are the roots of the equation

$$y = -\left[my + \frac{1}{4m} - 2\right]^2$$

$$\Rightarrow m^2y^2 + \left(\frac{3}{2} - 4m\right)y + \frac{1}{16m^2} + 4 - \frac{1}{m} = 0 \qquad (4.118)$$

This line touches the parabola $y = -(x - 2)^2$

$\Leftrightarrow$ The roots of Eq. (4.118) are equal

$$\Leftrightarrow \left(\frac{3}{2} - 4m\right)^2 - 4m^2\left(\frac{1}{16m^2} + 4 - \frac{1}{m}\right) = 0$$

$$\Leftrightarrow \frac{9}{4} - 12m + 16m^2 - \frac{1}{4} - 16m^2 + 4m = 0$$

$$\Leftrightarrow 2 - 8m = 0$$

$$\Leftrightarrow m = \frac{1}{4}$$

Hence,

$$x = \frac{y}{4} + 1 \Rightarrow y = 4x - 4$$

Also the line $y = 0$ (i.e., x-axis) touches $y = x^2$ at $(0, 0)$ and also $y = -(x - 2)^2$ at $(2, 0)$.

Answer: (A), (B)

8. Consider the parabola with its vertex at origin and the axis along the x-axis. The line $y = 2x + c$, where $c > 0$, is a common tangent to the parabola and the circle $x^2 + y^2 = 5$. Then the

(A) directrix is $x = -10$

(B) focus is $(10, 0)$

(C) latus rectum is 20

(D) directrix is $x = 10$

Solution: The line $y = 2x + c$ touches the circle. This implies

$$c^2 = 5(1 + 2^2) = 25$$

$$\Rightarrow c = 5 \ (\because c > 0)$$

Let $y^2 = 4ax$ be the parabola. Since the line $y = 2x + 5$ touches the parabola we have

$$5 = \frac{a}{2} \quad \text{or} \quad a = 10.$$

Therefore, the parabola is $y^2 = 40x$. Hence, the focus is $(10, 0)$.

Answer: (A), (B)

9. Let P be the parabola which is the locus of the midpoints of focal radii of the parabola $y^2 = 4ax$. For P, which of the following are true?

(A) Vertex is $\left(\frac{a}{2}, 0\right)$

(B) Latus rectum is $2a$

(C) Directrix is $x = \frac{a}{2}$

(D) Directrix $x = 0$

Solution: Let Q $(at^2, 2at)$ be a point on the parabola and $S = (a, 0)$. $M(x, y)$ is the midpoint of SQ. This implies

$$x = \frac{at^2 + a}{2} \text{ and } y = at$$

Now

$$\frac{2x - a}{a} = t^2 = \left(\frac{y}{a}\right)^2$$

$$\Rightarrow y^2 = a(2x - a) = 2a\left(x - \frac{a}{2}\right)$$

Therefore, the vertex is $(a/2, 0)$, the latus rectum is $2a$ and the directrix equation is

$$x - \frac{a}{2} = -\frac{a}{2} \Rightarrow x = 0$$

Answer: (A), (B), (D)

10. Consider the circle C: $x^2 + y^2 - 6y + 4 = 0$ and the parabola P: $y^2 = x$. Then

(A) the number of common tangents to C and P is 3.

(B) the number of common tangents to C and P is 2.

(C) $x - 2y + 1 = 0$ is one of the common tangents.

(D) $x + 2y + 1 = 0$ is also one of the common tangents.

Solution: C: $x^2 + (y - 3)^2 = 5$. Let $Q(t^2, t)$ be any point on the parabola $y^2 = x$ so that the equation of the tangent at Q is $x - 2ty + t^2 = 0$ which touches the circle C. So

$$\left|\frac{0 - 2t(3) + t^2}{\sqrt{1 + 4t^2}}\right| = \sqrt{5}$$

$$\Rightarrow (t^2 - 6t)^2 = 5(1 + 4t^2)$$

$$\Rightarrow t^4 - 12t^3 + 16t^2 - 5 = 0$$

$$\Rightarrow (t - 1)^2(t^2 - 10t - 5) = 0$$

$$\Rightarrow t = 1, 5 \pm \sqrt{30}$$

Hence, the number of common tangents is 3 and $x - 2y + 1 = 0$ is a common tangent when $t = 1$.

Answer: (A), (C)

Matrix-Match Type Questions

1. Match the items of Column I with those of Column II.

Column I	*Column II*
(A) If the line $x - 1 = 0$ is the directrix of the parabola $y^2 - kx + 8 = 0$, then the value of k is	(p) 2
(B) If l is the length of one side of an equilateral triangle inscribed in the parabola $y^2 = 4x$ with one vertex at the origin, then $l/2\sqrt{3} =$	(q) 4
(C) The latus rectum of a parabola having (3, 5) and (3, −3) as extremities of the latus rectum is	(r) 8
	(s) −4
(D) If (2, 0) is the vertex and y-axis as the directrix, then its focus is (a, 0) when a equals	(t) −8

(Continued)

Solution:

(A) The parabola $y^2 - kx + 8 = 0$ is written as

$$y^2 = k\left(x - \frac{8}{k}\right)$$

$$= 4\left(\frac{k}{4}\right)\left(x - \frac{8}{k}\right)$$

Hence, the directrix is

$$x - \frac{8}{k} = \frac{-k}{4}$$
$$\Rightarrow x + \frac{k}{4} - \frac{8}{k} = 0$$
$$\Rightarrow x + \frac{k^2 - 32}{4k} = 0$$

By hypothesis, $x = 1$ is the directrix. Therefore

$$\frac{k^2 - 32}{4k} = -1$$
$$\Rightarrow k^2 + 4k - 32 = 0$$
$$\Rightarrow (k + 8)(k - 4) = 0$$
$$\Rightarrow k = 4, -8$$

Answer: (A) → (q), (t)

(B) From Problem 8 of the section 'Subjective Problems', we have $l = 8a\sqrt{3}$, where $a = 1$. Therefore

$$\frac{l}{2\sqrt{3}} = 4$$

Answer: (B) → (q)

(C) The length of the latus rectum is $\sqrt{(3-3)+(5+3)^2} = 8$

Answer: (C) → (r)

(D) Distance of the vertex from the directrix (i.e., y-axis) is 2 and it is equal to half the distance of the focus from the directrix so that the focus is (4, 0). Therefore, $a = 4$.

Answer: (D) → (q)

2. Match the items of Column I with those of Column II.

Column I	*Column II*
(A) If point P is on the circle $x^2 + y^2 = 5$, then the equation of the chord of contact with respect to the parabola $y^2 = 4x$ is $y = 2(x - 2)$. The coordinates of P are	(p) $(9, -6)$ (q) $(1, 2)$
(B) Tangents are drawn from the point (2, 3) to a parabola $y^2 = 4x$. Then, the points of contact are	(r) $(-2, 1)$
(C) The common chord of the circle $x^2 + y^2 = 5$ and the parabola $6y = 5x^2 + 7x$ passes through	(s) $(4, 4)$
(D) Two points $P(4, -4)$ and Q are on the parabola $y^2 = 4x$ such that the area of ΔPOQ (O is the vertex) is 6 sq. unit. Then, the coordinates of Q are	(t) $(2, 1)$

(Continued)

Solution:

(A) Let $P(x_1, y_1)$ be on the circle $x^2 + y^2 = 5$. Then

$$x_1^2 + y_1^2 = 5 \tag{4.119}$$

The equation of the chord of contact of $P(x_1, y_1)$ with the parabola $y^2 = 4x$ is

$$yy_1 - 2(x + x_1) = 0$$
$$\Rightarrow 2x - y_1 y + 2x_1 = 0$$

However,

$$2x - y - 4 = 0 \tag{4.120}$$

is the chord of contact. Therefore, from Eqs. (4.119) and (4.120), we get

$$\frac{2}{2} = \frac{-y_1}{-1} = \frac{2x_1}{-4}$$
$$\Rightarrow x_1 = -2, y = 1$$
$$\Rightarrow P = (-2, 1)$$

Answer: (A) → (r)

(B) Tangent to the parabola $y^2 = 4x$ at $(t^2, 2t)$ is $ty = x + t^2$. This passes through the point (2, 3). So

$$3t = 2 + t^2$$
$$\Rightarrow t^2 - 3t + 2 = 0$$
$$\Rightarrow (t - 1)(t - 2) = 0$$
$$\Rightarrow t = 1, 2$$

Therefore, the points of contact are (1, 2) and (4, 4).

Answer: (B) → (q), (s)

(C) Substituting

$$y = \frac{5x^2 + 7x}{6}$$

in the circle equation $x^2 + y^2 = 5$, we get

$$x^2 + \left(\frac{5x^2 + 7x}{6}\right)^2 = 5$$
$$\Rightarrow x^2 + \frac{25x^4 + 70x^3 + 49x^2}{36} = 5$$
$$\Rightarrow 25x^4 + 70x^3 + 85x^2 - 180 = 0$$

which clearly implies that $x = 1$ is a root. So

$$(x-1)[25x^3+95x^2+180x+180]=0$$
$$\Rightarrow (x-1)(x+2)[25x^2+45x+90]=0$$
$$\Rightarrow (x-1)(x+2)(5x^2+9x+18)=0$$
$$\Rightarrow x = 1, -2$$

Therefore, the points of intersection are (1, 2) and (−2, 1).

Answer: (C) → (q), (r)

(D) Let $Q = (t^2, 2t)$. Therefore

$$6 = \text{Area of } \Delta POQ = \frac{1}{2}\begin{vmatrix} 4 & -4 & 1 \\ 0 & 0 & 1 \\ t^2 & 2t & 1 \end{vmatrix}$$

$$\Rightarrow 12 = |8t+4t^2|$$
$$\Rightarrow t^2+2t=\pm 3$$

Case 1: When $t^2 + 2t + 3 = 0$, we have

$$(t-1)(t+3)=0$$
$$\Rightarrow t = 1, -3$$

So $Q = (1, 2), (9, -6)$.

Case 2: When $t^2 + 2t + 3 = 0$, it has no real roots.

Answer: (D) → (p), (q)

3. Match the items of Column I with those of Column II.

Column I	*Column II*
(A) The point from which perpendicular tangents can be drawn to the parabola $y^2 = 8x$ is	(p) (−2, 1)
(B) The line $x + y + 3 = 0$ touches the parabola $y^2 = 12x$ at the point	(q) (4, 6) (r) (3, −6)
(C) $4x + 3y - 34 = 0$ is normal to the parabola $y^2 = 9x$ at the point	(s) (28, 14)
(D) The line parallel to $4y - x + 3 = 0$ touches the parabola $y^2 = 7x$ at the point	(t) (−2, 5)

Solution:

(A) It is known that the locus of the point from which perpendicular tangents is drawn to a parabola is the directrix of the parabola. For the parabola $y^2 = 8x$, the directrix is $x + 2 = 0$ on which the points (−2, 1) and (−2, 5) lie.

Answer: (A) → (p), (t)

(B) The line

$$y = mx + \frac{a}{m}$$

touches $y^2 = 4ax$ at

$$\left(\frac{a}{m^2}, \frac{2a}{m}\right)$$

The given line is $y=-x-3 (a=3, m=-1)$ and hence it touches the parabola $y^2 = 12x$ at

$$\left(\frac{a}{m^2}, \frac{2a}{m}\right) = (3, -6)$$

Answer: (B) → (r)

(C) It is known that the line $y = mx - 2am - am^3$ is a normal to the parabola $y^2 = 4ax$ at the point $(am^2, -2am)$. In the present case, $m = -4/3$ and $a = 9/4$. Therefore

$$(am^2, -2am) = \left(\frac{9}{4}\times\frac{16}{9}, -2\left(\frac{9}{4}\right)\left(\frac{-4}{3}\right)\right) = (4, 6)$$

Answer: (C) → (q)

(D) The line parallel to $4y - x + 3 = 0$ is $4y - x + c = 0$. The line with slope m touches the parabola $y^2 = 4ax$ at the point

$$\left(\frac{a}{m^2}, \frac{2a}{m}\right)$$

Here, $a = \frac{7}{4}$ and $m = \frac{1}{4}$. Therefore, the point of contact is

$$\left(\frac{a}{m^2}, \frac{2a}{m}\right) = \left(\frac{7}{4}\times 16, 2\left(\frac{7}{4}\right)\times 4\right) = (28, 14)$$

Answer: (D) → (s)

Comprehension Type Questions

1. Passage: For the parabola $y^2 = 4ax$, the vertex is (0, 0), the focus is $(a, 0)$ and the directrix is $x + a = 0$. Answer the following three questions.

(i) The vertex of the parabola $(y-1)^2 = 2(x-1)$ is

(A) (1,0) (B) (2,0)
(C) (1,1) (D) (0,1)

(ii) Focus of the parabola $y^2 = 4(x-1)$ is

(A) $(1,0)$ (B) $(2,0)$
(C) $(1,1)$ (D) $(2,2)$

(iii) The directrix of the parabola $(y-2)^2 = 4(x-1)$ is

(A) $x=1$ (B) $x=-1$
(C) $x=2$ (D) $x=0$

Solution:

(i) The parabola is

$$Y^2 = 4\left(\frac{1}{2}\right)(X)$$

where $X = x - 1$ and $Y = y - 1$. Therefore, the vertex is

$$(X=0, Y=0) = (x-1=0, y-1=0) = (1,1)$$

Answer: (C)

(ii) $y^2 = 4(x-1)$ is $Y^2 = 4X$ where $X = x-1$ and $Y = y$ and $a = 1$. The focus is

$$(X=1, Y=0) = (x-1=1, y=0) = (2,0)$$

Answer: (B)

(iii) $(y-2)^2 = 4(x-1) \Rightarrow Y^2 = 4X$ where $X = x - 1$ and $Y = y - 2$. The directrix equation is

$$X = -a = -1$$
$$\Rightarrow x - 1 = -1$$
$$\Rightarrow x = 0$$

Answer: (D)

2. Passage: $P(2t^2, 4t)$ is a point on the parabola $y^2 = 8x$ and $Q(h, k)$ is a point on the tangent at P and external to the circle $x^2 + y^2 = 8$. Answer the following questions.

(i) As Q moves on the tangent at P, the locus of the point of intersection of the chord of contact of Q with respect to the circle at the tangent at P is

(A) $y^2 - x^2 = 4$ (B) $y^2 = 2x$
(C) $y^2 - 2x^2 = 4$ (D) $y^2 = -4x$

(ii) The point in the second quadrant from which perpendicular tangents can be drawn to both the parabola and the circle is

(A) $(-2, 2\sqrt{3})$ (B) $(-1, \sqrt{2})$
(C) $(-\sqrt{2}, \sqrt{2})$ (D) $(-3, 2\sqrt{3})$

(iii) If AB is the chord of contact of $Q(h, k)$ with respect to the circle $x^2 + y^2 = 8$, then the circumcentre of ΔAQB lies on the curve (when $t = 2$)

(A) $x + 2y = 4$ (B) $x - 2y = 4$
(C) $2y = x + 4$ (D) $x + 2y + 4 = 0$

Solution:

(i) The tangent at P is

$$ty = x + 2t^2 \quad (4.121)$$

which passes through $Q(h, k)$. This implies

$$tk = h + 2t^2 \quad (4.122)$$

The equation of the chord of contact of $Q(h, k)$ with respect to the circle $x^2 + y^2 = 8$ is

$$hx + ky - 8 = 0 \quad (4.123)$$

From Eqs. (4.122) and (4.123), we have

$$hx + y\frac{(h+2t^2)}{t} - 8 = 0$$
$$\Rightarrow 2(ty-4) + h\left(x + \frac{y}{t}\right) = 0$$

This line passes through the point

$$y = \frac{4}{t}, x = -\frac{y}{t}$$

Therefore, the locus is

$$\frac{y}{4} = \frac{-x}{y}$$
$$\Rightarrow y^2 = -4x$$

Answer: (D)

(ii) The required point lies on the director circle of the circle and directrix of the parabola. The directrix of the parabola is $x + 2 = 0$ and the director circle of the given circle is $x^2 + y^2 = (2\sqrt{2} \times \sqrt{2})^2 = 16$. Now

$$x = -2 \Rightarrow 4 + y^2 = 16$$
$$\Rightarrow y = \pm 2\sqrt{3}$$

Therefore, the required point $= (-2, 2\sqrt{3})$.

Answer: (A)

(iii) The equation of the circumcircle of ΔAQB is

$$(x^2 + y^2 - 8) + \lambda(hx + ky - 8) = 0$$

This should pass through $(0,0)$ (centre of the circle) which implies that $\lambda = -1$. Therefore, the circumcircle of ΔAQB is

$$x^2 + y^2 - hx - ky = 0$$

so that $(h/2, k/2)$ is its centre. If (x, y) is the circumcentre of ΔAQB, then $x = h/2$, $y = k/2$. Substituting the values of h and k in Eq. (4.122), when $t = 2$, we have

$$2(2y) = 2x + 8$$
$$\Rightarrow x - 2y + 4 = 0$$

Answer: (C)

3. Passage: Consider the circle $x^2 + y^2 = 9$ and the parabola $y^2 = 8x$. They intersect at P and Q in the first and the fourth quadrants, respectively. The tangents to the circle at P and Q intersect x-axis at R and tangents to the parabola at P and Q intersect x-axis at S. Answer the following questions.

(i) The ratio of the areas of ΔPQS and ΔPQR is

(A) $1:\sqrt{2}$ (B) 1:2 (C) 1:4 (D) 1:8

(ii) The radius of the circumcircle of ΔPRS is

(A) 5 (B) $3\sqrt{3}$ (C) $3\sqrt{2}$ (D) $2\sqrt{3}$

(iii) The radius of the incircle of ΔPQR is

(A) 4 (B) 3 (C) $\dfrac{8}{3}$ (D) 2

Solution: Solving the two equations $x^2 + y^2 = 9$ and $y^2 = 8x$, we get $P = (1, 2\sqrt{2})$ and $Q = (1, -2\sqrt{2})$ (see Fig. 4.33). Tangents to the parabola at P and Q, respectively, are

$$y(2\sqrt{2}) = 4(x+1)$$

and
$$y(-2\sqrt{2}) = 4(x+1)$$

Hence, $S = (-1, 0)$. The tangent to the circle at P is

$$x(1) + y(2\sqrt{2}) = 9$$

Hence, $R = (9, 0)$. Therefore, we have $P = (1, 2\sqrt{2})$, $Q = (1, -2\sqrt{2})$, $R = (9, 0)$ and $S = (-1, 0)$.

(i) The area of ΔPQR is given by

$$\frac{1}{2}\left|1(-2\sqrt{2} - 0) + 1(0 - 2\sqrt{2}) + 9(2\sqrt{2} + 2\sqrt{2})\right|$$
$$= \frac{1}{2}\left|-4\sqrt{2} + 36\sqrt{2}\right| = 16\sqrt{2}$$

The area of ΔPQS is given by

$$\frac{1}{2}\left|1(-2\sqrt{2} - 0) + 1(0 - 2\sqrt{2}) - 1(2\sqrt{2} + 2\sqrt{2})\right| = 4\sqrt{2}$$

Therefore, $\Delta PQS : \Delta PQR = 4 : 16 = 1 : 4$.

Answer: (C)

(ii) The area of ΔPRS is given by

$$\frac{1}{2}\left|1(0-0) + 9(0 - 2\sqrt{2}) - 1(2\sqrt{2} - 0)\right| = 10\sqrt{2}$$

Now $PS = 2\sqrt{3}, RS = 10, PR = 6\sqrt{2}$. Therefore, the circumradius of ΔPQS is

$$\frac{(PR)(PS)(RS)}{4\Delta PRS} = \frac{(6\sqrt{2})(2\sqrt{3})(10)}{4(10\sqrt{2})} = 3\sqrt{3}$$

Answer: (B)

(iii) For ΔPQR, the area $= 16\sqrt{2}$. Now

$$PQ = 4\sqrt{2}, PR = 6\sqrt{2}, QR = 6\sqrt{2}$$

Therefore, the inradius of ΔPQR is given by

$$\frac{\text{Area}}{\text{Semi-perimeter}} = \frac{16\sqrt{2}}{8\sqrt{2}} = 2$$

Answer: (D)

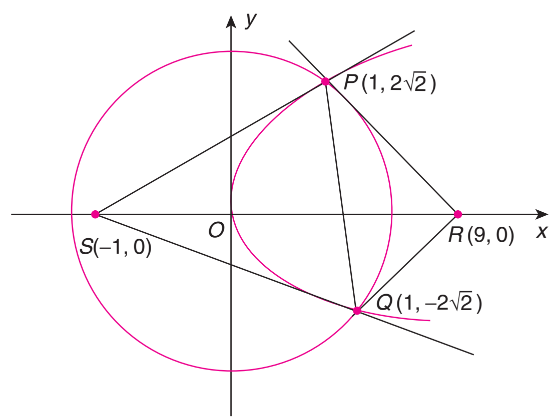


FIGURE 4.33

Integer Answer Type Questions

1. The number of points at which the parabola $y^2 = 4x$ and the circle $x^2 + y^2 - 6x + 1 = 0$ touch each other is ______.

Solution: Substituting $y^2 = 4x$ in the given circle equation, we have $x^2 - 2x + 1 = 0$ and hence $(1, \pm 2)$ are common points of the two curves. Also at $P(1, 2)$, equation of the tangent to the parabola is

$$y(2) - 2(x+1) = 0$$
$$\Rightarrow y - x - 1 = 0 \qquad (4.124)$$

The centre and the radius of the circle are $(3, 0)$ and $2\sqrt{2}$, respectively. Now, the distance of the centre (3, 0) from Eq. (4.124) is

$$\frac{|3 - 0 + 1|}{\sqrt{2}} = \frac{4}{\sqrt{2}} = 2\sqrt{2}$$

Hence, the line provided in Eq. (4.124) also touches the circle in a similar manner as the two curves touch at Q.

Answer: 2

2. Slope of the common tangent of the curves $y^2 = 8x$ and $xy = -1$ is ______.

Solution: We have

$$y^2 = 8x = 4(2)x \Rightarrow y = mx + \frac{2}{m}$$

is a tangent of $y^2 = 8x$. Substituting

$$y = mx + \frac{2}{m}$$

in $xy = -1$, we get

$$x\left(mx + \frac{2}{m}\right) = -1$$

$$\Rightarrow m^2x^2 + 2x + m = 0 \quad (4.125)$$

The line touches the parabola $\Leftrightarrow$ the discriminant of the quadratic equation provided in Eq. (4.125) is equal to zero

$$\Rightarrow 4 - 4m^3 = 0 \quad \Rightarrow m = 1$$

Hence, the common tangent is $y = x + 1$.

Answer: 1

3. If $x + y = a$ is normal to the parabola $y^2 = 12x$, then the value of a is equal to ______.

Solution: We have

$$y^2 = 12x = 4(3)x$$

which implies that every point on the parabola is of the form $(3t^2, 6t)$. The normal at $(3t, 6t)$ is

$$tx + y = 6t + 3t^3 \quad (4.126)$$

However,

$$x + y = a \quad (4.127)$$

is the normal. That is, Eqs. (4.126) and (4.127) represent the same line. Therefore

$$\frac{t}{1} = \frac{1}{1} = \frac{6t + 3t^3}{a} \Rightarrow t = 1 \text{ and } a = 9$$

Answer: 9

4. Length of the sub-normal to the parabola $y^2 = 8x$ at any point is ______.

Solution: In fact, we show that the length of the sub-normal to the curve $y^2 = 4ax$ at any point on the curve is equal to $2a$. Suppose that the tangent and normal to $y^2 = 4ax$ at $P(at^2, 2at)$ meet axis of the curve at T and N, respectively. Draw PG perpendicular to the axis (see Fig. 4.34). Hence, TG = sub-tangent and GN = sub-normal (see Definition 3.4, Chapter 3, Vol. 3, pg 225).

The tangent at $P(at^2, 2at)$ is $ty = x + at^2$. This meets axis at $T(-at^2, 0)$ Normal at $P(at^2, 2at)$ is $tx + y = 2at + at^3$. This meets the axis at $N\,(2a + at^2, 0)$. Therefore, the semi-latus rectum is

$$GN = ON - OG = (2a + at^2) - at^2 = 2a$$

Hence, sub-normal $= 2a = 4$ $(\because a = 2)$

Answer: 4

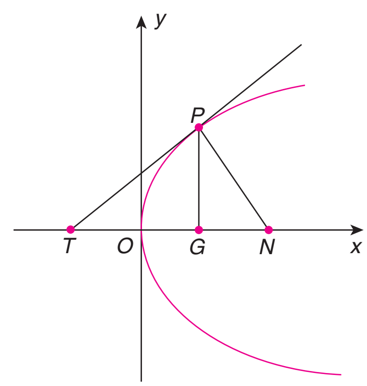


FIGURE 4.34

5. The locus of the midpoint of chord of $y^2 = 4ax$ which subtends right angle at the vertex is another parable with latus rectum ka, where the integral part of k is equal to ______.

Solution: See Problem 13 in the section 'Subjective Problems'.

Answer: 2

6. If the normals at t_1 and t_2 of the parabola $y^2 = 4ax$ meet again on the curve, then the value of $t_1 t_2$ is equal to ______.

Solution: See Theorem 4.13.

Answer: 2

7. The normals at $P(x_1, y_1)$ and $Q(x_2, y_2)$ of the parabola $y^2 = 4x$ meet on the curve again. Then $x_1 x_2$ is equal to ______.

Solution: By Theorem 4.13, we have

$$x_1x_2 = 4a^2 = 4 \; (\because a = 1)$$

Answer: 4

8. The equation of the parabola with focus $(-1, -1)$ and directrix $2x - 3y + 6 = 0$ is $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$. Then, $|a - b|$ is equal to ______.

Solution: By definition,

$$\frac{|2x - 3y + 6|}{\sqrt{2^2 + 3^2}} = \sqrt{(x+1)^2 + (y+1)^2}$$

$$\Rightarrow (2x - 3y + 6)^2 = 13[x^2 + y^2 + 2x + 2y + 2]$$

$$\Rightarrow 9x^2 + 12xy + 4y^2 + 2x + 62y - 10 = 0$$

Therefore, $|a - b| = 9 - 4 = 5$.

Answer: 5

9. The latus rectum of the parabola whose vertex is at (3, 2) and focus at (5, 2) is ______.

Solution: The vertex is $A = (3, 2)$ and the focus is $S = (5, 2)$. Therefore, $AS = 2$ (which is equal to a) and AS is horizontal. Hence, the equation is of the form

$$(y-k)^2 = 4a(x-h)$$
$$\Rightarrow (y-2)^2 = 8(x-3)$$

Thus, the latus rectum is 8.

Answer: 8

10. The locus of the midpoints of chords of the parabola $y^2 = 16x$ which passes through the vertex is a parabola whose length of latus rectum is ______.

Solution: Let $M(x_1, y_1)$ be the midpoint of a chord of $y^2 = 16x$. Hence, the equation of the chord is

$$yy_1 - 8(x + x_1) = y_1^2 - 16x_1$$

This chord passes through the vertex (0, 0). This implies

$$-8x_1 = y_1^2 - 16x_1$$
$$\Leftrightarrow y_1^2 = 8x_1$$

Therefore, the locus of $M(x_1, y_1)$ is the parabola $y^2 = 8x$. Hence, the length of latus rectum is 8.

Answer: 8

11. The locus of the midpoints of the chords of the parabola $2y^2 = 7x$ which are parallel to the line $3x - 2y = 0$ is the line $px + qy + r = 0$, where $|p+q+r|$ equals ______.

Solution: $M(x_1, y_1)$ is the midpoint of a chord so that its equation is

$$yy_1 - \frac{7}{4}(x + x_1) = y_1^2 - \frac{7}{2}x_1$$
$$\Rightarrow 7x - 4y_1 y = 7x_1 - 4y_1^2$$

which is parallel to

$$3x - 2y = 0$$

This again implies

$$\frac{3}{2} = \frac{7}{4y_1} \Rightarrow 6y_1 = 7$$

The locus of point $M(x_1, y_1)$ is the line $6y - 7 = 0$. Hence, $p = 0, q = 6, r = -7$. Thus,

$$|p+q+r| = 1$$

Answer: 1

12. If the line $y = mx + c$ is normal to the parabola $y^2 = 8x$, then $c + pm + qm^3 = 0$ where $p + q$ is equal to ______.

Solution: Consider $y^2 = 4ax$. The normal at $(at^2, 2at)$ is

$$tx + y = 2at + at^3 \tag{4.128}$$

Suppose $y = mx + c$ is normal at $(at^2, 2at)$. We have

$$\frac{m}{t} = \frac{-1}{1} = \frac{c}{-2at - at^3}$$
$$\Rightarrow t = -m \text{ and } c = 2at + at^3 = -2am - am^3$$
$$\Rightarrow c + 2am + am^3 = 0$$

Here, $a = 2$. Hence, $p = 4$, $q = 2$ so that $p + q = 6$.

Answer: 6

13. If $\sqrt{3}y = bx + 3, b > 0$ is the equation of a common tangent of the circle $(x-3)^2 + y^2 = 9$ and the parabola $y^2 = 4x$, then the value of b is ______.

Solution: We know that

$$y = mx + \frac{1}{m}$$

touches $y^2 = 4x$ for all $m \neq 0$. This also touches the given circle. So

$$\frac{|3m - 0 + (1/m)|}{\sqrt{1+m^2}} = 3$$
$$\Rightarrow \left(3m + \frac{1}{m}\right)^2 = 9(1+m^2)$$
$$\Rightarrow 6 + \frac{1}{m^2} = 9$$
$$\Rightarrow \frac{1}{m^2} = 3$$
$$\Rightarrow m = \pm\frac{1}{\sqrt{3}}$$

Therefore, the equation of the common tangents is

$$y = \pm\frac{x}{\sqrt{3}} + \sqrt{3}$$
$$\Rightarrow y\sqrt{3} = (\pm x) + 3$$

Hence, $b = 1$ ($\because b > 0$).

Answer: 1

14. Through the vertex O of the parabola $y^2 = 4x$, the chords OP at OQ are drawn at right angles to each other. Then, the equation of the locus of the midpoint of chord PQ is $y^2 = k(x + b)$ where $|k+b|$ is equal to ______.

Solution: Let $P = (t_1^2, 2t_1)$ and $Q = (t_2^2, 2t_2)$. Now $\angle POQ = 90°$. So

$$\left(\frac{2t_1}{t_1^2}\right)\left(\frac{2t_2}{t_2^2}\right) = -1$$

$$\Rightarrow t_1t_2 = -4 \quad (4.129)$$

Let $M(x, y)$ be the midpoint of PQ so that

$$x = \frac{t_1^2 + t_2^2}{2}$$

and $$y = t_1 + t_2$$

Therefore

$$y^2 = (t_1 + t_2)^2 = t_1^2 + t_2^2 + 2t_1t_2 = 2x - 8 = 2(x - 4)$$

So, $|k + b| = |2 - 4| = 2.$

Answer: 2

SUMMARY

4.1 ***Definition:*** Let L be a straight line and S be a point on the plane of the line L and not on L. Then, the locus of the point which is equidistant from the point S and the line L is called a parabola. For this parabola, S is called focus and L is called directrix.

4.2 ***Theorem:*** Standard equation of a parabola is $y^2 = 4ax$ $(a > 0)$. The focus is $(a, 0)$ and the directrix equation is $x + a = 0$.

4.3 The other standard forms of parabola are as follows:

1. $y^2 = -4ax$ [focus is $(-a, 0)$ and directrix is $x - a = 0$]
2. $x^2 = 4ay$ [focus is $(0, a)$ and directrix is $y + a = 0$]
3. $x^2 = -4ay$ [focus is $(0, -a)$ and directrix is $y - a = 0$]
4. $(y-k)^2 = 4a(x-h)$ [vertex is (h, k), focus is $(h + a, k)$ and directrix is $x = h - a$]
5. $(y-k)^2 = -4a(x-h)^2$ [vertex is (h, k), focus is $(h - a, k)$ and directrix is $x = h + a$]
6. $(x-h)^2 = 4a(y-k)$ [vertex is (h, k), focus is $(h, k + a)$ and directrix is $y = k - a$]
7. $(x-h)^2 = -4a(y-k)$ [vertex is (h, k), focus is $(h, k - a)$ and directrix is $y = k + a$]

4.4 The line about which the parabola is symmetric is called the axis of the parabola.

4.5 ***Note:***

1. x-axis is the axis of $y^2 = 4ax$ and $y^2 = -4ax$.
2. y-axis is the axis of $x^2 = 4ay$ and $x^2 = -4ay$.
3. The line $y = k$ is the axis of $(y-k)^2 = \pm 4a(x-h)$.
4. The line $x = h$ is the equation of the axis of $(x-h)^2 = \pm 4a(y-k)$.

4.6 ***Double ordinate and latus rectum:*** If a line perpendicular to the axis meets the curve at P and P', then PP' is called double ordinate. A double ordinate passing through the focus is called latus rectum.

4.7 Length of the latus rectum of the parabola $y^2 = 4ax$ is $4a$ (i.e., the coefficient of x).

4.8 ***Parametric equations:*** $x = at^2$, $y = 2at$, $t \in \mathbb{R}$ are called the parametric equations of the parabola $y^2 = 4ax$. That is, for all real values of t, the point $(at^2, 2at)$ lies on the parabola $y^2 = 4ax$.

4.9 ***Notation:***

$$\begin{aligned} S &\equiv y^2 - 4ax \\ S_1 &\equiv y_1 - 2a(x + x_1) \\ S_{12} &= S_{21} = y_1y_2 - 2a(x_1 + x_2) \\ S_{kk} &= y_k^2 - 4ax_k \end{aligned}$$

4.10 ***Theorem:*** Point (x_1, y_1) lies outside or inside the parabola $S \equiv y^2 - 4ax$ according as $S_{11} > 0$ or $S_{11} < 0$.

4.11 ***Theorem (Equation of chord joining two points):*** If $P(x_1, y_1)$ and $Q(x_2, y_2)$ are points on the parabola $y^2 = 4ax$, then the equation of the chord PQ is

$$S_1 + S_2 = S_{12}$$

$$\Rightarrow [yy_1 - 2a(x + x_1)] + [yy_2 - 2a(x + x_2)] = y_1y_2 - 2a(x_1 + x_2)$$

4.12. ***Theorem (Tangent):*** The equation of the tangent to $S \equiv y^2 - 4ax = 0$ at (x_1, y_1) is

$$S_1 \equiv yy_1 - 2a(x + x_1) = 0$$

Tangent (Parametric form): Equation of the tangent at $(at^2, 2at)$ is $ty = x + at^2$.

4.13 Point of intersection of tangents to $y^2 = 4ax$ at the points t_1 and t_2 is

$$[at_1t_2, a(t_1 + t_2)]$$

Note:

1. The tangents at t_1 and t_2 are at right angles $\Leftrightarrow t_1t_2 = -1$.
2. The locus of the point of intersection of perpendicular tangents is the directrix.

4.14. ***Theorem:*** The line $y = mx + c$ $(m \neq 0)$ touches the parabola

$$y^2 = 4ax \Leftrightarrow c = \frac{a}{m}$$

and in the case of tangency, the point of contact is

$$\left(\frac{a}{m^2}, \frac{2a}{m}\right)$$

4.15. ***Theorem (Normal):*** Equation of the normal at $(at^2, 2at)$ is $tx + y = 2at + at^3$. The normal in terms of its slope m is $y = mx - 2am - am^3$ at the point $(am^3, -2am)$.

4.16. ***Theorem:*** Point of intersection of normals at t_1 and t_2 is

$$[2a + a(t_1^2 + t_1t_2 + t_2^2), -at_1t_2(t_1 + t_2)]$$

Note: The normals at t_1 and t_2 are at right angles $\Leftrightarrow t_1t_2 = -1$.

4.17. **(1)** If the normal at t_1 meets the parabola again at t_2 then

$$t_2 = -t_1 - \frac{2}{t_1}$$

(2) If the normals at t_1 and t_2 intersect on the parabola, then $t_1t_2 = 2$ and in such a case product of the abscissa = $4a^2$ and product of the ordinates = $8a^2$.

4.18. ***Theorem (Number of normals):*** From any point in the plane of a parabola, in general, three normals can be drawn such that the algebraic sum of the ordinates of the feat of the normals is zero.

Note: If t_1, t_2 and t_3 are the parameters of the feet of normals drawn from the point (h, k), then t_1, t_2 and t_3 are the roots of the cubic equation

$$at^3 + (2a - h)t - k = 0$$

and hence

$$t_1 + t_2 + t_3 = 0$$

$$\sum t_1t_2 = \frac{2a - h}{a}$$

and $$t_1t_2t_3 = \frac{k}{a}$$

4.19. ***Definition (Conormal points):*** Points on a parabola are called conormal points if the nomals at those points are concurrent at a point.

4.20. ***Procedure to determine the number of normals from a given point:*** Let (h, k) be a given point in the plane of the parabola $y^2 = 4ax$. Let

$$G = \frac{-k}{a}, H = \frac{2a - h}{3a} \text{ and } \Delta = G^2 + 4H^3$$

Then we have

1. only one normal from (h, k) if $\Delta > 0$.
2. only two normals from (h, k) if $.\Delta = 0$.
3. three normals from (h, k) if $\Delta < 0$.

4.21. ***Definition (Focal chord and focal radius):*** If a chord of a parabola passes through the focus, then it is called the focal chord. If P is a point on a parabola having focus S, then SP is called focal radius of P with respect to the parabola.

4.22. ***Theorem (Properties of focal chords):*** Let PSQ be a focal chord of $y^2 = 4ax$. Then

(i) If $P = (at_1^2, 2at_1)$ and $Q = (at_2^2, 2at_2)$, then $t_2 = -1/t_1$. Equivalently, for all $t \neq 0$, $(at^2, 2at)$ and $(a/t^2, -2a/t)$ are the ends of a focal chord.

(ii) $$\frac{1}{SP} + \frac{1}{SQ} = \frac{1}{a} = \frac{2}{2a}$$

In fact, semi-latus rectum is HM between the focal radii of a focal chord.

(iii) If PQ is a focal chord and $P = (at^2, 2at)$, then the length PQ is equal to $a[t + (1/t)]^2$.

(iv) The tangents drawn at the extremities of focal chords intersect on the directrix and they are at right angles to each other.

(v) The circle described on a focal chord as diameter touches the directrix.

(vi) The circle described on a focal radius of a point as diameter touches the tangent at the vertex.

(vii) The circle described on a focal radius SP of a point P makes an intercept of length $a\sqrt{1+t^2}$ on the normal at $P(at^2, 2at)$.

4.23. ***Theorem:*** The orthocentre of a triangle formed by three tangents to a parabola lies on the directrix of the parabola. Also the circumcircle of a triangle formed by three tangents to a parabola passes through the focus.

4.24. ***Theorem:*** In general, parabola and a circle intersect in four points. If t_1, t_2, t_3 and t_4 are the points of intersection of the parabola $y^2 = 4ax$ and the circle $x^2 + y^2 + 2gx + 2fy + c = 0$, then t_1, t_2, t_3 and t_4 are roots of the equation.

$$a^2t^4 + 2a(g + 2a)t^2 + 4aft + c = 0$$

so that

$$\Sigma t_1 = 0$$

$$\Sigma t_1 t_2 = \frac{2(g+2a)}{a}$$

$$\Sigma t_1 t_2 t_3 = \frac{-4f}{a}$$

and $$t_1 t_2 t_3 t_4 = \frac{c}{a^2}$$

EXERCISES

Single Correct Choice Type Questions

1. The point of intersection of the tangents drawn at the ends of the latus rectum of the parabola $y^2 = 4x$ is

(A) $(-1, 1)$ (B) $(-1, -1)$
(C) $(-1, 2)$ (D) $(-1, 0)$

2. The equation of the directrix of the parabola $y^2 + 4y + 4x + 2 = 0$ is

(A) $x = -1$ (B) $x = 1$
(C) $x = -\dfrac{3}{2}$ (D) $x = \dfrac{3}{2}$

3. Two common tangents to the circle $x^2 + y^2 = 2a^2$ and the parabola $y^2 = 8ax$ are

(A) $x = \pm(y + 2a)$ (B) $y = \pm(x + 2a)$
(C) $x = \pm(y + a)$ (D) $y = \pm(x + a)$

4. The line $2bx + 3cy + 4d = 0$ passes through the points of intersection of the parabolas $x^2 = 4ay$ and $y^2 = 4ax$ $(a \neq 0)$. Then

(A) $a^2 + (3b - 2c)^2 = 0$ (B) $d^2 + (3b + 2c)^2 = 0$
(C) $d^2 + (2b - 3c)^2 = 0$ (D) $d^2 + (2b + 3c)^2 = 0$

5. Let P be the point $(1, 0)$ and Q be a variable point on the parabola $y^2 = 8x$. Then, the locus of the midpoint of the segment $\overline{PQ}$ is

(A) $y^2 - 4x + 2 = 0$ (B) $y^2 + 4x + 2 = 0$
(C) $x^2 + 4y + 2 = 0$ (D) $x^2 - 4y + 2 = 0$

6. The equation of a tangent to the parabola $y^2 = 8x$ is $y = x + 2$. The point on this line from which we can draw the other tangent to the parabola which is perpendicular to the given tangent is

(A) $(2, 4)$ (B) $(-2, 0)$
(C) $(-1, -1)$ (D) $(2, 0)$

7. The angle between the tangents drawn to the curve $y = x^2 - 5x + 6$ at the points $(2, 0)$ and $(3, 0)$ is

(A) π (B) $\dfrac{\pi}{2}$ (C) $\dfrac{\pi}{6}$ (D) $\dfrac{\pi}{4}$

8. A parabola has origin as its focus and the line $x = 2$ as the directrix. Then, the vertex of the parabola is at

(A) $(0, 2)$ (B) $(1, 0)$
(C) $(0, 1)$ (D) $(2, 0)$

9. The locus of the vertices of the family of parabolas

$$y = \frac{a^3x^2}{3} + \frac{a^2x}{2} - 2a$$

is

(A) $xy = \dfrac{3}{4}$ (B) $xy = \dfrac{35}{16}$
(C) $xy = \dfrac{105}{64}$ (D) $xy = \dfrac{64}{105}$

10. Suppose the normals at three distinct points on the parabola $y^2 = 4ax$ are concurrent at point (h, k). Then

(A) $0 < h < 1$ (B) $1 < h < 2$
(C) $h > 2$ (D) $h < -1$

11. If normals are drawn from the point $P(h, k)$ to the parabola $y^2 = 4ax$, then the sum of the intercepts which the normals cut off from the axis of the parabola is

(A) $h + a$ (B) $3(h + a)$
(C) $2(h + a)$ (D) 0

12. P, Q and R are the feet of the normals drawn to the parabola $(y - 3)^2 = 8(x - 2)$. Then, the circumcircle of ΔPQR passes through the point

(A) $(2, 3)$ (B) $(3, 2)$
(C) $(0, 3)$ (D) $(2, 0)$

13. If the normals at the extremities of the latus rectum of the parabola $y^2 = 4ax$ meet the parabola at N and N', then the length NN' is

(A) $10a$ (B) $20a$ (C) $4a$ (D) $12a$

14. A line having slope m, and passing through the focus of the parabola $y^2 = 4(x - 1)$ intersects the curve in two distinct points. Then

(A) $|m| > 1$

(B) $|m| < 1$

(C) m can be real number $\neq 0$

(D) m must be rational

15. The maximum value of a such that the circle $x^2 + y^2 = a^2$ completely lies within the parabola $y^2 = 4(x + 4)$ is

(A) $2\sqrt{3}$ (B) 4

(C) $4\sqrt{3}$ (D) $4\sqrt{6}$

16. The length of the shortest normal chord of the parabola $y^2 = 4ax$ is

(A) $9a$ (B) $a\sqrt{54}$

(C) $a\sqrt{3}$ (D) $6a$

17. Tangent and normal are drawn at the point $P(16, 16)$ to the parabola $y^2 = 16x$, which cut the axis of the parabola at A and B, respectively. Then, the line joining the point P and the circumcentre of ΔPAB makes the angle with the axis of the parabola whose value is

(A) $\text{Tan}^{-1} 2$ (B) $\text{Tan}^{-1} \dfrac{1}{2}$

(C) $\text{Tan}^{-1}\left(\dfrac{4}{3}\right)$ (D) $\text{Tan}^{-1}\left(\dfrac{3}{4}\right)$

18. The normals at the points (x_1, y_1) and (x_2, y_2) to the parabola $y^2 = 4x$ meet at a point on the curve. If $x_1 + x_2 = 4$, then the value of $|y_1 + y_2|$ is equal to

(A) 2 (B) $2\sqrt{2}$ (C) 4 (D) $4\sqrt{2}$

19. PQ and $P'Q'$ are normal chords of the parabola $y^2 = 4ax$. If the four points P, Q, P' and Q' are concyclic, then the tangents at Q and Q' intersect on the

(A) tangent at the vertex

(B) axis of the parabola

(C) directrix

(D) latus rectum

20. The locus of the centre of the circle which cuts the parabola $y^2 = 4x$ orthogonally at $(1, 2)$ is the

(A) line $y = x + 1$

(B) line $y = 2x + 1$

(C) parabola $y^2 = 2x$

(D) circle $(x-1)^2 + (y-2)^2 = 5$

21. Let P be a point on the curve $y^2 = 8ax$ and T be the foot of the perpendicular drawn from the focus onto the tangent at P. Then, the locus of the midpoint of TP is a parabola whose equation is

(A) $y^2 = 4ax$ (B) $y^2 = 9ax$

(C) $y^2 = 16ax$ (D) $y^2 = 12ax$

22. A tangent at $P(h, k)$ $(1 < h < 4)$ to the parabola $y^2 = 4ax$ meets the axis at T. The line PN is drawn perpendicular to the axis. Then, the maximum possible area of ΔPTN is

(A) 16 (B) 24 (C) 8 (D) 32

23. If $y + 3 = m(x + 2)$ touches the parabola $y^2 = 8x$, then m has two values m_1 and m_2 such that

(A) $m_1 + m_2 = 0$ (B) $m_1 m_2 = 2$

(C) $m_1 m_2 = -1$ (D) $m_1 + m_2 = 2$

24. $y^2 = 4ax$ is a variable parabola (i.e., a is a parameter) with focus S_a is such that for any point P on the parabola, the distance PS_a is equal to a constant k. Then, the locus of point P is

(A) $4x^2 - y^2 + 4kx = 0$ (B) $4x^2 + y^2 - 4kx = 0$

(C) $x^2 + y^2 - 4kx = 0$ (D) $x^2 + 2y^2 - 4kx = 0$

(***Hint:*** $SP = x + a$)

25. The line $lx + my + n = 0$ meets the parabola at points P and Q. The lines joining the points P and Q to the focus meet the parabola at P' and Q'. Then, the equation of the line $P'Q'$ is

(A) $nx - ly + ma = 0$ (B) $nx - my + la = 0$

(C) $nx + my + la = 0$ (D) $nx - my - la = 0$

26. If (x_1, y_1) and (x_2, y_2) are extremities of a local chord of the parabola $y^2 = 4x$, then $x_1x_2 + y_1 y_2$ is equal to

(A) $3a^2$ (B) $2a^2$ (C) $-2a^2$ (D) $-3a^2$

27. The locus of the foot of the perpendicular from the vertex onto a chord of the parabola $y^2 = 4x$ subtending an angle 45° at the vertex is the curve $(x^2 + y^2 - 4x)^2 = k(x^3 + xy^2 + y^2)$ where the value of k is

(A) 4 (B) 8 (C) 16 (D) 2

28. Slope of a chord PQ of the parabola $y^2 = 4ax$ is m (constant). Then, the normals at P and Q intersect on a normal to the parabola $y^2 = 4ax$ at a point whose coordinates are

(A) $\left(\dfrac{4a}{m^2}, \dfrac{-4a}{m}\right)$ (B) $\left(\dfrac{2a}{m^2}, \dfrac{-4a}{m}\right)$

(C) $\left(\dfrac{4a}{m^2}, \dfrac{-2a}{m}\right)$ (D) $\left(\dfrac{4a}{m^2}, \dfrac{-4a}{m}\right)$

29. The normals at P and Q on $y^2 = 4ax$ meet again on the parabola at R. Then, the locus of the orthocentre of ΔPQR is

(A) $y^2 = a(x + 6a)$ (B) $y^2 = a(x - 6a)$
(C) $y^2 = a(x + 4a)$ (D) $x^2 + y^2 = 2a^2$

30. If two different tangents of $y^2 = 4x$ are normals to the parabola $x^2 = 4ay$, then

(A) $|a| < \frac{1}{\sqrt{2}}$ (B) $|a| < \frac{1}{2\sqrt{2}}$
(C) $|a| > \frac{1}{\sqrt{2}}$ (D) $|a| > \frac{1}{2\sqrt{2}}$

31. The length of the normal chord of $y^2 = 4x$ which subtends a right angle at the vertex is

(A) $8\sqrt{2}$ (B) $8\sqrt{3}$ (C) $6\sqrt{3}$ (D) $4\sqrt{2}$

32. Consider the following two statements:

S_1: The curve $y = -x^2/2 + x + 1$ is symmetric about the line $x = 1$.
S_2: A parabola is always symmetric about its axis.
Then, which one of the following is true?

(A) Both S_1 and S_2 are true.
(B) Both S_1 and S_2 are false.
(C) S_1 is true and S_2 is false.
(D) S_1 is false and S_2 is true.

33. If the normals to $y^2 = 4ax$ at points P and Q intersect at a point $R(h, k)$ on the parabola, then the ordinates of the point P and Q are roots of the equation

(A) $y^2 + ky - 8a^2 = 0$ (B) $y^2 + ky + 8a^2 = 0$
(C) $y^2 - ky + 8a^2 = 0$ (D) $y^2 + ky + 4a^2 = 0$

34. The normal at point A on the parabola $y^2 = 4ax$ cuts the parabola again at point B. If the chord AB subtends a right angle at the vertex of the parabola, then the square of the slope of the chord AB is

(A) 1 (B) 2 (C) 8 (D) $\frac{1}{2}$

35. PQ is a variable focal chord of the parabola $y^2 = 8x$ and $O(0, 0)$ is its vertex. Then the locus of the centroid of ΔOPQ is a parabola whose latus rectum is

(A) $\frac{8}{3}$ (B) 3 (C) $\frac{4}{3}$ (D) $\frac{2}{3}$

36. Which of the following equations represent a common tangent to the parabolas $y^2 = 4ax$ and $x^2 = 32ay$?

(A) $x + 2y + 4a = 0$ (B) $x + 2y - 4a = 0$
(C) $x - 2y - 4a = 0$ (D) $x - 2y + 4a = 0$

37. Image of the directrix of the parabola $y^2 = 4(x + 1)$ in the line $x + 2y - 3 = 0$ is

(A) $3x + 4y - 16 = 0$ (B) $3x - 4y + 16 = 0$
(C) $x + 2 = 0$ (D) $x - 2 = 0$

38. Let PSP' be a focal chord, M be the foot of the perpendicular drawn from P onto the directrix of the parabola $y^2 = 4ax$ and R be the midpoint of SM. Then, the angle between PR and SM is

(A) 90° (B) 60° (C) 45° (D) 30°

39. PSQ is the focal chord of the parabola $y^2 = 4ax$, where S is the focus. If $SP = 4$, $SQ = 5$, then the latus rectum of the parabola is

(A) 80 (B) 9 (C) $\frac{9}{80}$ (D) $\frac{80}{9}$

40. The parabola $y = (a - b)x^2 + (b - c)x + (c - a) = 0$ is having the line $x - 1 = 0$ as a tangent. Then the equation $ax + by + c = 0$ represents

(A) a family of lines passing through a fixed point.
(B) a family of parallel lines.
(C) a family of lines concurrent on the directrix.
(D) a family of lines concurrent at the focus of the parabola.

Multiple Correct Choice Type Questions

1. PQ is a normal chord (normal at P) of the parabola $y^2 = 4x$ such that PQ subtends right angle at the vertex. Then the co-ordinates of P are

(A) $(2, 2\sqrt{2})$ (B) $(2, -2\sqrt{2})$
(C) $(3, 2\sqrt{3})$ (D) $(3, -2\sqrt{3})$

2. Consider the parabola $y^2 = 8x$. Then which of the following are true?

(A) The length of the focal chord having $(2, 4)$ as one extremity is 8.
(B) Common tangent to all the circles described on focal chords as diameters is $x + 2 = 0$.
(C) Focus of the parabola is $(2, 0)$.
(D) Directrix of the parabola is $x - 2 = 0$.

3. For the parabola having $(2, 0)$ as its vertex and y-axis as directrix, which of the following are true?

(A) Focus is $(4, 0)$ (B) Focus is $(-4, 0)$
(C) Focus $(8, 0)$ (D) Focus $(-8, 0)$

4. If the circle $x^2 + y^2 + 2bx = 0$ touches the parabola $y^2 = 4ax$, then

(A) $a > 0, b > 0$ (B) $a < 0, b < 0$
(C) $a > 0, b < 0$ (D) $a < 0, b > 0$

5. The straight line $x + y - 1 = 0$ touches the parabola

(A) $x^2 = -4y$

(B) $\left(x - \frac{1}{2}\right)^2 = -\left(y - \frac{1}{4}\right)$

(C) $(x-1)^2 = -2\left(y - \frac{1}{2}\right)$

(D) $4x^2 - 3x + y = 0$

6. PQ is a double ordinate of the parabola $y^2 = 4x$. If the normal at P meets the line passing through Q and parallel to the axis at G, then the locus of G is a parabola with

(A) Latus rectum 4
(B) Vertex at (4, 0)
(C) Directrix as $x - 3 = 0$
(D) Focus (5, 0)

7. If PQ is the shortest normal chord of the parabola $y^2 = 4ax$, then

(A) PQ makes angle $\text{Tan}^{-1}\sqrt{2}$ with the axis
(B) PQ makes angle $\text{Tan}^{-1} 2$ with the axis
(C) Length of the chord PQ is $6a\sqrt{3}$
(D) Length of the chord PQ is $4a\sqrt{2}$

8. The locus of the point of intersection of perpendicular normals to the parabola $y^2 = 4ax$ is another parabola with

(A) Vertex at $(2a, 0)$ (B) Vertex at $(3a, 0)$
(C) Latus rectum a (D) Directrix as $x = 3a$

9. PQ is a normal chord of $y^2 = 4ax$, normal at P. If the point P lies on the line $y = x$, then

(A) $P = (4a, 4a)$
(B) $Q = (9a, 6a)$
(C) $Q = (9a, -6a)$
(D) PQ subtends right angle at the focus

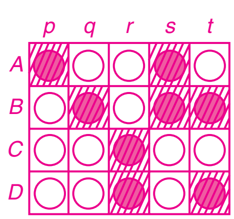
10. A square has one vertex at the vertex of the parabola $y^2 = 4ax$ and the diagonal through this vertex lies along the axis of the parabola. If the ends of the other diagonal lie on the parabola, then these two vertices are

(A) (0, 0) (B) $(8a, a)$
(C) $(4a, 4a)$ (D) $(4a, -4a)$

Matrix-Match Type Questions

In each of the following questions, statements are given in two columns, which have to be matched. The statements in *column I* are labeled as **(A)**, **(B)**, **(C)** and **(D)**, while those in *column II* are labeled as **(p)**, **(q)**, **(r)**, **(s)** and **(t)**. Any given statement in *column I* can have correct matching with *one or more* statements in *column II*. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following example.

Example: If the correct matches are (A) → (p), (s), (B) → (q), (s), (t), (C) → (r), (D) → (r), (t), that is if the matches are (A) → (p) and (s); (B) → (q), (s) and (t); (C) → (r); and (D) → (r), then the correct darkening of bubbles will look as follows:



1. PQ is a double ordinate of the parabola $y^2 = 4x$. If the normal at P meets the line passing through Q and parallel to axis at G, then the locus of G is a parabola. For this parabola, match the items of Column I with those of Column II.

Column I	*Column II*
(A) Length of the latus rectum of the locus of G	(p) 5
(B) Abscissa of the vertex	(q) 3
(C) Abscissa of the focus	(r) 4
(D) The directrix is $x = a$ where a is equal to	(s) 6
	(t) 2

2. The focus of the parabola $x^2 - \lambda y + 3 = 0$ is (0, 2). Match the items of Column I with those of Column II.

Column I	*Column II*
(A) The value of λ is	(p) 2
(B) Latus rectum of the parabola is	(q) 1

(Continued)

Column I	*Column II*
(C) If k is the ordinate of the vertex, then $2k$ is equal to	(r) 4
	(s) 3
(D) The directrix equation is $y = b$, where $3b$ is equal to	(t) 6

3. Suppose the line $y = 2$ is the directrix and the point (0, 1) is the vertex of the parabola $x^2 + by + c = 0$. Match the items of Column I with those of Column II.

Column I	*Column II*
(A) Value of b is	(p) 0
(B) Value of c is	(q) −4
(C) Length of the latus rectum is	(r) 2
(D) The ordinate of the focus is	(s) −2
	(t) 4

4. Consider the parabola $y^2 = 12x$ and match the items of Column I with those of Column II.

Column I	*Column II*
(A) LL' is the latus rectum of the parabola. The tangent and normal at L meet the axis at T and N, respectively. Then, the coordinates of the midpoint of TN is	(p) (2,0)
(B) All variable chords of the parabola subtending right angle at the vertex are concurrent at the point	(q) (2,1)
(C) If variable chords of the parabola pass through a fixed point Q on the axis such that the sum of the squares of the reciprocals of the two parts of chords through Q is constant then the coordinates of point Q are	(r) (12,0)
	(s) (6,0)
(D) Focus of the parabola is	(t) (3,0)

Comprehension Type Questions

1. Passage: Consider the parabola $y^2 = 8x$ and answer the following questions.

(i) TP and TQ are tangents to the parabola and the normals at points P and Q meet at a point R on the curve. Then, the circumcentre of ΔTPQ lies on the parabola whose equation is

(A) $y^2 = 2(x-2)$ (B) $y^2 = x-2$
(C) $y^2 = 4(x-2)$ (D) $y^2 = 2(x-4)$

(ii) If t_1 and t_2 are the parameters of points P and Q, then the sum of the products of their abscissae and ordinates is equal to

(A) 48 (B) 32 (C) 16 (D) 64

(iii) The abscissa of the point of intersection of the tangents at points P and Q is

(A) 2 (B) 6 (C) 8 (D) 4

2. Passage: Consider the parabola $y^2 = 4x$ and answer the following questions:

(i) PQ is a double ordinate of length 8 units. If O is the vertex of the parabola, then $\angle POQ$ is

(A) 45° (B) 65° (C) 90° (D) 105°

(ii) The length of the side of an equilateral triangle inscribed in the parabola with one vertex at $(0, 0)$ is

(A) $2\sqrt{2}$ (B) $2\sqrt{3}$ (C) $4\sqrt{2}$ (D) $8\sqrt{2}$

(iii) The radius of the maximum circle touching the parabola at the vertex which is having centre on the axis of the parabola and lies completely within the parabola is

(A) 2 (B) 4 (C) 4.5 (D) 2.5

3. Passage: Consider the parabola $y^2 = 16x$. Answer the following questions.

(i) Let $P(1, 4)$, $Q(1/4, 2)$ and $R(4, 8)$ be three points on a parabola. Then the area of the triangle formed by the tangents to the parabola at points P, Q and R is

(A) $\frac{3}{2}$ (B) $\frac{3}{4}$ (C) 1 (D) 2

(ii) The locus of the midpoints of chords of the parabola which subtend right angle at the vertex is

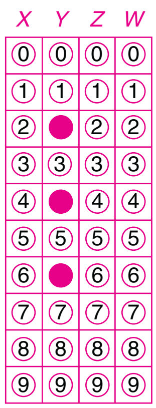
(A) $y^2 = 4(x-16)$ (B) $y^2 = 8(x-4)$
(C) $y^2 = 8(x-16)$ (D) $y^2 = 8(x-8)$

(iii) P is a point on a parabola, M is the foot of the perpendicular drawn from P onto the directrix and S is the focus. If ΔSPM is equilateral, then the focal radius SP is equal to

(A) 16 (B) 32 (C) 48 (D) 4

Integer Answer Type Questions

The answer to each of the questions in this section is a non-negative integer. The appropriate bubbles below the respective question numbers have to be darkened. For example, as shown in the figure, if the correct answer to the question number Y is 246, then the bubbles under Y labeled as 2, 4, 6 are to be darkened.



1. If the normal to a parabola at P meets the curve again at Q and if PQ and the normal at Q make angles α and β, respectively, with the axis, then $\tan^2\alpha + \tan\alpha\tan\beta + k = 0$ where k is equal to ______.

2. In the parabola $y^2 = 4ax$, the tangent at P whose abscissa is equal to the latus rectum meets the axis at T and the normal at P meets the curve again at Q. Then the ratio $PT:PQ = m:n$, where $m + n$ is equal to ______.

3. A chord PQ is normal to the parabola $y^2 = 4ax$ at P and subtends a right angle at the vertex. If S is the focus, then the ratio SQ/SP is ______.

4. PQ is a focal chord of a parabola $y^2 = 4ax$. The normals at P and Q meet the curve again at P' and Q', respectively. Then $P'Q'$ is parallel to PQ and k times PQ where k is equal to ______.

5. The length of the normal chord of $y^2 = 4ax$ which subtends a right angle at the focus is $ka\sqrt{5}$, where k is ______.

6. For the parabola $y^2 = 6x$, chords are drawn through the fixed point $(9, 5)$. Then the locus of the midpoints of these chords is a parabola whose latus rectum is ______.

7. QR is a chord of $y^2 = 4ax$ and is bisected by a diameter through a point P at V. Then

$$\frac{QV^2}{SP \cdot PV} = \_\_\_\_\_\_$$

where S is the focus.

8. The locus of the midpoints of system of parallel chords having slope m of the parabola $y^2 = 4ax$ is the straight line $y = ka/m$, where k is ______.

9. The number of normals to $y^2 = 4x$ drawn from the point $(1, 2)$ is ______.

10. The tangents to the parabola $y^2 = 4x$ at the points (1, 2) and (4, 4) meet on the line $y = k$, where k is ______.

ANSWERS

Single Correct Choice Type Questions

1. (D)
2. (D)
3. (B)
4. (D)
5. (A)
6. (B)
7. (B)
8. (B)
9. (C)
10. (C)
11. (C)
12. (A)
13. (D)
14. (C)
15. (A)
16. (C)
17. (C)
18. (D)
19. (B)
20. (A)

21. (B)
22. (A)
23. (C)
24. (B)
25. (B)
26. (D)
27. (C)
28. (A)
29. (A)
30. (A)
31. (C)
32. (A)
33. (B)
34. (B)
35. (A)
36. (A)
37. (B)
38. (A)
39. (D)
40. (A)

Multiple Correct Choice Type Questions

1. (A), (B)
2. (A), (B), (C)
3. (A), (B)
4. (A), (B)
5. (A), (B), (D)
6. (A), (B), (C), (D)
7. (A), (C)
8. (B), (C)
9. (A), (C), (D)
10. (C), (D)

Matrix-Match Type Questions

1. (A) → (r); (B) → (r); (C) → (p); (D) → (q)
2. (A) → (p), (t); (B) → (p); (C) → (s); (D) → (s)
3. (A) → (t); (B) → (q); (C) → (t); (D) → (p)
4. (A) → (t); (B) → (r); (C) → (s); (D) → (t)

Comprehension Type Questions

1. (i) (B); (ii) (A); (iii) (D)
2. (i) (C); (ii) (D); (iii) (A)
3. (i) (B); (ii) (C); (iii) (A)

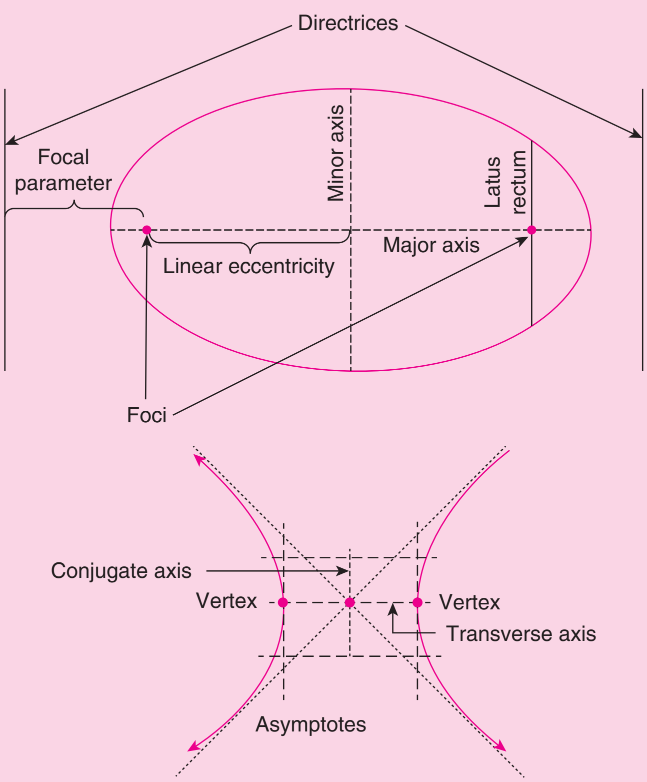
Integer Answer Type Questions

1. 2
2. 9
3. 3
4. 3
5. 5
6. 3
7. 4
8. 2
9. 1
10. 3

Ellipse and Hyperbola

5

Ellipse and Hyperbola



Contents

An **ellipse** is a section of a cone by a plane such that the plane parallel to it through the vertex meets the plane of the base in a line external to it.

A conic whose eccentricity is greater than unity is known as **hyperbola**.

In this chapter, we discuss about ellipse, hyperbola and their properties. 'Subjective Problems' sections provide worked-out subjective problems for the preceding sections. Students are advised to solve each and every problem to grasp the topics.

5.1 | Ellipse

An ellipse is a section of a cone by a plane such that the plane parallel to it through the vertex meets the plane of the base in a line external to it. Definition 5.1 is the plane geometrical definition.

DEFINITION 5.1 Let l be a line and S be a point on the plane of l, but not on l. Let $0 < e < 1$ be a fixed number. Then the locus of the point whose distance from the point S is e times the distance of the point from the line l is called *ellipse*. The point S is called the *focus* and l is called the *directrix* corresponding to S.

THEOREM 5.1 The standard equation of an ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

PROOF See Fig. 5.1. Let l be the directrix and S be the focus. Draw SZ perpendicular to l and divide $\overline{SZ}$ internally and externally at A and A' in the ratio $e:1$ (where $0 < e < 1$). Let C be the midpoint of $\overline{A'A}$ and $A'C = CA = a$. We consider $\overleftrightarrow{CS}$ as x-axis and line through C and perpendicular to $\overleftrightarrow{CS}$ as y-axis (see Fig. 5.1). We can see that $C = (0, 0)$, $A = (a, 0)$ and $A' = (-a, 0)$. Also $SA = e(AZ)$ and $SA' = eA'Z$. Therefore,

$$\begin{aligned}
&SA + SA' = e(AZ + A'Z)\\
&\Rightarrow A'A = e(CZ - CA + CZ + CA') = e(2CZ)\\
&\Rightarrow 2a = (2e)CZ\\
&\Rightarrow CZ = \frac{a}{e}
\end{aligned}$$

Hence, the equation of the directrix l is

$$x=\frac{a}{e} \tag{5.1}$$

Also

$$\begin{aligned}
&SA' - SA = e(A'Z - AZ)\\
&\Rightarrow (CS + CA') - (CA - CS) = e(A'A)\\
&\Rightarrow 2(CS) = e(2a)\\
&\Rightarrow CS = ea\\
&\Rightarrow S = (ae, 0)
\end{aligned}$$

So the focus is

$$S = (ae, 0) \tag{5.2}$$

Let $P(x, y)$ be a point on the ellipse. Draw PN perpendicular to x-axis and PM perpendicular to the directrix l. Hence, by definition,

$$\begin{aligned}
&SP = e(PN)\\
&\Rightarrow \sqrt{(x-ae)^2+y^2} = e\left|x-\frac{a}{e}\right|\\
&\Rightarrow (x-ae)^2+y^2 = (ex-a)^2\\
&\Rightarrow (1-e^2)x^2+y^2 = a^2(1-e^2)\\
&\Rightarrow \frac{x^2}{a^2}+\frac{y^2}{a^2(1-e^2)} = 1
\end{aligned}$$

Since $1 - e^2 > 0$, we write $a^2(1 - e^2)$ as b^2 where $b > 0$. Hence, the locus of P is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

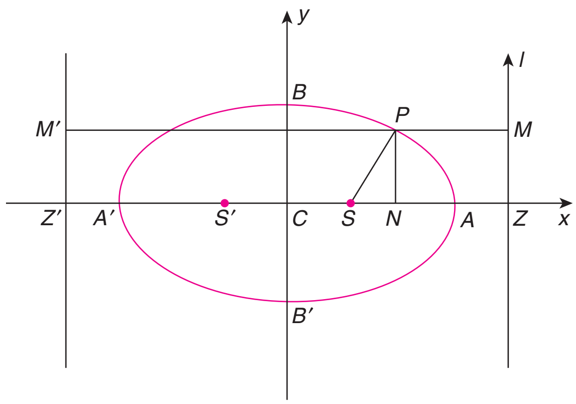


FIGURE 5.1

QUICK LOOK 1

The properties of the curve $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are as follows:

1. The curve is symmetric about both axes.
2. For any point (x, y) on the curve, we have $-a \le x \le a$ and $-b \le y \le b$.
3. The x-axis meets the curve at $A(a, 0)$ and $A'(-a, 0)$. The y-axis meets the curve at $B(0, b)$ and $B'(0, -b)$.
4. For each value of x,
$$y = \pm b\sqrt{1-\frac{x^2}{a^2}}$$
and for each value of y,
$$x = \pm a\sqrt{1-\frac{y^2}{b^2}}$$
5. $b < a$.
6. If $P(x, y)$ is a point on the ellipse
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}$$
then we have
$$SP = e(PM) = e(NZ) = e(CZ - CN)$$
$$= e\left(\frac{a}{e}-x\right) = a - ex$$
7. Since the curve is symmetric about both axes, there must be second focus and directrix. Another focus $S'(-ae, 0)$ and its corresponding directrix is
$$x=\frac{-a}{e}$$

DEFINITION 5.2 **Major Axis, Minor Axis and Vertices** In Fig. 5.1, AA' is called the *major axis* and BB' [where $B = (0, b)$ and $B' = (0, -b)$] is called the *minor axis*. The two points A and A' are called vertices of the ellipse. $2a$ is called the length of the major axis and $2b$ is called the length of the minor axis.

5.2 | Inverted Ellipse

If $b^2 > a^2$, then the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

is called an *inverted ellipse*. That is, if the roles of major and minor axes are interchanged, the ellipse is called an inverted ellipse. For the inverted ellipse, the vertices are $(0, b)$ and $(0, -b)$, major axis length is $2b$ and minor axis length is $2a$ (see Fig. 5.2). Foci are $(0, be)$ and $(0, -be)$ and directrices are

$$y = \pm\frac{b}{e}$$

The eccentricity is given by $a^2 = b^2(1 - e^2)$.

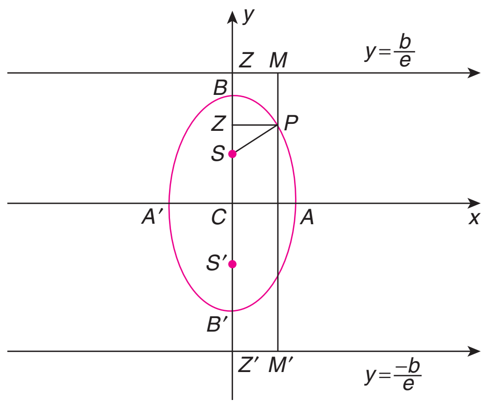


FIGURE 5.2

DEFINITION 5.3 **Double Ordinate and Latus Rectum** If a line perpendicular to major axis meets the curve at P and Q, then PQ is called *double ordinate*. Double ordinate through focus is called *latus rectum*.

THEOREM 5.2 The length of the latus rectum of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

is

$$\frac{2b^2}{a}$$

PROOF Let $L(ae, y)$ be one end of the latus rectum through the focus $(ae, 0)$. Hence

$$\frac{a^2e^2}{a^2}+\frac{y^2}{b^2}=1$$

so that

$$y^2 = b^2(1-e^2) = \frac{b^4}{a^2} \qquad [\because b^2 = a^2(1-e^2)]$$

Thus

$$y = \frac{b^2}{a}$$

and

$$L = \left(ae, \frac{b^2}{a}\right), L' = \left(ae, \frac{-b^2}{a}\right)$$

■

QUICK LOOK 2

If $b^2 > a^2$, then the length of the latus rectum of $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}$ is $\dfrac{2a^2}{b}$.

DEFINITION 5.4 **Centre** For any conic, point C is called its centre if C bisects every chord of the conic passing through it. For the *standard ellipse, $C(0, 0)$ is its centre.*

Example 5.1

Find the equation of the ellipse whose focus is at (4, –3), directrix is $x + 1 = 0$ and eccentricity is 2/3.

Solution: Let $P = (x, y)$. Now

$$SP = ePM$$

$$\Rightarrow (SP)^2 = \left(\frac{2}{3}\right)^2 (PM)^2$$

$$\Rightarrow (x-4)^2 + (y+3)^2 = \frac{4}{9}(x+1)^2$$

$$\Rightarrow 9(x-4)^2 + 9(y+3)^2 = 4(x+1)^2$$

$$\Rightarrow 5x^2 + 9y^2 - 80x + 54y = -221$$

$$\Rightarrow 5(x^2 - 16x + 64) + 9(y^2 + 6y + 9) = -221 + 320 + 81$$

$$\Rightarrow 5(x-8)^2 + 9(y+3)^2 = 180$$

$$\Rightarrow \frac{(x-8)^2}{36} + \frac{(y+3)^2}{20} = 1$$

Hence, the centre is (8, –3). Foci are

$$\left(x-8 = \pm 6\left(\frac{2}{3}\right),\ y+3=0\right) = (12,-3) \text{ and } (4,-3)$$

Directrices are

$$x - 8 = \pm\frac{a}{e} = \pm 9$$

$$\Rightarrow x = 17 \text{ and } x = -1$$

Examples 5.2

For the ellipse $9x^2 + 16y^2 = 576$, find the centre, foci, directrices and latus rectum.

Solution: The given equation can be written as

$$\frac{x^2}{64} + \frac{y^2}{36} = 1$$

Here, $a^2 = 64$, $b^2 = 36$. Now

$$36 = b^2 = a^2(1 - e^2) = 64(1 - e^2)$$

$$\Rightarrow e^2 = 1 - \frac{36}{64} \Rightarrow e = \frac{\sqrt{7}}{4}$$

Centre is (0, 0). Foci are

$$(\pm ae, 0) = \left(\pm 8\left(\frac{\sqrt{7}}{4}\right), 0\right) = (\pm 2\sqrt{7}, 0)$$

The directrices are

$$x = \pm\frac{a}{e} \Rightarrow x = \pm 8\left(\frac{4}{\sqrt{7}}\right) = \pm\frac{32}{\sqrt{7}}$$

The latus rectum is

$$\frac{2b^2}{a} = \frac{2(36)}{8} = 9$$

QUICK LOOK 3 (IMPORTANT OBSERVATION)

From the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

of the ellipse, we have the following observation: If the perpendicular distances p_1, p_2 of a moving point P from two perpendicular lines, say, l and l', respectively, are connected by the equation

$$\frac{p_1^2}{a^2}+\frac{p_2^2}{b^2}=1$$

where $a > b$, the locus of P is an ellipse with centre at the intersection of the lines l and l' and having major and minor axes along the lines l' and l, respectively, and they are of lengths $2a$ and $2b$, respectively.

The following example illustrates the concept explained in Quick Look 3.

Example 5.3

If the focus, centre and eccentricity of an ellipse are (2, 3), (3, 4) and 1/2, respectively, then find the equation of the ellipse.

Solution: It is known that the major axis is along the line joining the centre (2, 3) and the focus (3, 4). Its equation is

$$x - y + 1 = 0 = l' \quad \text{(say)}$$

The minor axis is the line through (2, 3) and perpendicular to the major axis. Its equation is

$$x + y - 5 = 0 = l \quad \text{(say)}$$

Now $C(2, 3)$ is the centre and $S(3, 4)$ is the focus. Therefore

$$CS = ae \Rightarrow CS = \frac{a}{2}$$

Hence

$$(3-2)^2+(4-3)^2=CS^2=\frac{a^2}{4}$$

$$\Rightarrow a=2\sqrt{2}$$

Now,

$$b^2=a^2(1-e^2)=8\left(1-\frac{1}{4}\right)=6 \Rightarrow b=\sqrt{6}$$

Let p_1 and p_2 be the distances of $P(x, y)$ from l' to l. Then

$$\frac{[(x-y+1)/\sqrt{2}]^2}{6}+\frac{[(x+y-5)/\sqrt{2}]^2}{8}=1$$

$$\Rightarrow \frac{(x-y+1)^2}{12}+\frac{(x+y-5)^2}{16}=1$$

Example 5.4

Find the centre and eccentricity of the ellipse

$$3(3x-2y+4)^2+2(2x+3y-5)^2=26$$

Solution: The given equation can be written as

$$\frac{(3x-2y+4)^2}{(26/3)}+\frac{(2x+3y-5)^2}{(26/2)}=1$$

Therefore, the centre of the ellipse is the point of intersection of the lines

$$3x - 2y + 4 = 0$$

and

$$2x + 3y - 5 = 0$$

which is

$$\left(\frac{-2}{13}, \frac{23}{13}\right)$$

Here,

$$b^2=\frac{26}{2}>a^2=\frac{26}{3}$$

so that

$$a^2=b^2(1-e^2) \Rightarrow \frac{1}{3}=\frac{1}{2}(1-e^2) \Rightarrow e=\frac{1}{\sqrt{3}}$$

DEFINITION 5.5 **Internal and External Points of Ellipse** Since an ellipse is a closed curve, any point on the plane of the ellipse belonging to the foci region is called *internal point*. Any point outside this is called an *external point*.

Notations Used:

$$S \equiv \frac{x^2}{a^2}+\frac{y^2}{b^2}-1$$

$$S_1 = \frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1$$

$$S_2 = \frac{xx_2}{a^2} + \frac{yy_2}{b^2} - 1$$

$$S_{12} = S_{21} = \frac{x_1x_2}{a^2} + \frac{y_1y_2}{b^2} - 1$$

$$S_{11} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1$$

QUICK LOOK 4

As in the case of circle and parabola, for the case of ellipse also, we have the following:

1. Point (x_1, y_1) is an external point to

$$S \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0 \Leftrightarrow S_{11} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1 > 0$$

2. Point (x_1, y_1) is an internal point

$$\Leftrightarrow S_{11} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1 < 0$$

DEFINITION 5.6 **Auxiliary Circle** The circle described on the major axis as diameter is called *auxiliary circle*. That is, for the ellipse,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where $a > b$, the circle $x^2 + y^2 = a^2$ is called the auxiliary circle (see Fig. 5.3).

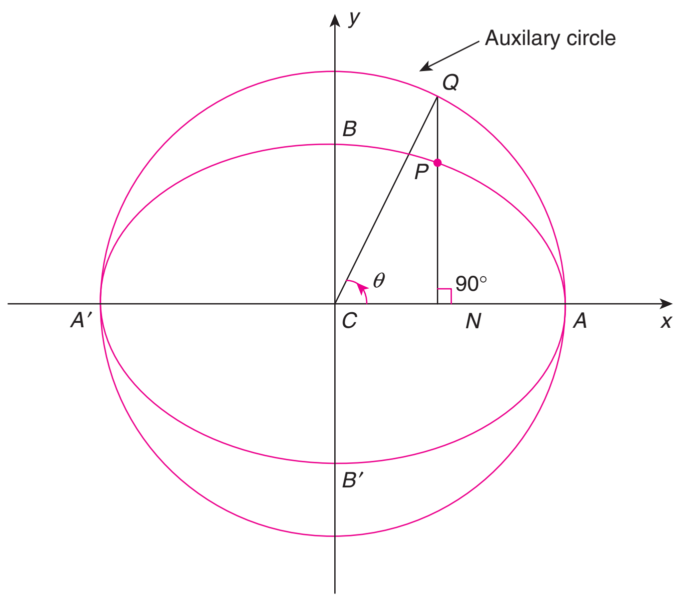


FIGURE 5.3

THEOREM 5.3 (PARAMETRIC EQUATIONS) Every point on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is represented by $(a \cos \theta, b \sin \theta)$, where θ is real.

PROOF Let $P(x, y)$ be a point on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Draw PN perpendicular to the major axis (i.e., x-axis) and produce NP to meet the auxiliary circle at Q (see Fig. 5.3). Let θ be $\angle ACQ$. Hence, by Theorem 3.4, Chapter 3, $Q = (a \cos \theta, a \sin \theta)$. Thus, $P = (a \cos \theta, y)$ and P lies on the curve given by

$$\frac{a^2\cos^2\theta}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow y=\pm b\sin\theta$$

Hence, $P = (a \cos \theta, b \sin \theta)$. Since the ellipse is symmetric about x-axis, $P'(a \cos \theta, -b \sin \theta)$ is also a point on the curve. Conversely, if $x = a \cos \theta$ and $y = b \sin \theta$, then

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

The equations $x = a \cos \theta$, $y = b \sin \theta$ are called the parametric equation of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

■

DEFINITION 5.7 **Eccentric Angle** The angle $\theta=\angle ACQ$ in Theorem 5.3 is called the eccentric angle of the point P where $P = (a \cos \theta, b \sin \theta)$.

Hereafter, when we say that θ is a point on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

it means that its coordinates are $(a \cos \theta, b \sin \theta)$ (we can assume that $0\le\theta\le 2\pi$).

THEOREM 5.4 (CHORD EQUATION)

1. The equation of the chord joining two points $A(x_1, y_1)$ and $B(x_2, y_2)$ on the ellipse

$$S\equiv\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$$

is $S_1+S_2=S_{12}$ and hence the equation of the tangent at (x_1, y_1) is

$$S_1\equiv\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1=0$$

2. The equation of the chord joining two points α and β on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

is given by

$$\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right)+\frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right)=\cos\left(\frac{\alpha-\beta}{2}\right)$$

and the equation of the tangent at α is given by

$$\frac{x}{a}\cos\alpha+\frac{y}{b}\sin\alpha=1$$

PROOF 1. Points $A(x_1, y_1)$ and $B(x_2, y_2)$ are on the ellipse. Therefore

$$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1=\frac{x_2^2}{a^2}+\frac{y_2^2}{b^2}$$

$$\Rightarrow \frac{x_1^2-x_2^2}{a^2}+\frac{y_1^2-y_2^2}{b^2}=0$$

$$\Rightarrow \frac{-b^2(x_1+x_2)}{a^2(y_1+y_2)}=\frac{y_1-y_2}{x_1-x_2}=\text{Slope of the chord } \overline{AB}$$

Therefore, the equation of the chord $\overline{AB}$ is

$$y-y_1=\frac{-b^2(x_1+x_2)}{a^2(y_1+y_2)}(x-x_1)$$

$$\Rightarrow b^2(x-x_1)(x_1+x_2)+a^2(y-y_1)(y_1+y_2)=0$$

$$\Rightarrow (b^2xx_1+a^2yy_1)+(b^2xx_2+a^2yy_2)-(b^2x_1x_2+a^2y_1y_2)-(b^2{x_1}^2+a^2{y_1}^2)=0$$

$$\Rightarrow\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}\right)+\left(\frac{xx_2}{a^2}+\frac{yy_2}{b^2}\right)-\left(\frac{x_1x_2}{a^2}+\frac{y_1y_2}{b^2}\right)-\left(\frac{{x_1}^2}{a^2}+\frac{{y_2}^2}{b^2}\right)=0$$

$$\Rightarrow\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1\right)+\left(\frac{xx_2}{a^2}+\frac{yy_2}{b^2}-1\right)-\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}\right)=\frac{x_1x_2}{a^2}+\frac{y_1y_2}{b^2}-1$$

$$\Rightarrow S_1+S_2-1=S_{12}-1$$

$$\Rightarrow S_1+S_2=S_{12}$$

When $x_2=x_1$ and $y_2=y_1$, the equation of the tangent at (x_1, y_1) is

$$S_1\equiv\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1=0$$

2. Let $P=(a\cos\alpha, b\sin\alpha)$ and $Q=(a\cos\beta, b\sin\beta)$. The equation of the chord PQ is

$$y-b\sin\alpha=\frac{b(\sin\alpha-\sin\beta)}{a(\cos\alpha-\cos\beta)}(x-a\cos\alpha)$$

$$=\frac{2b\cos[(\alpha+\beta)/2]\sin[(\alpha-\beta)/2]}{-2a\sin[(\alpha+\beta)/2]\sin[(\alpha-\beta)/2]}(x-a\cos\alpha)$$

$$=\frac{-b}{a}\times\frac{\cos[(\alpha+\beta)/2]}{\sin[(\alpha+\beta)/2]}(x-a\cos\alpha)$$

Therefore

$$xb\cos\left(\frac{\alpha+\beta}{2}\right)+ya\sin\left(\frac{\alpha+\beta}{2}\right)=ab\left[\cos\alpha\cos\left(\frac{\alpha+\beta}{2}\right)+\sin\alpha\sin\left(\frac{\alpha+\beta}{2}\right)\right]$$

$$=ab\cos\left(\frac{\alpha-\beta}{2}\right)$$

Dividing both sides by ab, we get

$$\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right)+\frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right)=\cos\left(\frac{\alpha-\beta}{2}\right)$$

If $\beta=\alpha$, then the equation of tangent at $(a\cos\alpha, a\sin\alpha)$ is

$$\frac{x}{a}\cos\alpha+\frac{y}{b}\sin\alpha=1$$

DEFINITION 5.8 **Normal** If P is any point on an ellipse, then the line passing through point P and perpendicular to the tangent at P to the ellipse is called *normal at point P*.

THEOREM 5.5 Let

$$S = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$$

be an ellipse. Then

1. The equation of the normal at $P(x_1, y_1)$ is

$$\frac{a^2(x - x_1)}{x_1} = \frac{b^2(y - y_1)}{y_1}$$

2. The equation of the normal at $P(a \cos \alpha, b \sin \alpha)$ is $ax \sec \alpha - by \operatorname{cosec} \alpha = a^2 - b^2$.

PROOF

1. From Theorem 5.4, the equation of the tangent at (x_1, y_1) is

$$S \equiv \frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1 = 0$$

Its slope is

$$\frac{-b^2 x_1}{a^2 y_1}$$

Hence, the equation of the normal at (x_1, y_1) is

$$y - y_1 = \frac{a^2 y_1}{b^2 x_1}(x - x_1)$$

$$\Rightarrow \frac{a^2(x - x_1)}{x_1} = \frac{b^2(y - y_1)}{y_1}$$

2. Since the tangent at $P(a \cos \alpha, b \sin \alpha)$ is

$$\frac{x}{a}\cos\alpha + \frac{y}{b}\sin\alpha = 1$$

its slope is

$$-\frac{b}{a}\cot\alpha$$

Hence, the equation of the normal at P is

$$y - b\sin\alpha = \frac{a\sin\alpha}{b\cos\alpha}(x - a\cos\alpha)$$

$$\Rightarrow \frac{ax}{\cos\alpha} - \frac{by}{\sin\alpha} = a^2 - b^2$$

$$\Rightarrow ax\sec\alpha - by\operatorname{cosec}\alpha = a^2 - b^2$$

THEOREM 5.6

1. The condition for the line $y = mx + c$ to touch the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is $c^2 = a^2m^2 + b^2$.

2. The condition for the line $x\cos\theta + y\sin\theta = p$ to touch the ellipse is

$$p = \sqrt{a^2\cos^2\theta + b^2\sin^2\theta}$$

PROOF

1. Suppose the line $y = mx + c$, $c \neq 0$ touches the ellipse at (x_1, y_1). However, the equation of the tangent at (x_1, y_1) (by Theorem 5.4) is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1 = 0$$

Since both equations represent the same line, we have

$$\frac{(x_1/a^2)}{m} = \frac{(y_1/b^2)}{-1} = \frac{-1}{c}$$

$$\Rightarrow x_1 = \frac{-a^2m}{c} \text{ and } y_1 = \frac{b^2}{c}$$

Since (x_1, y_1) lies on the curve, we have

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1$$

$$\Rightarrow \frac{a^4m^2}{c^2a^2} + \frac{b^4}{c^2b^2} = 1$$

$$\Rightarrow a^2m^2 + b^2 = c^2$$

So

$$(x_1, y_1) = \left(\frac{-a^2m}{c}, \frac{b^2}{c}\right)$$

Aliter: The abscissae of the points of intersection of the line and the ellipse are given by the quadratic equation

$$\frac{x^2}{a^2} + \frac{(mx+c)^2}{b^2} = 1$$

$$\Rightarrow (b^2 + a^2m^2)x^2 + 2a^2mcx + c^2a^2 - a^2b^2 = 0 \tag{5.3}$$

The line $y = mx + c$ touches the ellipse if and only if the roots of Eq. (5.3) are equal. That is if and only if the discriminant is zero.

$$\Rightarrow 4a^4m^2c^2 - 4a^2(c^2 - b^2)(b^2 + a^2m^2) = 0$$

$$\Rightarrow a^2m^2c^2 - (c^2 - b^2)(b^2 + a^2m^2) = 0$$

$$\Rightarrow -b^2c^2 + b^2(b^2 + a^2m^2) = 0$$

$$\Rightarrow c^2 = a^2m^2 + b^2$$

2. In the above condition, if we replace c with $p/\sin\theta$ and m with $-\cot\theta$, we have

$$\frac{p^2}{\sin^2\theta} = a^2\cot^2\theta + b^2 = \frac{a^2\cos^2\theta + b^2\sin^2\theta}{\sin^2\theta}$$

$$\Rightarrow p = \sqrt{a^2\cos^2\theta + b^2\sin^2\theta}$$

■

QUICK LOOK 5

1. For all values of m, the line $y = mx \pm \sqrt{a^2m^2 + b^2}$ touches the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

at $(-a^2m/c, b^2/c)$, where $c=\pm\sqrt{a^2m^2+b^2}$

2. For all values of θ, the line

$$x\cos\theta+y\sin\theta=\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}$$

touches the ellipse at

$$\left(\frac{a^2\cos\theta}{p}, \frac{b^2\sin\theta}{p}\right)$$

THEOREM 5.7 From an external point, two tangents can be drawn to an ellipse.

PROOF Let $P(x_1, y_1)$ be an external point to the ellipse

$$S\equiv\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$$

so that

$$S_{11}=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1>0 \tag{5.4}$$

From Quick Look 5, we have that the line $y=mx+\sqrt{a^2m^2+b^2}$ touches the ellipse. This line passes through the point (x_1, y_1)

$$\begin{aligned}&\Leftrightarrow y_1-mx_1=\sqrt{a^2m^2+b^2}\\&\Leftrightarrow (y_1-mx_1)^2=a^2m^2+b^2\\&\Leftrightarrow (a^2-x_1{}^2)m^2+2x_1y_1m+b^2-y_1{}^2=0\end{aligned} \tag{5.5}$$

This being a quadratic in m has two roots, say, m_1 and m_2, so that

$$m_1+m_2=\frac{-2x_1y_1}{a^2-x^2} \text{ and } m_1m_2=\frac{b^2-y^2}{a^2-x_1{}^2} \tag{5.6}$$

Correspondingly, there are two tangents through (x_1, y_1). You can see that the discriminant of Eq. (5.5) is

$$\begin{aligned}4x_1^2y_1^2-4(a^2-x_1^2)(b^2-y_1^2) &= 4[-a^2b^2+a^2y_1^2+b^2x_1^2]\\&=4a^2b^2\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1\right)\\&=4a^2b^2\,S_{11}>0 \qquad \text{[from Eq. (5.4)]}\end{aligned}$$

■

THEOREM 5.8 The locus of the point through which perpendicular tangents are drawn to the ellipse

$$S\equiv\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$$

is the circle $x^2+y^2=a^2+b^2$. This circle is called the *director circle of the ellipse*.

PROOF We have

The tangents are at right angles

$$\begin{aligned}&\Leftrightarrow -1=m_1m_2=\frac{b^2-y_1^2}{a^2-x_1^2} \qquad \text{[from Eq. (5.6) of Theorem 5.7]}\\&\Leftrightarrow -a^2+x_1^2=b^2-y_1^2\\&\Leftrightarrow x_1^2+y_1^2=a^2+b^2\end{aligned}$$

Therefore, the locus of (x_1, y_1) is $x^2 + y^2 = a^2 + b^2$.

Aliter: The perpendicular tangents to the ellipse are

$$y = mx + \sqrt{a^2m^2 + b^2} \Rightarrow y - mx = \sqrt{a^2m^2 + b^2}$$

and

$$y = \frac{-x}{m} + \sqrt{\frac{a^2}{m^2} + b^2} \Rightarrow my + x = \sqrt{a^2 + b^2m^2}$$

Squaring and adding on both sides, we have

$$(1 + m^2)(x^2 + y^2) = (a^2 + b^2)(1 + m^2)$$

Therefore, the locus of (x, y) is $x^2 + y^2 = a^2 + b^2$.

DEFINITION 5.9 **Director Circle** The locus of the point through which perpendicular tangents are drawn to an ellipse is a circle concentric with ellipse. This circle is called the *director circle* of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

QUICK LOOK 6

The equation of the director circle of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is given by

$$x^2 + y^2 = a^2 + b^2$$

Example 5.5

Find the director circle of the ellipse $9x^2 + 16y^2 = 144$.

Solution: The given ellipse is

$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$

Hence, the director circle equation is

$$x^2 + y^2 = 16 + 9 = 25$$

Example 5.6

1. Prove that the condition for the line $lx + my + n = 0$ may touch the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is that $a^2l^2 + b^2m^2 = n^2$.

2. If

$$\frac{a^2}{l^2} + \frac{b^2}{m^2} = \frac{(a^2 - b^2)^2}{n^2}$$

then prove that the line $lx + my = n$ is normal to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Solution:

1. The line $lx + my + n = 0$ can be written as

$$y = \left(\frac{-l}{m}\right)x + \left(\frac{-n}{m}\right)$$

where $m \neq 0$, and this line touches the ellipse. So from Theorem 5.6, we have

$$\frac{n^2}{m^2} = a^2\left(\frac{-l}{m}\right)^2 + b^2$$

$$\Rightarrow n^2 = a^2l^2 + b^2m^2$$

2. Suppose $lx + my = n$ is normal at $(a \cos \theta, b \sin \theta)$ at which the normal equation is $ax\sec\theta - by\operatorname{cosec}\theta = a^2 - b^2$. Therefore

$$\frac{a\sec\theta}{l} = \frac{-b\operatorname{cosec}\theta}{m} = \frac{a^2-b^2}{n}$$

$$\Rightarrow \frac{a}{l} = \left(\frac{a^2-b^2}{n}\right)\cos\theta \text{ and } \frac{-b}{m} = \left(\frac{a^2-b^2}{n}\right)\sin\theta$$

$$\Rightarrow \frac{a^2}{l^2} + \frac{b^2}{m^2} = \left(\frac{a^2-b^2}{n}\right)^2 (\cos^2\theta + \sin^2\theta) = \frac{(a^2-b^2)^2}{n^2}$$

THEOREM 5.9 The feet of the perpendiculars drawn from the foci onto a tangent of an ellipse lie on the auxiliary circle of the ellipse.

PROOF It is known that

$$y = mx + \sqrt{a^2m^2 + b^2}$$

is a tangent to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

The line perpendicular to the above tangent is of the form

$$y = \left(\frac{-1}{m}\right)x + c$$

This passes through the focus $S(ae, 0)$, as shown in Fig. 5.4, which implies that

$$0 = \frac{-ae}{m} + c \Rightarrow c = \frac{ae}{m}$$

Therefore, the line is

$$y = \frac{-x}{m} + \frac{ae}{m} \Rightarrow x + my = ae$$

So

$$(y - mx)^2 + (x + my)^2 = a^2m^2 + b^2 + a^2e^2$$

$$\Rightarrow (1 + m^2)(x^2 + y^2) = a^2m^2 + a^2 \qquad [\because b^2 = a^2(1 - e^2)]$$

$$\Rightarrow x^2 + y^2 = a^2$$

Therefore, the foot of the perpendicular from $S(ae, 0)$ onto a tangent lies on the auxiliary circle

$$x^2 + b^2 = a^2$$

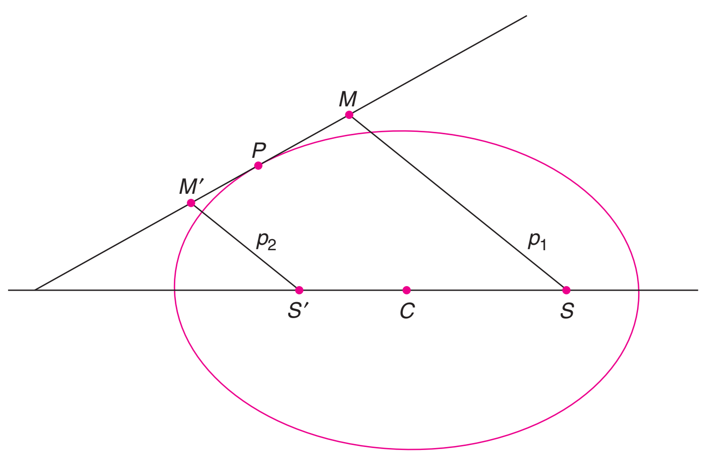


FIGURE 5.4 ■

THEOREM 5.10 The product of the perpendiculars drawn from the foci onto a tangent of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

is a constant and is equal to b^2.

PROOF See Fig. 5.4. Let $SM = p_1$ and $S'M' = p_2$ be the perpendiculars drawn from $S(ae, 0)$ and $S'(-ae, 0)$, respectively, onto a tangent

$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$$

Therefore

$$p_1=\frac{|1-e\cos\theta|ab}{\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}}$$

and
$$p_2=\frac{|1-e\cos\theta|ab}{\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}}$$

Now

$$\begin{aligned}
p_1p_2&=\frac{|1-e^2\cos^2\theta|a^2b^2}{a^2\sin^2\theta+b^2\cos^2\theta}\\
&=\frac{(1-e^2\cos^2\theta)a^2b^2}{a^2\sin^2\theta+a^2(1-e^2)\cos^2\theta} \qquad (\because e^2\cos^2\theta<1)\\
&=\frac{(1-e^2\cos^2\theta)b^2}{\sin^2\theta+(1-e^2)\cos^2\theta}\\
&=\frac{(1-e^2\cos^2\theta)b^2}{1-e^2\cos^2\theta} \qquad (\because \sin^2\theta+\cos^2\theta=1)\\
&=b^2
\end{aligned}$$

■

THEOREM 5.11 Except the point of contact, every point on a tangent line to an ellipse is an external point to the ellipse.

PROOF From Theorem 5.4, part (2), it is known that the equation of a tangent to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

at $(a\cos\theta, b\sin\theta)$ is

$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1 \tag{5.7}$$

Let (x_1, y_1) be any point on this tangent so that

$$\frac{x_1}{a}\cos\theta+\frac{y_1}{b}\sin\theta=1 \tag{5.8}$$

Now, from Eq. (5.8), we get

$$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-\left(\frac{x_1}{a}\cos\theta+\frac{y_1}{b}\sin\theta\right)^2$$

$$= \left(\frac{x_1}{a}\sin\theta - \frac{y_1}{b}\cos\theta \right)^2 \geq 0$$

and the equality holds if and only if

$$\frac{x_1}{a}\sin\theta - \frac{y_1}{b}\cos\theta = 0 \tag{5.9}$$

Now, from Eqs. (5.9) and Eq. (5.8), we have

$$x_1 = a\cos\theta$$

and $$y_1 = b\sin\theta$$

That is,

$$S_{11} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1 \geq 0$$

and the equality holds only when (x_1, y_1) is the point of contact. ■

Theorem 5.12 (Equation of a Chord in Terms of its Midpoint) If $M(x_1, y_1)$ is the midpoint of a chord of the ellipse

$$S \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$$

then the equation of the chord is

$$S_1 = S_{11}$$

$$\Rightarrow \frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1 = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1$$

Proof Let $A(x_2, y_2)$ and $B(x_3, y_3)$ be the extremities of the chord whose midpoint is $M(x_1, y_1)$ so that

$$x_2 + x_3 = 2x_1 \tag{5.10 a}$$

$$y_2 + y_3 = 2y_1 \tag{5.10 b}$$

Since $A(x_2, y_2)$ and $B(x_3, y_3)$ are the points on the curve, we have

$$\frac{x_2^2}{a^2} + \frac{y_2^2}{b^2} = 1 = \frac{x_3^2}{a^2} + \frac{y_3^2}{b^2}$$

$$\Rightarrow \frac{(x_2 - x_3)(x_2 + x_3)}{a^2} = \frac{(y_3 - y_2)(y_3 + y_2)}{b^2}$$

$$\Rightarrow \frac{-b^2(x_2 + x_3)}{a^2(y_2 + y_3)} = \frac{y_3 - y_2}{x_3 - x_2}$$

which is the slope of the chord AB. From Eqs. (5.10a) and (5.10b), we have

$$\frac{-b^2(2x_1)}{a^2(2y_1)} = \frac{y_3 - y_2}{x_3 - x_2}$$

which is the slope of the chord AB which is written as

$$\frac{-b^2 x_1}{a^2 y_1}$$

Therefore, the equation of the chord AB is

$$y-y_1=\frac{-b^2x_1}{a^2y_1}(x-x_1)$$
$$\Rightarrow \frac{xx_1}{a^2}+\frac{yy_1}{b^2}=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}$$
$$\Rightarrow \frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1$$
$$\Rightarrow S_1=S_{11}$$

■

THEOREM 5.13 The midpoints of parallel chords of an ellipse are collinear and this line passes through the center of the ellipse.

PROOF See Fig. 5.5. Let $M(x_1, y_1)$ be the midpoint of a chord of the ellipse having slope m. Since the equation of the chord is $S_1 = S_{11}$, we have

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}$$

The slope of this chord is

$$\frac{-b^2x_1}{a^2y_1}=m$$
$$\Rightarrow b^2x_1+a^2my_1=0$$

Therefore, $M(x_1, y_1)$ lies on the line $b^2x+a^2my=0$ which passes through the centre (0, 0) of the ellipse.

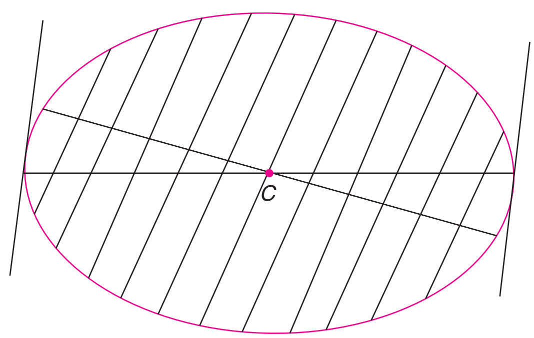


FIGURE 5.5

■

DEFINITION 5.10 The line of collinearity of the midpoints of parallel chords of an ellipse is called the *diameter of the ellipse*.

THEOREM 5.14 If one diameter of an ellipse bisects the chords parallel to another, then the second one bisects the chords parallel to the first. These types of diameters are called *conjugate diameters*.

PROOF From Theorem 5.13, the midpoints of chords parallel to $y = mx$ lies on the line $b^2x + a^2my = 0$. Hence, the midpoints of the chords parallel to $b^2x + a^2my = 0$ whose slope is

$$\frac{-b^2}{a^2m}$$

lie on the line

$$b^2x + a^2\left(\frac{-b^2}{a^2m}\right)y = 0$$

That is, the midpoints of the parallel chords lie on the line $y = mx$. ■

THEOREM 5.15 (PAIR OF TANGENTS) The combined equation of the pair of tangents drawn from an external point (x_1, y_1) to the ellipse

$$S \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$$

is

$$S_1^2 = SS_{11}$$

$$\Rightarrow \left(\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1\right)^2 = \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)\left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1\right)$$

PROOF See Fig. 5.6. Let $Q(x_2, y_2)$ be a point on one of the tangents from $P(x_1, y_1)$. Let R divide $\overline{PQ}$ in the ratio $1:\lambda$, where $\lambda + 1 \neq 0$, so that

$$R = \left(\frac{\lambda x_1 + x_2}{\lambda + 1}, \frac{\lambda y_1 + y_2}{\lambda + 1}\right)$$

Now

$$R \text{ lies on the ellipse} \Leftrightarrow \frac{[(\lambda x_1 + x_2)/(\lambda + 1)]^2}{a^2} + \frac{[(\lambda y_1 + y_2)/(\lambda + 1)]^2}{b^2} = 1$$

On simplification, we have

$$\lambda^2 S_{11} + 2\lambda S_{12} + S_{22} = 0 \tag{5.11}$$

$\overleftrightarrow{PQ}$ touches the ellipse if and only if Eq. (5.11) has equal roots

$$\Rightarrow 4S_{12}^2 - 4S_{11}S_{22} = 0$$
$$\Rightarrow S_{12}^2 = S_{11}S_{22}$$

Therefore, the locus of $Q(x_2, y_2)$ is $S_1^2 = S_{11}S$.

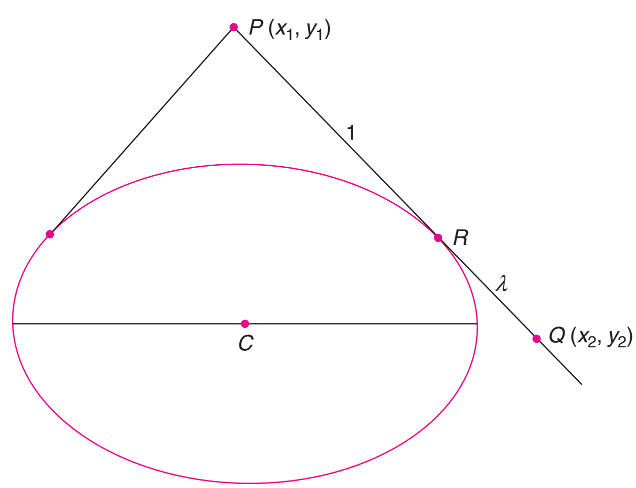


FIGURE 5.6 ■

DEFINITION 5.11 **Chord of Contact** If the tangents drawn from an external point P to an ellipse touch the curve at A and B, then the chord AB is called *chord of contact* of P with respect to the given ellipse.

QUICK LOOK 7

As in the case of circle and parabola, we can see that the equation of the chord of contact of $P(x_1, y_1)$ with respect to the ellipse

$$S \equiv \frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$$

is given by

$$S_1 \equiv \frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1=0$$

Example 5.7

Two points A and B are on the ellipse $2x^2+7y^2=14$ at which the tangents drawn to the ellipse intersect at P. If the equation of line AB is $5x + 7y = 7$, find the coordinates of P.

Solution: Suppose $P = (x_1, y_1)$. By hypothesis,

$$5x+7y=7 \tag{5.12}$$

is the chord of contact of P. By Quick Look 7, we have that

$$\frac{xx_1}{7}+\frac{yy_1}{2}=1 \tag{5.13}$$

is the chord of contact $P(x_1, y_1)$. Therefore, from Eqs. (5.12) and (5.13), we get

$$\frac{(x_1/7)}{5}=\frac{(y_1/2)}{7}=\frac{1}{7}$$

$$\Rightarrow x_1=5,\ y_1=2$$

$$\Rightarrow P=(5,2)$$

Example 5.8

Find the coordinates of the midpoint of the portion of the straight line $x + y = 2$ intercepted by the ellipse $3x^2 + 2y^2 = 6$.

Solution: Suppose $M(x_1, y_1)$ is the midpoint of the chord. Therefore, by Theorem 5.12, we have

$$\frac{xx_1}{2}+\frac{yy_1}{3}=\frac{x_1^2}{2}+\frac{y_1^2}{3}$$

However, $x+y=2$ is the equation of the chord. Therefore

$$\frac{(x_1/2)}{1}=\frac{(y_1/3)}{1}=\frac{(x_1^2/2)+(y_1^2/3)}{2}=\lambda \quad \text{(say)}$$

Therefore, $x_1=2\lambda,\ y_1=3\lambda$ and $3x_1^2+2y_1^2=12\lambda$ implies that

$$3(4\lambda^2)+2(9\lambda^2)=12\lambda$$

$$\Rightarrow (30)\lambda^2=12\lambda$$

$$\Rightarrow \lambda=0 \text{ or } \lambda=\frac{2}{5}$$

Now

$$\lambda \neq 0 \Rightarrow \lambda=\frac{2}{5}$$

Hence, the midpoint is

$$(x_1,y_1)=(2\lambda,3\lambda)=\left(\frac{4}{5},\frac{6}{5}\right)$$

THEOREM 5.16 Four normals can be drawn to an ellipse from a point on the plane of the ellipse. That is, if point P is located on the plane of an ellipse, there exist four points (not necessary all distinct) on the ellipse at which the normals drawn to the ellipse are concurrent at P.

PROOF Let

$$S \equiv \frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$$

be the ellipse and $P(h, k)$ be a point in the plane of the ellipse. We know that [from Theorem 5.5, part (2)], the normal to the ellipse at θ is given by

$$ax\sec\theta - by\operatorname{cosec}\theta = a^2 - b^2$$

$$\Rightarrow \frac{ax[1+\tan^2(\theta/2)]}{1-[\tan^2(\theta/2)]} - \frac{by[1+\tan^2(\theta/2)]}{2\tan(\theta/2)} = a^2 - b^2$$

$$\Rightarrow \frac{ax(1+t^2)}{1-t^2} - \frac{by(1+t^2)}{2t} = a^2 - b^2 \quad \left(\text{where } t = \tan\frac{\theta}{2}\right)$$

$$\Rightarrow ax(1+t^2)2t - by(1-t^4) = 2t(1-t^2)(a^2-b^2) = 2t(1-t^2)a^2e^2$$

$$\Rightarrow byt^4 + 2(ax + a^2e^2)t^3 + 2(ax - a^2e^2)t - by = 0$$

This normal passes through $P(h, k)$. This implies

$$bkt^4 + 2(ah + a^2e^2)t^3 + 2(ah - a^2e^2)t - bk = 0 \tag{5.14}$$

Since Eq. (5.14) is a fourth-degree equation in t, it has four roots t_1, t_2, t_3 and t_4 (not necessarily distinct) so that there exist four points on the ellipse (t_r, $r = 1, 2, 3, 4$) at which the normals drawn are concurrent at $P(h, k)$. Also, we have the set of equations

$$t_1 + t_2 + t_3 + t_4 = -\frac{2(ah + a^2e^2)}{bk}$$

$$\Sigma t_1 t_2 = 0$$

$$\Sigma t_1 t_2 t_3 = -\frac{2(ah - a^2e^2)}{bk}$$

and

$$t_1 t_2 t_3 t_4 = \frac{-bk}{bk} = -1$$

THEOREM 5.17 A circle and an ellipse intersect at four points (real or imaginary), which are not necessarily be distinct, such that the algebraic sum of the eccentric angles of the four points is an even multiple of π.

PROOF Let

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

be a circle. Every point on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is of the form $(a\cos\theta, b\sin\theta)$. Substituting $x = a\cos\theta$ and $y = b\sin\theta$ in the equation of the circle, we have

$$a^2\cos^2\theta + b^2\sin^2\theta + 2ga\cos\theta + 2fb\sin\theta + c = 0 \tag{5.15}$$

Let

$$t = \tan\frac{\theta}{2}$$

so that

$$\cos\theta = \frac{1-t^2}{1+t^2} \quad \text{and} \quad \sin\theta = \frac{2t}{1+t^2}$$

Substituting the values of $\cos\theta$ and $\sin\theta$ in Eq. (5.15) and simplifying, we obtain the following fourth-degree equation in t:

$$(a^2+c-2ag)t^4+4bft^3+2(c+2b^2-a^2)t^2+4bft+(c+a^2+2ag)=0$$

This equation being a fourth-degree equation has four roots, say,

$$t_1=\tan\frac{\alpha}{2},\ \ t_2=\tan\frac{\beta}{2},\ \ t_3=\tan\frac{\gamma}{2}\ \text{and}\ t_4=\tan\frac{\delta}{2}$$

Hence, we have

$$\Sigma\tan\frac{\alpha}{2}=t_1+t_2+t_3+t_4=\frac{-4bf}{a^2+c-2ag} \tag{5.16}$$

$$\Sigma\tan\frac{\alpha}{2}\tan\frac{\beta}{2}=\Sigma t_1t_2=\frac{2(c+2b^2-a^2)}{c+a^2-2ag} \tag{5.17}$$

$$\Sigma\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}=\Sigma t_1t_2t_3=\frac{-4bf}{c+a^2-2ag} \tag{5.18}$$

$$\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}\tan\frac{\delta}{2}=t_1t_2t_3t_4=\frac{c+a^2+2ag}{c+a^2-2ag} \tag{5.19}$$

Now from Eqs. (5.16) and (5.18), we have

$$\sum\tan\frac{\alpha}{2}=\sum\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}$$

However,

$$\tan\left(\frac{\alpha}{2}+\frac{\beta}{2}+\frac{\gamma}{2}+\frac{\delta}{2}\right)=\frac{\sum\tan(\alpha/2)-\sum\tan(\alpha/2)\tan(\beta/2)\tan(\gamma/2)}{1-\left[\sum\tan(\alpha/2)\tan(\beta/2)\right]+\tan(\alpha/2)\tan(\beta/2)\tan(\gamma/2)\tan(\delta/2)}=0$$

Therefore

$$\frac{\alpha}{2}+\frac{\beta}{2}+\frac{\gamma}{2}+\frac{\delta}{2}=n\pi$$

or

$$\alpha+\beta+\gamma+\delta=2n\pi$$

■

Subjective Problems (Sections 5.1 and 5.2)

1. Find the equation of the ellipse of eccentricity 2/3 whose focus is (3, −1) and whose directrix is the straight line $2x - y + 16 = 0$.

Solution: If $P(x, y)$ is a point on the ellipse, we have

$$\sqrt{(x-3)^2+(y+1)^2}=\left(\frac{2}{3}\right)\left|\frac{2x-y+16}{\sqrt{5}}\right|$$

$$\Rightarrow 45[x^2+y^2-6x+2y+10]=4[4x^2+y^2-4xy+64x-32y+256]$$

$$\Rightarrow 29x^2+16xy+41y^2-526x+218y-574=0$$

2. Find the eccentricity of the ellipse

$$\frac{(x-1)^2}{9}+\frac{(y-2)^2}{1}=4$$

Also find the foci.

Solution: The given ellipse can be written as

$$\frac{(x-1)^2}{36}+\frac{(y-2)^2}{4}=1$$

We have $a^2 = 36$ and $b^2 = 4$. Now

$$b^2 = a^2(1 - e^2)$$

$$\Rightarrow 4 = 36(1-e)^2$$

$$\Rightarrow e = \frac{2\sqrt{2}}{3}$$

Hence

$$S(ae, 0) = \left(x - 1 = \frac{6(2\sqrt{2})}{3}, y - 2 = 0\right) = (1 + 4\sqrt{2}, 2)$$

and $$S'(-ae, 0) = (1 - 4\sqrt{2}, 0)$$

3. An ellipse has its centre at $O(0, 0)$ and axes along OX and OY, respectively. If the ellipse passes through $P(2, 7)$ and $Q(4, 3)$, find the coordinates of foci.

Solution: Let the equation of the ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

It passes through (2, 7) and (4, 3). Therefore

$$\frac{4}{a^2} + \frac{49}{b^2} = 1 \qquad (5.20)$$

and $$\frac{16}{a^2} + \frac{9}{b^2} = 1 \qquad (5.21)$$

Solving Eqs. (5.20) and (5.21) for a^2 and b^2, we have

$$a^2 = \frac{187}{10} \quad \text{and} \quad b^2 = \frac{187}{3}$$

Hence, the ellipse equation is $10x^2 + 3y^2 = 187$ and the eccentricity e is given by

$$\frac{187}{10} = \frac{187}{3}(1 - e^2)$$

$$\Rightarrow e^2 = 1 - \frac{3}{10} = \frac{7}{10}$$

Therefore, foci are given by

$$\left(0, \pm \frac{\sqrt{187 \times 7}}{10}\right)$$

4. A bar of length 20 units moves with its ends on two fixed lines which are at right angles. Prove that a marked point on the bar describes an ellipse. If the point is at a distance of 8 units from one end, find the eccentricity.

Solution: See Fig. 5.7. Take OX and OY as coordinate axes and O as (0, 0). Let AB be the bar such that $AB = 20$. Let point P be marked on AB such that $AP = a$ and $PB = 20 - a$. Suppose $\angle OAB = \theta$ (see Fig. 5.7). We have $AP = 20 - a$ and $PB = a \Rightarrow x = a\cos\theta$ and $y = (20 - a)$ since $P = (x, y)$. This implies

$$\frac{x^2}{a^2} + \frac{y^2}{(20-a)^2} = 1$$

If $PB = 8$, then $a^2 = 144$ and $b^2 = 64$ so that

$$\frac{64}{144} = 1 - e^2$$

$$\Rightarrow e^2 = 1 - \frac{4}{9} = \frac{5}{9}$$

$$\Rightarrow e = \frac{\sqrt{5}}{3}$$

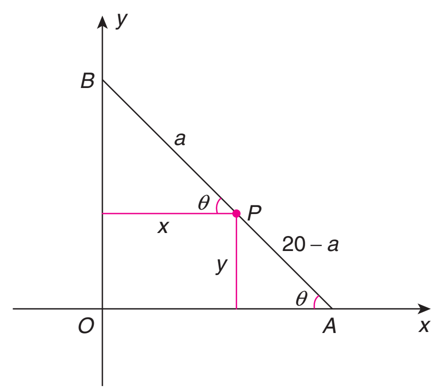


FIGURE 5.7

5. Let A and B be two fixed points and $2c$ be the distance between them. Then, show that the locus of the point P such that $PA + PB = 2a$, where $a > c$ is an ellipse having A and B as its foci.

Solution: Take the midpoint of $\overline{AB}$ as origin O at $\overrightarrow{OA}$ as positive x-axis. Since $AB = 2c$, we have $A = (c, 0)$ and $B = (-c, 0)$. Now, $P(x, y)$ is a point such that

$$PA + PB = 2a$$

$$\Leftrightarrow \sqrt{(x-c)^2 + y^2} + \sqrt{(x+c)^2 + y^2} = 2a$$

$$\Leftrightarrow (x+c)^2 + y^2 = 4a^2 + (x-c)^2 + y^2 - 4a\sqrt{(x-c)^2 + y^2}$$

$$\Leftrightarrow (cx - a^2)^2 = a^2[(x-c)^2 + y^2]$$

$$\Leftrightarrow x^2(a^2 - c^2) + y^2a^2 = a^4 - a^2c^2 = a^2(a^2 - c^2)$$

$$\Leftrightarrow \frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1 \qquad (a > c \Rightarrow a^2 - c^2 > 0)$$

This equation represents ellipse with major axis of length $2a$ and minor axis of length $2\sqrt{a^2 - c^2}$. Eccentricity

$$a^2 - c^2 = a^2(1 - e^2)$$

$$\Rightarrow e = \frac{c}{a}$$

6. For the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

let the vertices be $A(a, 0)$ and $A'(-a, 0)$. Suppose tangent at any point of the ellipse meets the tangents at the vertices at L and L', respectively. Then, show that the product $AL \cdot A'L'$ is constant and it is equal to b^2.

Proof: See Fig. 5.8. Tangent at θ is

$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$$

Tangents at A and A' are $x = a$ and $x = -a$, respectively, so that

$$AL\times A'L'=\frac{b(1-\cos\theta)}{\sin\theta}\times\frac{b(1+\cos\theta)}{\sin\theta}=b^2$$

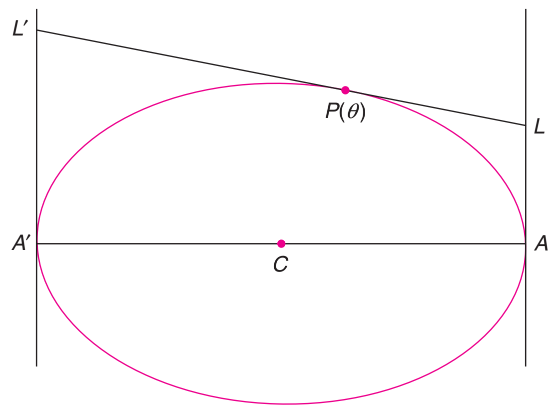


FIGURE 5.8

7. Point P is on the circle $x^2+y^2=a^2$. $\overline{AB}$ is the chord of contact of P with respect to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Then, show that the locus of the midpoint of $\overline{AB}$ is the curve

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2=\frac{x^2+y^2}{a^2}$$

Solution: Let $P(a\cos\theta, a\sin\theta)$ be a point on $x^2+y^2=a^2$. Then, the chord of contact $\overline{AB}$ of P with respect to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

by Quick Look 7, is

$$\frac{x(a\cos\theta)}{a^2}+\frac{y(a\sin\theta)}{b^2}=1$$

$$\Rightarrow \frac{x\cos\theta}{a}+\left(\frac{a}{b^2}\right)y\sin\theta=1 \tag{5.22}$$

Suppose $M(x_1, y_1)$ is the midpoint of $\overline{AB}$ so that the equation of chord $\overline{AB}$ is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2} \tag{5.23}$$

From Eqs. (5.22) and (5.23)

$$\frac{x_1/a^2}{(\cos\theta)/a}=\frac{y_1/b^2}{(a/b^2)\sin\theta}=\frac{(x_1^2/a^2)+(y_1^2/b^2)}{1}$$

$$\Rightarrow \frac{x_1}{a\cos\theta}=\frac{y_1}{a\sin\theta}=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}$$

$$\Rightarrow \frac{x_1^2+y_1^2}{a^2(\cos^2\theta+\sin^2\theta)}=\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}\right)^2$$

$$\Rightarrow \frac{x_1^2+y_1^2}{a^2}=\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}\right)^2$$

Therefore, the locus of $M(x_1, y_1)$ is

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)=\frac{x^2+y^2}{a^2}$$

8. Prove that the locus of the midpoint of the portion of the tangent to

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

lying between the axes is

$$\frac{a^2}{x^2}+\frac{b^2}{y^2}=2^2$$

Solution: The tangent at $(a\cos\theta, b\sin\theta)$ is

$$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$$

Therefore, $A(a\sec\theta, 0)$ and $B(0, b\operatorname{cosec}\theta)$ are the points of intersection of the tangent with the axes. If (x_1, y_1) is the midpoint of $\overline{AB}$, then

$$x_1=\frac{a\sec\theta}{2}\Rightarrow \cos\theta=\frac{a}{2x_1}$$

$$y_1=\frac{b\operatorname{cosec}\theta}{2}\Rightarrow \sin\theta=\frac{b}{2y_1}$$

Hence

$$\frac{a^2}{x_1^2}+\frac{b^2}{y_1^2}=4(\cos^2\theta+\sin^2\theta)=4$$

Thus, the locus of (x_1, y_1) is the curve

$$\frac{a^2}{x^2}+\frac{b^2}{y^2}=4=(2)^2$$

9. The normal to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

at point P meets the major axis at G. Find the locus of the midpoint of $\overline{PG}$.

Solution: Equation of the normal at $(a\cos\theta, b\sin\theta)$ is $ax\sec\theta - by\operatorname{cosec}\theta = a^2 - b^2$ [see Theorem 5.5, part (2)]. Substituting $y = 0$, we have

$$G=\left(\frac{(a^2-b^2)\cos\theta}{a}, 0\right)$$

Let $M(x_1, y_1)$ be the midpoint of $\overline{PG}$. Therefore

$$2x_1 = a\cos\theta+\frac{(a^2-b^2)\cos\theta}{a} \quad \text{and} \quad 2y_1 = b\sin\theta$$

$$\Rightarrow 2x_1 = \frac{\cos\theta}{a}(a^2+a^2e^2) \quad \text{and} \quad 2y_1 = b\sin\theta$$

$$\Rightarrow 2x_1 = a\cos\theta(1+e^2) \quad \text{and} \quad 2y_1 = b\sin\theta$$

Therefore,

$$\frac{4x_1^2}{a^2(1+e^2)^2}+\frac{4y_1^2}{b^2}=\cos^2\theta+\sin^2\theta=1$$

Hence, the locus of the midpoint of $\overline{PG}$ is the ellipse

$$\frac{4x^2}{a^2(1+e^2)^2}+\frac{4y^2}{b^2}=1$$

10. If the normal at one end of the latus rectum of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

passes through one end of the minor axis, then show that the eccentricity e satisfies the relation

$$e^4 + e^2 - 1 = 0$$

Solution: See Fig. 5.9. Let

$$L\left(ae, \frac{b^2}{a}\right)$$

be one end of the latus rectum through the focus $S(ae, 0)$. The normal at (x_1, y_1) [by Theorem 5.5, part (1)] is

$$\frac{a^2(x-x_1)}{x_1}=\frac{b^2(y-y_1)}{y_1}$$

Substituting $x_1 = ae$ and $y_1 = b^2/a$, the equation of the normal at $L(ae, b^2/c)$ is given by

$$\frac{a^2(x-ae)}{ae}=\frac{b^2[y-(b^2/a)]}{b^2/a}$$

$$\Rightarrow \frac{a}{e}(x-ae)=ay-b^2$$

Suppose this passes through $(0, -b)$. So we have

$$-a^2=-ab-b^2$$

$$\Rightarrow a^2-b^2=ab$$

$$\Rightarrow a^2e^2=a^2\sqrt{1-e^2} \qquad (\because b=a\sqrt{1-e^2})$$

$$\Rightarrow e^4=1-e^2$$

$$\Rightarrow e^4+e^2-1=0$$

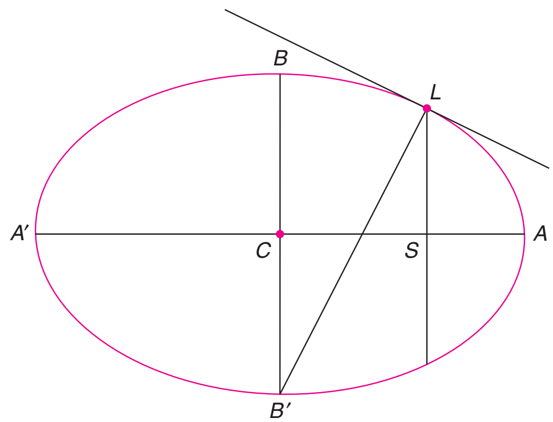


FIGURE 5.9

11. Show that the locus of the midpoints of chords of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

which pass through a fixed point (h, k) is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{hx}{a^2}+\frac{ky}{b^2}$$

Solution: If (x_1, y_1) is the midpoint of a chord, then its equation is

$$S_1 = S_{11} \qquad \text{(see Theorem 5.12)}$$

$$\Rightarrow \frac{xx_1}{a^2}+\frac{yy_1}{b^2}=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}$$

This passes through (h, k). This implies that

$$\frac{hx_1}{a^2}+\frac{ky_1}{b^2}=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}$$

Hence, the locus of (x_1, y_1) is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{hx}{a^2}+\frac{ky}{b^2}$$

12. The tangent to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

at point $P(a\cos\alpha, b\sin\alpha)$ meets the auxiliary circle $x^2+y^2=a^2$ in two points Q and R such that the chord $\overline{QR}$ subtends right angle at the centre of the ellipse. Show that the eccentricity of the ellipse is $(1+\sin^2\alpha)^{-1/2}$.

Solution: See Fig. 5.10. The tangent at $P(a\cos\alpha, b\sin\alpha)$ is

$$\frac{x\cos\alpha}{a}+\frac{y\sin\alpha}{b}=1$$

Therefore, the combined equation of the pair of lines CQ and CR (where C is the centre) is

$$x^2+y^2-a^2\left(\frac{x\cos\alpha}{a}+\frac{y\sin\alpha}{b}\right)^2=0$$

(see Theorem 2.33, Chapter 2). Now $\angle RCQ=90°$. This implies that in the above equation

Coefficient of x^2 + Coefficient of $y^2=0$

$$\Rightarrow (1-\cos^2\alpha)+\left(1-\frac{a^2\sin^2\alpha}{b^2}\right)=0$$

$$\Rightarrow \sin^2\alpha+1-\frac{a^2\sin^2\alpha}{a^2(1-e^2)}=0$$

$$\Rightarrow (1-e^2-1)\sin^2\alpha+1-e^2=0$$

$$\Rightarrow -e^2(1+\sin^2\alpha)+1=0$$

$$\Rightarrow e^2=\frac{1}{1+\sin^2\alpha} \text{ or } e=(1+\sin^2\alpha)^{-1/2}$$

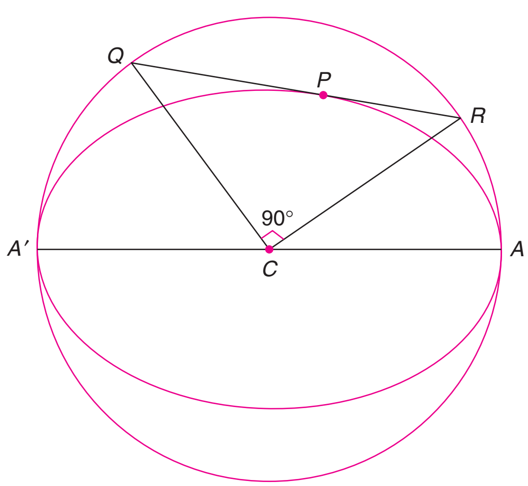


FIGURE 5.10

13. If P is a point on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

with foci S and S', then show that

$$\tan\frac{PSS'}{2}\times\tan\frac{PS'S}{2}=\frac{1-e}{1+e}$$

Solution: See Fig. 5.11. We know that $SP+S'P=2a$ and $S'S=2ae$. Therefore, the semi-perimeter (s) of $\Delta PSS'$ is

$$\frac{2a+2ae}{2}=a(1+e)$$

Now (see Theorem 4.6, Chapter 4, Vol. 2, p. 195), we have

$$\tan\frac{PSS'}{2}=\sqrt{\frac{(s-SP)(S-SS')}{s(S-S'P)}}$$

and
$$\tan\frac{PS'S}{2}=\sqrt{\frac{(s-S'P)(S-SS')}{s(S-SP)}}$$

Therefore

$$\tan\frac{PSS'}{2}\times\tan\frac{PS'S}{2}=\frac{s-SS'}{s}$$

$$=\frac{a+ae-2ae}{a+ae} \quad (\because s=a+ae)$$

$$=\frac{1-e}{1+e}$$

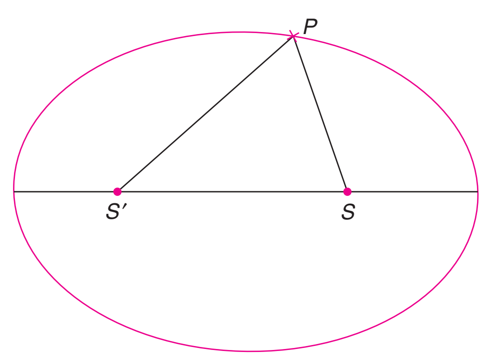


FIGURE 5.11

14. If a chord joining the points α and β on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

passes through a focus, then show that

$$\tan\frac{\alpha}{2}\tan\frac{\beta}{2}=\frac{e-1}{e+1}$$

Solution: The equation of the chord joining α and β [by Theorem 5.4, part (2)] is given by

$$\frac{x\cos[(\alpha+\beta)/2]}{a}+\frac{y\sin[(\alpha+\beta)/2]}{b}=\cos\left(\frac{\alpha-\beta}{2}\right)$$

Suppose, this chord passes through $(ae, 0)$. Therefore

$$e\cos\left(\frac{\alpha+\beta}{2}\right)=\cos\left(\frac{\alpha-\beta}{2}\right)$$

$$\Rightarrow \frac{\cos[(\alpha+\beta)/2]}{\cos[(\alpha-\beta)/2]}=\frac{1}{e}$$

$$\Rightarrow \frac{\cos[(\alpha+\beta)/2]+\cos[(\alpha-\beta)/2]}{\cos[(\alpha+\beta)/2]-\cos[(\alpha-\beta)/2]}=\frac{1+e}{1-e}$$

$$\Rightarrow \frac{2\cos(\alpha/2)\cos(\beta/2)}{-2\sin(\alpha/2)\sin(\beta/2)}=\frac{1+e}{1-e}$$

$$\Rightarrow \tan\frac{\alpha}{2}\tan\frac{\beta}{2}=\frac{e-1}{e+1}$$

15. Find the locus of the foot of the perpendicular drawn from centre of the ellipse onto any tangent to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Solution: The line

$$y=mx+\sqrt{a^2m^2+b^2} \tag{5.24}$$

is a tangent to the ellipse. The line passing through the centre (0, 0) and perpendicular to the line which is given in Eq. (5.24) is

$$x + my = 0 \tag{5.25}$$

From Eqs. (5.24) and (5.25), we have

$$y=x\left(\frac{-x}{y}\right)+\sqrt{\frac{a^2x^2}{y^2}+b^2}$$

$$\Rightarrow x^2+y^2=\sqrt{a^2x^2+b^2y^2}$$

$$\Rightarrow (x^2+y^2)^2=a^2x^2+b^2y^2$$

which is the required locus.

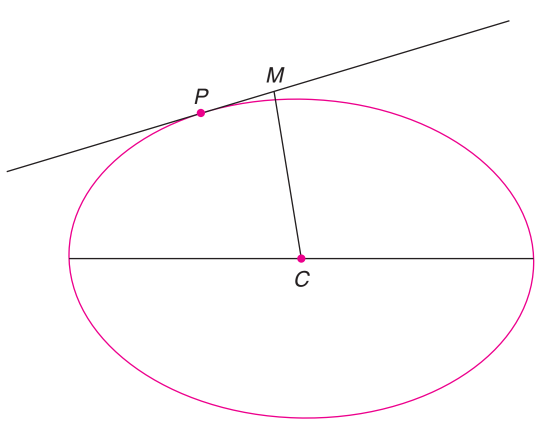


FIGURE 5.12

16. A tangent to an ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

is cut by the tangents at the vertices T and T'. Prove that the circle with T and T' as extremities of a diameter passes through the foci of the ellipse.

Solution: See Fig. 5.13. The tangent at $P(a\cos\theta, b\sin\theta)$ is

$$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$$

The tangents at $A(a, 0)$ and $A'(-a, 0)$ are $x = a$ and $x = -a$, respectively. Therefore

$$T=\left(a, \frac{b(1-\cos\theta)}{\sin\theta}\right)$$

and $$T'=\left(-a, \frac{b(1+\cos\theta)}{\sin\theta}\right)$$

Hence, the circle with T and T' as the ends of a diameter is

$$(x-a)(x+a)+\left(y-\frac{b(1-\cos\theta)}{\sin\theta}\right)\left(y-\frac{b(1+\cos\theta)}{\sin\theta}\right)$$

$$\Rightarrow x^2+y^2-\frac{by}{\sin\theta}(1+\cos\theta+1-\cos\theta)-a^2+\frac{b^2(1-\cos^2\theta)}{\sin^2\theta}=0$$

$$\Rightarrow x^2+y^2-\frac{2by}{\sin\theta}+b^2-a^2=0$$

which passes through $S(ae, 0)$ and $S'(-ae, 0)$.

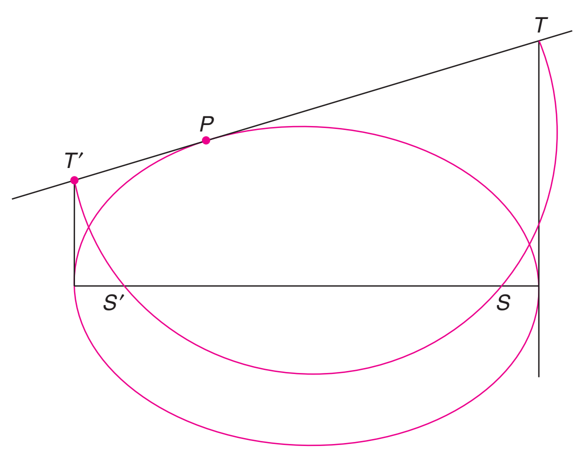


FIGURE 5.13

17. The tangent and the normal at point P on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{a^2}=1$$

meet the minor axis in T and G, respectively. Prove that $\overline{TG}$ subtends right angle at each of the foci.

Solution: See Fig. 5.14. Let coordinates of P be (x_1, y_1) so that

$$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1 \tag{5.26}$$

The tangent at $P(x_1, y_1)$ is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$$

so that

$$T=\left(0, \frac{b^2}{y_1}\right)$$

and the normal at P is

$$\frac{a^2(x-x_1)}{x_1}=\frac{b^2(y-y_1)}{y_1}$$

so that

$$G=\left(0, y_1-\frac{a^2 y_1}{b^2}\right) \quad \text{and} \quad S=(ae, 0)$$

Now,

$$\begin{aligned}\text{Slope of } ST \times \text{Slope of } SG &= \frac{(b^2/y_1)-0}{0-ae}\times\frac{y_1-(a^2y_1/b_2)}{0-ae}\\ &= \frac{b^2}{-aey_1}\times\frac{(b^2-a^2)y_1}{-aeb^2}\\ &= \frac{b^2-a^2}{a^2e^2}=-\frac{a^2e^2}{a^2e^2}=-1\end{aligned}$$

Hence, $\angle GST = 90°$.

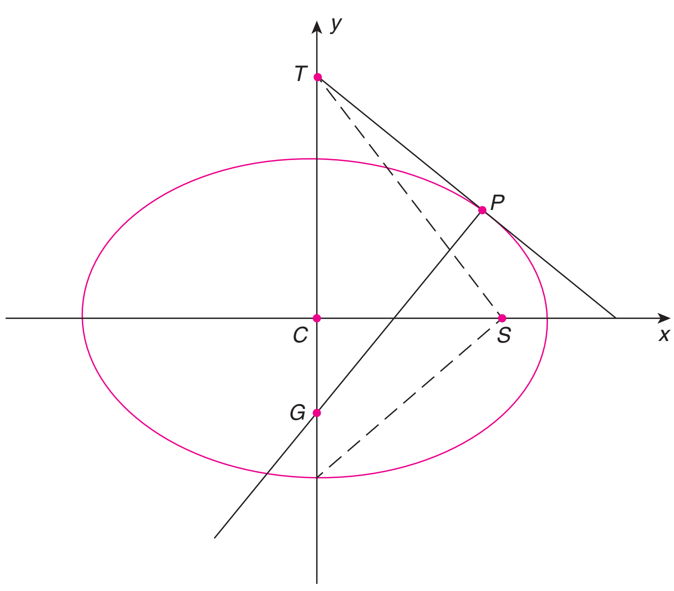


FIGURE 5.14

18. If the chord joining the points $P(a\cos\alpha, b\sin\alpha)$ and $Q(a\cos\beta, b\sin\beta)$ cuts the major axis of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

at a distance d units from the centre, then show that

$$\tan\frac{\alpha}{2}\tan\frac{\beta}{2}=\frac{d-a}{d+a}$$

Solution: The equation of the chord PQ is

$$\frac{x\cos(\alpha+\beta)/2}{a}+\frac{y\sin(\alpha+\beta)/2}{b}=\cos\left(\frac{\alpha-\beta}{2}\right)$$

The chord PQ meets the major axis at

$$\left(\frac{a\cos(\alpha-\beta)/2}{\cos(\alpha+\beta)/2}, 0\right)$$

By hypothesis, we have

$$\begin{aligned}&\frac{a\cos[(\alpha-\beta)/2]}{\cos[(\alpha+\beta)/2]}=d\\ \Rightarrow\ &\frac{\cos[(\alpha-\beta)/2]}{\cos[(\alpha+\beta)/2]}=\frac{d}{a}\\ \Rightarrow\ &\frac{\cos[(\alpha-\beta)/2]+\cos[(\alpha+\beta)/2]}{\cos[(\alpha-\beta)/2]-\cos[(\alpha+\beta)/2]}=\frac{d+a}{d-a}\\ \Rightarrow\ &\frac{2\cos(\alpha/2)\cos(\beta/2)}{2\sin(\alpha/2)\sin(\beta/2)}=\frac{d+a}{d-a}\\ \Rightarrow\ &\tan\frac{\alpha}{2}\tan\frac{\beta}{2}=\frac{d-a}{d+a}\end{aligned}$$

QUICK LOOK 8

If the chord passes through the focus $(ae, 0)$, then $d = ae$ so that (see Problem 14)

$$\tan\frac{\alpha}{2}\tan\frac{\beta}{2}=\frac{e-1}{e+1}$$

19. If the intercept of the line $x\cos\alpha + y\sin\alpha = p$ made by the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

subtends right angle at the centre, then show that the line touches the circle

$$x^2+y^2=\frac{a^2b^2}{a^2+b^2}$$

Solution: See Fig. 5.15. Let QR be the intercept of the line $x\cos\alpha + y\sin\alpha = p$ made by the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

The combined equation of the pair of lines CQ and CR (where C is the centre) is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\left(\frac{x\cos\alpha+y\sin\alpha}{p}\right)^2=0$$

Now $\angle PCQ=90°$. This implies that

Coefficient of x^2 + Coefficient of $y^2 = 0$

$$\Rightarrow\left(\frac{1}{a^2}-\frac{\cos^2\alpha}{p^2}\right)+\left(\frac{1}{b^2}-\frac{\sin^2\alpha}{p}\right)=0$$

$$\Rightarrow\frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{p^2}$$

$$\Rightarrow p^2=\frac{a^2b^2}{a^2+b^2}$$

However, p is the distance of (0, 0) from the line $x\cos\alpha + y\sin\alpha = p$. Hence, the line touches the circle

$$x^2+y^2=\frac{a^2b^2}{a^2+b^2}$$

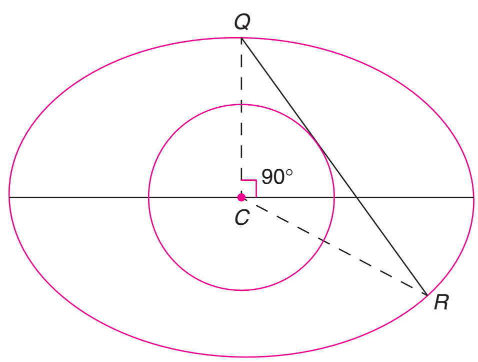


FIGURE 5.15

20. Let P be any point on the ellipse whose centre is $C(0, 0)$. If θ is the angle between the chord CP and the normal at P, then find $\tan\theta$ and show that as P moves on the curve, the maximum value of $\tan\theta$ is

$$\frac{a^2-b^2}{2ab}$$

Solution: Let $P = (a\cos\alpha, b\sin\alpha)$ so that the slope (m_1) of CP is

$$m_1=\frac{b\sin\alpha}{a\cos\alpha}=\frac{b}{a}\tan\alpha$$

The normal at P is

$$ax\sec\alpha - by\operatorname{cosec}\alpha = a^2-b^2$$

so that the slope (m_2) of the normal is

$$m_2=\frac{a}{b}\tan\alpha$$

Therefore

$$\begin{aligned}\tan\theta&=\left|\frac{m_1-m_2}{1+m_1m_2}\right|\\&=\frac{[(b/a)-(a/b)]\tan\alpha}{1+\tan^2\alpha}\\&=\left(\frac{a^2-b^2}{ab}\right)\sin\alpha\cos\alpha\\&=\left(\frac{a^2-b^2}{2ab}\right)\sin 2\alpha\\&\le\frac{a^2-b^2}{2ab}\end{aligned}$$

Hence, the maximum value of $\tan\theta$ is

$$\frac{a^2-b^2}{2ab}$$

and it is attained when

$$2\alpha=\frac{\pi}{2}\Rightarrow\alpha=\frac{\pi}{4}$$

21. Two points P and Q are corresponding points on an ellipse and its auxiliary circle. Prove that the normal to the ellipse at P and normal to the auxiliary circle at Q intersect on the circle

$$x^2+y^2=(a+b)^2$$

Solution: See Fig. 5.16. Let

$$P=(a\cos\alpha, b\sin\alpha)$$

and

$$Q=(a\cos\alpha, a\sin\alpha)$$

The normal at P to the ellipse is

$$ax\sec\alpha - by\operatorname{cosec}\alpha = a^2-b^2 \tag{5.27}$$

and the normal at Q to the circle $x^2+y^2=a^2$ is

$$y=(\tan\alpha)x \tag{5.28}$$

From Eq. (5.28), we have

$$\sin\alpha=\frac{y}{\sqrt{x^2+y^2}}\quad\text{and}\quad\cos\alpha=\frac{x}{\sqrt{x^2+y^2}}$$

Substituting these values of $\sin\alpha$ and $\cos\alpha$ in Eq. (5.27), we have

$$(ax)\frac{\sqrt{x^2+y^2}}{x}-(by)\frac{\sqrt{x^2+y^2}}{y}=a^2-b^2$$

$$\Rightarrow\sqrt{x^2+y^2}\,(a-b)=a^2-b^2 \text{ or } x^2+y^2=(a+b)^2$$

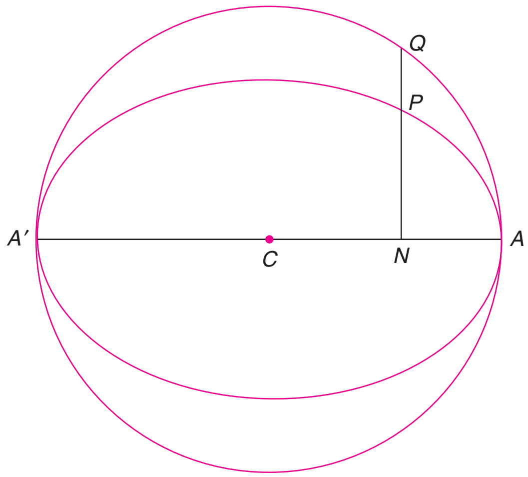


FIGURE 5.16

22. The tangent and the normal at point P of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

meet the major axis in T and T' such that $TT' = a$. Prove that the eccentric angle θ of the point P is given by

$$e^2\cos^2\theta + \cos\theta - 1 = 0$$

where e is the eccentricity.

Solution: See Fig. 5.17. Let $P = (a\cos\theta, b\sin\theta)$. The tangent at P is

$$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1 \tag{5.29}$$

Normal at P is

$$ax\sec\theta - by\operatorname{cosec}\theta = a^2-b^2 \tag{5.30}$$

Substituting $y = 0$ in Eqs. (5.29) and (5.30), we get

$$T=(a\sec\theta, 0) \text{ and } T'=\left(\frac{(a^2-b^2)}{a}\cos\theta, 0\right)$$

Now,

$$TT'=a$$

$$\Rightarrow \left|\frac{a}{\cos\theta}-\frac{(a^2-b^2)}{a}\cos\theta\right|=a$$

$$\Rightarrow a\left|\frac{1}{\cos\theta}-e^2\cos\theta\right|=a \qquad (\because a^2-b^2=a^2e^2)$$

$$\Rightarrow |1-e^2\cos^2\theta|=|\cos\theta|$$

$$\Rightarrow 1-e^2\cos^2\theta=\cos\theta \qquad (\because 1\ge e^2\cos^2\theta)$$

$$\Rightarrow e^2\cos^2\theta+\cos\theta-1=0$$

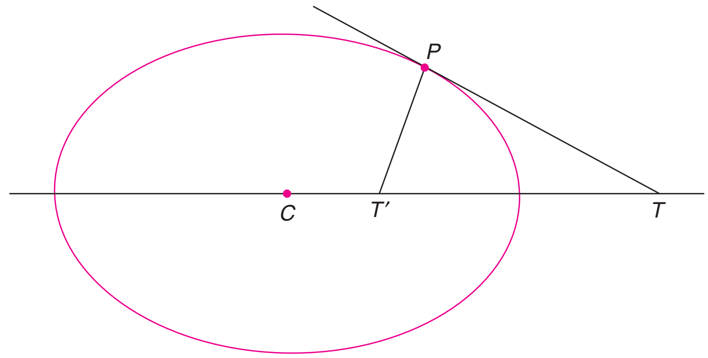


FIGURE 5.17

23. Let P be a point on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

and A and A' be the vertices of the ellipse. If the lines $A'P$ and AP meet a directrix at Q and R, respectively, then prove that QR subtends right angle at the corresponding focus.

Solution: See Fig. 5.18. We have $P = (a\cos\theta, b\sin\theta)$, $A' = (-a, 0)$ and $A = (a, 0)$. The equation of $A'P$ is

$$y=\frac{b\sin\theta}{a(1+\cos\theta)}(x+a)$$

$$\Rightarrow ay=\left(b\tan\frac{\theta}{2}\right)x+\left((ab)\tan\frac{\theta}{2}\right)$$

Substituting $x=a/e$ in the above equation, we get

$$Q=\left(\frac{a}{e},\frac{b(1+e)}{e}\tan\frac{\theta}{2}\right)$$

The equation of AP is

$$y=\frac{b\sin\theta}{a(\cos\theta-1)}(x-a)$$

In this equation, substituting $x=a/e$, we get

$$R=\left[\frac{a}{e},\frac{b}{e}(e-1)\cot\frac{\theta}{2}\right]$$

Now, the slope of SQ is given by

$$m_1=\frac{(b/e)(1+e)\tan(\theta/2)-0}{(a/e)-ae}=\frac{b(1+e)\tan(\theta/2)}{a(1-e^2)}$$

$$\Rightarrow m_1=\frac{b\tan(\theta/2)}{a(1-e)} \tag{5.31}$$

and the slope of SR is given by

$$m_2=\frac{(b/e)(1-e)\cot(\theta/2)}{(a/e)-ae}=\frac{b(e-1)\cot(\theta/2)}{a(1-e^2)}$$

$$\Rightarrow m_2 = \frac{-b\cot(\theta/2)}{a(1+e)} \tag{5.32}$$

From Eqs. (5.31) and (5.32), we get

$$m_1 m_2 = \frac{b\tan\theta/2}{a(1-e)} \times \frac{-b\cot\theta/2}{a(1+e)}$$
$$= \frac{-b^2}{a^2} \times \frac{1}{(1-e^2)}$$
$$= \frac{-b^2}{b^2} = -1$$

Hence, $\angle RSQ = 90°$.

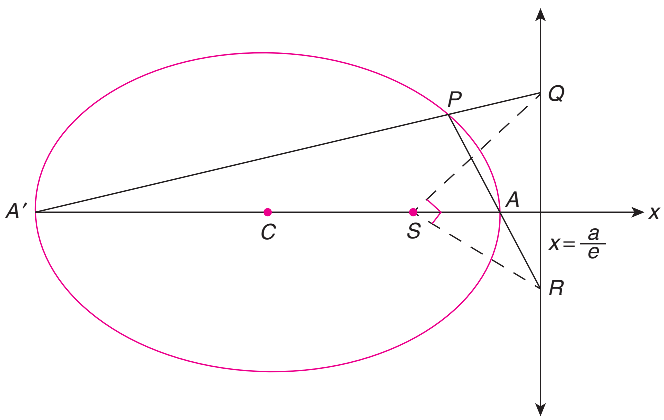


FIGURE 5.18

24. P is a point on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

and Q is the corresponding point on the auxiliary circle

$$x^2+y^2=a^2$$

If the tangent at P meets the major axis at T, then show that QT touches the auxiliary circle.

Solution: See Fig. 5.19. $P = (a\cos\theta, b\sin\theta)$ and $Q = (a\cos\theta, a\sin\theta)$. The tangent at P to the ellipse is

$$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$$

Therefore, $T = (a\sec\theta, 0)$ and the equation of the line TQ is

$$y = \frac{a\sin\theta - 0}{a\cos\theta - a\sec\theta}(x - a\sec\theta)$$
$$= \frac{\sin\theta\cos\theta}{(\cos^2\theta - 1)} \frac{(x\cos\theta - a)}{\cos\theta}$$
$$= \frac{-(x\cos\theta - a)}{\sin\theta}$$
$$= -x\cot\theta + a\,\mathrm{cosec}\,\theta$$

This line touches the auxiliary circle $x^2+y^2=a^2$. This implies

$$a^2\mathrm{cosec}^2\,\theta = a^2(1+\cot^2\theta) = a^2\,\mathrm{cosec}^2\theta$$

which is true. Hence, TQ touches the auxiliary circle.

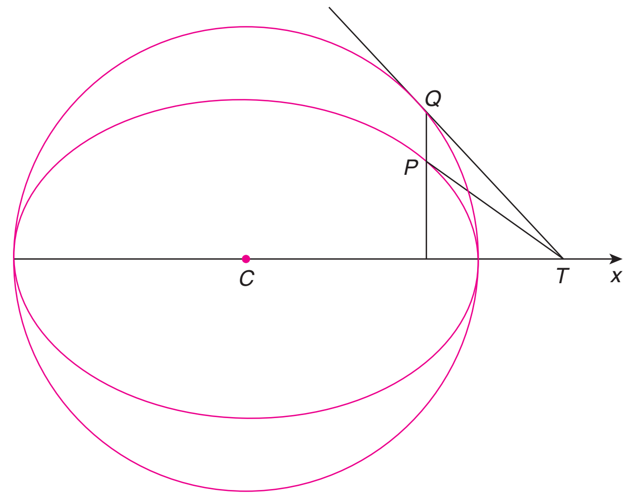


FIGURE 5.19

25. If the tangent at point P on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

meets a directrix at Q, then prove that $\overline{PQ}$ subtends right angle at the corresponding focus.

Solution: See Fig. 5.20. The tangent at $P(a\cos\theta, b\sin\theta)$ is

$$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$$

Substituting $x = a/e$, we have

$$Q = \left(\frac{a}{e}, \frac{b}{e}\frac{(e-\cos\theta)}{\sin\theta}\right)$$

Now

$$(\text{Slope of } SP)\times(\text{Slope of } SQ) = \left(\frac{b\sin\theta}{a(\cos\theta - e)}\right) \times \frac{[b(e-\cos\theta)/e\sin\theta]}{(a/e)-ae}$$
$$= \frac{b\sin\theta}{a(\cos\theta-e)} \times \frac{b(e-\cos\theta)e}{ae(1-e^2)\sin\theta}$$
$$= \frac{-b^2}{a^2(1-e^2)} = \frac{-b^2}{b^2} = -1$$

Therefore, $\angle QSP = 90°$.

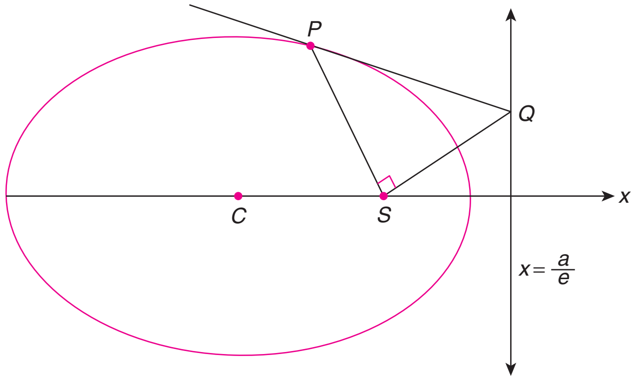


FIGURE 5.20

26. Prove that the locus of the midpoints of the chords of contact of points on the director circle of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

with respect to the ellipse is

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2(a^2+b^2)=x^2+y^2$$

Solution: $M(x_1, y_1)$ be the midpoint of the chord of contact of a point $P(h, k)$ on the director circle $x^2+y^2=a^2+b^2$. Therefore

$$h^2+k^2=a^2+b^2 \tag{5.33}$$

and
$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2} \tag{5.34}$$

Also the chord of contact of $P(h, k)$ with respect to the ellipse is

$$\frac{hx}{a^2}+\frac{ky}{b^2}=1 \tag{5.35}$$

From Eqs. (5.34) and (5.35), we get

$$\frac{(h/a^2)}{(x_1/a^2)}=\frac{(k/b^2)}{(y_1/b^2)}=\frac{1}{(x_1^2/a^2)+(y_1^2/b^2)}$$

$$\Rightarrow \frac{h}{x_1}=\frac{k}{y_1}=\frac{1}{(x_1^2/a^2)+(y_1^2/b^2)}$$

$$\Rightarrow h=\frac{x_1}{(x_1^2/a^2)+(y_1^2/b^2)},\quad k=\frac{y_1}{(x_1^2/a^2)+(y_1^2/b^2)}$$

Substituting the values of h and k in Eq. (5.33), we have

$$\frac{x_1^2}{[(x_1^2/a^2)+(y_1^2/b^2)]^2}+\frac{y_1^2}{[(x_1^2/a^2)+(y_1^2/b^2)]^2}=a^2+b^2$$

Therefore, the locus of (x_1, y_1) is

$$x^2+y^2=\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2(a^2+b^2)$$

27. Show that the locus of the midpoints of the chords of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

which subtend right angle at the centre of the ellipse is

$$\frac{a^2+b^2}{a^2b^2}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2=\frac{x^2}{a^4}+\frac{y^2}{b^4}$$

Solution: See Fig. 5.21. Let PQ be a chord whose midpoint is (x_1, y_1) which is subtending right angle at the centre. Hence, the equation of the chord PQ is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}$$

Now, the combined equation of the pair of lines CP and CQ is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\left[\frac{(xx_1/a^2)+(yy_1/b^2)}{(x_1^2/a^2)+(y_1^2/b^2)}\right]^2=0$$

Now $\angle PCQ=90°$ which implies that in the above equation

Coefficient of x^2 + Coefficient of $y^2 = 0$

$$\Rightarrow \frac{1}{a^2}-\frac{(x_1^2/a^4)}{[(x_1^2/a^2)+(y_1^2/b^2)]^2}+\frac{1}{b^2}-\frac{(y_1^2/b^4)}{[(x_1^2/a^2)+(y_1^2/b^2)]^2}=0$$

$$\Rightarrow \frac{1}{a^2}+\frac{1}{b^2}=\frac{(x_1^2/a^4)+(y_1^2/b^4)}{[(x_1^2/a^2)+(y_1^2/b^2))]^2}$$

Hence, the locus of (x_1, y_1) is

$$\frac{x^2}{a^4}+\frac{y^2}{b^4}=\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2$$

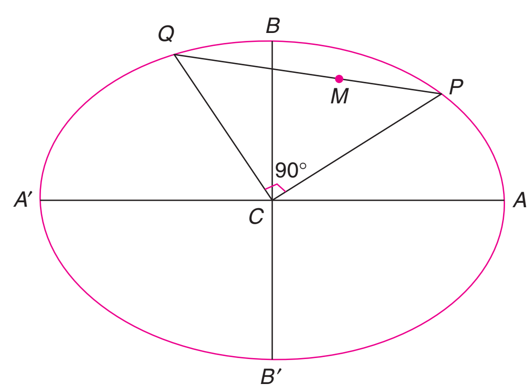


FIGURE 5.21

28. Two tangents are drawn from a point on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

to the circle $x^2+y^2=c^2$. Prove that the chord of contact touches the ellipse

$$a^2x^2+b^2y^2=c^4$$

Solution: See Fig. 5.22. Let $P\ (h, k)$ be a point on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

so that

$$\frac{h^2}{a^2}+\frac{k^2}{b^2}=1 \tag{5.36}$$

Chord of contact of $P(h, k)$ with respect to the circle $x^2+y^2=c^2$ is

$$hx+ky=c^2 \tag{5.37}$$

From Eq. (5.37), we have

$$y=\left(\frac{-h}{k}\right)+\frac{c^2}{k}$$

which touches the ellipse

$$a^2x^2+b^2y^2=c^4$$

or

$$\frac{x^2}{(c^2/a)^2}+\frac{y^2}{(c^2/a)^2}=1$$

This implies

$$\left(\frac{c^2}{k}\right)^2=\frac{c^4}{a^2}\cdot\left(\frac{-h}{k}\right)^2+\frac{c^4}{b^2}$$

$$\Rightarrow \frac{c^4}{k^2}=\frac{c^4h^2}{a^2k^2}+\frac{c^4}{b^2}$$

$$\Rightarrow \frac{1}{k^2}=\frac{h^2}{a^2k^2}+\frac{1}{b^2}$$

$$\Rightarrow 1=\frac{h^2}{a^2}+\frac{k^2}{b^2}$$

which is true by hypothesis.

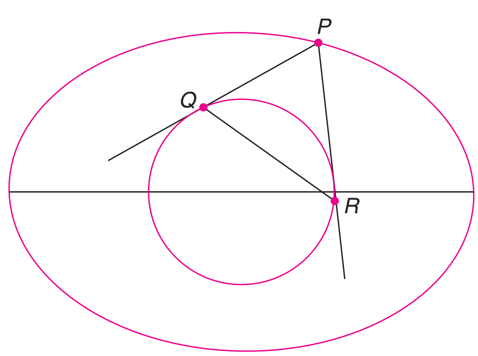


FIGURE 5.22

29. Let d be the distance of a tangent at a point P of the ellipse from the centre. If S and S' are the foci, then show that

$$(PS-PS')^2=4a^2\left(1-\frac{b^2}{d^2}\right)$$

Solution: Let P be $(a\cos\theta, b\sin\theta)$ so that the tangent at P is

$$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$$

Therefore

$$d=\left|\frac{0+0-1}{\sqrt{\cos^2\theta/a^2+\sin^2\theta/b^2}}\right|$$

$$\Rightarrow \frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}=\frac{1}{d^2}$$

$$\Rightarrow \frac{b^2\cos^2\theta}{a^2}+\sin^2\theta=\frac{b^2}{d^2}$$

$$\Rightarrow \frac{b^2\cos^2\theta}{a^2}-\cos^2\theta=\frac{b^2}{d^2}-1$$

$$\Rightarrow \cos^2\theta\left(1-\frac{b^2}{a^2}\right)=1-\frac{b^2}{d^2}$$

$$\Rightarrow \cos^2\theta\frac{(a^2-b^2)}{a^2}=1-\frac{b^2}{d^2}$$

$$\Rightarrow e^2\cos^2\theta=1-\frac{b^2}{d^2} \qquad (\because a^2-b^2=a^2e^2) \tag{5.38}$$

Also $SP=a-ea\cos\theta$ and $S'P=a+ea\cos\theta$. This implies

$$\begin{aligned}(SP-S'P)^2&=(2ea\cos\theta)^2\\&=4a^2e^2\cos^2\theta\\&=4a^2\left(1-\frac{b^2}{d^2}\right)\end{aligned}$$

30. P is a point on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

whose foci are S and S'. Show that the incentre of $\Delta PSS'$ lies on a concentric ellipse whose eccentricity is

$$\sqrt{\frac{2e}{1+e}}$$

where e is the eccentricity of the given ellipse.

Solution: Let $P=(a\cos\theta, b\sin\theta)$, $S=(ae, 0)$ and $S'=(-ae, 0)$. We know that

$$SS'=2ae,\ SP=a-ae\cos\theta,\ S'P=a+ae\cos\theta$$

so that the perimeter of $\Delta PSS' = 2a(1+e)$. Let (h, k) be the incentre of $\Delta PSS'$. Therefore

$$h=\frac{2ae(a\cos\theta)+aea(1+e\cos\theta)-aea(1-e\cos\theta)}{2a(1+e)}$$

$$=\frac{2a^2e\cos\theta(1+e)}{2a(1+e)}$$

$$\Rightarrow h=(ae)\cos\theta \qquad (5.39)$$

Similarly

$$k=\frac{be\sin\theta}{1+e} \qquad (5.40)$$

From Eqs. (5.39) and (5.40), we get

$$\frac{h^2}{(ae)^2}+\frac{k^2}{[be/(1+e)]^2}=\cos^2\theta+\sin^2\theta=1$$

Therefore, (h, k) lies on the ellipse

$$\frac{x^2}{p^2}+\frac{y^2}{q^2}=1$$

where

$$p^2=a^2e^2 \text{ and } q^2=\frac{b^2e^2}{(1+e)^2}$$

If the eccentricity of this is e', then

$$\frac{b^2e^2}{(1+e)^2}=a^2e^2(1-e'^2)$$

$$\Rightarrow \frac{b^2}{a^2(1+e)^2}=1-e'^2$$

$$\Rightarrow \frac{1-e^2}{(1+e)^2}=1-e'^2$$

$$\Rightarrow \frac{1-e}{1+e}=1-e'^2$$

$$\Rightarrow e'^2=1-\frac{1-e}{1+e}=\frac{2e}{1+e}$$

$$\Rightarrow e'=\sqrt{\frac{2e}{1+e}}$$

31. Prove that a common tangent of the ellipses

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{2x}{c}=0 \quad \text{and} \quad \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{2x}{c}=0$$

subtends right angle at the origin.

Solution: See Fig. 5.23. The given ellipses pass through the origin and touch at origin. Their equations can be rewritten as

$$\frac{[(x-a^2)/c]^2}{a^4/c^2}+\frac{y^2}{a^2b^2/c^2}=1$$

and

$$\frac{[(x+b^2)/c]^2}{b^4/c^2}+\frac{y^2}{a^2b^2/c^2}=1$$

The centres of the ellipses are $(a^2/c, 0)$ and $(-b^2/c, 0)$. Also for both curves, the length of the minor axis is same. Hence, the common tangent touches the ellipses at

$$P\left(\frac{a^2}{c},\frac{ab}{c}\right) \quad \text{and} \quad Q\left(\frac{-b^2}{c},\frac{ab}{c}\right)$$

Hence, we have

$$\text{Slope of } OP\times\text{Slope of } OQ=\left[\frac{(ab/c)-0}{(a^2/c)-0}\right]\left[\frac{(ab/c)-0}{-(b^2/c)}\right]$$

$$=\left(\frac{b}{a}\right)\left(\frac{-a}{b}\right)=-1$$

Thus, $\angle QOP=90°$.

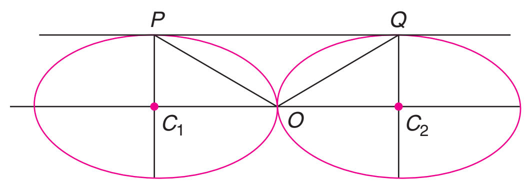


FIGURE 5.23

32. Point P is on

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

and point Q is on the auxiliary circle

$$x^2+y^2=a^2$$

corresponding to P. The line through P parallel to OQ (O is the centre) meets the major axis at A and minor axis at A'. Prove that $PA = b$ and $PA' = a$.

Solution: See Fig. 5.24. $P = (a\cos\theta, b\sin\theta)$ and $Q = (a\cos\theta, a\sin\theta)$. The line through point P and parallel to OQ is

$$y-b\sin\theta=\frac{\sin\theta}{\cos\theta}(x-a\cos\theta)$$

Therefore, $A=(-b\cos\theta+a\cos\theta,0)$ and $A'=(0,b\sin\theta-a\sin\theta)$. Now,

$$(PA)^2=(a\cos\theta+b\cos\theta-a\cos\theta)^2+b^2\sin^2\theta$$

$$=b^2(\cos^2\theta+\sin^2\theta)=b^2$$

$$\Rightarrow PA=b$$

Again

$$(PA')^2=a^2\cos^2\theta+(b\sin\theta-b\sin\theta+a\sin\theta)^2$$

$$=a^2(\cos^2\theta+\sin^2\theta)$$

$$= a^2$$
$$\Rightarrow PA' = a$$

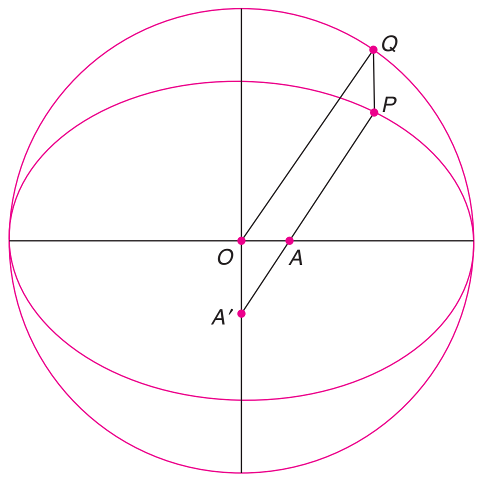


FIGURE 5.24

33. Find the equation of the common tangent in the first quadrant to the circle $x^2 + y^2 = 16$ and the ellipse

$$\frac{x^2}{25} + \frac{y^2}{4} = 1$$

Also find the length of the intercept of the tangent between the coordinate axes. **(IIT-JEE 2005)**

Solution: We know that

$$\frac{x\cos\theta}{5} + \frac{y\sin\theta}{2} = 1$$

is a tangent to the given ellipse for all real values of θ. This line also touches the given circle which implies that

$$\left|\frac{-10}{\sqrt{4\cos^2\theta + 25\sin^2\theta}}\right| = 4$$
$$\Rightarrow 4(4\cos^2\theta + 25\sin^2\theta) = 25$$
$$\Rightarrow 84\cos^2\theta - 75 = 0$$
$$\Rightarrow \cos^2\theta = \frac{75}{84} \text{ and } \sin^2\theta = \frac{9}{84}$$

Since the tangent is in the first quadrant, its slope must be negative. Hence, the equation of the tangent is

$$y = \left(\frac{-2}{5}\cot\theta\right)x + \frac{2}{\sin\theta}$$
$$\Rightarrow y = \frac{-2}{5}\sqrt{\frac{75}{9}}x + \frac{2}{3}\sqrt{84}$$
$$\Rightarrow y = \frac{-2}{5}\cdot\frac{5}{\sqrt{3}}x + \frac{2\times 2\sqrt{21}}{3}$$
$$\Rightarrow 2x + \sqrt{3}y = 4\sqrt{7}$$

Therefore, the common tangent is $2x + \sqrt{3}y = 4\sqrt{7}$. The common tangent meets the axes in the points $A(2\sqrt{7}, 0)$ and $B(0, 4\sqrt{7}/\sqrt{3})$. Hence, the length AB is

$$\sqrt{28 + \frac{112}{3}} = \frac{\sqrt{196}}{\sqrt{3}} = \frac{14}{\sqrt{3}}$$

34. Find the point on the ellipse $x^2 + 2y^2 = 6$ which is nearest to the line $x + y - 7 = 0$. **(IIT-JEE 2003)**

Solution: The given ellipse is

$$\frac{x^2}{6} + \frac{y^2}{3} = 1$$

and every point on it is of the form $P(\sqrt{6}\cos\theta, \sqrt{3}\sin\theta)$. The nearest point on the ellipse is the point at which the normal to the ellipse is perpendicular to the line $x + y - 7 = 0$ or equivalently, the tangent is parallel to the line (see Fig. 5.25). The tangent at $P(\sqrt{6}\cos\theta, \sqrt{3}\sin\theta)$ is

$$\frac{x\cos\theta}{\sqrt{6}} + \frac{y\sin\theta}{\sqrt{3}} = 1$$

whose slope is given by

$$\frac{-\sqrt{3}}{\sqrt{6}}\cot\theta = \frac{-\cot\theta}{\sqrt{2}} = -1$$
$$\Rightarrow \cot\theta = \sqrt{2}$$
$$\Rightarrow \cos\theta = \sqrt{2}\sin\theta$$
$$\Rightarrow \cos^2\theta = 2\sin^2\theta$$
$$\Rightarrow 3\sin^2\theta = 1$$
$$\Rightarrow \sin\theta = \pm\frac{1}{\sqrt{3}} \text{ and } \cos\theta = \pm\sqrt{\frac{2}{3}}$$

Therefore, the nearest point in the first quadrant is

$$P = \left[\sqrt{6}\sqrt{\frac{2}{3}}, \sqrt{3}\left(\frac{1}{\sqrt{3}}\right)\right] = (2, 1)$$

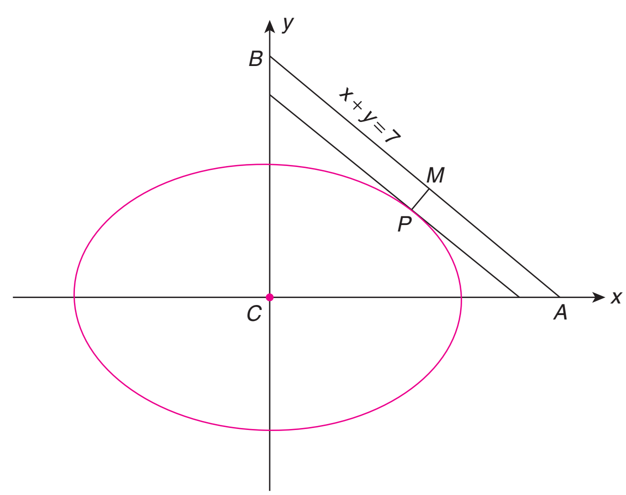


FIGURE 5.25

35. Prove that in an ellipse, the perpendicular from a focus upon any tangent and the line joining the centre of the ellipse to the point of contact meet on the corresponding directrix. **(IIT-JEE 2002)**

Solution: See Fig. 5.26. Let P be $(a\cos\theta, b\sin\theta)$ so that the tangent at P is

$$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$$

Let $S = (ae, 0)$ be one focus. The equation of the line through S and perpendicular to the tangent is

$$y=\frac{a}{b}\frac{\sin\theta}{\cos\theta}(x-ae)$$

This line meets the directrix $x=a/e$ corresponding to the focus $s(ae,0)$ which is given by

$$\left(\frac{a}{e},\frac{a^2}{be}\frac{\sin\theta}{\cos\theta}(1-e^2)\right)=\left(\frac{a}{e},\frac{b}{e}\tan\theta\right)\quad [\because b^2=a^2(1-e^2)]$$

Now, the equation of the line CP where $C = (0, 0)$ is

$$y=\left(\frac{b\sin\theta}{a\cos\theta}\right)x$$

which clearly passes through

$$\left(\frac{a}{e},\frac{b}{e}\tan\theta\right)$$

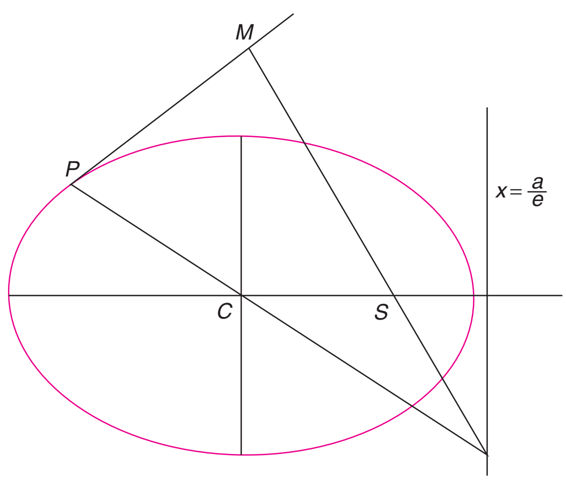


FIGURE 5.26

36. Consider the family of circles $x^2+y^2=r^2$, where $2<r<5$. If in the first quadrant, the common tangent to a circle of this family and the ellipse

$$\frac{x^2}{25}+\frac{y^2}{4}=1$$

meets the coordinate axes at A and B, then find the equation of the focus of the midpoint of AB. **(IIT-JEE 1999)**

Solution: The tangent to the ellipse is of the form $y=mx+\sqrt{25m^2+4}$. This line also touches the circle. So

$$25m^2+4=r^2(1+m^2)$$

$$\Rightarrow m^2=\frac{r^2-4}{25-r^2} \tag{5.41}$$

Therefore, the common tangent is $y=mx+\sqrt{25m^2+4}$ where

$$m=\sqrt{\frac{r^2-4}{25-r^2}}$$

Therefore,

$$A=\left(\frac{-\sqrt{25m^2+4}}{m},0\right)$$

and $$B=(0,\sqrt{25m^2+4})$$

Let $M(x_1, y_1)$ be the midpoint of AB so that

$$2x_1=\frac{-\sqrt{25m^2+4}}{m}\quad\text{and}\quad 2y_1=\sqrt{25m^2+4}$$

$$\Rightarrow 4m^2x_1^2=4y_1^2$$

$$\Rightarrow m^2=\frac{y_1^2}{x_1^2}$$

Therefore,

$$2y_1=\sqrt{25m^2+4}\quad\text{and}\quad m^2=\frac{y_1^2}{x_1^2}$$

$$\Rightarrow 4y_1^2=25\left(\frac{y_1^2}{x_1^2}\right)+4$$

$$\Rightarrow 4x_1^2+25y_1^2=4x_1^2y_1^2$$

Hence, the locus of (x_1, y_1) is

$$4x^2+25y^2=4x^2y^2$$

37. A tangent to the ellipse $x^2+4y^2=4$ meets the ellipse $x^2+2y^2=6$ at P and Q. Prove that the tangents at P and Q of the ellipse intersect at right angles.

Solution: See Fig. 5.27. The two given ellipses can be written as

$$\frac{x^2}{4}+\frac{y^2}{1}=1 \tag{5.42}$$

and $$\frac{x^2}{6}+\frac{y^2}{3}=1 \tag{5.43}$$

Let us consider that the tangent to ellipse which is given in Eq. (5.42) meets the ellipse which is given in Eq. (5.43) at P and Q. Let T be the point of intersection of the tangents to Eq. (5.43) at P and Q. Let T be (h, k). Since PQ is the chord of contact of T with respect to the ellipse which is given in Eq. (5.43), the equation of PQ is

$$\frac{hx}{6}+\frac{ky}{3}=1$$

$$\Rightarrow hx+2ky=6$$

$$\Rightarrow y=\left(\frac{-h}{2k}\right)x+\frac{3}{k} \qquad (5.44)$$

The line in Eq. (5.44) touches the ellipse in Eq. (5.42). So

$$\frac{9}{k^2}=4\left(\frac{-h}{2k}\right)^2+1$$

$$\Rightarrow h^2+k^2=9=6+3$$

Thus, $T(h, k)$ lies on the director circle of Eq. (5.43) so that the tangents at P and Q to Eq. (5.43) are at right angles.

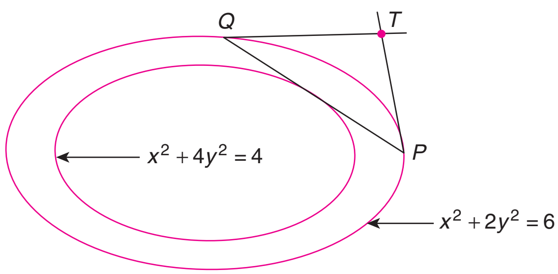


FIGURE 5.27

38. If α, β and γ are the eccentric angles of three points on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

at which the normals are concurrent, then show that

$$\sin(\alpha+\beta)+\sin(\beta+\gamma)+\sin(\gamma+\alpha)=0$$

Solution: Suppose the normals at α, β and γ are concurrent at (h, k) and let δ be the foot of the fourth normal from (h, k) (see Theorem 5.16). Then we have

$$\sum\tan\frac{\alpha}{2}\tan\frac{\beta}{2}=0$$

and $$\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}\tan\frac{\delta}{2}=-1$$

Eliminating $\tan\delta/2$ from the above two equations gives

$$0=\sum\tan\left(\frac{\alpha}{2}\right)\tan\left(\frac{\beta}{2}\right)$$

$$=\tan\left(\frac{\alpha}{2}\right)\tan\left(\frac{\beta}{2}\right)+\tan\left(\frac{\alpha}{2}\right)\tan\left(\frac{\gamma}{2}\right)+\tan\left(\frac{\beta}{2}\right)\tan\left(\frac{\gamma}{2}\right)$$

$$+\tan\left(\frac{\alpha}{2}\right)\left[\frac{-1}{\tan(\alpha/2)\tan(\beta/2)\tan(\gamma/2)}\right]$$

$$+\tan\left(\frac{\beta}{2}\right)\left[\frac{-1}{\tan(\alpha/2)\tan(\beta/2)\tan(\gamma/2)}\right]$$

$$+\tan\left(\frac{\gamma}{2}\right)\left[\frac{-1}{\tan(\alpha/2)\tan(\beta/2)\tan(\gamma/2)}\right]$$

$$=\left[\tan\left(\frac{\alpha}{2}\right)\tan\left(\frac{\beta}{2}\right)-\cot\left(\frac{\alpha}{2}\right)\cot\left(\frac{\beta}{2}\right)\right]$$

$$+\left[\tan\left(\frac{\beta}{2}\right)\tan\left(\frac{\gamma}{2}\right)-\cot\left(\frac{\beta}{2}\right)\cot\left(\frac{\gamma}{2}\right)\right]$$

$$+\left[\tan\left(\frac{\gamma}{2}\right)\tan\left(\frac{\alpha}{2}\right)-\cot\left(\frac{\gamma}{2}\right)\cot\left(\frac{\alpha}{2}\right)\right]$$

$$=-\frac{[\cos^2(\alpha/2)\cos(\beta/2)]-[\sin^2(\alpha/2)\sin(\beta/2)]}{[\cos(\alpha/2)\cos(\beta/2)\sin(\beta/2)(\alpha/2)\sin(\beta/2)}$$

$$-\frac{[\cos^2(\beta/2)\cos^2(\gamma/2)]-[\sin^2(\beta/2)\sin^2(\gamma/2)]}{[\cos(\beta/2)\cos(\gamma/2)]\cdot[\sin(\beta/2)\sin(\gamma/2)]}$$

$$-\frac{[\cos^2(\gamma/2)\cos^2(\alpha/2)]-[\sin^2(\gamma/2)\sin^2(\alpha/2)]}{[\cos(\gamma/2)\cos(\alpha/2)].[\sin(\gamma/2)\sin(\alpha/2)]}$$

Therefore

$$\sum\frac{4\cos(\alpha+\beta)/2\cos(\alpha-\beta)/2}{\sin\alpha\sin\beta}=0$$

$$\Rightarrow\sum\frac{2(\cos\alpha+\cos\beta)}{\sin\alpha\sin\beta}=0$$

$$\Rightarrow\sin\gamma(\cos\alpha+\cos\beta)+\sin\alpha(\cos\beta+\cos\gamma)$$
$$+\sin\beta(\cos\gamma+\cos\alpha)=0$$

$$\Rightarrow\sin(\beta+\gamma)+\sin(\gamma+\alpha)+\sin(\alpha+\beta)=0$$

39. If the normals at (x_r, y_r) $(r = 1, 2, 3, 4)$ to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

are concurrent, then show that

$$(x_1+x_2+x_3+x_4)\left(\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_4}\right)=4$$

Solution: Normal at (α, β) is

$$\frac{a^2(x-\alpha)}{\alpha}=\frac{b^2(y-\beta)}{\beta}$$

This passes through (h, k). This implies

$$\frac{a^2(h-\alpha)}{\alpha}=\frac{b^2(k-\beta)}{\beta}$$

$$\Rightarrow a^2\beta(h-\alpha)=b^2\alpha(k-\beta)$$

$$\Rightarrow[(h-\alpha)a^2+b^2\alpha]\beta=b^2k\alpha$$

$$\Rightarrow[a^2h-\alpha(a^2-b^2)]^2\beta^2=b^4k^2\alpha^2 \quad (5.45)$$

Also (α, β) lies on the ellipse, which means

$$\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}=1 \quad (5.46)$$

Eliminating β from Eqs. (5.45) and (5.46) gives

$$[a^4h^2+\alpha^2(a^2-b^2)^2-2a^2h\alpha(a^2-b^2)]\left(\frac{a^2-\alpha^2}{a^2}\right)b^2$$

$$=b^4k^2\alpha^2$$

$$\Rightarrow\frac{\alpha^4(a^2-b^2)^2b^2}{a^2}-2\alpha^3h(a^2-b^2)b^2+\alpha^2[a^2b^2h^2+b^4k^2-b^2(a^2-b^2)]$$

$$+\alpha[2a^2b^2h(a^2-b^2)]-a^4b^2h^2=0$$

This being a fourth-degree equation in α, we get four values of α, say, x_1, x_2, x_3 and x_4. Therefore,

$$x_1+x_2+x_3+x_4=\frac{2b^2h(a^2-b^2)a^2}{b^2(a^2-b^2)^2}=\frac{2a^2h}{(a^2-b^2)}$$

$$\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_4}=\frac{2a^2b^2h(a^2-b^2)}{a^4b^2h^2}=\frac{2(a^2-b^2)}{a^2h}$$

Hence

$$(x_1+x_2+x_3+x_4)\left(\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_4}\right)=\frac{2a^2h}{a^2-b^2}\times\frac{2(a^2-b^2)}{a^2h}=4$$

40. Two tangents to an ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

are drawn which intersect at right angles. Let l_1 and l_2 be the intercepts on these tangents made by the auxiliary circle $x^2+y^2=a^2$. Show that $l_1^2+l_2^2$ is equal to $(SS')^2$, where S and S' are the foci.

Solution: See Fig. 5.28. Let

$$y=mx+\sqrt{a^2m^2+b^2} \quad (5.47)$$

and
$$y=\frac{-x}{m}+\sqrt{\frac{a^2}{m^2}+b^2}$$

$$\Rightarrow x+my=\sqrt{a^2+b^2m^2} \quad (5.48)$$

Let the intercept of Eqs. (5.47) and (5.48) on the auxillary circle be $l_1 = PQ$ and $l_2 = RT$. Draw CM and CN perpendicular to PQ and RT, respectively, where C (0, 0) is the centre of the ellipse. Now,

$$(CM)^2=\frac{a^2m^2+b^2}{1+m^2}$$

and
$$(CN)^2=\frac{a^2+b^2m^2}{1+m^2}$$

Now,

$$l_1=PQ=2PM$$

$$\Rightarrow l_1{}^2=4(PM)^2$$

$$=4(CP^2-CM^2)$$

$$=4\left[a^2-\frac{a^2m^2+b^2}{1+m^2}\right]$$

$$=\frac{4}{1+m^2}(a^2-b^2) \quad (5.49)$$

Also

$$l_2^2=4\left[a^2-\frac{a^2+b^2m^2}{1+m^2}\right] \quad (5.50)$$

$$=\frac{4m^2}{1+m^2}(a^2-b^2)$$

Therefore, from Eqs. (5.49) and (5.50), we get

$$l_1^2+l_2^2=4(a^2-b^2)\left[\frac{1}{1+m^2}+\frac{m^2}{1+m^2}\right]$$

$$=4(a^2-b^2)$$

$$=4a^2e^2$$

$$=(2ae)^2$$

$$=(SS')^2$$

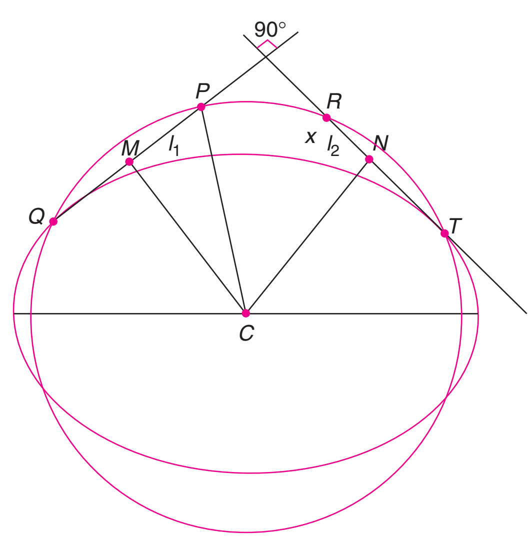


FIGURE 5.28

5.3 | Hyperbola

Hyperbola is a conic whose eccentricity is greater than unity. In this section, we discuss the principal properties that are common to all hyperbolas. Some of these properties are same as that of the ellipse and can be proved in a similar way. However, the additional feature that the hyperbola possess is a pair of *asymptotes* (tangents with points of contact at infinity) which gives the curve its own properties. Let us start the discussion with the formal definition of hyperbola.

DEFINITION 5.12 Let l be a straight line (called the directrix), S be a point which is not located on l (called the focus) and $e>1$ be a fixed real number (called the eccentricity). Then, the locus of the point P such that SP is equal to $(e\,PM)$, where PM is the perpendicular distance of P from the directrix, is called a hyperbola.

Example 5.9

Find the equation of the hyperbola whose directrix is $x-y+1=0$, focus is $(-1,-1)$ and eccentricity is 2.

Solution: $P(x,y)$ is a point on hyperbola

$$\Leftrightarrow SP=2(PM)$$

$$\Leftrightarrow (SP)^2=4(PM)^2$$

$$\Leftrightarrow (x+1)^2+(y+1)^2=4\left|\frac{x-y+1}{\sqrt{2}}\right|^2$$

$$\Leftrightarrow x^2+y^2+2x+2y+2=2x^2-4xy+2y^2+4x-4y+2$$

$$\Leftrightarrow x^2-4xy+y^2+2x-6y=0$$

Thus, the equation of the hyperbola is

$$x^2-4xy+y^2+2x-6y=0$$

THEOREM 5.18 The standard equation of the hyperbola is

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

where $b^2=a^2(e^2-1)$.

PROOF See Fig. 5.29. Draw SZ perpendicular to the directrix l. Divide SZ internally and externally at points A and A' in the ratio e:1. Let C be the midpoint of AA' and $A'C = CA = a$. Since $SA:AZ=e:1$ and $SA':A'Z=e:1$, we have

$$
\begin{aligned}
&SA=e(AZ) \text{ and } SA'=e(A'Z)\\
&\Rightarrow SA+A'S=e(AZ+A'Z)\\
&\Rightarrow (CS-CA)+(A'C+CS)=e(AA')\\
&\Rightarrow 2(CS)=e(2a)\\
&\Rightarrow CS=ae \qquad (5.51)
\end{aligned}
$$

Also

$$
\begin{aligned}
&SA'-AS=e(A'Z-AZ)\\
&\Rightarrow A'A=e(A'C+CZ-AC+CZ)\\
&\Rightarrow 2a=2e(CZ)\\
&\Rightarrow CZ=\frac{a}{e} \qquad (5.52)
\end{aligned}
$$

Now, choose $\overrightarrow{CS}$ as positive x-axis and C as origin. Hence from Eqs. (5.51) and (5.52), we have $S=(ae,0)$ and the equation of CZ (i.e., directrix) is $x=a/e$. Now, $P(x,y)$ is a point on the curve. This implies

$$SP=ePM$$

$$\Rightarrow (x-ae)^2+y^2=e^2\left(x-\frac{a}{e}\right)^2$$

$$\Rightarrow (x-ae)^2 + y^2 = (ex-a)^2$$
$$\Rightarrow (e^2-1)x^2 - y^2 = a^2(e^2-1)$$
$$\Rightarrow \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

where $b^2 = a^2(e^2 - 1)$.

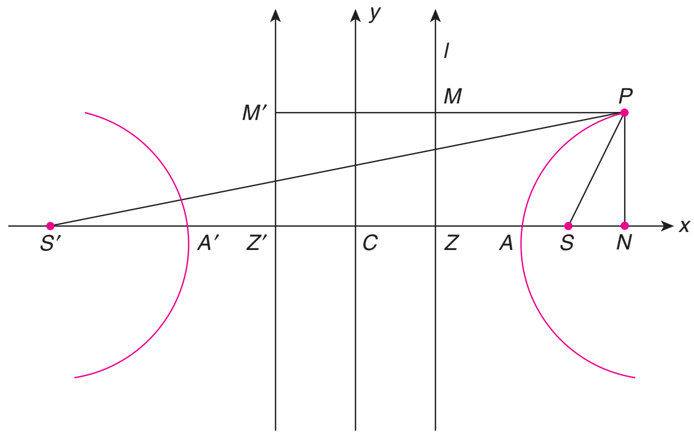


FIGURE 5.29

QUICK LOOK 9

For any point $P(x, y)$ on the hyperbola, we have

$$|S'P - SP| = e|PM' - PM| = e(MM') = e(ZZ') = e\left(\frac{2a}{e}\right) = 2a = A'A$$

5.3.1 Features of the Curve $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

1. The curve is symmetric about both axes.
2. To each value of x, such that $|x| > a$, there are two values for y which are equal in magnitude but opposite in sign. To each value of y, there are two values for x.
3. The abscissae a of any point should not be between $-a$ and $+a$.
4. The curve cuts the x-axis at $(-a, 0)$ and $(a, 0)$.
5. The curve cannot intersect y-axis.
6. As y increases infinitely, x also increases infinitely.
7. Due to the symmetry of the curve, there is another focus $S'(-ae, 0)$; the corresponding directrix is $x = -a/e$.

DEFINITIONS 5.13

1. $A'(-a, 0)$ and $A(a, 0)$ are called the *vertices* of the curve and $A'A$ is called the *transeverse axis* whose length is $2a$.
2. If $B = (0, -b)$ and $B' = (0, b)$, then BB' is called the *conjugate axis* whose length is $2b$.
3. The double ordinate passing through the focus is called *latus rectum*.
4. $x^2 + y^2 = a^2$ is called the *auxiliary circle* which touches the curve at vertices A and A'.

THEOREM 5.19 The length of the latus rectum of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $\frac{2b^2}{a}$.

PROOF Let SL be the semi-latus rectum where $S = (ae, 0)$ and $L = (ae, y)$. Now L is a point on the curve, so

$$\frac{a^2e^2}{a^2} - \frac{y^2}{b^2} = 1$$
$$\Rightarrow y^2 = b^2(e^2 - 1)$$
$$\Rightarrow y^2 = \frac{b^4}{a^2}$$
$$\Rightarrow SL = \frac{b^2}{a}$$

DEFINITION 5.14 In a hyperbola, if the lengths of transeverse and conjugate axes are equal, then it is called *rectangular hyperbola*.

QUICK LOOK 10

In a rectangular hyperbola, $b^2 = a^2 \Rightarrow a^2 = a^2(e^2 - 1) \Rightarrow e = \sqrt{2}$. The eccentricity of any rectangular hyperbola is $\sqrt{2}$ and the *standard form of a rectangular hyperbola* is $x^2 - y^2 = a^2$.

Example 5.10

For the hyperbola $4x^2 - 9y^2 = 36$, find the foci, eccentricity, the directrices and the lengths of the axis.

Solution: The given hyperbola is

$$\frac{x^2}{9} - \frac{y^2}{4} = 1$$

so that $a^2 = 9, b^2 = 4$. We have

$$b^2 = a^2(e^2 - 1)$$
$$\Rightarrow 4 = 9(e^2 - 1)$$
$$\Rightarrow e^2 = \frac{4}{9} + 1 \text{ or } e = \frac{\sqrt{13}}{3}$$

Foci are given by

$$(\pm ae, 0) = \left[\pm 3\left(\frac{\sqrt{13}}{3}\right), 0\right] = (+\sqrt{13}, 0)$$

Directrices are given by

$$x = \pm\frac{a}{e} \Rightarrow x = \pm 3\left(\frac{3}{\sqrt{13}}\right) = \pm\frac{9}{\sqrt{13}}$$

Transeverse axis length is $2a = 6$ and conjugate axis length $= 2b = 4$.

Example 5.11

Find the eccentricity and foci of the hyperbola $9x^2 - 16y^2 + 72x - 32y - 16 = 0$.

Solution: The given equation can be written as

$$9(x^2 + 8x) - 16(y^2 + 2y) = 16$$
$$\Rightarrow 9(x+4)^2 - 16(y+1)^2 = 16 + 144 - 16 = 144$$
$$\Rightarrow \frac{(x+4)^2}{16} - \frac{(y+1)^2}{9} = 1$$

Therefore, the centre is $(-4, -1)$. Now

$$b^2 = a^2(e^2 - 1)$$
$$\Rightarrow 9 = 16(e^2 - 1)$$
$$\Rightarrow e = \frac{5}{4}$$

The foci are given by

$$(x + 4 = \pm ae,\ y + 1 = 0) = (-4 \pm 5, -1) = (1, -1) \text{ and } (-9, -1)$$

DEFINITION 5.15 **Parametric Equations** For all real $\theta \neq (2n+1)\pi/2$, the point (a secθ, b tanθ) lies on the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

The equations

$$x=a\sec\theta \text{ and } y=b\tan\theta$$

are called the *parametric equations* of the hyperbola.

DEFINITION 5.16 **Position of a Point** For any hyperbola, the foci region is called *internal region* and any point which is not (a) inside the curve and (b) on the curve is called *external point*.

Notations used:

$$S\equiv\frac{x^2}{a^2}-\frac{y^2}{b^2}-1$$

$$S_1\equiv\frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1$$

$$S_2\equiv\frac{xx_2}{a^2}-\frac{yy_2}{b^2}-1$$

$$S_{12}\equiv S_{21}=\frac{x_1x_2}{a^2}-\frac{y_1y_2}{b^2}-1$$

$$S_{11}=\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1$$

The following can be proved similar to the case of ellipse by replacing b^2 with $-b^2$ and hence only statements are given without proofs.

THEOREM 5.20 Let $S\equiv\frac{x^2}{a^2}-\frac{y^2}{b^2}-1=0$. Then

1. A point in the plane of the hyperbola is external or internal to the curve according as $S_{11}<0$ or $S_{11}>0$, respectively. Note that *in the case of circle, parabola and ellipse, $S_{11}>0$ is the condition for an external point and $S_{11}<0$ is the condition for an internal point. However, in the case of hyperbola, it is the reverse.*
2. The equation of the chord joining two points (x_1, y_1) and (x_2, y_2) on the curve is $S_1+S_2=S_{12}$.
3. The equation of the tangent at (x_1, y_1) is $S_1\equiv\frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1=0$.
4. The equation of the normal at (x_1, y_1) is $\frac{a^2(x-x_1)}{x_1}=\frac{-b^2(y-y_1)}{y_1}$.
5. The equation of tangent at (a sec θ, b tan θ) is $\frac{x\sec\theta}{a}-\frac{y\tan\theta}{b}=1$.
6. The equation of the normal at (a sec θ, b tan θ) is

$$\frac{x\tan\theta}{b}+\frac{y\sec\theta}{a}=\left(\frac{a^2+b^2}{ab}\right)\sec\theta\tan\theta$$

$$\Rightarrow ax\cos\theta+by\cot\theta=a^2+b^2$$

7. The condition for the line $y=mx+c$ to touch the hyperbola is that $c^2=a^2m^2-b^2$ or the line $y=mx\pm\sqrt{am^2-b^2}$ touches the hyperbola.
8. The equation of the director circle is $x^2+y^2=a^2-b^2$. This will exist only when $a>b$.

9. The equation of a tangent is $x\cos\alpha + y\sin\alpha = p$, where $p=\sqrt{a^2\cos^2\alpha - b^2\sin^2\alpha}$.
10. All points on a tangent except the point of contact are external to the hyperbola.
11. Equation of the chord in terms of its midpoint (x_1, y_1) is $S_1 = S_{11}$.
12. From any external point, two tangents can be drawn to a hyperbola and the equation of the pair of tangents from the point (x_1, y_1) is $S_1^2 = SS_{11}$.

DEFINITION 5.17 **Asymptotes** The two lines given by

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=0 \text{ or } y=\pm\frac{b}{a}x$$

are called the *asymptotes* of the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$.

Algebraically, the two lines $y=\pm(b/a)x$ satisfy the condition of a tangent, but without the existence of points of contact. Due to this reason, asymptotes are also stated as lines through the centres which touch the curve at infinity. In general, a line l is called an asymptote to a curve if the line l goes nearer and nearer to the curve, but never meets the curve.

THEOREM 5.21 For the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$, the following properties hold good with regard to asymptotes:

1. The chord of contact of any point on an asymptote is parallel to the same asymptote.
2. The product of the perpendiculars drawn from any point on the curve onto its asymptotes is constant.
3. The foot of the perpendicular from a focus onto an asymptote is one of the points of intersection of the corresponding directrix with the auxiliary circle.
4. If a straight line cuts the curve at P and Q and the asymptotes at R and S, then $PR = QS$.
5. The angle between the two asymptotes is $2\sec^{-1}(e)$.

PROOF 1. $P(x_1, y_1)$ is a point on the asymptote $y=(b/a)x$ so that

$$y_1=\frac{b}{a}x_1 \tag{5.53}$$

The chord of contact of $P(x_1, y_1)$ with respect to the hyperbola is

$$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$$

From Eq. (5.53), the slope of the chord of contact of

$$P=\frac{b^2x_1}{a^2y_1}=\frac{b^2}{a^2}\left(\frac{a}{b}\right)=\frac{b}{a}$$

which is the slope of the asymptote. Hence, the chord of contact of P is parallel to the asymptote on which point P lies.

2. Let $P(x_1, y_1)$ be a point on the curve so that

$$\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}=1 \tag{5.54}$$

The product of the perpendiculars drawn from $P(x_1, y_1)$ onto the asymptotes is

$$\frac{|(x_1/a)-(y_1/b)|}{\sqrt{(1/a^2)+(1/b^2)}}\times\frac{|(x_1/a)+(y_1/b)|}{\sqrt{(1/a^2)+(1/b^2)}}=\left|\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}\right|\frac{a^2b^2}{a^2+b^2}$$

Therefore, from Eq. (5.54), we have

$$\frac{|(x_1/a)-(y_1/a)|}{\sqrt{(1/a^2)+(1/b^2)}}\times\frac{|(x_1/a)+(y_1/a)|}{\sqrt{(1/a^2)+(1/b^2)}}=\frac{a^2b^2}{a^2+b^2}$$

3. Let $S=(ae,0)$ and $y=(b/a)x$ be an asymptote. The line passing through S and perpendicular to the asymptote $y=(b/a)x$ is

$$y-0=\frac{-a}{b}(x-ae)$$

Solving the equations

$$y=\left(\frac{b}{a}\right)x \quad \text{and} \quad y=\frac{-a}{b}(x-ae)$$

we have

$$x\left(\frac{a}{b}+\frac{b}{a}\right)=\frac{a^2e}{b}$$

$$\Rightarrow x=\frac{a^2e}{b}\times\frac{ab}{a^2+b^2}=\frac{a^3e}{a^2e^2}=\frac{a}{e}$$

and

$$y=\frac{b}{a}\left(\frac{a}{e}\right)=\frac{b}{e}$$

That is, the foot of the perpendicular from S $(ae, 0)$ onto the asymptote $y=(b/a)x$ is $(a/e, b/e)$ which lies on the auxiliary circle $x^2+y^2=a^2$.

4. See Fig. 5.30. Let $M(x_1, y_1)$ be the midpoint of PQ. We show that $M(x_1, y_1)$ is also the midpoint of RS. The equation of the chord PQ is

$$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2} \tag{5.55}$$

Substituting $y=(b/a)x$ and $y=-(b/a)x$ in Eq. (5.55), we have

$$R=\left[a\left(\frac{x_1}{a}+\frac{y_1}{b}\right), b\left(\frac{x_1}{a}+\frac{y_1}{b}\right)\right]$$

and

$$S=\left[a\left(\frac{x_1}{a}-\frac{y_1}{b}\right), -b\left(\frac{x_1}{a}-\frac{y_1}{b}\right)\right]$$

Hence, the midpoint of RS is (x_1, y_1).

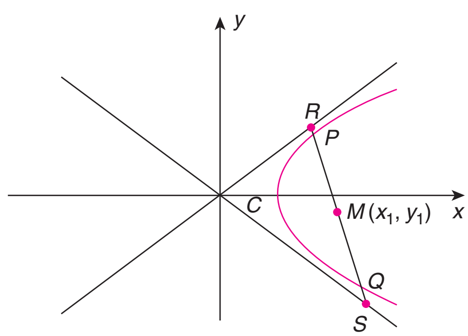


FIGURE 5.30

5. The asymptotes are $y=(b/a)x$ and $y=-(b/a)x$. They are equally inclined to the x-axis. If 2θ is the angle between them, then

$$\tan\theta = \frac{b}{a}$$

so that

$$\sec^2\theta = \frac{a^2+b^2}{a^2} = \frac{a^2e^2}{a^2} = e^2$$

and $$2\theta = 2\,\mathrm{Sec}^{-1}(e)$$

QUICK LOOK 11

The combined equation of the pair of asymptotes of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$ which shows that the combined equation of the asymptotes of a hyperbola and the equation of a hyperbola differ by a constant.

Example 5.12 (Method to Find Asymptotes)

Find the asymptotes of the hyperbola $3x^2-5xy-2y^2+17x+y+14=0$.

Solution: Since the hyperbola equation and the pair of asymptotes equation differ by a constant, let the combined equation of the pair of asymptotes be

$$3x^2-5xy-2y^2+17x+y+k=0 \tag{5.56}$$

Since Eq. (5.56) represents a pair of lines, from Theorem 2.31, Chapter 2, we have

$$3(-2)k+2\left(\frac{1}{2}\right)\left(\frac{17}{2}\right)\left(\frac{-5}{2}\right)-3\left(\frac{1}{2}\right)^2-(-2)\left(\frac{17}{2}\right)^2-k\left(\frac{-5}{2}\right)^2=0$$

$$\Rightarrow \frac{-49k}{4}+\frac{490}{4}=0$$

$$\Rightarrow k=10$$

Hence, the combined equation of the asymptotes is

$$3x^2-5xy-2y^2+17x+y+10=0$$

Since $3x^2-5xy-2y^2=(3x+y)(x-2y)$, let us consider that

$$3x^2-5xy-2y^2+17x+y+10=(3x+y+n_1)(x-2y+n_2)$$

Equating the coefficients of x and y on both sides, we have

$$n_1+3n_2=17$$

and $$-2n_1+n_2=1$$

Solving for n_1 and n_2, we have $n_1=2$ and $n_2=5$. Hence, the asymptotes are $3x+y+2=0$ and $x-2y+5=0$.

Example 5.13

The asymptotes of a hyperbola are the lines $2x+3y-8=0$ and $3x-2y+1=0$, and the curve passes through the point (5, 3). Find the equation of the hyperbola.

Solution: Let the equation of the hyperbola be

$$(2x+3y-8)(3x-2y+1)+k=0$$

Since this passes through the point (5, 3), we have

$$(10+9-8)(15-6+1)+k=0$$

$$\Rightarrow k=-110$$

Hence, the equation of the hyperbola is

$$6x^2+5xy-6y^2-22x+19y-118=0$$

DEFINITION 5.18 **Conjugate Hyperbola** A hyperbola whose transverse and conjugate axes are conjugate and transverses of another hyperbola is called *conjugate hyperbola*. For example, the conjugate hyperbola of $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $\frac{y^2}{b^2}-\frac{x^2}{a^2}=1$ or $\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$

QUICK LOOK 12

For a hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ and its conjugate $\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$, the asymptotes are the same lines

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$$

In fact, one can observe that

$$\text{Hyperbola} + \text{Conjugate hyperbola } =2\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)$$
$$= 2 \times \text{Asymptote}$$

Example 5.14

Find the conjugate hyperbola of $3x^2-5xy-2y^2+17x+y+14=0$.

Solution: The asymptote of the given hyperbola (see Example 5.12) is

$$3x^2-5xy-2y^2+17x+y+10=0$$

Therefore, according to Quick Look 12,

$$(3x^2-5xy-2y^2+17x+14)+\text{Conjugate}$$
$$=2\times \text{Assymptotes}$$
$$=2\times(3x^2-5xy-2y^2+17x+y+10)$$

This gives

$$3x^2-5xy-2y^2+17x+y+6=0$$

which is the equation of the conjugate hyperbola.

THEOREM 5.22 If e_1 and e_2 are eccentricities of a hyperbola and its conjugate, then $\frac{1}{e_1^2}+\frac{1}{e_2^2}=1$.

PROOF Suppose e_1 is the eccentricity of

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

and e_2 is the eccentricity of

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$$

Therefore

$$b^2=a^2(e_1^2-1) \text{ and } a^2=b^2(e_2^2-1)$$

Now

$$(e_1^2-1)(e_2^2-1)=1$$
$$\Rightarrow e_1^2e_2^2-e_1^2-e_2^2-0$$
$$\Rightarrow \frac{1}{e_1^2}+\frac{1}{e_2^2}=1$$

■

Now, we recall the definition of rectangular hyperbola from Definition 5.14. The hyperbola in which transverse and conjugate axes having equal lengths is called rectangular hyperbola. Hence, the general standard equation of the rectangular hyperbola is $x^2-y^2=a^2$. To study some of the properties of the curve $x^2-y^2=a^2$, we transform this equation to simplest form $xy=c^2$, by rotation of the axes through a certain angle.

THEOREM 5.23 The equation of the rectangular hyperbola with its asymptotes as the coordinate axes is $xy=c^2$, where c is constant, is called the *simplest form of a rectangular hyperbola*.

PROOF Let $x^2-y^2=a^2$ be rectangular hyperbola. Now, rotate the axes about the origin through $\pi/4$ in the clockwise direction so that the asymptotes become coordinate axes [because the angle between the asymptotes of $x^2-y^2=a^2$ is $2\text{Sec}^{-1}(\sqrt{2})$ which equal to $\pi/2$]. Then

$$X = x\,\mathrm{Cos}^{-1}\left(\frac{-\pi}{4}\right) - y\sin\left(\frac{-\pi}{4}\right)$$

and
$$Y = x\sin\left(\frac{-\pi}{4}\right) + y\cos\left(\frac{-\pi}{4}\right)$$

That is,

$$X = \frac{x+y}{\sqrt{2}} \text{ and } Y = \frac{y-x}{\sqrt{2}}$$

Hence, the equation of the hyperbola is

$$\left(\frac{x+y}{\sqrt{2}}\right)^2 - \left(\frac{y-x}{\sqrt{2}}\right)^2 = a^2$$

$$\Rightarrow 4xy = 2a^2$$

$$\Rightarrow xy = \frac{a^2}{2}$$

We write $c = a/\sqrt{2}$ so that $xy = c^2$. Also $xy = c^2 \Rightarrow$ either both x and y are positive or both x and y are negative. ■

QUICK LOOK 13

If $S \equiv xy - c^2 = 0$, then $S_1 \equiv xy_1 + x_1y - 2c^2 = 0$ and $S_{11} = x_1y_1 - c^2$.

THEOREM 5.24 (PARAMETRIC EQUATIONS OF $xy = c^2$) The point (x, y) is on the curve $xy = c^2$ if and only if $x = ct$ and $y = c/t$ for some real number t. These equations are called parametric equation of $xy = c^2$.

PROOF The point $(ct, c/t)$ clearly lies on the curve $xy = c^2$. Conversely, if (x, y) is any point on the curve $xy = c^2$, then take either $t = x/c$ or c/y. So $x = ct$ and $y = c/t$. ■

Note: The parametric equations of $x^2 - y^2 = a^2$ are $x = a\sec\theta$, $y = a\tan\theta$, where $\theta \in R$.

THEOREM 5.25 Let $S \equiv xy - c^2 = 0$. Then

1. $xy_1 + x_1y = 2c^2$ is the equation of the tangent at (x_1, y_1). In particular, if $x_1 = ct$ and $y_1 = c/t$, then the equation of the tangent in the parametric form is

$$\frac{cx}{t} + cty = 2c^2$$

or
$$\frac{x}{t} + ty = 2c$$

2. The equation of the normal at $(ct, c/t)$ is $t^3x - ty + c - ct^4 = 0$.

PROOF We have $S \equiv 2xy - 2c^2 = 0$. Hence

1. The tangent at (x_1, y_1) is
$$xy_1 + x_1 y - 2c^2 = 0$$
In particular, if $x_1 = ct$ and $y_1 = c/t$, the equation of the tangent at $(ct, c/t)$ is
$$\frac{x}{t} + ty = 2c$$
2. The slope of the tangent at $(ct, c/t)$ is $-1/t^2$. Hence, the equation of the normal at $(ct, c/t)$ is given by
$$y - \frac{c}{t} = t^2(x - ct)$$
$$\Rightarrow ty - c = t^3(x - ct)$$
$$\Rightarrow t^3x - ty + c - ct^4 = 0$$

THEOREM 5.26 If the normal at $(ct, c/t)$ for $xy = c^2$ meets the curve again at $(ct', c/t')$, then $t^3t' = -1$.

PROOF The normal at $(ct, c/t)$ is
$$t^3x - ty + c - t^4c = 0$$
This passes through $(ct', c/t')$. This implies
$$t^3(ct') - t\left(\frac{c}{t'}\right) + c - ct^4 = 0$$
$$\Rightarrow t^3t' - \frac{t}{t'} - t^4 + 1 = 0$$
$$\Rightarrow t^3t'^2 - t - t't^4 + t' = 0$$
$$\Rightarrow t^3t'(t' - t) + (t' - t) = 0$$
$$\Rightarrow (t^3t' + 1)(t' - t) = 0$$
$$\Rightarrow t^3t' = -1 \qquad (\because t \neq t')$$

Example 5.15

The tangent at a point P of a rectangular hyperbola $xy = c^2$ meets the asymptotes at L and M. Prove that $PL = PM = PO$, where O is the centre of the hyperbola. Also show that the area of ΔLOM is constant.

Solution: See Fig. 5.31. For $xy = c^2$, the asymptotes are the coordinate axes. Tangent at $(ct, c/t)$ is
$$\frac{x}{t} + ty = 2c$$
$$\Rightarrow x + t^2y = 2ct$$
Therefore, $L = (2ct, 0)$ and $M = (0, 2c/t)$. The midpoint of LM is
$$\left(\frac{2ct + 0}{2}, \frac{0 + (2c/t)}{2}\right) = \left(ct, \frac{c}{t}\right) = P \quad \text{(say)}$$
Since $\angle MOL = 90°$ and P is the midpoint of LM, we have $OP = PL = PM$. Also the area of ΔLOM is given by
$$\frac{1}{2}|OL \times OM| = \frac{1}{2}\left|(2ct)\left(\frac{2c}{t}\right)\right| = 2c^2$$
which is constant.

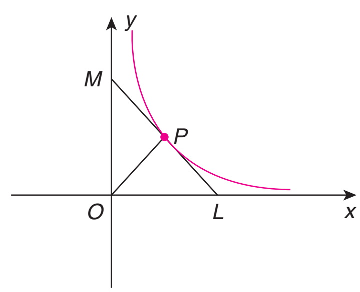


FIGURE 5.31

Try it out Try Example 5.15 for the hyperbola $x^2 - y^2 = a^2$ whose asymptotes are $x \pm y = 0$.

THEOREM 5.27 The portion of the tangent to a hyperbola intercepted between its asymptotes is bisected at the point of contact.

PROOF Let $P(a\sec\theta, b\tan\theta)$ be a point on

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

The tangent at P is

$$\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1 \tag{5.57}$$

We know that the equations of the asymptotes are

$$y = \pm\left(\frac{b}{a}\right)x$$

Substituting $y = (b/a)x$ in Eq. (5.57), we get

$$\frac{x\sec\theta}{a} - \frac{b\tan\theta x}{ab} = 1 \Rightarrow x = \frac{a}{\sec\theta - \tan\theta} = a(\sec\theta + \tan\theta)$$

and $$y = b(\sec\theta + \tan\theta)$$

Let $Q = [a(\sec\theta + \tan\theta), b(\sec\theta + \tan\theta)]$. Similarly, by substituting $y = -(b/a)x$ in Eq. (5.57), we have

$$R = [a(\sec\theta - \tan\theta), b(\tan\theta - \sec\theta)]$$

Now, it is easy to see that the midpoint of QR is $P(a\sec\theta, b\tan\theta)$. ■

Subjective Problems (Section 5.3)

1. Find the eccentricity and the coordinates of the hyperbola $2x^2 - 3y^2 = 5$.

Solution: The equation of hyperbola can be written as

$$\frac{x^2}{5/2} - \frac{y^2}{5/3} = 1$$

so that

$$\frac{5}{3} = \frac{5}{2}(e^2 - 1)$$

$$\Rightarrow \text{Eccentricity, } e = \sqrt{\frac{5}{3}}$$

and the foci is

$$(\pm ae, 0) = \left(\pm\sqrt{\frac{5}{2}} \times \sqrt{\frac{5}{3}}, 0\right) = \left(\pm\frac{5}{\sqrt{6}}, 0\right)$$

2. Find the centre, vertices, eccentricity and foci of the hyperbola $9x^2 - 18x - 16y^2 - 64y + 89 = 0$.

Solution: The given equation is

$$9(x^2 - 2x) - 16(y^2 + 4y) + 89 = 0$$

$$\Rightarrow 9(x-1)^2 - 16(y+2)^2 + 89 - 9 + 64 = 0$$

$$\Rightarrow \frac{(x-1)^2}{16} - \frac{(y+2)^2}{9} = -1$$

which is the conjugate hyperbola of

$$\frac{(x-1)^2}{16} - \frac{(y+2)^2}{9} = 1$$

The centre is $(x-1=0, y+2=0) = (1, -2)$. The eccentricity e is given by

$$16 = 9(e^2 - 1) \Rightarrow e = 5/3$$

so that

$$be = 3\left(\frac{5}{3}\right) = 5$$

The foci is $(x-1=0, y+2=\pm be)$ which is given by

$$(1, y = -2 \pm 5) = (1, 3) \text{ and } (1, -7)$$

3. If every tangent to the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

meets the director circle at two points P and Q, and C is the centre of the curve, then prove that the product of the slopes of CP and CQ is equal to b^2/a^2.

Solution: The equation of the director circle is $x^2+y^2=a^2-b^2$. It is known that the line $y=mx+\sqrt{a^2m^2-b^2}$ is a tangent to the hyperbola so that the combined equation of the lines CP and CQ is

$$x^2+y^2-(a^2-b^2)\left(\frac{y-mx}{\sqrt{a^2m^2-b^2}}\right)^2=0$$

This being a second-degree equation in x and y representing pair of lines, the product of their slopes (see Theorem 2.32 clubbed with Theorem 2.27, Note 2, Chapter 2) is given by

$$\frac{\text{Coefficient of } x^2}{\text{Coefficient of } y^2}=\frac{1-[(a^2-b^2)m^2/(a^2m^2-b^2)]}{1-[(a^2-b^2)/(a^2m^2-b^2)]}$$

$$=\frac{-b^2+b^2m^2}{a^2m^2-a^2}$$

$$=\frac{b^2(m^2-1)}{a^2(m^2-1)}$$

$$=\frac{b^2}{a^2}$$

Try it out What happens if $m=\pm 1$?

4. The chord QQ' of a hyperbola is parallel to the tangent at P. PN, QM and $Q'M'$ are the perpendiculars to either of the asymptotes. Show that $QM\cdot Q'M'=(PN)^2$.

Solution: Let the hyperbola be

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \tag{5.58}$$

See Fig. 5.32. Let $P=(a\sec\alpha, b\tan\alpha)$, $Q=(a\sec\beta, b\tan\beta)$ and $Q'=(a\sec\gamma, b\tan\gamma)$. Let us consider the asymptote

$$y=\left(\frac{b}{a}\right)x \tag{5.59}$$

The slope m of QQ' is

$$\frac{b(\tan\beta-\tan\gamma)}{a(\sec\beta-\sec\gamma)}=\frac{b}{a}\cdot\left[\frac{\cos[(\gamma-\beta)/2]}{\sin[(\beta-\gamma)/2]}\right]$$

The tangent at P is

$$\frac{x\sec\alpha}{a}-\frac{y\tan\alpha}{b}=1 \tag{5.60}$$

Since QQ' is parallel to the tangent P [Eq. (5.60)], we have

$$\frac{b\sec\alpha}{a\tan\alpha}=\frac{b}{a}\left[\frac{\cos[(\gamma-\beta)/2]}{\cos[(\gamma+\beta)/2]}\right]$$

$$\Rightarrow \sin\left(\frac{\beta+\gamma}{2}\right)=\sin\alpha\cos\left(\frac{\beta-\gamma}{2}\right) \tag{5.61}$$

The perpendicular PN from P onto the line provided in Eq. (5.59) is

$$\frac{|ba\sec\alpha-ab\tan\alpha|}{\sqrt{b^2+a^2}}$$

The perpendicular QM is

$$\frac{|ba\sec\beta-ab\tan\beta|}{\sqrt{b^2+a^2}}$$

The perpendicular $Q'M'$ is

$$\frac{|ba\sec\gamma-ab\tan\gamma|}{\sqrt{b^2+a^2}}$$

Therefore

$$QM\cdot Q'M'=\frac{a^2b^2}{a^2+b^2}|(\sec\beta-\tan\beta)(\sec\gamma-\tan\gamma)|$$

$$=\frac{a^2b^2}{a^2+b^2}\left|\frac{(1-\sin\beta)(1-\sin\gamma)}{\cos\beta\cos\gamma}\right|$$

$$=\frac{a^2b^2}{a^2+b^2}\left|\frac{\{1-\cos[(\pi/2)-\beta]\}\cdot\{1-\cos[(\pi/2)-\gamma]\}}{\sin[(\pi/2)-\beta]\cdot\sin[(\pi/2)-\gamma]}\right|$$

$$=\frac{a^2b^2}{a^2+b^2}\left|\tan\left(\frac{\pi}{4}-\frac{\beta}{2}\right)\tan\left(\frac{\pi}{4}-\frac{\gamma}{2}\right)\right| \tag{5.62}$$

Now, from Eq. (5.61), we have

$$\frac{\sin[(\beta+\gamma)/2]}{\cos[(\beta-\gamma)/2]}=\sin\alpha$$

$$\Rightarrow \frac{\cos\{(\pi/2)-[(\beta+\gamma)/2]\}}{\cos[(\beta-\gamma)/2]}=\cos\left(\frac{\pi}{2}-\alpha\right)$$

Using componendo and dividendo, we have

$$\frac{\cos[(\beta-\gamma)/2)]-\cos\{(\pi/2)-[(\beta+\gamma)/2]\}}{\cos[(\beta-\gamma)/2)]+\cos\{(\pi/2)-[(\beta+\gamma)/2]\}}$$

$$=\frac{1-\cos[(\pi/2)-\alpha]}{1+\cos[(\pi/2)-\alpha]}$$

$$\Rightarrow \frac{2\sin[(\pi/4)-(\gamma/2)]\sin[(\pi/4)-(\beta/2)]}{2\cos[(\pi/4)-(\gamma/2)]\cos[(\pi/4)-(\beta/2)]}$$

$$=\frac{2\sin^2[(\pi/4)-(\alpha/2)]}{2\cos^2[(\pi/4)-(\alpha/2)]}$$

$$\Rightarrow \tan\left(\frac{\pi}{4}-\frac{\gamma}{2}\right)\tan\left(\frac{\pi}{4}-\frac{\beta}{2}\right)=\tan^2\left(\frac{\pi}{4}-\frac{\alpha}{2}\right) \quad (5.63)$$

Therefore, from Eqs. (5.59) and (5.60), we have

$$QM\cdot Q'M'=\frac{a^2b^2}{a^2+b^2}\tan^2\left(\frac{\pi}{4}-\frac{\alpha}{2}\right) \quad (5.64)$$

Now,

$$PN=\frac{|ba\sec\alpha-ab\tan\alpha|}{\sqrt{b^2+a^2}}=\frac{ab}{\sqrt{a^2+b^2}}|\sec\alpha-\tan\alpha|$$

$$=\frac{ab}{\sqrt{a^2+b^2}}\left|\frac{1-\sec\alpha}{\cos\alpha}\right|$$

$$=\frac{ab}{\sqrt{a^2+b^2}}\left|\frac{1-\cos[(\pi/2)-\alpha]}{\sin[(\pi/2)-\alpha]}\right|$$

$$=\frac{ab}{\sqrt{a^2+b^2}}\tan\left(\frac{\pi}{4}-\frac{\alpha}{2}\right) \quad (5.65)$$

From Eqs. (5.64) and (5.65), we have $QM\cdot Q'M'=PN^2$.

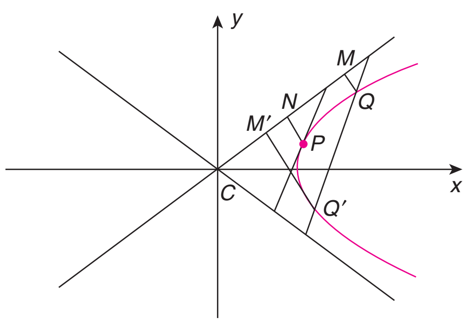


FIGURE 5.32

5. Prove that the area of a triangle formed by the two asymptotes and any tangent to the hyperbola is constant.

Solution: See Fig. 5.33. The tangent at $P(a\sec\theta, b\tan\theta)$ is

$$\frac{x}{a}\sec\theta-\frac{y}{b}\tan\theta=1 \quad (5.66)$$

Asymptotes are

$$y=\pm\left(\frac{b}{a}\right)x$$

Substitute $y=(b/a)x$ in Eq. (5.66) and let $Q=[a(\sec\theta+\tan\theta), b(\sec\theta+\tan\theta)]$. Also, substitute $y=(-b/a)x$ and let $R=[a(\sec\theta-\tan\theta), b(\tan\theta-\sec\theta)]$. Therefore, the area of ΔCQR is

$$\frac{1}{2}\left|ab(\tan^2\theta-\sec^2\theta)-ab(\sec^2\theta-\tan^2\theta)\right|$$

$$=\frac{ab}{2}\left|2(\tan^2\theta-\sec^2\theta)\right|$$

$$=ab$$

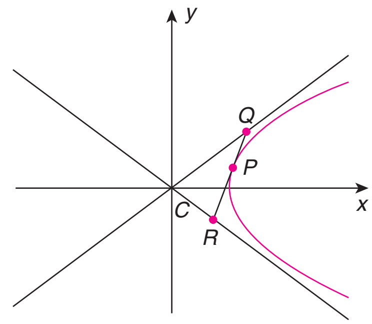


FIGURE 5.33

6. If $m\neq 0$, then prove that the point of intersection of the lines

$$\frac{x}{a}+\frac{y}{b}=m \text{ and } \frac{x}{a}-\frac{y}{b}=\frac{1}{m}$$

lies on a hyperbola.

Solution: Solving the given two equations, we have

$$x=\frac{a}{2}\left(m+\frac{1}{m}\right)$$

and

$$y=\frac{b}{2}\left(m-\frac{1}{m}\right)$$

So

$$\left(\frac{2x}{a}\right)^2-\left(\frac{2y}{b}\right)^2=4$$

$$\Rightarrow \frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

Hence, the lines intersect the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

7. Find the common tangent of the parabola $y^2=8x$ and the hyperbola $\frac{x^2}{1}-\frac{y^2}{3}=1$.

Solution: We have

$$y^2=8x=4(2)x$$

so that $a=2$. Hence, the line

$$y=mx+\frac{2}{m}$$

is a tangent to the parabola $y^2=8x$ for all real values of $m\neq 0$. This also touches the given hyperbola. So

$$\left(\frac{2}{m}\right)^2 = 1(m^2) - 3$$

$$\Rightarrow m^4 - 3m^2 - 4 = 0$$

$$\Rightarrow (m^2 - 4)(m^2 + 1) = 0$$

$$\Rightarrow m = \pm 2$$

Hence, the common tangents are $y = 2x + 1$ and $y = -2x - 1$.

8. P and Q are two points on the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

and $C(0,0)$ is its centre. If PQ subtends right angle at the centre C and $a < b$, then show that

$$\frac{1}{CP^2} + \frac{1}{CQ^2} = \frac{1}{a^2} - \frac{1}{b^2}$$

Solution: Let $CP = r_1$ and θ be the angle made by CP with the positive direction of the x-axis so that

$$P = (r_1 \cos\theta, r_1 \sin\theta)$$

Since $\angle PCQ = 90°$, we have

$$Q = (r_2 \cos(\pi/2 + \theta),\ r_2 \sin(\pi/2 + \theta))$$

$$Q = (-r_2 \sin\theta, r_2 \cos\theta)$$

where $r_2 = CQ$. Now, P and Q lie on the curve. This implies

$$r_1^2\left(\frac{\cos^2\theta}{a^2} - \frac{\sin^2\theta}{b^2}\right) = 1$$

and
$$r_2^2\left(\frac{\sin^2\theta}{a^2} - \frac{\cos^2\theta}{b^2}\right) = 1$$

Therefore

$$\frac{1}{r_1^2} + \frac{1}{r_2^2} = \frac{(\cos^2\theta + \sin^2\theta)}{a^2} - \frac{(\sin^2\theta + \cos^2\theta)}{b^2}$$
$$= \frac{1}{a^2} - \frac{1}{b^2}$$

Hence

$$\frac{1}{CP^2} + \frac{1}{CQ^2} = \frac{1}{a^2} - \frac{1}{b^2}$$

9. P is a point on the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

The tangent at P meets the asymptotes at Q and R. If C is the centre of the curve, then show that

$$CP \cdot CQ = a^2 + b^2$$

Solution: Let $P = (a\sec\theta,\ b\tan\theta)$ so that the tangent at P is

$$\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1 \tag{5.67}$$

and the two asymptotes are

$$\frac{x}{a} - \frac{y}{b} = 0 \tag{5.68}$$

and
$$\frac{x}{a} + \frac{y}{b} = 0 \tag{5.69}$$

Solving Eqs. (5.67) and (5.68), we have

$$Q = [a(\sec\theta + \tan\theta),\ b(\sec\theta + \tan\theta)]$$

and solving Eqs. (5.67) and (5.69), we have

$$R = [a(\sec\theta - \tan\theta),\ b(\tan\theta - \sec\theta)]$$

Therefore

$$(CP)^2(CQ)^2 = [a^2(\sec\theta + \tan\theta)^2 + b^2(\sec\theta + \tan\theta)^2]$$
$$\times [a^2(\sec\theta - \tan\theta)^2 + b^2(\tan\theta - \sec\theta)^2]$$
$$= (a^2 + b^2)^2(\sec\theta + \tan\theta)^2(\sec\theta - \tan\theta)^2$$
$$= (a^2 + b^2)^2(\sec^2\theta - \tan^2\theta)^2$$
$$= (a^2 + b^2)^2$$

Hence, $CP \cdot CQ = a^2 + b^2$.

10. A tangent to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

cuts the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

at P and Q. Show that the locus of the midpoint of PQ is the curve

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^2 = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

Solution: Let $M(x_1, y_1)$ be the midpoint of PQ so that the equation of the chord PQ is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}$$

This chord touches the hyperbola. This implies

$$\left(\frac{b^2}{y_1}\right)^2\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}\right)^2 = a^2\left(\frac{-b^2x_1}{a^2y_1}\right)^2 - b^2 \quad (\because c^2 = a^2m^2 - b^2)$$

$$\Rightarrow \frac{b^2}{y_1^2}\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}\right)^2 = \frac{b^2x_1^2}{a^2y_1^2} - 1$$

$$\Rightarrow \left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}\right)^2 = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2}$$

Hence, the required locus is

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2 = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

11. Prove that the line $y = x + 2$ touches the hyperbola $5x^2 - 9y^2 = 45$ and also find the point of contact.

Solution: The given hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

For the line $y = x + 2$, we have $c = 2$ and $m = 1$. Now,

$$a^2m^2 - b^2 = 9(1) - 5 = 4 = c^2$$

Hence, the line $y = x + 2$ touches the hyperbola. Suppose (x_1, y_1) is the point of contact. So

$$\frac{xx_1}{9} - \frac{yy_1}{5} = 1$$

This line and $y = x + 2$ represent the same line. Therefore

$$\frac{x_1/9}{1} = \frac{-y_1/5}{-1} = \frac{-1}{2}$$

Hence

$$x_1 = \frac{-9}{2} \text{ and } y_1 = \frac{-5}{2}$$

The point of contact is $(-9/2, -5/2)$.

12. The tangent at a point P on the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

meets one of the directrices at Q. Show that PQ subtends right angle at the corresponding focus.

Solution: Let $P = (a\sec\theta, b\tan\theta)$ so that the tangent at P is

$$\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1$$

This meets the directrix $x = a/e$ at point

$$Q = \left[\frac{a}{e}, \frac{b}{e}\frac{(1-e\cos\theta)}{\sin\theta}\right]$$

Also $S = (ae, 0)$ is the corresponding focus. Now,

$$\text{Slope of } SP \times \text{Slope of } SQ$$

$$= \left[\frac{b\tan\theta - 0}{a\sec\theta - ae}\right]\left[\frac{[(b/e)(1-e\sec\theta)] - 0}{[(a/e) - ae]\sin\theta}\right]$$

$$= \frac{b\sin\theta}{a(1-e\cos\theta)} \times \frac{b(1-e\cos\theta)}{a(1-e^2)\sin\theta}$$

$$= \frac{b^2}{a^2(1-e^2)} = \frac{b^2}{-b^2} = -1 \quad [\because b^2 = a^2(e^2-1)]$$

Therefore, PQ subtends right angle at $S(ae, 0)$.

13. A variable chord of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

touches the circle $x^2 + y^2 = k^2$. Show that the locus of the midpoint of the chord is

$$\left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^2 = k^2\left(\frac{x^2}{a^4} + \frac{y^2}{b^4}\right)$$

Solution: Let $M(x_1, y_1)$ be the midpoint of a chord of

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

so that its equation is

$$S_1 = S_{11}$$

$$\Rightarrow \frac{xx_1}{a^2} - \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2}$$

This line touches the circle $x^2 + y^2 = k^2$. So

$$\frac{\left|0 - 0 - (x_1^2/a^2) + (y_1^2/b^2)\right|}{\sqrt{(x_1^2/a^4) + (y_1^2/b^4)}} = k$$

$$\Rightarrow \left(\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2}\right)^2 = k^2\left(\frac{x_1^2}{a^4} + \frac{y_1^2}{b^4}\right)$$

Therefore, the locus of (x_1, y_1) is

$$\left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^2 = k^2\left(\frac{x^2}{a^4} + \frac{y^2}{b^4}\right)$$

14. A variable tangent to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

meets the tangent at the vertex $(a, 0)$ at R and the transverse axis at Q. Find the locus of the midpoint of QR.

Solution: Let $P=(a\sec\theta, b\tan\theta)$ so that the tangent at P is

$$\frac{x\sec\theta}{a}-\frac{y\tan\theta}{b}=1 \tag{5.70}$$

Transverse axis is

$$y=0 \tag{5.71a}$$

The tangent at $(a, 0)$ is

$$x=a \tag{5.71b}$$

Therefore,

$$Q=(a\cos\theta, 0) \text{ and } R=\left[a, \frac{b(1-\cos\theta)}{\sin\theta}\right]$$

Let (h, k) be the midpoint of QR. Therefore

$$2h=a(1+\cos\theta)=2a\cos^2\frac{\theta}{2}$$

$$\Rightarrow h=a\cos^2\frac{\theta}{2} \tag{5.72}$$

and $k=\dfrac{b(1-\cos\theta)}{2\sin\theta}=\dfrac{2b\sin^2(\theta/2)}{4\sin(\theta/2)\cos(\theta/2)}=\dfrac{b}{2}\tan\left(\dfrac{\theta}{2}\right)$ (5.73)

From Eqs. (5.72) and (5.73), we get

$$\frac{a}{h}-\left(\frac{2k}{b}\right)^2=\sec^2\frac{\theta}{2}-\tan^2\frac{\theta}{2}=1$$

Therefore, the locus of (h, k) is

$$\frac{a}{x}-\frac{4y^2}{b^2}=1$$

15. The chords of the hyperbola $x^2-y^2=a^2$ touch the parabola $y^2=4ax$. Find the locus of their midpoint.

Solution: The equation of the chord of $x^2-y^2=a^2$ whose midpoint is (x_1, y_1) is

$$xx_1-yy_1=x_1^2-y_1^2$$

This chord touches the parabola $y^2=4ax$. This implies

$$\frac{x_1^2-y_1^2}{y_1}=\frac{-a}{x_1/y_1} \qquad \left(\because c=\frac{a}{m}\right)$$

$$\Rightarrow \frac{x_1^2-y_1^2}{y_1}=-\frac{ay_1}{x_1}$$

$$\Rightarrow x_1^3-x_1y_1^2=-ay_1^2$$

$$\Rightarrow x_1^3=y_1^2(x_1-a)$$

Therefore, the locus of (x_1, y_1) is $y^2(x-a)=x^3$.

16. Find the locus of the midpoint of the normal chords of a rectangular hyperbola.

Solution: The equation of the chord of $x^2-y^2=a^2$ with midpoint (x_1, y_1) is

$$xx_1-yy_1=x_1^2-y_1^2$$

Suppose this is normal to $x^2-y^2=a^2$ at $(a\sec\theta, a\tan\theta)$ so that the equation of the normal is

$$x\cos\theta+y\cot\theta=2a$$

Hence

$$\frac{x_1}{\cos\theta}=\frac{-y_1}{\cot\theta}=\frac{x_1^2-y_1^2}{2a}$$

Therefore

$$\sec\theta=\frac{x_1^2-y_1^2}{2ax_1} \quad \text{and} \quad \tan\theta=\frac{-(x_1^2-y_1^2)}{2ay_1}$$

Now

$$\begin{aligned}1&=\sec^2\theta-\tan^2\theta\\&=\frac{(x_1^2-y_1^2)^2}{4a^2x_1^2}-\frac{(x_1^2-y_1^2)^2}{4a^2y_1^2}\\&=(x_1^2-y_1^2)^2\left[\frac{y_1^2-x_1^2}{4a^2x_1^2y_1^2}\right]\\&\Rightarrow (x_1^2-y_1^2)^3+4a^2x_1^2y_1^2=0\end{aligned}$$

Therefore, the locus of (x_1, y_1) is

$$(x^2-y^2)^3+4a^2x^2y^2=0$$

17. From any point on a hyperbola, tangents are drawn to another hyperbola having the same asymptotes. Show that the chord of contact cuts off a triangle of constant area from the asymptotes.

Solution: Let the hyperbolas be

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \tag{5.74}$$

and

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=k \tag{5.75}$$

Let $P(a\sec\theta, b\tan\theta)$ be a point on the hyperbola provided in Eq. (5.74). The chord of contact of P with hyperbola provided in Eq. (5.75) is

$$\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = k \tag{5.76}$$

The asymptotes are

$$y = \pm\left(\frac{b}{a}\right)x$$

These asymptotes meet the line provided in Eq. (5.76) at the points

$$[ak(\sec\theta+\tan\theta), bk(\sec\theta+\tan\theta)]$$

and $$[ak(\sec\theta-\tan\theta), bk(\tan\theta-\sec\theta)]$$

and hence the area of the triangle cut off is given by

$$y = \frac{2}{2}abk^2\left|\sec^2\theta - \tan^2\theta\right| = abk^2$$

which is constant.

18. Show that the locus of the midpoint of normal chords of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is

$$(a^6y^2-b^6x^2)(a^2y^2-b^2x^2)^2 = (a^2+b^2)^2\, a^4b^4x^2y^2$$

Solution: The equation of a chord with midpoint (x_1, y_1) is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} \tag{5.77}$$

Suppose the equation of the chord provided in Eq. (5.77) is normal to the hyperbola at $P(a\sec\theta, b\tan\theta)$ whose equation is

$$ax\cos\theta + by\cot\theta = a^2+b^2 \tag{5.78}$$

From Eqs. (5.77) and (5.78), we get

$$\frac{x_1/a^2}{a\cos\theta} = \frac{-y_1/a^2}{b\cos\theta} = \frac{(x_1^2/a^2)-(y_1^2/b^2)}{a^2+b^2}$$

Hence

$$\sec\theta = \frac{a^3[(x_1^2/a^2)-(y_1^2/b^2)]}{x_1(a^2+b^2)}$$

and $$\tan\theta = -\frac{b^3[(x_1^2/a^2)-(y_1^2/b^2)]}{y_1(a^2+b^2)}$$

Now

$$1 = \sec^2\theta - \tan^2\theta$$

$$= \frac{a^6[(x_1^2/a^2)-(y_1^2/b^2)]^2}{x_1^2(a^2+b^2)^2} - \frac{b^6[(x_1^2/a^2)-(y_1^2/b^2)]^2}{y_1^2(a^2+b^2)^2}$$

$$\Rightarrow 1 = \frac{[(x_1^2/a^2)-(y_1^2/b^2)]^2}{(a^2+b^2)^2}\left[\frac{a^6}{x_1^2} - \frac{b^6}{y_1^2}\right]$$

$$\Rightarrow 1 = \frac{(b^2x_1^2-a^2y_1^2)^2(a^6y_1^2-b^6x_1^2)}{a^4b^4(a^2+b^2)^2x_1^2y_1^2}$$

Hence, the locus of (x_1, y_1) is

$$a^4b^4(a^2+b^2)^2x^2y^2 = (b^2x^2-a^2y^2)^2(a^6y^2-b^6x^2)$$

19. Prove that the locus of the midpoint of a chord of the hyperbola $ax^2-by^2=1$ which subtends right angle at the centre of the curve is

$$(a-b)(ax^2-by^2)^2 = a^2x^2+b^2y^2$$

Solution: The given conic is

$$\frac{x^2}{(1/a)} - \frac{y^2}{(1/b)} = 1$$

The equation of the chord PQ in terms of its midpoint (x_1, y_1) is

$$axx_1 - byy_1 = ax_1^2 - by_1^2 \tag{5.79}$$

Hence, the combined equation of the pair of lines CP and CQ (C is the origin) is

$$ax^2 - by^2 - 1\left[\frac{axx_1-byy_1}{ax_1^2-by_1^2}\right]^2 = 0$$

Since $\angle PCQ = 90°$ in the above equation

$$\text{Coefficient of } x^2 + \text{Coefficient of } y^2 = 0$$

$$\Rightarrow \left(a - \frac{a^2x_1^2}{(ax_1^2-by_1^2)^2}\right) + \left(-b - \frac{b^2y_1^2}{(ax_1^2-by_1^2)^2}\right) = 0$$

$$\Rightarrow a - b - \frac{(a^2x_1^2+b^2y_1^2)}{(ax_1^2-by_1^2)^2} = 0$$

$$\Rightarrow (a-b)(ax_1^2-by_1^2)^2 = a^2x_1^2+b^2y_1^2$$

Hence, the locus of (x_1, y_1) is

$$(a-b)(ax^2-by^2)^2 = a^2x^2+b^2y^2$$

20. The chords of the circle $x^2+y^2=a^2$ touch the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. Prove that the midpoints of the chords lie on the curve $(x^2+y^2)^2 = a^2x^2-b^2y^2$.

Solution: The chords of the circle $x^2+y^2=a^2$ whose midpoint (x_1, y_1) is $xx_1+yy_1=x_1^2+y_1^2$. This chord touches the hyperbola. This implies

$$\left(\frac{x_1^2+y_1^2}{y_1}\right)^2=a^2\left(\frac{-x_1}{y_1}\right)^2-b^2$$

$$\Rightarrow (x_1^2+y_1^2)^2=a^2x_1^2-b^2y_1^2$$

Therefore, (x_1, y_1) lies on the curve

$$(x^2+y^2)^2=a^2x^2-b^2y^2$$

21. Prove that the locus of the midpoints of the chords of

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

passing through a fixed point (h, k) is also a hyperbola with centre at $(h/2, k/2)$.

Solution: The equation of chord in terms of its midpoint (x_1, y_1) is

$$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}$$

This passes through the point (h, k). So

$$\frac{hx_1}{a^2}-\frac{ky_1}{b^2}=\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}$$

$$\Rightarrow \frac{x_1^2-hx_1}{a^2}-\frac{y_1^2-ky_1}{b^2}=0$$

$$\Rightarrow \frac{(x_1-h/2)^2}{a^2}-\frac{(y_1-k/2)^2}{b^2}=\frac{h^2}{4a^2}-\frac{k^2}{4b^2}$$

Therefore, the locus of (x_1, y_1) is the hyperbola

$$\frac{[x^2-(h/2)]^2}{a^2}-\frac{[y-(k/2)]^2}{b^2}=\frac{1}{4}\left(\frac{h^2}{a^2}-\frac{k^2}{b^2}\right)$$

22. On the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

P is a point and S is one of the foci. If the tangent at P, the latus rectum through S and one of the asymptotes are concurrent, then show that focal radius SP is parallel to the other asymptote.

Solution: The tangent at $P(a\sec\theta, b\tan\theta)$ is

$$\frac{x\sec\theta}{a}-\frac{y\tan\theta}{b}=1 \tag{5.80}$$

The latus rectum through $S(ae, 0)$ is

$$x=ae \tag{5.81}$$

The asymptote is

$$\frac{x}{a}-\frac{y}{b}=0$$

$$\Rightarrow y=\left(\frac{b}{a}\right)x \tag{5.82}$$

The point of intersection of Eqs. (5.81) and (5.82) is (ae, be). By hypothesis, the point (ae, be) lies on the tangent provided in Eq. (5.80) which implies that

$$e(\sec\theta-\tan\theta)=1$$

$$\Rightarrow e=\sec\theta+\tan\theta \tag{5.83}$$

Now, the slope of SP is

$$\frac{b\tan\theta-0}{a\sec\theta-ae}=\frac{b\sin\theta}{a(1-e\cos\theta)}$$

Using Eq. (5.83), we get

$$\frac{b\sin\theta}{a(1-e\cos\theta)}=\frac{b\sin\theta}{a[1-\cos\theta(\sec\theta+\tan\theta)]}$$

$$=\frac{b\sin\theta}{a[1-1-\sin\theta]}=\frac{-b}{a}$$

which is the slope of the other asymptote

$$\frac{x}{a}+\frac{y}{b}=0$$

23. A normal to the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

meets the axes at Q and R. Lines QL and RL are drawn at right angles to the axes and meet at L. Prove that the locus of L is the hyperbola

$$a^2x^2-b^2y^2=(a^2+b^2)$$

Solution: See Fig. 5.34. The normal at $P(a\sec\theta, b\tan\theta)$ is $ax\cos\theta+by\cot\theta=a^2+b^2$. Therefore

$$Q=\left[\left(\frac{a^2+b^2}{a}\right)\sec\theta, 0\right]$$

and $$R=\left[0, \left(\frac{a^2+b^2}{a}\right)\tan\theta\right]$$

If $L=(x, y)$, then

$$x=\left(\frac{a^2+b^2}{a}\right)\sec\theta$$

and $$y = \left(\frac{a^2+b^2}{b}\right)\tan\theta$$

Therefore

$$\left(\frac{ax}{a^2+b^2}\right)^2 - \left(\frac{by}{a^2+b^2}\right)^2 = \sec^2\theta - \tan^2\theta = 1$$

$$\Rightarrow a^2x^2 - b^2y^2 = (a^2+b^2)^2$$

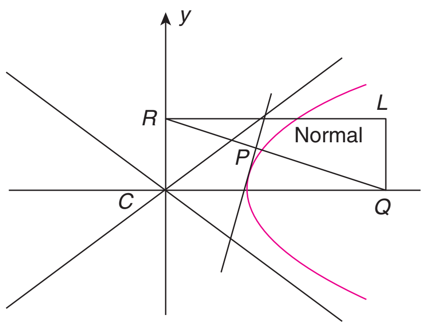


FIGURE 5.34

24. If the tangent at (α, β) of

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

meets the auxiliary circle at (x_1, y_1) and (x_2, y_2), then show that β is harmonic between y_1 and y_2.

Solution: The tangent at (α, β) is

$$\frac{x\alpha}{a^2} - \frac{y\beta}{b^2} = 1 \tag{5.84}$$

and the auxillary circle is

$$x^2 + y^2 = a^2 \tag{5.85}$$

Substituting

$$x = \frac{a^2}{\alpha}\left(1 + \frac{y\beta}{b^2}\right)$$

in Eq. (5.85), we get

$$\frac{a^4}{\alpha^2}\frac{(b^2+\beta y)^2}{b^4} + y^2 = a^2$$

$$\Rightarrow \frac{a^4[b^4 + 2\beta b^2 y + \beta^2 y^2]}{\alpha^2 b^4} + y^2 = a^2$$

$$\Rightarrow (a^4\beta^2 + \alpha^2 b^4)y^2 + 2\beta a^4 b^2 y - a^4 b^4 + a^2\alpha^2 b^4 = 0$$

Let y_1 and y_2 be the roots of this quadratic equation. Then

$$y_1 + y_2 = \frac{-2\beta a^4 b^2}{a^4\beta^2 + \alpha^2 b^4}$$

and $$y_1 y_2 = \frac{b^4 a^2(a^2 - \alpha^2)}{a^4\beta^2 + \alpha^2 b^4}$$

Therefore

$$\frac{1}{y_1} + \frac{1}{y_2} = \frac{y_1 + y_2}{y_1 y_2}$$

$$= \frac{-2\beta a^4 b^2}{b^4 a^2(a^2 - \alpha^2)}$$

$$= \frac{(-2\beta)a^2}{b^2(a^2 - \alpha^2)}$$

$$= \frac{-2\beta}{(b^2/a^2)(a^2 - \alpha^2)}$$

$$= \frac{-2\beta}{-\beta^2} = \frac{2}{\beta} \qquad \left(\because \frac{\alpha^2}{a^2} - \frac{\beta^2}{b^2} = 1\right)$$

Hence, β is harmonic between y_1 and y_2.

25. Prove that the chord of the hyperbola which touches the conjugate hyperbola is bisected at the point of contact.

Solution: Let the hyperbola and its conjugate be

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

and $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$$

respectively. Let $P(h, k)$ be a point on

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$$

Hence

$$\frac{h^2}{a^2} - \frac{k^2}{b^2} = -1 \tag{5.86}$$

The equation of the tangent at (h, k) is

$$\frac{hx}{a^2} - \frac{ky}{b^2} = -1 \tag{5.87}$$

Let the tangent provided in Eq. (5.87) meet the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

at Q and R. We have to show that (h, k) is the midpoint of the chord QR. Now, the equation of the chord QR in terms of its midpoint, say, (x_1, y_1) is

$$\frac{xx_1}{a^2}-\frac{yy_1}{b_2}=\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2} \tag{5.88}$$

Thus, Eqs. (5.87) and (5.88) represent the same straight line. Hence

$$\frac{h/a^2}{x_1/a^2}=\frac{k/a^2}{y_1/b^2}=\frac{-1}{(x_1^2/a^2)-(y_1^2/b^2)}$$

$$\Rightarrow \frac{h}{x_1}=\frac{k}{y_1}=\frac{-1}{(x_1^2/a^2)-(y_1^2/b^2)} \tag{5.89}$$

Therefore,

$$h=\frac{-x_1}{(x_1^2/a^2)-(y_1^2/b^2)}$$

and $$k=\frac{-y_1}{(x_1^2/a^2)-(y_1^2/b^2)}$$

Hence, from Eq. (5.86), we have

$$-1=\frac{h^2}{a^2}-\frac{k^2}{b^2}=\frac{(x_1^2/a^2)-(y_1^2/b^2)}{[(x_1^2/a^2)-(y_1^2/b^2)]^2}=\frac{1}{(x_1^2/a^2)-(y_1^2/b^2)}$$

$$\Rightarrow \frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}=-1$$

Hence, from Eq. (5.89), we have

$$\frac{h}{x_1}=\frac{k}{y_1}=\frac{(-1)}{(-1)}=1$$

Thus, $h=x_1$ and $k=y_1$.

26. Let P be a point such that the chord of contact of P with respect to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ touches the circle described on the line joining the foci as ends of a diameter. Show that P lies on the curve $\frac{x^2}{a^4}-\frac{y^2}{b^4}=\frac{1}{a^2+b^2}$.

Solution: Let $P=(x_1, y_1)$ so that its chord of contact with respect to the hyperbola is

$$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1 \tag{5.90}$$

The equation of the circle with $S(ae, 0)$ and $S'(-ae, 0)$ as ends of a diameter is

$$(x-ae)(x+ae)+y^2=0$$

$$\Rightarrow x^2+y^2=(ae)^2 \tag{5.91}$$

The line provided in Eq. (5.90) touches the circle given by Eq. (5.91). This implies

$$\frac{\left|[(0)x_1/a^2]-[(0)y_1/b^2-1]\right|}{\sqrt{(x_1^2/a^4)+(y_1^2/b^4)}}=ae$$

$$\Rightarrow \frac{x_1^2}{a^4}+\frac{y_1^2}{b^4}=\frac{1}{a^2e^2}=\frac{1}{a^2+b^2}$$

27. Show that the locus of the point of intersection of tangents to the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

whose sum of the slopes is a constant p is

$$p(x^2-a^2)=2xy$$

Solution: Let (x_1, y_1) be a point through which tangent can be drawn. It is known that $y=mx+\sqrt{a^2m^2-b^2}$ touches the hyperbola. This tangent passes through (x_1, y_1). This implies

$$(y_1-mx_1)^2=a^2m^2-b^2$$

$$\Rightarrow (x_1^2-a^2)m^2-2mx_1y_1+y_1^2+b^2=0$$

This equation being a quadratic in m has two roots which are the slopes of the tangents from (x_1, y_1). Therefore

$$p = \text{Sum of slopes } =\frac{2x_1y_1}{x_1^2-a^2}$$

Hence, the locus of the point (x_1, y_1) is $p(x^2-a^2) = 2xy$.

28. Find the locus of the midpoints of the chords of $ax^2-by^2=1$ which touch the parabola $y^2=4px$.

Solution: The given hyperbola is

$$\frac{x^2}{1/a}-\frac{y^2}{1/b}=1$$

The equation of the chord of the hyperbola whose midpoint is (x_1, y_1) is

$$axx_1-byy_1=ax_1^2-by_1^2$$

This line touches the parabola which implies

$$\frac{-(ax_1^2-by_1^2)}{by_1}=\frac{p}{ax_1/by_1}$$

$$-ax_1(ax_1^2-by_1^2)=pb^2y_1^2$$

$$\Rightarrow ax_1(ax_1^2-by_1^2)+pb^2y_1^2=0$$

Thus, the locus of (x_1, y_1) is $ax(ax^2-by^2)+pb^2y^2=0$.

29. The chord of the circle $x^2+y^2=a^2$ touches the rectangular hyperbola $x^2-y^2=a^2$. Show that the locus of the midpoint of the chord is $(x^2+y^2)^2 = a^2(x^2-y^2)$.

Solution: The equation of the chord of $x^2+y^2=a^2$ with (x_1,y_1) as its midpoint is $xx_1+yy_1=x_1^2+y_1^2$. This chord touches the hyperbola $x^2-y^2=a^2$. So

$$\left(\frac{x_1^2+y_1^2}{y_1}\right)^2=a^2\left(\frac{-x_1}{y_1}\right)^2-a^2$$

$$\Rightarrow (x_1^2+y_1^2)^2 = a^2(x_1^2-y_1^2)$$

Hence, the locus of (x_1,y_1) is $(x^2+y^2)^2=a^2(x^2-y^2)$.

30. Tangents are drawn to the circle $x^2+y^2=9$ from point on the hyperbola

$$\frac{x^2}{9}-\frac{y^2}{4}=1$$

Find the locus of the midpoint of the chord of contact. **(IIT-JEE 2005)**

Solution: Let $M(x_1,y_1)$ be the midpoint of chord of contact of the circle $x^2+y^2=9$. Hence, its equation is

$$xx_1+yy_1=x_1^2+y_1^2 \tag{5.92}$$

Suppose the line provided in Eq. (5.92) is the chord of contact of (h,k) on the hyperbola. Therefore

$$\frac{h^2}{9}-\frac{k^2}{4}=1 \tag{5.93}$$

The equation of the chord of contact of (h, k) with respect to the circle $x^2+y^2=9$ is

$$hx+ky=9 \tag{5.94}$$

Now, Eqs. (5.92) and (5.94) represent the same line. Therefore

$$\frac{h}{x_1}=\frac{k}{+y_1}=\frac{9}{x_1^2+y_1^2}$$

$$\Rightarrow h=\frac{9x_1}{x_1^2+y_1^2},k=\frac{9y_1}{x_1^2+y_1^2}$$

From Eq. (5.93), since

$$\frac{h^2}{9}-\frac{k^2}{4}=1$$

we have

$$\frac{81x_1^2}{9(x_1^2+y_1^2)^2}-\frac{81y_1^2}{4(x_1^2+y_1^2)^2}=1$$

$$\Rightarrow 9x_1^2-\frac{81y_1^2}{4}=(x_1^2+y_1^2)^2$$

Therefore, the locus of (x_1,y_1) is

$$x^2-\frac{9y^2}{4}=\frac{1}{9}(x^2+y^2)^2$$

31. In a rectangular hyperbola $xy=c^2$, if m is the slope of a chord, then show that the equation of the circle described on this chord as diameter is of the form

$$x^2+y^2-c^2\left(m+\frac{1}{m}\right)+\lambda(x-my)=0$$

for some real number λ.

Solution: Let

$$A\left(ct_1,\frac{c}{t_1}\right) \text{ and } B\left(ct_2,\frac{c}{t_2}\right)$$

be the two points on $xy=c^2$ such that

$$\text{Slope of } \overline{AB}=m$$

$$\Rightarrow \frac{c/t_1-c/t_2}{ct_1-ct_2}=m$$

$$\Rightarrow t_1t_2=-\frac{1}{m} \tag{5.95}$$

The circle having A and B as extremities of a diameter is

$$(x-ct_1)(x-ct_2)+\left(y-\frac{c}{t_1}\right)\left(y-\frac{1}{t_2}\right)=0$$

$$\Rightarrow x^2+y^2-c(t_1+t_2)x-c\left(\frac{1}{t_1}+\frac{1}{t_2}\right)y+c^2t_1t_2+\frac{c^2}{t_1t_2}=0$$

$$\Rightarrow x^2+y^2-c(t_1+t_2)x-cy(t_1+t_2)(-m)+c^2\left(-\frac{1}{m}-m\right)=0$$

$$\Rightarrow x^2+y^2-c^2\left(m+\frac{1}{m}\right)-c(t_1+t_2)(x-my)=0$$

Take $\lambda=-c(t_1+t_2)$ to get the required result.

32. Prove that the orthocentre of a triangle inscribed in a rectangular hyperbola lies on the rectangular hyperbola.

Solution: Suppose

$$P\left(ct_1,\frac{c}{t_1}\right),\ Q\left(ct_2,\frac{c}{t_2}\right) \text{ and } R\left(ct_3,\frac{c}{t_3}\right)$$

be the three points on the curve $xy=c^2$. Since, the slope of QR is $-1/t_2t_3$, the equation of the altitude through P is

$$y-\frac{c}{t_1}=t_2t_3(x-ct_1)$$

$$\Rightarrow y-\frac{c}{t_1}=t_2t_3x-ct_1t_2t_3 \qquad (5.96)$$

The equation of the altitude through Q is

$$y-\frac{c}{t_2}=t_3t_1x-ct_1t_2t_3 \qquad (5.97)$$

Subtracting Eq. (5.97) from Eq. (5.96), we have

$$c\left(\frac{1}{t_2}-\frac{1}{t_1}\right)=t_3(t_2-t_1)x$$

$$\Rightarrow \frac{c(t_1-t_2)}{t_1t_2}=t_3(t_2-t_1)x$$

Therefore

$$x=\frac{-c}{t_1t_2t_3}$$

and $$y=-ct_1t_2t_3$$

so that $xy=c^2$. This implies that the orthocentre lies on the curve.

33. If a circle intersects $xy = 1$ at four points (x_r, y_r) (where $r=1, 2, 3$ and 4), then show that

$$x_1x_2x_3x_4=y_1y_2y_3y_4=1$$

Solution: Let the equation of the circle be

$$S\equiv x^2+y^2+2gx+2fy+c=0$$

In $S=0$, substituting $y=1/x$ we get

$$x^4+2gx^3+cx^2+2fx+1=0$$

This is fourth-degree equation in x. So it has four roots, say, x_1, x_2, x_3 and x_4. Therefore,

$$x_1x_2x_3x_4=\text{Product of the roots}=1$$

and $$y_1y_2y_3y_4=\frac{1}{x_1x_2x_3x_4}=1$$

34. If the normals at (x_r, y_r) $(r=1, 2, 3$ and $4)$ on the rectangular hyperbola $xy = c^2$ meet in a point (h, k), then prove that

$$h=x_1+x_2+x_3+x_4$$
$$k=y_1+y_2+y_3+y_4$$
$$x_1x_2x_3x_4=y_1y_2y_3y_4=-c^4$$

Solution: The normal at $(ct, c/t)$, according to Theorem 5.25, part (2), is

$$xt^3-ty+c-ct^4=0$$

It passes through (h, k). This implies

$$ht^3-tk+c-ct^4=0$$

$$\Rightarrow ct^4-ht^3+kt-c=0 \qquad (5.98)$$

If $x_r=ct_r$ and $y_r=c/t_r$ for r = 1, 2, 3 and 4, then t_1, t_2, t_3 and t_4 are the roots of Eq. (5.98). Therefore

$$t_1+t_2+t_3+t_4=\frac{h}{c}$$

and $$t_1t_2t_3t_4=\frac{-c}{c}=-1$$

Hence

$$x_1+x_2+x_3+x_4=c(t_1+t_2+t_3+t_4)=h$$

$$y_1+y_2+y_3+y_4=c\left(\frac{1}{t_1}+\frac{1}{t_2}+\frac{1}{t_3}+\frac{1}{t_4}\right)=\frac{c\,\Sigma\, t_2t_3t_4}{t_1t_2t_3t_4}$$

$$=c\frac{-k/c}{-1}=k$$

Therefore

$$x_1x_2x_3x_4=c^4(t_1t_2t_3t_4)=c^4(-1)=-c^4$$

and $$y_1y_2y_3y_4=\frac{c^4}{t_1t_2t_3t_4}=-c^4$$

35. Let P_1, P_2, P_3 and P_4 be four points on $xy = c^2$ such that the chord P_1P_2 is perpendicular to the chord P_3P_4. If C is the centre and the lines CP_1, CP_2, CP_3 and CP_4 make angles α, β, γ and δ, respectively, with an asymptote, then show that

$$\tan\alpha\,\tan\beta\,\tan\gamma\,\tan\delta=1$$

Solution: The line through the centre is $y=x\tan\theta$ so that it meets the curve $xy=c^2$ at point $(c\sqrt{\cot\theta}, c\sqrt{\tan\theta})$. Hence, the four points are $(c\sqrt{\cot\theta}, c\sqrt{\tan\theta})$ (where θ = α, β, γ and δ). The lines P_1P_2 and P_3P_4 are at right angles. This means

$$\left(\frac{c\sqrt{\tan\beta}-c\sqrt{\tan\alpha}}{c\sqrt{\cot\beta}-c\sqrt{\cot\alpha}}\right)\left(\frac{c\sqrt{\tan\delta}-c\sqrt{\tan\gamma}}{c\sqrt{\cot\delta}-c\sqrt{\cot\gamma}}\right)=-1$$

$$\Rightarrow(-\sqrt{\tan\beta}\sqrt{\tan\alpha})(-\sqrt{\tan\delta}\sqrt{\tan\gamma})=-1$$

$$\Rightarrow\tan\alpha\tan\beta\tan\gamma\tan\delta=(-1)^2=1$$

36. A circle cuts a rectangualar hyperbola $xy=c^2$ at A, B, C and D. If H is the orthocentre of ΔABC, then show that H and D are the extremities of a diameter of the curve.

Solution: If $(ct_r, c/t_r)$ (where r = 1, 2, 3 and 4) are points A, B, C and D, respectively, then by Problem 33, the orthocentre H of triangle ABC is

$$\left(-\frac{e}{t_1t_2t_3}, -ct_1t_2t_3\right)$$

which also lies on the curve. Now, since $t_1t_2t_3t_4$ = 1, we have

$$t_4 = \frac{1}{t_1t_2t_3}$$

so that

$$D = \left(ct_4, \frac{c}{t_4}\right) = \left(\frac{c}{t_1t_2t_3}, ct_1t_2t_3\right)$$

and hence (0, 0) is the midpoint of HD.

37. From any point P of a rectangular hyperbola $x^2 - y^2 = a$ with centre O, the perpendiculars PM and PN are drawn to the principal axes (i.e., usual axes). Show that the tangent at P is perpendicular to MN and that the distance of O from the tangent varies inversely as OP.

Solution: Let $P = (a \sec\theta, a \tan\theta)$ so that $M = (a \sec\theta, 0)$ and $N = (0, a \tan\theta)$. The tangent at P is

$$x\sec\theta - y\tan\theta = a$$

so that the slope of the tangent at P is

$$\frac{\sec\theta}{\tan\theta} = \operatorname{cosec}\theta$$

Slope of MN is

$$\frac{-a\tan\theta}{a\sec\theta} = -\sin\theta$$

Now,

Slope of the tangent at $P \times$ Slope of $MN = \operatorname{cosec}\theta \times (-\sin\theta) = -1$

Hence, the tangent at P is perpendicular to MN. Also d is the distance of O from the tangent at P which is given by

$$d = \frac{|a|}{\sqrt{\sec^2\theta + \tan^2\theta}} = \frac{|a|^2}{OP}$$

38. If the tangent and normal to a rectangular hyperbola $x^2 - y^2 = a^2$ cut off intercepts a_1 and a_2 on one axis and b_1 and b_2 on another axis, then show that

$$a_1a_2 + b_1b_2 = 0$$

Solution: The tangent and normal at $P\,(a \sec\theta, a \tan\theta)$ are

$$x\sec\theta - y\tan\theta = a \tag{5.99}$$

and

$$x\cos\theta + y\cot\theta = 2a \tag{5.100}$$

Therefore, $a_1 = a\cos\theta$, $a_2 = 2a\sec\theta$, $b_1 = -a\cot\theta$ and $b_2 = 2a\tan\theta$. Hence

$$a_1a_2 + b_1b_2 = 2a^2 - 2a^2 = 0$$

39. A rectangular hyperbola, with centre C, is cut by a circle of radius r at four points P_1, P_2, P_3 and P_4. Prove that

$$CP_1^2 + CP_2^2 + CP_3^2 + CP_4^2 = 4r^2$$

Solution: Let the hyperbola be $xy = c^2$ and the circle be

$$S \equiv x^2 + y^2 + 2gx + 2fy + k = 0 \tag{5.101}$$

Substituting $x = ct$ and $y = c/t$ in Eq. (5.101), we get

$$c^2t^4 + 2gct^3 + kt^2 + 2fct + c^2 = 0$$

whose roots are t_1, t_2, t_3 and t_4 where

$$P_r = \left(ct_r, \frac{c}{t_r}\right)$$

(where $r = 1, 2, 3$ and 4). Therefore

$$t_1 + t_2 + t_3 + t_4 = \frac{-2g}{c}$$

$$\sum t_1t_2 = \frac{k}{c^2}$$

$$\sum t_1t_2t_3 = \frac{-2f}{c}$$

and

$$t_1t_2t_3t_4 = \frac{c^2}{c^2} = 1$$

Now,

$$\begin{aligned}
\sum_{i=1}^{4} CP_i^2 &= c^2\left[t_1^2 + t_2^2 + t_3^2 + t_4^2 + \frac{1}{t_1^2} + \frac{1}{t_2^2} + \frac{1}{t_3^2} + \frac{1}{t_4^2}\right] \\
&= c^2\left[(\Sigma t_1)^2 - 2\Sigma t_1t_2 + \left(\sum \frac{1}{t_1}\right)^2 - 2\sum \frac{1}{t_1t_2}\right] \\
&= c^2\left[\frac{4g^2}{c^2} - \frac{2k}{c^2} + \left(\frac{\Sigma t_1t_2t_3}{t_1t_2t_3t_4}\right)^2 - \frac{2\Sigma t_3t_4}{t_1t_2t_3t_4}\right] \\
&= c^2\left[\frac{4g^2}{c^2} - \frac{2k}{c^2} + \frac{4f^2}{c^2} - \frac{2k}{c^2}\right] \qquad (\because t_1t_2t_3t_4 = 1) \\
&= 4(g^2 + f^2 - k) \\
&= 4r^2
\end{aligned}$$

40. Show that if a rectangular hyperbola cuts a circle at four points, the centre of mean position of the four points is in midway between the centres of the two curves.

Solution: Let $xy=k$ and

$$S\equiv x^2+y^2+2gx+2fy+c=0 \qquad (5.102)$$

Substituting $y=k/x$ in Eq. (5.102), we get

$$x^4+2gx^3+cx^2+2fkx+k^2=0$$

whose roots are the abscissae of the common points of the circle and the rectangular hyperbola. If (x_1, y_1), (x_2, y_2), (x_3, y_3) and (x_4, y_4) are the points of intersection, then

$$x_1+x_2+x_3+x_4=-2g$$

and $$y_1+y_2+y_3+y_4=k\left(\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_4}\right)$$

$$=\frac{k\sum x_1x_2x_3}{x_1x_2x_3x_4}=\frac{k(-2fk)}{k^2}=-2f$$

Now, the mean position of the four points is

$$\left(\frac{x_1+x_2+x_3+x_4}{4}, \frac{y_1+y_2+y_3+y_4}{4}\right)=\left(\frac{-g}{2}, \frac{-f}{2}\right)$$

which is the midpoint of the line joining the centre $(-g,-f)$ of the circle and $C(0,0)$ of the rectangular hyperbola.

WORKED-OUT PROBLEMS

Single Correct Choice Type Questions

1. If m_1 and m_2 are the slopes of the tangents to the hyperbola

$$\frac{x^2}{25}-\frac{y^2}{16}=1$$

from the point $(6,2)$, then

$$\frac{1}{m_1}+\frac{1}{m_2}=$$

(A) $\frac{5}{6}$ (B) $\frac{6}{5}$ (C) $\frac{4}{5}$ (D) $\frac{5}{4}$

Solution: Suppose the tangent $y=mx+\sqrt{25m^2-16}$ passes through the point $(6,2)$. That is,

$$(6m-2)^2=25m^2-16$$

$$\Rightarrow 11m^2-24m+20=0$$

The roots are m_1 and m_2. Therefore

$$m_1+m_2=\frac{24}{11}$$

and $$m_1m_2=\frac{20}{11}$$

Hence

$$\frac{1}{m_1}+\frac{1}{m_2}=\frac{24}{20}=\frac{6}{5}$$

Answer: (B)

2. If $2x+\sqrt{6}y=2$ touches the hyperbola $x^2-2y^2=4$, then the point of contact is

(A) $\left(\frac{1}{2}, \frac{1}{\sqrt{6}}\right)$ (B) $(4, -\sqrt{6})$

(C) $(4, \sqrt{6})$ (D) $(-2, \sqrt{6})$

(IIT-JEE 2005)

Solution: Suppose $2x+\sqrt{6}y=2$ touches at (x_1, y_1). At (x_1, y_1), the equation of the tangent to the hyperbola is

$$\frac{xx_1}{4}-\frac{yy_1}{2}=1$$

That is, both equations represent the tangent at (x_1, y_1). Hence

$$\frac{(x_1/4)}{2}=\frac{(-y_1/2)}{\sqrt{6}}=\frac{1}{2}$$

$$\Rightarrow x_1=4, \quad y_1=-\sqrt{6}$$

Hence point of contact $=(4, -\sqrt{6})$.

Answer: (B)

3. If the foci of the ellipse

$$\frac{x^2}{16}+\frac{y^2}{b^2}=1$$

and the hyperbola

$$\frac{x^2}{144}-\frac{y^2}{81}=\frac{1}{25}$$

coincide, then the value of b^2 is

(A) 1 (B) 5 (C) 7 (D) 9

Solution: From the ellipse equation, we have $b^2 = a^2(1-e^2)$. From the hyperbola equation, we have

$$\frac{81}{25} = \frac{144}{25}(e_1{}^2 - 1)$$

$$\Rightarrow \frac{81}{144} + 1 = e_1{}^2$$

$$\Rightarrow e_1 = \frac{15}{12}$$

Foci of the hyperbola are

$$\left(\pm\frac{12}{5}\left(\frac{15}{12}\right), 0\right) = (\pm 3, 0)$$

Now for the ellipse $(3,0) = (4e, 0) \Rightarrow e = 3/4$. Therefore

$$b^2 = a^2(1-e^2) = 16\left(1 - \frac{9}{16}\right) = 7$$

Answer: (C)

4. The eccentricity of the hyperbola $9x^2 - 16y^2 - 72x + 96y - 144 = 0$ is

(A) $\frac{3}{2}$ (B) $\frac{4}{3}$ (C) $\frac{6}{5}$ (D) $\frac{5}{4}$

Solution: The given equation can be written as

$$9(x^2 - 8x) - 16(y^2 + 6y) - 144 = 0$$

$$\Rightarrow 9(x-4)^2 - 16(y+3)^2 = 144 + 144 - 144$$

$$\Rightarrow \frac{(x-4)^2}{16} - \frac{(y+3)^2}{9} = 1$$

Therefore, the eccentricity is given by

$$9 = 16(e^2 - 1)$$

$$\Rightarrow e = \frac{5}{4}$$

Answer: (D)

5. The number of values of c such that the line $y = 4x + c$ touches the ellipse $\frac{x^2}{4} + \frac{y^2}{1} = 1$ is

(A) 0 (B) 1
(C) 2 (D) infinite

Solution: We have

$$c^2 = a^2m^2 + b^2$$

$$\Rightarrow c^2 = 4(16) + 1 = 65$$

$$\Rightarrow c = \pm\sqrt{65}$$

Hence, there are two values of c, namely, $+\sqrt{65}$ and $-\sqrt{65}$.

Answer: (C)

6. Tangents are drawn to $x^2 + 2y^2 = 2$. The locus of the midpoint of the intercept made by tangents between the axis is

(A) $\frac{1}{x^2} + \frac{1}{2y^2} = 1$ (B) $\frac{1}{4x^2} + \frac{1}{2y^2} = 1$

(C) $\frac{1}{2x^2} + \frac{1}{4y^2} = 1$ (D) $\frac{1}{2x^2} + \frac{1}{y^2} = 1$

Solution: Let (x_1, y_1) be the midpoint of intercept of the tangent between the axes. The given ellipse is

$$\frac{x^2}{2} + \frac{y^2}{1} = 1$$

The tangent at $(\sqrt{2}\cos\theta, \sin\theta)$ is

$$\frac{x\cos\theta}{\sqrt{2}} + \frac{y\sin\theta}{1} = 1$$

so that it meets the axes at $A(\sqrt{2}\sec\theta, 0)$ and $B(0, \operatorname{cosec}\theta)$
Since (x_1, y_1) is the midpoint of $\overline{AB}$, we have

$$2x_1 = \sqrt{2}\sec\theta \text{ and } 2y_1 = \operatorname{cosec}\theta$$

$$\Rightarrow \cos\theta = \frac{1}{x_1\sqrt{2}} \text{ and } \sin\theta = \frac{1}{2y_1}$$

$$\Rightarrow \frac{1}{2x_1^2} + \frac{1}{4y_1^2} = \cos^2\theta + \sin^2\theta = 1$$

Therefore, the locus of (x_1, y_1) is

$$\frac{1}{2x^2} + \frac{1}{4y^2} = 1$$

Answer: (C)

7. If $x\cos\alpha + y\sin\alpha = p$ touches the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then

(A) $p^2 = a^2\sin^2\alpha + b^2\cos^2\alpha$

(B) $p^2 = a^2\cos^2\alpha + b^2\sin^2\alpha$

(C) $p^2 = a^2\operatorname{cosec}^2\alpha + b^2\sec^2\alpha$

(D) $p^2 = a^2\cot^2\alpha + b^2\tan^2\alpha$

Solution: The given line equation can be written as

$$y = (-\cot\alpha)x + p\operatorname{cosec}\alpha$$

It touches the ellipse. So

$$p^2\operatorname{cosec}^2\alpha = a^2(-\cot\alpha)^2 + b^2$$

$$\Rightarrow \frac{p^2}{\sin^2 \alpha} = \frac{a^2 \cos^2 \alpha + b^2 \sin^2 \alpha}{\sin^2 \alpha}$$

$$\Rightarrow p^2 = a^2 \cos^2 \alpha + b^2 \sin^2 \alpha$$

Answer: (B)

8. The tangents are drawn to the ellipse

$$\frac{x^2}{9} + \frac{y^2}{5} = 1$$

at the ends of latus recta. The area of the quadrilateral formed is

(A) 27 (B) $\frac{27}{2}$ (C) $\frac{27}{11}$ (D) $\frac{27}{55}$

Solution: The eccentricity e is given by

$$5 = 9(1 - e^2)$$

$$\Rightarrow e = \frac{2}{3}$$

Therefore, the foci are $(-ae,\ 0) = (-2,\ 0)$ and $(ae,\ 0) = (2, 0)$. We have

$$L = \left(ae,\ \frac{b^2}{a}\right) = \left(2,\ \frac{5}{3}\right)$$

See Fig. 5.35. The tangent at $L(2, 5/3)$ is

$$\frac{2x}{9} + \frac{(5/3)\,y}{5} = 1$$

$$\Rightarrow \frac{2x}{9} + \frac{y}{3} = 1$$

so that $P = (0,\ 3)$ and $S = (9/2, 0)$. Therefore, the area of the parallelogram is

$$4(\Delta\, POS) = 4\left(\frac{1}{2} \times 3 \times \frac{9}{2}\right) = 27$$

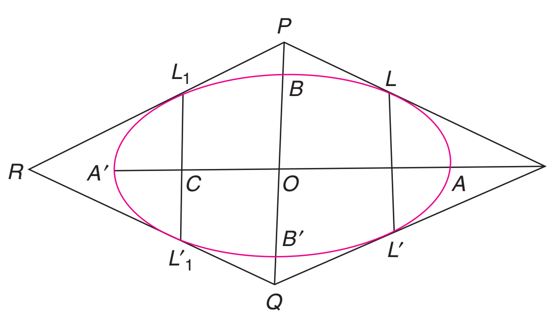


FIGURE 5.35

Answer: (A)

9. The radius of the largest circle inscribed in the ellipse $\frac{x^2}{16} + \frac{y^2}{4} = 1$ and having centre at $(1, 0)$ is

(A) $\frac{11}{3}$ (B) $\sqrt{\frac{11}{3}}$ (C) $\frac{7}{3}$ (D) $\sqrt{\frac{7}{3}}$

Solution: See Fig. 5.36. Since the circle is to be largest, it must touch the ellipse (internally) at some point, say, $P(4\cos\theta, 2\sin\theta)$. The tangent to the ellipse P is

$$\frac{x\cos\theta}{4} + \frac{y\sec\theta}{2} = 1$$

whose slope is $(-1/2)\cot\theta$. Also the line joining (1, 0) and P is perpendicular to the tangent. So

$$\left(\frac{-1}{2}\cot\theta\right)\frac{(2\sin\theta - 0)}{4\cos\theta - 1} = -1$$

$$\Rightarrow \frac{\cos\theta}{4\cos\theta - 1} = 1$$

$$\Rightarrow \cos\theta = \frac{1}{3} \text{ and } \sin\theta = \pm\frac{2\sqrt{2}}{3}$$

The radius of the circle is

$$[(4\cos\theta - 1)^2 + (2\sin\theta)^2]^{1/2} = \left[\left(\frac{4}{3} - 1\right)^2 + 4\left(\frac{8}{9}\right)\right]^{1/2}$$

$$= \sqrt{\frac{33}{9}} = \sqrt{\frac{11}{3}}$$

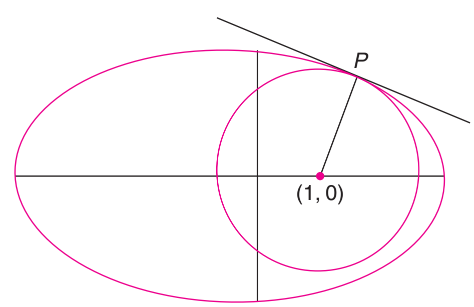


FIGURE 5.36

Answer: (B)

10. Any ordinate NP of an ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

meets the auxiliary circle at Q. Then, the locus of the normals at P and Q is

(A) $x^2 + y^2 = (a+b)^2$ (B) $x^2 + y^2 = (a-b)^2$

(C) $x^2 + y^2 = (a^2 + b^2)^{1/2}$ (D) $x^2 + y^2 = (a^2 - b^2)^{1/2}$

Solution: Let $P = (a\cos\theta, b\sin\theta)$ so that $Q = (a\cos\theta, a\sin\theta)$. The normal at P to the ellipse is

$$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2 \qquad (5.103)$$

and the normal Q to the circle is

$$y = x\tan\theta \tag{5.104}$$

Solving Eqs. (5.103) and (5.104), we have

$$\frac{ax}{\cos\theta} - \frac{bx\tan\theta}{\sin\theta} = a^2 - b^2$$

$$\Rightarrow (a-b)x = (a^2 - b^2)\cos\theta$$

$$\Rightarrow x = (a+b)\cos\theta \text{ and } y = x\tan\theta = (a+b)\sin\theta$$

$$\Rightarrow x^2 + y^2 = (a+b)^2$$

Therefore, the locus of the point of intersection is $x^2 + y^2 = (a+b)^2$.

Answer: (A)

11. The smallest possible positive slope of a line whose y-intercept is 5 and which has a common point with the ellipse $9x^2 + 16y^2 = 144$ is

(A) $\frac{3}{4}$ (B) 1 (C) $\frac{4}{3}$ (D) $\frac{9}{16}$

Solution: Let the line be $y = mx + 5$. Since $m > 0$ and is to be least, the line should touch the ellipse. Hence

$$5^2 = 16m^2 + 9 \Rightarrow m = \pm 1$$

However, $m > 0 \Rightarrow m = 1$.

Answer: (B)

12. Consider an ellipse with foci at (5, 15) and (21, 15). If the x-axis touches the ellipse, then the length of the major axis is

(A) 17 (B) 34 (C) 13 (D) $\sqrt{416}$

Solution: The major axis is $y = 15$. The distance between the foci is

$$2ae = 16 \Rightarrow ae = 8$$

Since x-axis is touching the ellipse, we have $b = 15$ so that

$$15^2 = b^2 = a^2(1-e^2) = a^2 - 64$$

$$\Rightarrow a = 17 \text{ and } e = \frac{8}{17}$$

Therefore, the length of the major axis is $2a = 2 \times 17 = 34$.

Answer: (B)

13. Let $P = (\sqrt{12}\cos\theta, \sqrt{8}\sin\theta)$ and $Q = (-\sqrt{12}\sin\theta, \sqrt{8}\cos\theta)$ be the points on the ellipse

$$\frac{x^2}{12} + \frac{y^2}{8} = 1$$

Then, the locus of the point of intersection of the tangents at P and Q is

(A) $\frac{x^2}{12} + \frac{y^2}{9} = 1$ (B) $\frac{x^2}{12} + \frac{y^2}{8} = 1$

(C) $\frac{x^2}{12} + \frac{y^2}{8} = 2$ (D) $\frac{x^2}{8} + \frac{y^2}{12} = 2$

Solution: The tangents at P and Q, respectively, are

$$\frac{x\cos\theta}{\sqrt{12}} + \frac{y\sin\theta}{\sqrt{8}} = 1 \tag{5.105}$$

and

$$\frac{-x\sin\theta}{\sqrt{12}} + \frac{y\cos\theta}{\sqrt{8}} = 1 \tag{5.106}$$

Squaring Eqs. (5.105) and (5.106) and adding, we have

$$\frac{x^2}{12} + \frac{y^2}{8} = 2$$

Answer: (C)

14. The sum of the squares of the perpendiculars drawn onto any tangent to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

from the two points on the minor axis, each at a distance of $\sqrt{a^2 - b^2}$ from the centre, is

(A) a^2 (B) $\frac{1}{2}a^2$ (C) $\frac{3}{2}a^2$ (D) $2a^2$

Solution: We have

$$a^2 - b^2 = a^2 - a^2(1-e^2)$$
$$= a^2e^2$$

Let $P = (0, ae)$ and $Q = (0, -ae)$. Let p_1 and p_2 be the perpendiculars drawn from P and Q onto

$$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$$

Therefore

$$p_1^2 = \frac{[(ae\sin\theta/b) - 1]^2}{(\cos^2\theta/a^2) + (\sin^2\theta/b^2)}$$

and

$$p_2^2 = \frac{[(-ae\sin\theta/b) - 1]^2}{[(\cos^2\theta)/a^2] + [(\sin^2\theta)/b^2]}$$

Hence

$$p_1^2 + p_2^2 = \frac{a^2(ae\sin\theta - b)^2 + a^2(ae\sin\theta + b)^2}{b^2\cos^2\theta + a^2\sin^2\theta}$$

$$= \frac{2a^2[a^2e^2\sin^2\theta + b^2]}{b^2\cos^2\theta + a^2\sin^2\theta}$$

$$=\frac{2a^2[a^2e^2\sin^2\theta+b^2]}{b^2(1-\sin^2\theta)+a^2\sin^2\theta}$$

$$=\frac{2a^2[a^2e^2\sin^2\theta+b^2]}{b^2+(a^2-b^2)\sin^2\theta}$$

$$=\frac{2a^2[a^2e^2\sin^2\theta+b^2]}{b^2+a^2e^2\sin^2\theta} \qquad (\because a^2-b^2=a^2e^2)$$

$$=2a^2$$

Answer: (D)

15. The tangent and normal to the ellipse $\frac{x^2}{4}+\frac{y^2}{1}=1$ at the point $P(2\cos\theta, \sin\theta)$ meet the major axis at Q and R, respectively. If $QR=2$, then $\cos\theta$ is equal to

(A) $\frac{1}{3}$ (B) $\frac{\sqrt{2}}{3}$ (C) $\frac{2}{3}$ (D) $\frac{2\sqrt{3}}{3}$

Solution: The tangent at P is

$$\frac{x\cos\theta}{2}+\frac{y\sin\theta}{1}=1$$

so that $Q=(2\sec\theta, 0)$. The normal at P is

$$\frac{2x}{\cos\theta}-\frac{y}{\sin\theta}=2^2-1^2=3$$

Hence

$$R=\left(\frac{3\cos\theta}{2},\ 0\right)$$

Now,

$$QR=2$$

$$\Rightarrow 2\sec\theta-\frac{3}{2}\cos\theta=2$$

$$\Rightarrow 4-3\cos^2\theta=4\cos\theta$$

$$\Rightarrow 3\cos^2\theta+4\cos\theta-4=0$$

$$\Rightarrow 3\cos^2\theta+6\cos\theta-2\cos\theta-4=0$$

$$\Rightarrow 3\cos\theta(\cos\theta+2)-2(\cos\theta+2)=0$$

$$\Rightarrow (3\cos\theta-2)(\cos\theta+2)=0$$

$$\Rightarrow \cos\theta=\frac{2}{3}$$

Answer: (C)

16. The line passing through the extremity A of the major axis and extremity B of the minor axis of the ellipse

$$\frac{x^2}{9}+\frac{y^2}{1}=1$$

meets the auxiliary circle at point M. Then the area of the triangle with vertices at A, M and the origin O is

(A) $\frac{31}{10}$ (B) $\frac{29}{10}$ (C) $\frac{21}{10}$ (D) $\frac{27}{10}$

(IIT-JEE 2009)

Solution: See Fig. 5.37. We have

$$A=(3, 0),\ B=(0, 1)$$

The equation of the line AB is

$$y=\frac{0-1}{3-0}(x-3)$$

$$\Rightarrow x+3y=3 \qquad (5.107)$$

The equation of the auxiliary circle is

$$x^2+y^2=9 \qquad (5.108)$$

Substituting $x=3-3y$ in Eq. (5.108), we get

$$(3-3y)^2+y^2=9$$

$$\Rightarrow 10y^2-18y=0$$

$$\Rightarrow y=0,\ \frac{9}{5}$$

Now, $y=0 \Rightarrow x=3$ so that the point is $A(3, 0)$ and

$$y=\frac{9}{5}\Rightarrow x=\frac{-12}{5}$$

so that

$$M=\left(\frac{-12}{5},\ \frac{9}{5}\right)$$

Now, $A=(3, 0)$, $O=(0, 0)$ and $M=(-12/5, 9/5)$. Therefore, area of ΔAOB is given by

$$\frac{1}{2}\left|3\left(\frac{9}{5}\right)-0\left(\frac{-12}{5}\right)\right|=\frac{27}{10}$$

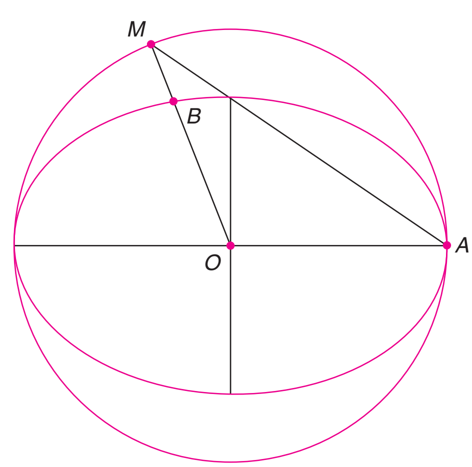


FIGURE 5.37

Answer: (D)

17. If $P(x,y)$ be any point on $16x^2+25y^2=400$ and $F_1(3, 0)$, $F_2(-3, 0)$ are the foci, then

$$PF_1+PF_2=$$

(A) 8 (B) 6 (C) 10 (D) 12

Solution: The given ellipse is

$$\frac{x^2}{25}+\frac{y^2}{16}=1$$

In an ellipse, the sum of the focal distances of a point on the ellipse is equal to the length of the major axis. Hence

$$PF_1+PF_2=2(5)=10$$

Answer: (C)

18. The radius of the circle passing through the foci of the ellipse

$$\frac{x^2}{16}+\frac{y^2}{9}=1$$

and having its centre at $(0, 3)$ is

(A) 4 (B) 3 (C) $\dfrac{1}{\sqrt{2}}$ (D) $\dfrac{7}{2}$

(IIT-JEE 1995)

Solution: Let the radius be r so that the circle equation is

$$x^2+(y-3)^2=r^2 \tag{5.109}$$

The eccentricity of the ellipse is given by

$$9=16(1-e^2)$$

$$\Rightarrow e^2=1-\frac{9}{16}=\frac{7}{16}$$

$$\Rightarrow e=\frac{\sqrt{7}}{4}$$

Therefore

$$ae=4\left(\frac{\sqrt{7}}{4}\right)=\sqrt{7}$$

Hence, the foci are $(\pm\sqrt{7}, 0)$. Since the circle passes through $(ae,0)=(\sqrt{7}, 0)$, we have

$$7+9=r^2 \Rightarrow r=4$$

Answer: (A)

19. Let E be the ellipse

$$\frac{x^2}{9}+\frac{y^2}{4}=1$$

and C be the circle $x^2+y^2=9$. Let P and Q be the points $(1, 2)$ and $(2, 1)$, respectively. Then

(A) Q lies inside C but lies outside E

(B) Q lies outside both C and E

(C) P lies inside both C and E

(D) P lies inside C but outside E

Solution: We have

$$S\equiv\frac{x^2}{9}+\frac{y^2}{4}-1$$

and $$S'\equiv x^2+y^2-9$$

Also $P=(x_1, y_1)=(1, 2)$ and $(x_2, y_2)=(2, 1)=Q$. Now,

$$S_{11}=\frac{1}{9}+\frac{4}{4}-1=\frac{1}{9}>0$$

$$S'_{11}=1^2+2^2-9=-5<0$$

Therefore, P lies outside E and inside C. Therefore

$$S_{22}=\frac{4}{9}+\frac{1}{4}-1<0$$

and $$S'_{22}=4+1-9<0$$

Hence, Q lies inside both E and C.

Answer: (D)

20. The locus of a variable point whose distance from $(-2, 0)$ is $2/3$ times its distance from the line $x=-9/2$ is

(A) ellipse (B) parabola

(C) hyperbola (D) pair of lines

Solution: By the definition of ellipse, the locus is an ellipse with focus $(-2, 0)$, directrix $x=-9/2$ and eccentricity $=2/3<1$.

Answer: (A)

21. The eccentricity of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

whose latus rectum is half of the major axis is

(A) $\dfrac{1}{\sqrt{3}}$ (B) $\dfrac{1}{\sqrt{2}}$ (C) $\dfrac{1}{2}$ (D) $\dfrac{1}{3}$

Solution: we have

$$\frac{2b^2}{a}=a$$

$$\Rightarrow 2b^2=a^2$$

$$\Rightarrow a^2=2b^2=2a^2(1-e^2)$$

$$\Rightarrow e^2=\frac{1}{2}$$

$$\Rightarrow e=\frac{1}{\sqrt{2}}$$

Answer: (B)

22. The director circle of the ellipse, which is having its axes along the coordinate axes, has foci at $(\pm 2, 0)$ and eccentricity $1/2$. Its equation is

(A) $x^2+y^2=25$ (B) $x^2+y^2=26$
(C) $x^2+y^2=24$ (D) $x^2+y^2=28$

Solution: Let the ellipse be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

where the eccentricity $e=1/2$.
Also

$$\text{Focus} = (2, 0) \Rightarrow ae = 2 \Rightarrow a = 4$$

Again

$$b^2=a^2(1-e^2)=16\left(1-\frac{1}{4}\right)=12$$

Hence, the equation of the ellipse is

$$\frac{x^2}{16}+\frac{y^2}{12}=1$$

and the equation of the director circle is

$$x^2+y^2=a^2+b^2=16+12=28$$

Answer: (D)

23. The points $(5, 12)$ and $(24, 7)$ are the foci of an ellipse passing through the origin. Then the eccentricity of the ellipse is

(A) $\dfrac{\sqrt{386}}{38}$ (B) $\dfrac{\sqrt{286}}{38}$
(C) $\dfrac{\sqrt{286}}{28}$ (D) $\dfrac{\sqrt{386}}{35}$

Solution: We have

$$OS+OS'=\sqrt{25+144}+\sqrt{576+49}=13+25=38=2a$$

Now

$$SS'=2ae$$
$$\Rightarrow \sqrt{386}=(38)e$$
$$\Rightarrow e=\frac{\sqrt{386}}{38}$$

Answer: (A)

24. If

$$\frac{x}{a}+\frac{y}{b}=\sqrt{2}$$

touches the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

then the eccentric angle of the point of contact is

(A) $\dfrac{\pi}{3}$ (B) $\dfrac{\pi}{4}$ (C) $\dfrac{\pi}{6}$ (D) $\dfrac{\pi}{2}$

Solution: $\dfrac{x}{a}+\dfrac{y}{b}=\sqrt{2}$ can be written as

$$\frac{x}{a\sqrt{2}}+\frac{y}{b\sqrt{2}}=1$$
$$\Rightarrow \frac{x}{a}\cos\frac{\pi}{4}+\frac{y}{b}\sin\frac{\pi}{4}=1$$

so that $\theta=\pi/4$.

Answer: (B)

25. If S_1 and S_2 are the foci of an ellipse with major axis of length 10 and P is any point on the ellipse such that the perimeter of $\Delta\, PS_1S_2$ is 15, then the eccentricity of the ellipse is

(A) $\dfrac{1}{2}$ (B) $\dfrac{1}{4}$ (C) $\dfrac{7}{25}$ (D) $\dfrac{3}{4}$

Solution: By hypothesis, we have

$$S_1S_2 = 2ae \Rightarrow S_1S_2 = 10e \qquad (\because\ 2a = 10)$$

Also

$$15 = PS_1 + PS_2 + S_1S_2 = 10 + 10e$$

Therefore

$$e=\frac{5}{10}=\frac{1}{2}$$

Answer: (A)

26. If (h, k) is a variable point on the circle $x^2+y^2=1$, then the locus of the point $(3h + 2, k)$ is an ellipse whose eccentricity is

(A) $\dfrac{1}{3}$ (B) $\dfrac{2\sqrt{2}}{3}$ (C) $\dfrac{1}{\sqrt{3}}$ (D) $\dfrac{\sqrt{2}}{3}$

Solution: Let $x=3h+2$ and $y=k$ so that

$$\left(\frac{x-2}{3}\right)^2+y^2=h^2+k^2=1$$

Hence, the locus of $(3h + 2, k)$ is an ellipse with centre $(2, 0)$ and

$$1=a(1-e^2)\Rightarrow e=\frac{2\sqrt{2}}{3}$$

Answer: (B)

27. If P is a point on the ellipse

$$\frac{x^2}{25}+\frac{y^2}{16}=1$$

lying in the first quadrant and PN is drawn perpendicular to major axis and produced to Q such that NQ is equal to SP where S in a focus, then the focus of Q is

(A) $2x - 3y - 25 = 0$ (B) $3x - 5y - 25 = 0$
(C) $3x - 5y + 25 = 0$ (D) $3x + 5y + 25 = 0$

Solution: See Fig. 5.38. Let P be $(5\cos\theta, 4\sin\theta)$ and $0<\theta<\pi/2$. Now,

$$PN = 4\sin\theta \Rightarrow N = (5\cos\theta, 0)$$

Let Q be (h, k). By hypothesis, $NQ = SP$. Also observe that $k < 0$. Now

$$NQ = SP$$

$$\Rightarrow |k| = a + ex = 5+\frac{3}{5}(5\cos\theta) \tag{5.110}$$

Since the abscissa of Q is $5 \cos\theta$, we have $h = 5 \cos\theta$. From Eq. (5.110), we get

$$-k = 5+3\cos\theta$$

$$\Rightarrow -k = 3\left(\frac{h}{5}\right)+5$$

$$\Rightarrow 3h + 5k + 25 = 0$$

Therefore, the locus of Q is $3x + 5y + 25 = 0$.

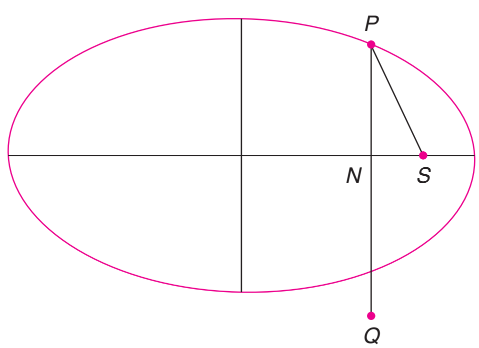


FIGURE 5.38

Answer: (D)

28. The product of the slopes of the common tangents of the ellipse

$$\frac{x^2}{4}+\frac{y^2}{1}=1$$

and the circle $x^2 + y^2 = 3$ is

(A) 1 (B) −2 (C) −3 (D) −1

Solution: The line $y = mx+\sqrt{4m^2+1}$ is a tangent to the ellipse

$$\frac{x^2}{4}+y^2=1$$

This line touches the circle $x^2 + y^2 = 3$. This means

$$\frac{\sqrt{4m^2+1}}{\sqrt{1+m^2}}=\sqrt{3}$$

$$\Rightarrow 3m^2+3=4m^2+1$$

$$\Rightarrow m=\pm\sqrt{2}$$

Therefore, the product of the slopes = −2.

Answer: (B)

29. Two lines PQ and PR are drawn from a point on the circle $x^2 + y^2 = 25$ to the ellipse

$$\frac{x^2}{4^2}+\frac{y^2}{b^2}=1$$

where $b < 4$. If the parallelogram $PQSR$ is completed and S lies on the circumcircle of ΔPQR, then the eccentricity of ellipse is

(A) $\frac{\sqrt{7}}{3}$ (B) $\frac{\sqrt{5}}{4}$ (C) $\frac{\sqrt{5}}{3}$ (D) $\frac{\sqrt{7}}{4}$

Solution: Since the points P, Q, S and R concyclic and $PQSR$ is a parallelogram, the angles of the parallelogram are equal to 90°. Hence, P lies on the director circle of

$$\frac{x^2}{4}+\frac{y^2}{b^2}=1$$

which is given by $x^2 + y^2 = 25$. Hence, $b = 3$. Therefore

$$9=16(1-e^2)$$

$$\Rightarrow e=\sqrt{1-\frac{9}{16}}=\frac{\sqrt{7}}{4}$$

Answer: (D)

30. Point O is the centre of an ellipse with major axis $\overline{AB}$ and minor axis $\overline{CD}$. Point F is one of the foci. If $OF = 6$ and the diameter of the inscribed circle of ΔOCF is 2, then $(AB)(CD) =$

(A) 35 (B) 45 (C) 55 (D) 65

Solution: See Fig. 5.39. Let $OC = b$, $OF = 6$ and $CF = a$. Since

$$1 = \text{Inradius of } \Delta OCF$$
$$= \frac{(1/2)(b)(6)}{(1/2)(b+6+a)}$$

we have

$$b + 6 + a = 6b$$

$$\Rightarrow 5b = 6 + a \tag{5.111}$$

Also

$$36 = a^2e^2 = a^2 - b^2 \quad (5.112)$$

From Eqs. (5.111) and (5.112), we have

$$(5b-6)^2 = a^2 = 36 + b^2$$

$$\Rightarrow 24b^2 - 60b = 0$$

$$\Rightarrow b = \frac{60}{24} = \frac{5}{2}$$

and hence

$$a = 5b - 6 = \frac{25}{2} - 6 = \frac{13}{2}$$

Thus,

$$(AB)(CD) = (2a)(2b) = 13 \times 5 = 65$$

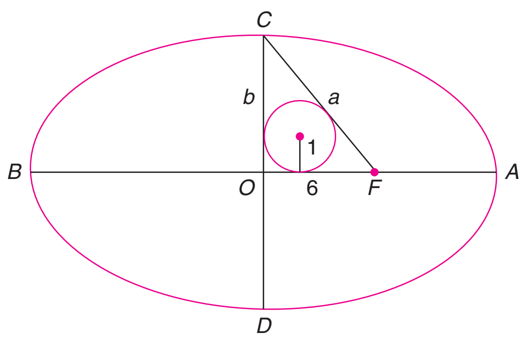


FIGURE 5.39

Answer: (D)

31. An ellipse is having its axes along the x-axis and y-axis and its latus rectum is of length 10 units. If the distance between the foci is equal to the length of the semi-minor axis, then the equation of the ellipse is

(A) $x^2 + 2y^2 = 100$ (B) $2x^2 + y^2 = 100$
(C) $x^2 + 2y^2 = 200$ (D) $2x^2 + y^2 = 200$

Solution: Let the ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

so that

$$\frac{2b^2}{a} = 10$$

$$\Rightarrow 2b^2 = 10a \quad (5.113)$$

Also

$$2ae = 2b \Rightarrow b = ae \quad (5.114)$$

Now, from Eq. (5.114), we have

$$b^2 = a^2(1-e^2)$$

$$\Rightarrow a^2e^2 = a^2 - a^2e^2$$

$$\Rightarrow e = \frac{1}{\sqrt{2}}$$

Therefore, from Eq. (5.114), we get $a = b\sqrt{2}$. Hence from Eq. (5.113), we get

$$2b^2 = 10a = 10(b\sqrt{2})$$

$$\Rightarrow b = 5\sqrt{2} \text{ and } a = b\sqrt{2} = 10$$

So, the equation of the ellipse is

$$\frac{x^2}{100} + \frac{y^2}{50} = 1$$

$$\Rightarrow x^2 + 2y^2 = 100$$

Answer: (A)

32. The centre of an ellipse is the origin and the x-axis is the major axis. If it passes through the points $(-3, 1)$ and $(2, 2)$, then the eccentricity of the ellipse is

(A) $\frac{2}{3}$ (B) $\frac{3}{4}$ (C) $\sqrt{\frac{2}{5}}$ (D) $\frac{\sqrt{3}}{2}$

Solution: Let the ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

This passes through $(-3, 1)$ and $(2, -2)$. So

$$\frac{9}{a^2} + \frac{1}{b^2} = 1 \quad (5.115)$$

and

$$\frac{4}{a^2} + \frac{4}{b^2} = 1 \quad (5.116)$$

Solving Eqs. (5.115) and (5.116) for a^2 and b^2, we get

$$a^2 = \frac{32}{3} \quad \text{and} \quad b^2 = \frac{32}{5}$$

Therefore

$$b^2 = a^2(1-e^2) \Rightarrow e = \sqrt{\frac{2}{5}}$$

Answer: (C)

33. S and S' are the foci of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

and B is one end of the minor axis. If $\Delta BSS'$ is equilateral triangle, then the eccentricity of the ellipse is equal to

(A) $\frac{\sqrt{3}}{2}$ (B) $\frac{1}{3}$ (C) $\frac{1}{2}$ (D) $\frac{1}{\sqrt{2}}$

Solution: We have

$$SS' = SB = S'B$$
$$\Rightarrow SB + S'B = 2SS'$$
$$\Rightarrow 2a = 2(2ae)$$
$$\Rightarrow e = \frac{1}{2}$$

Answer: (C)

34. If the eccentricity of an ellipse

$$\frac{x^2}{\lambda^2+1}+\frac{y^2}{\lambda^2+2}=1$$

is $1/\sqrt{6}$, then its latus rectum is

(A) $\frac{8}{\sqrt{6}}$ (B) $\frac{10}{\sqrt{6}}$ (C) $\frac{5}{\sqrt{6}}$ (D) $\frac{7}{\sqrt{6}}$

Solution: We have

$$\lambda^2+1=(\lambda^2+2)\left(1-\frac{1}{6}\right)$$
$$\Rightarrow 6(\lambda^2+1)=5\lambda^2+10$$
$$\Rightarrow \lambda^2=4$$

Hence, the ellipse is

$$\frac{x^2}{5}+\frac{y^2}{6}=1$$

and the latus rectum is

$$\frac{2a^2}{b}=\frac{2(5)}{\sqrt{6}}=\frac{10}{\sqrt{6}}$$

Answer: (B)

35. A tangent is drawn to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

to cut the ellipse

$$\frac{x^2}{c^2}+\frac{y^2}{d^2}=1$$

at P and Q. If the tangents drawn at P and Q to the ellipse

$$\frac{x^2}{c^2}+\frac{y^2}{d^2}=1$$

are at right angles, then

(A) $\frac{a^2}{c^2}+\frac{b^2}{d^2}=1$ (B) $\frac{c^2}{a^2}+\frac{d^2}{b^2}=1$

(C) $a^2d^2=b^2c^2$ (D) $a^2d^2+c^2d^2=1$

Solution: See Fig. 5.40. Clearly, if $R(h, k)$ is the intersection of tangents at P and Q, then R lies on the director circle and hence

$$h^2+k^2=c^2+d^2 \tag{5.117}$$

Since PQ is the chord of contract of R, its equation is

$$\frac{hx}{c^2}+\frac{ky}{d^2}=1 \tag{5.118}$$

However, the line provided in Eq. (5.118) touches the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Therefore

$$\left(\frac{d^2}{k}\right)^2=a^2\left(\frac{-d^2h}{c^2k}\right)^2+b^2$$
$$\Rightarrow d^4=\frac{a^2d^4h^2}{c^4}+b^2k^2$$
$$\Rightarrow 1=\frac{a^2h^2}{c^4}+\frac{b^2k^2}{d^4} \tag{5.119}$$

Equations (5.117) and (5.119) hold if $h^2 = c^2$ and $k^2 = d^2$. Therefore

$$\frac{a^2}{c^2}+\frac{b^2}{d^2}=1$$

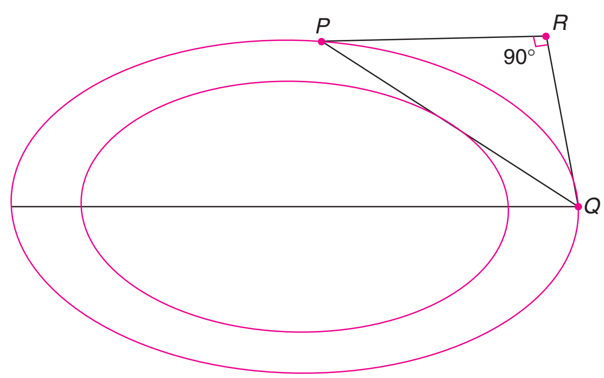


FIGURE 5.40

Answer: (A)

36. If a tangent of slope 2 of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

is normal to the circle $x^2 + y^2 + 4x + 1 = 0$, then the maximum value of ab is equal to

(A) 2 (B) 1 (C) 4 (D) 3

Solution: $y=2x+\sqrt{4a^2+b^2}$ is a tangent to the ellipse. This passes through the centre $(-2, 0)$ of the circle. Hence

$$0=-4+\sqrt{4a^2+b^2}$$
$$\Rightarrow 4a^2+b^2=16$$

Since $AM \geq GM$, we have

$$8=\frac{4a^2+b^2}{2} \geq \sqrt{4a^2b^2}=2ab$$

Equality holds when $2a = b$. Hence, the maximum value of ab is 4.

Answer: (C)

37. An ellipse has OB as a semi-minor and F, F' are its foci and the angle FBF' is a right angle. Thus, the eccentricity of the ellipse is

(A) $\frac{1}{2}$ (B) $\frac{\sqrt{2}}{3}$ (C) $\frac{2\sqrt{2}}{3}$ (D) $\frac{1}{\sqrt{2}}$

Solution: We have $B = (0, b)$, $F = (ae, 0)$ and $F' = (-ae, 0)$. Now

$$\angle FBF'=90^\circ$$
$$\Rightarrow (FF')^2=(FB)^2+(F'B)^2$$
$$\Rightarrow 4a^2e^2=(a^2e^2+b^2)+(a^2e^2+b^2)=2a^2e^2+2b^2$$
$$\Rightarrow 2a^2e^2=2b^2$$
$$\Rightarrow a^2e^2=a^2(1-e^2)$$
$$\Rightarrow 2e^2=1$$
$$\Rightarrow e=\frac{1}{\sqrt{2}}$$

Answer: (D)

38. A circle has the same centre as the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

and passes through two foci F_1 and F_2. P is one of the four points of their intersection. If the major axis is of length 17 and the area of the $\Delta\, PF_1F_2$ is 30, then the distance between the foci is

(A) 11 (B) 15 (C) 13 (D) 12

Solution: See Fig. 5.41. Let $PF_1=x$ and $PF_2 = y$. Then

$$F_1F_2 = 2ae = 17e$$

Now

$$\angle F_1PF_2 = 90^\circ \Rightarrow 30 = \text{Area of } \Delta\, PF_1F_2 = \frac{1}{2} PF_1 \cdot PF_2$$

$$\Rightarrow 60 = PF_1 \cdot PF_2 = xy \quad (5.120)$$

Also

$$x + y = PF_1 + PF_2 = 2a = 17 \quad (5.121)$$

From Eqs. (5.120) and (5.121), we have

$$x^2 - 17x + 60 = 0 \Rightarrow (x - 12)(x - 5) = 0$$

Therefore, $x = 12$ or 5 so that $y = 5$ or 17. Hence

$$F_1F_2 = \sqrt{x^2+y^2} = \sqrt{12^2+5^2} = 13$$

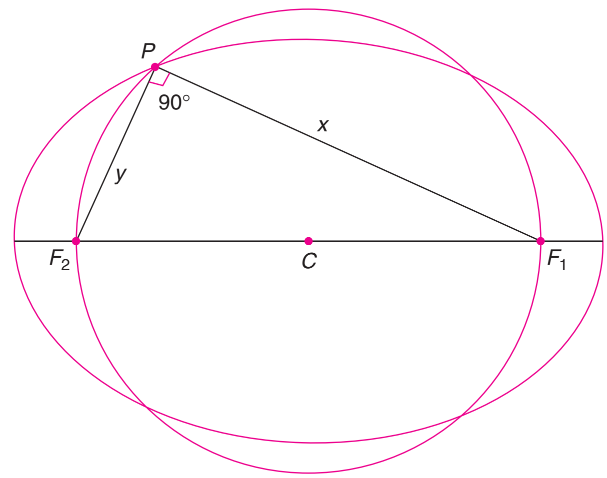


FIGURE 5.41

Answer: (C)

39. The line PQ is a focal cord of the ellipse

$$\frac{x^2}{25}+\frac{y^2}{16}=1$$

If the focus $S = (3, 0)$ and $SP = 2$, then the length PQ is equal to

(A) 8 (B) 4 (C) 10 (D) 6

Solution: Major axis length is $2 \times 5 = 10$. $S = (3, 0)$ and $SP = 2$. Since the vertex A (see Fig. 5.42) is $(5, 0)$, it follows that

$$SA = 2 \text{ and } A = P$$

Therefore, PQ is nothing but the major axis and hence $PQ = 10$.

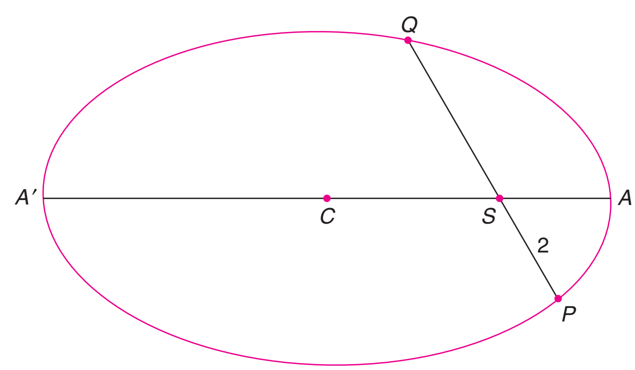


FIGURE 5.42

Answer: (C)

40. Point S is one of the foci of the ellipse

$$\frac{x^2}{25}+\frac{y^2}{16}=1$$

With S as centre and r as radius, a circle is drawn such that it touches the ellipse and completely lies within the ellipse. Then, the value of the radius r is

(A) 4 (B) 3 (A) $\sqrt{3}$ (D) 2

Solution: The ellipse is

$$\frac{x^2}{25}+\frac{y^2}{16}=1 \Rightarrow a=5, b=4$$

which implies that $a = 5$ and $b = 4$. Also

$$16=25(1-e^2) \Rightarrow e=\frac{3}{5}$$

Hence, $S = (3, 0)$ (see Fig. 5.43). Also we know that the vertex $A = (5, 0)$. This implies that $SA = 2$. Since the circle is touching the ellipse and lies completely within the ellipse, it follows that its radius r must be equal to 2.

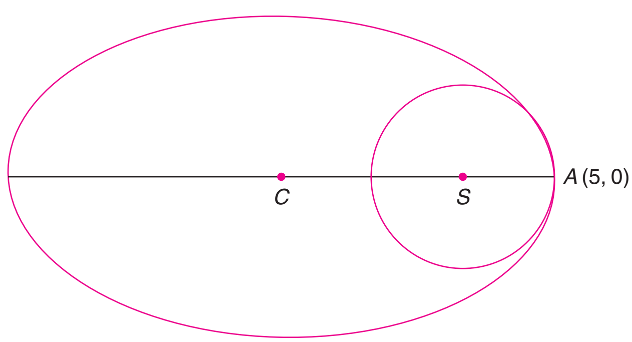


FIGURE 5.43

Answer: (D)

41. The angle subtended by the common tangent of the two ellipses

$$\frac{(x-4)^2}{25}+\frac{y^2}{4}=1$$

and
$$\frac{(x+1)^2}{1}+\frac{y^2}{4}=1$$

at the origin is

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{6}$

Solution: See Fig. 5.44. $B(4, 2)$ is one end of the minor axis of the ellipse

$$\frac{(x-4)^2}{25}+\frac{y^2}{4}=1$$

and $(-1, 2)$ is one end of the major axis of the second ellipse. Therefore,

$$AB=5,\ OB=\sqrt{16+4}=\sqrt{20},\ OA=\sqrt{1+4}=\sqrt{5}$$

We have

$$(OA)^2+(OB)^2=25=(AB)^2$$

Therefore

$$\angle AOB=\frac{\pi}{2}$$

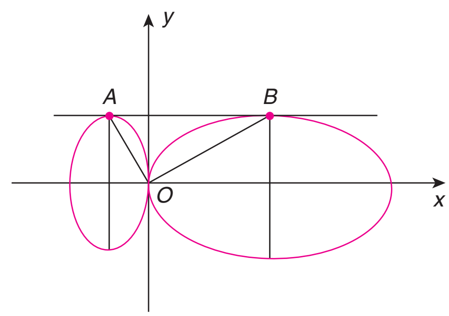


FIGURE 5.44

Answer: (B)

42. If the latus rectum of the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

subtends a right angle at the centre, then the eccentricity of the hyperbola is

(A) $\frac{\sqrt{5}+1}{2}$ (B) $\sqrt{3}+1$

(C) $\frac{2+\sqrt{3}}{2}$ (D) $\frac{3+\sqrt{2}}{2}$

Solution: The ends of a latus rectum are

$$L'=\left(ae, \frac{-b^2}{a}\right)$$

and
$$L=\left(ae, \frac{b^2}{a}\right)$$

Since LL' subtends right angle at the centre, then

$$-1=\left(\frac{-b^2/a}{ae}\right)\left(\frac{b^2/a}{ae}\right)$$

$$=\frac{-b^4}{a^2(a^2e^2)}$$

$$=-\left(\frac{b^2}{a^2}\right)^2\frac{1}{e^2}=\frac{-(e^2-1)^2}{e^2}$$

Therefore

$$(e^2-1)^2=+e^2$$

Hence

$$e^4 - 3e^2 + 1 = 0$$

$$\Rightarrow e^2 = \frac{3+\sqrt{5}}{2} \qquad (\because e > 1)$$

Hence

$$e = \frac{\sqrt{5}+1}{2}$$

Answer: (A)

43. The distance between two parallel tangents of the hyperbola

$$\frac{x^2}{9} - \frac{y^2}{49} = 1$$

is 2. Then their slopes are

(A) $\pm\frac{5}{2}$ (B) $\pm\frac{7}{2}$ (C) $\pm\frac{4}{5}$ (D) $\pm\frac{5}{4}$

Solution: The two parallel tangents are

$$y = mx \pm \sqrt{9m^2 + 49}$$

Therefore, by hypothesis

$$\frac{2\sqrt{9m^2 - 49}}{\sqrt{1+m^2}} = 2$$

$$\Rightarrow 9m^2 - 49 = 1 + m^2$$

$$\Rightarrow 8m^2 = 50$$

$$\Rightarrow m = \pm\frac{5}{2}$$

Answer: (A)

44. A hyperbola has its axes along the coordinate axes, latus rectum is 8 and conjugate axis is half of the distance between the foci. Then the eccentricity is

(A) $\frac{2}{\sqrt{3}}$ (B) $\frac{3}{\sqrt{2}}$ (C) $\frac{4}{\sqrt{3}}$ (D) $\frac{4}{3}$

Solution: Hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

By hypothesis, we have

$$\frac{2b^2}{a} = 8$$

$$\Rightarrow b^2 = 4a \tag{5.122}$$

and

$$2b = ae$$

$$\Rightarrow b = \frac{ae}{2} \tag{5.123}$$

From Eq. (5.121), we have

$$a^2(e^2 - 1) = 4a$$

$$\Rightarrow a = \frac{4}{e^2 - 1} \tag{5.124}$$

From Eqs. (5.122) and (5.123), we have

$$\frac{a^2e^2}{4} = b^2 = 4a$$

$$\Rightarrow a = \frac{16}{e^2} \tag{5.125}$$

From Eqs. (5.124) and (5.125), we have

$$\frac{4}{e^2-1} = \frac{16}{e^2}$$

$$\Rightarrow e^2 = 4e^2 - 4$$

$$\Rightarrow e = \frac{2}{\sqrt{3}}$$

Answer: (A)

45. Consider a branch of the hyperbola

$$x^2 - 2y^2 - 2\sqrt{2}x - 4\sqrt{2}y - 6 = 0$$

with A as one vertex. B is one of the end points of the latus rectum and C is the focus nearest to A. Then, the area of ΔABC is

(A) $1 - \sqrt{\frac{2}{3}}$ (B) $\sqrt{\frac{3}{2}} - 1$

(C) $1 + \sqrt{\frac{2}{3}}$ (D) $\sqrt{\frac{2}{3}} + 1$

(IIT-JEE 2008)

Solution: The equation of the given hyperbola can be written as

$$(x-\sqrt{2})^2 - (\sqrt{2}y + 2)^2 = 6 + 2 - 4 = 4$$

$$\Rightarrow \frac{(x-\sqrt{2})^2}{4} - \frac{2(y+\sqrt{2})^2}{4} = 1$$

$$\Rightarrow \frac{(x-\sqrt{2})^2}{4} - \frac{(y+\sqrt{2})^2}{2} = 1$$

$$\Rightarrow \frac{X^2}{4} - \frac{Y^2}{2} = 1$$

where $X = x - \sqrt{2}$ and $Y = y + \sqrt{2}$. Now

$$\begin{aligned} A &= (X = 2, Y = 0) = (x - \sqrt{2} = 2, y + \sqrt{2} = 0) \\ &= (2+\sqrt{2}, -\sqrt{2}) \end{aligned}$$

The centre is $(\sqrt{2},-\sqrt{2})$. Let e be the eccentricity. Therefore

$$2=4(e^2-1)\Rightarrow e=\sqrt{\frac{3}{2}}$$

Hence,

$$C=(X=ae,\ Y=0)$$
$$\Rightarrow C=(\sqrt{2}+\sqrt{6},-\sqrt{2})$$

Also

$$BC=\frac{b^2}{a}=\frac{2}{2}=1$$

and

$$B=\left(X=ae,\ Y=\frac{b^2}{a}\right)=(\sqrt{2}+\sqrt{2}+\sqrt{6},1-\sqrt{2})$$
$$=(2\sqrt{2}+\sqrt{6},1-\sqrt{2})$$

Therefore, the area of ΔABC is

$$\frac{1}{2}\begin{vmatrix} 2+\sqrt{2} & -\sqrt{2} & 1 \\ 2\sqrt{2}+\sqrt{6} & 1-\sqrt{2} & 1 \\ \sqrt{2}+\sqrt{6} & -\sqrt{2} & 1 \end{vmatrix}$$

$$=\frac{1}{2}\begin{vmatrix} 2+\sqrt{2} & -\sqrt{2} & 1 \\ -2+\sqrt{2}+\sqrt{6} & 1 & 0 \\ \sqrt{6}-2 & 0 & 0 \end{vmatrix} \quad (\text{by } R_2-R_1, R_3-R_1)$$

$$=\frac{1}{2}|\sqrt{6}-2|$$

$$=\sqrt{\frac{3}{2}}-1$$

Answer: (B)

46. A hyperbola having transverse axis of length $2\sin\theta$ is confocal with the ellipse $3x^2+4y^2=12$. Then its equation is

(A) $x^2\operatorname{cosec}^2\theta-y^2\sec^2\theta=1$
(B) $x^2\sec^2\theta-y^2\operatorname{cosec}^2\theta=1$
(C) $x^2\sin^2\theta-y^2\cos^2\theta=1$
(D) $x^2\cos^2\theta-y^2\sin^2\theta=1$

(IIT-JEE 2007)

Solution: The equation of the ellipse is

$$\frac{x^2}{4}+\frac{y^2}{3}=1$$

and its eccentricity is given by

$$\frac{3}{4}=1-e^2\ \Rightarrow\ e=\frac{1}{2}$$

Hence, the foci are $(\pm ae,0)=(\pm 1,0)$. Now, let the hyperbola be

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

so that $a=\sin\theta$ and the eccentricity e' is given by

$$b^2=a^2(e'^2-1)=\sin^2\theta(e'^2-1) \qquad (5.126)$$

Also

$$ae'=1\Rightarrow\ e'=\operatorname{cosec}\theta \qquad (\because\ a=\sin\theta)$$

Therefore, from Eq. (5.126),

$$b^2=\sin^2\theta(\operatorname{cosec}^2\theta-1)=1-\sin^2\theta=\cos^2\theta$$

Therefore, the equation of the hyperbola is

$$\frac{x^2}{\sin^2\theta}-\frac{y^2}{\cos^2\theta}=1$$

Answer: (A)

47. If the line $2x+\sqrt{6}\,y=2$ touches the hyperbola $x^2-2y^2=4$, then the point of contact is

(A) $(-2,\sqrt{2})$ (B) $(-5,2\sqrt{6})$
(C) $\left(\frac{1}{2},\frac{1}{\sqrt{6}}\right)$ (D) $(4,\sqrt{6})$

Solution: Suppose the line $2x+\sqrt{6}\,y=2$ touches the curve at (x_1,y_1). However, at (x_1,y_1), the equation of the tangent is

$$xx_1-2yy_1=4$$

Hence

$$\frac{x_1}{2}=\frac{-2y_1}{\sqrt{6}}=\frac{4}{2}$$
$$\Rightarrow x_1=4,\ y_1=-\sqrt{6}$$

Thus, the point of contact is $(4,-\sqrt{6})$.

Answer: (D)

48. For the hyperbola

$$\frac{x^2}{\cos^2\alpha}-\frac{y^2}{\sin^2\alpha}=1$$

where the value of α changes, which of the following remain constant?

(A) Abscissae of the vertices
(B) Abscissae of the foci
(C) Eccentricity
(D) Directrices

(IIT-JEE 2004)

Solution: The eccentricity e is given by

$$\sin^2\alpha = \cos^2\alpha(e^2-1)$$
$$\Rightarrow e = \sqrt{\tan^2\alpha+1} = |\sec\alpha|$$

Therefore, the foci are $(\pm\cos\alpha|\sec\alpha|, 0) = (\pm 1, 0)$.

Answer: (B)

49. If $x = 9$ is a chord of contact of the hyperbola $x^2 - y^2 = 9$, then the equation of the corresponding pair of tangents is

(A) $9x^2 - 8y^2 + 18x - 9 = 0$

(B) $9x^2 - 8y^2 - 18x - 9 = 0$

(C) $9x^2 - 8y^2 + 18x + 9 = 0$

(D) $9x^2 - 8y^2 - 18x + 9 = 0$

(IIT-JEE 1999)

Solution: Substitute $x = 9$ in the curve equation

$$x^2 - y^2 = 9$$

so that $y^2 = 72$ or $y = \pm 6\sqrt{2}$. Hence, the extremities of the chord of contact are $(9, \pm 6\sqrt{2})$. The tangent at $(9, 6\sqrt{2})$ is

$$9x - 6\sqrt{2}y - 9 = 0$$

and the tangent at $(9, -6\sqrt{2})$ is

$$9x + 6\sqrt{2}y - 9 = 0$$

Hence the combined equation of the tangents drawn at the ends of the chord of contact is

$$(9x - 6\sqrt{2}y - 9)(9x + 6\sqrt{2}y - 9) = 0$$
$$\Rightarrow 81x^2 - 72y^2 - 9(9x - 6\sqrt{2} + 9x + 6\sqrt{2}y) + 81 = 0$$
$$\Rightarrow 81x^2 - 72y^2 - 9(18x) + 81 = 0$$
$$\Rightarrow 9x^2 - 8y^2 - 18x + 9 = 0$$

Answer: (D)

50. Let $P(a\sec\alpha, b\tan\alpha)$ and $Q(a\sec\beta, a\tan\beta)$, where $\alpha + \beta = \pi/2$, be two points on the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

If (h, k) is the point of intersection of the normals at P and Q, then k is equal to

(A) $\dfrac{a^2+b^2}{a}$ (B) $-\left(\dfrac{a^2+b^2}{a}\right)$

(C) $\dfrac{a^2+b^2}{b}$ (D) $-\left(\dfrac{a^2+b^2}{b}\right)$

(IIT-JEE 1999)

Solution: The normal at $P(a\sec\alpha, b\tan\alpha)$ is

$$ax\cos\alpha + by\cot\alpha = a^2 + b^2 \qquad (5.127)$$

and the normal at

$$Q = \left[a\sec\left(\frac{\pi}{2}-\alpha\right), b\tan\left(\frac{\pi}{2}-\alpha\right)\right] = (a\,\text{cosec}\,\alpha, b\cot\alpha)$$
$$\left(\because \alpha+\beta = \frac{\pi}{2}\right)$$

is

$$ax\sin\alpha + by\tan\alpha = a^2 + b^2 \qquad (5.128)$$

Now,

$$\{\sin\alpha \times [\text{Eq. }(5.127)]\} - \{\cos\alpha \times [\text{Eq. }(5.128)]\}$$
$$\Rightarrow by(\cos\alpha - \sin\alpha) = (a^2+b^2)(\sin\alpha - \cos\alpha)$$
$$\Rightarrow y = \frac{(a^2+b^2)}{b}$$

Therefore,

$$k = -\frac{(a^2+b^2)}{b}$$

Answer: (D)

51. The distance between the foci of a hyperbola is 16 and its eccentricity is $\sqrt{2}$. Then, the equation of the hyperbola is

(A) $x^2 - y^2 = 16$ (B) $x^2 - y^2 = 8$

(C) $x^2 - y^2 = 32$ (D) $x^2 - y^2 = 24$

Solution: Since $\sqrt{2}$ is the eccentricity, the curve is a rectangular hyperbola, say, $x^2 - y^2 = a^2$.

Again $2ae = 16$, and $e = \sqrt{2} \Rightarrow a = 4\sqrt{2}$. Hence the equation of the hyperbola is

$$x^2 - y^2 = 32$$

Answer: (C)

52. The locus of the point of intersection of two lines

$$\sqrt{3}x - y - 4\sqrt{3}k = 0 \text{ and } k\sqrt{3}x + ky - 4\sqrt{3} = 0$$

is a hyperbola whose eccentricity is

(A) 2 (B) $\sqrt{2}$ (C) $\dfrac{1}{2}$ (D) $\dfrac{3}{4}$

Solution: We have

$$\sqrt{3}x - y = 4\sqrt{3}k \qquad (5.129)$$
$$k\sqrt{3}x + ky = 4\sqrt{3} \qquad (5.130)$$

From Eq. (5.130), we have

$$k = \frac{4\sqrt{3}}{\sqrt{3}x + y}$$

Substituting the value of k in Eq. (5.129), we have

$$\sqrt{3}x - y = 4\sqrt{3}\left(\frac{4\sqrt{3}}{\sqrt{3}x + y}\right)$$

$$\Rightarrow 3x^2 - y^2 = 48$$

$$\Rightarrow \frac{x^2}{16} - \frac{y^2}{48} = 1$$

Now,

$$48 = 16(e^2 - 1)$$

$$\Rightarrow e = \sqrt{\frac{48}{16} + 1} = 2$$

Answer: (A)

53. The equations of the pair of straight lines parallel to the x-axis and tangent to the ellipse

$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

are

(A) $y = -3, y = 6$ (B) $y = 3, y = -6$
(C) $y = \pm 6$ (D) $y = \pm 3$

Solution: Let $y = k$ be a line parallel to x-axis. The points of intersections of the line with the ellipse are given by the equation

$$9x^2 + 4k^2 = 36$$

which touches the ellipse. Solving we get

$$4k^2 = 36 \quad \text{or} \quad k = \pm 3$$

Therefore, the tangents are $y = \pm 3$.

Answer: (D)

54. The area of the triangle formed by a tangent of slope m to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

and the two axes is

(A) $\frac{|m|}{2}(a^2 + b^2)$ (B) $\frac{1}{2|m|}(a^2 + b^2)$
(C) $\frac{|m|}{2}(a^2m^2 + b^2)$ (D) $\frac{1}{2|m|}(a^2m^2 + b^2)$

Solution: The tangent with slope m is of the form

$$y = mx + \sqrt{a^2m^2 + b^2}$$

Therefore, the intercepts of the tangent on the axes are

$$-\frac{\sqrt{a^2m^2 + b^2}}{m}$$

and $$\sqrt{a^2m^2 + b^2}$$

Hence, the area of the triangle is

$$\frac{1}{2}\left|-\frac{\sqrt{a^2m^2 + b^2}}{m}\right|\sqrt{a^2m^2 b^2} = \frac{1}{2|m|}(a^2m^2 + b^2)$$

Answer: (D)

55. In an ellipse, the distance between the foci is 6 and minor axis is of length 8. Then, the eccentricity is

(A) $\frac{3}{5}$ (B) $\frac{1}{2}$ (C) $\frac{4}{5}$ (D) $\frac{1}{\sqrt{5}}$

Solution: We have $2ae = 6$ and $b = 4$. Therefore

$$16 = b^2 = a^2(1 - e^2) = a^2 - a^2e^2 = a^2 - 9$$

$$\Rightarrow a = 5$$

Therefore

$$e = \frac{3}{5} \quad (\because ae = 3)$$

Answer: (A)

56. The focus of an ellipse is at the origin. The directrix is the line $x = 4$ and the eccentricity is 1/2. Then the length of the semi-major axis is

(A) $\frac{8}{3}$ (B) $\frac{2}{3}$ (C) $\frac{4}{3}$ (D) $\frac{5}{3}$

Solution: The directrix corresponding to the focus is $x = 4$. Therefore, the perpendicular distance of the directrix from the focus is

$$\frac{a}{e} - ae = 4$$

$$\Rightarrow 2a - \frac{a}{2} = 4$$

$$\Rightarrow a = \frac{8}{3}$$

Answer: (A)

57. PQ is a double ordinate of

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

such that ΔOPQ is an equilateral triangle. If e is the eccentricity of the hyperbola, then

(A) $e = \frac{2}{\sqrt{3}}$ (B) $e > \frac{2}{\sqrt{3}}$
(C) $1 < e < \frac{2}{\sqrt{3}}$ (D) $e > \frac{4}{\sqrt{3}}$

Solution: See Fig. 5.45. Let $P = (a \sec\theta, b\tan\theta)$ and $Q = (a\sec\theta, -b\tan\theta)$. Now,

$$\tan 30^\circ = \frac{b\tan\theta}{a\sec\theta}$$

$$\Rightarrow \frac{1}{\sqrt{3}} = \frac{b}{a}\sin\theta$$

$$\Rightarrow \frac{3b^2}{a^2} = \operatorname{cosec}^2\theta > 1$$

$$\Rightarrow 3(e^2 - 1) > 1$$

$$\Rightarrow e^2 > 1 + \frac{1}{3} = \frac{4}{3}$$

$$\Rightarrow e > \frac{2}{\sqrt{3}}$$

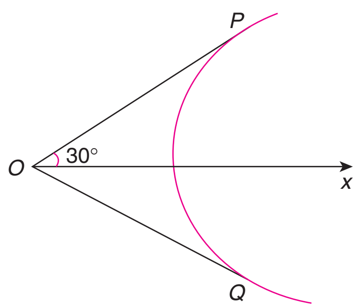


FIGURE 5.45

Answer: (B)

58. If $ax + by = 1$ is a tangent to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

then $a^2 - b^2 =$

(A) $b^2 e^2$ (B) $\dfrac{1}{b^2e^2}$

(C) $a^2 e^2$ (D) $\dfrac{1}{a^2e^2}$

Solution: We know that

$$y = \left(\frac{-a}{b}\right)x + \frac{1}{b}$$

is a tangent which implies that

$$\frac{1}{b^2} = a^2\left(\frac{-a}{b}\right)^2 - b^2 = \frac{a^4}{b^2} - b^2$$

$$\Rightarrow a^4 - b^4 = 1$$

$$\Rightarrow (a^2 - b^2) = \frac{1}{a^2 + b^2} = \frac{1}{a^2 + a^2(e^2 - 1)} = \frac{1}{a^2e^2}$$

Answer: (D)

59. If the eccentricity of the hyperbola $x^2 - y^2\sec^2\theta = 5$ is $\sqrt{3}$ times the eccentricity of the ellipse

$$x^2\sec^2\theta + y^2 = 25$$

then one of the value of θ is

(A) $\dfrac{\pi}{2}$ (B) $\dfrac{\pi}{4}$ (C) $\dfrac{\pi}{3}$ (D) $\dfrac{\pi}{6}$

Solution: The hyperbola is

$$\frac{x^2}{5} - \frac{y^2}{5\cos^2\theta} = 1$$

and the ellipse is

$$\frac{x^2}{25\cos^2\theta} + \frac{y^2}{25} = 1$$

Let e_1 and e_2 be the eccentricities of the hyperbola and the ellipse. Therefore

$$5\cos^2\theta = 5(e_1^2 - 1)$$

$$\Rightarrow e_1^2 = 1 + \cos^2\theta$$

Also

$$25\cos^2\theta = 25(1 - e_2^2)$$

$$\Rightarrow e_2^2 = 1 - \cos^2\theta = \sin^2\theta$$

By hypothesis,

$$e_1 = \sqrt{3}e_2$$

$$\Rightarrow e_1^2 = 3e_2^2 \Rightarrow 1 + \cos^2\theta = 3\sin^2\theta$$

$$\Rightarrow 1 + 1 - \sin^2\theta = 3\sin^2\theta$$

$$\Rightarrow \sin^2\theta = \frac{1}{2} \text{ or } \theta = \frac{\pi}{4}$$

Answer: (B)

60. The distance between the tangents to the hyperbola $4x^2 - 9y^2 = 1$ parallel to the line $5x - 4y + 7 = 0$ is

(A) $\dfrac{1}{3}\sqrt{\dfrac{161}{41}}$ (B) $\dfrac{1}{2}\sqrt{\dfrac{161}{41}}$

(C) $\dfrac{1}{3}\sqrt{\dfrac{151}{41}}$ (D) $\dfrac{1}{2}\sqrt{\dfrac{151}{41}}$

Solution: The given hyperbola is

$$\frac{x^2}{1/4} - \frac{y^2}{1/9} = 1$$

and the line is

$$y = \left(\frac{5}{4}\right)x + \frac{7}{4}$$

Any line parallel to this line is

$$y = \left(\frac{5}{4}\right)x + c$$

This will touch the hyperbola which implies that

$$c^2 = a^2m^2 - b^2 = \left(\frac{1}{4}\right)\left(\frac{5}{4}\right)^2 - \frac{1}{9} = \frac{25}{64} - \frac{1}{9}$$

Therefore

$$c = \pm\frac{\sqrt{161}}{24}$$

Hence, the tangents are

$$y = \frac{5x}{4} \pm \frac{\sqrt{161}}{24}$$

$$\Rightarrow 30x - 24y \pm \sqrt{161} = 0$$

Hence, the distance between these tangents is

$$\frac{2\sqrt{161}}{\sqrt{30^2+24^2}} = \frac{\sqrt{161}}{3\sqrt{41}}$$

Answer: (A)

Multiple Correct Choice Type Questions

1. For the hyperbola $9x^2 - 16y^2 - 18x + 32y - 151 = 0$, which of the following are true?

(A) Eccentricity is $\frac{5}{4}$

(B) Foci are $(-4, 1)$ and $(6, 1)$

(C) Centre is $(1, -1)$

(D) Length of the latus rectum is $\frac{9}{2}$

Solution: The given equation is

$$9(x^2 - 2x) - 16(y^2 + 2y) - 151 = 0$$

$$\Rightarrow 9(x-1)^2 - 16(y+1)^2 = 151 + 9 - 16 = 144$$

$$\Rightarrow \frac{(x-1)^2}{16} - \frac{(y+1)^2}{9} = 1$$

$$\Rightarrow \frac{X^2}{16} - \frac{Y^2}{9} = 1$$

where $X = x - 1$, $Y = y + 1$. Here $a^2 = 16, b^2 = 9$. The eccentricity e is given by

$$9 = 16(e^2 - 1) \text{ or } e^2 = 1 + \frac{9}{16} = \frac{25}{16}$$

so that

$$e = \frac{5}{4}$$

The centre is given by

$$(X = 0, Y = 0) = (x - 1 = 0, y + 1 = 0) = (1, -1)$$

The foci is given by

$$(X = \pm ae,\ Y = 0) = (x - 1 = \pm 5,\ -1) = (6, -1) \text{ and } (-4, -1)$$

The latus rectum is given by

$$\frac{2b^2}{a} = \frac{2(9)}{4} = \frac{9}{2}$$

Answers: (A), (C), (D)

2. If the circle $x^2 + y^2 = a^2$ cuts the hyperbola $xy = c^2$ at four points (x_k, y_k) (where $k = 1, 2, 3$ and 4), then

(A) $x_1 + x_2 + x_3 + x_4 = 0$ (B) $y_1 + y_2 + y_3 + y_4 = 0$

(C) $x_1 x_2 x_3 x_4 = c^4$ (D) $y_1 y_2 y_3 y_4 = c^4$

Solution: The abscissa x_k (where $k = 1, 2, 3$ and 4) are the roots of the equation

$$x^2 + \frac{c^4}{x^2} = a^2$$

$$\Rightarrow x^4 - a^2x^2 + c^4 = 0$$

Therefore

$$x_1 + x_2 + x_3 + x_4 = 0$$

Since the coefficient of x_3 is zero, we have

$$\sum x_1x_2 = -a^2,\ \sum x_1x_2x_3 = 0,\ \sum x_1x_2x_3x_4 = c^4$$

Now,

$$y_1 + y_2 + y_3 + y_4 = c^2\left(\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4}\right)$$

$$= \frac{c^2(\sum x_2x_3x_4)}{x_1x_2x_3x_4}$$

$$= \frac{c^2(0)}{c^4} = 0$$

Finally

$$y_1y_2y_3y_4 = \frac{c^2}{x_1}\cdot\frac{c^2}{x_2}\cdot\frac{c^2}{x_3}\cdot\frac{c^2}{x_4} = \frac{c^8}{c^4} = c^4$$

Answers: (A), (B), (C), (D)

3. On the ellipse $4x^2 + 9y^2 = 1$, the points at which the tangents are parallel to the line $9y = 8x$ are

(A) $\left(\frac{2}{5}, \frac{1}{5}\right)$ (B) $\left(\frac{-2}{5}, \frac{1}{5}\right)$

(C) $\left(\frac{-2}{5},\frac{-1}{5}\right)$ (D) $\left(\frac{2}{5},\frac{-1}{5}\right)$

Solution: The ellipse is

$$\frac{x^2}{(1/4)}+\frac{y^2}{(1/9)}=1$$

The tangent at (x_1, y_1) is $4xx_1 + 9yy_1 = 1$ which is parallel to $9y = 8x$. This implies

$$\frac{-4x_1}{9y_1}=\frac{8}{9}$$

$$\Rightarrow x_1=-2y_1 \tag{5.131}$$

Since (x_1, y_1) lies on the ellipse, we have

$$4x_1^2+9y_1^2=1 \tag{5.132}$$

Substituting $x_1 = -2y_1$ in Eq. (5.132), we get

$$16y_1^2+9y_1^2=1 \text{ or } y_1=\pm\frac{1}{5}$$

and $$x_1=\mp\frac{2}{5}$$

Therefore, the points are

$$\left(-\frac{2}{5},\frac{1}{5}\right) \text{or} \left(\frac{2}{5},-\frac{1}{5}\right)$$

Answers: (B), (D)

4. Let a hyperbola pass through the focus of the ellipse

$$\frac{x^2}{25}+\frac{y^2}{16}=1$$

The transeverse and conjugate axes of this hyperbola coincide with the major and minor axes of the ellipse. Also the product of the eccentricities of the ellipse and hyperbola is 1. Then

(A) the equation of the hyperbola is $\frac{x^2}{9}-\frac{y^2}{16}=1$

(B) the equation of the hyperbola is $\frac{x^2}{9}-\frac{y^2}{25}=1$

(C) the focus of the hyperbola is $(5, 0)$

(D) the vertex of hyperbola is $(5\sqrt{3},0)$

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Solution: The eccentricity e of the ellipse is given by

$$16 = 25\,(1 - e^2)$$

which gives that

$$e=\frac{3}{5}$$

Hence, the eccentricity of the hyperbola is $5/3$. Let

$$\frac{x^2}{\alpha^2}-\frac{y^2}{\beta^2}=1$$

be the hyperbola. Now,

$$\beta^2=\alpha^2\left(\frac{25}{9}-1\right)=\frac{16\alpha^2}{9}$$

This implies that the equation is

$$\frac{x^2}{\alpha^2}-\frac{9y^2}{16\alpha^2}=1$$

Also the hyperbola passes through the focus $(3, 0)$ of the ellipse. This implies that

$$\frac{9}{\alpha^2}=1 \Rightarrow \alpha^2=9$$

Also $$\beta^2=\frac{16\alpha^2}{9} \Rightarrow \beta^2=16$$

Hence, the equation of the hyperbola is

$$\frac{x^2}{9}-\frac{y^2}{16}=1$$

One vertex is $(3, 0)$ and the focus is

$$(\pm ae,0)=\left(\pm 3\left(\frac{5}{3}\right),0\right)=(\pm 5,0)$$

Answers: (A), (C)

5. Suppose the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

is confocal with the ellipse

$$\frac{x^2}{25}+\frac{y^2}{9}=1$$

and has eccentricity is 2. Then

(A) $a^2 + b^2 = 16$

(B) the hyperbola has no director circle

(C) the length of the latus rectum is 12

(D) $a^2 - b^2 = 16$

Solution: For the ellipse $a^2 = 25$, $b^2 = 9$ and the eccentricity is given by

$$9 = 25(1 - e^2) \Rightarrow e=\frac{4}{5}$$

Therefore, the foci of the ellipse are

$$(\pm ae,0)=\left(\pm 5\left(\frac{4}{5}\right),0\right)=(\pm 4,0)$$

Now, for the hyperbola, the foci is $(\pm ae, 0)$. So

$$(\pm 4,0)=(\pm 2a,0) \quad (\because e=2)$$
$$\Rightarrow a=2$$

Therefore

$$b^2 = a^2(e^2-1) = 4(4-1) = 12$$

Hence, the hyperbola is

$$\frac{x^2}{4} - \frac{y^2}{12} = 1$$

so that

$$a^2 + b^2 = 4 + 12 = 16$$

Director circle: $x^2 + y^2 = a^2 - b^2 = -8$ which is not possible. Hence, there is no director circle to the hyperbola. The length of the latus rectum is

$$\frac{2b^2}{a} = \frac{2(12)}{2} = 12$$

Answers: (A), (B), (C)

6. Suppose the normal at any point P on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

meets the major axis at G and the minor axis at g. CF is drawn perpendicular to the normal where C is the centre of the ellipse. Then

(A) $PF \cdot PG = b^2$
(B) $PF \cdot Pg = a^2$
(C) $PG \cdot Pg = SP \cdot S'P$
(D) $CG \cdot CT = CS^2$

where S is the focus

Solution: See Fig. 5.46. Let P be $(a\cos\theta, b\sin\theta)$. The normal at P is

$$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$$

Therefore

$$G = \left(\frac{(a^2-b^2)}{a}\cos\theta, 0\right) = (ae^2\cos\theta, 0)$$

and $$g = \left(0, \left(\frac{b^2-a^2}{b}\right)\sin\theta\right)$$

Now,

$$PG^2 = \left(a\cos\theta - \left(\frac{a^2-b^2}{a}\right)\cos\right)^2 + (b\sin\theta - 0)^2$$

$$= \frac{b^4\cos^2\theta}{a^2} + b^2\sin^2\theta$$

$$= \frac{b^2}{a^2}(b^2\cos^2\theta + a^2\sin^2\theta) \tag{5.133}$$

We have $PF = CT =$ perpendicular distance of C from the tangent at P which is given by

$$\frac{1}{\sqrt{(\cos^2\theta)/a^2 + (\sin^2\theta)/b^2}} = \frac{ab}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}} \tag{5.134}$$

(A) We have

$$PF \cdot PG = \frac{b}{a}\sqrt{b^2\cos^2\theta + a^2\sin^2\theta} \times \frac{ab}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}} = b^2$$

Also

$$Pg^2 = a^2\cos^2\theta + \left(b\sin\theta - \frac{(b^2-a^2)}{b}\sin\theta\right)^2$$

$$= a^2\cos^2\theta + \frac{a^4\sin^2\theta}{b^2}$$

$$= \frac{a^2}{b^2}(b^2\cos^2\theta + a^2\sin^2\theta)$$

Therefore

$$Pg = \frac{a}{b}\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}$$

(B) We have

$$PF \cdot Pg = \frac{ab}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}} \cdot \frac{a}{b}\sqrt{b^2\cos^2\theta + a^2\sin^2\theta} = a^2$$

(C) We have

$$PG \cdot Pg = \frac{b}{a}\sqrt{b^2\cos^2\theta + a^2\sin^2\theta} \times \frac{a}{b}\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}$$
$$= b^2\cos^2\theta + a^2\sin^2\theta$$

Also

$$SP \cdot S'P = e^2\left(a\cos\theta + \frac{a}{e}\right)\left(\frac{a}{e} - a\cos\theta\right)$$
$$= a^2(e\cos\theta + 1)(1 - e\cos\theta)$$
$$= a^2[1 - e^2\cos^2\theta]$$
$$= a^2 - (a^2 - b^2)\cos^2\theta$$
$$= a^2\sin^2\theta + b^2\cos^2\theta$$

Therefore

$$PG \cdot Pg = SP \cdot S'P$$

(D) We have

$$CG \cdot CT = \left(\frac{a^2-b^2}{a}\right)\cos\theta \cdot \frac{ab}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}$$

$$= \frac{b(a^2-b^2)\cos\theta}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}} = (CS)^2$$

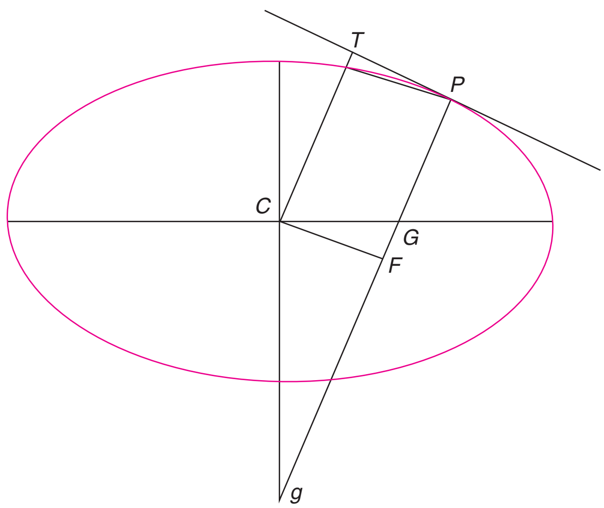


FIGURE 5.46

Answers: (A), (B), (C), (D)

7. The normal to a curve at $P(x, y)$ meets the x-axis at G. If the distance of G from the origin is twice the abscissa of P, then the curve is a

(A) circle (B) hyperbola
(C) ellipse (D) parabola

Solution: The normal at $P(x_1, y_1)$ is

$$y - y_1 = -\frac{1}{m}(x - x_1)$$

where

$$m\left(\frac{dy}{dx}\right)_{(x_1,y_1)}$$

Now, $y = 0 \Rightarrow x = x_1 + my_1$ so that by hypothesis, we have

$$|x_1 + my_1| = 2|x_1|$$

Therefore, $x_1 + my_1 = \pm 2x_1$.

Case 1:

$$n_1 + my_1 = 2x_1$$

$$\Rightarrow my_1 = x_1$$

$$\Rightarrow \frac{dy}{dx} = \frac{x}{y}$$

$$\Rightarrow xdx - ydy = 0$$

Therefore, the solution is

$$\frac{x^2}{2} - \frac{y^2}{2} = c$$

$$\Rightarrow x^2 - y^2 = 2c$$

which is a hyperbola.

Case 2:

$$x_1 + my_1 = -2x_1$$

$$\Rightarrow y\frac{dy}{dx} + 3x = 0$$

$$\Rightarrow ydy + (3x)dx = 0$$

Therefore

$$\frac{y^2}{2} + \frac{3x^2}{2} = c$$

$$\Rightarrow \frac{x^2}{(2/3)c} + \frac{y^2}{2c} = 1$$

which is an ellipse.

Answer: (B), (C)

Matrix-Match Type Questions

1. Consider the ellipse

$$\frac{x^2}{14} + \frac{y^2}{5} = 1$$

The normal at P meets the major axis at G and the minor axis at g. CF is drawn perpendicular to the normal at P. Match the items of Column I with those of Column II.

Column I	*Column II*
(A) $PF \cdot PG$ is equal to	(p) $5 + 9\sin^2\theta$
(B) $PF \cdot Pg$ is equal to	(q) 5
(C) $PG \cdot Pg$ is equal to	(r) $9 + 5\cos^2\theta$
(D) $CG \cdot CT$ is equal to	(s) 9
	(t) 16

Solution: According to Problem 6 of previous section (Multiple Correct Choice Type Questions), we have

$$PF \cdot PG = b^2$$
$$PF \cdot Pg = a^2$$
$$PG \cdot Pg = SP \cdot S'P$$

and $$CG \cdot CT = (CS)^2.$$

Here, $a^2 = 14$, $b^2 = 5$, eccentricity $e = 3/\sqrt{14}$ and the foci are $(\pm 3, 0)$.

Answer: (A) → (q); (B) → (t); (C) → (p); (D) → (s)

2. Consider the ellipse

$$\frac{x^2}{25} + \frac{y^2}{9} = 1$$

with centre C and P is a point on the ellipse with eccentric angle 45°. The normal drawn at P meets the major and the minor axes at G and g, respectively. F is the foot of the perpendicular drawn from the centre C onto the normal at P. The tangent at P meets the major axis at T. M and N are the feet of the perpendiculars drawn from the foci S and S' onto the tangent at P. Match the items of Column I with those of Column II.

Column I	*Column II*
(A) $CG \cdot CT$ is equal to	(p) 9
(B) $PF \cdot Pg$ is equal to	(q) 16
	(r) 17
(C) $SM \cdot S'N$ is equal to	(s) 15
(D) $SP \cdot S'P$ is equal to	(t) 25

Solution: See Fig. 5.47. We have

$$P = (a\cos\theta, b\sin\theta) = \left(\frac{5}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)$$

The tangent at P is

$$\frac{x}{5\sqrt{2}} + \frac{y}{3\sqrt{2}} = 1$$

$$\Rightarrow 3x + 5y = 15\sqrt{2} \tag{5.135}$$

The normal at P is

$$\frac{25[x-(5/\sqrt{2})]}{5/\sqrt{2}} = \frac{9[y-(3/\sqrt{2})]}{3/\sqrt{2}}$$

$$\Rightarrow 5\sqrt{2}x - 3\sqrt{2}y = 25 - 9 = 16$$

$$\Rightarrow 5\sqrt{2}x - 3\sqrt{2}y = 16 \tag{5.136}$$

From Eq. (5.135), we get

$$T = (5\sqrt{2}, 0) \tag{5.137}$$

The eccentricity e is given by

$$9 = 25(1-e^2) \Rightarrow e = \frac{4}{5}$$

so that

$$S = (ae, 0) = (4, 0) \text{ and } S' = (-4, 0)$$

Now,

$$SM = \left|\frac{12 + 0 - 15\sqrt{2}}{\sqrt{3^2 + 5^2}}\right| = \frac{15\sqrt{2} - 12}{\sqrt{34}}$$

$$S'N = \frac{15\sqrt{2} + 12}{\sqrt{34}}$$

Hence

$$SM \cdot S'N = \frac{306}{34} = 9$$

Also note that $SM \cdot S'N$ is always equal to b^2. In Problem 6 of previous section (Multiple Correct Choice Type Questions), we have proved that $PF \cdot Pg = a^2$ so that

$$PF \cdot Pg = a^2 = 25$$

Now,

$$CG \cdot CT = \frac{8\sqrt{2}}{5} \cdot 5\sqrt{2} = 16$$

Finally,

$$SP \cdot S'P = \frac{4}{5}\left[\left(\frac{25}{4}\right) - \frac{5}{\sqrt{2}}\right] \cdot \frac{4}{5}\left[\frac{25}{4} + \frac{5}{\sqrt{2}}\right]$$
$$= \frac{16}{25}\left[\frac{25^2}{16} - \frac{25}{2}\right]$$
$$= 16\left[\frac{25}{16} - \frac{1}{2}\right]$$
$$= 16\left(\frac{17}{16}\right) = 17$$

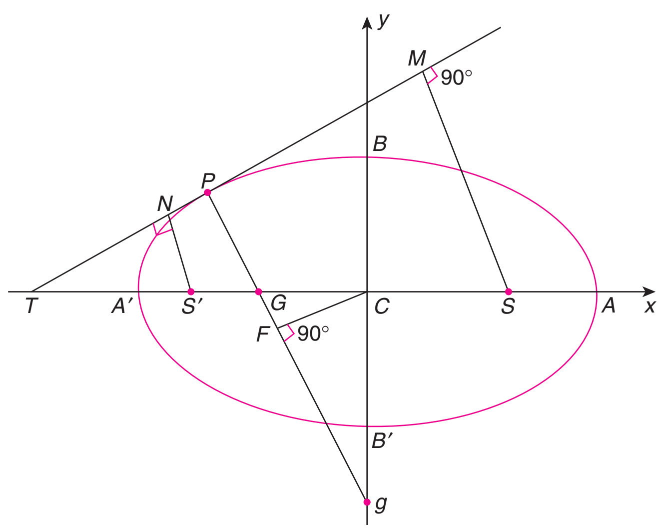


FIGURE 5.47

Answer: (A) → (q); (B) → (t); (C) → (p); (D) → (r)

3. Some ellipses are given in Column I and their eccentricities are given in Column II. Match the items of Column I with those of Column II.

Column I	*Column II*
(A) $9x^2 + 25y^2 - 18x - 100y - 116 = 0$	(p) $\frac{1}{2}$
(B) $9x^2 + 4y^2 = 36$	(q) $\frac{3}{5}$
(C) $3x^2 + 4y^2 + 6x - 8y - 5 = 0$	(r) $\frac{\sqrt{2}}{3}$
(D) $\frac{(x-1)^2}{9} + \frac{(y-2)^2}{4} = 1$	(s) $\frac{4}{5}$
	(t) $\frac{\sqrt{5}}{3}$

Solution: The eccentricity is given by the relation

$$b^2 = a^2(1 - e^2) \text{ if } a > b$$

and
$$a^2 = b^2(1 - e^2) \text{ if } a < b$$

(A) We have

$$9(x-1)^2 + 25(y-2)^2 = 116 + 9 + 100 = 225$$

$$\Rightarrow \frac{(x-1)^2}{25} + \frac{(y-2)^2}{9} = 1$$

Also

$$9 = 25(1-e^2)$$

$$\Rightarrow e^2 = 1 - \frac{9}{25} = \frac{16}{25}$$

$$\Rightarrow e = \frac{4}{5}$$

Answer: (A) → (s)

(B) We have

$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

Now

$$4 = 9(1-e^2)$$

$$\Rightarrow e^2 = 1 - \frac{4}{9} = \frac{5}{9}$$

$$\Rightarrow e = \frac{\sqrt{5}}{3}$$

Answer: (B) → (t)

(C) We have

$$3(x+1)^2 + 4(y-1)^2 = 5 + 3 + 4 = 12$$

$$\Rightarrow \frac{(x+1)^2}{4} + \frac{(y-1)^2}{3} = 1$$

Now

$$3 = 4(1-e^2)$$

$$\Rightarrow e^2 = 1 - \frac{3}{4} = \frac{1}{4}$$

$$\Rightarrow e = \frac{1}{2}$$

Answer: (C) → (p)

(D) We have

$$4 = 9(1-e^2)$$

$$\Rightarrow e^2 = 1 - \frac{4}{9} = \frac{5}{9}$$

$$\Rightarrow e = \frac{\sqrt{5}}{3}$$

Answer: (D) → (t)

4. Match the items of Column I with those of Column II.

Column I	*Column II*
(A) The locus of the point whose chord of contact with respect to the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$ touches the circle described on the line joining the foci is	(p) $(x^2 + y^2)^2 = 4x^2 - 3y^2$
(B) The chords of the circle $x^2 + y^2 = 4$ touch the hyperbola $\frac{x^2}{4} - \frac{y^2}{3} = 1.$ Then the locus of the midpoints of these chords is	(q) $x^2 + y^2 = 9$
(C) The director circle of the hyperbola $\frac{x^2}{25} - \frac{y^2}{16} = 1$ is	(r) $x^2 - y^2 = 32$

(Continued)

Column I	*Column II*
(D) The distance between the foci of a hyperbola is 16 and its eccentricity is $\sqrt{2}$. Then the equations of the hyperbola is	(s) $\dfrac{x^2}{256}+\dfrac{y^2}{81}=\dfrac{1}{25}$ (t) $x^2-y^2=64$

Solution:

(A) $P(x_1, y_1)$ be a point on the locus. That is

$$\frac{xx_1}{16}-\frac{yy_1}{9}=1$$

touches the circle described on the line joining the foci $S(5, 0)$ and $S'(-5, 0)$ whose equation is

$$x^2+y^2=25$$

$$\Rightarrow \left|\frac{0-0-1}{\sqrt{x_1^2/16^2+y_1^2/9^2}}\right|=5$$

$$\Rightarrow \frac{x_1^2}{16^2}+\frac{y_1^2}{9^2}=\frac{1}{25}$$

Therefore, the locus is

$$\frac{x^2}{16^2}+\frac{y^2}{9^2}=\frac{1}{25}$$

Answer: (A) → (s)

(B) (x_1, y_1) is the midpoint of a chord $x^2 + y^2 = 4$ touching the hyperbola

$$\frac{x^2}{4}-\frac{y^2}{3}=1$$

That is, the line $xx_1 + yy_1 = x^2_1 + y^2_1$ touches the hyperbola. This means

$$\left(\frac{x_1^2+y_1^2}{y_1}\right)^2=4\left(\frac{-x_1}{y_1}\right)^2-3$$

$$\Rightarrow (x_1^2+y_1^2)^2=4x_1^2-3y_1^2$$

Therefore, the locus is

$$(x^2+y^2)^2=4x^2-3y^2$$

Answer: (B) → (p)

(C) Director circle of

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

is $x^2 + y^2 = a^2 - b^2$. Hence, $a^2 = 25$ and $b^2 = 16$. Hence, the director circle is $x^2 + y^2 = 9$.

Answer: (C) → (q)

(D) Since $\sqrt{2}$ is the eccentricity of the hyperbola, it must be a rectangular hyperbola. Hence, it is of the form $x^2 - y^2 = a^2$. By hypothesis,

$$2ae=16 \Rightarrow 2a(\sqrt{2})=16 \Rightarrow a=4\sqrt{2}$$

Hence, the hyperbola is $x^2 - y^2 = 32$.

Answer: (D) → (r)

Comprehension Type Questions

1. Passage: Tangents are drawn from the point $P(3, 4)$ to the ellipse

$$\frac{x^2}{9}+\frac{y^2}{4}=1$$

touching the ellipse at A and B. Answer the following questions.

(i) The coordinates of A and B are

(A) $\left(\dfrac{-8}{5},\dfrac{2\sqrt{161}}{15}\right)$ and $\left(\dfrac{-9}{5},\dfrac{8}{5}\right)$

(B) $\left(\dfrac{-8}{5},\dfrac{2\sqrt{161}}{15}\right)$ and $(0,2)$

(C) $(3,0)$ and $(0,2)$

(D) $(3,0)$ and $\left(\dfrac{-9}{5},\dfrac{8}{5}\right)$

(ii) The orthocentre of ΔPAB is

(A) $\left(5,\dfrac{8}{7}\right)$ (B) $\left(\dfrac{7}{5},\dfrac{25}{8}\right)$

(C) $\left(\dfrac{11}{5},\dfrac{8}{5}\right)$ (D) $\left(\dfrac{8}{25},\dfrac{7}{5}\right)$

(iii) The equation of the locus of the point, whose distance from the point P and the line AB are equal, is

(A) $9x^2 + y^2 - 6xy - 54x - 62y + 241 = 0$

(B) $x^2 + 9y^2 + 6xy - 54x - 62y - 241 = 0$

(C) $9x^2 + 9y^2 - 6xy - 54x - 62y - 241 = 0$

(D) $x^2 + y^2 - 2xy + 27x + 31y - 120 = 0$

(IIT-JEE 2010)

Solution:

(i) $y = mx \pm \sqrt{9m^2 + 4}$ is a tangent to the ellipse. This passes through (3, 4). So

$$(4 - 3m)^2 = 9m^2 + 4$$
$$\Rightarrow 16 - 24m - 9m^2 = 9m^2 + 4$$
$$\Rightarrow m = \frac{1}{2}$$

Therefore, the tangent from (3, 4) is

$$y = \frac{x}{2} + \sqrt{\frac{9}{4} + 4}$$
$$\Rightarrow x - 2y = -5$$

Suppose $x - 2y + 5 = 0$ touches at (x_1, y_1). Hence

$$x - 2y + 5 = 0$$

and
$$\frac{xx_1}{9} + \frac{yy_1}{4} = 1$$

represent the same straight line. Therefore

$$\frac{(x_1/9)}{1} = \frac{(y_1/4)}{-2} = \frac{1}{-5}$$
$$\Rightarrow x_1 = \frac{-9}{5}, y_1 = \frac{8}{5}$$

The point of contact is $(-9/5, 8/5)$. Also the line $x = 3$ is a tangent at the vertex $A(3, 0)$. Therefore, the points of contact are $(-9/5, 8/5)$ and $(3, 0)$.

Aliter: The tangent at $(3 \cos \theta, 2 \sin \theta)$ is

$$\frac{x\cos\theta}{3} + \frac{y\sin\theta}{2} = 1$$

This passes through (3, 4) which implies

$$\cos\theta + 2\sin\theta = 1$$
$$\Rightarrow (1 - \cos\theta)^2 = 4\sin^2\theta$$
$$\Rightarrow 1 - 2\cos\theta + \cos^2\theta = 4(1 - \cos^2\theta)$$
$$\Rightarrow 5\cos^2\theta - 2\cos\theta - 3 = 0$$
$$\Rightarrow 5\cos^2\theta - 5\cos\theta + 3\cos\theta - 3 = 0$$
$$\Rightarrow (5\cos\theta + 3)(\cos\theta - 1) = 0$$
$$\Rightarrow \cos\theta = 1, -3/5$$

When

$$\cos\theta = 1 \Rightarrow \sin\theta = 0$$

and
$$\cos\theta = \frac{-3}{5} \Rightarrow \sin\theta = \pm\frac{4}{5}$$

Therefore, the points of contact are (3, 0) and $(-9/5, 8/5)$. Note that for

$$\cos\theta = \frac{-3}{5} \text{ and } \sin\theta = \frac{-4}{5}$$

the corresponding point does not lie on the tangent at P.

Answer: (D)

(ii) The vertices are $P(3, 4)$, $A(3, 0)$ and $B(-9/5, 8/5)$. Since AP in a vertical line, the altitude through B is $y = 8/5$. As orthocentre lies on the line $y = 8/5$, the orthocentre of ΔPAB is $(11/5, 8/5)$.

Answer: (C)

(iii) The equation of the line AB is

$$\Rightarrow y = \frac{-1}{3}(x - 3)$$
$$\Rightarrow x + 3y - 3 = 0$$

Hence, the required locus is the parabola

$$(x - 3)^2 + (y - 4)^2 = \left(\frac{x + 3y - 3}{\sqrt{10}}\right)^2$$
$$\Rightarrow 10(x^2 + y^2 - 6x - 8y + 25)$$
$$= x^2 + 9y^2 + 6xy - 6x - 18y + 9$$
$$\Rightarrow 9x^2 + y^2 - 6xy - 54x - 62y + 241 = 0$$

Answer: (A)

2. Passage: Tangents are drawn to the hyperbola $x^2 - 9y^2 = 9$ from (3, 2). Answer the following questions.

(i) The area of the triangle formed by the tangents and the chord contact of (3, 2) is

(A) 6 (B) 8 (C) 10 (D) 12

(ii) The area of the triangle formed by the tangent to the hyperbola at (3, 0) and the two asymptotes is

(A) 3 (B) 6 (C) 9 (D) 2

(iii) The midpoint of the intercept of the tangent at (3, 0) between the asymptotes is

(A) (1, 0) (B) (2, 0) (C) (3, 1) (D) (3, 0)

Solution:

(i) $y = mx + \sqrt{9m^2 - 1}$ is a tangent to the hyperbola

$$\frac{x^2}{9} - \frac{y^2}{1} = 1$$

This passes through the point (3, 2). This implies

$$(3m - 2)^2 = 9m^2 - 1$$
$$\Rightarrow -12m = -5$$
$$\Rightarrow m = \frac{5}{12}$$

Therefore, one tangent is $5x - 12y + 9 = 0$. Also the tangent at the vertex (3, 0) passes through (3, 2). Hence, the other tangent through (3, 2) is $x = 3$. The chord of contact is

$$\frac{3x}{9} - \frac{2y}{1} = 1$$
$$\Rightarrow x - 6y = 3$$

Therefore, the sides of the triangle are

$$5x - 12y + 9 = 0$$
$$x = 3 \text{ and } x - 6y = 3$$

Solving these equations, the vertices of the triangle are (3, 2), (3, 0) and $(-5, -4/3)$. Hence, the area of the triangle is

$$\frac{1}{2}\begin{vmatrix} 3 & 2 & 1 \\ 3 & 0 & 1 \\ -5 & \frac{-4}{3} & 1 \end{vmatrix} = \frac{1}{2}\left|3\left(0+\frac{4}{3}\right) - 2(3+5) + 1(-4-0)\right|$$

$$= \frac{1}{2}|4 - 16 - 4| = 8$$

Answer: (B)

(ii) The area of the triangle formed by two asymptotes and a tangent to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

is always constant which is equal to ab (see Problem 5 of the section 'Subjective Problems (Section 5.3)').

Answer: (A)

(iii) We know that the portion of the tangent is intercepted between the asymptote is bisected at the point of contact (see Theorem 5.28). In fact the asymptotes are $x = \pm 3y$ and the tangent at (3, 0) is $x = 3$. Hence, the tangent at (3,0) intersects the asymptotes at points (3, 1) and (3, $-$ 1) so that the midpoint of the segment is (3, 0).

Answer: (D)

3. Passage: The locus of the foot of the perpendicular drawn from a focus onto a tangent to the hyperbola is the auxiliary circle. Consider the hyperbola whose foci are (5, 6) and ($-$ 3, $-$ 2). The foot of the perpendicular from the focus upon a tangent is the point (2, 5). Answer the following questions.

(i) Length of the conjugate axis of the hyperbola is

(A) $4\sqrt{22}$ (B) $2\sqrt{22}$ (C) $2\sqrt{11}$ (D) $4\sqrt{11}$

(ii) The directrix of the hyperbola corresponding to the focus (5, 6) is

(A) $2x + 2y - 11 = 0$
(B) $2x + 2y - 1 = 0$
(C) $2x + 2y - 9 = 0$
(D) $2x + 2y - 7 = 0$

(iii) Length of the latus rectum of the hyperbola

(A) $\frac{44}{\sqrt{10}}$ (B) $\frac{22}{\sqrt{10}}$ (C) $\frac{32}{\sqrt{10}}$ (D) $\frac{42}{\sqrt{10}}$

Solution:

(i) The centre of the hyperbola is

$$\left(\frac{-3+5}{2}, \frac{6-2}{2}\right) = (1, 2)$$

Since (2, 5) lies on the auxiliary circle, its radius is

$$a = \sqrt{(1-2)^2 + (2-5)^2} = \sqrt{10}$$

The distance between the foci is $2ae$. From the coordinates of the foci, the distance between them is

$$\sqrt{(5+3)^2 + (6+2)^2} = \sqrt{128} = 8\sqrt{2}$$

Now

$$2\sqrt{10}e = 8\sqrt{2}$$
$$\Rightarrow e = \frac{4}{\sqrt{5}}$$

We have

$$b^2 = a^2(e^2 - 1) = 10\left(\frac{16}{5} - 1\right)$$
$$\Rightarrow 2b = 2\sqrt{22}$$

Answer: (B)

(ii) Let the corresponding directrix be $x + y = \lambda$ (because the directrix is perpendicular to transverse axis). The distance of this from the centre is a/e

$$\left|\frac{1+2-\lambda}{\sqrt{2}}\right| = \sqrt{10} \times \frac{\sqrt{5}}{4} = \frac{5}{2\sqrt{2}}$$
$$\Rightarrow \lambda = 3 + \frac{5}{2} = \frac{11}{2}$$

Therefore, the directrix equation is $2x + 2y - 11 = 0$.

Answer: (A)

(iii) The length of the latus rectum is

$$\frac{2b^2}{a} = \frac{2(22)}{\sqrt{10}} = \frac{44}{\sqrt{10}}$$

Answer: (A)

Integer Answer Type Questions

1. Let $S = (1, 2)$ and $S' = (5, 5)$ be the foci of a hyperbola. For any point P on the curve, it is given that $|S'P - SP| = 3$. If e' is the eccentricity of the conjugate hyperbola, then $4e'$ is equal to _______.

Solution: We have

$$|S'P - SP| = 3 \Rightarrow 2a = 3 \text{ or } a = \frac{3}{2}$$

Therefore

$$2ae = SS' = 5 \Rightarrow 3e = 5 \Rightarrow e = \frac{5}{3}$$

Hence

$$\frac{1}{e^2} + \frac{1}{e'^2} = 1$$

$$\Rightarrow \frac{1}{e'^2} = 1 - \frac{1}{e^2} = 1 - \frac{9}{25} = \frac{16}{25}$$

Answer: 5

2. If the angle between the asymptotes of a hyperbola is $\pi/3$, then the eccentricity of its conjugate hyperbola is _______.

Solution: From Theorem 5.21, part (2), let e' be the eccentricity of the conjugate hyperbola. Therefore

$$\frac{1}{e^2} + \frac{1}{e'^2} = 1 \Rightarrow \frac{1}{e'^2} = 1 - \frac{3}{4} = \frac{1}{4}$$

Hence, $e' = 2$.

Answer: 2

3. If p_1 and p_2 are perpendiculars drawn from any point on the curve $x^2 - 2y^2 = 2$ to its asymptotes, then $3p_1p_2$ is equal to _______.

Solution: By Theorem 5.21, part (2), we have

$$p_1p_2 = \frac{a^2b^2}{a^2 + b^2} = \frac{2(1)}{2+1} = \frac{2}{3}$$

Therefore, $3\, p_1p_2 = 2$.

Answer: 2

4. If the equation of the asymptotes of the hyperbola

$$3x^2 + 10xy + 8y^2 + 14x + 22y + 7 = 0$$

is

$$3x^2 + 10xy + 8y^2 + 14x + 22y + c = 0$$

then the value of $c - 7$ is _______.

Solution: Since $3x^2 + 10xy + 8y^2 + 14x + 22y + c = 0$ represent pair of lines, we have

$$\begin{vmatrix} 3 & 5 & 7 \\ 5 & 8 & 11 \\ 7 & 11 & c \end{vmatrix} = 0$$

$$\Rightarrow 3(8c - 121) - 5(5c - 77) + 7(55 - 56) = 0$$

$$\Rightarrow -c - 363 + 385 - 7 = 0$$

$$\Rightarrow -c + 15 = 0$$

$$\Rightarrow c = 15$$

Hence, $c - 7 = 8$.

Answer: 8

5. Four times the eccentricity of the hyperbola $9x^2 - 16y^2 - 18x + 32y - 151 = 0$ is equal to _______.

Solution: The equation of the hyperbola can be written as

$$9(x - 1)^2 - 16(y + 1)^2 = 151 + 9 - 16 = 144$$

$$\Rightarrow \frac{(x-1)^2}{16} - \frac{(y-1)^2}{9} = 1$$

Therefore, the eccentricity e is given by

$$9 = 16(e^2 - 1)$$

$$\Rightarrow e^2 = 1 + \frac{9}{16} = \frac{25}{16}$$

$$\Rightarrow e = \frac{5}{4}$$

$$\Rightarrow 4e = 5$$

Answer: 5

6. The radius of the director circle of the hyperbola $\frac{x^2}{4} - \frac{y^2}{3} = 1$ is _______.

Solution: The director circle of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

is

$$x^2 + y^2 = a^2 - b^2$$

$$\Rightarrow x^2 + y^2 = 4 - 3 = 1$$

Answer: 1

7. For the hyperbola

$$\frac{x^2}{25} - \frac{y^2}{16} = 1$$

the tangent at (5, 0) is intercepted by the asymptotes of the hyperbola at P and Q. Then, the sum of the coordinates of the midpoint of the segment $\overline{PQ}$ is _______.

Solution: A portion of the intercept of any tangent to the hyperbola between the asymptotes is bisected by the point of contact (see Theorem 5.28).

Answer: 5

8. If $\alpha - \beta = \pi/2$, then the chord joining the points whose eccentric angles are α and β of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

touches the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{1}{k}$$

where k is equal to _______.

Solution: The chord joining α and β is

$$\frac{x\cos[(\alpha+\beta)/2]}{a}+\frac{y\sin[(\alpha+\beta)/2]}{b}=\cos\left(\frac{\alpha-\beta}{2}\right)$$
$$=\cos\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}$$

If

$$\frac{\alpha+\beta}{2}=\theta$$

then the equation of the chord is

$$\frac{x}{(a/\sqrt{25})}\cos\theta+\frac{y}{(b/\sqrt{2})}\sin\theta=1$$

Hence, the chord touches the ellipse

$$\frac{x^2}{(a/\sqrt{2})^2}+\frac{y^2}{(b/\sqrt{2})^2}=1$$

or
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{1}{2}$$

Therefore, $k=2$.

Answer: 2

9. The locus of the midpoint of the portion of the tangent to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

included between the axes is the curve

$$\frac{a^2}{x^2}+\frac{b^2}{y^2}=k$$

where k is equal to _______.

Solution: The tangent at $(a\cos\theta, b\sin\theta)$ to the ellipse is

$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$$

This meets the axes at the points $A(a\sec\theta, 0)$ and $B(0, b\operatorname{cosec}\theta)$. Let (x_1, y_1) be the midpoint of $\overline{AB}$. Therefore

$$2x_1=a\sec\theta \text{ and } 2y_1=b\operatorname{cosec}\theta$$
$$\Rightarrow 2\cos\theta=\frac{a}{x_1} \text{ and } 2\sin\theta=\frac{b}{y_1}$$
$$\Rightarrow 4(\cos^2\theta+\sin^2\theta)=\frac{a^2}{x_1^2}+\frac{b^2}{y_1^2}$$

Therefore, the locus of (x_1, x_2) is

$$\frac{a^2}{x^2}+\frac{b^2}{y^2}=4$$

Answer: 4

10. Through the point (3,5), the number of tangents drawn to the ellipses $3x^2+5y^2=32$ and $25x^2+9y^2=450$ is _______.

Solution: Let

$$S \equiv 3x^2+5y^2-32$$

and
$$S' \equiv 25x^2+9y^2-450$$

Also $(3, 5) = (x_1, y_1)$. Now,

$$S_{11}=3(9)+5(25)-32>0$$
$$S'_{11}=25(9)+9(25)-450=0$$

Therefore, (3, 5) is an external point to $S = 0$ and (3, 5) lies on $S'=0$. Hence, the number of tangents drawn to the ellipses through (3, 5) is 3.

Answer: 3

11. The number of distinct normals that can be drawn from the point (0, 6) to the ellipse

$$\frac{x^2}{169}+\frac{y^2}{25}=1$$

is _______.

Solution: Observe that y-axis is normal to the ellipse at (0, 5) and y-axis is passing through (0, 6). Now, a normal to the ellipse at $(13\cos\theta, 5\sin\theta)$ is

$$\frac{13x}{\cos\theta}-\frac{5y}{\sin\theta}=169-25=144$$

This passes through the point (0, 6). So

$$\frac{-30}{\sin\theta}=144$$

$$\Rightarrow \sin\theta = \frac{-15}{72} = \frac{-5}{24}$$

$$\Rightarrow \theta = \pi + \text{Sin}^{-1}\left(\frac{5}{24}\right) \quad \text{or} \quad 2\pi - \text{Sin}^{-1}\left(\frac{5}{24}\right)$$

Hence, the number of normals that can pass through $(0, 6)$ is 3.

Answer: 3

12. The number of following statements which are true in the case of the ellipse $x^2 + 4y^2 - 2x - 16y + 13 = 0$ is _______.

(A) The latus rectum of the ellipse is 1.
(B) The distance between the foci is $4\sqrt{3}$.
(C) The sum of the focal distances of a point $P(x, y)$ on the ellipse is 4.
(D) The line $y = 3$ meets the tangents drawn at the vertices of the ellipse at points P and Q. Then $\overline{PQ}$ subtends a right angle at either of the foci.

Solution: The given ellipse equation is

$$(x-1)^2 + 4(y-2)^2 = -13 + 1 + 16 = 4$$

$$\Rightarrow \frac{(x-1)^2}{4} + \frac{(y-2)^2}{1} = 1$$

We have $a^2 = 4, b^2 = 1$. The eccentricity e is given by

$$1 = 4(1-e^2) \Rightarrow e = \frac{\sqrt{3}}{2}$$

The latus rectum is

$$\frac{2b^2}{a} = \frac{2(1)}{2} = 1$$

Hence, Statement (A) is *true*. Also, the ellipse is

$$\frac{X^2}{4} + \frac{Y^2}{1} = 1$$

where $X = x - 1, Y = y - 2$. The foci is

$$(X = \pm ae, Y = 0) = (x - 1 = \pm\sqrt{3}, y - 2 = 0) = (1 \pm \sqrt{3}, 2)$$

Hence, the distance between the foci is

$$(1+\sqrt{3}) - (1-\sqrt{3}) = 2\sqrt{3}$$

Hence, Statement (B) is *not true*. The sum of the focal distance is

$$SP + S'P = 2a = 2(2) = 4$$

Therefore, Statement (C) is *true*. The tangents at the vertices are $x - 1 = \pm 2$ or $x = 3, -1$. The line $y = 3$ meets these tangents at $(3, 3)$ and $(-1, 3)$. We have $P = (3, 3)$ and $Q = (-1, 3) S = (1+\sqrt{3}, 2)$. Therefore

$$\text{Slope of } SP \times \text{Slope of } SQ = \left(\frac{3-2}{3-1-\sqrt{3}}\right)\left(\frac{3-2}{-1-1-\sqrt{3}}\right)$$

$$= \frac{1}{2-\sqrt{3}} \times \frac{-1}{2+\sqrt{3}} = \frac{-1}{4-3} = -1$$

Therefore, $\overline{PQ}$ subtends right angle at the focus $(1, +\sqrt{3}, 2)$ and also at the other focus. Hence, Statement (D) is *true*.

Answer: 3

SUMMARY

5.1. ***Definition:*** The locus of the point whose distance from a fixed point S is equal to e times $(0 < e < 1)$ the distance of the same point from a fixed line l, which is not passing through S, is called an *ellipse*. The fixed point S is called a *focus* and the fixed line l is called the *directrix*. The fixed number e is called the *eccentricity*.

5.2. ***Standard form:*** The equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \, (a^2 > b^2)$$

is called *standard equation* to the ellipse where $b^2 = a^2(1 - e^2)$.

5.3. ***Properties of curve:***

1. Curve is symmetric about both x-axis and y axis. That is, if (x, y) is a point on a curve, then $(x, -y)$, $(-x, -y)$ and $(-x, y)$ are also on the curve.
2. For any point (x, y) on a curve, $-a \le x \le a$ and $-b \le y \le b$.
3. x-axis meets a curve at $A(a, 0)$ and $A'(-a, 0)$ (called the *vertices*) and y-axis meets the curve at $B(0, b)$ and $B'(0, -b)$.
4. For each real value of $x(-a \le x \le a)$, there are two corresponding values for y which are given by

$$y = \pm \frac{b}{a}\sqrt{a^2 - x^2}$$

and for each value of $y(-b \le y \le b)$, there are two corresponding values for x which are given by

$$x = \pm\frac{a}{b}\sqrt{b^2 - y^2}$$

5. Since a curve is symmetric about axes, there are another focus and another directrix.

5.4. ***Foci:*** For the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

the foci are $S\,(ae, 0)$ and $S'(-ae, 0)$.

5.5. ***Directrices:*** $x = a/e$ is the directrix corresponding to the focus $(ae, 0)$ and $x = -a/e$ is the directrix corresponding to the focus $(-ae, 0)$.

5.6 ***Major and minor axes:*** In Fig. 5.1, AA' is called major axis and its length is $2a$. BB' is called minor axis and its length is $2b$.

5.7. ***Double ordinate:*** Any chord of the ellipse which is perpendicular to the major axis is called double ordinate.

5.8. ***Latus rectum:*** The double ordinate passing through the foci is called the latus rectum and its length is $2b^2/a$.

5.9. If $S(ae, 0)$ and $S'(-ae, 0)$ are the two foci of an ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

then for any point $P\,(x, y)$ on the curve,

$$SP = e(PM) = a - ex$$

and $$S'P = a + ex$$

and hence $SP + S'P = 2a$ (constant).

5.10. ***Inverted ellipse:*** If $b^2 > a^2$, then the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

represents an ellipse called inverted ellipse. That is, in the standard form, the role of a and b are interchanged.

5.11. For an inverted ellipse, the eccentricity e is given by $a^2 = b^2(1 - e^2)$. For an inverted ellipse, $(0, b)$ and $(0, -b)$ are the vertices, the foci are $(0, \pm be)$ and the corresponding directrices are $y = \pm\, b/e$. The length of latus rectum is $2a^2/b$.

5.12. $$\frac{(x - x_1)^2}{a^2} + \frac{(y - y_2)^2}{b^2} = 1$$

where $a^2 > b^2$ or $b^2 > a^2$ are ellipses with (x_1, y_1) as centre.

5.13. ***Position of a point:*** Any point belonging to the foci region is called an *internal point*. A point which is neither internal nor on the curve is called *external point*.

5.14. ***Notation:***

$$S \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1$$

$$S_1 = \frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1$$

$$S_2 = \frac{xx_2}{a^2} + \frac{yy_2}{b^2} - 1$$

$$S_{12} = S_{21} = \frac{x_1x_2}{a^2} + \frac{y_1y_2}{b^2} - 1$$

$$S_{11} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1$$

5.15. ***Theorem:*** A point (x_1, y_1) is external or internal to the ellipse

$$S \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$$

according as $S_{11} > 0$ or $S_{11} < 0$.

5.16. ***Auxiliary circle and parametric equations:*** The circle $x^2 + y^2 = a^2$ is called auxiliary circle for the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (a^2 > b^2)$$

For an inverted ellipse, the auxiliary circle is $x^2 + y^2 = b^2$. The equations $x = a\cos\theta$ and $y = b\sin\theta$ are called parametric equations of the ellipse and θ is called the eccentric angle of the point.

5.17. ***Equations of the chord:***

1. The equations of the chord joining $P(x_1, y_1)$ and $Q(x_2, y_2)$ on the ellipse

$$S \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$$

is $S_1 + S_2 = S_{12}$. That is,

$$\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1\right)+\left(\frac{xx_2}{a^2}+\frac{yy_2}{b^2}-1\right)=\frac{x_1x_2}{a^2}+\frac{y_1y_2}{b^2}-1$$

2. The equation of the chord joining two points $(a\cos\alpha, b\sin\alpha)$ and $(a\cos\beta, b\sin\beta)$ is

$$\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right)+\frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right)=\cos\left(\frac{\alpha-\beta}{2}\right)$$

5.18. ***Tangent and normal:***

1. The equation of the tangent at (x_1, y_1) is

$$S_1 \equiv \frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1=0$$

and the normal at (x_1, y_1) is

$$\frac{a^2(x-x_1)}{x_1}=\frac{b^2(y-y_1)}{y_1}$$

2. The equation of the tangent at $(a\cos\alpha, b\sin\alpha)$ is

$$\frac{x}{a}\cos\alpha+\frac{y}{b}\sin\alpha=1$$

and the normal is

$$ax\sec\alpha - by\,\mathrm{cosec}\,\alpha = a^2-b^2$$

5.19. ***Condition for tangency:***

1. The line $y=mx+c,\ c\neq 0$, touches the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

if and only if $c^2=a^2m^2+b^2$.

2. For all real values of α, the line

$$x\cos\alpha + y\sin\alpha = p$$

where

$$p=\sqrt{a^2\cos^2\alpha+b^2\sin^2\alpha}$$

touches the ellipse. If $y = mx + c$ touches the ellipse, then the point of contact is

$$\left(\frac{-a^2m}{c}, \frac{b^2}{c}\right)$$

5.20. ***Theorem:*** From any external point, two tangents can be drawn to an ellipse.

5.21. ***Director circle:*** The locus of the point from which perpendicular tangents can be drawn to an ellipse is a circle called *director circle* of the ellipse. The equation of the director circle of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $x^2+y^2=a^2+b^2$.

5.22. **1.** The condition that the line $lx+my+n=0$ touches the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

$$a^2l^2+b^2m^2=n^2$$

2. The condition for the line $lx + my = n$, where $lmn \neq 0$ may be a normal to the ellipse, is

$$\frac{a^2}{l^2}+\frac{b^2}{m^2}=\frac{(a^2-b^2)^2}{n^2}$$

5.23. ***Theorem:*** The feet of the perpendiculars drawn from the foci onto a tangent of an ellipse lie on the auxiliary circle of the ellipse.

5.24. ***Theorem:*** The product of the perpendiculars drawn from the foci onto a tangent of the ellipse is constant and equals to square of the semi-minor axis.

5.25. ***Theorem:*** Except the point of contact, every point on a tangent to an ellipse lies outside the ellipse.

5.26. ***Theorem (Equation of the chord in terms of its midpoint):*** If (x_1, y_1) is the midpoint of a chord of the ellipse

$$S \equiv \frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$$

then its equation is

$$S_1 = S_{11}$$

$$\Rightarrow \frac{xx_1}{a^2}+\frac{yy_1}{b^2}=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}$$

5.27. ***Theorem:*** The midpoints of parallel chords of an ellipse always lie on a line that passes through the centre of the ellipse.

5.28. ***Diameter:*** The line on which the midpoints of a system of parallel chords of an ellipse lie is called a diameter of the ellipse. If m is the slope of the parallel chords, then their midpoints lie on the diameter $b^2x+a^2my=0$.

5.29. ***Theorem:*** If one diameter of an ellipse bisects the chords parallel to another diameter, then the second one bisects the chords parallel to the first diameter. Such diameters are called *conjugate diameters*.

5.30. ***Theorem (Pair of tangents):*** The combined equation of the pair of tangents drawn from a point (x_1, y_1) to the ellipse $S\equiv \frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$ is

$$S_1^2 = SS_{11}$$

$$\Rightarrow \left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1\right)^2=\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1\right)$$

5.31. ***Definition (Chord of contact):*** If the tangents drawn from an external point P touch the ellipse at Q and R, then the chord QR is called chord of contact of P with regard to the given ellipse. If $P = (x_1, y_1)$, the equation of its chord of contact is

$$S_1 \equiv \frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1 = 0$$

5.32. ***Theorem:*** Through a given point in the plane of an ellipse, in general, four normals can be drawn to the ellipse.

5.33. ***Theorem:*** In general, a circle and an ellipse intersect at four points such that the algebraic sum of the eccentric angles of the common points is an even multiple of π.

5.34. ***Definition:*** Let l be a fixed straight line called directrix, S be a fixed point (which is not located on l) called focus and e (which is greater than 1) is real number called the eccentricity. Then, the locus of the point whose distance from the fixed point S is equal to e times the distance of the point from the line l is called hyperbola.

5.35. ***Standard equation:*** $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is the standard equation of the hyperbola and $b^2 = a^2(e^2 - 1)$ if $a^2 > b^2$, otherwise $a^2 = b^2(e^2 - 1)$.

5.36. ***Features of curve:***

1. Curve is symmetric about both axes.
2. When x-axis intersects a curve at $A(a, 0)$ and $A'(-a, 0)$, then *y-axis cannot intersect* the curve.
3. Also (x, y) is a point as the curve $\Leftrightarrow |x| \geq a$.
4. $A'A$ is called transverse axis whose length is $2a$ and $B'B$ is called conjugate axis where $B' = (0, -b)$ and $B = (0, b)$. $B'B = 2b$ is called length of the conjugate axis.
5. Curve is an unbounded curve.
6. The two foci are S $(ae, 0)$ and $S'(-ae, 0)$ and their corresponding directrices are $x = \pm a/e$.
7. $x^2 + y^2 = a^2$ is called auxiliary circle.
8. The length of the latus rectum is $2b^2/a$.
9. $|SP - S'P| = 2a$.

5.37. ***Parametric equations:*** $x = a\sec\theta$ and $y = b\tan\theta$ are called parametric equations of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

5.38. ***External and internal points:*** The points belonging to the region where the foci lie is called internal region. The point which is neither internal nor on the curve is called external point.

5.39. ***Theorem:*** Point (x_1, y_1) in the plane of the hyperbola

$$S = \frac{x^2}{a^2} - \frac{y^2}{a^2} - 1 = 0$$

is an external point or an internal point according as $S_{11} < 0$ or $S_{11} > 0$, respectively.

5.40.

1. The equation of the chord joining two points (x_1, y_1) and (x_2, y_2) on the hyperbola
$$S = \frac{x^2}{a^2} - \frac{y^2}{a^2} - 1 = 0$$
is $S_1 + S_2 = S_{12}$.
2. The equation of the tangent at (x_1, y_1) is
$$S_1 \equiv \frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1 = 0$$
3. The equation of the tangent at $(a\sec\theta, b\tan\theta)$ is
$$\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1$$
4. The equation of the normal at (x_1, y_1) is
$$\frac{a^2(x - x_1)}{x_1} = \frac{-b^2(y - y_1)}{y_1}$$
5. The equation of the normal at $(a\sec\theta, b\tan\theta)$ is $ax\cos\theta + by\cot\theta = a^2 + b^2$.
6. The condition for the line $y = mx + c$ to touch the hyperbola
$$\frac{x^2}{a^2} - \frac{y^2}{a^2} = 1$$
is that $c^2 = a^2m^2 - b^2$. That is, the line $y = mx \pm \sqrt{a^2m^2 - b^2}$ touches the hyperbola.
7. For all values of α, the line $x\cos\alpha + y\sin\alpha = p$, where $p = \sqrt{a^2\cos^2\alpha - a^2\sin^2\alpha}$, touches the hyperbola.
8. The equation of the *director circle* is $x^2 + y^2 = a^2 - b^2$ when $a \geq b$.
9. The equation of the chord in terms of its midpoint (x_1, y_1) is $S_1 = S_{11}$.
10. Equation of the pair of tangents from an external point (x_1, y_1) is $S_1^2 = SS_{11}$.

5.41. ***Asymptotes:*** The lines

$$\frac{x}{a} \pm \frac{y}{b} = 0$$

$$\Rightarrow y = \pm\frac{b}{a}x$$

look like tangents, but actually they are not the tangents. These two lines are called asymptotes of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{a^2} = 1$$

5.42. ***Angle between asymptotes:*** The angle between the two asymptotes is $2\,\text{Sec}^{-1}e$.

5.43. ***Few properties of asymptotes:***

1. The chord of contact of any point on an asymptote with respect to the hyperbola is parallel to the same asymptote.
2. The product of perpendiculars drawn from a point on the curve onto its asymptotes is constant which is equal to $\dfrac{a^2b^2}{a^2+b^2}$.
3. The foot of the perpendicular from a focus onto an asymptote is one of the points of intersection of the corresponding directrix with the auxiliary circle.
4. If a line cuts the hyperbola at P and Q and the asymptotes at R and S, then $PR = QS$.
5. The combined equation of the pair of asymptotes and the equation of the hyperbola differ by a constant.

5.44. ***Conjugate hyperbola:*** If the roles of transverse and conjugate axes are interchanged, then the resulting hyperbola is called conjugate hyperbola.

5.45. $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = -1$ is the conjugate hyperbola of $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = -1$.

5.46. Hyperbola + Conjugate hyperbola = 2 (Asymptotes).

5.47. If e_1 and e_2 are the eccentricities of a hyperbola and its conjugate, then $\dfrac{1}{e_1^{\,2}} + \dfrac{1}{e_2^{\,2}} = 1$.

5.48. ***Rectangular hyperbola:*** A hyperbola in which length of the transverse axis is equal to the length of conjugate axis is called rectangular or equilateral hyperbola.

5.49. The standard equation of rectangular hyperbola is $x^2 - y^2 = a^2$.

5.50. The eccentricity a rectangular hyperbola is $\sqrt{2}$ and hence the angle between the asymptotes is

$$2\,\text{Sec}^{-1}\sqrt{2} = 2\left(\frac{\pi}{4}\right) = \frac{\pi}{2}$$

5.51. The *director circle* of a rectangular hyperbola is the centre itself [i.e., {(0,0)}].

5.52. ***Simplest form:*** $xy = c^2$ is the simplest form of a rectangular hyperbola.

5.53. ***Parametric equation of $xy = c^2$:*** $x = ct$ and $y = c/t$ are the parametric equations of $xy = c^2$. Hence, every point on the curve $xy = c^2$ is of the form $(ct, c/t)$ for some non-zero, real t.

5.54. The equation of the tangent to $xy = c^2$ at t is

$$\frac{x}{t} + ty = 2c \Rightarrow x + t^2y = 2ct$$

5.55. The normal at $(ct, c/t)$ is $t^3x - ty + c - ct^4 = 0$.

5.56. If the normal at $(ct, c/t)$ to $xy = c^2$ meets the curve again at $(ct', c/t')$, these $t^3t' = -1$.

5.57. A portion of a tangent to $xy = c^2$ lying between the coordinate area is bisected by the point of contact.

EXERCISES

Single Correct Choice Type Questions

1. For any real t, the point $\left(\dfrac{e^t + e^{-t}}{2}, \dfrac{e^t - e^{-t}}{2}\right)$ is a point on the

(A) hyperbola $x^2 - y^2 = 1$

(B) hyperbola $\dfrac{x^2}{2} - \dfrac{y^2}{1} = 1$

(C) ellipse $\dfrac{x^2}{2} + \dfrac{y^2}{1} = 1$

(D) circle $x^2 + y^2 = 2$

2. Which one of the following equations represents the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$?

(A) $x = a\tan\theta, y = b\sec\theta$

(B) $x = \dfrac{a}{2}\left(t+\dfrac{1}{t}\right), y = \dfrac{b}{2}\left(t-\dfrac{1}{t}\right)$

(C) $x = \cos\theta, y = \operatorname{cosec}\theta$

(D) $x = a\cos\theta, y = b\tan\theta$

3. If the chord joining $(a\cos\alpha, b\sin\alpha)$ and $(a\cos\beta, b\sin\beta)$ on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{a^2}=1$$

passes through $(ae, 0)$, then eccentricity e is equal to

(A) $\dfrac{\cos(\alpha+\beta)/2}{\cos(\alpha-\beta)/2}$ (B) $\dfrac{\sin(\alpha-\beta)/2}{\sin(\alpha+\beta)/2}$

(C) $\dfrac{\cos(\alpha-\beta)/2}{\cos(\alpha+\beta)/2}$ (D) $\dfrac{\cos(\alpha+\beta)/2}{\sin(\alpha-\beta)/2}$

4. The latus rectum of the ellipse $\dfrac{x^2}{9}+\dfrac{y^2}{5}=1$ is

(A) $\dfrac{5}{3}$ (B) $\dfrac{2\sqrt{5}}{3}$ (C) $\dfrac{\sqrt{5}}{3}$ (D) $\dfrac{10}{3}$

5. The radius of the circle passing through the foci of the ellipse

$$\frac{x^2}{16}+\frac{y^2}{9}=1$$

which is having its centre at $(0, 3)$ is

(A) $\dfrac{7}{2}$ (B) 8 (C) $2\sqrt{3}$ (D) 4

6. The axes of an ellipse are along the coordinates axes and its latus rectum and eccentricity are equal to $1/2$. Then the equation of the ellipse is

(A) $9x^2 + 12y^2 = 1$ (B) $3x^2 + 12y^2 = 1$

(C) $6x^2 + 12y^2 = 1$ (D) $12x^2 + 6y^2 = 1$

7. The normal at point P on the ellipse $x^2 + 4y^2 = 16$ meets the x-axis at Q. If M is the midpoint of the line segment PQ, then the locus of M intersects the latus rectums of the given ellipse at the points

(A) $\left(\pm\dfrac{3\sqrt{5}}{2}, \pm\dfrac{2}{7}\right)$ (B) $\left(\pm 2\sqrt{3}, \pm\dfrac{1}{7}\right)$

(C) $\left(\pm\dfrac{3\sqrt{5}}{2}, \pm\dfrac{\sqrt{19}}{4}\right)$ (D) $\left(\pm 2\sqrt{3}, \pm\dfrac{4\sqrt{3}}{7}\right)$

8. Let S and S' be the foci of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

PSQ and $PS'R$ are the focal chords of the ellipse. Then, the equation of chord QR is

(A) $\dfrac{x}{a}\cos\theta+\dfrac{y}{b}\left(\dfrac{1+e^2}{1-e^2}\right)\sin\theta=1$

(B) $\dfrac{x}{a}\left(\dfrac{1+e^2}{1-e^2}\right)\cos\theta+\dfrac{y}{b}\sin\theta=1$

(C) $\dfrac{x}{a}\left(\dfrac{1-e^2}{1+e^2}\right)\cos\theta+\dfrac{y}{b}\sin\theta=1$

(D) $\dfrac{x}{a}\cos\theta+\dfrac{y}{b}\left(\dfrac{1-e^2}{1+e^2}\right)\sin\theta=1$

9. S and S' are the foci of

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

and P is any point on the curve. The lines PS and PS' meet the curve again at Q and Q'. The tangents at Q and Q' meet at T. As P moves on the curve, the locus of P is

(A) $(1+e^2)\dfrac{x^2}{a^2}+(1-e^2)\dfrac{y^2}{a^2}=1$

(B) $(1+e^2)^2\dfrac{x^2}{a^2}+(1-e^2)^2\dfrac{y^2}{a^2}=(1+e^2)^2$

(C) $(1-e^2)^2\dfrac{x^2}{a^2}+(1+e^2)^2\dfrac{y^2}{a^2}=(1-e^2)^2$

(D) $(1+e^2)^2\dfrac{x^2}{a^2}+(1-e^2)^2\dfrac{y^2}{a^2}=1$

10. Let H and H' be a hyperbola and its conjugate, respectively. Let e and e' be the intercept of a line on the positive coordinate axis where $e/2$ and $e'/2$ are the eccentricities of H and H', respectively. Then the line touches the circle

(A) $x^2 + y^2 = 4$ (B) $x^2 + y^2 = 3$

(C) $x^2 + y^2 = 2$ (D) $x^2 + y^2 = 1$

11. The angle between the asymptotes of the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

is 60° and the product of the perpendiculars drawn from the foci onto any tangent is 9. Then the equation of the director circle of the hyperbola is

(A) $x^2+y^2=9$ (B) $x^2+y^2=6$
(C) $x^2+y^2=18$ (D) $x^2+y^2=12$

12. $P(\alpha)$ and $Q(\beta)$ are points on the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

where $\alpha+\beta=\pi$. Then the chord PQ passes through

(A) the centre of the hyperbola
(B) a focus
(C) the cone of the vertices of the hyperbola
(D) either through $(0, b)$ or through $(0, -b)$

13. The locus of the foot of the perpendicular drawn from the centre of the hyperbola $xy = c^2$ onto any tangent of the same curve is

(A) $(x^2+y^2)^2=4c^2xy$ (B) $x^2+y^2=4c^2x^2y^2$
(C) $(x^2+y^2)^2=2c^2xy$ (D) $(x^2-y^2)^2=4c^2xy$

14. If the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

where $a > b$, and the hyperbola $x^2-y^2=c^2$ cut orthogonally, then

(A) $a^2+b^2=2c^2$ (B) $a^2-b^2=2c^2$
(C) $a^2-b^2=c^2$ (D) $a^2b^2=2c^2$

15. The tangent at (x_0, y_0) to the hyperbola $xy=c^2$ meets the coordinate axis at P and Q. If O is the centre of the curve, then the circumcentre of the ΔOPQ lies on the curve

(A) $x^2-y^2=c^2$ (B) $xy=c^2$
(C) $xy=\dfrac{c^2}{2}$ (D) $x^2-y^2=2c^2$

16. The director circles of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

and the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

respectively, are $x^2+y^2=4r^2$ and $x^2+y^2=r^2$ and their eccentricities are e_1 and e_2, respectively. Then

(A) $e_1^2-4e_2^2=6$ (B) $2e_1^2-e_2^2=6$
(C) $4e_2^2-e_1^2=6$ (D) $e_1^2+e_2^2=6$

17. The locus of the midpoint of chords of the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

which pass through a fixed point (h, k) is

(A) an ellipse with centre $\left(\dfrac{h}{2},\dfrac{k}{2}\right)$
(B) hyperbola with centre $\left(\dfrac{h}{2},\dfrac{k}{2}\right)$
(C) a circle with centre $\left(\dfrac{h}{2},\dfrac{k}{2}\right)$
(D) a parabola with focus at $\left(\dfrac{h}{2},\dfrac{k}{2}\right)$

18. The ordinate MP of the ellipse

$$\frac{x^2}{25}+\frac{y^2}{9}=1$$

meets the auxiliary circle at Q. The locus of the point of intersection of the normals at P and Q to the ellipse and the circle, respectively, is

(A) $x^2+y^2=64$ (B) $x^2+y^2=34$
(C) $x^2+y^2=15$ (D) $x^2+y^2=8$

19. The locus of the midpoint of the portion of the tangent lying between the coordinate axes of the ellipse

$$\frac{x^2}{16}+\frac{y^2}{9}=1$$

is

(A) $9x^2+16y^2=x^2y^2$ (B) $\dfrac{16}{y^2}+\dfrac{9}{x^2}=4$
(C) $\dfrac{4}{x^2}+\dfrac{3}{y^2}=4$ (D) $\dfrac{16}{x^2}+\dfrac{9}{y^2}=4$

20. $P(\alpha), Q(\beta)$ and $R(\gamma)$ are vertices of one equilateral triangle which lie on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

If (x_1, y_1) is the circumcentre of ΔPQR, then

$$\sum\cos(\alpha-\beta) =$$

(A) $\dfrac{9x_1^2}{a^2}+\dfrac{9y_1^2}{b^2}+\dfrac{3}{2}$ (B) $9x_1^2-9y_1^2+a^2b^2$

(C) $\frac{9x_1^2}{a}+\frac{9y_1^2}{b}+3$ (D) $\frac{9x_1^2}{2a^2}+\frac{9y_1^2}{2b^2}-\frac{3}{2}$

21. The coordinates of the point P on the ellipse

$$\frac{x^2}{16}+\frac{y^2}{9}=1$$

for which the area of ΔPON is maximum where O is the centre and N is the foot of the perpendicular from O onto the tangent at P are

(A) $\left(\pm\frac{16}{5},\pm\frac{9}{5}\right)$ (B) $\left(\pm\frac{9}{5},\pm\frac{16}{5}\right)$

(C) $\left(\pm\frac{16}{25},\pm\frac{9}{25}\right)$ (D) $(\pm 16,\pm 9)$

Multiple Correct Choice Type Questions

1. If e_1 and e_2, respectively, are the eccentricities of the ellipse $9x^2+4y^2=36$ and the hyperbola $9x^2-4y^2=36$, then

(A) $e_1=\frac{\sqrt{5}}{3}$ (B) $e_2=\frac{\sqrt{13}}{2}$

(C) $e_1^2+e_2^2=\frac{137}{36}$ (D) $e_1^2+e_2^2=\frac{127}{36}$

2. Which of the following are true to the ellipse $4x^2+y^2-8x+2y+1=0$?

(A) Foci are $(1,-1\pm\sqrt{3})$

(B) Directrices are $y=-1\pm\frac{4}{\sqrt{3}}$

(C) Eccentricity is $\frac{3}{4}$

(D) Latus rectum is 8

3. Let $P(x_1, y_1)$ and $Q(x_2, y_2)$, $y_1<0$, $y_2<0$ be the end points of a latus rectum of the ellipse $x^2+4y^2=4$. The equation of the parabola with latus rectum PQ is

(A) $x^2+2\sqrt{3}y=3+\sqrt{3}$

(B) $x^2-2\sqrt{3}y=3+\sqrt{3}$

(C) $x^2+2\sqrt{3}y=3-\sqrt{3}$

(D) $x^2-2\sqrt{3}y=3-\sqrt{3}$

(IIT-JEE 2008)

4. On the ellipse $4x^2+9y^2=1$, the points at which the tangent are parallel to the line $9y=8x$ are

(A) $\left(\frac{2}{5},\frac{1}{5}\right)$ (B) $\left(\frac{-2}{5},\frac{1}{5}\right)$

(C) $\left(\frac{-2}{5},\frac{-1}{5}\right)$ (D) $\left(\frac{2}{5},\frac{-1}{5}\right)$

5. An ellipse intersects the hyperbola $2x^2-2y^2=1$ orthogonally. The eccentricity of the ellipse is reciprocal of that of the hyperbola. If the axes of the ellipse are along the coordinate axes, then

(A) Equation of the ellipse $x^2+2y^2=2$

(B) The foci of the ellipse are $(\pm 1,0)$

(C) The equation of the ellipse is $x^2+2y^2=4$

(D) The foci of the ellipse are $(\pm\sqrt{2},0)$

(***Hint:*** Conics which intersect orthogonally have the same foci.)

6. Let a hyperbola passes through foci of the ellipse

$$\frac{x^2}{25}+\frac{y^2}{16}=1$$

The transverse and conjugate axes of the hyperbola coincide with the major and minor axes of the ellipse. If the product of the eccentricities of the given ellipse and hyperbola is 1, then

(A) The equation of the hyperbola is $\frac{x^2}{9}-\frac{y^2}{16}=1$

(B) The equation of the hyperbola is $\frac{x^2}{9}-\frac{y^2}{25}=1$

(C) A focus of the hyperbola is $(5,0)$

(D) A vertex of the hyperbola is $(5\sqrt{3},0)$

(IIT-JEE 2006)

7. If the normals at P to the rectangular hyperbola $x^2-y^2=4$ meet the major and minor axes at G and g, respectively, and C is the centre of the hyperbola, then

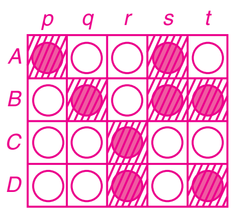
(A) $PG=PC$ (B) $Pg=PC$

(C) $PG=Pg$ (D) $Gg=2PC$

Matrix-Match Type Questions

In each of the following questions, statements are given in two columns, which have to be matched. The statements in *column I* are labeled as **(A)**, **(B)**, **(C)** and **(D)**, while those in *column II* are labeled as **(p)**, **(q)**, **(r)**, **(s)** and **(t)**. Any given statement in *column I* can have correct matching with *one or more* statements in *column II*. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following example.

Example: If the correct matches are (A) $\to$ (p), (s), (B) $\to$ (q), (s), (t), (C) $\to$ (r), (D) $\to$ (r), (t), that is if the matches are (A) $\to$ (p) and (s); (B) $\to$ (q), (s) and (t); (C) $\to$ (r); and (D) $\to$ (r), then the correct darkening of bubbles will look as follows:



1. Some conics are given in Column I and their directrices are given in Column II. Match the items of Column I with those of Column II.

Column I	*Column II*
(A) $3x^2 - 4y^2 + 6x + 24y - 135 = 0$	(p) $x = -1 + 2\sqrt{\frac{34}{7}}$
(B) $2x^2 + 3y^2 - 8x + 6y - 7 = 0$	(q) $x = 1 \pm 17\sqrt{\frac{3}{23}}$
(C) $2x^2 + 3y^2 - 4x + 12y - 20 = 0$	(r) $x = -2 + \frac{2}{\sqrt{7}}$
(D) $3x^2 - 4y^2 + 12x + 8y - 4 = 0$	(s) $x = 2 \pm 3\sqrt{3}$
	(t) $-1 - 2\sqrt{\frac{34}{7}} = x$

2. Match the items of Column I with those of Column II.

Column I	*Column II*
(A) The locus of the point (h, k) such that the line $hx + ky = 1$ touches the circles $x^2 + y^2 = 4$ is	(p) a circle
(B) Point Z in the complex plane satisfying $\lvert Z+2\rvert - \lvert Z-2\rvert = 3$ is	(q) a parabola
(C) The conic represented by the equation $$x = \sqrt{3}\left(\frac{1-t^2}{1+t^2}\right), y = \frac{2t}{1+t^2}$$ is	(r) an ellipse
(D) The conic whose eccentricity is greater than or equal to 1 is	(s) a hyperbola

(Continued)

3. Let the foci of the hyperbola

$$\frac{x^2}{c^2} - \frac{y^2}{d^2} = 1$$

be the vertices of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

and the foci of the ellipse are the vertices of the hyperbola. Let e_1 and e_2 be the eccentricities of the ellipse and hyperbola, respectively. Match the items of Column I with those Column II.

Column I	*Column II*
(A) $\frac{b}{d}$ is equal to	(p) 4
(B) $e_1 + e_2$ is always greater than	(q) 3
(C) If the angle between the asymptotes is $2\pi/3$, then $4e_1 =$	(r) 2
(D) If $e_1 = 1/\sqrt{2}$ and (h, k) is a point of intersection of the ellipse and hyperbola, then $9h^2/2k^2 =$	(s)1

Comprehension Type Questions

1. Passage: Consider the ellipse

$$\frac{x^2}{9} + \frac{y^2}{4} = 2$$

Answer the following questions.

(i) Which line among the following lines touches the ellipse?

(A) $2x + 3y = 12$ (B) $x + 3y = 12$

(C) $2x + y = 12$ (D) $x + y = 6$

(ii) The product of the perpendiculars drawn from the foci onto a tangent of the ellipse is

(A) 4 (B) 8 (C) 16 (D) $4\sqrt{2}$

(iii) The locus of the feet of the perpendiculars drawn from the foci onto a tangent of the ellipse is

(A) $\frac{x^2}{18}+\frac{y^2}{8}=1$ (B) $x^2+y^2=9$
(C) $x^2+y^2=81$ (D) $x^2+y^2=91$

2. Passage: Consider the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Answer the following questions.

(i) If the latus rectum is half of the major axis, then the eccentricity of the ellipse is

(A) $\frac{1}{2}$ (B) $\frac{2}{3}$ (C) $\frac{1}{\sqrt{3}}$ (D) $\frac{1}{\sqrt{2}}$

(ii) If the angle between the lines joining the foci to an extremity of the minor axis is 90°, then the eccentricity of the ellipse is equal to

(A) $\frac{1}{\sqrt{2}}$ (B) $\frac{\sqrt{3}}{2}$ (C) $\frac{\sqrt{3}}{4}$ (D) $\frac{1}{2}$

(iii) If the latus rectum is equal to half of the minor axis then the eccentricity is equal to

(A) $\frac{\sqrt{2}}{3}$ (B) $\frac{1}{3}$ (C) $\frac{\sqrt{3}}{2}$ (D) $\frac{1}{\sqrt{2}}$

3. Passage: Consider the hyperbola $9x^2 - 16y^2 + 18x + 32y - 151 = 0$. Answer the following questions.

(i) The eccentricity of the curve is

(A) $\frac{4}{3}$ (B) $\frac{5}{4}$ (C) $\frac{4}{\sqrt{3}}$ (D) $\frac{5}{2}$

(ii) The eccentricity of the conjugate hyperbola is

(A) $\frac{4}{\sqrt{7}}$ (B) $\frac{5}{3}$ (C) $\frac{4}{\sqrt{13}}$ (D) $\frac{5}{\sqrt{21}}$

(iii) The product of the perpendiculars drawn from any point on the hyperbola onto its asymptote is equal to

(A) $\frac{144}{25}$ (B) $\frac{25}{144}$ (C) $\frac{9}{25}$ (D) $\frac{25}{7}$

4. Passage: The circle $x^2+y^2-8x=0$ and the hyperbola

$$\frac{x^2}{9}-\frac{y^2}{4}=1$$

intersect at points A and B. Answer the following questions.

(i) The equation of a common tangent with positive slope to the circle as well as to the hyperbola is

(A) $2x-\sqrt{5}y-20=0$

(B) $2x-\sqrt{5}y+4=0$

(C) $3x-4y+8=0$

(D) $4x-3y+4=0$

(ii) The equation of the circle with AB as diameter is

(A) $x^2+y^2-12x+24=0$

(B) $x^2+y^2+12x+24=0$

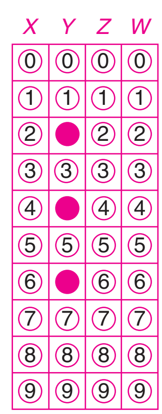
(C) $x^2+y^2+24x-12=0$

(D) $x^2+y^2-24x+12=0$

(IIT-JEE 2000)

Integer Answer Type Question

The answer to each of the questions in this section is a non-negative integer. The appropriate bubbles below the respective question numbers have to be darkened. For example, as shown in the figure, if the correct answer to the question number Y is 246, then the bubbles under Y labeled as 2, 4, 6 are to be darkened.



1. The line $2x + y = 1$ is a tangent to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

If this line passes through the point of intersection of the nearest directrix with x-axis, then the eccentricity of the hyperbola is _________.

[***Hint:*** The line $2x + 3 = 1$ meets the x-axis at $(1/2, 0)$.]

2. If the line $2x+\sqrt{6}y=2$ touches the hyperbola $x^2 - 2y^2 = 4$ at (x_1, y_1), then $y_1^2 - x_1$ is equal to _________.

3. The circle $x^2 + y^2 = 16$ cuts the hyperbola $xy = 1$ at four points (x_r, y_r) (where $r = 1, 2, 3$ and 4). Then the value of $x_1x_2x_3x_4 + y_1y_2y_3y_4$ is equal to _________.

4. If e_1 and e_2 are the eccentricities of the hyperbolas

$$\frac{x^2}{16} - \frac{y^2}{9} = 1 \text{ and } \frac{x^2}{16} - \frac{y^2}{9} = -1$$

respectively, then $\dfrac{e_1^2 e_2^2}{e_1^2 + e_2^2}$ is equal to _________.

5. The tangent to a rectangular hyperbola $xy = 1$ at a point meets the coordinate axis at P and Q. If O is the center of the hyperbola, then the event ΔOPQ is _________ sq. unit.

6. The tangent to the ellipse $x^2 + 4y^2 = 4$ meets the ellipse

$$\frac{x^2}{6}+\frac{y^2}{3}=1$$

at P and Q. If θ is the angle between the tangents drawn to the ellipse

$$\frac{x^2}{6} + \frac{y^2}{3} = 1$$

at P and Q, then the value of $\cos\theta$ is_________.

7. The number of points on the ellipse

$$\frac{x^2}{50} + \frac{y^2}{20} = 1$$

from which a pair of perpendicular tangents are drawn to the ellipse

$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$

is_________.

8. If the equation of the pair of asymptotes of the hyperbola $3x^2 - 5xy - 2y^2 + 17x + y + 14 = 0$ is $3x^2 - 5xy - 2y^2 + 17x + y + 1 + k = 0$, then the value of k is _________.

9. If p_1 and p_2 are the lengths of the perpendiculars drawn from the point $(\sqrt{5}, 1)$ onto the asymptotes of the hyperbola $x^2 - y^2 = 4$, then p_1p_2 is equal to _________.

10. If θ is the angle of rotation required to transform the equation $x^2 - y^2 = a^2$ into $xy = c^2$, then the value of $|\tan\theta|$ is _________.

11. If the foci of the ellipse

$$\frac{x^2}{16} + \frac{y^2}{b^2} = 1$$

and the hyperbola

$$\frac{x^2}{144} - \frac{y^2}{81} = \frac{1}{25}$$

coincide, then b^2 is equal to _________.

12. A and A' are the vertices of the ellipse

$$\frac{x^2}{25} + \frac{y^2}{16} = 1$$

and P is a point on the ellipse. The tangent at P meets y-axis at point Q. Let R be the image of point Q in the line $y = x$. Then, the circle described on the segment QR as diameter passes through a fixed point whose abscissa is _________.

ANSWERS

Single Correct Choice Type Questions

1. (A)

2. (B)

3. (C)

4. (D)

5. (D)

6. (A)

7. (B)
8. (A)
9. (B)
10. (A)
11. (C)
12. (A)
13. (A)
14. (B)
15. (B)
16. (C)
17. (B)
18. (A)
19. (D)
20. (D)
21. (A)

Multiple Correct Choice Type Questions

1. (A), (B), (C)
2. (A), (B)
3. (B), (C)
4. (B), (D)
5. (A), (B)
6. (A), (C)
7. (A), (B), (C), (D)

Matrix-Match Type Questions

1. (A) $\to$ (p), (t); (B) $\to$ (s); (C) $\to$ (q); (D) $\to$ (r)
2. (A) $\to$ (p); (B) $\to$ (s); (C) $\to$ (r); (D) $\to$ (q), (s)
3. (A) $\to$ (s); (B) $\to$ (s), (r); (C) $\to$ (s); (D) $\to$ (p)

Comprehension Type Questions

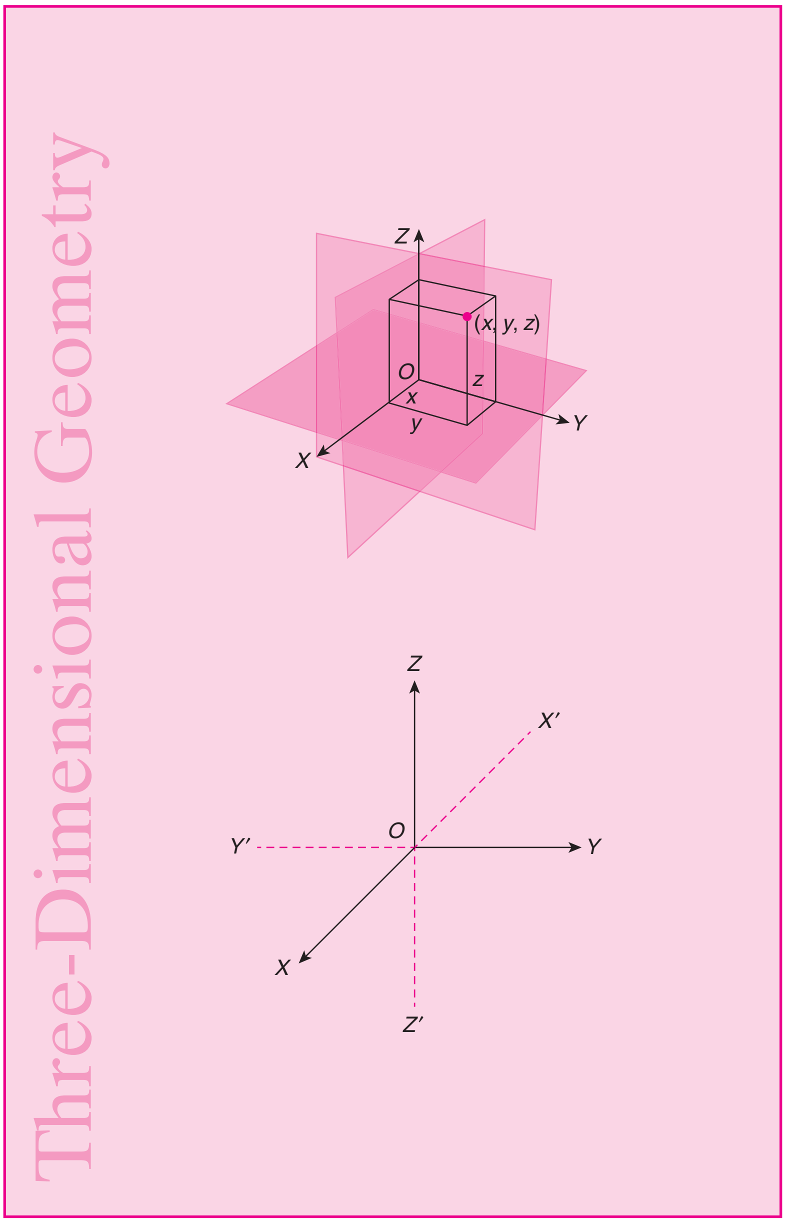
1. (i) $\to$ (A); (ii) $\to$ (B); (iii) $\to$ (B)
2. (i) $\to$ (D); (ii) $\to$ (A); (iii) $\to$ (C)
3. (i) $\to$ (B); (ii) $\to$ (B); (iii) $\to$ (A)

Integer Answer Type Questions

1. 2
2. 2
3. 2
4. 1
5. 2
6. 0
7. 4
8. 9
9. 2
10. 1
11. 7
12. 0

Three-Dimensional Geometry

6



Contents

Geometry is divided into two major parts, namely, (a) plane geometry and (b) solid geometry.

The branch of elementary geometry that is dealt in two dimensions is called plane geometry. Solid geometry is called **three-dimensional geometry**. The three coordinate axes divide the space $\mathbb{R}^3$ into eight parts called **octants**.

Geometry is a fascinating and thought-provoking topic since its origin. Greek philosopher Plato once pronounced that "God always geometrizes" and he went on this philosophy to such an extent that he denied entry into his academy to anyone who knew nothing about geometry. Fundamentally, geometry is divided into two parts such as plane geometry (two-dimensional geometry) and solid geometry (three-dimensional geometry). In this chapter, we learn three-dimensional geometry for which we have to know some fundamentals of Euclidean pure geometry, solid geometry and vectors.

The aim of this chapter is to present standard properties of lines and planes with vector algebra approach.

6.1 | Pre-Requisites

Though we have discussed about vector algebra in Vol. 2, we list out some of the concepts and results (without proofs) of vector algebra in this section for the convenience of the students. In this chapter, though coordinate methods are explained, we give significance to vector methods since they are easy to understand. We begin the discussion with non-empty set S ($\mathbb{R}^3$) called *space* and the elements of S are called *points*. Lines and planes are certain sub-sets of S. They would be the central point of discussion in this chapter. First, let us discuss about the axioms on which Euclidean geometry was built. This approach was propounded by Euclid which was further developed by Hilbert. Throughout this chapter, $\mathbb{R}$ represents the set of all real numbers.

Axiom 6.1 Given any two distinct points, there is one and only one line passing through them. If the two points are A and B, then the line is denoted by $\overleftrightarrow{AB}$. ■

Axiom 6.2 Three non-collinear points determine unique plane. The plane determined by three non-collinear points A, B and C is denoted by $\overleftrightarrow{ABC}$. ■

Axiom 6.3 If two distinct points lie in a plane, then the line passing through them also lies in the same plane. ■

Axiom 6.4 If two planes intersect, they intersect in a unique straight line (see the proof using coordinates in Section 6.2). ■

Axiom 6.5 On every straight line, there are at least two points. In every plane, there are at least three non-collinear points and in the space there are at least four non-coplanar points. ■

Axioms 6.1–6.5 are called incidence postulates.

Axiom 6.6 For any two points, there corresponds a non-zero real number called *distance*. The function which spell this correspondence is called *distance function* which is denoted by d and the distance between the points A and B is denoted by $d(A, B)$. The distance function satisfies the following conditions: $d: S \times S \to \mathbb{R}$ is a function such that

1. $d(A, B) \geq 0$.
2. $d(A, B) = 0$ if and only if $A = B$.
3. $d(A, B) = d(B, A)$.

Hereafter, in this chapter we will denote $d(A, B)$ by AB. ■

DEFINITION 6.1 **Coordinate System** Let L be a straight line and $f: L \to \mathbb{R}$ an injection such that

$$|f(P) - f(Q)| = PQ \quad \forall\, P, Q \in L$$

Then f is called *coordinate system* on the line L and $f(P)$ is called x-coordinate of P.

Axiom 6.7 Every line has a coordinate system. This is called **Rooler axiom**. ■

Note:

1. Every line may have many coordinate systems.
2. If A and B are two points on line L, then there exists a coordinate system on L such that the coordinate of A is zero and the coordinate of B is positive.

Axiom 6.8 (Betweenness) If A, B and C are collinear points such that

$$AB + BC = AC$$

then B is said to be in between A and C and is denoted by $A - B - C$. If A and B are two points, then $\{P \in \overleftrightarrow{AB} \mid A - P - B\}$ is called segment joining A and B and is denoted by $\overline{AB}$. Generally $\overline{AB}$ is also denoted by AB; however, as per the context, we can know whether it refers to the distance between A and B or segment $\overline{AB}$. ■

DEFINITION 6.2 **Ray** Let A and B be two distinct points. Then $\{P \in \overleftrightarrow{AB} \mid A \notin \overline{BP}\}$ is called a ray from A through point B.

Generally, there exists confusion while mentioning a ray and a vector. The ray is denoted by $\overrightarrow{AB}$. The vector with A as initial point and B as terminal point is also denoted by $\overrightarrow{AB}$. Generally $\overrightarrow{AB}$ denotes vector $\overrightarrow{AB}$. Hence, while mentioning $\overrightarrow{AB}$ as ray, we mention that $\overrightarrow{AB}$ is a ray. Further, vector $\overrightarrow{AB}$ has a finite length whereas ray $\overrightarrow{AB}$ has no finite length.

DEFINITION 6.3 **Parallel Lines and Parallel Rays**

1. Let L_1 and L_2 be two coplanar lines (i.e., in the same planes). Then L_1 and L_2 are said to be parallel if either $L_1 = L_2$ or L_1 and L_2 do not have a common point. If L_1 and L_2 are parallel, then we write $L_1 \parallel L_2$ and if they are not parallel, then we write $L_1 \nparallel L_2$.
2. Let $\overrightarrow{AB}$ and $\overrightarrow{CD}$ be two rays. They are said to be parallel either if both $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are on the same line and one is a sub-set of the other or if the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are distinct parallel lines and B and D lie on the same side of the line $\overleftrightarrow{AC}$.

Axiom 6.9 (Parallel Axiom) There will be one and only one line passing through a given point and parallel to a given line. ■

Using the above definitions and axioms, we can prove the Euclidean geometry theorems. However, for this process, we need some of the solid geometry theorems which we list below without their proofs.

Theorem 6.1 If two distinct lines intersect, then they intersect at a unique point. ■

Theorem 6.2 If a line, which is not in a given plane, intersects the plane, then it intersects at one and only one point. ■

Theorem 6.3 There is one and only one plane containing a given line and a point which is not located on the line. ■

Theorem 6.4 Two distinct intersecting lines determine unique plane. ■

DEFINITION 6.4 Suppose L is a line intersecting a plane E at point P. If L is perpendicular to every line in the plane E which passes through point P, then L is said to be perpendicular to E and we write $L \perp E$.

THEOREM 6.5 If a line is perpendicular to two intersecting lines at their point of intersection, then it is perpendicular to the plane determined by the two intersecting lines (see Fig. 6.1).

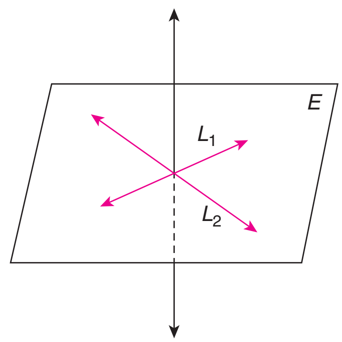


FIGURE 6.1

THEOREM 6.6 There is only one plane perpendicular to a given line at a given point on it.

THEOREM 6.7 The locus of the point in the space which is equidistant from two given points is a plane bisecting the segment joining the two points perpendicularly.

THEOREM 6.8 The lines perpendicular to the same plane are parallel.

THEOREM 6.9 For a given plane, there is one and only one line perpendicular to the plane at a given point in it.

THEOREM 6.10 The set of all lines perpendicular to a given line at a given point constitutes a plane which is perpendicular to the line at that given point.

THEOREM 6.11 (THEOREM OF THREE PERPENDICULARS) Let E be a plane and P be a point in the space, but not located in E. Let M be the foot of the perpendicular drawn from P onto E. Let N be the foot of the perpendicular drawn from M onto a line l in E. Then, $\overleftrightarrow{PN}$ is perpendicular to the line l (see Fig. 6.2).

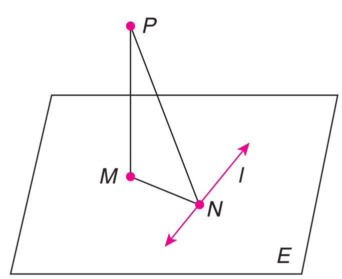


FIGURE 6.2

THEOREM 6.12 If a plane intersects two parallel planes, then it intersects in two parallel lines (see Fig. 6.3).

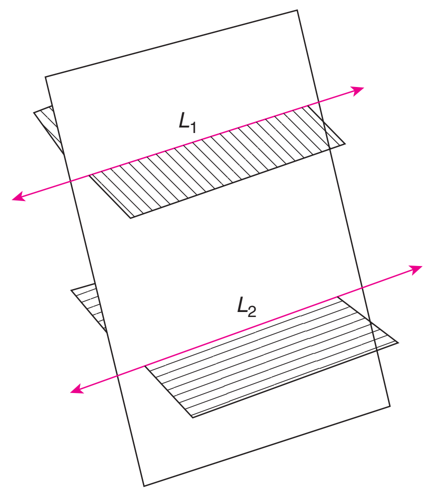


FIGURE 6.3

THEOREM 6.13 If a line is perpendicular to one of the parallel planes, then it is also perpendicular to the other planes.

THEOREM 6.14 Planes perpendicular to a line are parallel.

THEOREM 6.15 If L_1, L_2 and L_3 are three lines such that $L_1 \parallel L_2$ and $L_2 \parallel L_3$, then $L_1 \parallel L_3$.

THEOREM 6.16 The intercepts made by parallel planes on the lines perpendicular to them are equal. The length of those equal intercepts is called the *distance between the parallel planes* (see Fig. 6.4).

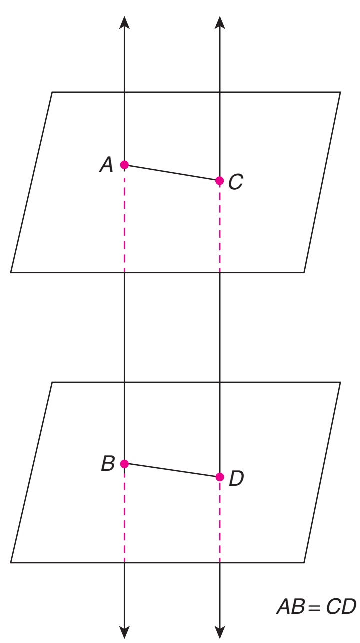


FIGURE 6.4

THEOREM 6.17 If a line is parallel to another line in a plane, then the line is parallel to the plane itself (see Fig. 6.5).

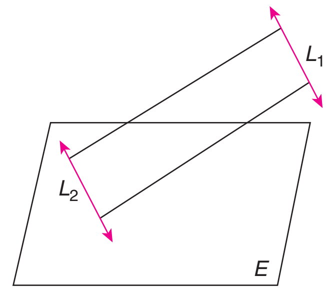


FIGURE 6.5

THEOREM 6.18 Three planes, in which no two planes are parallel, intersect either in parallel lines or in concurrent lines (see Fig. 6.6).

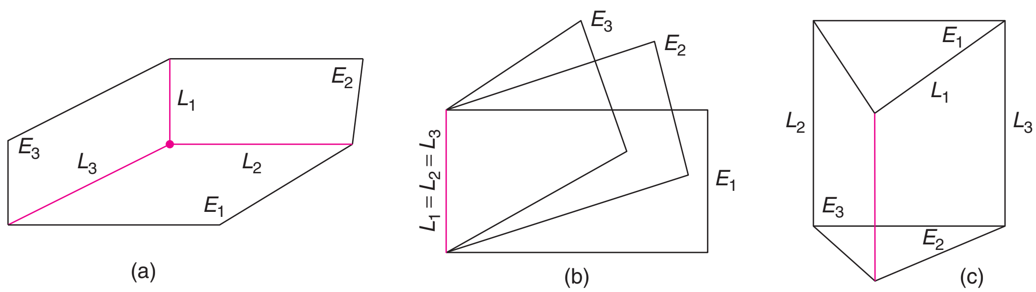


FIGURE 6.6

DEFINITION 6.5 **Skew Lines** Two lines are said to be skew lines if there exists no plane containing both the lines.

Note: Skew lines never intersect and they are also non-coplanar.

THEOREM 6.19 If L_1 and L_2 are two skew lines, there exists a unique plane containing L_1 and parallel to L_2.

THEOREM 6.20 (ORTHOGONAL PROJECTION) Let E be a plane and L be a line in the space. Then, the locus of the feet of the perpendiculars drawn from the point on L onto the plane E is a straight line and this line is called orthogonal projection of L on E.

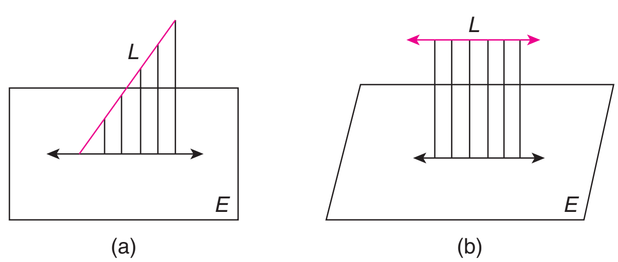


FIGURE 6.7

Note: If a line is parallel to a plane, then the line is parallel to its orthogonal projection in the plane.

DEFINITION 6.6 **Angle between Line and Plane** Let E be a plane and L be a line which is not parallel to E. Let L' be the projection of L on E. Then, the angle between L and L' is called the angle between the line L and the plane E. If θ is the angle between L and a normal to E, then either $90°-\theta$ or $\theta-90°$ is the angle between L and E (see Fig. 6.8).

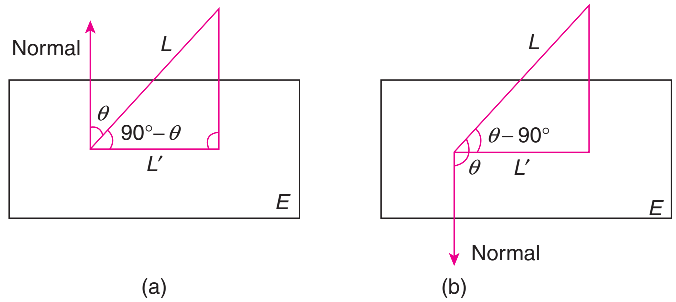


FIGURE 6.8

THEOREM 6.21 Let L_1 and L_2 be two skew lines. Then, there exists a unique line L perpendicular to both L_1 and L_2. The intercept of L between L_1 and L_2 is called the shortest distance between L_1 and L_2 (see Theorem 6.43, Chapter 6, Vol. 2). ■

6.1.1 Vectors

We shall recall some of the concepts and results of vector algebra.

THEOREM 6.22 Let $\vec{a}$ and $\vec{b}$ be the position vectors of points A and B. If point P divides the segment $\overline{AB}$ in the ratio $l{:}m$ $(l+m \neq 0)$, then the position vector of P is

$$\frac{m\vec{a}+l\vec{b}}{l+m}$$

In particular, the position vector of the midpoint of $\overline{AB}$ is

$$\frac{\vec{a}+\vec{b}}{2}$$

■

THEOREM 6.23

1. The position vector of the centroid of a triangle whose vertices are $\vec{a}, \vec{b}$ and $\vec{c}$ is

$$\frac{\vec{a}+\vec{b}+\vec{c}}{3}$$

2. Let $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ be the position vectors of four points A, B, C and D. Let G_1, G_2, G_3 and G_4 be the centroids of ΔBCD, ΔCAD, ΔABD and ΔABC, respectively. Then lines $\overleftrightarrow{AG_1}$, $\overleftrightarrow{BG_2}$, $\overleftrightarrow{CG_3}$ and $\overleftrightarrow{DG_4}$ are concurrent. The point of concurrence is denoted by G and is called the *centre* or the *centroid* of the tetrahedron $ABCD$. Its position vector is

$$\frac{\vec{a}+\vec{b}+\vec{c}+\vec{d}}{4}$$

■

Theorem 6.24 If $\vec{a}$ and $\vec{b}$ are non-collinear vectors and $\vec{r}$ is any vector in the plane determined by $\vec{a}$ and $\vec{b}$, then there exists a unique pair of scalars x and y such that $\vec{r}=x\vec{a}+y\vec{b}$ (see Fig. 6.9). In particular, if $\vec{i}$ and $\vec{j}$ are unit vectors along the positive directions of the coordinate axes in the given plane, then every vector $\vec{r}$ in the coordinate plane is expressed as $\vec{r}=x\vec{i}+y\vec{j}$ in one and only one way.

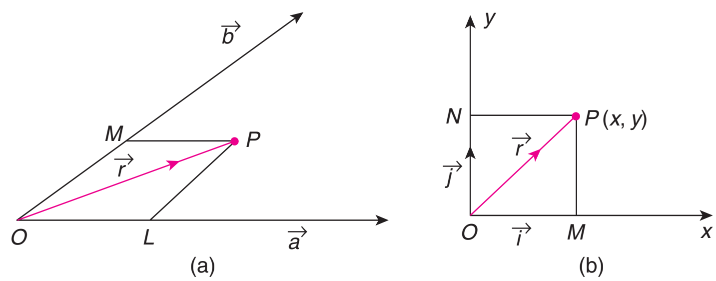


FIGURE 6.9

Theorem 6.25 Suppose $\vec{a},\vec{b}$ and $\vec{c}$ are non-coplanar vectors and $\vec{r}$ is any vector in the space (Fig. 6.10). Then, there exist a unique triad of scalars (x, y, z) such that $\vec{r}=x\vec{a}+y\vec{b}+z\vec{c}$. In the space if $\vec{i},\vec{j},\vec{k}$ are unit vectors along three mutually perpendicular lines, then $\vec{r}=x\vec{i}+y\vec{j}+z\vec{k}$ for some unique triad of real numbers (x, y, z).

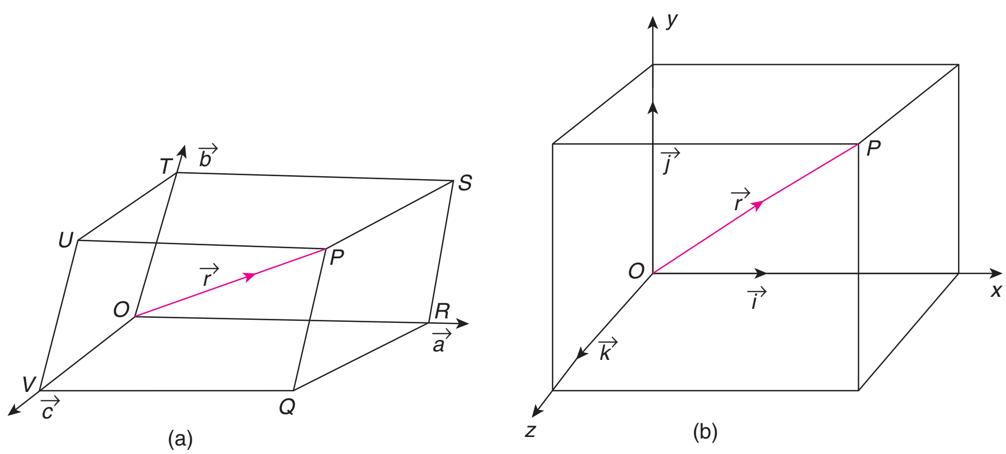


FIGURE 6.10

In Section 6.2, we show that this (x, y, z) are the coordinates of the point whose position vector is $\vec{r}$.

Theorem 6.26 Let $\vec{a},\vec{b}$ and $\vec{c}$ be the position vectors of three points A, B and C, respectively. Then the points A, B and C are collinear if and only if there exist scalars l, m and n (where all scalars are not equal to zero) such that $l\vec{a}+m\vec{b}+n\vec{c}=\vec{0}$ and $l+m+n=0$.

Theorem 6.27

1. The equation of the line passing through the point $A(\vec{a})$ and parallel to a vector $\vec{b}$ is $\vec{r}=\vec{a}+t\vec{b}$, where $t\in\mathbb{R}$ (see Fig. 6.11).
2. The equation of the line passing through two points $A(\vec{a})$ and $B(\vec{b})$ is $\vec{r}=(1-t)\vec{a}+t\vec{b}$, where $t\in\mathbb{R}$.

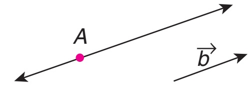


FIGURE 6.11

THEOREM 6.28 (PARAMETRIC VECTORIAL EQUATIONS OF A PLANE)

1. The equation of the plane passing through the point $A(\vec{a})$ and parallel to the vectors $\vec{b}$ and $\vec{c}$ is $\vec{r}=\vec{a}+t\vec{b}+s\vec{c}$, where $t, s \in \mathbb{R}$.
2. The equation of the plane passing through two points $\vec{a}$ and $\vec{b}$ and parallel to vector $\vec{c}$ is $\vec{r}=(1-t)\vec{a}+t\vec{b}+s\vec{c}$, where $t, s \in \mathbb{R}$.
3. The equation of the plane passing through three non-collinear points $\vec{a}, \vec{b}$ and $\vec{c}$ is $\vec{r}=(1-t-s)\vec{a}+t\vec{b}+s\vec{c}$. ■

THEOREM 6.29 Four points $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ in the space are coplanar if and only if there exist scalars x, y, z and u (where all scalars are not equal to zero) such that $x\vec{a}+y\vec{b}+z\vec{c}+u\vec{d}=\vec{0}$ and $x+y+z+u=0$. ■

DEFINITION 6.7 **Angle between Vectors** Let $\vec{a}=\overline{OA}$ and $\vec{b}=\overline{OB}$ be two non-zero vectors (see Fig. 6.12). Then, the measure of $\angle AOB$ which does not exceed 180° is called the *angle between $\vec{a}$ and $\vec{b}$* which is denoted by $(\vec{a},\vec{b})$.

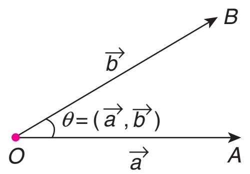


FIGURE 6.12

Note: See Fig. 6.13.

1. $0° \le (\vec{a},\vec{b}) \le 180°$. Unlike the angle between the two lines, the angle between two non-zero vectors is only one angle so that $(\vec{a},\vec{b})=(\vec{b},\vec{a})=(-\vec{a},-\vec{b})$.
2. Let λ and μ be two positive real numbers. Then
   (i) $(\lambda\vec{a},\mu\vec{b})=(\vec{a},\vec{b})=(-\lambda\vec{a},-\mu\vec{b})$
   (ii) $(-\vec{a},\vec{b})=(\vec{a},-\vec{b})=180°-(\vec{a},\vec{b})$
   (iii) $(-\lambda\vec{a},\mu\vec{b})=(\lambda\vec{a},-\mu\vec{b})=180°-(\vec{a},\vec{b})$

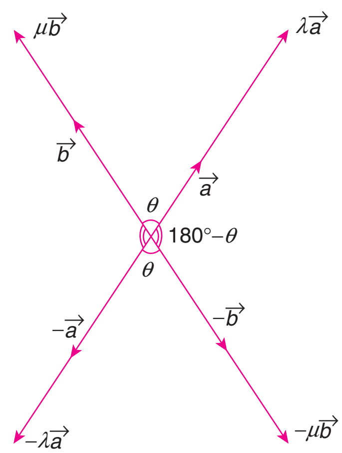


FIGURE 6.13

DEFINITION 6.8 **Right-Handed and Left-Handed Systems** Let $\vec{a}=\overline{OA}$, $\vec{b}=\overline{OB}$ and $\vec{c}=\overline{OC}$ (see Fig. 6.14). When observed from point C, if the rotation of $\vec{a}$ towards $\vec{b}$ is in anticlockwise sense not exceeding 180°, then the triad $(\vec{a},\vec{b},\vec{c})$ is said to be in *right-handed system*. If $(\vec{a},\vec{b},\vec{c})$ is not a right-handed system, then it is called *left-handed system*.

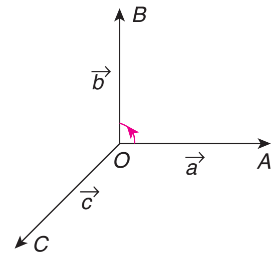


FIGURE 6.14

Note:

1. Triads $(\vec{a}, \vec{b}, \vec{c})$, $(\vec{b}, \vec{c}, \vec{a})$ and $(\vec{c}, \vec{a}, \vec{b})$ all exist in one system only. That is, all exist in right-handed system or all exist in left-handed system.
2. In a system, if two vectors are interchanged, then the system changes.
3. In a system, if one vector is replaced by its additive inverse, then the system changes.

DEFINITION 6.9 **Scalar Product or Dot Product** Let $\vec{a}$ and $\vec{b}$ be two vectors. Then the scalar product (or dot product) denoted by $\vec{a}\cdot\vec{b}$ of $\vec{a}$ and $\vec{b}$ is defined as follows:

1. $\vec{a}\cdot\vec{b}=0$ if one of $\vec{a}, \vec{b}$ is a null vector
2. If $\vec{a}, \vec{b}$ are non-zero vectors and θ is the angle between them, then $\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos\theta$.

Note:

1. $\vec{a}\cdot\vec{b}$ is a real number (scalar).
2. When $\vec{a}\cdot\vec{b} >=< 0$, the angle θ between $\vec{a}$ and $\vec{b}$ becomes acute or right angle or obtuse accordingly.
3. $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}$, $\vec{a}\cdot(-\vec{b})=(-\vec{a})\cdot\vec{b}=-(\vec{a}\cdot\vec{b})$, $(\lambda\vec{a})\cdot(\mu\vec{b})=\lambda\mu(\vec{a}\cdot\vec{b})$, where λ and μ are scalars.

DEFINITION 6.10 **Orthogonal Projection** Let $\vec{0}\neq\vec{a}=\overrightarrow{OA}$ and $\vec{b}=\overrightarrow{LM}$ in the space (see Fig. 6.15). Draw LP and MQ perpendicular to the support of the vector $\vec{a}$. Then $\overrightarrow{PQ}$ is called the *projection vector* of $\vec{b}$ on $\vec{a}$ and $|\overrightarrow{PQ}|$ is called the *orthogonal projection* of $\vec{b}$ on $\vec{a}$.

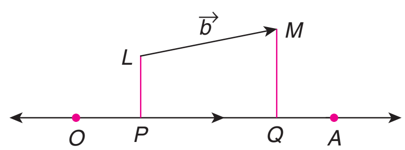


FIGURE 6.15

Note: In the definition of projection of $\vec{b}$ on $\vec{a}$, we considered the supporting line of $\vec{a}$ and hence the projection of $\vec{b}$ on $\vec{a}$ is equal to the projection of $\vec{b}$ on any vector which is collinear with $\vec{a}$.

THEOREM 6.30 Projection of $\vec{b}$ on $\vec{a}$ is

$$\left(\frac{\vec{b}\cdot\vec{a}}{|\vec{a}|^2}\right)\vec{a}=\left(\frac{\vec{b}\cdot\vec{a}}{|\vec{a}|}\right)\left(\frac{\vec{a}}{|\vec{a}|}\right)$$
$$=\left(\frac{\vec{b}\cdot\vec{a}}{|\vec{a}|}\right)\vec{e}$$

where $\vec{e}$ is the unit vector along the direction of $\vec{a}$. The magnitude of the projection of $\vec{b}$ on $\vec{a}$ is

$$\frac{|\vec{b}\cdot\vec{a}|}{|\vec{a}|}$$

DEFINITION 6.11 Suppose $\vec{a}$ and $\vec{b}$ are not collinear vectors. Then the vector

$$\left(\frac{\vec{b}\cdot\vec{a}}{|\vec{a}|^2}\right)\vec{a}$$

is called the component of $\vec{b}$ along $\vec{a}$ and

$$\vec{b}-\frac{(\vec{b}\cdot\vec{a})}{|\vec{a}|^2}\vec{a}$$

is called the component of $\vec{b}$ perpendicular to $\vec{a}$.

Hereafter, $(\vec{i},\vec{j},\vec{k})$ denotes the right-handed system of mutually perpendicular unit vectors.

FORMULA 6.1: If $\vec{a}=a_1\vec{i}+a_2\vec{j}+a_3\vec{k}$ and $\vec{b}=b_1\vec{i}+b_2\vec{j}+b_3\vec{k}$, then

$$\vec{a}\cdot\vec{b}=a_1b_1+a_2b_2+a_3b_3$$

and

$$\cos\theta=\frac{\vec{a}\cdot\vec{b}}{|\vec{a}\,\|\,\vec{b}|}=\frac{a_1b_1+a_2b_2+a_3b_3}{\sqrt{a_1^2+a_2^2+a_3^2}\sqrt{b_1^2+b_2^2+b_3^2}}$$

Hereafter, we denote $\vec{a}\cdot\vec{a}$ by $\vec{a}^2$.

6.1.2 Equations of a Plane Using Scalar Product

THEOREM 6.31 Let E be a plane, O be its origin and ON be perpendicular drawn to E (Fig. 6.16). If $ON=p$ and $\vec{n}$ is the unit vector in the direction of $\overline{ON}$, then the equation of the plane E is $\vec{r}\cdot\vec{n}=p$. In particular, if the plane passes through the origin, then its equation is $\vec{r}\cdot\vec{n}=0$.

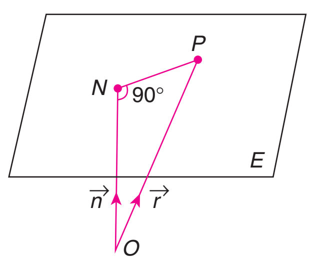


FIGURE 6.16

■

THEOREM 6.32 If q is any real number and $\vec{m}\neq\vec{0}$, then the equation $\vec{r}\cdot\vec{m}=q$ represents the plane perpendicular to the direction of $\pm\vec{m}$ and the plane is at a distance of $|q|/|\vec{m}|$ from its origin. ■

THEOREM 6.33 The equation $(\vec{r}-\vec{a})\cdot\vec{m}=0$ represents a plane passing through the point $\vec{a}$ and the plane is perpendicular to the direction of $\vec{m}$. ■

Note: The distance of the plane $(\vec{r}-\vec{a})\cdot\vec{m}=0$ from the origin is $\dfrac{|\vec{a}\cdot\vec{m}|}{|\vec{m}|}$.

THEOREM 6.34 The distance of a point $A(\vec{\alpha})$ from the plane $\vec{r}\cdot\vec{n}=p$ is $|\vec{\alpha}\cdot\vec{n}-p|$. ■

DEFINITION 6.12 The angle between two planes is defined to be the angle between their normals. If θ is the angle between the planes $\vec{r}\cdot\vec{m}_1 = q_1$ and $\vec{r}\cdot\vec{m}_2 = q_2$, then

$$\cos\theta = \frac{\vec{m}_1\cdot\vec{m}_2}{|\vec{m}_1||\vec{m}_2|}$$

6.1.3 Cross Product

DEFINITION 6.13 Consider two vectors $\vec{a}$ and $\vec{b}$. Then the cross product of $\vec{a}$ and $\vec{b}$ (in this order), denoted by $\vec{a}\times\vec{b}$, is defined as follows:

1. $\vec{a}\times\vec{b}=\vec{0}$ if either $\vec{a}$ or $\vec{b}$ is a null vector or they are collinear vectors.
2. If $\vec{a}$ and $\vec{b}$ are non-zero and non-collinear vectors, then

$$\vec{a}\times\vec{b} = (|\vec{a}||\vec{b}|\sin\theta)\vec{n}$$

where $\theta=(\vec{a},\vec{b})$ and $\vec{n}$ is the unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ such that $(\vec{a},\vec{b},\vec{n})$ is a right-handed triad.

Note:

1. Since $\sin\theta > 0$ (for $0 < \theta < 180°$), the vector $\vec{a}\times\vec{b}$ is in the direction of $\vec{n}$ and also $\vec{a}\times\vec{b}$ is perpendicular to the plane of $\vec{a}$ and $\vec{b}$.
2. The unit vectors $\pm\dfrac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$ are perpendicular to the plane of $\vec{a}$ and $\vec{b}$.
3. $\vec{b}\times\vec{a} = -(\vec{a}\times\vec{b})$.
4. $\vec{a}\times(-\vec{b}) = (-\vec{a})\times\vec{b} = -(\vec{a}\times\vec{b})$.
5. $(\lambda\vec{a})\times(\mu\vec{b}) = (\lambda\mu)(\vec{a}\times\vec{b})\ \forall\ \lambda,\mu\in\mathbb{R}$.

THEOREM 6.35 The distributive laws are as follows:

1. $\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b}+\vec{a}\times\vec{c}$
2. $(\vec{a}+\vec{b})\times\vec{c} = \vec{a}\times\vec{c}+\vec{b}\times\vec{c}$

Also, if $(\vec{i},\vec{j},\vec{k})$ is a right-handed system of mutually perpendicular unit vectors triad, then we have $\vec{i}\times\vec{j}=\vec{k}=-(\vec{j}\times\vec{i}), \vec{j}\times\vec{k}=\vec{i}=-(\vec{k}\times\vec{j})$ and $\vec{k}\times\vec{i}=\vec{j}=-(\vec{i}\times\vec{k})$. ■

FORMULA 6.2: If $\vec{a}=a_1\vec{i}+a_2\vec{j}+a_3\vec{k}$ and $\vec{b}=b_1\vec{i}+b_2\vec{j}+b_3\vec{k}$, then

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Note:

1. $|\vec{a}\times\vec{b}| = \sqrt{\sum (a_2b_3 - a_3b_2)^2}$.
2. If θ is the angle between $\vec{a}$ and $\vec{b}$, then

$$\sin\theta = \frac{|\vec{a}\times\vec{b}|}{\sqrt{a_1^2+a_2^2+a_3^2}+\sqrt{b_1^2+b_2^2+b_3^2}}$$

DEFINITION 6.14 **Vector Area** Let D be a plane region enclosed by a curve C (see Fig. 6.17). Let P_1, P_2 and P_3 be three points on the closed curve C and Δ be the real area of D. If $\vec{n}$ is the unit vector perpendicular to the plane region D, such that from the side of $\vec{n}$, the points P_1, P_2 and P_3 (in this order) are in anticlockwise sense, then $\Delta\vec{n}$ is called the *vector area* of D.

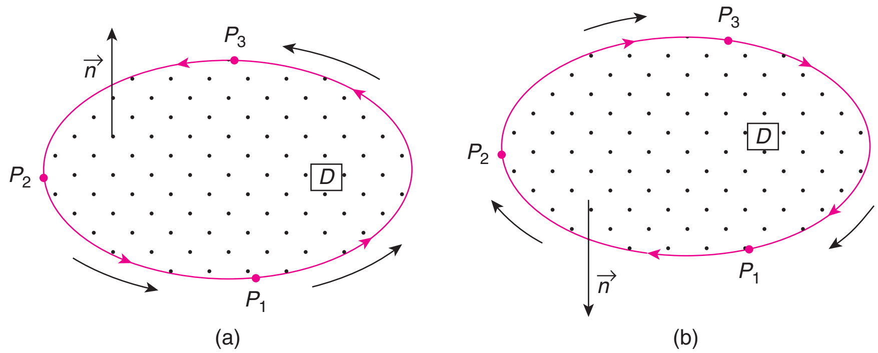


FIGURE 6.17

Note: If $\vec{n}$ is a unit vector perpendicular to the plane of D and Δ is the real area of D, then $\pm(\Delta\vec{n})$ is the vector area of D. Magnitude of the vector area is the real area.

THEOREM 6.36 If A, B and C are three non-collinear points (see Fig. 6.18), then the vector area of ΔABC is

$$\frac{1}{2}(\overrightarrow{AB}\times\overrightarrow{AC})=\frac{1}{2}(\overrightarrow{BC}\times\overrightarrow{BA})=\frac{1}{2}(\overrightarrow{CA}\times\overrightarrow{CB})$$

Further, if $\vec{a},\vec{b}$ and $\vec{c}$ are the position vectors of the points A, B and C, respectively, then its vector area is

$$\frac{1}{2}(\vec{b}\times\vec{c}+\vec{c}\times\vec{a}+\vec{a}\times\vec{b})$$

and its real area is

$$\frac{1}{2}\left|\vec{b}\times\vec{c}+\vec{c}\times\vec{a}\times\vec{a}\times\vec{b}\right|$$

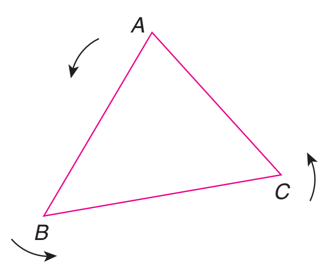


FIGURE 6.18

Note:

1. The vector area of a parallelogram with $\vec{a}$ and $\vec{b}$ as adjacent side vectors is $\vec{a}\times\vec{b}$ and its real area is $\left|\vec{a}\times\vec{b}\right|$.
2. The real area of a parallelogram (in fact, any quadrilateral $ABCD$) in terms of its diagonal vectors is $\frac{1}{2}\left|\overrightarrow{AC}\times\overrightarrow{BD}\right|$.
3. Three points $\vec{a},\vec{b}$ and $\vec{c}$ are collinear $\Leftrightarrow \vec{b}\times\vec{c}+\vec{c}\times\vec{a}+\vec{a}\times\vec{b}=\vec{0}$.

DEFINITION 6.15 **Scalar Triple Product** $\vec{a}\cdot(\vec{b}\times\vec{c})$ is called the scalar triple product of vectors $\vec{a}, \vec{b}$ and $\vec{c}$ which is denoted by $[\vec{a}\,\vec{b}\,\vec{c}]$.

THEOREM 6.37

1. $\vec{a}\cdot(\vec{b}\times\vec{c})=(\vec{a}\times\vec{b})\cdot\vec{c}$ (i.e., placement of dot and cross can be interchanged).
2. If any two of three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are collinear, then $[\vec{a}\,\vec{b}\,\vec{c}]=0$.
3. The vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar $\Leftrightarrow [\vec{a}\,\vec{b}\,\vec{c}]=0$.
4. The volume V of the parallelepiped with coterminous edges $\vec{a}, \vec{b}$ and $\vec{c}$ is $V=\pm[\vec{a}\,\vec{b}\,\vec{c}]$. Accordingly $(\vec{a},\vec{b},\vec{c})$ is considered as a right-handed system or a left-handed system.
5. The value of $[\vec{a}\,\vec{b}\,\vec{c}]$ changes if any two vectors are interchanged or any of the vectors is replaced by its additive inverse.
6. $[\vec{a}\,\vec{b}\,\vec{c}]=[\vec{b}\,\vec{c}\,\vec{a}]=[\vec{c}\,\vec{a}\,\vec{b}]$. That is, the value does not change, by cyclically permuting the vectors. ■

FORMULA 6.3

1. If $\vec{a}=a_1\vec{i}+a_2\vec{j}+a_3\vec{k},\ \vec{b}=b_1\vec{i}+b_2\vec{j}+b_3\vec{k}$ and $\vec{c}=c_1\vec{i}+c_2\vec{j}+c_3\vec{k}$, then

$$[\vec{a}\,\vec{b}\,\vec{c}]=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

2. If the vectors $\vec{l}, \vec{m}$ and $\vec{n}$ are non-coplanar and $\vec{a}=a_1\vec{l}+a_2\vec{m}+a_3\vec{n},\ \vec{b}=b_1\vec{l}+b_2\vec{m}+b_3\vec{n}$ and $\vec{c}=c_1\vec{l}+c_2\vec{m}+c_3\vec{n}$, then

$$[\vec{a}\,\vec{b}\,\vec{c}]=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}[\vec{l}\,\vec{m}\,\vec{n}]$$

Note: The vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar if and only if

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}=0$$

THEOREM 6.38

1. Let V be the volume of the tetrahedron $ABCD$. Then

$$V=\frac{1}{6}\left|[\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}]\right|$$
$$=\frac{1}{6}\left|[\vec{b}-\vec{a}\ \ \vec{c}-\vec{a}\ \ \vec{d}-\vec{a}]\right|$$

where $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ are the position vectors of the points A, B, C and D, respectively.

2. The volume of triangular prism $ABCDEF$ is given by

$$\frac{1}{2}\left|[\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}]\right|$$

3. Four points A, B, C and D are coplanar if and only if $[\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}]=0$. ■

6.2 | Coordinates, Direction Cosines and Direction Ratios

Let O be a point in the space called origin of reference and $\overleftrightarrow{x'x}$ and $\overleftrightarrow{y'y}$ be the two perpendicular lines passing through O called x- and y-axes, respectively (see Fig. 6.19). The plane determined by the lines $\overleftrightarrow{x'x}$ and $\overleftrightarrow{y'y}$ (see Theorem 6.4) is called xy-plane. Let $\overleftrightarrow{z'z}$ be the line perpendicular to xy-plane at point O (Theorem 6.9). $\overleftrightarrow{z'z}$ is called z-axis. On $\overrightarrow{Ox}, \overrightarrow{Oy}$ and $\overrightarrow{Oz}$, mark points i, j and k, respectively, such that $Oi = Oj = Ok$. Those points are called unit points.

According to Rooler axiom (Axiom 6.7) on $\overleftrightarrow{x'x}$, $\overleftrightarrow{y'y}$ and $\overleftrightarrow{z'z}$, there exist coordinate systems with O as origin and i, j and k as unit points. We call the coordinates of points on x-, y- and z-axes as x-coordinate, y-coordinate point and z-coordinate, respectively, and they are denoted by x, y and z, respectively. The plane determined by $\overleftrightarrow{x'x}$ and $\overleftrightarrow{y'y}$ is called xy-plane. The plane determined by $\overleftrightarrow{y'y}$ and $\overleftrightarrow{z'z}$ is called yz-plane and the plane determined by $\overleftrightarrow{z'z}$ and $\overleftrightarrow{x'x}$ is called zx-plane. These are also called coordinate planes. $Oxyz$ is called frame of reference.

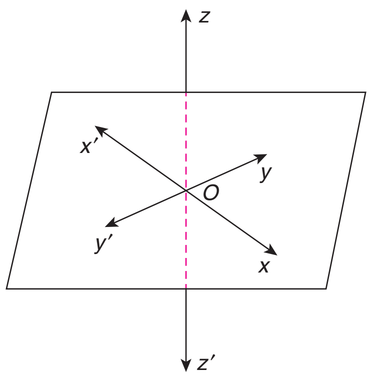


FIGURE 6.19

6.2.1 Coordinates of a Point

Let P be any point in the space. Let the feet of the perpendiculars drawn from P onto x-, y- and z-axes be L, M and L, respectively. Suppose coordinates of L, M and N are x, y and z, respectively. These are called x-, y- and z- coordinates of P, respectively, and we write $P = (x, y, z)$. That is, with reference to a frame of axes, to each point P in the space, there corresponds an ordered triad (x, y, z) of real numbers. Conversely, suppose (x, y, z) is an ordered triad of real numbers. Let L, M and N be the points on x-, y- and z-axes such the x is the coordinate of L, y is the coordinate of M and z is the coordinate of N. At L, M and N, draw the planes perpendicular to x-, y- and z-axes, respectively, meeting at point P (see Fig. 6.20). Since the line PL lies in the plane perpendicular to x-axis at L, it follows that PL is perpendicular to x-axis (see Definition 6.4). Similarly, the lines PM and PN are perpendicular to y-axis and z-axis, respectively. Hence x, y and z are the coordinates of P where $P = (x, y, z)$. That is, to each ordered trial (x, y, z) of real numbers, there corresponds unique point P such that $P = (x, y, z)$ with reference to $Oxyz$. Thus, with reference to a frame of reference, we established a one-to-one correspondence (bijection) between the set of all points in the space and the set or all ordered triad (x, y, z) of real numbers. This space is called *three-dimensional space* which is denoted by $\mathbb{R}^3$.

The three coordinate axes divide the space $\mathbb{R}^3$ into eight parts called *octants*. Table 6.1 shows the signs of the coordinates of any point in the space. Since PQ is perpendicular to zx-plane at Q, by Theorem 6.5, it is perpendicular to both QL and QN so that $OLQN$ is a rectangle and hence $OL = NQ$ and $ON = QL$. Also $OQPN$ is a rectangle, so we have $Ox = PQ$. Thus,

$$QN = OL = |x|,\ PQ = ON = |y| \text{ and } LQ = ON = |z|$$

That is, if $P = (x, y, z)$, then $|x|, |y|$ and $|z|$ are the distances of the point P from yz-, zx- and xy-planes, respectively. Also by Pythagoras theorem,

$$PL^2 = PQ^2 + QL^2 = y^2 + z^2$$

$$\Rightarrow PL = \sqrt{y^2 + z^2}$$

That is, the distance of $P(x, y, z)$ from x-axis $= PL = \sqrt{y^2+z^2}$. Similarly, the distance $P(x,y,x)$ from y- and z-axes, respectively, are $\sqrt{z^2+y^2}$ and $\sqrt{x^2+y^2}$. Also PL is perpendicular to X-axis which implies that $\angle PLO$ is $90°$ so that

$$OP^2 = PL^2 + OL^2 = y^2 + z^2 + x^2$$

and

$$OP = \sqrt{x^2+y^2+z^2}$$

Table 6.1

Octant with bonding lines	$Oxyz$	$Ox'yz$	$Oxy'z$	$Oxyz'$	$Ox'y'z$	$Ox'yz'$	$Oxy'z'$	$Ox'y'z'$
Sign of the coordinates (x, y, z)	$+,+,+$	$-,+,+$	$+,-+$	$+,+,-$	$-,-,+$	$-,+,-$	$+,-,-$	$-,-,-$

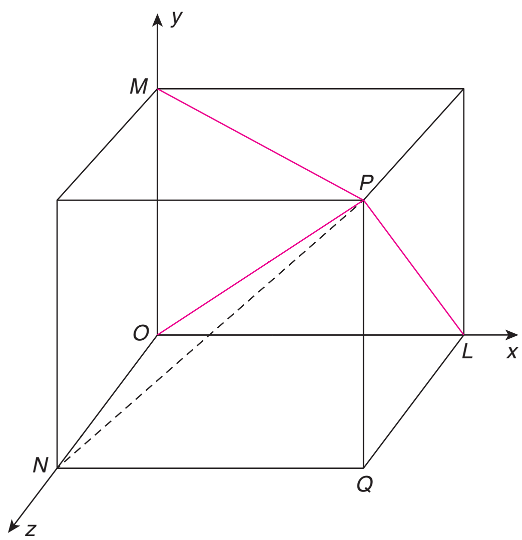


FIGURE 6.20

QUICK LOOK 1

We can describe the coordinate axis and the coordinate planes as follows:

1. x-axis $= \{P(x,y,z) | y=0, z=0\} = \{P(x,0,0) | x \in \mathbb{R}\}$
2. y-axis $= \{P(0,y,0 | y \in \mathbb{R})\}$
3. z-axis $= \{p(0,0,z) | z \in \mathbb{R}\}$
4. xy-plane $= \{p(x,y,z) | z=0\} = \{p(x,y,0) | x,y \in \mathbb{R}\}$
5. yz-plane $= \{p(0,y,z) | y,z \in \mathbb{R}\}$
6. zx-plane $= \{p(x,0,z) | x,z \in \mathbb{R}\}$

Note: If $\vec{i}, \vec{j}$ and $\vec{k}$ are unit vectors along $\overrightarrow{Ox}$, $\overrightarrow{Oy}$, and $\overrightarrow{Oz}$, then $\vec{i} = (1, 0, 0)$, $\vec{j} = (0, 1, 0)$ and $\vec{k} = (0, 0, 1)$ and $P = (x,y,z) \Rightarrow \overrightarrow{OP} = x\vec{i} + y\vec{j} + z\vec{k}$. Hereafter, we identify the vector $x\vec{i} + y\vec{j} + z\vec{k}$ with the point (x, y, z).

THEOREM 6.39 If $A = (x_1, y_1, z_1)$ and $B = (x_2, y_2, z_2)$, then the distance between A and B is

$$\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

PROOF We have

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (x_2-x_1)\vec{i} + (y_2-y_1)\vec{j} + (z_2-z_1)\vec{k}$$

and hence

$$|\overrightarrow{AB}| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$ ■

THEOREM 6.40 Let $A=(x_1, y_1, z_1)$ and $B=(x_2, y_2, z_2)$. Suppose point P divides $\overline{AB}$ in the ratio $l{:}m$ $(l+m \neq 0)$. Then

$$P=\left(\frac{lx_2+mx_1}{l+m}, \frac{ly_2+my_1}{l+m}, \frac{lz_2+mz_1}{l+m}\right)$$

In particular, the midpoint of $\overline{AB}$ is

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$$

PROOF We have $\overrightarrow{OA}=x_1\vec{i}+y_1\vec{j}+z_1\vec{k}$ and $\overrightarrow{OB}=x_2\vec{i}+y_2\vec{j}+z_2\vec{k}$ and also $AP{:}PB = l{:}m$. Hence, by Theorem 6.22, we have

$$x\vec{i}+y\vec{j}+z\vec{k}=\overrightarrow{OP}=\frac{l\overrightarrow{OB}+m\overrightarrow{OA}}{l+m}=\frac{(lx_2+mx_1)\vec{i}+(ly_2+my_1)\vec{j}+(lz_2+mz_1)\vec{k}}{l+m}$$

Therefore

$$x=\frac{lx_2+mx_1}{l+m}$$

$$y=\frac{ly_2+my_1}{l+m}$$

and

$$z=\frac{lz_2+mz_1}{l+m}$$

■

QUICK LOOK 2

1. If $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$ are the vertices of a triangle, then the coordinates of its centroid are given by

$$\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right)$$

2. If (x_r, y_r, z_r) (where $r=1, 2, 3$ and 4) are the vertices of a tetrahedron, then the coordinates of its centre are given by

$$\left(\frac{x_1+x_2+x_3+x_4}{4}, \frac{y_1+y_2+y_3+y_4}{4}, \frac{z_1+z_2+z_3+z_4}{4}\right)$$

Example 6.1

Show that the points $A=(-1,-3,4)$, $B=(-2,1,-4)$ and $C=(3,-11,5)$ form an isosceles triangle.

Solution: We have

$$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=(-1,4,-8)$$

$$\Rightarrow |\overrightarrow{AB}|=\sqrt{1^2+4^2+8^2}=9$$

Also $\overrightarrow{BC}=(5,-12,9)$. This implies

$$|\overrightarrow{BC}|=\sqrt{5^2+(-12)^2+9^2}=\sqrt{250}$$

Finally $\overrightarrow{CA}=(-4,8,-1)$. This implies

$$|\overrightarrow{AC}|=\sqrt{4^2+8^2+1^2}=9$$

Since $AB=9=AC$, ΔABC is isosceles.

Example 6.2

Show that the points $A=(1,-1,2)$, $B=(0,1,1)$ and $C=(2, -3, 3)$ are collinear and find the ratio in which B divides the segment $\overline{AC}$.

Solution: We have

$$(AB)^2=(1-0)^2+(-1-1)^2+(2-1)^2=6$$

$$(BC)^2=(2-0)^2+(-3-1)^2+(3-1)^2=24$$

$$(AC)^2=(2-1)^2+(-3+1)^2+(3-2)^2=6$$

Therefore

$$AB+AC=2\sqrt{6}=BC$$

This implies that A, B and C are collinear. Suppose B divides $\overline{AC}$ in the ratio $\lambda:1$. Therefore, from Theorem 6.40, we get

$$(0,1,1)=B=\left(\frac{2\lambda+1}{\lambda+1},\frac{-3\lambda-1}{\lambda+1},\frac{3\lambda+2}{\lambda+1},\right)$$

Therefore

$$2\lambda+1=0,\ \frac{-3\lambda-1}{\lambda+1}=1,\frac{-3\lambda+2}{\lambda+1}=1$$

From these three equations, we get $\lambda=-1/2$. Hence, B divides $\overline{AC}$ externally in the ratio 1:2.

Example 6.3

If $A=(2,3,4)$ and $B=(-3,5,-4)$, find the ratio in which yz-plane divides the segment $\overline{AB}$.

Solution: Suppose yz-plane meets the line $\overleftrightarrow{AB}$ at P and $AP:PB=\lambda:1$. Therefore, from Theorem 6.40, we get

$$P=\left(\frac{2-3\lambda}{\lambda+1},\frac{5\lambda-3}{\lambda+1},\frac{4-4\lambda}{\lambda+1}\right)$$

Since P belongs to yz-plane, its x-coordinate should be zero so that

$$2-3\lambda=0 \Rightarrow \lambda=2/3$$

The ratio is 2:3.

Example 6.4

Find the centre and radius of the sphere passing through the points $O(0, 0, 0)$, $A(a, 0, 0)$, $B(0, b, 0)$ and $C(0, 0, c)$.

Solution: Let $H(x, y, z)$ be the centre of the sphere so that $HO = HA = HB = HC$ = Radius. Now

$$HO=HA\Rightarrow(x-a)^2+y^2+z^2=x^2+y^2+z^2$$
$$\Rightarrow x=\frac{a}{2}$$

Similarly

$$HO=HB\Rightarrow y=\frac{b}{2}$$

and

$$HO=HC\Rightarrow z=\frac{c}{2}$$

Therefore, the centre H is given by

$$\left(\frac{a}{2},\frac{b}{2},\frac{c}{2}\right)$$

and the radius is given by

$$HO=\frac{1}{2}\sqrt{a^2+b^2+c^2}$$

Example 6.5

Suppose $A=(-2,2,3)$ and $B=(13,-3,13)$. If P is a variable point such that $3PA=2PB$, then find the equation of the locus.

Solution: Suppose $P=(x, y, z)$. Now

$$3PA=2PB$$

$$\Rightarrow 9PA^2=4PB^2$$
$$\Rightarrow 9[(x+2)^2+(y-2)^2+(z-3)^2]$$
$$=4[(x-13)^2+(y+3)^2+(z-13)^2]$$
$$\Rightarrow 5(x^2+y^2+z^2)+140x-60y+50z-1235=0$$
$$\Rightarrow x^2+y^2+z^2+28x-12y+10z-247=0$$

DEFINITION 6.16 **Direction Ratios and Direction Cosines** Let L be a line in the space and L' be a line passing through the origin and parallel to L (see Fig. 6.21). If $P\ (a, b, c) \neq (0, 0, 0)$ is a point on L', then a, b and c are called *direction ratios* of the line L or the vector (a, b, c), where at least one of a, b and c is non-zero, is called the *direction ratio* of L. Generally, "DRs" denotes the direction ratios. If $Q \neq (0, 0, 0)$ is any point on L', then $\overrightarrow{OQ} = \lambda\overrightarrow{OP} \Rightarrow (\lambda a, \lambda b, \lambda c)$ are also DRs. Hence, a straight line has many DRs and it is known that the vectors (a, b, c) and $(\lambda a, \lambda b, \lambda c)$ are parallel. In particular, if $P(a, b, c) \neq (0, 0, 0)$ is on L' and $OP = 1$, then a, b and c are called the *direction cosines* (DCs) of the line L (a, b and c are DCs of L). If a, b and c are DCs of L, then $-a$, $-b$, and $-c$ are also DCs of L. Generally, DCs of a line are denoted by (l, m, n). Remember that $(-l, -m, -n)$ are also DCs.

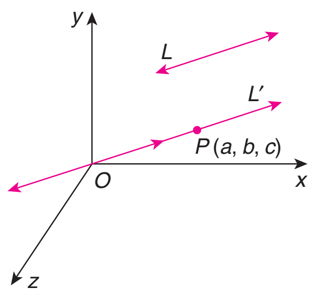


FIGURE 6.21

QUICK LOOK 3

1. DRs of a line L means any vector in the space parallel to L.
2. DCs of L means any unit vector parallel to the line.
3. If $(l, m\ n)$ are DCs of a line L, then $l^2 + m^2 + n^2 = 1$.

DEFINITION 6.17 **DCs of Array** Suppose $\overrightarrow{AB}$ is an ray in the space. Then any unit vector parallel to $\overrightarrow{AB}$ in the direction of the ray $\overrightarrow{AB}$ is called DC of the ray AB.

QUICK LOOK 4

See Fig. 6.22.

1. A line has two sets of DCs.
2. A ray has only one set of DCs.
3. If (l, m, n) are the DCs of a line, then we have $l^2 + m^2 + n^2 = 1$ and hence if we know any two values of l, m and n, and the sign of the third, then we can write the DCs of the line.
4. If (a, b, c) are the DRs of a line L, then the DCs of L are

$$\left(\frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}, \frac{c}{\sqrt{a^2+b^2+c^2}}\right)$$

and $$\left(\frac{-a}{\sqrt{a^2+b^2+c^2}}, \frac{-b}{\sqrt{a^2+b^2+c^2}}, \frac{-c}{\sqrt{a^2+b^2+c^2}}\right)$$

5. The DCs of the positive coordinate axes $\overrightarrow{Ox}, \overrightarrow{Oy}$ and $\overrightarrow{Oz}$ are $\vec{i} = (1,0,0)$, $\vec{j} = (0,1,0)$ and $\vec{k} = (0,0,1)$, respectively.
6. The DRs of parallel lines are proportional.

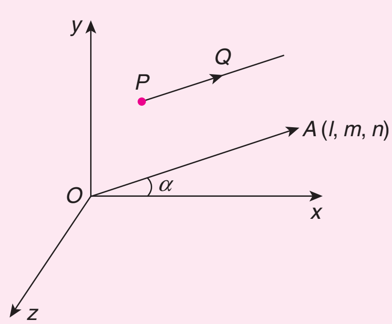


FIGURE 6.22

THEOREM 6.41 If a ray $\overrightarrow{PQ}$ makes angles α, β and γ with the positive directions of the axes, then the DCs of the ray $\overrightarrow{PQ}$ are $(\cos\alpha, \cos\beta, \cos\gamma)$.

PROOF Suppose the DCs of the ray $\overrightarrow{PQ}$ are (l, m, n). Let $A = (l, m, n)$ so that $\overrightarrow{OA}$ is parallel to $\overrightarrow{PQ}$ and $|\overrightarrow{OA}|=1$. Further, the vector $\overrightarrow{OA}$ makes angles α, β and γ with $\overrightarrow{Ox}$, $\overrightarrow{Oy}$ and $\overrightarrow{Oz}$. Since $\vec{i}$ is the unit vector in the direction of $\overrightarrow{Ox}$, we have

$$\overrightarrow{OA}\cdot\vec{i}=|\overrightarrow{OA}||\vec{i}|\cos\alpha=\cos\alpha \Rightarrow l=\cos\alpha$$

Similarly $m=\cos\beta$ and $n=\cos\gamma$. Also

$$l^2+m^2+n^2=1$$
$$\Rightarrow \cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$$

■

Note: If (l, m, n) are the DCs of the line $\overleftrightarrow{PQ}$, then $(-l, -m, -n)$ are the DCs of the ray $\overrightarrow{QP}$. The angles made by $\overrightarrow{QP}$ with $\overrightarrow{Ox}, \overrightarrow{Oy}$ and $\overrightarrow{Oz}$ are $\pi-\alpha, \pi-\beta$ and $\pi-\gamma$ so that the DCs of $\overrightarrow{QP}$ are $-\cos\alpha, -\cos\beta$ and $-\cos\gamma$. Hence, for the line $\overleftrightarrow{PQ}$, both $(\cos\alpha, \cos\beta, \cos\gamma)$ and $(-\cos\alpha, -\cos\beta, -\cos\gamma)$ are DCs.

THEOREM 6.42 The DRs of the line joining $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ are $(x_2-x_1, y_2-y_1, z_2-z_1)$.

PROOF Let $O(0, 0, 0)$ be the origin (see Fig. 6.23). Now, complete the parallelogram $OABC$ so that

$$\overrightarrow{OC}=\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=(x_2-x_1, y_2-y_1, z_2-z_1)$$

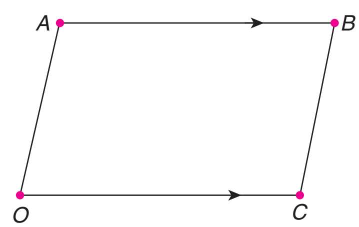


FIGURE 6.23

■

THEOREM 6.43 Suppose (l, m, n) are DCs of a ray $\overrightarrow{AB}$. If $P=(x_1, y_1, z_1)$ and $Q=(x_2, y_2, z_2)$, then the projection vector of $\overrightarrow{PQ}$ on $\overrightarrow{AB}$ is

$$[l(x_2-x_1)+m(y_2-y_1)+n(z_2-z_1)](l\vec{i}+m\vec{j}+n\vec{k})$$

and it modulus is $|l(x_2-x_1)+m(y_2-y_1)+n(z_2-z_1)|$.

PROOF The vector $\vec{e}=l\vec{i}+m\vec{j}+n\vec{k}$ is the unit vector in the direction of $\overrightarrow{AB}$. Therefore, from Theorem 6.30, the projection vector of $\overrightarrow{PQ}$ on $\overrightarrow{AB}$ is given by $=(\overrightarrow{PQ}\cdot\vec{e})\vec{e}$. That is

$$[l(x_2-x_1)+m(y_2-y_1)+n(z_2-z_1)](l\vec{i}+m\vec{j}+n\vec{k})$$

■

THEOREM 6.44 Suppose (l_1, m_1, n_1) and (l_2, m_2, n_2) are the DCs of the lines L_1 and L_2, respectively. If θ is the angle between L_1 and L_2, then $\cos\theta=l_1l_2+m_1m_2+n_1n_2$.

PROOF Let $A=(l_1, m_1, n_1)$ and $B=(l_2, m_2, n_2)$ so that $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are unit vectors parallel to L_1 and L_2, respectively. Hence, θ is also the angle between $\overrightarrow{OA}$ and $\overrightarrow{OB}$. Therefore

$$\cos\theta=\frac{\overrightarrow{OA}\cdot\overrightarrow{OB}}{|\overrightarrow{OA}||\overrightarrow{OB}|}=l_1l_2+m_1m_2+n_1n_2 \qquad (\because |\overrightarrow{OA}|=1=|\overrightarrow{OB}|)$$

■

Example 6.6

If (l_1, m_1, n_1) and (l_2, m_2, n_2) are the DCs of the lines L_1 and L_2, respectively, then show that the DRs of a line perpendicular to both L_1 and L_2 are $(m_1n_2 - m_2n_1, l_2n_1 - l_1n_2, l_1m_2 - l_2m_1)$.

Solution: Let $A = (l_1, m_1, n_1)$ and $B = (l_2, m_2, n_2)$ so that the DRs of the line perpendicular to both L_1 and L_2 are given by

$$\overline{OA} \times \overline{OB} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}$$

$$= (m_1n_2 - m_2n_1)\vec{i} - (l_1n_2 - l_2n_1)\vec{j} + (l_1m_2 - l_2m_1)\vec{k}$$

$$= (m_1n_2 - m_2n_1, l_2n_1 - l_1n_2, l_1m_2 - l_2m_1)$$

Example 6.7

If the DCs of two lines satisfy relations $l + m + n = 0$ and $mn - 2nl - 2lm = 0$, then find them. Also show that one of the angles between the lines is $2\pi/3$.

Solutions: We have

$$l + m + n = 0 \qquad (6.1)$$

$$mn - 2nl - 2lm = 0 \qquad (6.2)$$

From Eq. (6.10), we have $l = -(m+n)$ and substituting the value of l in Eq. (6.2), we have

$$mn + 2n(m+n) + 2m(m+n) = 0$$

$$\Rightarrow 2m^2 + 5mn + 2n^2 = 0$$

$$\Rightarrow (2m+n)(m+2n) = 0$$

$$\Rightarrow 2m = -n \text{ or } m = -2n$$

If we substitute $2m = -n$ in $l = -(m+n)$, we have

$$l = -(m - 2m) = m$$

and $$m = -2n \Rightarrow l = -(m+n) = n$$

Case 1: $$l = m, 2m = -n \Rightarrow \frac{l}{1} = \frac{m}{1} = \frac{n}{-2}$$

Case 2: $$l = n, m = -2n \Rightarrow \frac{l}{1} = \frac{m}{-2} = \frac{n}{1}$$

Hence, the DRs of the lines are (1, 1, –2) and (1, –2, 1) and the DCs are given by

$$\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}\right)$$

and $$\left(\frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$$

If θ is the angle between them, we get

$$\cos\theta = \frac{1}{6} - \frac{2}{6} - \frac{2}{6} = -\frac{1}{2}$$

so that $2\pi/3$.

Example 6.8

The DCs of two lines satisfy the equations $2l + 2m - n = 0$ and $mn + nl + lm = 0$. Show that the two lines are at right angles to each other.

Solution: We have

$$2l + 2m - n = 0 \qquad (6.3)$$

$$mn + nl + lm = 0 \qquad (6.4)$$

From Eq. (6.3), we have $n = 2l + 2m$. Substituting the value of n in Eq. (6.4), we have

$$m(2l + 2m) + l(2l + 2m) + lm = 0$$

$$\Rightarrow 2l^2 + 5lm + 2m^2 = 0$$

$$\Rightarrow (2l + m)(l + 2m) = 0$$

$$\Rightarrow m = -2l, l = -2m$$

Now $m = -2l$ and $n = 2l + 2m = 2l - 4l = -2l$. This implies that

$$\frac{l}{1} = \frac{m}{-2} = \frac{n}{-2} \Rightarrow \frac{l}{-1} = \frac{m}{2} = \frac{n}{2}$$

Hence, $l = -2m$ and $n = 2l + 2m = -2m$. This implies that

$$\frac{l}{2} = \frac{m}{-1} = \frac{n}{2}$$

Therefore, the DRs of the lines are (–1, 2, 2) and (2, –1, 2) and the dot product is given by –2 – 2 + 4 = 0. Hence, the lines are at right angles to each other.

Example 6.9

If L_1 and L_2 are two straight lines whose DCs satisfy the equations $al+bm+cn=0$ and $ul^2+vm^2+wn^2=0$, then prove that

(a) $a^2(v+w)+b^2(w+v)+c^2(u+v)=0$ if L_1 and L_2 are at right angles

(b) $\dfrac{a^2}{u}+\dfrac{b^2}{v}+\dfrac{c^2}{w}=0$ if L_1 and L_2 are parallel.

Solution: Eliminating n from the given two equations, we have

$$(c^2u+a^2w)l^2+2abw\,lm+(c^2v+b^2w)m^2=0$$

Dividing this equation with m^2, we have

$$(c^2u+a^2w)\left(\frac{l}{m}\right)^2+2abw\left(\frac{l}{m}\right)+c^2v+b^2w=0 \quad (6.5)$$

which is a quadratic equation in l/m. If (l_1,m_1,n_1) and (l_2,m_2,n_2) are the DCs of the two given lines, then l_1/m_1 and l_2/m_2 are the roots of Eq. (6.5) so that

$$\left(\frac{l_1}{m_1}\right)\left(\frac{l_2}{m_2}\right)=\frac{c^2v+b^2w}{c^2u+a^2w}$$

Therefore

$$\frac{l_1l_2}{c^2v+b^2w}=\frac{m_1m_2}{c^2u+a^2w}$$

Similarly, we can show that

$$\frac{m_1m_2}{c^2u+a^2w}=\frac{n_1n_2}{b^2u+a^2v}$$

Therefore,

$$\frac{l_1l_2}{c^2v+b^2w}=\frac{m_1m_2}{c^2u+a^2w}=\frac{n_1n_2}{b^2u+a^2v}=k \quad \text{(say)}$$

(a) As L_1 is perpendicular to L_2, we have

$$l_1l_2+m_1m_2+n_1n_2=0$$
$$\Rightarrow k[c^2v+b^2w+c^2u+a^2w+b^2u+a^2v]=0$$
$$\Rightarrow a^2(v+w)+b^2(w+u)+c^2(u+v)=0$$

(b) As L_1 is parallel to L_2, we have

$$\frac{l_1}{m_1}=\frac{l_2}{m_2}$$

That is, the roots of Eq. (6.5) are equal. Therefore, we get

$$4a^2b^2w^2=4(c^2u+a^2w)(c^2v+b^2w)$$
$$\Rightarrow a^2b^2w^2=c^4uv+b^2c^2uw+a^2c^2vw+a^2b^2w^2$$
$$\Rightarrow c^2(c^2uv+b^2wu+a^2vw)=0$$
$$\Rightarrow a^2vw+b^2uw+c^2uv=0 \quad (6.6)$$

Dividing Eq. (6.6) by uvw, we have

$$\frac{a^2}{u}+\frac{b^2}{v}+\frac{c^2}{w}=0$$

Example 6.10

If (l_r,m_r,n_r)(where $r=1,2$ and 3) are the DCs of three concurrent lines, then show that the condition for them to be coplanar is that

$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}=0$$

Solution: It is known that the lines are coplanar if and only if the vectors (l_r,m_r,n_r) are coplanar. These vectors are coplanar if and only if their scalar triple product is zero. That is, if and only if

$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}=0$$

Example 6.11

If a line makes angles α, β, γ and δ with the diagonals of a cube, then show that

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma+\cos^2\delta=\frac{4}{3}$$

Solution: Suppose $A=(1,0,0), B=(0,1,0)$ and $C(0,0,1)$ are the unit points on the coordinate axes and consider the cube $OADCBEFG$ (see Fig 6.24) so that $D=(1,0,1)$, $E=(1,1,0)$, $F=(1,1,1)$ $G=(0,1,1)$. Suppose the given line makes an angle α with the diagonal OF. If (l,m,n) are the DCs of the given line, then

$$\cos\alpha = \frac{(l,m,n)\cdot(1,1,1)}{(1)\sqrt{1^2+1^2+1^2}} = \frac{l+m+n}{\sqrt{3}}$$

Also $\overline{BD} = (1,-1,1)$, $\overline{CE} = (1,1,-1)$, and $\overline{AG} = (-1,1,1)$. Suppose β is the angle made by $\overline{AG}$ with the line. We have

$$\cos\beta = \frac{-l+m+n}{\sqrt{3}}$$

$$\cos\gamma = \frac{l-m+n}{\sqrt{3}}$$

and $$\cos\delta = \frac{l+m-n}{\sqrt{3}}$$

Therefore

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta$$

$$= \frac{1}{3}[(l+m+n)^2 + (-l+m+n)^2 + (l-m+n)^2 + (l+m-n)^2]$$

$$= \frac{1}{3}[4(l^2+m^2+n^2)]$$

$$= \frac{4}{3} \quad (\because l^2+m^2+n^2=1)$$

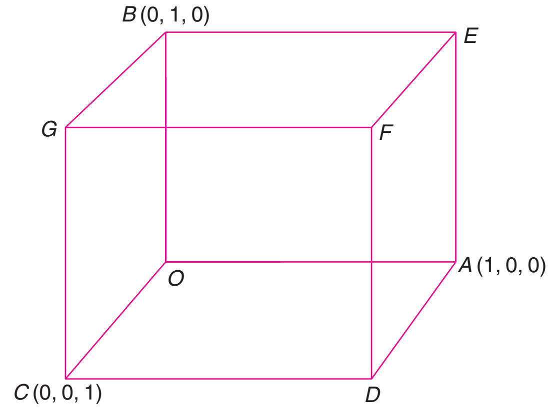


FIGURE 6.24

Example 6.12

If the DCs of the rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are (l_1, m_1, n_1) and (l_2, m_2, n_2), respectively, then show that DCs of the angle bisectors of the rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are $(l_1 \pm l_2,\ m_1 \pm m_2,\ n_1 \pm n_2)$.

Solution: Let $\overrightarrow{AM}$ and $\overrightarrow{AN}$ be the angle bisectors of the angle between $\overrightarrow{AB}$ and $\overrightarrow{AC}$. If $P=(l_1,m_1,n_1)$ and $Q=(l_2,m_2,n_2)$, then we know that $\overrightarrow{OP}$ is parallel to $\overrightarrow{AB}$ and $\overrightarrow{OQ}$ is parallel to $\overrightarrow{AC}$ and hence the angle between $\overrightarrow{AB}$ and $\overrightarrow{AC}$ is same as the angle between $\overrightarrow{OP}$ and $\overrightarrow{OQ}$. Let $(\overrightarrow{OP}, \overrightarrow{OQ}) = \theta$. Let $\overrightarrow{OM_1}$ be one of the angle bisectors of $\overrightarrow{OP}$ and $\overrightarrow{OQ}$. Suppose $\overrightarrow{AM}$ is parallel to $\overrightarrow{OM_1}$. Suppose PQ meets OM at R (see Fig. 6.25). Since $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ are unit vectors, ΔOPQ is isosceles so that R is the midpoint of PQ and hence

$$R = \left(\frac{l_1+l_2}{2}, \frac{m_1+m_2}{2}, \frac{n_1+n_2}{2}\right)$$

Since R lies on $\overrightarrow{OM_1}$, and the fact that $\overrightarrow{OM_1}$ is parallel to $\overrightarrow{AM}$ it follows that $(l_1+l_2, m_1+m_2, n_1+n_2)$ are the DRs of $\overrightarrow{AM}$.

Suppose $\overrightarrow{ON_1}$ is the second bisector of the angle θ, that is, bisector of $\pi-\theta$. Now

$$\overrightarrow{ON_1} \perp \overrightarrow{OM_1} \text{ and } \overrightarrow{OM_1} \perp \overrightarrow{PQ} \Rightarrow \overrightarrow{ON_1} \,\|\, \overrightarrow{PQ}$$

and $(l_2-l_1,\ m_2-m_1,\ n_2-n_1)$ are the DRs of $\overrightarrow{PQ}$. Hence, the DRs of $\overrightarrow{ON_1}$ are also $(l_2-l_1, m_2-m_1, n_2-n_1)$ or $(l_1-l_2, m_1-m_2, n_1-n_2)$.

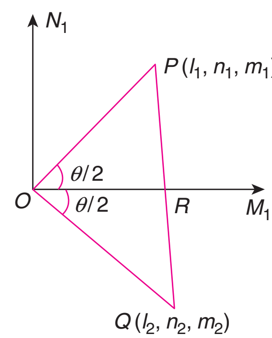


FIGURE 6.25

Example 6.13

If a, b and c are the lengths of the edges of a rectangular parallelepiped, then show that the angle between any two diagonals of it is

$$\mathrm{Cos}^{-1}\left(\frac{\pm a^2 \pm b^2 \pm c^2}{a^2+b^2+c^2}\right)$$

where all + signs or all are – signs should not be taken.

Solution: Let $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$ be coterminous edges of a rectangular parallelepiped (see Fig. 6.26). Suppose $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$ are x-, y- and z-axis, respectively. Also we have $OA = a$, $OB = b$ and $OC = c$. Therefore, $\overrightarrow{OA}=(a,0,0), \overrightarrow{OB}=(0,b,0)$ and $\overrightarrow{OC}=(0,0,c)$. Consider the angle between the diagonals BD and OE. Now, $\overrightarrow{OE}=(a,b,c)$ and $\overrightarrow{BD}=(a,-b,c)$. Let $(\overrightarrow{BD}, \overrightarrow{OE}) = \alpha$ so that

$$\cos\alpha = \frac{\overline{OE}\cdot\overline{BD}}{|\overline{OE}||\overline{BD}|} = \frac{a^2-b^2+c^2}{a^2+b^2+c^2}$$

$$\Rightarrow \alpha = \text{Cos}^{-1}\left(\frac{a^2-b^2+c^2}{a^2+b^2+c^2}\right)$$

Similarly, the other angles are

$$\text{Cos}^{-1}\left(\frac{a^2+b^2-c^2}{a^2+b^2+c^2}\right)$$

$$\text{Cos}^{-1}\left(\frac{-a^2+b^2+c^2}{a^2+b^2+c^2}\right)$$

$$\text{Cos}^{-1}\left(\frac{a^2-b^2-c^2}{a^2+b^2+c^2}\right)$$

$$\text{Cos}^{-1}\left(\frac{-a^2+b^2-c^2}{a^2+b^2+c^2}\right)$$

and

$$\text{Cos}^{-1}\left(\frac{-a^2-b^2+c^2}{a^2+b^2+c^2}\right)$$

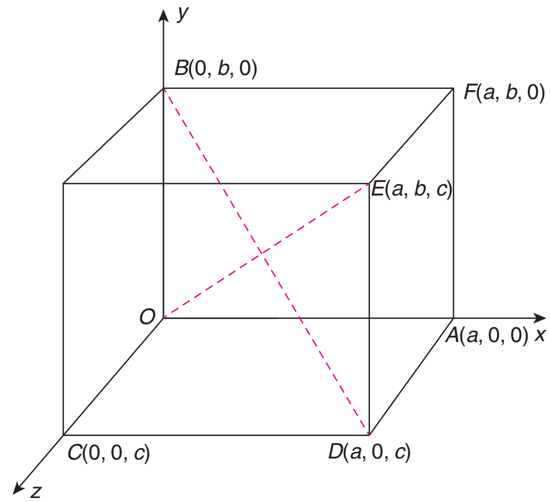


FIGURE 6.26

QUICK LOOK 5

Since two diagonals are not parallel, in the formula $\text{Cos}^{-1}\left(\frac{\pm a^2 \pm b^2 \pm c^2}{a^2+b^2+c^2}\right)$, we should not take all + signs or all − signs. Further, if $a = b = c$, then the angle is $\text{Cos}^{-1}(1/3)$.

Example 6.14

Suppose (l_r, m_r, n_r) (where $r = 1, 2$ and 3) are the DCs of the three mutually perpendicular lines. The DRs of another line L are proportional to $(l_1 + l_2 + l_3, m_1 + m_2 + m_3, n_1 + n_2 + n_3)$. Find the DCs of L and show that L is equally inclined to all the three lines.

Solution: Let L_1, L_2 and L_3 be the lines. Since these lines are mutually perpendicular to each other, we have

$$l_1l_2+m_1m_2+n_1n_2=0$$

$$l_2l_3+m_2m_3+n_2n_3=0$$

$$l_3l_1+m_3m_1+n_3n_1=0$$

$$l_1^2+m_1^2+n_1^2=1$$

$$l_2^2+m_2^2+n_2^2=1$$

and
$$l_3^2+m_3^2+n_3^2=1$$

Therefore,

$$(l_1+l_2+l_3)^2+(m_1+m_2+m_3)^2+(n_1+n_2+n_3)^2 = 3+2(0+0+0)=3$$

Hence, the DCs of L are

$$\left(\frac{l_1+l_2+l_3}{\sqrt{3}}, \frac{m_1+m_2+m_3}{\sqrt{3}}, \frac{n_1+n_2+n_3}{\sqrt{3}}\right)$$

Suppose θ is the angle between L and L_1. Therefore

$$\cos\theta = \frac{l_1(l_1+l_2+l_3)}{\sqrt{3}} + \frac{m_1(m_1+m_2+m_3)}{\sqrt{3}} + \frac{n_1(n_1+n_2+n_3)}{\sqrt{3}}$$

$$= \frac{l_1^2+m_1^2+n_1^2}{\sqrt{3}} + \frac{1}{\sqrt{3}}(0) = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \theta = \text{Cos}^{-1}\frac{1}{\sqrt{3}}$$

Similarly, the angles made by the line L with L_1 and L_2 are equal to $\text{Cos}^{-1}(1/\sqrt{3})$.

Example 6.15

Show that in a tetrahedron $OABC$, the opposite pairs of edges are at right angles if and only if $(OA)^2+(BC)^2=(OB)^2+(CA)^2=(OC)^2+(AB)^2$.

Solution: See Fig. 6.27. In the tetrahedron $OABC$, $(OC, AB),(AC, OB)$ and (BC, OA) are the pairs of opposite edges. Let $\overrightarrow{OA}=\vec{a}$, $\overrightarrow{OB}=\vec{b}$ and $\overrightarrow{OC}=\vec{c}$. Assume that

$$(OA)^2+(BC)^2=(OB)^2+(AC)^2$$
$$\Rightarrow \vec{a}^2+(\vec{c}-\vec{b})^2=\vec{b}^2+(\vec{c}-\vec{a})^2$$
$$\Rightarrow \vec{a}^2+\vec{b}^2+\vec{c}^2-2(\vec{b}\cdot\vec{c})=\vec{a}^2+\vec{b}^2+\vec{c}^2-2\vec{c}\cdot\vec{a}$$
$$\Rightarrow \vec{b}\cdot\vec{c}=\vec{c}\cdot\vec{a}$$
$$\Rightarrow \vec{c}\cdot(\vec{b}-\vec{a})=0$$

This implies that $\overrightarrow{OC}$ is perpendicular to $\overrightarrow{AB}$. Similarly, if

$$(OB)^2+(AC)^2=(OC)^2+(AB)^2$$

we have

$$\vec{b}^2+(\vec{c}-\vec{a})^2=\vec{c}^2+(\vec{b}-\vec{a})^2$$
$$\Rightarrow \vec{c}\cdot\vec{a}=\vec{a}\cdot\vec{b}$$

Therefore, $\vec{a}\cdot(\vec{c}-\vec{b})=0 \Rightarrow \overrightarrow{OA}$ and $\overrightarrow{BC}$ are at right angles.

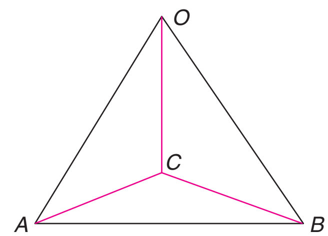


FIGURE 6.27

Example 6.16

Find the equation to the locus of a point whose distance from the z-axis is three times the distance from the point $(-1, 2, -3)$.

Solution: $P(x, y, z)$ is a point whose distance from the z-axis is three times its distances from the point $(-1, 2, -3)$. This implies

$$\sqrt{x^2+y^2}=3\sqrt{(x+1)^2+(y-2)^2+(z+3)^2}$$
$$\Rightarrow x^2+y^2=9\left[(x+1)^2+(y-2)^2+(z+3)^2\right]$$
$$\Rightarrow 8x^2+8y^2+9z^2+18-36y+54z+126=0$$

Example 6.17

Three vertices of a parallelogram $ABCD$ are $A\ (1, 2, 3)$, $B\ (-1, -2, -1)$ and $C\ (2, 3, 2)$. Find the coordinates of the fourth vertex D.

Solution: In a parallelogram, diagonals bisect each other. Therefore, if $D(x, y, z)$ is the fourth vertex, then we have

$$\left(\frac{1+2}{2},\frac{2+3}{2}+\frac{3+2}{2}\right)=\text{Midpoint of } AC$$
$$=\text{Midpoint of } BD$$
$$=\left(\frac{x-1}{2},\frac{y-2}{2},\frac{z-1}{2}\right)$$

Therefore

$$\frac{x-1}{2}=\frac{3}{2}\Rightarrow x=4$$
$$\frac{y-2}{2}=\frac{5}{2}\Rightarrow y=7$$

and $$\frac{z-1}{2}=\frac{5}{2}\Rightarrow z=6$$

Therefore, $D=(4, 7, 6)$.

Example 6.18

Find the length and the direction cosines of a line segment whose projections on the coordinate axes are 6, –3 and 2, respectively.

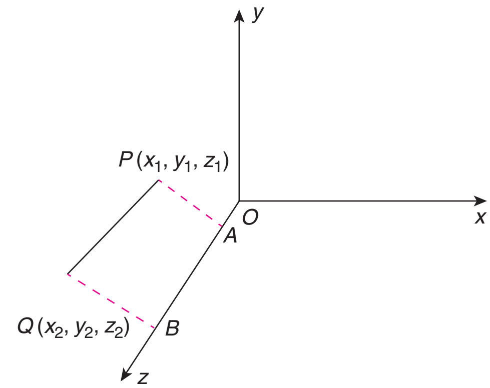


FIGURE 6.28

Solution: See Fig. 6.28. Let $P=(x_1, y_1, z_1)$ and Q (x_2, y_2, z_2) be the extremities of the given line segment. Let PA and PB be perpendicular to the z-axis. Now,

$$OA=z_1, OB=z_2 \Rightarrow \overline{AB}=z_2-z_1$$

Therefore, $z_2-z_1=2$. Similarly, $x_2-x_1=6$ and $y_2-y_1=-3$. Therefore

$$PQ=\sqrt{6^2+(-3)^2+2^2}=7$$

Hence, the DRs of PQ are $(6,-3,2)$ and the DCs of PQ are $\left(\frac{6}{7}, \frac{-3}{7}, \frac{2}{7}\right)$.

6.3 | Plane

Suppose $P(x, y, z)$ is a point with reference to the frame of reference $Oxyz$. In this case $\overrightarrow{OP}$ is a vector. Conversely, if $\vec{r}$ is any vector in the space, there corresponds a unique point P in the space such that $\overrightarrow{OP}=\vec{r}$. Hence, there is one-to-one correspondence between the set of space vectors and the set of all points in the space. Hence, hereafter, we identify space with vector space and denote it by $\mathbb{R}^3$.

DEFINITION 6.18 Let $f:\mathbb{R}^3 \to \mathbb{R}$ be a function. Then the set $S=\{P(x,y,z) \mid f(x,y,z)=0\}$ is called a *surface* represented by the equation $f(x,y,z)=0$.

One equation represents only one surface. However, a surface may be represented by more than one equations, because, if $f(x,y,z)=0$ represents a surface S and $\lambda \neq 0$, then $\lambda f(x,y,z)=0$ also represents the same surface S.

DEFINITION 6.19 If f is a polynomial in x, y and z, then the surface represented by $f(x,y,z)=0$ is called *algebraic surface*.

DEFINITION 6.20 If a, b, c and d are real numbers and at least one of a, b and c is not zero, then the equation $ax+by+cz+d=0$ is called *first-degree equation* in x, y and z. The surface represented by first-degree equation is called *first-degree surface*.

THEOREM 6.45 The equation of a plane is a first-degree equation in x, y and z (*normal form*).

PROOF See Fig. 6.29. Let E be a plane and O be its origin. Let N be the foot of the perpendicular drawn from the origin O onto E. Let $\vec{n}$ be the unit vector in the direction of $\overline{ON}$. If $\vec{n}=(l,m,n)$, then $\overline{ON}=\overline{pn}$ where $p=ON$.

$$\begin{aligned} P(x,y,z) \text{ is any point in } E &\Leftrightarrow \overrightarrow{NP}\cdot\overrightarrow{ON}=0 \\ &\Leftrightarrow (\vec{r}-p\vec{n})\cdot p\vec{n}=0 \\ &\Leftrightarrow \vec{r}\cdot\vec{n}=p \end{aligned}$$

Now $\overrightarrow{OP}=x\vec{i}+y\vec{j}+z\vec{k}$ and $\vec{n}=l\vec{i}+m\vec{j}+n\vec{k}$. Then

$$\vec{r}\cdot\vec{n}=p \Leftrightarrow lx+my+nz=p$$

Since $\vec{n}=(l,m,n)$ is a unit vector, $l,m,n \neq 0$. Here, the equation of the plane E is $lx+my+nz-p=0$, a first-degree equation in x, y and z and this equation is called *normal form* of the plane.

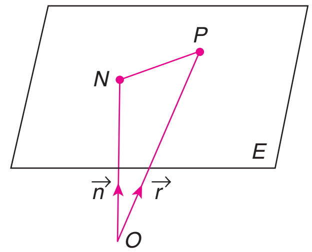


FIGURE 6.29

QUICK LOOK 6

1. If E passes through origin O, then $P = 0$ so that its equation is $lx+my+nz=0$.
2. $l\neq 0, m\neq 0 \Rightarrow (l,m,0)\cdot(0,0,1)=0$. Hence, the line having $(l,m,0)$ as DRs is perpendicular to the z-axis. Hence, the plane $lx+my=p$ is parallel to z-axis. Similarly, the plane $my+nz=p$ is parallel to x-axis and $lx+nz=p$ is parallel to y-axis.

DEFINITION 6.21 If a plane meets the axes at $(a,0,0),(0,b,0)$ and $(0,0,c)$, then a, b and c are called the *intercepts of the plane on the axes*.

THEOREM 6.46 If $abc\neq 0$, then the equation of the plane having intercepts on coordinate axes a, b and c is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

This form is called *intercept form*.

PROOF Suppose E is the plane meeting the axes at $A(a,0,0)$, $B(0,b,0)$ and $C(0,0,c)$. Let $\vec{n}=(l,m,n)$ be the unit vector perpendicular to the plane E so that the equation of the plane is of the form

$$lx+my+nz=p \tag{6.7}$$

Now $A(a,0,0)$ belongs to E. This implies

$$(a,0,0)\cdot(l,m,n)=p \Rightarrow al=p \Rightarrow l=\frac{p}{a}$$

Similarly $m=p/b$ and $n=p/c$. Hence, from Eq. (6.7), the equation of the plane is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

Aliter: $P(x,y,z)$ is a point in the plane $E \Leftrightarrow$ the four points A,B,C and P are coplanar. This is possible if and only if the vectors $\overrightarrow{AP}$, $\overrightarrow{AB}$ and $\overrightarrow{AC}=0$. This means

$$[\overrightarrow{AP}\ \overrightarrow{AB}\ \overrightarrow{AC}]=0$$

$$\Rightarrow \begin{vmatrix} x-a & y & z \\ -a & b & 0 \\ -a & 0 & c \end{vmatrix}=0$$

$$\Rightarrow bc(x-a)+cay+abz=0$$

$$\Rightarrow bcx+cay+abz=abc$$

$$\Rightarrow \frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

THEOREM 6.47 A first-degree equation in x, y and z represents a plane.

PROOF Suppose

$$ax+by+cz+d=0 \tag{6.8}$$

is a first-degree equation in x, y and z. Since both equations $ax+by+cz+d=0$ and $-(ax+by+cz+d)=0$ represent the same surface (see Quick Look 6), we can suppose that $-d \geq 0$. Now, we can write Eq. (6.8) as

$$\frac{a}{\sqrt{a^2+b^2+c^2}}x+\frac{b}{\sqrt{a^2+b^2+c^2}}y+\frac{c}{\sqrt{a^2+b^2+c^2}}z=\frac{-d}{\sqrt{a^2+b^2+c^2}}$$

Take

$$l=\frac{a}{\sqrt{a^2+b^2+c^2}}$$

$$m=\frac{b}{\sqrt{a^2+b^2+c^2}}$$

and

$$n=\frac{c}{\sqrt{a^2+b^2+c^2}}$$

If $\vec{n}=(l,m,n)$, then $\vec{n}$ is a unit vector. If $\vec{r}=(x,y,z)$ and

$$p=\frac{-d}{\sqrt{a^2+b^2+c^2}}$$

then Eq. (6.8) becomes $\vec{r}\cdot\vec{n}=p$, which represents a plane. ■

QUICK LOOK 7

1. Since

$$(l,m,n)=\left(\frac{a}{\sqrt{a^2+b^2+c^2}},\frac{b}{\sqrt{a^2+b^2+c^2}},\frac{c}{\sqrt{a^2+b^2+c^2}}\right)$$

is a normal to the plane provided in Eq. (6.8), it follows that (a, b, c) are the DRs of the plane $ax+by+cz+d=0$.

2. $p=\dfrac{|d|}{\sqrt{a^2+b^2+c^2}}$ is the distance of the plane from the origin.

DEFINITION 6.22 If one of a, b and c is not zero, then the equation $ax+by+cz+d=0$ is called *general equation* of a plane.

THEOREM 6.48 Two first-degree equations

$$a_1x+b_1y+c_1z+d_1=0 \tag{6.9}$$

and

$$a_2x+b_2y+c_2z+d_2=0 \tag{6.10}$$

represent the same plane if and only if $a_1:b_1:c_1:d_1=a_2:b_2:c_2:d_2$.

PROOF Suppose $a_1:a_2=b_1:b_2=c_1:c_2=d_1:d_2=k\neq 0$, then

$$a_2x+b_2y+c_2z+d_2=0 \Leftrightarrow k(a_1x+b_1y+c_1z+d_1)=0$$

Therefore, Eqs. (6.9) and (6.10) represent the same plane.

Conversely, suppose Eqs. (6.9) and (6.10) represent the same plane, then (a_1,b_1,c_1) and (a_2,b_2,c_2) are the DRs of the normal to the plane [Quick Look 8, part (1)]. Therefore

$$(a_2,b_2,c_2)=k(a_1,b_1,c_1)$$

for some $k \neq 0$. If (x_1, y_1, z_1) is a point of the plane represented by Eqs. (6.9) and (6.10), we have

$$d_2 = -(a_2x_1 + b_2y_1 + c_2z_1) = -k(a_1x_1 + b_1y_1 + c_1z_1) = -kd_1$$

Therefore, $a_2 : a_1 = b_2 : b_1 = c_2 : c_1 = d_2 : d_1$.

QUICK LOOK 8

The equation of the plane passing through the point $A(x_1, y_1, z_1)$ which is having $\vec{n} = (a, b, c)$ as normal is $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$ (see Theorem 6.33). Hence, we can write the equation of the plane if we know one point in the plane and DRs of the normal to the plane.

THEOREM 6.49 The equation of the plane determined by three non-collinear points $A(x_1, y_1, z_1), B(x_2, y_2, z_2)$, and $C(x_3, y_3, z_3)$ is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

PROOF We have

$$\begin{aligned} & P(x, y, z) \text{ is a point in the plane } \overline{ABC} \\ \Leftrightarrow\ & \overrightarrow{AP}, \overrightarrow{AB}, \text{ and } \overrightarrow{AC} \text{ coplanar vectors} \\ \Leftrightarrow\ & \left[\overrightarrow{AP}, \overrightarrow{AB}, \overrightarrow{AC}\right] = 0 \\ \Leftrightarrow\ & \begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0 \end{aligned}$$

The points (x_r, y_r, z_r) (where $r = 1, 2, 3$ and 4) are coplanar if and only if

$$\begin{vmatrix} x_4 - x_1 & y_4 - y_1 & z_4 - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

THEOREM 6.50 The distance of the plane $ax + by + cz + d = 0$ from a point $A(x_1, y_1, z_1)$ is

$$\frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

PROOF Let E be the given plane and $\vec{r} \cdot \vec{n} = p$ be its equation. Let $B(\vec{b})$ be a point which is not equal to N (see Fig. 6.30) where N is the foot of the perpendicular from A onto E so that $\vec{b} \cdot \vec{n} = p$. Therefore

$$\begin{aligned} AN = \left|\text{Projection of } \overline{AB} \text{ on } \overline{AN}\right| &= \frac{\left|\overrightarrow{AB} \cdot \overrightarrow{AN}\right|}{\left|\overrightarrow{AN}\right|} \\ &= \frac{\left|(\vec{b} - \vec{a}) \cdot \vec{n}\right|}{\vec{n}} \\ &= \frac{\left|\vec{p} - \vec{a} \cdot \vec{n}\right|}{|\vec{n}|} \end{aligned}$$

Now, $\vec{a}=(x_1,y_1,z_1)$, $\vec{n}=(a,b,c)$ and $p=-d$ imply

$$AN=\frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$$

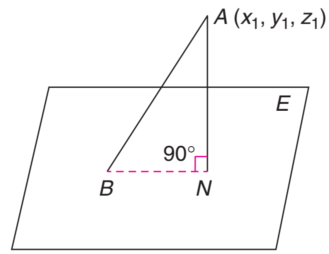


FIGURE 6.30

THEOREM 6.51 The equation of any two parallel planes are of the form $ax+by+cz+d_1=0$ and $ax+by+cz+d_2=0$ and the distance between them is equal to

$$\frac{|d_1-d_2|}{\sqrt{a^2+b^2+c^2}}$$

PROOF Let

$$E_1\equiv a_1x+b_1y+c_1z+d_1=0$$

and
$$E_2\equiv a_2x+b_2y+c_2z+d_2=0$$

be two parallel planes so that $\vec{n}_1=(a_1,b_1,c_1)$ and $\vec{n}_2=(a_2,b_2,c_2)$ are the normal vectors to E_1 and E_2, respectively (see Quick Look 7). Since E_1 and E_2 are parallel, their normals $\vec{n}_1$ and $\vec{n}_2$ are parallel. Hence, let $\overrightarrow{n_2}=k\overrightarrow{n_1}$ so that $a_2=ka_1, b_2=kb_1$ and $c_2=kc_1$. Thus, the equation of E_2 is $k(a_1x+b_1y+c_1z)+d_2=0$. Dividing by k, we have $E_2\equiv a_1x+b_1y+c_1z+d_2=0$ (here, in the place of d_2/k, we take d_2). Let $E_1\equiv ax+by+cz+d_1=0$ and $E_2\equiv ax+by+cz+d_2=0$ be the two parallel planes. Let $A(x_1,y_1,z_1)$ be a point in E_2 so that

$$ax_1+by_1+cz_1+d_2=0 \tag{6.11}$$

Now,

$$\begin{aligned}\text{Distance between } E_1 \text{ and } E_2 &= \text{Distance of } E_1 \text{ from point } A\\ &=\frac{|ax_1+by_1+cz_1+d_1|}{\sqrt{a^2+b^2+c^2}} \qquad \text{(Theorem 6.49)}\\ &=\frac{|-d_2+d_1|}{\sqrt{a^2+b^2+c^2}} \qquad \text{[by Eq. (6.11)]}\end{aligned}$$

Notation: If E denotes $ax+by+cz+d$, then we denote $ax_1+by_1+cz_1+d$ by E_{11} and $ax_2+by_2+cz_2+d_2$ by E_{22}.

THEOREM 6.52 The ratio in which the plane $E\equiv ax+by+cz+d=0$ divides the segment joining the points $A(x_1,y_1,z_1)$ and $B(x_2,y_2,z_2)$ is $-E_{11}:E_{22}$.

PROOF See Fig. 6.31. Let $E\equiv ax+by+cz+d=0$ be the plane meeting the line AB at P and suppose $AP:PB=\lambda_1:\lambda_2$ so that

$$P=\left(\frac{\lambda_1x_2+\lambda_2x_1}{\lambda_1+\lambda_2},\frac{\lambda_1y_2+\lambda_2y_1}{\lambda_1+\lambda_2},\frac{\lambda_1z_2+\lambda_2z_1}{\lambda_1+\lambda_2}\right)$$

Since $P \in E,$ we have

$$a\left(\frac{\lambda_1 x_2+\lambda_2 x_1}{\lambda_1+\lambda_2}\right)+b\left(\frac{\lambda_1 x_2+\lambda_2 x_1}{\lambda_1+\lambda_2}\right)+c\left(\frac{\lambda_1 x_2+\lambda_2 x_1}{\lambda_1+\lambda_2}\right)+d=0$$
$$\Rightarrow \lambda_1(ax_2+by_2+cz_2+d_2)+\lambda_2(ax_1+by_1+cz_1+d)=0$$
$$\Rightarrow \frac{\lambda_1}{\lambda_2}=-\frac{(ax_1+by_1+cz_1+d)}{ax_2+by_2+cz_2+d}=\frac{-E_{11}}{E_{22}}$$

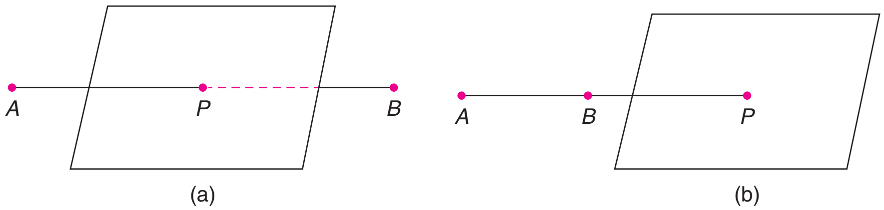


FIGURE 6.31

Note

1. Points A and B lie on the opposite sides of the plane $E=0 \Leftrightarrow$ the ratio $-E_{11}{:}E_{22}$ is positive $\Leftrightarrow E_{11}$ and E_{22} should have opposite signs. A and B lie on the same side of $E \Leftrightarrow$ the ratio $-E_{11}{:}E_{22} < 0 \Leftrightarrow E_{11}$ and E_2 should have the same sign.
2. Let $O=(0, 0, 0)$, $A=(x_1, y_1, z_1)$ and $E \equiv ax+by+cz+d=0,$ then A lies on non-origin side of the plane $E=0 \Leftrightarrow$ E_{11} and $E_{22}=d$ have opposite signs and A lies on the origin side $\Leftrightarrow E_{11},\ d$ have the same sign.

For example, if $E \equiv x+2y+3z-1$ and $A=(1,2,3),$ then $E_{11}=1+4+6-1=10$ and $d=-1$ so that the point A lies on the non-origin side of $E=0$.

6.3.1 System of Planes

We have discussed that the first-degree equation $ax+by+cz+d=0$ represents a plane. This equation is called *general equation of a plane*. Since one of a, b and c is not equal to zero, we can consider $a \neq 0$. Then, the plane equation can be written as

$$x+\frac{b}{a}y+\frac{c}{a}z+\frac{d}{a}=0$$

Hence, if the values of $b/a, c/a$ and d/a are given, then we can determine the plane equation. If the three ratios $b/a, c/a$ and d/a are to be determined uniquely, we need three conditions. If lesser number of conditions is given, then the ratios $b/a, c/a$ and d/a can be determined in many ways. The planes so obtained are called *system of planes*.

For example, if we fix a, b and c and vary d, then $ax+by+cz+d=0$ represents system of parallel planes. For all those planes (a, b, c) is the normal. If (x_1, y_1, z_1) is a fixed point, then for different values of a, b and c, the equation

$$a(x-x_1)+b(y-y_1)+c(z-z_1)=0$$

represents the planes passing through the point (x_1, y_1, z_1).

THEOREM 6.53 Let $E_1 \equiv a_1x+b_1y+c_1z+d_1=0,\ E_2 \equiv a_2x+b_2y+c_2z+d_2=0$ be the two intersecting planes which intersect in a line L (since there are two planes, when they intersect, they intersect in a line). If λ_1 and λ_2 are any two real numbers such that at least one of them is not zero, then the equation $\lambda_1E_1+\lambda_2E_2=0$ represents plane passing through the line L.

PROOF Let

$$E \equiv \lambda_1E_1+\lambda_2E_2=(\lambda_1a_1+\lambda_2a_2)x+(\lambda_1b_1+\lambda_2b_2)y+(\lambda_1c_1+\lambda_2c_2)z+(\lambda_1d_1+\lambda_2d_2)=0$$

Now

$$\lambda_1a_1+\lambda_2a_2=\lambda_1b_1+\lambda_2b_2=\lambda_1c_1+\lambda_2c_2=0$$
$$\Rightarrow a_1:a_2=b_1:b_2=c_1:c_2=-\lambda_1:\lambda_2$$

Therefore, $E_1 = 0$ and $E_2 = 0$ are parallel planes, which is not true in this case. Therefore

$$E \equiv \lambda_1 E_1 + \lambda_2 E_2 = 0$$

is a first-degree equation and hence it represents a plane. Since the line L is contained in both $E_1 = 0$ and $E_2 = 0$, it is also contained in the plane represented by $\lambda_1 E_1 + \lambda_2 E_2 = 0$. ■

QUICK LOOK 9

$\lambda_1 E_1 + \lambda_2 E_2 = 0$ represents $E_1 = 0$, if $\lambda_2 = 0$ and $\lambda_1 = 1$, and represents $E_2 = 0$, if $\lambda_1 = 0$ and $\lambda_2 = 1$. Hence, we can take the equation of any plane other than E_1 and E_2 passing through the line L as $E_1 + \lambda E_2 = 0$ (λ is a parameter).

Theorem 6.54 is the converse of Theorem 6.53.

THEOREM 6.54 If $E_1 \equiv a_1x + b_1y + c_1z + d_1 = 0, E_2 \equiv a_2x + b_2y + c_2z + d_2 = 0$ are two intersecting planes which intersect in a line L, then the equation of any plane passing through line L is of the form $\lambda_1 E_1 + \lambda_2 E_2 = 0$ where $|\lambda_1| + |\lambda_2| \neq 0$.

PROOF Suppose $E \equiv px + qy + rz + t = 0$ is a plane containing the line L. Let $\vec{n}_1 = (a_1, b_1, c_1)$ and $\vec{n}_2 = (a_2, b_2, c_2)$ be the normals to $E_1 = 0$ and $E_2 = 0$, respectively. Hence, $\vec{n}_1 \times \vec{n}_2$ is parallel to the line L so that $(b_1c_2 - b_2c_1, c_1a_2 - c_2a_1, a_1b_2 - a_2b_1)$ are the DRs of the line L. Since one of the DRs is not zero, we consider that $a_1b_2 - a_2b_1 \neq 0$. Therefore, the equations $a_1x + a_2y = p$ and $b_1x + b_2y = q$ are uniquely solvable for a non-zero solution and let $x = \lambda_1$ and $y = \lambda_2$ be the solution. Therefore, $a_1\lambda_1 + a_2\lambda_2 = p$ and $b_1\lambda_1 + b_2\lambda_2 = q$. Let $k = r - \lambda_1c_1 - \lambda_2c_2$ and $l = t - \lambda_1d_1 - \lambda_2d_2$ so that

$$E \equiv \lambda_1(a_1x + b_1y + c_1z + d_1) + \lambda_2(a_2x + b_2y + c_2z + d_2) + kz + l = 0 \tag{6.12}$$

Suppose $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$, where $z_1 \neq z_2$, are points on line L. Therefore,

$$a_1x_1 + b_1y_1 + c_1z_1 + d_1 = 0$$

and

$$a_2x_1 + b_2y_1 + c_2z_1 + d_2 = 0$$

so that point $P(x_1, y_1, z_1)$ satisfies the equation $E = 0$ and similarly point $Q(x_2, y_2, z_2)$ satisfies $E = 0$ [from Eq. (6.12)]. Therefore, $kz_1 + l = 0, kz_2 + l = 0$ and $z_1 \neq z_2 \Rightarrow k = 0$ and then $l = 0$. Hence, the equation of the plane $E = 0$ is of the form $\lambda_1 E_1 + \lambda_2 E_2 = 0$. ■

THEOREM 6.55 Let $E_1 \equiv a_1x + b_1y + c_1z + d_1 = 0$ and $E_2 \equiv a_2x + b_2y + c_2z + d_2 = 0$ be the two intersecting planes. Then, the equations of the planes bisecting between the angles are

$$\frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a^2 + b^2 + c^2}} = \pm \frac{a_2x + b_2y + c_2z + d_2}{\sqrt{a^2 + b^2 + c^2}}$$

PROOF We have

$P(x, y, z)$ is a point on a bisecting plane of the given planes

$\Leftrightarrow$ The distance of P from both the planes are equal

$$\Leftrightarrow \frac{|a_1x + b_1y + c_1z + d_1|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|a_2x + b_2y + c_2z + d_2|}{\sqrt{a^2 + b^2 + c^2}} \quad \text{(by Theorem 6.12)}$$

$$\Leftrightarrow \frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a^2 + b^2 + c^2}} = \pm \frac{a_2x + b_2y + c_2z + d_2}{\sqrt{a^2 + b^2 + c^2}}$$

■

QUICK LOOK 10

To determine the plane bisecting the acute angle between the planes, consider one of the planes E_1 or E_2 and one of the bisecting planes, say, E. If the angle between the plane E and one of E_1 and E_2 and one of the bisecting planes, say, is less than $\text{Tan}^{-1} 1$, then E is the acute angle bisector plane.

Note: As in the case of straight lines, on the similar lines, we can find acute and obtuse angle bisecting planes and plane bisecting the origin angle (see Theorems 2.25 and 2.26 and Quick Look 14, Chapter 2).

Example 6.19

Find a point on x-axis which is equidistant from points $A(4,-3,7)$ and $B(2,-1,1)$ and also find the equation of the plane passing through this point and perpendicular to the line AB.

Solution: Let $P(x,0,0)$ be a point on the x-axis. Therefore,

$$PA = PB$$
$$\Rightarrow (PA)^2 = (PB)^2$$
$$\Rightarrow (x-4)^2 + (-3)^2 + 7^2 = (x-2)^2 + (-1)^2 + 1^2$$
$$\Rightarrow 4x = 68$$
$$\Rightarrow x = 17$$

Hence, $P(17, 0, 0)$. Now, from Quick Look 8, the equation of the plane passing through $P(17, 0, 0)$ and perpendicular to $\overrightarrow{AB} = (-2, 2, -6)$ is $-2(x-17) + 2(y-0) - 6(x-0) = 0$

$$\Rightarrow 2x - 2y + 6z - 34 = 0$$
$$\Rightarrow x - y + 3z - 17 = 0$$

Example 6.20

Find the equation of the plane passing through the point $(-1, 3, 2)$ and perpendicular to both the planes $E_1 \equiv x + 2y + 2z - 5 = 0$ and $E_2 \equiv 3x + 3y + 2z - 8 = 0$.

Solution: Let $E = 0$ be the required plane. Since E is perpendicular to both $E_1 = 0$ and $E_2 = 0$, its normal is parallel to the cross product of the normals of $E_1 = 0$ and $E_2 = 0$. Therefore, the normal of E is

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 2 \\ 3 & 3 & 2 \end{vmatrix} = -2\vec{i} + 4\vec{j} - 3\vec{k}$$

Therefore

$$E \equiv -2(x+1) + 4(y-3) - 3(z-2) = 0$$
$$\equiv -2x + 4y - 3z - 8 = 0$$
$$\equiv 2x - 4y + 3z + 8 = 0$$

Example 6.21

Two sets of axes have the same origin. If a plane makes intercepts a, b and c on one set of the axes and a_1, b_1 and c_1 on another set of the axes, then show that $a^{-2} + b^{-2} + c^{-2} = a_1^{-2} + b_1^{-2} + c_1^{-2}$.

Solution: Let E be the given plane. Therefore, its equations are

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

and

$$\frac{x}{a_1} + \frac{y}{b_1} + \frac{z}{c_1} = 1$$

Since the perpendicular distance from origin onto the plane is the same, we have (see Theorem 6.50)

$$\frac{|-1|}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}} = \frac{|-1|}{\sqrt{\frac{1}{a_1^2} + \frac{1}{b_1^2} + \frac{1}{c_1^2}}}$$

$$\Rightarrow \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{a_1^2} + \frac{1}{b_1^2} + \frac{1}{c_1^2}$$

Example 6.22

Find the equation of the plane passing through the point $(2,-1,3)$ and parallel to the plane $E_1 \equiv 3x+4y-7z+5=0$.

Solution: Let E be the required plane. Since E is parallel to the plane $E_1=0$, E is of the form

$$3x+4y-7z+d=0$$

Since it passes through $(2,-1,3)$, we have

$$3(2)+4(-1)-7(3)+d=0$$

$$\Rightarrow d=19$$

So

$$E \equiv 3x+4y-7z+19=0$$

Example 6.23

A plane intersects the coordinate axes in A, B and C, respectively. If (α,β,γ) is the centroid of ΔABC, then show that the equation of the plane

$$\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}=3$$

Solution: Let $A=(a,0,0)$, $B=(0,b,0)$ and $C=(0,0,c)$ so that

$$G=\left(\frac{a}{3},\frac{b}{3},\frac{c}{3}\right)$$

and the equation of the plane is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \tag{6.13}$$

Since (α,β,γ) is the centroid, we have

$$\alpha=\frac{a}{3},\ \beta=\frac{b}{3},\ \gamma=\frac{c}{3}$$

$$\Rightarrow a=3\alpha,\ b=3\beta,\ c=3\gamma$$

so that from Eq. (6.13), the equation of the planes is

$$\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}=3$$

Example 6.24

If the sum of the squares of the intercepts of a variable plane on the coordinate axis is constant k^2, then show that the locus of the foot of the perpendicular drawn from the origin onto the plane is

$$(x^2+y^2+z^2)^2(x^{-2}+y^{-2}+z^{-2})=k^2$$

Solution: Let E be the variable plane and a, b and c be the intercepts of the plane E on the coordinate axes. Therefore, by hypothesis, we have

$$a^2+b^2+c^2=k^2 \tag{6.14}$$

Let $P(x_1,y_1,z_1)$ be the foot of the perpendicular from the origin onto E. Hence, by Quick Look 8, the equation of the plane is

$$x_1(x-x_1)+y_1(y-y_1)+z_1(z-z_1)=0$$

$$\Rightarrow xx_1+yy_1+zz_1=x_1^2+y_1^2+z_1^2$$

$$\Rightarrow \frac{x}{(x_1^2+y_1^2+z_1^2)/x_1}+\frac{y}{(x_1^2+y_1^2+z_1^2)/y_1}+\frac{z}{(x_1^2+y_1^2+z_1^2)/z_1}=1$$

Therefore, the intercepts of the plane on the axes are

$$\frac{x_1^2+y_1^2+z_1^2}{x_1},\frac{x_1^2+y_1^2+z_1^2}{y_1},\frac{x_1^2+y_1^2+z_1^2}{z_1}$$

Hence by Eq. (6.14), we have

$$\frac{(x_1^2+y_1^2+z_1^2)^2}{x_1^2}+\frac{(x_1^2+y_1^2+z_1^2)^2}{y_1^2}+\frac{(x_1^2+y_1^2+z_1^2)^2}{z_1^2}=k^2$$

Therefore, the locus is

$$(x^2+y^2+z^2)^2\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right)=k^2$$

Example 6.25

A variable plane is moving such that its distance from the origin is $3p$ (constant) and meets the axes at A, B and C, respectively. Show that the locus of the centroid of ΔABC is $x^{-2}+y^{-2}+z^{-2}=p^{-2}$.

Solution: Let $A=(a,0,0), B=(0,b,0), C=(0,0,c)$. Let $G=(x_1, y_1, z_1)$ be the centroid so that

$$x_1=\frac{a}{3}, y_1=\frac{b}{3}, z_1=\frac{c}{3}$$

The equation of the plane is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

so that

$$\frac{x}{3x_1}+\frac{y}{3y_1}+\frac{z}{3z_1}=1$$

By hypothesis, we have

$$\frac{|-1|}{\sqrt{(1/9x_1^2)+(1/9y_1^2)+(1/9z_1^2)}}=3p$$

$$\Rightarrow \frac{1}{9x_1^2}+\frac{1}{9y_1^2}+\frac{1}{9z_1^2}=\frac{1}{9p^2}$$

Therefore, the locus of $G(x_1, y_1, z_1)$ is

$$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{1}{p^2}$$

Example 6.26

A variable plane is passing through a fixed point (a, b, c) and meeting the coordinate axes at A, B and C. Then show that the locus of the point of intersection of the planes through A, B and C and parallel to the coordinate planes is

$$\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=1$$

Solution: Let $A=(\alpha,0,0), B=(0,\beta,0)$ and $C=(0,0,\gamma)$ so that equation of the plane is

$$\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}=1$$

Since this plane is passing through (a, b, c), we have

$$\frac{a}{\alpha}+\frac{b}{\beta}+\frac{c}{\gamma}=1 \tag{6.15}$$

Now, the equation of the plane passing through $A(\alpha,0,0)$ and parallel to yz-plane is $x=\alpha$. Similarly, the other two planes are $y=\beta$ and $z=\gamma$. Since the intersection of these planes is (α,β,γ), from Eq. (6.15), the locus of the point (α, β, γ) is

$$\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=1$$

Example 6.27

P is a variable point in the plane

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

The plane passing through P and perpendicular to $\overrightarrow{OP}$ (O is the origin) meets the coordinate axes at A, B and C. Show that the equation to the locus of the point of intersection of the planes through A, B and C and parallel to the coordinate planes is

$$x^{-2}+y^{-2}+z^{-2}=(ax)^{-1}+(by)^{-1}+(cz)^{-1}$$

Solution: Suppose $P=(\alpha,\beta,\gamma)$. Therefore, the equation of the plane passing through P and perpendicular to $\overrightarrow{OP}$ is

$$\alpha(x-\alpha)+\beta(y-\beta)+\gamma(x-\gamma)=0$$

$$\Rightarrow \alpha x+\beta y+\gamma z=\alpha^2+\beta^2+\gamma^2 \tag{6.16}$$

Since P belongs to the plane

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

we have

$$\frac{\alpha}{a}+\frac{\beta}{b}+\frac{\gamma}{c}=1 \tag{6.17}$$

The plane represented by Eq. (6.16) meets the coordinates axes at

$$A\left(\frac{\alpha^2+\beta^2+\gamma^2}{\alpha},0,0\right)$$

$$B\left(0,\frac{\alpha^2+\beta^2+\gamma^2}{\beta},\gamma\right)$$

and $$C\left(0,0,\frac{\alpha^2+\beta^2+\gamma^2}{\gamma}\right)$$

Hence, the equations of the planes passing through A, B and C and parallel to coordinate planes are

$$x=\frac{\alpha^2+\beta^2+\gamma^2}{\alpha}$$

$$y=\frac{\alpha^2+\beta^2+\gamma^2}{\beta}$$

and
$$z=\frac{\alpha^2+\beta^2+\gamma^2}{\gamma}$$

Suppose the point of intersection of these three planes is $Q(x_1, y_1, z_1)$ so that

$$x_1=\frac{\alpha^2+\beta^2+\gamma^2}{\alpha}$$

$$y_1=\frac{\alpha^2+\beta^2+\gamma^2}{\beta}$$

and
$$z_1=\frac{\alpha^2+\beta^2+\gamma^2}{\gamma}$$

Therefore, from Eq. (6.17), we have

$$\frac{1}{ax_1}+\frac{1}{by_1}+\frac{1}{cz_1}=\frac{(\alpha/a)+(\beta/b)+(\gamma/c)}{\alpha^2+\beta^2+\gamma^2}=\frac{1}{\alpha^2+\beta^2+\gamma^2}$$

Also

$$\frac{1}{x_1^2}+\frac{1}{y_1^2}+\frac{1}{z_1^2}=\frac{\alpha^2+\beta^2+\gamma^2}{(\alpha^2+\beta^2+\gamma^2)^2}=\frac{1}{\alpha^2+\beta^2+\gamma^2}$$

Therefore

$$\frac{1}{x_1^2}+\frac{1}{y_1^2}+\frac{1}{z_1^2}=\frac{1}{ax_1}+\frac{1}{by_1}+\frac{1}{cz_1}$$

Hence, the locus of $Q(x_1, y_1, z_1)$ is

$$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{1}{ax}+\frac{1}{by}+\frac{1}{cz}$$

Example 6.28

A plane E is intersecting three mutually perpendicular planes on the sides AB, BC and CA of ΔABC. If the angles between the plane E and the three planes are α, β and γ, respectively, then prove that $\cos^2\gamma=\cot B\cot C$, $\cos^2\beta=\cot C\cot A$ and $\cos^2\alpha=\cot A\cot B$.

Solution: Without loss of generality, we can consider that the three mutually perpendicular planes are the coordinate planes. Suppose the equation of the plane E is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

so that $A=(a,0,0)$, $B=(0,b,0)$ and $C=(0,0,c)$. Hence, the DRs of $\overline{AB}$ and $\overline{AC}$, respectively, are $(-a, b, 0)$ and $(-a, 0, c)$. Therefore, from the dot product and cross product of vectors, we have

$$\tan A=\frac{\sqrt{b^2c^2+c^2a^2+a^2b^2}}{a^2}$$

$$\Rightarrow \cot A=\frac{a^2}{\sqrt{b^2c^2+c^2a^2+a^2b^2}}$$

Similarly

$$\cot B=\frac{b^2}{\sqrt{b^2c^2+c^2a^2+a^2b^2}}$$

and
$$\cot C=\frac{c^2}{\sqrt{b^2c^2+c^2a^2+a^2b^2}}$$

If α is the angle between the plane E and yz-plane, then

$$\cos\alpha=\frac{(1/a)+0+0}{\sqrt{(1/a^2)+(1/b^2)+(1/c^2)}}=\frac{bc}{\sqrt{b^2c^2+c^2a^2+a^2b^2}}$$

Therefore

$$\cos^2\alpha=\frac{b^2c^2}{b^2c^2+c^2a^2+a^2b^2}=\cot B\cot C$$

Similarly, we can prove the other parts.

Example 6.29

Let $E_1\equiv 2x-7y+4z-3=0$ and $E_2\equiv 3x-5y+4z+11=0$ be two intersecting planes. Find the equation of the plane passing through the intersecting line of $E_1=0$ and $E_2=0$ and also through the point $(-2, 1, 3)$.

Solution: Let the required plane be E. Then by Theorem 6.54, we have $E=E_1+\lambda E_2$ for some λ. That is,

$$E\equiv(2x-7y+4z-3)+\lambda(3x-5y+4z+11)=0$$

Now, E passes through the point $(-2,1,3)$. So

$$(-4-7+12-3)+\lambda(-6-5+12+11)=0$$
$$\Rightarrow -2+12\lambda=0$$
$$\Rightarrow \lambda=\frac{1}{6}$$

Therefore, the plane

$$E\equiv(2x-7y+4z-3)+\frac{1}{6}(3x-5y+4z+11)=0$$
$$\Rightarrow E\equiv 15x-47y+28z-7=0$$

Example 6.30

Find the equation of the plane passing through the intersection of the planes $E_1\equiv x+2y+3z-4=0$ and $E_2\equiv 2x+y-z+5=0$ and perpendicular to the plane $5x+3y+6z+8=0$.

Solution: Let $E\equiv E_1+\lambda E_2=0$ be the required plane. That is,

$$E\equiv(1+2\lambda)x+(2+\lambda)y+(3-\lambda)z-4+5\lambda=0$$

$E=0$ is perpendicular to the plane $5x+3y+6z+8=0$. So their normals are at right angles. This means

$$5(1+2\lambda)+3\cdot(2+\lambda)+6(3-\lambda)=0$$
$$\Rightarrow 7\lambda=-29$$
$$\Rightarrow \lambda=\frac{-29}{7}$$

Therefore, $E\equiv 51x+15y-50z+173=0$.

Example 6.31

Let $E_1\equiv x+2y+2z-9=0$ and $E_2\equiv 4x-3y+12z+13=0$ be two planes. Find the following planes: (a) The bisecting planes of the angle between E_1 and E_2. (b) The acute angle bisector plane of E_1 and E_2. (c) The angle which contains the origin.

Solution:

(a) Bisecting planes are

$$\frac{x+2y+2z-9}{\sqrt{1^2+2^2+2^2}}=\pm\frac{4x-3y+12z+13}{\sqrt{4^2+(-3)^2+12^2}}$$

That is,

$$x+35y-10z-156=0 \qquad (6.18)$$

and

$$25x-17y+62z-78=0 \qquad (6.19)$$

(b) Let θ be the angle between E_1 and the plane provided in Eq. (6.18). Therefore

$$\cos\theta=\frac{1(1)+2(35)+2(-10)}{\sqrt{1^2+2^2+2^2}\sqrt{1^2+35^2+10^2}}=\frac{17}{\sqrt{1326}}$$

$$\Rightarrow \sec\theta=\frac{\sqrt{1326}}{17}$$

Now

$$\tan^2\theta=\sec^2\theta-1=\frac{1326}{289}-1=\frac{1027}{289}$$
$$\Rightarrow \tan\theta=\frac{\sqrt{1027}}{17}>1$$

Therefore, Eq. (6.18) represents the obtuse angle bisecting plane and hence Eq. (6.19) represents acute angle bisecting plane.

(c) The constant terms of the given planes are -9 and 13. Multiply $E_1=0$ with (-1) so that

$$E_1\equiv -x-2y-2z+9=0$$

and

$$E_2\equiv 4x-3y+12z+13=0$$

Now,

$$a_1a_2+b_1b_2+c_1c_2=-4+6-24=-18<0$$

Hence, the origin lies in the acute angle region.

6.4 | Line

It is said that if two planes intersect, they intersect in a straight line (Axiom 6.4) and the equation of a plane is a first-degree equation in x, y and z. Hence, a straight line can be represented by a pair of intersecting planes. For example, x-axis is the intersection of xy-plane and zx-plane and hence x-axis equations are $y=0$ and $z=0$. Similarly, y-axis equations are $z=0$ and $x=0$ and z-axis equations are $x=0$ and $y=0$.

THEOREM 6.56 (PARAMETRIC FORM OF A STRAIGHT LINE) The equations of the line passing through the point $A(x_1, y_1, z_1)$ (see Fig. 6.32) and having the DCs (l, m, n) are $x = x_1 + lr$, $y = y_1 + mr$ and $z = z_1 + nr$, where r is real.

PROOF It is known that the equation of the line passing through the point $\vec{a}$ and parallel to a vector $\vec{b}$ is $\vec{r} = \vec{a} + t\vec{b}, t \in \mathbb{R}$ (see Theorem 5.27, p. 333, Vol. 2). Now, take $\vec{a} = (x_1, y_1, z_1) = x_1\vec{i} + y_1\vec{j} + z_1\vec{k}$ and $\vec{b} = (l, m, n) = l\vec{i} + m\vec{j} + n\vec{k}$. Let $\vec{r} = (x, y, z) = x\vec{i} + y\vec{j} + z\vec{k}$ be any point on the line. Then

$$x\vec{i} + y\vec{j} + z\vec{k} = (x_1\vec{i} + y_1\vec{j} + z_1\vec{k}) + t(l\vec{i} + m\vec{j} + n\vec{k})$$

$$\Rightarrow x = x_1 + tl,\ y = y_1 + tm,\ z = z_1 + tn$$

Replacing t by r, we have

$$x = x_1 + lr,\ y = y_1 + mr \text{ and } z = z_1 + nr$$

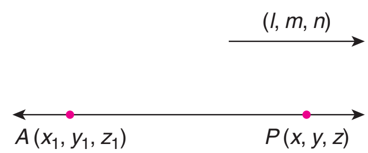


FIGURE 6.32

Before pronouncing the symmetric form of a line we adopt the following convention.

Convention: Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be real numbers. We write

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_n}{b_n}$$

only to mean that $a_1 : a_2 : a_3 : \cdots : a_n = b_1 : b_2 : b_3 : \cdots : b_n$. If some b_i's are zeros, then we mean that the corresponding a_i's are zeros. In this case the parametric form of the line can be written as

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$$

which is called the *symmetric form of a line*.

QUICK LOOK 11 (SYMMETRIC FORM OF A LINE)

The equation of the line passing through the point (x_1, y_1, z_1) having (l, m, n) as DCs is

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$$

Here, DCs may be replaced DRs as well. Also

$$(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 = r^2$$

Hence, $|r|$ gives us the distance of (x, y, z) from the given point (x_1, y_1, z_1).

THEOREM 6.57 If $E_1 \equiv a_1x + b_1y + c_1z + d_1 = 0$ and $E_2 \equiv a_2x + b_2y + c_2z + d_2 = 0$ be the two intersecting planes where L is their line of intersection, then determine the symmetric form of the line L.

PROOF Let $\vec{n}_1 = (a_1, b_1, c_1)$ and $\vec{n}_2 = (a_2, b_2, c_2)$ which are the normals of $E_1 = 0$ and $E_2 = 0$, respectively. Hence, the line L is parallel to $\vec{n}_1 \times \vec{n}_2$ so that the DRs of the line L are $(b_1c_2 - b_2c_1, c_1a_2 - c_2a_1, a_1b_2 - a_2b_1)$. Now, we find a point on L. Since one of the DRs is not zero, we consider that $(a_1b_2 - a_2b_1) \neq 0$. Therefore, the equations $a_1x + b_1y = -d_1$ and $a_2x + b_2y = -d_2$ have unique solutions

$$x = \frac{b_1 d_2 - b_2 d_1}{a_1 b_2 - a_2 b_1}$$

and
$$y = \frac{d_1 a_2 - d_2 a_1}{a_1 b_2 - a_2 b_1}$$

so that the point $(x, y, 0)$ lies on L. Therefore, the symmetric form of L is

$$\frac{x - [(b_1 d_2 - b_2 d_1)/(a_1 b_2 - a_2 b_1)]}{b_1 c_2 - b_2 c_1} = \frac{y - [(d_1 a_2 - d_2 a_1)/(a_1 b_2 - a_2 b_1)]}{c_1 a_2 - c_2 a_1} = \frac{z - 0}{a_1 b_2 - a_2 b_1}$$

THEOREM 6.58 The plane $E \equiv ax + by + cz + d = 0$ contains the line

$$L: \frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$$

if and only if (a) $al + bm + cn = 0$ and (b) $ax_1 + by_1 + cz_1 + d = 0$.

PROOF Suppose the line L is contained in the given plane. Since (x_1, y_1, z_1) lies on L and L is contained in $E = 0$ we have

$$ax_1 + by_1 + cz_1 + d = 0$$

Also the normal (a, b, c) of $E = 0$ is perpendicular to the line L. This implies

$$(a, b, c) \cdot (l, m, n) = 0$$
$$\Rightarrow al + bm + cn = 0$$

Conversely, assume that

$$ax_1 + by_1 + cz_1 + d = 0 \tag{6.20}$$

and
$$al + bm + cn = 0 \tag{6.21}$$

Since one of a, b and c is not zero, we have that the vector (a, b, c) is normal to the plane $E \equiv ax + by + cz + d = 0$. Also from Eqs. (6.20) and (6.21), we have that (x_1, y_1, z_1) lies in $E = 0$ and $al + bm + cn = 0$ which implies that (l, m, n) is perpendicular to the normal (a, b, c) or $E = 0$. Hence, L must lie in the plane $E = 0$.

QUICK LOOK 12

1. To show that a line lies in a plane, it is enough if we show one point of the line belongs to the plane and the normal to the plane is perpendicular to the line.
2. If $al + bm + cn = 0$ then the equation of the plane containing the line

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$$

is

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

In the relation $al + bm + cn = 0$, if a, b and c vary, then we obtain system of planes containing the line.

6.4.1 Discussion on Nature of Three Planes

If Π_1, Π_2 and Π_3 are three planes, any one of the following three cases may arise:

Case 1: All three planes Π_1, Π_2 and Π_3 are parallel to each other (i.e., their normals are parallel).

Case 2: Two planes are parallel and the third plane intersects them in parallel lines (see Fig. 6.3, Theorem 6.12).

Case 3: No two planes of Π_1, Π_2 and Π_3 are parallel. In this case, there are three sub-cases:

1. All three planes interest at a point. That is, the three lines of intersection of the planes, which are taken pairwise, are concurrent [see Fig 6.6(a), Theorem 6.18]. Example: every corner of a room is a point of intersection of three planes.
2. All three planes intersect in a single line [see Fig 6.6(b), Theorem 6.18].
3. Every two planes intersect in a line which is parallel to the third plane. In this case, the planes form a triangular prism [see Fig 6.6(c), Theorem 6.18]. That is, if $\Pi_1 \cap \Pi_2 = L_1, \Pi_2 \cap \Pi_3 = L_2$ and $\Pi_3 \cap \Pi_1 = L_3$, then $L_1 \parallel \Pi_3, L_2 \parallel \Pi_1$ and $L_3 \parallel \Pi_2$. Suppose the equation of the planes Π_1, Π_2 and Π_3 are

$$E_1 \equiv a_1x + b_1y + c_1z + d_1 = 0 \tag{6.22}$$

$$E_2 \equiv a_2x + b_2y + c_2z + d_2 = 0 \tag{6.23}$$

$$E_3 \equiv a_3x + b_3y + c_3z + d_3 = 0 \tag{6.24}$$

If $a_1 : a_2 : a_3 := b_1 : b_2 : b_3 = c_1 : c_2 : c_3$, then the three planes are parallel. Suppose Π_1 and Π_2 are parallel and they interest Π_3 so that L_2 and L_3 are parallel. Then, $E_2 = 0 = E_3$ represents L_2 and $E_1 = 0 = E_3$ represents L_3.

In the above three cases, let us discuss Case 3 (when no two planes out of Π_1, Π_2 and Π_3 are parallel) in detail. Let us consider that

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$\Delta_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

$$\Delta_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$$

and

$$\Delta_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

1. Suppose $\Delta \neq 0$. Then, by Cramer's rule (Theorem 8.46, p. 413, Vol. 1), the three Eqs. (6.22)–(6.24) have unique solution and hence the three planes are concurrent at a point.
2. Suppose $\Delta = 0$ and $\Delta_3 = 0$. Since $\Pi_2 \cap \Pi_3 = L_2$ the DRs of L_2 are

$$(a_2, b_2, c_2) \times (a_3, b_3, c_3) = (b_2c_3 - b_3c_2, c_2a_3 - a_2c_3, a_2b_3 - a_3b_2)$$

Now,

$$\Delta = 0 \Rightarrow a_1(b_2c_3 - b_3c_2) + b_1(c_2a_3 - c_3a_2) + c_1(a_2b_3 - a_3b_2) = 0$$

So the line L_2 is parallel to the plane Π_1.

Also if $a_2b_3 - a_3b_2 \neq 0$, then by Theorem 6.57, the point

$$\left(\frac{b_2d_3 - b_3d_2}{a_2b_3 - a_3b_2}, \frac{d_2a_3 - d_3a_2}{a_2b_3 - a_3b_2}, 0 \right)$$

lies on L_2. Therefore

$$a_1\left(\frac{b_2d_3 - b_3d_2}{a_2b_3 - a_3b_2} \right) + b_1\left(\frac{d_2a_3 - d_3a_2}{a_2b_3 - a_3b_2} \right) + c_1(0) + d_1 = \frac{\Delta_3}{a_2b_3 - a_3b_2} = 0 \qquad (\because \Delta_3 = 0)$$

That is, a point on the line L_2 also lies in the plane Π_1 and L_2 is parallel to Π_1 so that L_2 lies in Π_1. Therefore $L_1 = L_2 = L_3$. Similarly, $\Delta = 0 \Rightarrow \Delta_1 = 0$ and $\Delta = 0 \Rightarrow \Delta_2 = 0$.

3. Suppose $\Delta=0, \Delta_3 \neq 0$. $\Delta=0$ implies that the line L_2 is parallel to Π_1. Also

$$\Delta_3 \neq 0 \Rightarrow \frac{\Delta_3}{a_2b_3-a_3b_2} \neq 0$$

so that the line L_2 does not lie in Π_1. In this case, triangular prism is formed.

QUICK LOOK 13

If $\Delta=0$, then either all Δ_j (where $j=1$, 2 and 3) are zero or all Δ_j (where $j=1,2$ and 3) are non-zero.

6.4.2 Method to Solve Problems on Three Planes

1. Check whether all three planes are parallel. If all the three planes are parallel, then no common solution exists for the three equations.
2. If two planes are parallel and the third intersects both in parallel lines, then there exists no common solution.
3. If $\Delta \neq 0$, then all three planes meet at a unique point.
4. If $\Delta=0$ and one of Δ_1, Δ_2 and Δ_3 is not zero, then the three planes form a triangular prism.
5. If $\Delta=0$ and $\Delta_1=\Delta_2=\Delta_3=0$, then the equations have infinitely many solutions and the three planes intersect in a single line.

Example 6.32

Discuss the nature of the following three planes:

1. $2x+y+z+4=0,\ y-z+4=0,\ 3x+2y+z+8=0$
2. $4x-5y-2z-2=0,\ 5x-4y+2z+2=0,\ 2x+2y+8z-1=0$
3. $2x+3y-z-2=0,\ 3x+3y+z-4=0,\ x-y+2z-5=0$

Solution:

1. We have

$$\Delta=\begin{vmatrix} 2 & 1 & 1 \\ 0 & 1 & -1 \\ 3 & 2 & 1 \end{vmatrix}=2(1+2)-1(0+3)+1(0-3)=6-6=0$$

$$\Delta_1=\begin{vmatrix} 4 & 1 & 1 \\ 4 & 1 & -1 \\ 8 & 2 & 1 \end{vmatrix}=4\begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 2 & 2 & 1 \end{vmatrix}=4(0)=0$$

Hence, the three planes intersect in a single line (thus, the three equations have infinitely many common solutions).

2. We have

$$\Delta=\begin{vmatrix} 4 & -5 & -2 \\ 5 & -4 & 2 \\ 2 & 2 & 8 \end{vmatrix}\begin{aligned}&=4(-32-4)+5(40-4)-2(10+8)\\&=-144+180-36=0\end{aligned}$$

$$\Delta_1=\begin{vmatrix} -2 & -5 & -2 \\ 2 & -4 & 2 \\ -1 & 2 & 8 \end{vmatrix}\begin{aligned}&=-2(-32-4)+5(16+2)-2(4-4)\\&=72+90=162\neq 0\end{aligned}$$

Hence, the three planes form a triangular prism (in this case, the system of equation has no common solution).

3. We have

$$\Delta=\begin{vmatrix} 2 & 3 & -1 \\ 3 & 3 & 1 \\ 1 & -1 & 2 \end{vmatrix}\begin{aligned}&=2(6+1)-3(6-1)-1(-3-3)\\&=14-15+6=5\neq 0\end{aligned}$$

Hence, the three planes are concurrent at a point (i.e., the three equations have unique solution).

THEOREM 6.59 The equation of the plane passing through the line

$$\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}$$

and parallel to the line $\dfrac{x-x_2}{l_2}=\dfrac{y-y_2}{m_2}=\dfrac{z-z_2}{n_2}$

is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$$

PROOF Let L_1 and L_2 be the given lines and $A=(x_1,y_1,z_1)$ (see Fig. 6.33). Let $\vec{n}_1=(l_1,m_1,n_1)$ and $\vec{n}_2=(l_2,m_2,n_2)$ so that $\vec{n}_1\times\vec{n}_2$ is perpendicular to the required plane. Now

$$\begin{aligned} &P(x,y,z) \text{ is any point in the plane} \\ \Leftrightarrow\ &\overrightarrow{AP}\cdot(\vec{n}_1\times\vec{n}_2)=0 \\ \Leftrightarrow\ &\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0 \end{aligned}$$

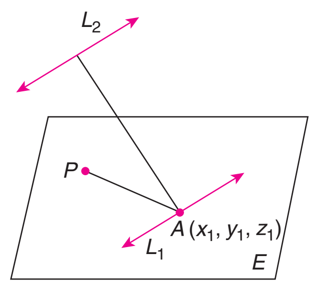


FIGURE 6.33

QUICK LOOK 14

The equation of the plane containing both the lines

$$\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1} \text{ and } \frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}$$

is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$$

Example 6.33

If the points $A(a,b,c)$, $B(a',b',c')$ and the origin are collinear, then show that

$$|aa'+bb'+cc'|=\sqrt{a^2+b^2+c^2}\ \sqrt{a'^2+b'^2+c'^2}$$

Solution: The equation of the line $\overleftrightarrow{AB}$ is

$$\frac{x-a}{a'-a}=\frac{y-b}{b'-b}=\frac{z-c}{c'-c}$$

This passes through origin which means

$$\frac{a}{a-a'}=\frac{b}{b-b'}=\frac{c}{c-c'}$$

$$\Rightarrow \frac{a-a'}{a}=\frac{b-b'}{b}=\frac{c-c'}{c}$$

$$\Rightarrow \frac{-a'}{a}=\frac{-b'}{b}=\frac{-c'}{c}$$

$$\Rightarrow \frac{a'}{a}=\frac{b'}{b}=\frac{c'}{c}=\lambda \quad \text{(say)}$$

Therefore

$$\begin{aligned}(a^2+b^2+c^2)(a'^2+b'^2+c'^2) &= \lambda^2(a^2+b^2+c^2)^2 \\ &= (\lambda aa+\lambda bb+\lambda cc)^2 \\ &= (a'a+b'b+c'c)^2\end{aligned}$$

Hence

$$|a'a+b'b+c'c|=\sqrt{a^2+b^2+c^2}\cdot\sqrt{a'^2+b'^2+c'^2}$$

Example 6.34

Find the shortest distance and the equation of the line containing the shortest distance segment of the skew lines

$$\frac{x-3}{-1}=\frac{y-4}{2}=\frac{z+2}{1}$$

and
$$\frac{x-1}{1}=\frac{y+7}{3}=\frac{z+2}{2}$$

Solution: See Fig. 6.34. We have $A=(3,4,-2)$, $\vec{b}=(-1,2,1)$, $C=(1,-7,-2)$ and $\vec{d}=(1,3,2)$. Let LM be the shortest distance between the lines. Hence, by Theorem 6.44, p. 413, Vol. 2, we have

$$LM=\frac{|\overline{AC}\cdot(\vec{b}\times\vec{d})|}{|\vec{b}\times\vec{d}|}=\sqrt{35}$$

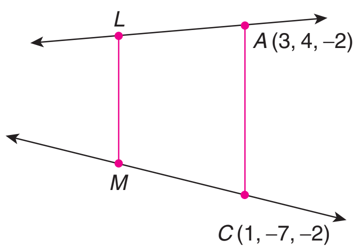


FIGURE 6.34

Let the given skew lines be L_1 and L_2, respectively. The plane containing L_1 and $\overleftrightarrow{LM}$ is

$$\begin{vmatrix} x-3 & y-4 & z+2 \\ -1 & 2 & 1 \\ 1 & 3 & 2 \end{vmatrix}=0$$

$$\Rightarrow 13x+4y+5z-45=0 \qquad (6.25)$$

Here we have used $(1,3,-5)=\vec{b}\times\vec{d}$.
Also, the equation of the plane containing L_2 and $\overleftrightarrow{LM}$ is

$$\begin{vmatrix} x-1 & y+7 & z+2 \\ 1 & 3 & 2 \\ 1 & 3 & -5 \end{vmatrix}=0$$

$$\Rightarrow 3x-y-10=0 \qquad (6.26)$$

Intersection of the planes provided in Eqs. (6.25) and (6.26) is $\overleftrightarrow{LM}$. Hence, the equation of the line containing the shortest distance segment is

$$13x+4y+5z-45=0=3x-y-10$$

Example 6.35

Suppose $5x-y-z=0=x-2y+z+3$ and $7x-4y-2z=0=x-y+z-3$ represent skew lines. Find the shortest distance between them and also find the equation of the shortest distance.

Solution: Let the lines

$$5x-y-z=0=x-2y+z+3 \qquad (6.27)$$

and
$$7x-4y-2z=0=x-y+z-3 \qquad (6.28)$$

be L_1 and L_2, respectively. The equation of any plane passing through L_1, by Theorem 6.54, is

$$(5x-y-z)+\lambda(x-2y+z+3)=0 \qquad (6.29)$$

and the equation of the plane passing through L_2 is

$$(7x-4y-2z)+\mu(x-y+z-3)=0 \qquad (6.30)$$

Equations (6.29) and (6.30) represent parallel planes. This means

$$\frac{5+\lambda}{7+\mu}=\frac{-1-2\lambda}{-4-\mu}=\frac{\lambda-1}{\mu-2}$$

Now,

$$\frac{5+\lambda}{7+\mu}=\frac{(1+2\lambda)}{(4+\mu)}\Rightarrow 10\lambda-4\mu+\lambda\mu=13 \qquad (6.31)$$

$$\frac{1+2\lambda}{4+\mu}=\frac{\lambda-1}{\mu-2}\Rightarrow 8\lambda-2\mu-\lambda\mu=2 \qquad (6.32)$$

$$\frac{5+\lambda}{7+\mu}=\frac{\lambda-1}{\mu-2}\Rightarrow 9\lambda-6\mu=-3 \qquad (6.33)$$

Adding Eqs. (6.31) and (6.32), we get

$$18\lambda-6\mu=15$$

$$\Rightarrow 6\lambda-2\mu=5 \qquad (6.34)$$

Solving Eqs. (6.33) and (6.34) for λ and μ, we have $\lambda=2$ and $\mu=7/2$. Substituting the values of λ and μ in Eq. (6.29), we get the planes passing through L_1 and L_2, which are parallel to each other, as

$$7x-5y+z+6=0 \qquad (6.35)$$

and
$$7x-5y+z-7=0 \qquad (6.36)$$

The distance between the planes provided in Eqs. (6.35) and (6.36) is

$$\frac{|6+7|}{\sqrt{7^2+(-5)^2+1^2}}=\frac{13}{\sqrt{75}}$$

The planes represented by Eqs. (6.29) and (6.35) are at right angles. This implies

$$7(5+\lambda)-5(-1-2\lambda)+1(\lambda-1)=0$$

$$\Rightarrow \lambda=\frac{-13}{6}$$

That is the equation of the plane passing through L_1 and perpendicular to the plane provided in Eq. (6.35) is

$$\left(5-\frac{13}{6}\right)x-\left(1-\frac{26}{6}\right)y+\left(\frac{-13}{6}-1\right)z-\frac{39}{6}=0$$

$$\Rightarrow 17x+20y-19z-39=0 \qquad (6.37)$$

Also the planes represented by Eq. (6.30) is perpendicular to the plane represented by Eq. (6.27). So

$$7(7+\mu)-5(-4-\mu)+1(\mu-2)=0$$

$$\Rightarrow 13\mu=-67$$

$$\Rightarrow \mu=\frac{-67}{13}$$

Substituting the value of $\mu=-67/13$ in Eq. (6.30), we have

$$8x+5y-31z+67=0 \qquad (6.38)$$

which is the plane passing through the line L_2 and perpendicular to the plane represented by Eq. (6.30). From Eqs. (6.37) and (6.38), the equation of the line having the shortest distance is

$$17x+20y-19z-39=0=8x+5y-31z+67$$

Example 6.36

Find the equation of the line intersecting the lines $2x+y-1=0=x-2y+3z$, $3x-y+z+2=0=4x+5y-2z-3$ and parallel to the line $\frac{x-1}{1}=\frac{y-2}{2}=\frac{z-3}{3}$.

Solution: The required line equation is

$$(2x+y-1)+\lambda(x-2y+3z)=0$$

$$=(3x-y+z+2)+\mu(4x+5y-2z-3)$$

so we have

$$(2+\lambda)x+(1-2\lambda)y+3\lambda z-1=0$$

$$=(3+4\mu)x+(-1+5\mu)y+(1-2\mu)z+2-3\mu \qquad (6.39)$$

is parallel to the line

$$\frac{x-1}{1}=\frac{y-2}{2}=\frac{z-3}{3}$$

Therefore

$$1(2+\lambda)+2(1-2\lambda)+3(3\lambda)=0=1(3+4\mu)+2(-1+5\mu)+3(1-2\mu)$$

$$\Rightarrow 6\lambda=-4 \text{ and } 8\mu=-4$$

$$\Rightarrow \lambda=\frac{-2}{3} \text{ and } \mu=\frac{-1}{2}$$

Substituting the values of $\lambda=-2/3$ and $\mu=-1/2$ in Eq. (6.39), the required line equation is

$$4x+7y-6z-3=0=2x-7y+4z+7$$

Example 6.37

Find the angle between the planes $x-y-2z+9=0$ and $x+2y+z-5=0$.

Solution: The angle between the two planes is equal to the angle between their normals (see Definition 6.12). $\vec{n}_1=(1,-1,-2)$ and $\vec{n}_2=(1,2,1)$ are normals to the given two planes, respectively. Suppose θ is the angle between the normals. Hence

$$\cos\theta=\frac{|\vec{n}_1\cdot\vec{n}_2|}{|\vec{n}_1|\cdot|\vec{n}_2|}=\frac{|1-2-2|}{\sqrt{6}\sqrt{6}}=\frac{1}{2}$$

Therefore, $\theta=\pi/3$.

Example 6.38

Find the foot of the perpendicular and also the perpendicular distance of the point $(5,9,3)$ from the line

$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$$

Solution: See Fig. 6.35. Let $M\ (x, y, z)$ be the foot of the perpendicular drawn from P onto the given line. Therefore, $x = 1 + 2\lambda$, $y = 2 + 3\lambda$ and $z = 3 + 4\lambda$ for some real λ. Therefore,

$$\overrightarrow{PM} = (2\lambda - 4, 3\lambda - 7, 4\lambda)$$

$\overrightarrow{PM}$ is perpendicular to the line

$$\Rightarrow \overrightarrow{PM} \cdot (2,3,4) = 0$$
$$\Rightarrow 2(2\lambda - 4) + 3(3\lambda - 7) + 4(4\lambda) = 0$$
$$\Rightarrow 29\lambda = 29$$
$$\Rightarrow \lambda = 1$$

Therefore, the foot of the perpendicular $M = (1+2, 2+3, 3+4) = (3, 5, 7)$. Also

$$|\overrightarrow{PM}| = \sqrt{(5-3)^2 + (9-5)^2 + (3-7)^2}$$
$$= \sqrt{4 + 16 + 16}$$
$$= 6$$

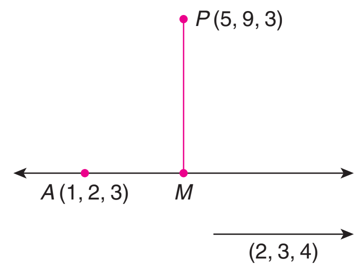


FIGURE 6.35

Example 6.39

Find the symmetric form of the line $x + 2y - z - 6 = 0 = 2x + 3y - z - 8$.

Solution: Let $E_1 \equiv 2x + 2y - z - 6 = 0 = 2x + 3y - z - 8 \equiv E_2$ and L be the line represented by the planes $E_1 = 0 = E_2$. Therefore, the DRs of L is

$$(2, 2, -1) \times (2, 3, -1) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 2 & -1 \\ 2 & 3 & -1 \end{vmatrix} = \vec{i} + 2\vec{k}$$

Also, substitute $z = 0$ in the plane equations so that $2x + 2y = 6$ and $2x + 3y = 8$. Solving these equations, we get $x = 1$ and $y = 2$. Therefore, $(1, 2, 0)$ is a point on the line L. Hence, the equation of the line is

$$\frac{x-1}{1} = \frac{y-2}{0} = \frac{z-0}{+2}$$

Example 6.40

Find the angle between the lines

$$x + 2y - 2z - 11 = 0 = x - 2y + z - 9$$

and
$$\frac{x-3}{1} = \frac{y+5}{-3} = \frac{z-1}{2}$$

Solution: Let L_1 be the line

$$x + 2y - 2z - 11 = 0 = x - 2y + z - 9$$

so that DRs of L_1 is

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & -2 \\ 1 & -2 & 1 \end{vmatrix} = -2\vec{i} - 3\vec{j} - 4\vec{k}$$

Let L_2 be the line

$$\frac{x-3}{1} = \frac{y+5}{-3} = \frac{z-1}{2}$$

Let θ be the acute angle between L_1 and L_2. Therefore,

$$\cos\theta = \frac{|(-2, -3, -4) \cdot (1, -3, 2)|}{\sqrt{2^2 + 3^2 + 4^2}\sqrt{1^2 + 3^2 + 2^2}} = \frac{1}{\sqrt{406}}$$

Example 6.41

Find the equation of the plane through the point $(1, 4, -2)$ and perpendicular to the line of intersection of the planes $x + y + z - 10 = 0$ and $2x - y + 3z - 18 = 0$.

Solution: Let L be the line of intersection of the planes. Hence, the DRs of L is

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 2 & -1 & 3 \end{vmatrix} = 4\vec{i} - \vec{j} - 3\vec{k}$$

Since the required plane is perpendicular to L, the DRs of the normal of the plane are the DRs of L. That is (4, −1, − 3). Hence the equation of the plane is

$$4(x-1) - 1(y-4) - 3(z+2) = 0$$
$$\Rightarrow 4x - y - 3z - 6 = 0$$

Example 6.42

Find the point of intersection of the line

$$\frac{x-2}{-3} = \frac{y-1}{2} = \frac{z-3}{2}$$

and the plane $2x + y - z - 3 = 0$.

Solution: Any point on the given line is of the form $(2 - 3\lambda, 1 + 2\lambda, 3 + 2\lambda)$. This point lies on the given plane. So

$$2(2-3\lambda)+(1+2\lambda)-(3+2\lambda)-3=0$$

$$\Rightarrow -1-6\lambda=0$$
$$\Rightarrow \lambda=\frac{-1}{6}$$

Hence, the required point is

$$\left[2-3\left(\frac{-1}{6}\right), 1+2\left(\frac{-1}{6}\right), 3+2\left(\frac{-1}{6}\right)\right] = \left(\frac{5}{2}, \frac{2}{3}, \frac{8}{3}\right)$$

WORKED-OUT PROBLEMS

Single Correct Choice Type Questions

1. The angle between the lines whose direction cosines satisfy the relations $l + m + n = 0$ and $3lm - 5mn + 2nl = 0$ is

(A) $\frac{\pi}{4}$ (B) $\frac{\pi}{2}$

(C) $\frac{\pi}{3}$ (D) $\text{Cos}^{-1}\left(\frac{3}{\sqrt{16}}\right)$

Solution: We have

$$l + m + n = 0 \quad (6.40)$$

$$3lm - 5mn + 2nl = 0 \quad (6.41)$$

From Eq. (6.40), we have $n = -(l + m)$. Substituting this value of n in Eq. (6.41), we have

$$3lm + 5m(l + m) - 2l(l + m) = 0$$

$$\Rightarrow 2l^2 - 6lm - 5m^2 = 0$$

$$\Rightarrow 2\left(\frac{l}{m}\right)^2 - 6\left(\frac{l}{m}\right) - 5 = 0$$

If l_1/m_1 and l_2/m_2 are the roots of this quadratic in l/m, then

$$\left(\frac{l_1}{m_1}\right)\left(\frac{l_2}{m_2}\right) = \frac{-5}{2} \quad (6.42)$$

Substituting $m = -(l + n)$ in Eq. (6.42), we have

$$\left(\frac{l_1}{n_1}\right)\left(\frac{l_2}{n_2}\right) = \frac{-5}{3} \quad (6.43)$$

Therefore, from Eqs. (6.42) and (6.43), we have

$$\frac{l_1 l_2}{-5} = \frac{m_1 m_2}{2} = \frac{n_1 n_2}{3} = \lambda \quad \text{(say)}$$

Therefore

$$\begin{aligned}(l_1, m_1, n_1)\cdot(l_2, m_2, n_2) &= l_1 l_2 + m_2 m_2 + n_1 n_2 \\ &= \lambda(-5+2+3) \\ &= \lambda(0) = 0\end{aligned}$$

Therefore, the two lines are at right angles.

Answer: (B)

2. The direction cosines of the line joining the points $P(4, 3, -5)$ and $Q(-2, 1, -8)$ are

(A) $\left(\frac{6}{7}, \frac{2}{7}, \frac{3}{7}\right)$ or $\left(\frac{-6}{7}, \frac{-2}{7}, \frac{-3}{7}\right)$

(B) $\left(\frac{6}{7}, \frac{-2}{7}, \frac{3}{7}\right)$ or $\left(\frac{-6}{7}, \frac{2}{7}, \frac{-3}{7}\right)$

(C) $\left(\frac{6}{7}, \frac{-2}{7}, \frac{3}{7}\right)$ or $\left(\frac{6}{7}, \frac{2}{7}, \frac{-3}{7}\right)$

(D) $\left(\frac{6}{7}, \frac{-2}{7}, \frac{-3}{7}\right)$ or $\left(\frac{-6}{7}, \frac{2}{7}, \frac{3}{7}\right)$

Solution: The DRs of the line $\overrightarrow{PQ}$ are

$$(-2-4, 1-3, -8+5) = (-6, -2, -3)$$

Therefore

$$|\overrightarrow{PQ}| = \sqrt{6^2 + 2^2 + 3^2} = 7$$

Therefore, the DCs of the line $\overrightarrow{PQ}$ is

$$\left(\frac{-6}{7}, \frac{-2}{7}, \frac{-3}{7}\right) \quad \text{or} \quad \left(\frac{6}{7}, \frac{2}{7}, \frac{3}{7}\right)$$

Answer: (A)

3. If $A = (3, 6, 4)$, $B = (2, 5, 2)$, $C = (6, 4, 4)$ and $D = (0, 2, 1)$, then the length of the projection of AB on CD is equal to

(A) 4 (B) 3 (C) 1 (D) 2

Solution: $\overrightarrow{AB} = (-1, -1, -2)$ and $\overrightarrow{CD} = (-6, -2, -3)$ are the DRs of the lines $\overrightarrow{AB}$ and $\overrightarrow{CD}$. Therefore, the magnitude of the projection of $\overrightarrow{AB}$ on $\overrightarrow{CD}$ (see Theorem 6.29) is

$$\frac{|\overrightarrow{AB} \cdot \overrightarrow{CD}|}{|\overrightarrow{CD}|} = \left|\frac{(-1)(-6) + (-1)(-2) + (-2)(-3)}{\sqrt{(-6)^2 + (-2)^2 + (-3)^2}}\right| = \frac{14}{7} = 2$$

Answer: (D)

4. The DRs of the line $3x + 1 = 6y - 2 = 1 - z$ are

(A) $\left(\frac{1}{3}, 6, -1\right)$ (B) $\left(3, \frac{1}{6}, -1\right)$

(C) $\left(\frac{1}{3}, \frac{1}{6}, -1\right)$ (D) $\left(\frac{1}{3}, \frac{-1}{6}, 1\right)$

Solution: We have

$$3x + 1 = 6y - 2 = 1 - z \Rightarrow \frac{x + (1/3)}{1/3} = \frac{y - (1/3)}{1/6} = \frac{z - 1}{-1}$$

Therefore, the DRs are

$$\left(\frac{1}{3}, \frac{1}{6}, -1\right)$$

Answer: (C)

5. The DRs of the line perpendicular to the plane determined by the lines

$$\frac{x}{2} = \frac{y-1}{-3} = \frac{z+2}{-2} \quad \text{and} \quad \frac{x+3}{-1} = \frac{y+2}{2} = \frac{z-1}{3}$$

are

(A) (5, 4, 1) (B) (5, 4, –1)

(C) (–5, 4, –1) (D) (5, –4, 1)

Solution: We have $\vec{n}_1 = (2, -3, -2)$ and $\vec{n}_2 = (-1, 2, 3)$ as the DRs of the two given lines, respectively. Hence, $\vec{n}_1 \times \vec{n}_2$ is the normal to the plane determined by the given lines is

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -3 & -2 \\ -1 & 2 & 3 \end{vmatrix} = -5\vec{i} - 4\vec{j} + \vec{k}$$

Therefore, the DRs of the normal are (–5, –4, 1) or (5, 4, –1).

Answer: (B)

6. The point of intersection of the lines

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} \quad \text{and} \quad \frac{x-4}{5} = \frac{y-1}{2} = z$$

is

(A) (1, 1, 1) (B) (1, 1, – 1)

(C) (– 1, 1, – 1) (D) (– 1, – 1, – 1)

Solution: Let the given lines be L_1 and L_2, respectively. Every point on L_1 is of the form $(1 + 2t, 2 + 3t, 3 + 4t)$ and every point on L_2 is of the form $(4 + 5s, 1 + 2s, s)$. These two lines intersect at same point P. This implies

$$1 + 2t = 4 + 5s \Rightarrow 2t - 5s = 3 \tag{6.44}$$

$$2 + 3t = 1 + 2s \Rightarrow 3t - 2s = -1 \tag{6.45}$$

$$3 + 4t = s \Rightarrow 4t - s = -3 \tag{6.46}$$

Solving Eqs. (6.44) and (6.45), we have $t = -1$ and $s = -1$ which also satisfy Eq. (6.46). Therefore, the point of intersection is (–1, –1, –1).

Answer: (D)

7. The foot of the perpendicular drawn from the point (1, 0, 2) onto the line

$$\frac{x+1}{3} = \frac{y-2}{-2} = \frac{z+1}{-1}$$

is

(A) $\left(\frac{1}{2}, 1, \frac{-3}{2}\right)$ (B) $\left(\frac{2}{3}, 1, -1\right)$

(B) $\left(\frac{2}{3}, \frac{-1}{2}, -2\right)$ (D) $(1, -2, -1)$

Solution: Any point on the given line is of the form $Q = (-1 + 3\lambda, 2 - 2\lambda, -1 - \lambda)$. Let $P = (1, 0, 2)$. $\overrightarrow{PQ}$ is perpendicular to the given line $\Rightarrow \overrightarrow{PQ}$ and the vector (3, −2, −1) are at right angles. So

$$(3\lambda-2, 2-2\lambda, -3-\lambda)\cdot(3,-2,-1)=0$$
$$\Rightarrow 9\lambda-6-2(2-2\lambda)-1(-3-\lambda)=0$$
$$\Rightarrow 14\lambda=7$$
$$\Rightarrow \lambda=\frac{1}{2}$$

Therefore

$$Q=\left(\frac{1}{2},1,\frac{-3}{2}\right)$$

is the foot of the perpendicular from P onto the line.

Answer: (D)

8. The lines

$$\frac{x-1}{3}=\frac{y+1}{2}=\frac{z-1}{5}$$

and $$\frac{x-2}{4}=\frac{y-1}{3}=\frac{z+1}{-2}$$

intersect at

(A) (1, 3, 2) (B) (2, 3, 1)
(C) no point (D) (2, −3, −1)

Solution: Let L_1 and L_2 be the two given lines so that the general points on L_1 and L_2 are $(1+3t, -1+2t, 1+5t)$ and $(2+4s, 1+3s, -1-2s)$, respectively. The two lines L_1 and L_2 intersect. This implies for same t and s we have

$$1 + 3t = 2 + 4s \Rightarrow 3t - 4s = 1 \quad (6.47)$$
$$-1 + 2t = 1 + 3s \Rightarrow 2t - 3s = 2 \quad (6.48)$$
$$1 + 5t = -1 - 2s \Rightarrow 5t + 2s = -2 \quad (6.49)$$

Solving Eqs. (6.47) and (6.48), we have $t = -5$ and $s = -4$ which do not satisfy Eq. (6.49). Hence L_1 and L_2 do not intersect.

Aliter: Let $A = (1, -1, 1)$, $B = (2, 1, -1)$, $\vec{n}_1 = (3, 2, 5)$ and $\vec{n}_2 = (4, 3, -2)$. Also $\vec{n}_1$ and $\vec{n}_2$ are not parallel $\Rightarrow$ the lines are not parallel. Now

$$[\overrightarrow{AB}\ \vec{n}_1\ \vec{n}_2]=\begin{vmatrix} 1 & 2 & -2 \\ 3 & 2 & 5 \\ 4 & 3 & -2 \end{vmatrix}$$
$$=1(-4-15)-2(-6-20)-2(9-8)$$
$$=-19+52-2\neq 0$$

Therefore, the lines are not coplanar and hence they are skew lines.

Answer: (C)

9. The image of the point (1, 6, 3) in the line $\frac{x}{1}=\frac{y-1}{2}=\frac{z-2}{3}$ is

(A) (1, 0, 7) (B) (3, −2, 3)
(C) (3, −1, 1) (D) (−3, 0, 1)

Solution: Let $P = (1, 6, 3)$ and $Q(\lambda, 1 + 2\lambda, 2 + 3\lambda)$ be two points on the line so that

$$\overrightarrow{PQ}=(\lambda-1, 2\lambda-5, 3\lambda-1)$$

Now, $\overrightarrow{PQ}$ is perpendicular to the given line. This implies

$$(\lambda-1, 2\lambda-5, 3\lambda-1)\cdot(1,2,3)=0$$
$$\Rightarrow(\lambda-1)+2(2\lambda-5)+3(3\lambda-1)=0$$
$$\Rightarrow 14\lambda=14$$
$$\Rightarrow \lambda=1$$

Therefore, the foot of the perpendicular from P onto the line is $Q(1, 3, 5)$. If $P'(x, y, z)$ is the image of P in the given line, then $Q(1, 3, 5)$ must be the midpoint of PP'. Hence, we get

$$\frac{x+1}{2}=1, \frac{y+6}{2}=3, \frac{z+3}{2}=5$$

Therefore, $P' = (1, 0, 7)$.

Answer: (A)

10. The angle between the lines whose DRs are (1, 1, 2) and $(\sqrt{3}, -\sqrt{3}, 0)$ is

(A) 60° (B) 45° (C) 90° (D) 75°

Solution: Let $\vec{n}_1=(1,1,2)$ and $\vec{n}_2=(\sqrt{3},-\sqrt{3},0)$ and θ be the acute angle between the lines. Therefore

$$\cos\theta=\frac{|\vec{n}_1\cdot\vec{n}_2|}{|\vec{n}_1||\vec{n}_2|}=\frac{|\sqrt{3}-\sqrt{3}+0|}{\sqrt{6}\sqrt{18}}=0$$

Therefore, $\theta = 90°$.

Answer: (C)

11. If $A=(1,-1,2)$, $B=(3,4,-2)$, $C=(0,3,2)$ and $D=(3,5,6)$, then the angle between the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ is

(A) 60° (B) 90° (C) 75° (D) 45°

Solution: The DRs of two lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ be (2, 5, −4) and (3, 2, 4), respectively. Let $\vec{n}_1=(2,5,-4)$ and $\vec{n}_2=(3,2,4)$. If θ is the angle between the lines, then

$$\cos\theta=\frac{\vec{n}_1\cdot\vec{n}_2}{|\vec{n}_1||\vec{n}_2|}=\frac{2(3)+5(2)+(-4)4}{\sqrt{2^2+5^2+4^2}\sqrt{3^2+2^2+4^2}}=0$$

Therefore, $\theta = 90°$.

Answer: (B)

12. The points $A(4, 2, 4)$, $B(10, 2, -2)$ and $C(2, 0, -4)$ are the vertices of

(A) an isosceles, but not an equilateral triangle
(B) an isosceles and a right-angled triangle
(C) right-angled triangle only
(D) an equilateral triangle

Solution: We have

$$(AB)^2 = (10-4)^2 + (2-2)^2 + (-2-4)^2 = 36+36 = 72$$
$$(BC)^2 = (10-2)^2 + (2-0)^2 + (-2+4)^2 = 64+4+4 = 72$$
$$(CA)^2 = (4-2)^2 + 2^2 + (4+4)^2 = 4+4+64 = 72$$

Since $AB = BC = CA$, the triangle is equilateral.

Answer: (D)

13. The locus of the point equidistant from the points $(1, -2, 3)$ and $(-3, 4, 2)$ is

(A) $8x - 12y + 2z + 15 = 0$
(B) $8x + 12y - 2z + 15 = 0$
(C) $8x - 12y - 2z + 15 = 0$
(D) $8x - 12y + 2z - 15 = 0$

Solution: Let $A = (1, -2, 3)$ and $B = (-3, 4, 2)$. $P(x, y, z)$ is a point on the locus $\Leftrightarrow PA = PB$. So

$$(PA)^2 = (PB)^2$$
$$\Rightarrow (x-1)^2 + (y+2)^2 + (z-3)^2 = (x+3)^2 + (y-4)^2 + (z-2)^2$$
$$\Rightarrow 8x - 12y + 2z + 15 = 0$$

Answer: (A)

14. Given the points $A(3, 2, 0)$ and $B(2, 1, -5)$, the locus of the point $P(x, y, z)$ such that PA is perpendicular to PB is

(A) $x^2 + y^2 + z^2 - 5x - 3y - 5z + 8 = 0$
(B) $x^2 + y^2 + z^2 - 5x - 3y + 5z + 8 = 0$
(C) $x^2 + y^2 + z^2 - 5x - 3y + 5z - 8 = 0$
(D) $x^2 + y^2 + z^2 + 5x + 3y + 5z - 8 = 0$

Solution: We can write

$$\overrightarrow{AP} = (x-3, y-2, z)$$

and

$$\overrightarrow{BP} = (x-2, y-1, z+5)$$

Therefore

$$\overrightarrow{AP} \cdot \overrightarrow{BP} = 0$$
$$\Rightarrow (x-3)(x-2) + (y-2)(y-1) + z(z+5) = 0$$
$$\Rightarrow x^2 + y^2 + z^2 - 5x - 3y + 5z + 8 = 0$$

Answer: (B)

15. The equation of the locus of a point whose distance from the yz-plane is twice its distance from the point $(4, -2, 1)$ is

(A) $3x^2 + 4y^2 + 4z^2 - 32x + 16y - 8z - 84 = 0$
(B) $3x^2 + 4y^2 + 4z^2 - 32x + 16y + 8z + 84 = 0$
(C) $3x^2 + 4y^2 + 4z^2 - 32x + 16y - 8z + 84 = 0$
(D) $3x^2 + 4y^2 + 4z^2 + 32x + 16y - 8z - 84 = 0$

Solution: The distance of a point from yz-plane is the numerical value of the x-coordinate of the point. $P(x, y, z)$ is a point on the locus

$$\Rightarrow 4[(x-4)^2 + (y+2)^2 + (z-1)^2] = |x|^2$$
$$\Rightarrow 3x^2 + 4y^2 + 4z^2 - 32x + 16y - 8z + 84 = 0$$

Answer: (C)

16. The equation of the locus of a point whose distance from $(0, 0, -2)$ is one-third of its distance from the plane $z + 18 = 0$ is

(A) $9x^2 + 9y^2 + 8z^2 = 188$
(B) $9x^2 + 9y^2 + 9z^2 = 288$
(C) $9x^2 + 9y^2 + 8z^2 = 288$
(D) $9x^2 + 9y^2 - 8z^2 = 288$

Solution: $P(x, y, z)$ is a point on the locus. So

$$\sqrt{x^2 + y^2 + (z+2)^2} = \frac{1}{3}|z+18|$$
$$\Rightarrow 9x^2 + 9y^2 + 9(z+2)^2 = (z+18)^2$$
$$\Rightarrow 9x^2 + 9y^2 + 8z^2 = 288$$

Answer: (C)

17. The equation of the plane passing through the points $(1, -2, 2)$ and $(-3, 1, -2)$ and perpendicular to the plane $2x + y - z + 6 = 0$ is

(A) $x - 12y - 10z - 5 = 0$
(B) $x - 12y - 10z + 5 = 0$
(C) $x - 12y - 10z - 10 = 0$
(D) $x - 12y - 10z + 10 = 0$

Solution: Let $E \equiv ax + by + cz + d = 0$ be the required plane. It passes through $(1, -2, 2)$ and $(-3, 1, -2)$. This implies

$$a - 2b + 2c + d = 0 \tag{6.50}$$

and

$$-3a + b - 2c + d = 0 \tag{6.51}$$

$E = 0$ is perpendicular to the plane $2x + y - z + 6 = 0$. This gives

$$2a + b - c = 0 \tag{6.52}$$

Subtracting Eq. (6.51) from Eq. (6.50), we get

$$4a - 3b + 4c = 0 \tag{6.53}$$

From Eqs. (6.52) and (6.53), we get

$$12a + b = 0 \Rightarrow b = -12a$$

Therefore,

$$c = 2a + b = -10a$$

and $\quad d = -a + 2b - 2c = -a + 2(-12a) + 20a = -5a$

Hence,

$$E \equiv ax + by + cz + d \equiv ax - 12ay - 10az - 5a = 0$$

$$\Rightarrow E \equiv x - 12y - 10z - 5 = 0$$

Answer: (A)

18. The equation of the plane parallel to the plane $2x - 3y - 6z - 14 = 0$ and at a distance of 5 units from the origin is

(A) $2x-3y-6z\pm 25=0$

(B) $2x-3y-6z\pm 35=0$

(C) $2x+3y+6z\pm 35=0$

(D) $2x-3y-6z\pm 45=0$

Solution: Required plane is of the form $2x - 3y - 6z + d = 0$. Since its distance from origin is 5, we have

$$\frac{|d|}{\sqrt{2^2+3^2+6^2}} = 5 \Rightarrow d = \pm 35$$

Answer: (B)

19. If the equation of the plane parallel to the plane $2x - 3y - 5z + 6 = 0$ and passing through the point $(-1, 2, 4)$ is $2x - 3y - 5z + d = 0$, then the value of d is

(A) 8 (B) 18 (C) -18 (D) 28

Solution: Since $2x - 3y - 5z + d = 0$ passes through $(-1, 2, 4)$, we have

$$2(-1) - 3(2) - 5(4) + d = 0$$

$$\Rightarrow d = 28$$

Answer: (D)

20. The equation of the plane passing through the point $(3, -2, 4)$ and perpendicular to each of the planes $7x - 3y + z - 5 = 0$ and $4x - y - z + 9 = 0$ is

(A) $4x+11y+5z-10=0$

(B) $x+y+z-5=0$

(C) $2x-y+2z-16=0$

(D) $3x+2y-z-1=0$

Solution: Let $\vec{n}_1 = (7,-3,1)$ and $\vec{n}_2 = (4,-1,-1)$ so that $\vec{n}_1$ and $\vec{n}_2$ are the normals to the respective planes. Hence

$$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 7 & -3 & 1 \\ 4 & -1 & -1 \end{vmatrix} = 4\vec{i} + 11\vec{j} + 5\vec{k}$$

is the normal to the required plane. Therefore, the required plane equation is

$$4(x-3)+11(y+2)+5(z-4)=0$$

$$\Rightarrow 4x+11y+5z-10=0$$

Answer: (A)

21. Let $P = (-3, 1, 1)$ and $Q = (3, 4, 2)$. R divides $\overrightarrow{PQ}$ in the ratio $PR: PQ = 1:3$. Then, the equation of the plane perpendicular to $\overrightarrow{PQ}$ at R is

(A) $18x+9y+3z=8$ (B) $18x+9y+3z=4$

(C) $9x+18y+3z=4$ (D) $3x+9y+18z=8$

Solution: See Fig. 6.36.

$$PR : PQ = 1:3 \Rightarrow 3PR = PQ$$

$$\Rightarrow 3PR = PR + RQ$$

$$\Rightarrow 2PR = RQ$$

Therefore, $PR:RQ = 1:2$. Hence

$$R = \left(\frac{-6+3}{1+2}, \frac{2+4}{3}, \frac{2+2}{3}\right) = \left(-1, 2, \frac{4}{3}\right)$$

The normal to the required plane is $\overrightarrow{PQ} = (6,3,1)$. Hence, the equation of the required plane is

$$6(x+1)+3(y-2)+1\left(z-\frac{4}{3}\right)=0$$

$$\Rightarrow 18x+9y+3z-4=0$$

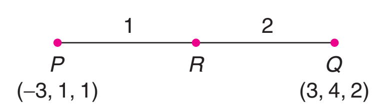


FIGURE 6.36

Answer: (B)

22. From the point $P(a, b, c)$, perpendiculars PL, PM, and PN are drawn to the coordinate planes. If the equation of the plane LMN is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = k$$

then the value of k is

(A) 1 (B) -1 (C) -2 (D) 2

Solution: It is known that $L = (a, b, 0)$, $M = (0, b, c)$ and $N = (a, 0, c)$ so that the equation of LMN is

$$\begin{vmatrix} x-a & y-b & z \\ -a & b & 0 \\ 0 & -b & c \end{vmatrix} = 0$$

$$\Rightarrow bc(x-a)+ca(y-b)+(ab)z=0$$
$$\Rightarrow bcx+cay+abz=2abc$$
$$\Rightarrow \frac{x}{a}+\frac{y}{b}+\frac{z}{c}=2$$

Answer: (D)

23. If the planes $\lambda x+3y-7z-1=0$ and $5x+6y-\mu z=0$ are parallel, then the integral part of $\lambda+\mu$ (i.e., $\lambda+\mu$) is

(A) 15 (B) 17 (C) 14 (D) 16

Solution: The planes are parallel. So

$$\frac{\lambda}{5}=\frac{3}{6}=\frac{-7}{-\mu}$$
$$\Rightarrow \lambda=\frac{5}{2}, \mu=14$$

Therefore, $[\lambda+\mu]=\left[\frac{5}{2}+14\right]=\left[\frac{32}{2}\right]=16.$

Answer: (D)

24. The distance between the parallel planes $3x-2y+6z+8=0$ and $3x-2y+6z-6=0$ is

(A) $\frac{3}{7}$ (B) $\frac{4}{7}$ (C) $\frac{3}{7}$ (D) $\frac{1}{7}$

Solution: The distance between the planes, by Theorem 6.51, is

$$\frac{|8-(-6)|}{\sqrt{3^2+(-2)^2+6^2}}=\frac{14}{7}=2$$

Answer: (C)

25. If the sum of the reciprocals of the intercepts of a plane on the coordinate axis is the constant $1/k$, then the plane passes through the point

(A) $\left(\frac{1}{k},\frac{1}{k},\frac{1}{k}\right)$ (B) (k,k,k)

(C) (k^{-2},k^{-2},k^{-2}) (D) $\left(\frac{2}{k},\frac{2}{k},\frac{2}{k}\right)$

Solution: Let the plane be

$$\frac{x}{c}+\frac{y}{b}+\frac{z}{c}=1$$

and
$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{k}$$

$$\Rightarrow \frac{k}{a}+\frac{k}{b}+\frac{k}{c}=1$$

Therefore, the plane passes through the fixed point (k,k,k).

Answer: (B)

26. The value of k such that the line

$$\frac{x-4}{1}=\frac{y-2}{1}=\frac{z-k}{2}$$

lies in the plane $2x-4y+z-7=0$ is

(A) 7 (B) -7

(C) no real value (D) 4

(IIT-JEE 2003)

Solution: Every point on the given line is of the form $(4+\lambda, 2+\lambda, k+2\lambda)$, where $\lambda \in \mathbb{R}$. This point lies in the plane

$$2x-4y+z-7=0$$

So

$$2(4+\lambda)-4(2+\lambda)+l(k+2\lambda)-7=0$$

for all $\lambda \in \mathbb{R}$.
In particular, for $\lambda=0$, we have

$$8-8+k-7=0 \Rightarrow k=7$$

Note that the point $(4, 2, k)$ lies on the given line and hence it lies in the given plane. So

$$2(4)-4(2)+k-7=0$$
$$\Rightarrow k=7$$

Answer: (A)

27. If the lines

$$\frac{x-1}{2}=\frac{y+1}{3}=\frac{z-1}{4}$$

and
$$\frac{x-3}{1}=\frac{y-k}{2}=\frac{z}{1}$$

intersect, then the value of k is

(A) $\frac{3}{2}$ (B) $\frac{9}{2}$ (C) $\frac{-2}{9}$ (D) $\frac{-3}{2}$

(IIT-JEE 2004)

Solution: Let $A=(1,-1,1)$, $B=(3,k,0)$, $\vec{n}_1=(2,3,4)$ and $\vec{n}_2=(1,2,1)$. The lines intersect $\Leftrightarrow \overline{AB}, \vec{n}_1, \vec{n}_2$ are coplanar. Hence

$$\begin{vmatrix} 2 & k+1 & -1 \\ 2 & 3 & 4 \\ 1 & 2 & 1 \end{vmatrix} = 0$$

$$\Rightarrow 2(3-8)-(k+1)(2-4)-1(4-3)=0$$

$$\Rightarrow -10+2k+2-1=0$$
$$\Rightarrow 2k=9$$
$$\Rightarrow k=\frac{9}{2}$$

Answer: (B)

28. The value of λ such that the planes $2x - y + 2z = 2$, $x - 2y + z = -4$, $x + y + \lambda z = 4$ form a triangular prism is

(A) 3 (B) 1 (C) 0 (D) –3

(IIT-JEE 2004)

Solution: Since the planes are not parallel, from Section 6.4.1, the planes form a triangular prism if

$$\Delta = \begin{vmatrix} 2 & -1 & 2 \\ 1 & -2 & 1 \\ 1 & 1 & \lambda \end{vmatrix} = 0$$

and
$$\Delta_3 = \begin{vmatrix} 2 & -1 & 2 \\ 1 & -2 & -4 \\ 1 & 2 & 4 \end{vmatrix} = 8 \neq 0$$

Also

$$\Delta = 0 \Rightarrow 2(-2\lambda-1)+1(\lambda-1)+2(1+2)=0$$
$$\Rightarrow -3\lambda-2-1+6=0$$
$$\Rightarrow \lambda=1$$

Aliter: The planes form a triangular prism if and only if the planes are not parallel and the planes are not having common solutions. We have

$$\begin{bmatrix} 2 & -1 & 2 \\ 1 & -2 & 1 \\ 1 & 1 & \lambda \end{bmatrix} X = \begin{bmatrix} 2 \\ -4 \\ 4 \end{bmatrix}$$

where

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Interchanging R_1 and R_2, we get

$$\begin{bmatrix} 1 & -2 & 1 \\ 2 & -1 & 2 \\ 1 & 1 & \lambda \end{bmatrix} X = \begin{bmatrix} -4 \\ 2 \\ 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 1 \\ 0 & 3 & 0 \\ 0 & 3 & \lambda-1 \end{bmatrix} X = \begin{bmatrix} -4 \\ 10 \\ 8 \end{bmatrix} \mathrm{R}_2 - 2\mathrm{R}_1, \mathrm{R}_3 - \mathrm{R}_1$$

$$\sim \begin{bmatrix} 1 & -2 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & \lambda-1 \end{bmatrix} X = \begin{bmatrix} -4 \\ 10 \\ -2 \end{bmatrix} \mathrm{R}_3 - \mathrm{R}_2$$

The system is inconsistent if $\lambda = 1$.

Answer: (B)

29. A plane is parallel to two lines whose DRs are (1, 0, –1) and (–1, 1, 0) and it contains the point (1, 1, 1). If it cuts coordinate axis at A, B and C, then the volume of the tetrahedron $OABC$ (in cubic units) is

(A) $\frac{11}{2}$ (B) $\frac{9}{2}$ (C) $\frac{7}{2}$ (D) 6

(IIT-JEE 2004)

Solution: Let $\vec{n}_1 = (1, 0, -1)$ and $\vec{n}_2 = (-1, 1, 0)$. Since the required plane is parallel to $\vec{n}_1$ and $\vec{n}_2$, its normal

$$\vec{n}_1 \times \vec{n}_2 = \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} = \vec{i} + \vec{j} + \vec{k} = (1, 1, 1)$$

Since the plane also passes through the point (1, 1, 1), its equation is

$$1(x-1) + 1(y-1) + 1(z-1) = 0$$
$$\Rightarrow x + y + z = 3$$
$$\Rightarrow \frac{x}{3} + \frac{y}{3} + \frac{z}{3} = 1$$

Therefore, $A = (3, 0, 0)$, $B = (0, 3, 0)$ and $C = (0, 0, 3)$. Hence, the volume of the tetrahedron $OABC$ is

$$\frac{1}{6}\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \frac{27}{6} = \frac{9}{2}$$

Answer: (B)

30. The equation of the plane containing the straight line

$$\frac{x}{2} = \frac{y}{3} = \frac{z}{4}$$

which is perpendicular to the plane containing the lines

$$\frac{x}{3} = \frac{y}{4} = \frac{z}{2} \quad \text{and} \quad \frac{x}{4} = \frac{y}{2} = \frac{z}{3}$$

is

(A) $x + 2y - 2z = 0$ (B) $3x + 2y - 2z = 0$
(C) $x - 2y + z = 0$ (D) $5x + 2y - 4z = 0$

(IIT-JEE 2010)

Solution: The normal to the plane containing the given lines is

$$(3, 4, 2) \times (4, 2, 3) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 4 & 2 \\ 4 & 2 & 3 \end{vmatrix} = 8\vec{i} - \vec{j} - 10\vec{k}$$

Let E be the required plane so that the normal to E is perpendicular to $(8, -1, -10)$. Let $E \equiv ax + by + cz + d = 0$. $E = 0$ passes through $(0, 0, 0) \Rightarrow d = 0$. Also $E = 0$ contains

$$\frac{x}{2} = \frac{y}{3} = \frac{z}{4}$$

Therefore

$$2a + 3b + 4c = 0 \qquad (6.54)$$

Further, the normal (a, b, c) is perpendicular to $(8, -1, -10)$. Therefore

$$8a - b - 10c = 0 \qquad (6.55)$$

From Eqs. (6.54) and (6.55), we get

$$\frac{a}{1} = \frac{b}{-2} = \frac{c}{1}$$

Therefore, the equation of E is $x - 2y + z = 0$.

Answer: (C)

31. If the distance of the point $P(1, -2, 1)$ from the plane $x + 2y - 2z = \alpha$, where $\alpha > 0$, is 5, then the foot of the perpendicular from point P to the plane is

(A) $\left(\frac{8}{3}, \frac{4}{3}, \frac{-7}{3}\right)$ (B) $\left(\frac{4}{3}, \frac{-4}{3}, \frac{1}{3}\right)$

(C) $\left(\frac{1}{3}, \frac{2}{3}, \frac{10}{3}\right)$ (D) $\left(\frac{2}{3}, \frac{1}{3}, \frac{5}{2}\right)$

(IIT-JEE 2010)

Solution: By hypothesis, we get

$$\left|\frac{1 + 2(-2) - 2(1) - \alpha}{\sqrt{1^2 + 2^2 + (-2)^2}}\right| = 5$$

$$\Rightarrow \alpha + 5 = \pm 15$$

$$\Rightarrow \alpha = 10, -20$$

Since $\alpha > 0$, we have $\alpha = 10$. Therefore, the plane is

$$x + 2y - 2z = 10 \qquad (6.56)$$

Let $Q(x_1, y_1, z_1)$ be the foot of the perpendicular from $P(1, -2, 1)$ onto the plane provided in Eq. (6.56). Therefore

$$x_1 + 2y_1 - 2z_1 = 10 \qquad (6.57)$$

Also $\overline{PQ}$ is normal to the plane provided in Eq. (6.57), so that

$$(x_1 - 1, y_1 + 2, z_1 - 1) = \lambda(1, 2, -2)$$

Since $(1, 2, -2)$ is normal to the plane provided in Eq. (6.56), we have

$$x_1 = 1 + \lambda, \quad y_1 = -2 + 2\lambda, \quad z_1 = 1 - 2\lambda$$

Substituting the values of x_1, y_1 and z_1 in Eq. (6.56), we have

$$(1 + \lambda) + 2(-2 + 2\lambda) - 2(1 - 2\lambda) = 10$$

$$\Rightarrow 9\lambda - 5 = 10$$

$$\Rightarrow \lambda = \frac{5}{3}$$

Therefore

$$x_1 = 1 + \lambda = 1 + \frac{5}{3} = \frac{8}{3}$$

$$y_1 = -2 + 2\lambda = -2 + \frac{10}{3} = \frac{4}{3}$$

and

$$z_1 = 1 - 2\lambda = 1 - \frac{10}{3} = \frac{-7}{3}$$

So

$$Q(x_1, y_1, z_1) = \left(\frac{8}{3}, \frac{4}{3}, \frac{-7}{3}\right)$$

Answer: (A)

32. A line with positive direction cosines passes through the point $P(2, -1, 2)$ and makes equal angles with the coordinate axes. The line meets the plane $2x + y + z = 9$ at Q. The length of the line segment PQ is equal to

(A) 1 (B) $\sqrt{2}$ (C) $\sqrt{3}$ (D) 2

(IIT-JEE 2009)

Solution: Since the line makes equal angles with the axes and the DCs are positive, the DCs are

$$\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

Hence, the equation of the line is

$$\frac{x-2}{1/\sqrt{3}} = \frac{y+1}{1/\sqrt{3}} = \frac{z-2}{1/\sqrt{3}}$$

Therefore, every point on the line is of the form

$$x = 2 + \frac{t}{\sqrt{3}},\ y = -1 + \frac{t}{\sqrt{3}},\ z = 2 + \frac{t}{\sqrt{3}}$$

This point lies on the plane $2x + y + z = 9$. So we have

$$2\left(2 + \frac{t}{\sqrt{3}}\right) + \left(-1 + \frac{t}{\sqrt{3}}\right) + \left(2 + \frac{t}{\sqrt{3}}\right) = 9$$

$$\Rightarrow 5+\frac{4t}{\sqrt{3}}=9$$
$$\Rightarrow t=\sqrt{3}$$

Therefore, $Q = (3, 0, 3)$. This gives

$$PQ=\sqrt{(3-2)^2+(0+1)^2+(3-2)^2}=\sqrt{3}$$

Answer: (C)

33. Let $P(3, 2, 6)$ be a point in space and Q be a point on the line $\vec{r}=(\vec{i}-\vec{j}+2\vec{k})+\lambda(-3\vec{i}+\vec{j}+5\vec{k})$. Then the value of λ for which the vector $\overrightarrow{PQ}$ is parallel to the plane $x - 4y + 3z = 1$ is

(A) $\frac{1}{4}$ (B) $\frac{-1}{4}$ (C) $\frac{1}{8}$ (D) $\frac{-1}{8}$

(IIT-JEE 2009)

Solution: The given line

$$\vec{r} = (\vec{i} - \vec{j} + 2\vec{k}) + \lambda(-3\vec{i} + \vec{j} + 5\vec{k})$$

is written in the parametric form as

$$x = 1 - 3\lambda, y = -1 + \lambda, z = 2 + 5\lambda$$

Therefore

$$\overrightarrow{PQ}=(1-3\lambda-3,-1+\lambda-2,2+5\lambda-6)$$
$$=(-3\lambda-2,\lambda-3,5\lambda-4)$$

$\overrightarrow{PQ}$ is parallel to the plane $x - 4y + 3z = 1$

$\Rightarrow$ $\overrightarrow{PQ}$ is perpendicular to the normal of the plane

$$\Rightarrow 1(-3\lambda-2)-4(\lambda-3)+3(5\lambda-4)=0$$
$$\Rightarrow 8\lambda-2=0$$
$$\Rightarrow \lambda=\frac{1}{4}$$

Answer: (B)

34. A plane which is perpendicular to two planes $2x - 2y + z = 0$ and $x - y + 2z = 4$ passes through $(1, -2, 1)$. The distance of the plane from the point $(1, 2, 2)$ is

(A) 0 (B) 1 (C) $\sqrt{2}$ (D) $2\sqrt{2}$

(IIT-JEE 2006)

Solution: We have $\vec{n}_1=(2, -2, 1)$ and $\vec{n}_2=(1, -1, 2)$. The normal to the required plane is

$$\vec{n}_1\times\vec{n}_2=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 2 & -2 & 1\\ 1 & -1 & 2\end{vmatrix}$$
$$=-3\vec{i}-3\vec{j}$$

Since the plane passes through the point $(1, -2, 1)$, its equation is

$$-3(x-1)-3(y+2)+0(z-1)=0$$
$$3x+3y-3+6=0$$
$$\Rightarrow x+y+1=0 \qquad (6.58)$$

The distance of the point $(1, 2, 2)$ from the plane provided in Eq. (6.58) is

$$\left|\frac{1+2+1}{\sqrt{1^2+1^2}}\right|=2\sqrt{2}$$

Answer: (D)

35. If the equation of the plane passing through the intersection of the planes $x + 2y + z - 1 = 0$ and $2x + y + 3z - 2 = 0$ and perpendicular to the plane $x + y + z - 1 = 0$ is $x + ky + 3z - 1 = 0$, then the value of k is

(A) 4 (B) -4 (C) 2 (D) -2

Solution: Let $E_1 = 0$ and $E_2 = 0$ be the given two planes and $E = 0$ be the required plane. Since $E = 0$ passes through the line of intersection of $E_1 = 0$ and $E_2 = 0$,

$$E = E_1 + \lambda E_2 = 0$$
$$\Rightarrow E \equiv (x + 2y + z - 1) + \lambda(2x + y + 3z - 2) = 0$$
$$\Rightarrow E \equiv (1 + 2\lambda)x + (2 + \lambda)y + (1 + 3\lambda)z - 1 - 2\lambda = 0$$

This is perpendicular to the plane $x + y + z - 1 = 0$. So

$$1(1 + 2\lambda) + 1(2 + \lambda) + 1(1 + 3\lambda) = 0$$
$$\Rightarrow 6\lambda + 4 = 0$$
$$\Rightarrow \lambda=\frac{-2}{3}$$

Therefore

$$E\equiv\left(1-\frac{4}{3}\right)x+\left(2-\frac{2}{3}\right)y+(1-2)z-1+\frac{4}{3}=0$$
$$\Rightarrow E\equiv -x+4y-3z+1=0$$
$$\Rightarrow E\equiv x-4y+3z-1=0$$

Hence, $k = -4$.

Answer: (B)

36. The equation of the plane passing through the line of intersection of the planes $x + 2y + 3z = 2$ and $x - y + z = 3$ and at a distance $2/\sqrt{3}$ from the point $(3, 1, -1)$ is

(A) $5x - 11y + z = 17$ (B) $\sqrt{2}x+y-3\sqrt{2}+1=0$

(C) $x + y + z = \sqrt{3}$ (D) $x-\sqrt{2}y+\sqrt{2}-1=0$

(IIT-JEE 2012)

Solution: The required plane is of the form

$$(x + 2y + 3z - 2) + \lambda(x - y + z - 3) = 0$$

$$\Rightarrow (1 + \lambda)\, x + (2 - \lambda)\, y + (3 + \lambda)\, z - 2 - 3\lambda = 0 \quad (6.59)$$

The plane provided in Eq. (6.59) is at a distance of $2/\sqrt{3}$ from $(3, 1, -1)$. So

$$\left|\frac{(1+\lambda)(3)+(2-\lambda)(1)+(3+\lambda)(-1)-2-3\lambda}{\sqrt{(1+\lambda)^2+(2-\lambda)^2+(3+\lambda)^2}}\right| = \frac{2}{\sqrt{3}}$$

$$\Rightarrow \frac{|3+2-3-2-2\lambda|}{\sqrt{3\lambda^2+4\lambda+14}} = \frac{2}{\sqrt{3}}$$

$$\Rightarrow 12\lambda^2 = 4(3\lambda^2 + 4\lambda + 14)$$

$$\Rightarrow 16\lambda = -56$$

$$\Rightarrow \lambda = \frac{-7}{2}$$

From Eq. (6.59), the required plane is

$$x+2y+3z-2-\frac{7}{2}(x-y+z-3)=0$$

$$\Rightarrow -5x + 11y - z + 17 = 0$$

$$\Rightarrow 5x - 11y + z - 17 = 0$$

Answer: (A)

37. The point P is the intersection of the line joining $Q(2, 3, 5)$ and $R(1, -1, 4)$ with the plane $5x - 4y - z - 1 = 0$. If S is the foot of the perpendicular drawn from the point $T(2, 1, 4)$ to QR, then the length of the line segment PS is

(A) $\frac{1}{\sqrt{2}}$ (B) $\sqrt{2}$ (C) 2 (D) $2\sqrt{2}$

(IIT-JEE 2012)

Solution: We have $\overrightarrow{QR} = (-1, -4, -1)$. The equation of the line $\overleftrightarrow{QR}$ is

$$\frac{x-1}{-1} = \frac{y+1}{-4} = \frac{z-4}{-1} = \lambda \quad \text{(say)}$$

Therefore, every point on the line QR is given by

$$x = 1 - \lambda,\ y = -1 - 4\lambda,\ z = 4 - \lambda$$

Point P lies in the plane $5x - 4y - z - 1 = 0$. This implies

$$5(1 - \lambda) - 4(-1 - 4\lambda) - (4 - \lambda) - 1 = 0$$

$$\Rightarrow 5 + 4 - 4 - 5\lambda + 16\lambda + \lambda - 1 = 0$$

$$\Rightarrow 12\lambda = -4$$

$$\Rightarrow \lambda = \frac{-1}{3}$$

Therefore

$$P = (1-\lambda, -1-4\lambda, 4-\lambda)$$

$$= \left(1+\frac{1}{3}, -1+\frac{4}{3}, 4+\frac{1}{3}\right) = \left(\frac{4}{3}, \frac{1}{3}, \frac{13}{3}\right)$$

Now, $S = (1 - \lambda, -1 - 4\lambda, 4 - \lambda)$ is a point on QR and $T = (2, 1, 4)$ and $\overrightarrow{TS}$ is perpendicular to $\overrightarrow{QR}$. This implies

$$\overrightarrow{TS} \cdot \overrightarrow{QR} = 0$$

$$\Rightarrow (1 - \lambda - 2, -1 - 4\lambda - 1, 4 - \lambda - 4) \cdot (-1, -4, -1) = 0$$

$$\Rightarrow (-\lambda - 1, -2 - 4\lambda, -\lambda) \cdot (-1, -4, -1) = 0$$

$$\Rightarrow (\lambda + 1, 2 + 4\lambda, \lambda) \cdot (1, 4, 1) = 0$$

$$\Rightarrow \lambda + 1 + 8 + 16\lambda + \lambda = 0$$

$$\Rightarrow 18\lambda = -9$$

$$\Rightarrow \lambda = -\frac{1}{2}$$

Therefore

$$S = \left(1+\frac{1}{2}, -1+\frac{4}{2}, 4+\frac{1}{2}\right) = \left(\frac{3}{2}, 1, \frac{9}{2}\right)$$

Hence

$$(PS)^2 = \left(\frac{4}{3}-\frac{3}{2}\right)^2 + \left(\frac{1}{3}-1\right)^2 + \left(\frac{13}{3}-\frac{9}{2}\right)^2$$

$$= \frac{1}{36}+\frac{4}{9}+\frac{1}{36} = \frac{18}{36} = \frac{1}{2}$$

$$\Rightarrow PS = \frac{1}{\sqrt{2}}$$

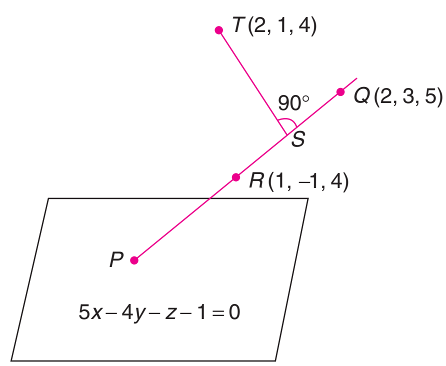


FIGURE 6.37

Answer: (A)

38. Suppose a, b and c are real and

$$[a\,b\,c]\begin{bmatrix} 1 & 9 & 7 \\ 8 & 2 & 7 \\ 7 & 3 & 7 \end{bmatrix} = [0, 0, 0]$$

If the point $P(a, b, c)$, which is satisfying the above condition, lies in the plane $2x + y + z = 1$, then the value of $7a + b + c$ is

(A) 0 (B) 12 (C) 7 (D) 6

(IIT-JEE 2011)

Solution: We have

$$[a\,b\,c]\begin{bmatrix} 1 & 9 & 7 \\ 8 & 2 & 7 \\ 7 & 3 & 7 \end{bmatrix} = [0, 0, 0]$$

$$\Rightarrow [a + 8b + 7c, 9a + 2b + 3c, 7a + 7b + 7c] = [0, 0, 0]$$

Therefore

$$a + 8b + 7c = 0 \quad (6.60)$$

$$9a + 2b + 3c = 0 \quad (6.61)$$

$$a + b + c = 0 \quad (6.62)$$

Hence

$$9a + 2b + 3(-a - b) = 0 \qquad (\because c = -a - b)$$

$$\Rightarrow 6a - b = 0, c = -7a$$

$$\Rightarrow \frac{a}{1} = \frac{b}{6} = \frac{c}{-7}$$

$P(a, b, c)$ lies on the plane $2x + y + z = 1$. This implies

$$2a + b + c = 1$$

$$\Rightarrow 2a + 6a - 7a = 1 \qquad (\because\ b = 6a, c = -7a)$$

$$\Rightarrow a = 1$$

Therefore

$$7a + b + c = 7(1) + 6 - 7 = 6$$

Answer: (D)

39. A line through origin meets the lines

$$\frac{x-2}{1} = \frac{y-1}{-2} = \frac{z+1}{1} \quad \text{and} \quad \frac{x-(8/3)}{2} = \frac{y+3}{-1} = \frac{z-1}{1}$$

at P and Q, respectively, then the length PQ is

(A) $\sqrt{5}$ (B) $\sqrt{6}$ (C) $\sqrt{7}$ (D) $2\sqrt{2}$

(IIT-JEE 2010)

Solution: The line $\overleftrightarrow{PQ}$ passes through (0, 0, 0). This implies $\overrightarrow{OP} = \lambda(\overrightarrow{OQ})$. Therefore

$$\underbrace{\frac{2+t}{(8/3)+2s}}_{\text{(i)}} = \underbrace{\frac{2t-1}{3+s}}_{\text{(ii)}} = \underbrace{\frac{t-1}{s+1}}_{\text{(iii)}}$$

(i) = (ii) $\Rightarrow (2 + t)(3 + s)3 = (2t - 1)(8 + 6s)$

$$\Rightarrow 3ts + 9t + 6s + 18 = 12ts - 6s + 16t - 8$$

$$\Rightarrow 9ts = -7t + 12s + 26 \quad (6.63)$$

(ii) = (iii) $\Rightarrow (2t - 1)(s + 1) = (t - 1)(s + 3)$

$$\Rightarrow 2ts + 2t - s - 1 = ts + 3t - s - 3$$

$$\Rightarrow ts = t - 2$$

$$\Rightarrow s = \frac{t-2}{t} \quad (6.64)$$

Substituting the value of s in Eq. (6.63), we have

$$2t^2 - 7t + 3 = 0$$

Hence $t = 3$ or $1/2$.

Case 1: When $t = 3$ and $s = \frac{t-2}{3} = \frac{1}{3}$

$$\text{(i)} = \text{(iii)} \Rightarrow 3ts = -5t + 12s + 14 \quad (6.65)$$

Now, $t = 3$ and $s = 1/3$ also satisfy Eq. (6.65). Therefore, $P = (5, -5, 2)$ and $Q = (10/3, -10/3, 4/3)$. Hence

$$(PQ)^2 = \frac{25}{9} + \frac{25}{9} + \frac{4}{9} = \frac{54}{9} = 6$$

Case 2: When $t = 1/2$ and $s = -3$ do not satisfy Eq. (6.65), $P = (5, -5, 2)$ and $Q = (10/3, -10/3, 4/3)$. Hence,

$$PQ = \sqrt{6}$$

Answer: (B)

40. A variable plane at a distance of 1 unit from the origin cuts the coordinate axes at A, B and C. If the centroid $D(x, y, z)$ of ΔABC satisfies the relation

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = k$$

then the value of k is

(A) 9 (B) $\frac{1}{3}$ (C) 1 (D) 3

Solution: Let $A = (a, 0, 0)$, $B = (0, b, 0)$ and $C = (0, 0, c)$. The equation of the plane ABC is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

Since the distance of the plane ABC from origin is 1, we have

$$\frac{1}{\sqrt{(1/a^2) + (1/b^2) + (1/c^2)}} = 1$$

$$\Rightarrow \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = 1 \quad (6.66)$$

The centroid $D(x, y, z)$ is $(a/3, b/3, c/3)$. This gives

$$x = \frac{a}{3}, y = \frac{b}{3}, z = \frac{c}{3}$$

So

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = k$$
$$\Rightarrow \frac{9}{a^2} + \frac{9}{b^2} + \frac{9}{c^2} = k$$
$$\Rightarrow 9 = k$$

Answer: (A)

41. The normal to the plane passing through the points $A = (2, 1, -1)$, $B = (1, 1, 1)$ and $C = (3, 3, 0)$ is

(A) $(4, -3, 2)$ (B) $(4, 3, 2)$
(C) $(-4, 3, 2)$ (D) $(4, 3, -2)$

Solution: The normal to the plane ABC is

$$\overline{AB} \times \overline{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 0 & 2 \\ 1 & 2 & 1 \end{vmatrix} = -4\vec{i} + 3\vec{j} - 2\vec{k} = -(4, -3, 2)$$

Answer: (A)

42. The image of the point $P(3, 5, 7)$ in the plane $2x + y + z = 0$ is

(A) $(9, 1, -1)$ (B) $(-9, -1, 1)$
(C) $(-9, 1, -1)$ (D) $(9, -1, -1)$

Solution: See Fig. 6.38. Let $Q(x_1, y_1, z_1)$ be the foot of the perpendicular from $P(3, 5, 7)$ onto the plane. Therefore, $\overrightarrow{PQ}$ is parallel to the normal to the plane. So

$$(x_1 - 3, y_1 - 5, z_1 - 7) = \lambda(2, 1, 1)$$
$$\Rightarrow x_1 = 3 + 2\lambda, y_1 = 5 + \lambda \text{ and } z_1 = 7 + \lambda$$

(x_1, y_1, z_1) lies in the plane. This implies

$$2(3 + 2\lambda) + (5 + \lambda) + (7 + 8) = 0$$
$$\Rightarrow 6\lambda = -18$$
$$\Rightarrow \lambda = -3$$

Therefore, $Q = (x_1, y_1, z_1) = (-3, 2, 4)$. If $P'(\alpha, \beta, \gamma)$ is the image of $P(3, 5, 7)$, then Q is the midpoint of $\overline{PP'}$ which implies that

$$\frac{\alpha + 3}{2} = -3 \Rightarrow \alpha = -9$$
$$\frac{\beta + 5}{2} = 2 \Rightarrow \beta = -1$$

and $$\frac{\gamma + 7}{2} = 4 \Rightarrow \gamma = +1$$

Hence, the image of $P = P' = (\alpha, \beta, \gamma) = (-9, -1, 1)$.

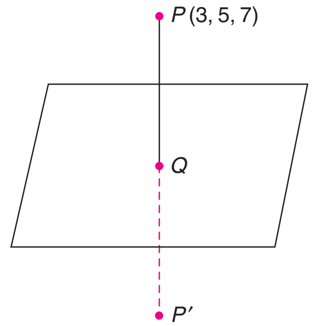


FIGURE 6.38

Answer: (B)

43. The planes $E_1 \equiv x - y - z - 4 = 0$ and $E_2 \equiv x + y + 2z - 4 = 0$ intersect in line L. The plane $E_1 = 0$ is rotated about the line L through 90°. The equation of $E_1 = 0$ in its new position is

(A) $5x + y - 4z - 20 = 0$
(B) $5x - y + 4z - 20 = 0$
(C) $5x + y + 4z - 20 = 0$
(D) $5x + y + 4z + 20 = 0$

Solution: See Fig. 6.39. Let E be the required plane. Therefore,

$$E \equiv E_1 + \lambda E_2 = 0$$
$$\Rightarrow E \equiv (x - y - z - 4) + \lambda(x + y + 2z - 4) = 0$$
$$\equiv (1 + \lambda)x + (\lambda - 1)y + (2\lambda - 1)z - 4 - 4\lambda = 0$$

E is perpendicular to E_1. This implies

$$(1 + \lambda)(1) + (\lambda - 1)(-1) + (2\lambda - 1)(-1) = 0$$
$$\Rightarrow -2\lambda + 1 + 1 + 1 = 0$$
$$\Rightarrow \lambda = \frac{3}{2}$$

Therefore

$$E \equiv \left(1 + \frac{3}{2}\right)x + \left(\frac{3}{2} - 1\right)y + (3 - 1)z - 4 - 6 = 0$$
$$\Rightarrow E \equiv 5x + y + 4z - 20 = 0$$

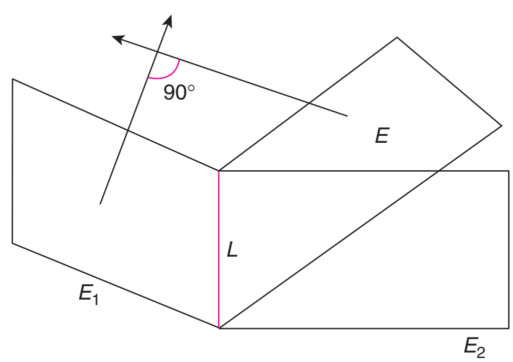


FIGURE 6.39

Answer: (C)

44. The equation of the plane passing through the points (3, 4, 1) and (0, 1, 0) and parallel to the line

$$\frac{x+3}{2}=\frac{y-3}{7}=\frac{z-2}{5}$$

is

(A) $8x + 13y + 15z + 13 = 0$
(B) $8x + 13y - 15z - 13 = 0$
(C) $8x - 13y - 15z + 13 = 0$
(D) $8x - 13y + 15z + 13 = 0$

Solution: Let the required plane be $E \equiv ax + by + cz + d = 0$. Since it passes through (3, 4, 1) and (0, 1, 0), we have

$$3a + 4b + c + d = 0 \qquad (6.67)$$

and

$$b + d = 0 \qquad (6.68)$$

Also since the plane $E = 0$ is parallel to the given line, we have

$$2a + 7b + 5c = 0 \qquad (6.69)$$

From Eq. (6.68), we have $d = -b$. Substituting this value in Eq. (6.67), we have

$$3a + 3b + c = 0 \qquad (6.70)$$

From Eqs. (6.69) and (6.70), we get

$$2a + 7b + 5(-3a - 3b) = 0$$

$$\Rightarrow -13a - 8b = 0$$

Therefore

$$a=\frac{-8}{13}b$$

$$c=-3a-3b=\frac{24}{13}b-3b=\frac{-15b}{13}$$

Hence

$$a=\left(\frac{-8}{13}\right)b,\ c=\frac{-15}{13}b,\ d=-b$$

Therefore, the equation of the plane is

$$\frac{-8}{13}bx+by-\left(\frac{15}{13}b\right)z-b=0$$

$$\Rightarrow 8x - 13y + 15z + 13 = 0$$

Answer: (D)

45. The symmetric form of the line of intersection of the two planes $4x + 4y - 5z - 12 = 0$ and $8x + 12y - 13z - 32 = 0$ is

(A) $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z}{4}$ (B) $\frac{x+1}{2}=\frac{y-2}{3}=\frac{z-1}{4}$

(C) $\frac{x}{2}=\frac{y-2}{3}=\frac{z}{4}$ (D) $\frac{x+1}{2}=\frac{y+2}{3}=\frac{z}{4}$

Solution: Let L be the line of intersection. We know that $\vec{n}_1=(4, 4, -5)$ and $\vec{n}_2=(8, 12, -13)$ are the normals of the given planes. Therefore, the DRs of the line L is

$$\vec{n}_1\times\vec{n}_2=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 4 & 4 & -5\\ 8 & 12 & -13\end{vmatrix}$$

$$=8\vec{i}+12\vec{j}+16\vec{k}$$

Therefore, the DRs of L is (2, 3, 4). Also, substituting $z = 0$ in both plane equations, we get

$$4x + 4y = 12 \text{ and } 8x + 12y = 32$$

Solving these equations, we get $x = 1$ and $y = 2$. Hence, (1, 2, 0) is a point on the line L. Thus, the equation of the line is

$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-0}{4}$$

Answer: (A)

46. The equation of the plane passing through the point (1, –1, 2) and perpendicular to each of the planes $2x + 3y - 2z = 5$ and $x + 2y - 3z = 8$ is

(A) $5x + 4y - z = 7$ (B) $5x - 4y + z = 7$
(C) $5x - 4y - z = 7$ (D) $5x + 4y + z = 7$

Solution: Let $\vec{n}_1=(2, 3, -2)$ and $\vec{n}_2=(1, 2, -3)$ which are the normals of the given planes, respectively, so that the normal to the required plane is

$$\vec{n}_1\times\vec{n}_2=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 2 & 3 & -2\\ 1 & 2 & -3\end{vmatrix}$$

$$=-5\vec{i}+4\vec{j}+\vec{k}$$

The required plane passes through the point (1, –1, 2). Hence its equation is

$$-5(x - 1) + 4(y + 1) + 1(z - 2) = 0$$

$$\Rightarrow 5x - 4y - z = 7$$

Answer: (C)

47. The coordinates of the point where the line

$$\frac{x+1}{2}=\frac{y+2}{3}=\frac{z+3}{4}$$

meets the plane $x + y + 4z - 6 = 0$ are

(A) (–1, –1, –1) (B) (1, –1, –1)
(C) (–1, 1, –1) (D) (1, 1, 1)

Solution: $P = (-1 + 2\lambda, -2 + 3\lambda, -3 + 4\lambda)$ is a point on the given line. Point P belongs to the given plane implies that

$$(-1 + 2\lambda) + (-2 + 3\lambda) + 4(-3 + 4\lambda) = 6$$

$$\Rightarrow 21\lambda = 21$$

$$\Rightarrow \lambda = 1$$

Therefore, the required point is $(-1 + 2, -2 + 3, -3 + 4) = (1, 1, 1)$.

Answer: (D)

48. The equation of the plane containing both lines

$$\frac{x-3}{2} = \frac{y+1}{-3} = \frac{z+2}{1} \text{ and } \frac{x-7}{-3} = \frac{y}{1} = \frac{z+7}{2}$$

is

(A) $x + y - z = 0$ (B) $x + y + z = 0$
(C) $x + y + z = 1$ (D) $x + y + z = 2$

Solution: The DRs of the required plane is

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -3 & 1 \\ -3 & 1 & 2 \end{vmatrix} = -7\vec{i} - 7\vec{j} - 7\vec{k}$$

Therefore, (1, 1, 1) is the normal vector which passes through the point (7, 0, −7). So we have

$$1(x - 7) + 1(y - 0) + 1(z + 7) = 0$$

$$\Rightarrow x + y + z = 0$$

Answer: (B)

49. The equation of the plane passing through the intersection of the planes $x + 2y + 3z + 4 = 0$, $4x + 3y + 2z + 1 = 0$ and the origin is

(A) $x + y + z = 0$ (B) $3x + 2y + z + 1 = 0$
(C) $2x + 3y + z = 0$ (D) $3x + 2y + z = 0$

Solution: The required plane is of the form

$$(x + 2y + 3z + 4) + \lambda(4x + 3y + 2z + 1) = 0$$

This passes through (0, 0, 0). This implies

$$4 + \lambda = 0 \Rightarrow \lambda = -4$$

Therefore, the required plane is

$$-15x - 10y - 5y = 0$$

$$\Rightarrow 3x + 2y + z = 0$$

Answer: (D)

50. A line with DCs proportional to 2, 1, 2 meets each of the lines $x = y + a = z$ and $x + a = 2y = 2z$. The coordinates of each of the points of intersection are

(A) $(3a, 2a, 3a), (a, a, 2a)$
(B) $(3a, 2a, 3a), (a, a, a)$
(C) $(3a, 3a, 3a), (a, a, a)$
(D) $(2a, 3a, 3a), (2a, a, a)$

Solution: The given line L_1 is

$$\frac{x}{1} = \frac{y+a}{1} = \frac{z}{1}$$

P is a point on L_1. Therefore

$$P = (t, -a + t, t) \tag{6.71}$$

and the given line L_2 is

$$\frac{x+a}{2} = \frac{y}{1} = \frac{z}{1}$$

Q is a point on L_2. Therefore

$$Q = (-a + 2s, s, s) \tag{6.72}$$

From Eqs. (6.71) and (6.72), the DRs of $\overrightarrow{PQ}$ are

$$(2s - a - t, s - t + a, s - t)$$

Therefore, by hypothesis, we have

$$\frac{2s-a-t}{2} = \frac{s-t+a}{1} = \frac{s-t}{2}$$

Hence

$$\frac{2s-a-t}{2} = \frac{s-t+a}{1} \Rightarrow t = 3a$$

$$\frac{s-t+a}{1} = \frac{s-t}{2} \Rightarrow s - t = -2a \Rightarrow s = a \quad (\because t = 3a)$$

Therefore, $P = (3a, 2a, 3a)$ and $Q = (a, a, a)$.

Answer: (B)

51. If the angle θ between the line

$$\frac{x+1}{1} = \frac{y-1}{2} = \frac{z-2}{2}$$

and plane $2x - y + \sqrt{\lambda}z + 4 = 0$

is such that $\sin\theta = 1/3$, then the value of λ is

(A) $\frac{5}{3}$ (B) $\frac{-3}{5}$ (C) $\frac{3}{4}$ (D) $\frac{-4}{3}$

Solution: The DRs of the line $= (1, 2, 2) = \vec{n}_1$. The DRs of the normal to the plane $= (2, -1, \sqrt{\lambda}) = \vec{n}_2$. By hypothesis, $90° - \theta$ is the angle between $\vec{n}_1$ and $\vec{n}_2$. Therefore

$$\cos(90° - \theta) = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|} = \frac{2 - 2 + 2\sqrt{\lambda}}{3\sqrt{2^2 + 1^2 + \lambda}}$$

Hence

$$\frac{1}{3}=\frac{2\sqrt{\lambda}}{3\sqrt{5+\lambda}}\Rightarrow 5+\lambda=4\lambda\Rightarrow \lambda=\frac{5}{3}$$

Answer: (A)

52. If the lines

$$\frac{x-1}{k}=\frac{y-2}{2}=\frac{z-3}{3}$$

and $$\frac{x-2}{3}=\frac{y-3}{k}=\frac{z-1}{2}$$

intersect at a point, then the integer value of k is

(A) 2 (B) −2 (C) −5 (D) 5

Solution: $P = (1 + kt, 2 + 2t, 3 + 3t)$ is a point on the first line and $Q = (2 + 3s, 3 + ks, 1 + 2s)$ is a point on the second line. The lines intersect. So

$$1 + kt = 2 + 3s \Rightarrow kt - 3s = 1 \quad (6.73)$$

$$2 + 2t = 3 + ks \Rightarrow 2t - ks = 1 \quad (6.74)$$

and $$3 + 3t = 1 + 2s \Rightarrow 3t - 2s = -2 \quad (6.75)$$

From Eqs. (6.73)–(6.75) and eliminating t and s, we have

$$2k^2 + 5k - 25 = 0$$

whose roots are −5 and $5/2$. Integer value of k is −5.

Aliter: We know that the points $A(1, 2, 3)$ and $B(2, 3, 1)$ are points on the first and the second line, respectively. Also the DRs of the lines are $(k, 2, 3)$ and $(3, k, 2)$, respectively. The lines intersect. So $\overrightarrow{AB}$, $(k, 2, 3)$ and $(3, k, 2)$ are coplanar vectors. Therefore

$$\begin{vmatrix} 1 & 1 & -2 \\ k & 2 & 3 \\ 3 & k & 2 \end{vmatrix}=0$$

$$\Rightarrow 1(4-3k)-1(2k-9)-2(k^2-6)=0$$

$$\Rightarrow -2k^2-5k+25=0$$

$$\Rightarrow 2k^2+5k-25=0$$

Therefore, the integer value of $k = -5$ or $5/2$.

Answer: (C)

53. The line joining the points $(5, 1, a)$ and $(3, b, 1)$ crosses the yz-plane at the point $(0, 17/2, -13/2)$. Then

(A) $a=6, b=4$ (B) $a=8, b=2$

(C) $a=2, b=8$ (D) $a=4, b=6$

Solution: Let $P=(5,1,a)$, $Q=(3,b,1)$ and $R=(0,17/2, -13/2)$. The line $\overrightarrow{PQ}$ meets yz-plane in $R(0,17/2,-13/2)$. So the vectors $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ are parallel vectors. This gives

$$\overrightarrow{PQ}=\lambda\overrightarrow{QR}$$

$$\Rightarrow(-2,b-1,1-a)=\lambda\left(-3,\frac{17}{2}-b,\frac{-13}{2}-1\right)$$

$$\Rightarrow\frac{-2}{-3}=\frac{b-1}{17/2-b}=\frac{1-a}{-13/2-1}$$

$$\Rightarrow\frac{2}{3}=\frac{2(b-1)}{17-2b}=\frac{2(a-1)}{15}$$

$$\Rightarrow a=6, b=4$$

Answer: (A)

Multiple Correct Answer Type Questions

1. The equation of a plane containing the line

$$\frac{x-1}{1}=\frac{y+1}{-2}=\frac{z}{1}$$

is

(A) $x+y+z=0$ (B) $4x+y-2z=3$

(C) $3x+2y+z=2$ (D) $3x+2y+z=0$

Solution: The required plane is of the form

$$a(x-1)+b(y+1)+cz=0$$

Since (a, b, c) is normal to the plane, we have

$$a-2b+c=0 \quad (6.76)$$

We know that $a = 1, b = 1, c = 1$; $a = 4, b = 1, c = -2$ and $a = 3, b = 2, c = 1$ satisfy Eq. (6.76). Substituting these in the equation of the plane we get

$$x+y+z=0$$

$$4x+y-2z=3$$

$$3x+2y+z=2$$

Answer: (A), (B), (C)

2. Which of the following equations represent the line of intersection of the planes $4x - 4y - z + 11 = 0 = x + 2y - z - 1$?

(A) $\dfrac{x-2}{1}=\dfrac{y}{1}=\dfrac{z-3}{4}$ (B) $\dfrac{2x-13}{4}=\dfrac{4y-15}{4}=\dfrac{z}{4}$

(C) $\dfrac{x}{2}=\dfrac{y-2}{1}=\dfrac{z-3}{4}$ (D) $\dfrac{x-4}{-2}=\dfrac{y-4}{2}=\dfrac{z-11}{2}$.

Solution: We have

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & -4 & -1 \\ 1 & 2 & -1 \end{vmatrix} = 6\vec{i} + 3\vec{j} + 12k$$

Hence, the DRs of the line are (2, 1, 4). In the given plane equations, substitute $x = 0$ so that we have $4y + 8 = 11$, $2y - z = 1$. On solving these equations, we have (0, 2, 3) is a point on this line. Hence, the line equation is

$$\frac{x}{2} = \frac{y-2}{1} = \frac{z-1}{4}$$

Similarly, taking $z = 0$ and solving the two equations for x and y, we have $x = -13/2$ and $y = -15/4$ so that $(-13/2, -15/4, 0)$ is a point on the line. Hence, the equation of the line is

$$\frac{x + (13/2)}{2} = \frac{y + (15/4)}{1} = \frac{z}{4}$$

$$\Rightarrow \frac{2x + 13}{4} = \frac{4y + 15}{4} = \frac{z}{4}$$

It is clear that (2, 0, 3) is not a point on the line and (–2, 2, 2) is not parallel to (2, 1, 4).

Answer: (B), (C)

3. A point on the line

$$\frac{x-1}{1} = \frac{y-2}{2} = \frac{z+1}{3}$$

at a distance of $\sqrt{6}$ units from the origin is

(A) $\left(\frac{5}{7}, \frac{10}{7}, \frac{-13}{7}\right)$ (B) $(-1, -2, 1)$

(C) $(1, 2, -1)$ (D) $\left(\frac{-5}{7}, \frac{-10}{7}, \frac{13}{7}\right)$

Solution: Any point P on the given line is of the form $(1 + t, 2 + 2t, -1 + 3t)$. If O is the origin and $OP = \sqrt{6}$, we have

$$(1+t)^2 + 4(1+t)^2 + (3t-1)^2 = 6$$

$$\Rightarrow 14t^2 + 4t + 6 = 6$$

$$\Rightarrow t = 0, \frac{-2}{7}$$

Case 1: $t = 0 \Rightarrow P = (1, 2, -1)$.

Case 2: $t = \frac{-2}{7} \Rightarrow P = \left(1 - \frac{2}{7}, 2 - \frac{4}{7}, -1 - \frac{6}{7}\right) = \left(\frac{5}{7}, \frac{10}{7}, \frac{-13}{7}\right)$.

Answer: (A), (C)

4. Consider the planes $P_1: x - y + z = 1$, $P_2: x + y - z = -1$ and $P_3: x - 3y + 3z = 2$. Let $L_1 = P_2 \cap P_3$, $L_2 = P_3 \cap P_1$ and $L_3 = P_1 \cap P_2$ be their lines of intersection two by two. Then

(A) L_1, L_2, L_3 are parallel.

(B) At least two of L_1, L_2, L_3 are parallel.

(C) L_1, L_2, L_3 intersect in a point.

(D) The three planes do not have a common point.

Solution: The DRs of L_1 are

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & -1 \\ 1 & -3 & 3 \end{vmatrix} = -4\vec{j} - 4\vec{k}$$

The DRs of L_2 are

$$L_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -3 & +3 \\ 1 & -1 & 1 \end{vmatrix} = 2\vec{j} + 2\vec{k}$$

The DRs of L_3 are

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = 2\vec{j} + 2\vec{k}$$

Therefore,

$$-2(\text{DRs of } L_2) = \text{DRs of } L_1 = -2(\text{DRs of } L_3)$$

Hence, the lines L_1, L_2 and L_3 are parallel.

Answer: (A), (B), (D)

5. Consider the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$. Then

(A) Vector $14\vec{i} + 2\vec{j} + 15\vec{k}$ is parallel to their line of intersection.

(B) The two planes are intersecting.

(C) The symmetric form of their line of intersection is $\frac{x-3}{14} = \frac{y+1}{2} = \frac{z}{15}$.

(D) The line of intersection meets the line $\frac{x}{1} = \frac{y}{1} = \frac{z}{1}$ at (1, 1, 0).

Solution: The DRs of the line of intersection of the planes are

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -6 & -2 \\ 2 & 1 & -2 \end{vmatrix} = 14\vec{i} + 2\vec{j} + 15\vec{k}$$

Clearly, the planes are not parallel. If we substitute $z = 0$ in both plane equations and solve for x and y, we have

$x = 3$ and $y = -1$ so that the equation of the line of intersection is

$$\frac{x-3}{14}=\frac{y+1}{2}=\frac{2}{15}$$

Answer: (A), (B), (C)

Try it out Try that Choice (D) is not correct in Problem 5 above.

6. Let P be the point $(1, 2, 3)$ and $E \equiv x + y + z + 1 = 0$ be a plane. Then

(A) P lies on the origin side of the plane $E = 0$.

(B) Distance of P from the plane E is $\sqrt{7/2}$.

(C) Foot of the perpendicular from P on the plane $E = 0$ is $(-4/3, -1/3, 2/3)$.

(D) The line joining origin to P meets the plane $E = 0$ at the point $(1, 1, 1)$.

Solution:

(A) We have

$$P=(x_1,y_1,z_1)\Rightarrow E_{11}=1+2+3+1=7>0$$
$$O=(0,0,0)\Rightarrow E_{22}=1>0$$

Hence, from the Note under Theorem 6.52, it is clear that P lies on the origin side of the plane.

(B) Distance of $P(1, 2, 3)$ from the plane is

$$\frac{|1+2+3+1|}{\sqrt{1^2+2^2+3^2}}=\frac{7}{\sqrt{14}}=\sqrt{\frac{7}{2}}$$

(C) Let $Q(x_1, y_1, z_1)$ be the foot of the perpendicular from P onto the plane $E = 0$. Therefore

$$x_1+y_1+z_1+1=0 \tag{6.77}$$

Also $\overrightarrow{PQ}$ is normal to the plane $E = 0 \Rightarrow \overrightarrow{PQ} = \lambda(1,1,1)$. So

$$x_1-1=\lambda, y_1-2=\lambda, z_1-3=\lambda$$
$$\Rightarrow x_1=\lambda+1, y_1=\lambda+2, z_1=3+\lambda$$

Substituting the values of x_1, y_1 and z_1 in Eq. (6.77), we have

$$3\lambda+7=0\Rightarrow \lambda=-7/3$$

Therefore

$$Q=(x_1,y_1,z_1)=(1+\lambda,2+\lambda,3+\lambda)$$
$$=\left(1-\frac{7}{3},2-\frac{7}{3},3-\frac{7}{3}\right)$$
$$=\left(\frac{-4}{3},\frac{-1}{3},\frac{2}{3}\right)$$

Answer: (A), (B), (C)

7. The DRs of the normal to the plane passing through $(1, 0, 0), (0, 1, 0)$ which makes an angle $\pi/4$ with plane $x + y = 3$ are

(A) $(1,\sqrt{2},1)$ (B) $(1,1,\sqrt{2})$

(C) $(1, 1, 2)$ (D) $(-1,-1,-\sqrt{2})$

Solution: The equation of the plane is

$$a(x-1)+by+cz=0 \tag{6.78}$$

It passes through $(0, 1, 0)$. So

$$-a+b=0$$
$$\Rightarrow a=b \tag{6.79}$$

Also

$$\frac{1}{\sqrt{2}}=\cos\frac{\pi}{4}=\frac{|a+b|}{\sqrt{1^2+1^2+0^2}\sqrt{a^2+b^2+c^2}}$$
$$\Rightarrow\frac{1}{\sqrt{2}}=\frac{|2a|}{\sqrt{2}\sqrt{2a^2+c^2}}$$
$$\Rightarrow 2a^2+c^2=4a^2$$
$$\Rightarrow 2a^2=c^2$$

Therefore, $a = b$ and $c=\sqrt{2}a$. Hence, $a:b:c=1:1:\sqrt{2}$. The DRs of the normal of the plane are $(1,1,\sqrt{2})$ or $(-1,-1,-\sqrt{2})$.

Answer: (B), (D)

8. Consider the planes $x - 2y + 3z = 1$, $-x + y - 2z = k$ and $x - 3y + 4z = 1$. Then

(A) The three planes have no common point for $k \neq 3$.

(B) The three planes have infinitely many common points for $k \neq 3$.

(C) The three planes have infinitely many common points for $k = 3$.

(D) The three planes have unique common point, for $k = 3$

Solution: Consider the matrix equation

$$\begin{bmatrix}1 & -2 & 3\\ -1 & 1 & -2\\ 1 & -3 & 4\end{bmatrix}X=\begin{bmatrix}-1\\ k\\ 1\end{bmatrix}$$

where

$$X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}$$

Add R_1 to R_2 and subtract R_1 from R_3 to get

$$\begin{bmatrix}1 & -2 & 3\\ 0 & -1 & 1\\ 0 & -1 & 1\end{bmatrix}X=\begin{bmatrix}-1\\ k-1\\ 2\end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} X = \begin{bmatrix} -1 \\ k-1 \\ 3-k \end{bmatrix} \quad (R_3 \to R_3 - R_2) \quad (6.80)$$

If $k = 3$, then the system is consistent and it can be written as

$$\begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} X = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$$

Also $R_1 - 2R_2$ gives

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} X = \begin{bmatrix} -5 \\ 2 \\ 0 \end{bmatrix}$$

$$\Rightarrow x + z = -5, -y + z = 2$$
$$\Rightarrow x = -5 - z, \; y = z - 2$$

Therefore

$$X = \begin{bmatrix} -5-z \\ z-2 \\ z \end{bmatrix} = z\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -5 \\ -2 \\ 0 \end{bmatrix}$$

is a solution. That is, in this case, all three equations have infinitely many solutions. That is, they intersect in a line. If $k \neq 3$, then $0\lambda + 0y + 0z = 3 - k \neq 0$, which is illogical by Eq. (6.80). Hence, there is no common point.

Answer: (B), (C)

Matrix-Match Type Questions

1. Match the items of Column I with those of Column II.

Column I	*Column II*
(A) The foot of the perpendicular drawn from the point (2, −1, 5) onto the line $\dfrac{x-11}{10} = \dfrac{y+2}{-4} = \dfrac{z+8}{-11}$ is	(p) $(-1, 2, -3)$
(B) The coordinates of the point on the line joining the points $A(1, 2, 3)$ and $B(3, 5, 9)$ at a distance 14 units from the midpoint of $\overline{AB}$ are	(q) $(2, 0, 1)$
(C) The point of intersection of the planes $x + 2y - z = 6$, $2x - y + 3z = -13$ and $3x - 2y + 3z = -16$ is	(r) $(1, 2, 3)$
	(s) $\left(6, \dfrac{19}{2}, 18\right)$
(D) $(1, 3, -2)$ are the DRs of line L_1 and $(-2, 2, 4)$ are the DRs of another line L_2. If a line L is perpendicular to both L_1 and L_2, then the DRs of L are	(t) $\left(-2, \dfrac{-5}{2}, -6\right)$

Solution:

(A) Let $P = (2, -1, 5)$ and $Q = (x_1, y_1, z_1)$ be the foot of the perpendicular from P onto the line

$$\frac{x-11}{10} = \frac{y+2}{-4} = \frac{z+8}{-11}$$

Therefore,

$$x_1 = 11 + 10t$$
$$y_1 = -2 - 4t$$
$$z_1 = -8 - 11t$$

for some $t \in \mathbb{R}$. $\overrightarrow{PQ}$ is perpendicular to the given line. So

$$\overrightarrow{PQ} \cdot (10, -4, -11) = 0$$
$$\Rightarrow (9 + 10t, -1 - 4t, -13 - 11t) \cdot (10, -4, -11) = 0$$
$$\Rightarrow 10(9 + 10t) - 4(-1 - 4t) + 121t + 143 = 0$$
$$\Rightarrow 237t + 237 = 0$$
$$\Rightarrow t = -1$$

and hence

$$Q = (x_1, y_1, z_1) = (1, 2, 3)$$

Answer: (A) → (r)

(B) The equation of the line $\overleftrightarrow{AB}$ is

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{6}$$

and hence any point on this line is of the form $P(1 + 2t, 2 + 3t, 3 + 6t)$. Also

$$Q = \text{Midpoint of } \overline{AB} = \left(2, \frac{7}{2}, 6\right)$$

Now

$$PQ = 14$$
$$\Rightarrow (PQ)^2 = (14)^2$$
$$\Rightarrow (2t-1)^2 + \left(3t - \frac{3}{2}\right)^2 + (6t-3)^2 = 196$$
$$\Rightarrow (2t-1)^2 + \frac{(6t-3)^2}{4} + (6t-3)^2 = 196$$
$$\Rightarrow (16t^2 - 16t + 4) + 5(6t-3)^2 = 784$$
$$\Rightarrow 196t^2 - 196t - 735 = 0$$

whose roots are $5/2, -3/2$. Now

$$t = \frac{-3}{2} \Rightarrow P = \left(-2, \frac{-5}{2}, -6\right)$$

and $$t = \frac{5}{2} \Rightarrow P = \left(6, \frac{19}{2}, 18\right)$$

Answer: (B) → (s), (t)

(C) By solving the three equations simultaneously, we get that $x = -1, y = 2, z = -3$.

Answer: (C) → (p)

(D) The DRs of L are givne by

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 3 & -2 \\ -2 & 2 & 4 \end{vmatrix} = 16\vec{i} + 8\vec{k} = 8(2\vec{i} + \vec{k})$$

Therefore, the DRs of $L = (2, 0, 1)$.

Answer: (D) → (q)

2. Match the items of Column I with those of Column II.

Column I	*Column II*
(A) The equation of the locus of a point whose distance from the z-axis is equal to its distance from the xy-plane is	(p) $x^2 + y^2 + z^2 - 6x + 2y - 4z + 5 = 0$
(B) The equation of the sphere with centre at (3, –1, 2) and touching yz-plane is	(q) $y^2 - 2y - 4x + 4z + 6 = 0$

(Continued)

Column I	*Column II*
(C) The equation of the locus of the point whose distance from (2, –1, 3) is 4 is	
(D) The equation of the locus of the point whose distance from the y-axis is equal to its distance from the point (2, 1, –1) is	(s) $x^2 + y^2 - z^2 = 0$ (t) $x^2 + y^2 + z^2 - 4x + 2y - 6z - 2 = 0$

Solution:

(A) The distance of a point from z-axis is $\sqrt{x^2 + y^2}$. The distance of the point from xy-plane is $|z|$. Therefore

$$x^2 + y^2 = z^2 \text{ or } x^2 + y^2 - z^2 = 0$$

Answer: (A) → (s)

(B) Since the sphere touches yz-plane, its radius is $|x|$. Hence, the equation of the sphere is

$$(x-3)^2 + (y+1)^2 + (z-2)^2 = 3^2$$
$$\Rightarrow x^2 + y^2 + z^2 - 6x + 2y - 4z + 5 = 0$$

Answer: (B) → (p)

(C) The locus is

$$(x-2)^2 + (y+1)^2 + (z-3)^2 = 16$$
$$x^2 + y^2 + z^2 - 4x + 2y - 6z - 2 = 0$$

Answer: (C) → (t)

(D) We have

$$\sqrt{x^2 + z^2} = \sqrt{(x-2)^2 + (y-1)^2 + (z+1)^2}$$
$$\Rightarrow x^2 + z^2 = (x-2)^2 + (y-1)^2 + (z+1)^2$$
$$\Rightarrow y^2 - 4x - 2y + 2z + 6 = 0$$

Answer: (D) → (q)

3. Match the items of Column I with those of Column II.

Column I	*Column II*
(A) The equation of the plane parallel to the plane passing through the points (1, 1, 1), (2, 3, 5) and (–1, 0, 2) which is at a distance of 2 units from the given plane is	(p) $x + 2y + 3z = 3$

(Continued)

Column I	*Column II*
(B) The equation of the plane passing through the intersection of the lines $L_1: \frac{x-1}{3}=\frac{y-2}{1}=\frac{z-3}{2}$ $L_2: \frac{x-3}{1}=\frac{y-1}{2}=\frac{z-2}{3}$ and perpendicular to the line joining the point of their intersection and origin is	(q) $19(x+5y-4z)-300=0$ (r) $2x-3y+z+2\sqrt{14}=0$
(C) A plane containing the line $\frac{x-1}{1}=\frac{y-2}{2}=\frac{z-3}{3}$ is	(s) $2x-3y+z-2\sqrt{14}=0$
(D) The equation of the plane which is parallel to the plane $x+5y-4z+5=0$ and whose sum of the intercepts on the coordinate axes is 15 is	(t) $x+2y+3z-14=0$

Solution:

(A) The equation of the plane passing through the points (1, 1, 1) (2, 3, 5) and (–1, 0, 2), from Theorem 6.49, is

$$\begin{vmatrix} x+1 & y & z-2 \\ 1 & 2 & 4 \\ -2 & -1 & 1 \end{vmatrix}=0$$

$$\Rightarrow 6(x+1)-9y+3(z-2)=0$$

$$\Rightarrow 6x-9y+3z=0$$

$$\Rightarrow 2x-3y+z=0$$

Any plane parallel to the above plane is of the form $2x-3y+z+d=0$. Since the distance between those two planes is 2, we have

$$\frac{|d-0|}{\sqrt{2^2+(-3)^2+1^2}}=2$$

$$\Rightarrow d=\pm 2\sqrt{14}$$

Answer: (A) → (r), (s)

(B) Every point on the lines L_1 and L_2 is of the form $P(1+3t, 2+t, 3+2t)$ and $Q(3+s, 1+2s, 2+3s)$, respectively. Now $P=Q$ gives

$$1+3t=3+s \Rightarrow 3t-s=2 \tag{6.81}$$

$$2+t=1+2s \Rightarrow t-2s=-1 \tag{6.82}$$

$$3+2t=2+3s \Rightarrow 2t-3s=-1 \tag{6.83}$$

From the above equations $t=1$ and $s=1$ and the point of intersection of the lines is (4, 3, 5). Now $\overrightarrow{OP}=(4,3,5)$. The required plane is

$$4(x-4)+3(y-3)+5(z-5)=0$$

$$\Rightarrow 4x+3y+5z-50=0$$

Answer: (B) → (t)

(C) The line

$$\frac{x-1}{1}=\frac{y-2}{2}=\frac{z-3}{3}$$

passes through (1, 2, 3). Since the plane passes through (1, 2, 3) which is having normal DRs (1, 2, 3), its equation is

$$1(x-1)+2(y-2)+3(z-3)=0$$

$$\Rightarrow x+2y+3z-14=0$$

Answer: (C) → (p)

(D) The plane parallel to the plane $x+5y-4z+5=0$ is $x+5y-4z=k$. Now,

$$15 = \text{Sum of the intercepts on the axes}$$

$$=k+\frac{k}{5}+\frac{k}{-4}$$

$$=\frac{20k+4k-5k}{20}=\frac{19k}{20}$$

Therefore, $k=300/19$. Hence, the equation of the plane

$$x+5y-4z=\frac{300}{19}$$

Answer: (D) → (q)

Comprehension Type Questions

1. Passage: The equation of the plane passing through a point (x_0, y_0, z_0) and having the vector (a, b, c) as normal is

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$

and this plane is also parallel to the plane $ax+by+cz+d=0$. Answer the following questions.

(i) The equation of the plane passing through the points $A(2,1,0)$, $B(5,0,1)$ and $C(4,1,1)$ is

(A) $x+y-2z-3=0$
(B) $x-y+2z-3=0$
(C) $x+y+2z-3=0$
(D) $x+y-2z+3=0$

(ii) The equation of the plane passing through the point (2, –3, 1) and perpendicular to the vector $3\vec{i}+4\vec{j}+7\vec{k}$ is

(A) $3x + 4y + 7z + 11 = 0$
(B) $3x + 4y + 7y - 1 = 0$
(C) $3x + 4y + 7z + 12 = 0$
(D) $3x + 4y + 7z - 12 = 0$

(iii) The equation of the plane through the point (–3, –3, 1) and normal to the line joining the points (2, 6, 1) and (1, 3, 0) is

(A) $x + 3y + z - 11 = 0$
(B) $2x + y + z + 11 = 0$
(C) $x + 3y + z + 11 = 0$
(D) $x + 2y + 3z - 11 = 0$

Solution:

(i) The normal to the required plane is

$$\overline{AB}\times\overline{AC}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 3 & -1 & 1\\ 2 & 0 & 1\end{vmatrix}=-\vec{i}-\vec{j}+2\vec{k}$$

The plane is passing through $A(2, 1, 0)$. Hence, its equation is

$$-1(x-2) - 1(y-1) + 2(z-0) = 0$$
$$\Rightarrow -x - y + 2z + 3 = 0$$
$$\Rightarrow x + y - 2z - 3 = 0$$

Answer: (A)

(ii) The equation of the plane through (2, –3, 1) and perpendicular to the vector $3\vec{i}+4\vec{j}+7\vec{k}$ is

$$3(x-2) + 4(y+3) + 7(z-1) = 0$$
$$\Rightarrow 3x + 4y + 7z - 1 = 0$$

Answer: (B)

(iii) The normal to the plane is $(1-2, 3-6, -1)=(-1, -3, -1)$. Hence the equation of the plane is

$$-1(x+3) - 3(y+3) - 1(z-1) = 0$$
$$\Rightarrow -x - 3y - z - 12 = 0$$
$$\Rightarrow x + 3y + z + 11 = 0$$

Answer: (C)

2. Passage: The equation of the line passing through a point (x_0, y_0, z_0) and having DRs (l, m, n) is

$$\frac{x-x_0}{l}=\frac{y-y_0}{m}=\frac{z-z_0}{n}$$

Also the DRs of the line represented by the planes

$$a_1x + b_1y + c_1z + d_1 = 0 = a_2x + b_2y + c_2z + d_2$$

is the cross product of the normals (a_1, b_1, c_1) and (a_2, b_2, c_2) of the given planes. Answer the following questions.

(i) The equation of the line drawn from $(1, -1, 0)$ to intersect the lines

$$\frac{x-2}{2}=\frac{y-1}{3}=\frac{z-3}{4} \quad\text{and}\quad \frac{x-4}{4}=\frac{y}{5}=\frac{z+1}{2}$$

orthogonally is

(A) $\dfrac{x-1}{14}=\dfrac{y+1}{-12}=\dfrac{z-1}{2}$

(B) $\dfrac{x-1}{14}=\dfrac{y+1}{-12}=\dfrac{z}{2}$

(C) $\dfrac{x+1}{14}=\dfrac{y-1}{-12}=\dfrac{z}{2}$

(D) $\dfrac{x+1}{14}=\dfrac{y-1}{-12}=\dfrac{z-1}{2}$

(ii) The symmetric form of the equation of the line $4x - 4y - z + 11 = 0 = x + 2y - z - 1$ is

(A) $\dfrac{x-2}{1}=\dfrac{y}{1}=\dfrac{z-3}{4}$

(B) $\dfrac{x-2}{2}=\dfrac{y-2}{1}=\dfrac{z}{4}$

(C) $\dfrac{x}{2}=\dfrac{y-2}{1}=\dfrac{z-3}{4}$

(D) $\dfrac{x+3/2}{2}=\dfrac{y-5/4}{1}=\dfrac{z}{4}$

Solution:

(i) The DRs of the required line are given by

$$\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 2 & 3 & 4\\ 4 & 5 & 2\end{vmatrix}=-14\vec{i}+12\vec{j}-2\vec{k}$$

Therefore, the DRs of the line are $(14, -12, 2)$. The line passes through $(1, -1, 0)$. Hence, its equation is

$$\frac{x-1}{14}=\frac{y+1}{-12}=\frac{z}{2}$$

Answer: (B)

(ii) The DRs of the line of intersection are

$$\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 4 & -4 & -1\\ 1 & 2 & -1\end{vmatrix}=6\vec{i}+3\vec{j}+12\vec{k}=3(2\vec{i}+\vec{j}+4\vec{k})$$

The DRs are $(2, 1, 4)$. In the given equations, substituting $z = 0$, we get

$$4x - 4y = -11$$

and $$x + 2y = 1$$

Solving the equations, we get

$$x = \frac{-3}{2}, y = \frac{5}{4}$$

Hence, $(-3/2, 5/4, 0)$ is a point on the line. Therefore, its equation is

$$\frac{x+(3/2)}{2} = \frac{y-(5/4))}{1} = \frac{z}{4}$$

Answer: (D)

3. Passage: Consider the lines

$$L_1 : \frac{x+1}{3} = \frac{y+2}{1} = \frac{z+1}{2} \text{ and } L_2 : \frac{x-2}{1} = \frac{y+2}{2} = \frac{z-3}{3}$$

Answer the following questions.

(i) Unit vector perpendicular to both L_1 and L_2 is

(A) $\frac{-\vec{i}+7\vec{j}+7\vec{k}}{\sqrt{99}}$ (B) $\frac{-\vec{i}-7\vec{j}+5\vec{k}}{5\sqrt{3}}$

(C) $\frac{-\vec{i}+7\vec{j}+5\vec{k}}{5\sqrt{3}}$ (D) $\frac{7\vec{i}-7\vec{k}-\vec{k}}{\sqrt{99}}$

(ii) The shortest distance between L_1 and L_2 is

(A) 0 (B) $\frac{17}{\sqrt{3}}$ (C) $\frac{41}{5\sqrt{3}}$ (D) $\frac{17}{5\sqrt{3}}$

(iii) The distance of the point $(1, 1, 1)$ from the plane passing through $(-1, -2, -1)$ and whose normal is perpendicular to both L_1 and L_2 is

(A) $\frac{2}{\sqrt{75}}$ (B) $\frac{7}{\sqrt{75}}$ (C) $\frac{13}{\sqrt{75}}$ (D) $\frac{23}{\sqrt{75}}$

(IIT-JEE 2008)

Solution: L_1 and L_2 are parallel to the vectors $\vec{n_1} = (3,1,2)$ and $\vec{n_2} = (1,2,3)$, respectively.

(i) Unit vector perpendicular to both $\vec{n_1}$ and $\vec{n_2}$ is

$$\pm \frac{\vec{n_1} \times \vec{n_2}}{|\vec{n_1} \times \vec{n_2}|}$$

Now

$$\vec{n_1} \times \vec{n_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{vmatrix} = -\vec{i} - 7\vec{j} + 5\vec{k}$$

$$|\vec{n_1} \times \vec{n_2}| = \sqrt{(-1)^2 + (-7)^2 + (5)^2} = \sqrt{75} = 5\sqrt{3}$$

Therefore

$$\frac{\vec{n_1} \times \vec{n_2}}{|\vec{n_1} \times \vec{n_2}|} = \frac{-\vec{i} - 7\vec{j} + 5\vec{k}}{5\sqrt{3}}$$

Answer: (B)

(ii) $A = (-1,-2,-1)$ and $B = (2,-2,3)$ so that $\overrightarrow{AB} = (3,0,4)$. Now

$$\left[\overrightarrow{AB}\ \vec{n_1}\ \vec{n_2}\right] = \begin{vmatrix} 3 & 0 & 4 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{vmatrix} = 3(3-4) + 4(6-1)$$

$$= -3 + 20 \neq 0$$

Therefore, L_1 and L_2 are skew lines. Hence, the shortest distance is

$$\frac{|\overrightarrow{AB} \cdot (\vec{n}_1 \times \vec{n}_2)|}{|\vec{n}_1 \times \vec{n}_2|} = \frac{|(3,0,4)\cdot(-1,-7,5)|}{5\sqrt{3}} = \frac{17}{5\sqrt{3}}$$

Answer: (D)

(iii) The plane passing through the point $(-1, -2, -1)$ whose normal is $\vec{n_1} \times \vec{n_2}$ is

$$-1(x+1) - 7(y+2) + 5(z+1) = 0$$

$$\Rightarrow x + 7y - 5z + 10 = 0 \qquad (6.84)$$

Now, the distance of the plane provided in Eq. (6.84) from the point $(1, 1, 1)$ is

$$\frac{|1+7-5+10|}{\sqrt{1^2+7^2+5^2}} = \frac{13}{\sqrt{75}}$$

Answer: (C)

Integer Answer Type Questions

1. If the distance between the plane $ax - 2y + z = d$ and the plane containing the lines

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} \quad \text{and} \quad \frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$$

is $\sqrt{6}$, then the value $|d|$ is ______.

Solution: Let $\vec{n_1} = (2,3,4)$ and $\vec{n_2} = (3,4,5)$. So

$$\vec{n_1} \times \vec{n_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = -\vec{i} + 2\vec{j} - \vec{k}$$

Hence, the equation of the plane containing the given lines is

$$-1(x-1)+2(y-2)-1(z-3)=0$$
$$\Rightarrow x-2y+z=0$$

This plane is parallel to the given plane if $a = 1$. Hence, the distance between the plane is

$$\frac{|d|}{\sqrt{1^2+2^2+1^2}}=\sqrt{6}$$

Thus, $|d| = 6$.

Answer: 6

2. Point (α, β, γ) lies in the plane $x + y + z = 2$. Let $\vec{a}=\alpha\vec{i}+\beta\vec{j}+\gamma\vec{k}$ and $\vec{k}\times(\vec{k}\times\vec{a})=\vec{0}$, then $\gamma=$ _______.

(IIT-JEE 2006)

Solution: By hypothesis, we have

$$\alpha+\beta+\gamma=2 \qquad (6.85)$$

Therefore

$$\vec{0}=\vec{k}\times(\vec{k}\times\vec{a})=(\vec{k}\cdot\vec{a})\vec{k}-(\vec{k}\cdot\vec{k})\vec{a}$$
$$\Rightarrow \vec{0}=\gamma\vec{k}-\vec{a}$$
$$\Rightarrow \vec{a}=\gamma\vec{k}$$

so that $\alpha = 0$ and $\beta = 0$. Hence, from Eq. (6.85), we get $\gamma = 2$.

Answer: 2

3. The distance between the planes $2x - y + 2z - 4 = 0$ and $2x - y + 2z - 1 = 0$ is _______.

Solution: The distance is

$$\frac{|-4+1|}{\sqrt{2^2+(-1)^2+2^2}}=\frac{3}{3}=1$$

Answer: 1

4. A triangle is so placed that the midpoints of its sides are placed on the coordinate axes. If a, b and c are the sides of the triangle, then the equation of the triangle is

$$\frac{x}{x_1}+\frac{y}{y_1}+\frac{z}{z_1}=1$$

where $a^2+b^2+c^2=k(x_1^2+y_1^2+z_1^2)$. The value of k is equal to _______.

Solution: See Fig. 6.40. Let the intercepts of the plane on the axes be $(x_1, 0, 0)$, $(0, y_1, 0)$ and $(0, 0, z_1)$ so that the equation of the plane is

$$\frac{x}{x_1}+\frac{y}{y_1}+\frac{z}{z_1}=1$$

Let $D = (x_1, 0, 0)$, $E = (0, y_1, 0)$ and $F = (0, 0, z_1)$ which are the midpoints of the sides BC, CA and AB, respectively. Then

$$EF \parallel BC \text{ and } EF=\frac{1}{2}BC \Rightarrow 4(y_1^2+z_1^2)=a^2$$

Similarly

$$4(x_1^2+y_1^2)=c^2 \quad \text{and} \quad 4(z_1^2+x_1^2)=b^2$$

Therefore

$$8(x_1^2+y_1^2+z_1^2)=a^2+b^2+c^2$$

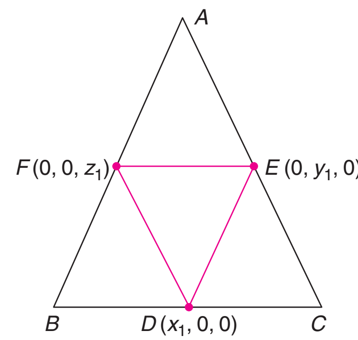


FIGURE 6.40

Answer: 8

5. If the point $S(3, 3, k)$ lies in the plane of the triangle with vertices $A(1,1,1)$, $B(0,-1,0)$ and $C(2,1,-1)$, then the value of k is equal to _______.

Solution: $\overrightarrow{AB}=(-1,-2,-1)$ and $\overrightarrow{AC}=(1,0,-2)$ so that

$$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ -1 & -2 & -1\\ 1 & 0 & -2\end{vmatrix}=4x-3y+2z$$

Since the plane having $(4,-3,2)$ as normal passes through the point $B(0,-1,0)$, its equation is

$$\Rightarrow 4(x-0)-3(y+1)+2(z-0)=0$$
$$\Rightarrow 4x-3y+2z-3=0$$

$S(3, 3, k)$ belongs to the plane. So

$$4(3)-3(3)+2k-3=0 \Rightarrow k=0$$

Answer: 0

6. If the coordinates of the points A, B, C and D are $(3,6,4)$, $(2,5,2)$, $(6,4,4)$ and $(0,2,1)$, respectively, then the length of the projection of $\overrightarrow{AB}$ on $\overrightarrow{CD}$ is _______.

Solution: $\overrightarrow{AB}=(-1,-1,-2)$ and $\overrightarrow{CD}=(-6,-2,-3)$. The length of the projection of $\overrightarrow{AB}$ on $\overrightarrow{CD}$ is

$$\frac{|\overrightarrow{AB}\cdot\overrightarrow{CD}|}{|\overrightarrow{CD}|}=\frac{|6+2+6|}{7}=2$$

Answer: 2

7. If the plane $2ax - 3ay + 4az + 6 = 0$ passes through the midpoint of the centres of the spheres

$$x^2 + y^2 + z^2 + 6x - 8y - 2z - 13 = 0$$

and $$x^2 + y^2 + z^2 - 10x + 4y - 2z - 8 = 0$$

then $|a|$ is equal to _______.

Solution: The centre of the sphere

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + c = 0$$

is $(-u, -v, -w)$. Hence, the centre of the given spheres are $A(-3, 4, 1)$ and $B(5, -2, 1)$ whose midpoint is $(1, 1, 1)$. Since $(1, 1, 1)$ lies on the plane $2ax - 3ay + 4az + 6 = 0$, we have

$$2a - 3a + 4a + 6 = 0 \Rightarrow a = -2$$

Answer: 2

8. If d is the shortest distance of the point $(1, 2, -1)$ from the surface $x^2 + y^2 + z^2 = 54$, then $d/\sqrt{6}$ is equal to _______.

Solution: See Fig. 6.41. The equation

$$x^2 + y^2 + z^2 = 54$$

is a sphere with centre $(0, 0, 0)$ and radius $=\sqrt{54}=3\sqrt{6}$. $P(1, 2, -1)$ is a point inside the sphere such that

$$OP=\sqrt{1^2+2^2+(-1)^2}=\sqrt{6}$$

Thus distance of P from the surface of the sphere is

$$d = 3\sqrt{6}-\sqrt{6}=2\sqrt{6}$$

Therefore

$$\frac{d}{\sqrt{6}}=\frac{2\sqrt{6}}{\sqrt{6}}=2$$

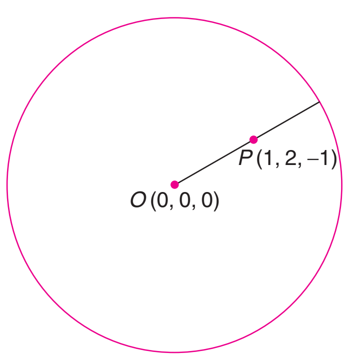


FIGURE 6.41

Answer: 2

9. If the equation of the plane through the intersection of the planes $x + 2y + 3z - 4 = 0$ and $2x + y - z + 5 = 0$ and perpendicular to the plane $5x + 3y + 6z + 8 = 0$ is $ax + by + cz + 173 = 0$, then $b - 9(a + c)$ is equal to _______.

Solution: The required plane is of the form

$$(x + 2y + 3z - 4) + \lambda(2x + y - z + 5) = 0$$

whose normal is $(1+2\lambda, 2+\lambda, 3-\lambda)$. This plane is perpendicular to the plane $5x + 3y + 6z + 8 = 0$. So we have

$$5(1+2\lambda)+3(2+\lambda)+6(3-\lambda)=0$$

$$\Rightarrow 7\lambda=-29$$

$$\Rightarrow \lambda=\frac{-29}{7}$$

Therefore, the required plane is

$$(x+2y+3z-4)-\frac{29}{7}(2x+y-z+5)=0$$

$$\Rightarrow 51x+15y-50z+173=0$$

comparing this with $ax + by + cz + 173 = 0$ we get $a = 51$, $b = 15$, $c = -50$.
so that

$$b - 9(a + c) = 15 - 9 = 6$$

Answer: 6

10. The equation of the planes passing through the line of intersection of the planes $x + 3y + 6 = 0 = 3x - y - 4z$ which is at a distance of 1 unit from the origin are $2x + y - 2z + d_1 = 0$ and $x - 2y - 2z + d_2 = 0$. Then $-d_1d_2$ is equal to _______.

Solution: The required plane is of the form

$$(x+3y+6)+\lambda(3x-y-4z)=0$$

$$\Rightarrow (1+3\lambda)x+(3-\lambda)y+-4\lambda z+6= \qquad (6.86)$$

The distance of this plane provided in Eq. (6.86) from the origin = 1. This gives

$$\frac{|6|}{\sqrt{(1+3\lambda)^2+(3-\lambda)^2+16\lambda^2}}=1$$

$$\Rightarrow 36=26\lambda^2+10$$

$$\Rightarrow \lambda=\pm 1$$

Therefore, the required planes are obtained by substituting $\lambda=\pm 1$ in Eq. (6.86) and they are $2x + y - 2z + 3 = 0$ and $x - 2y - 2z - 3 = 0$. Therefore

$$-d_1d_2 = -3(-3) = 9$$

Answer: 9

SUMMARY

6.1. ***Theorem:*** The distance between the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ is

$$\sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}$$

6.2. ***Theorem (Section formula):*** Let $A = (x_1, y_1, z_1)$ and $B = (x_2, y_2, z_2)$ be two points and $P(x, y, z)$ be a point on the line $\overleftrightarrow{AB}$ such that

$$AP:PB = l:m\,(l+m\neq 0)$$

Then

$$x=\frac{lx_2+mx_1}{l+m}, y=\frac{ly_2+my_1}{l+m}, z=\frac{lz_2+mz_1}{l+m}$$

The coordinates of midpoint are

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$$

6.3. The centroid of a triangle with vertices $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$ is

$$\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right)$$

and centroid of a tetrahedron with vertices (x_r, y_r, z_r) (where $r = 1, 2, 3$ and 4) is

$$\left(\frac{x_1+x_2+x_3+x_4}{4}, \frac{y_1+y_2+y_3+y_4}{4}, \frac{z_1+z_2+z_3+z_4}{4}\right)$$

6.4. ***Definition (DCs and DRs of a line):*** Let L be any line in the space. Any unit vector parallel to the line L is called DC (*direction cosine*) of L and any vector parallel to L is called DR (*direction ratio*) of the line L. If a line makes angles α, β and γ with the x-, y- and z-axes, respectively, and (l, m, n) are the DCs of L, then $l=\cos\alpha, m=\cos\beta, n=\cos\gamma$ or $l=-\cos\alpha, m=-\cos\beta, n=-\cos\gamma$. That is, $(\cos\alpha, \cos\beta, \cos\gamma), (-\cos\alpha, -\cos\beta, -\cos\gamma)$ are the DCs of L.

6.5. ***Theorem:*** If (l, m, n) are the DCs of a line, then $l^2 + m^2 + n^2 = 1$.

Note: If (a, b, c) are the DRs of a line, then its DCs are

$$\left(\frac{\pm a}{\sqrt{a^2+b^2+c^2}}, \frac{\pm b}{\sqrt{a^2+b^2+c^2}}, \pm\frac{c}{\sqrt{a^2+b^2+c^2}}\right)$$

where we have to take all + signs or all – signs.

6.6. ***Theorem:*** The direction ratios of the line joining the two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ are $(x_2 - x_1, y_2 - y_1, z_2 - z_1)$.

6.7. ***Theorem:*** If (l, m, n) are the direction cosines of a line $\overleftrightarrow{AB}$ and $P=(x_1, y_1, z_1)$ and $Q=(x_2, y_2, z_2)$, then the projection vector of $\overleftrightarrow{PQ}$ on $\overleftrightarrow{AB}$ is

$$l(x_2-x_1)\vec{i}+m(y_2-y_1)\vec{j}+n(z_2-z_1)\vec{k}$$

6.8. ***Theorem:*** If (l, m, n) are the DCs of the normal of a plane whose distance from the origin is p, then the equation of the plane is $lx + my + nz = p$. This equation of the plane is called *normal form*.

6.9. ***Definition:*** If a plane meets x-, y- and z-axes at the points $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$, respectively, then a, b, c are called the intercepts of the plane on the coordinate axes.

6.10. ***Theorem (Equation of the plane in intercept form):*** If a, b, c are the intercepts $(abc\neq 0)$ of a plane on the coordinate axes, then its equation is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

6.11. ***Definition:*** If a, b, c and d are real and at least one of a, b and c is not zero, then the equation $ax + by + cz + d = 0$ is called a *first-degree equation* in x, y and z.

6.12. ***Theorem:*** The equation of a plane is a first-degree equation in x, y and z and every first-degree equation in x, y and z represents a plane.

6.13. ***Theorem:*** (1) Two first-degree equations

$$a_1x + b_1y + c_1z + d_1 = 0 \quad\text{and}\quad a_2x + b_2y + c_2z + d_2 = 0$$

represent the same plane if and only if

$$a_1 : b_1 : c_1 : d_1 = a_2 : b_2 : c_2 : d_2$$

(2) Two first-degree equations

$$a_1x + b_1y + c_1z + d_1 = 0 \text{ and } a_2x + b_2y + c_2z + d_2 = 0$$

represent parallel planes if and only if $a_1 : b_1 : c_1 = a_2 : b_2 : c_2$.

6.14. ***Theorem:*** (1) The perpendicular distance of a point $P(x_1, y_1, z_1)$ from a plane $ax + by + cz + d = 0$ is

$$\frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$$

(2) The distance between two parallel planes $ax + by + cz + d_1 = 0$ and $ax + by + cz + d_2 = 0$ is

$$\frac{|d_1-d_2|}{\sqrt{a^2+b^2+c^2}}$$

6.15. ***Theorem:*** The equation of the plane passing through the point (x_1, y_1, z_1) and having $\vec{n}=(a,b,c)$ as normal is

$$a(x-x_1)+b(y-y_1)+c(z-z_1)=0$$

6.16. ***Theorem:*** The equation of the plane determined by three non-collinear points (x_r, y_r, z_r) (where $r = 1, 2$ and 3) is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix}=0$$

6.17. ***Notation:*** If $E \equiv ax+by+cz+d$, then E_{11} means $ax_1+by_1+cz_1+d$.

6.18. ***Theorem:*** If a plane $E=0$ cuts a line joining two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ at a point P, then the ratio

$$AP:PB=-E_{11}:E_{22}$$

Further, A and B are on opposite sides the plane $E=0$ if E_{11} and E_{22} are of opposite signs and A and B lie on the same side if E_{11} and E_{22} are of same signs. In particular, if $E=ax+by+cz+d=0$, then a point (x_1, y_1, z_1) lies in the origin side if E_{11} and d are of same signs and (x_1, y_1, z_1) lies on the non-origin side of the plane, if E_{11} and d have opposite signs.

6.19. ***Theorem:*** If $E_1=0$ and $E_2=0$ are two intersecting planes which intersect in a line L, then any plane passing through the line L is of the form

$$\lambda_1 E_1+\lambda_2 E_2=0$$

where $|\lambda_1|+|\lambda_2|\neq 0$. For practical purpose, we can consider such a plane as

$$E_1+\lambda E_2=0$$

The converse of this theorem is also true.

6.20. ***Theorem:*** If $E_1 \equiv a_1x+b_1y+c_1z+d_1=0$ and $E_2 \equiv a_2x+b_2y+c_2z+d_2=0$ are intersecting planes, then the equations of the plane bisecting the angle between the planes are

$$\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}}=\pm\frac{a_2x+b_2y+c_2z}{\sqrt{a_2^2+b_2^2+c_2^2}}$$

6.21. ***To find the acute angle bisector plane:*** Let $E_1 \equiv a_1x+b_1y+c_1z+d_1=0$ and $E_2 \equiv a_2x+b_2y+c_2z+d_2=0$ be two intersecting lines which do not pass through the origin. Make both d_1 and d_2 positive. Then, the origin angle bisector is always with + signs. That is,

$$\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}}=+\frac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}}$$

Case 1: If $a_1a_2+b_1b_2+c_1c_2>0$, then the origin lies on the obtuse angle region and hence the acute bisector is

$$\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}}=-\frac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}}$$

Case 2: If $a_1a_2+b_1b_2+c_1c_2<0$, then the origin lies on the acute angle region and hence the acute angle bisector is

$$\frac{a_1x+b_1y+c_1z}{\sqrt{a_1^2+b_1^2+c_1^2}}=+\frac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}}$$

6.22. ***Foot of the perpendicular and image:*** Let $E \equiv ax+by+cz+d=0$ be a plane and $P(x_1, y_1, z_1)$ be a point which is not on the plane.

(i) If (x_1', y_1', z_1') is the foot of the perpendicular drawn from $P(x_1, y_1, z_1)$ onto the plane, then

$$\frac{x_1'-x_1}{a}=\frac{y_1'-y_1}{b}=\frac{z_1'-z_1}{c}=\frac{-(ax_1+by_1+cz_1+d)}{a^2+b^2+c^2}$$

(ii) If (x_1'', y_1'', z_1'') is the image of $P(x_1, y_1, z_1)$ in the plane $E=0$, then

$$\frac{x_1''-x_1}{a}=\frac{y_1''-y_1}{b}=\frac{z_1''-z_1}{z}=-2\frac{(ax_1+by_1+cz_1+d)}{a^2+b^2+c^2}$$

6.23. ***Angle between two planes:*** The angle between two planes is defined to be the angle between their normal vectors. If $a_1x+b_1y+c_1z+d_1=0$ and $a_2x+b_2y+c_2z+d_2=0$ are two planes, then the angle between the two planes is

$$\text{Cos}^{-1}\left(\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}\right)$$

6.24. ***Angle between a line and a plane:*** If θ is the angle between a line and the normal to a plane, then $90°-\theta$ is called the angle between the line and the plane.

6.25. ***Equation of a line (Symmetric form):*** The equation of the line which is passing through a point (x_0, y_0, z_0) and having the DRs (a, b, c) is

$$\frac{x-x_0}{a}=\frac{y-y_0}{b}=\frac{z-z_0}{c}$$

6.26. ***Parametric form:*** If (x, y, z) is any point on the line

$$\frac{x-x_0}{a}=\frac{y-y_0}{b}=\frac{z-z_0}{c}$$

then $x=x_0+\lambda a, y=y_0+\lambda b$ and $z=z_0+\lambda c$, where $\lambda\in\mathbb{R}$. These equations are called parametric equations of the line passing through the point (x_0, y_0, z_0) and having DRs (a, b, c).

6.27. ***To write the line of intersection of two planes in symmetric form:*** Suppose the planes $E_1 \equiv a_1x + b_1y + c_1z + d_1 = 0$ and $E_2 \equiv a_2x + b_2y + c_2z + d_2 = 0$ intersect in the line L so that L is parallel to the cross product of the normals (a_1, b_1, c_1) and (a_2, b_2, c_2) of the planes. If $a_1b_2 - a_2b_1 \neq 0$, then substitute $z = 0$ in the plane equations and solve for x and y. Suppose $(x_0, y_0, 0)$ is a point on the line. Then, the symmetric form of the line L is

$$\frac{x-x_0}{b_1c_2-b_2c_1} = \frac{y-y_0}{a_2c_1-a_1c_2} = \frac{z-0}{a_1b_2-a_2b_1}.$$

6.28. ***Nature of three planes:*** Let $E_1 \equiv a_1x + b_1y + c_1z + d_1 = 0$, $E_2 \equiv a_2x + b_2y + c_2z + d_2 = 0$ and $E_3 \equiv a_3x + b_3y + c_3z + d_3 = 0$ be three planes and

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \Delta_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix},$$

$$\Delta_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}, \Delta_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Case 1: If $a_1 : a_2 : a_3 = b_1 : b_2 : b_3 = c_1 : c_2 : c_3$, then the planes are parallel and hence no common points.

Case 2: If $\Delta \neq 0$, then the three planes intersect in unique point.

Case 3: If $\Delta = 0$ and one of Δ_1, Δ_2 and Δ_3 is zero and hence all of them are zero, then the three planes intersect in a single line.

Case 4: If $\Delta = 0$ and at least one of Δ_1, Δ_2 and Δ_3 is not zero, then the three planes form a triangular prism. That is, the planes intersect pairwise in parallel lines.

EXERCISES

Single Correct Choice Type Questions

1. The equation of the plane passing through the line of intersection of the planes $x + y + z = 1$ and $2x + 3y - z = -4$ and parallel to x-axis is

(A) $y + 4z - 7 = 0$ (B) $2y - 4z - 7 = 0$
(C) $y - 2z + 7 = 0$ (D) $y - 4z + 7 = 0$

2. The equation of the plane determined by the points (0, 4, 3) (−2, − 2, 1) and (1, 1, −1) is

(A) $9x + 5y + 6z + 2 = 0$
(B) $9x - 5y + 6z + 2 = 0$
(C) $9x - 5y - 6z + 2 = 0$
(D) $9x - 5y + 6z - 2 = 0$

3. The image of the point (2, −1, 3) in the plane $3x - 2y + z = 9$ is

(A) (1, 1, 8) (B) (2, −1, 2)
(C) (−2, −2, −2) (D) $\left(\frac{8}{7}, \frac{-3}{7}, \frac{19}{7}\right)$

4. The equation of the plane through the line

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$$

and parallel to y-axis is

(A) $2x + z + 1 = 0$ (B) $2x - z + 1 = 0$
(C) $x - 2z + 1 = 0$ (D) $x + 2z - 1 = 0$

5. Point of intersection of the lines

$$\frac{x+4}{3} = \frac{y+6}{5} = \frac{z-1}{-2}$$

and $3x - 2y + z + 5 = 0 = 2x - 3y + 4z - 4$ is

(A) (2, 4, −3) (B) (−4, −6, 1)
(C) (−2, 4, 3) (D) (−2, 4, −3)

6. The planes $2x + y + z + 4 = 0$, $3x + 2y + z + 8 = 0$ and $y - z + 4 = 0$

(A) do not have common points.
(B) have a unique common point.
(C) form a triangular prism.
(D) intersect in a line.

(***Hint:*** Use item 6.26 of the section "Summary")

7. The vertices of a triangle ABC are $A(-1, -2, -3)$, $B(-1, 2, 3)$ and $O(0, 0, 0)$. The direction ratios of the internal bisector of the angle AOB are

(A) 0, 0, 1 (B) −1, 0, 0
(C) −1, 1, 1 (D) −1, −1, −1

8. If the line

$$\frac{x}{1} = \frac{y}{1} = \frac{z}{1}$$

meets the line

$$x\sin A + y\sin B + z\sin C - 2d^2 = 0$$
$$= x\sin 2A + y\sin 2B + zs\sin 2C - d^2$$

then the value of

$$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

where $A+B+C=\pi$ is

(A) $\frac{1}{16}$ (B) $\frac{1}{32}$ (C) $\frac{1}{8}$ (D) $\frac{1}{12}$

9. The image of the point $(-1, 3, 4)$ on the plane $x - 2y = 0$ is

(A) $(15, 11, 4)$ (B) $\left(\frac{-17}{3}, \frac{-19}{3}, 1\right)$

(C) $(8, 4, 4)$ (D) $\left(\frac{-17}{3}, \frac{-19}{3}, 4\right)$

10. The plane $2x + y + 2z - 9 = 0$ meets the coordinate axes at points A, B and C. Then the orthocentre of ΔABC is

(A) $\left(\frac{3}{2}, 3, \frac{3}{2}\right)$ (B) $(2, 1, 2)$

(C) $\left(\frac{9}{7}, \frac{27}{7}, \frac{9}{7}\right)$ (D) $\left(\frac{3}{13}, \frac{1}{13}, \frac{3}{13}\right)$

11. The foot of the perpendicular drawn from $(0, 0, 0)$ to the line joining the points $(-9, 4, 5)$ and $(10, 0, -1)$ is

(A) $(3, 4, 5)$ (B) $(1, 2, 2)$

(C) $(4, 5, 3)$ (D) $\left(\frac{58}{59}, \frac{112}{59}, \frac{109}{59}\right)$

12. The distance of the point $(-2, 3, 1)$ from the line through the point $(-3, 5, 2)$ and making equal angles with the coordinate axes is

(A) $\frac{2}{\sqrt{3}}$ (B) $\frac{16}{\sqrt{3}}$ (C) $\frac{5}{\sqrt{3}}$ (D) $\sqrt{\frac{14}{3}}$

13. The shortest distance between the lines $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ and $\frac{x-5}{4}=\frac{y-4}{4}=\frac{z-5}{5}$ is

(A) 0 (B) $\sqrt{2/3}$ (C) $2\sqrt{3}$ (D) $3\sqrt{2}$

14. The point of intersection of the lines whose parametric equations are $x = 3r + 3$, $y = -4r + 2$, $z = r - 1$ and $x = -6t + 1$, $y = 3t - 2$ and $z = 1$ is

(A) $(9, 6, -1)$ (B) $(9, -6, 1)$

(C) $(-9, 6, -1)$ (D) $(9, 6, 1)$

15. The coordinates of the point on the line joining the points $(-3, 7, -13)$ and $(-6, 1, -10)$ which is nearest to the intersection of the planes $3x - y - 3z + 32 = 0$ and $3x + 2y - 15z - 8 = 0$ are

(A) $(9, 1, 7)$ (B) $(7, 9, 1)$

(C) $(-7, -1, -9)$ (D) $(-7, -9, -1)$

16. The distance of the point $(-2, 3, -4)$ from the line

$$\frac{x+2}{3}=\frac{2y+3}{4}=\frac{3z+4}{5}$$

measured parallel to the plane $4x + 12y - 3z + 1 = 0$ is

(A) $\frac{15}{2}$ (B) $\frac{17}{2}$ (C) $\frac{13}{2}$ (D) 6

17. The line

$$\frac{x-2}{3}=\frac{y+1}{4}=\frac{z-2}{1}$$

meets the plane $x - y + z = 5$ at Q. If P is the point $(-1, -5, -10)$, then the distance PQ is

(A) 10 (B) 11 (C) $\frac{13}{2}$ (D) 13

18. If the lines

$$\frac{x-1}{-3}=\frac{y-2}{2\lambda}=\frac{z-3}{2}$$

and

$$\frac{x-1}{3\lambda}=\frac{y-5}{1}=\frac{z-6}{-5}$$

are at right angles to each other, then λ is equal to

(A) $\frac{9}{7}$ (B) $\frac{-9}{7}$ (C) $\frac{-10}{7}$ (D) $\frac{10}{7}$

19. The shortest distance between any two opposite edges of a tetrahedron formed by the planes $y + z = 0$, $z + x = 0$, $x + y = 0$ and $x + y + z = 1$ is

(A) $\frac{1}{\sqrt{6}}$ (B) $\sqrt{\frac{2}{3}}$ (C) $\frac{1}{\sqrt{3}}$ (D) $2\sqrt{3}$

20. The equation of the plane containing the line

$$\frac{y}{b}+\frac{z}{c}=1, \quad x=0$$

and parallel to the straight line

$$\frac{x}{a}-\frac{z}{c}=1, \quad y=0$$

is

(A) $\frac{x}{a}+\frac{y}{b}-\frac{z}{c}=1$ (B) $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}+1=0$

(C) $\frac{x}{a}-\frac{y}{b}-\frac{z}{c}+1=0$ (D) $\frac{-x}{a}+\frac{y}{b}-\frac{z}{c}=1$

Multiple Correct Choice Type Questions

1. For the lines

$$\frac{x-3}{3}=\frac{y-8}{-1}=\frac{z-3}{1}$$

and $$\frac{x-3}{3}=\frac{y+7}{2}=\frac{z-6}{4}$$

choose the correct choices.

(A) Coplanar and non-parallel.

(B) Skew lines.

(C) The shortest distance between them is $3\sqrt{30}$.

(D) Standard deviation is 0.

2. Consider the line

$$L:\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$$

and the point $P(5, 9, 3)$ in the space. Then

(A) the distance of P from the line is 6.

(B) the distance of P from the line is $\sqrt{6}$.

(C) the foot of the perpendicular from P onto L is $(3, 5, 7)$.

(D) the equation of the perpendicular from P onto the line L is

$$\frac{x-5}{-2}=\frac{y-9}{4}=\frac{z-3}{-4}$$

3. For the two lines

$$\frac{x-1}{3}=\frac{y-2}{-4}=\frac{z}{2}$$

and

$$x = 2 + 3t, y = 4t \quad\text{and}\quad z = 2$$

Identify the correct equations from the following:

(A) $4x + 3y - 10 = 0 = 2y + 4z - 4$

(B) $4x + 3y + 10 = 0 = 2y + 4z - 4$

(C) $4x - 3y - 8 = 0 = z - 2$

(D) $4x + 3y + 8 = 0 = z - 2$

4. The symmetric and the parametric forms of the line $2x - y + z - 5 = 0$ $x + y - 2z - 1$ are

(A) $\dfrac{x-2}{1}=\dfrac{y+1}{5}=\dfrac{z}{3}$

(B) $\dfrac{x-2}{-2}=\dfrac{y+1}{-10}=\dfrac{z}{-6}$

(C) $x = 2 + t, y = -1 + 5t, z = 3t$

(D) $x = 2 + 2t, y = 1 + 5t, z = -3t$

5. Consider the plane $2x + y + z = 0$ and the point $P(3, 5, 7)$. Then, the

(A) foot of the perpendicular from P onto the given plane is $(-3, 2, 4)$.

(B) image of P in the given plane is $(-9, -1, 1)$.

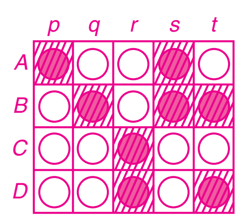
(C) equation of the line joining the points $(3, 5, 7)$ and $(-9, -1, 1)$ is $\dfrac{x-3}{6}=\dfrac{y-5}{6}=\dfrac{z-7}{4}$.

(D) distance of $(-9, -1, 1)$ from the plane $2x + y + z = 0$ is 9.

Matrix-Match Type Questions

In each of the following questions, statements are given in two columns, which have to be matched. The statements in *column I* are labeled as **(A)**, **(B)**, **(C)** and **(D)**, while those in *column II* are labeled as **(p)**, **(q)**, **(r)**, **(s)** and **(t)**. Any given statement in *column I* can have correct matching with *one or more* statements in *column II*. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following example.

Example: If the correct matches are (A) → (p), (s), (B) → (q), (s), (t), (C) → (r), (D) → (r), (t), that is if the matches are (A) → (p) and (s); (B) → (q), (s) and (t); (C) → (r); and (D) → (r), then the correct darkening of bubbles will look as followes:



1. Match the items of Column I with those of Column II.

Column I	*Column II*
(A) The equation of the plane which is at a distance 10 from the origin, and whose normal has DRs (3, 2, 6) is	(p) $2x+3y+2\sqrt{3}z+11=0$

(*Continued*)

Column I	*Column II*
(B) The equation of the plane which is at a distance of 5 units from the origin and is perpendicular to the vector $(2, -3, 6)$ is	(q) $3x + 2y + 6z = 70$
	(r) $2x + 3y + 2\sqrt{3}z + 1 = 0$
(C) The equation of the plane which is at a distance of one unit from the plane $2x + 3y + 2\sqrt{3}z + 6 = 0$ is	(s) $2x - y + 3z + 8 = 0$
(D) The equation of the plane passing through the point $(1, 4, -2)$ and parallel to the plane $2x - y + 3z = 0$ is	(t) $2x - 3y + 6z = 35$

2. Match the items of Column I with those of Column II.

Column I	*Column II*
(A) The equation of the plane passing through the point $(-1, -1, 2)$ and perpendicular to the planes $3x + 2y - 3z - 1 = 0$ and $5x - 4y + z - 5 = 0$ is	(p) $3x + 4y - 5z - 9 = 0$
(B) The equation of the plane passing through the points $(2, 2, 1)$ and $(9, 3, 6)$ and perpendicular to the plane $2x + 6y + 6z - 1 = 0$ is	(q) $x + 11y + 9z = 0$
(C) The equation of the plane passing through the point $(3, 1, 2)$ and perpendicular to the vector $(2, -1, 1)$ is	(r) $5x + 9y + 11z = 0$
(D) The equation of the plane passing through the point $(2, 1, -1)$ and passing through the line of intersection of the plane $x + 3y - z = 0 = y + 2z$ is	(s) $2x - y + z - 7 = 0$
	(t) $x + 9y + 11z = 0$

(*Continued*)

3. Match the items of Column I with those of Column II.

Column I	*Column II*
(A) The distance of the point $(2, 5, -3)$ from the plane $6x - 3y + 2z - 4 = 0$ is	(p) $\sqrt{105}$
(B) If the lines $\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-1}{4}$ and $\frac{x-3}{1} = \frac{y-k}{2} = \frac{z}{1}$ intersect, then the distance between the points $(1, -1, 1)$ and $(3, 2k, 0)$ is	(q) 7
	(r) $\frac{10}{3\sqrt{3}}$
(C) The distance of the point $(1, 0, -3)$ from the plane $x - y - z - 9 = 0$ measured parallel to the line $\frac{x-2}{2} = \frac{y+2}{3} = \frac{z-6}{-6}$ is	(s) $\frac{10}{3}$
(D) The distance of the line $\frac{x-2}{1} = \frac{y+2}{-1} = \frac{z-3}{4}$ from the plane $x + 5y + z - 5 = 0$ is	(t) $\frac{13}{7}$

Comprehension Type Questions

1. Passage: Consider the plane $x + y - z - 1 = 0$, the point $P(1, 2, -3)$ and the line

$$L: \frac{x-1}{3} = \frac{y-2}{-1} = \frac{z-3}{4}$$

Answer the following questions.

(i) Equation of the plane containing the line L and the point $P(1, 2, -3)$ is

(A) $x + 3y - 7 = 0$ (B) $3x + y - 5 = 0$
(C) $x - 3y + 5 = 0$ (D) $x + 3y - z - 7 = 0$

(ii) The distance between the points on the line L which are at a distance of $2/\sqrt{3}$ from the plane is

(A) $4\sqrt{3}$ (B) $\dfrac{2}{\sqrt{3}}$

(C) $\sqrt{104}$ (D) $2\sqrt{3}$

(iii) If Q is a point on the line L such that PQ is parallel to the plane, then the coordinates of Q are

(A) $(-8, 5, -9)$ (B) $(4, 1, 7)$

(C) $(-5, 4, -5)$ (D) $(10, -1, 15)$

2. Passage: Consider the planes $ax + by + cz = 0$, $bx + cy + az = 0$ and $cx + ay + bz = 0$. Answer the following questions.

(i) If $a + b + c \neq 0$ and $a^2+b^2+c^2=ab+bc+ca$, then the

(A) three planes have no non-zero solution.

(B) three planes intersect in a single point only.

(C) three planes intersect in a line.

(D) three planes are identical.

(ii) If $a+b+c=0$ and $a^2+b^2+c^2 \neq ab+bc+ca$, then the

(A) planes intersect in the line $x = y = z$.

(B) planes do not have common point.

(C) planes have infinitely many common points among which $(1, 2, 3)$ is one such common point.

(D) planes form a triangular prism.

(iii) If $a+b+c=0$ and $a^2+b^2+c^2=ab+bc+ca$, then the

(A) three planes have unique common point.

(B) common solutions line on a line only.

(C) three planes are identical.

(D) three equations represent $\mathbb{R}^3$.

3. Passage: If a plane meets the coordinate axes at the points $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$, then its equation is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

(i) A plane cuts the coordinate axes at $A(2, 0, 0)$, $B(0, 3, 0)$ and $C(0, 0, 1)$. Then the area of the triangle ABC is

(A) $\dfrac{11}{2}$ (B) $\dfrac{9}{2}$ (C) $\dfrac{7}{2}$ (D) $\dfrac{5}{2}$

$\left(\textbf{\textit{Hint:}}\ \text{Area}=\dfrac{1}{2}\sqrt{b^2c^2+c^2a^2+a^2b^2}\right)$

(ii) The volume of the tetrahedron $OABC$, where O is the origin, is

(A) 6 (B) 1 (C) 2 (D) $\dfrac{1}{6}$

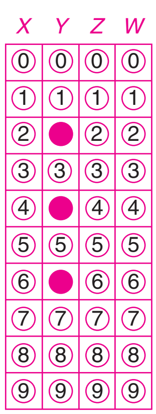
(iii) The centroid of tetrahedron $OABC$ is

(A) $\left(\dfrac{1}{2},\dfrac{3}{4},\dfrac{1}{4}\right)$ (B) $\left(\dfrac{1}{4},\dfrac{3}{4},\dfrac{1}{4}\right)$

(C) $\left(\dfrac{2}{3},1,\dfrac{1}{3}\right)$ (D) $\left(\dfrac{1}{3},\dfrac{1}{3},\dfrac{1}{3}\right)$

Integer Answer Type Questions

The answer to each of the questions in this section is a non-negative integer. The appropriate bubbles below the respective question numbers have to be darkened. For example, as shown in the figure, if the correct answer to the question number Y is 246, then the bubbles under Y labeled as 2, 4, 6 are to be darkened.



1. If a line makes angles α, β, γ and δ with the diagonals of a cube, then

$$3(\cos^2\alpha+\cos^2\beta+\cos^2\gamma+\cos^2\delta)$$

is ______.

2. $A(3, 2, -4)$, $B(5, 4, -6)$, and $P(9, 8, -10)$ are three collinear points. If the ratio $AP{:}PB = l{:}m$, then $|l|+|m|$ is equal to (l and m do not have common factor $s \neq 1$) ______.

3. If yz-plane divides the line joining the point $A(-2, 4, 7)$ and $B(3, -5, 8)$ in the ratio $2/k$, then the value of k is ______.

4. If the lines $x = ay + b$, $z = cy + d$ and $x = a'y + b'$, $z = c'y + d'$ are perpendicular to each other, then $aa' + cc' + k = 0$, where the value of k is______.

5. If p is the shortest distance between the lines

$$\frac{x-1}{2}=\frac{y-2}{1}=\frac{z-1}{2} \quad \text{and} \quad \frac{x-2}{2}=\frac{y+1}{1}=\frac{z+1}{2}$$

then the integral part of p ($[p]$) is______.

6. The equation of the plane passing through the points $(-1, 1, 1)$ and $(1, -1, 1)$ and perpendicular to the plane $x + 2y + 2z - 5 = 0$ is $2x + 2y - 3z + d = 0$, where the value of d is______.

ANSWERS

Single Correct Choice Type Questions

1. (A)
2. (B)
3. (D)
4. (B)
5. (A)
6. (D)
7. (B)
8. (A)
9. (D)
10. (B)
11. (D)
12. (D)
13. (A)
14. (B)
15. (C)
16. (B)
17. (D)
18. (C)
19. (B)
20. (C)

Multiple Correct Choice Type Questions

1. (B), (C)
2. (A), (C)
3. (A), (C)
4. (A), (B), (C)
5. (A), (B), (C)

Matrix-Match Type Questions

1. (A) $\rightarrow$ (q); (B) $\rightarrow$ (t); (C) $\rightarrow$ (p), (r); (D) $\rightarrow$ (s)
2. (A) $\rightarrow$ (r); (B) $\rightarrow$ (p); (C) $\rightarrow$ (s); (D) $\rightarrow$ (t)
3. (A) $\rightarrow$ (t); (B) $\rightarrow$ (p); (C) $\rightarrow$ (q); (D) $\rightarrow$ (r)

Comprehension Type Questions

1. (i) $\rightarrow$(A); (ii) $\rightarrow$ (C); (iii) $\rightarrow$ (A)
2. (i) $\rightarrow$(D); (ii) $\rightarrow$ (A); (iii) $\rightarrow$ (D)
3. (i) $\rightarrow$(C); (ii) $\rightarrow$ (B); (iii) $\rightarrow$ (A)

Integer Answer Type Questions

1. 4
2. 5
3. 3
4. 1
5. 3
6. 3

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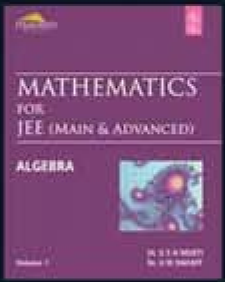

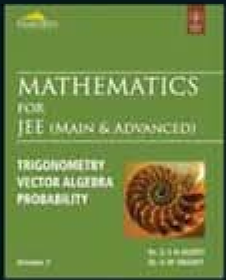

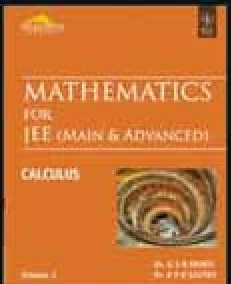

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About the Author

Dr. G.S.N. Murti

An eminent mathematician, accomplished author and a dedicated teacher, Dr. G.S.N. Murti brings vast knowledge and experience to teaching the fundamental concepts of pure Mathematics. Dr. Murti has taught Mathematics at degree level for 27 years before moving on to coach prospective candidates for IIT-JEE in the field of Mathematics. Over the last 27 years, Dr. Murti's name has become synonymous with the best quality IIT-JEE coaching in Mathematics. In addition to being a great teacher, Dr. Murti is also considered a master trainer and mentor by a huge community of Mathematics teachers who prepare students for engineering entrance examinations.

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