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Parameter Robustness in Data-Driven Estimation of Dynamical Systems

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Abstract— **We study the robustness of system estimation to parametric perturbations in system dynamics. In particular, we define the problem of sensitivity-based parametric uncertainty quantification in dynamical system estimation. The main contribution of this paper is the development of a novel robustness metric for estimation of parametrized linear dynamical systems with and without control actions. For the computation of this metric, we delineate the uncertainty contributions arising from control actions, system dynamics, and initial conditions. Furthermore, to validate our theoretical findings, we establish connections between our results and existing literature on model reduction robustness quantification. Our work provides guidance for selecting estimation methods based on tolerable levels of parametric uncertainty and paves the way for new cost functions in data-driven estimation that reward sensitivity to a desired subset of parameters while penalizing sensitivity to others.**

I. Introduction

Dynamical systems modeling and control methods have evolved with the rise of data-driven paradigms, enabled by rapid, large-scale access to data from complex physical systems. Estimation and control, the two fundamental problems in control theory, have significantly benefited from easily applicable data-driven methods. As these methods continue to advance, system designers must select the most appropriate technique to estimate system dynamics from an expanding range of options. We study this problem in the context of parametric robustness of system estimation. We start by discussing the existing literature on robustness quantification and data-driven estimation and control methods.

A. Data-driven estimation and control

Various data-driven system estimation and control methods have been developed in the control literature. Among these, the Dynamic Mode Decomposition (DMD) [1] has emerged as a popular choice to decompose complex system dynamics into identifiable modes that describe the system dynamics. DMD provides a linear estimate of discrete-time system dynamics from time-series snapshots and has been extended for control as DMD with control (DMDc) to handle actuated inputs [2], for systems with sensor noise [3], and many other extensions. Recently, a Python tool called pyDMD [4], [5] was developed that encompasses most of these DMD related tools in a user-friendly package. A related development in data-driven system estimation is the Sparse Identification of Nonlinear Dynamics (SINDy) method for continuous-time system dynamics. SINDy uses sparse regression to discover governing equations from time-series data, yielding interpretable models [6] which have been successfully applied for the estimation of fluid dynamic models [7], [8], for flight control [9], among other physical system examples. These methods are founded in an operator theoretic framework that applies Koopman theory [10] for finding embeddings of nonlinear dynamics using a high-dimensional linear operator. Beyond these operator theoretic data-driven methods, various model-free and learning-based control methods have become increasingly common in control applications. For example, reinforcement learning algorithms are now the go-to methods to learn close-to-optimal control policies from data for applications in robotics and beyond. RL algorithms can be iteratively tuned to achieve significant gains in performance (lower tracking error, better disturbance rejection), greater flexibility (adapting to new operating regimes without rederiving equations), and improved computational efficiency (enabling real-time implementation) but often lack formal guarantees. Towards that end, researchers are exploring the integration of neural network function approximators with physics-based models to improve generalization and interpretability. Although this is still an active and open area of research, a recent example [11] includes the use of deep autoencoders to discover coordinates for SINDy to inform the control decisions using a latent space where the dynamics are sparsely described by SINDy. Hybrid methods leverage neural networks to handle complex feature extraction (difficult in manual basis selection) while constraining the model structure using first-principles physics-based modeling. Along this line, the area of physics-informed neural networks, or gray-box modeling is growing rapidly and is impacting the ways in which control theorists implement system estimation using data [12]. Although the estimation and control performance in data-driven methods is greatly enhanced the lack of guarantees on robustness and safety is still an active area of research [13], [14], [15], [16], [17]. There is significant interest from the control theorists in bridging data-driven methods with formal methods and physics-informed approaches to embed safety and robustness guarantees in data-driven pipelines.

B. Robustness quantification

Both empirical and analytical benchmarking of robustness are common for data-driven estimation and control. Empirical approaches include multiple runs for the same system or dataset that can be compared and used for benchmarking, and visual illustrations using time-series plots or phase-plane trajectories. On the other hand, analytical approaches include the computation of signal to noise (or disturbance) ratio, or

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the computation of metrics such as the mean squared error (MSE). Classical control theory offers frequency-domain robustness metrics such as the $\mathbb{H}_\infty$ norm (the worst-case gain across frequencies) and μ-analysis (structured singular value for uncertainty). However, these frequency-domain metrics are less directly applicable in data-driven settings. For example, a recent study [18] demonstrates the use of fixed-structure controllers designed directly from frequency response data, minimizing $\mathbb{H}_2$ or $\mathbb{H}_\infty$ costs without explicit state-space models [19]. Despite such efforts, data-driven research predominantly emphasizes time-domain metrics due to inherent limitations like unclear error bounds, nonlinearities, high-dimensional models, and conservative robustness constraints.

In this paper, we take an approach motivated by sensitivity analysis, an area that has historically been integral to control theory but remains underexplored due to the success of frequency-domain methods. The key idea is to analyze the partial derivative of a dynamical quantity of interest with change in the model parameters. This provides a direct measure of how specific parameters influence the behavior of the system over time. This facilitates seamless integration into data-driven pipelines. But more importantly, with this approach, we can explore robustness measures against specific parameters of interest for a precise analysis that is often of interest in the design of physical systems. For example, in a rotor-driven robotic system, one may be particularly interested in ensuring stable hovering performance despite variations in motor constants or aerodynamic coefficients. Another motivating example comes from engineered biological systems, where a protein's concentration might be designed to respond strongly to certain dynamically varying parameters, while remaining robustly insensitive to fluctuations in other parameters within a specified range. Such precise, parameter-specific robustness analyses are challenging but critical, and our approach seeks to enable these capabilities within modern data-driven modeling and control pipelines.

Contributions: In this paper, we express the problem of robustness of error in data-driven system estimation and derive an easily computable bound for this robustness metric for linear systems with control. Specifically,

1) We define a new robustness metric for data-driven estimation of parametrized linear dynamical systems with and without actuation.
2) For the computation of our new robustness metric, the main theorem in this paper gives a bound to effectively quantify the robustness in the estimation of parameterized system dynamics.
3) To prove the validity of our results, we draw the equivalence of the main theorem in special cases to already known results in the literature.
4) The applications of our robustness metric provide a way forward in choosing estimation methods according to the levels of parametric uncertainties that may be tolerated in the estimated system.

Paper outline: We start by formally defining the problem in the next section. Then, in Section III, we present important preliminaries to derive the main result of the paper given in Section IV. We show two example applications of the proposed robustness metric in Section V.

II. Problem Formulation

A. System description

We consider the problem of estimation of a parametrized linear dynamical system with control inputs,

$$\begin{aligned} \dot{x} &= A(\theta)x + B(\theta)u, \\ y &= C(\theta)x + D(\theta)u \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $A \in \mathbb{R}^{n\times n}$ is a real-valued system matrix, $B \in \mathbb{R}^{n\times k}$ is the control matrix for the system with k inputs, $u \in \mathbb{R}^k$, $y \in \mathbb{R}^m$ is the output vector consisting of m measurements with the output matrix $C \in \mathbb{R}^{m\times n}$, and $D \in \mathbb{R}^{m\times k}$ is the feedforward matrix for the system. In the most general case of parametric dependence of the system dynamics, we have shown all matrices to be functions of a parameter vector $\theta \in \mathbb{R}^p$, which consists of all p parameters of the system.
An estimation of this system is written as

$$\begin{aligned} \dot{\tilde{x}} &= \tilde{A}(\theta)\tilde{x} + \tilde{B}(\theta)u, \\ \tilde{y} &= \tilde{C}(\theta)\tilde{x} + \tilde{D}(\theta)u \end{aligned} \tag{2}$$

where the $\tilde{\cdot}$ represents the corresponding estimated state variables while the control inputs u remain the same since these are usually external forces, which will be applied in the same way to the estimated system as well. Note that we assume here that the estimated dynamical system is also parametrized. This may not be the case for many 'off-the-shelf' data-driven methods for system estimation. However, enforcing such a structure on estimated dynamics is possible and has been used in the literature for many data-driven system estimation methods for parametrized systems [20], [21]. Thus, the assumption of this structure is not severely limiting.

B. Robustness metric

The key quantity that we are interested in is the quantification of a robustness metric to parametric uncertainties. For the system estimation problem defined above, we aim to compute a metric that bounds the change in estimation error as model parameters vary to give a robustness estimate for the estimation method. For this, we construct a robustness distance estimate, by adapting a parametric uncertainty-based robustness metric in the literature [22] that was defined for model reduction. For a parameter θ_i^*, we define a measure of robustness 'distance' for the estimation error as

$$d_R = \sum_{i=1}^{p} \frac{\theta_i^*}{\|\mathrm{err}(t,\theta_i^*)\|} \cdot \left\| \frac{\partial \mathrm{err}}{\partial \theta_i}\bigg|_{\theta_i=\theta_i^*} \right\|, \tag{3}$$

where $\text{err}(t, \theta_i^*)$ is the non-zero error in estimation for $t > 0$ and parameter $\theta_i = \theta_i^*$. Using d_R, we define the robustness of estimation error under parametric uncertainties as,

$$R = \frac{1}{1 + d_R} = \frac{1}{1 + \sum_{i=1}^{p} \frac{\theta_i^* \left\| \frac{\partial \text{err}}{\partial \theta_i} \Big|_{\theta_i = \theta_i^*} \right\|}{\left\| \text{err}(t, \theta_i^*) \right\|}}. \tag{4}$$

C. Augmented system dynamics

The main idea in the problem formulation is the description of an augmented system dynamics. By augmenting the ground truth system matrices with the estimated matrices, we obtain a form that enables the computation of the robustness metric. For this, we define the augmented state variables and matrices by using the $\bar{\cdot}$ notation as follows:

$$\bar{x} = \begin{bmatrix} x \\ \tilde{x} \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} A & 0 \\ 0 & \tilde{A} \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} B \\ \tilde{B} \end{bmatrix} \tag{5}$$

$$\bar{C} = \begin{bmatrix} C & -\tilde{C} \end{bmatrix}, \quad \bar{D} = D - \tilde{D}. \tag{6}$$

Note that we skipped the θ dependence for simplicity of notation. Unless otherwise stated, all system matrices are assumed to be dependent on the parameters θ.
Now, we can write the augmented system dynamics as

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u \tag{7}$$

$$\bar{y} = \bar{C}\bar{x} + \bar{D}u. \tag{8}$$

In this formulation, the output of the system $\bar{y}$ is the estimation error

$$\bar{y} = y - \tilde{y} = Cx - \tilde{C}\tilde{x} + Du - \tilde{D}u = \bar{C}\bar{x} + \bar{D}\bar{u} \tag{9}$$

Note that the total number of states in the augmented system is $2n$, the number of inputs is k, we have p outputs and $\bar{D} \in \mathbb{R}^{p \times k}$, similar to D and $\tilde{D}$.
X and $\tilde{X}$ are both $n \times 1$, but estimated state values will differ because $\tilde{A}$ differs. Thus, we have a new dynamical system:

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}u, \quad \tilde{y} = \tilde{C}\tilde{x} + \tilde{D}u \tag{10}$$

Here, u remains the same because u represents physical inputs for a system with forced response.

D. Parametric robustness of estimation error

To quantify the robustness of estimation, we are interested in the partial derivative of the estimation error with parameters θ. Therefore, we re-write the robustness metric R from equation (4) using $\bar{y}$ as

$$R = \left(1 + \sum_{i=1}^{p} \frac{\theta_i^* \left\| \frac{\partial \bar{y}}{\partial \theta_i} \Big|_{\theta_i = \theta_i^*} \right\|}{\left\| \bar{y}(t, \theta_i^*) \right\|} \right)^{-1}. \tag{11}$$

Thus, our goal is to compute a bound on $\|\partial \bar{y} / \partial \theta_i\|$ for all parameters θ_i. To develop this metric, we discuss the main lemmas and existing results related to parametric sensitivities in the next section.

III. Preliminary Results

Notation: In this paper, we consider the Euclidean 2-norm for vectors, $\|x\|_2$ for the 2-norm of $x \in \mathbb{R}^n$. For matrices, $\|\cdot\|$ represents the induced 2-norm. For all norms without the suffix, we assume the 2-norm. The l^∞-norm for vectors is defined as the sup-norm, in other words, this norm is used to compute the maximum absolute value of the vector. For matrices that describe system dynamics, we use the logarithm norm using the lemma below.
Assumptions: We assume that both the original and the estimated system dynamics are exponentially stable dynamical systems. This ensures that the estimation error is not unbounded, that is, we assume the correctness of the estimation algorithm. Thus, our method can be applied to any data-driven estimation algorithm to compare the robustness of the methods given that the methods all lead to bounded estimation errors and that the system descriptions can be reformulated in the structure required here. We also assume that $D = 0$, without loss of generality as it will add an additional term in the bound, which is easy to compute using $\|\bar{D}u\|_\infty$. Finally, we assume that the systems considered in this paper are input-to-state (ISS) stable that allows us to use the l^∞-norm bounds on the control input while developing the robustness metric.

Lemma 1 (See [23]). *For a matrix A, the norm of the matrix exponential (e^{At}) is bounded as*

$$\left\| e^{At} \right\| \leq e^{-|\mu| t},$$

where μ is the logarithm norm of A [24], under the assumption that A is Hurwitz. For the log-norm induced by the 2-norm, we have

$$\mu(A) = \frac{\lambda_{\max}(A + A^T)}{2},$$

Note that for Hurwitz A, the matrix exponential is decaying and thus, μ is always negative.

For partial parameter derivatives, the two lemmas that will be used in the derivation of the main result in the paper are:

Lemma 2. *For Hurwitz A, the partial derivative of e^{At} with respect to a parameter $\theta_i \in \theta$ is bounded above as*

$$\left\| \frac{\partial e^{At}}{\partial \theta_i} \right\| \leq \left\| \frac{\partial A}{\partial \theta_i} \right\| t e^{-|\mu| t}. \tag{12}$$

where $|\mu|$ is the absolute value of the log-norm of A as in Lemma 1.

Proof. Using the solution for linear dynamical system $x(t) = e^{At} x(0)$, we can write

$$\frac{\partial x(t)}{\partial \theta_i} = e^{At} \frac{\partial x(0)}{\partial \theta_i} + \frac{\partial e^{At}}{\partial \theta_i} x(0).$$

Then, using the convolution equation, the partial derivative of the matrix exponential is

$$\frac{\partial e^{At}}{\partial \theta_i} = \int_0^t e^{(t-\tau)A} \frac{\partial A}{\partial \theta_i} e^{\tau A} d\tau, \tag{13}$$

which leads to the upper bound in the lemma on solving the integral after using Lemma 1. □

Lemma 3. *The derivative of the negative matrix exponential e^{-As} with respect to the parameter θ_i satisfies the following bound:*

$$\left\| \frac{\partial e^{-As}}{\partial \theta_i} \right\| \leq \left\| \frac{\partial A}{\partial \theta_i} \right\| s e^{|\mu| s}, \tag{14}$$

where $|\mu|$ is the absolute value of the log-norm of the matrix A.

Proof. Using equation (13) and substituting the negative time in the integral we obtain

$$\frac{\partial e^{-As}}{\partial \theta_i} = \int_0^{-s} e^{A(-s-\tau)} \frac{\partial A}{\partial \theta_i} e^{A\tau} d\tau.$$

Using submultiplicativity, the substitution that $u = -\tau$, and Lemma 1, we get:

$$\left\| \frac{\partial e^{-As}}{\partial \theta_i} \right\| \leq \left\| \frac{\partial A}{\partial \theta_i} \right\| e^{|\mu| s} \int_0^s du, \tag{15}$$

which leads to the desired result on solving the integral. □

IV. Results

The main result of this paper is an upper bound on the robustness of the estimation error with respect to a model parameter θ_i. This bound enables the quantification of the robustness metric defined in equation (11) by computing the bound for each model parameter. The advantage of this approach is that the system modeler can individually evaluate the impact of physical parameters of interest and, accordingly, update the robustness computation. For example, if only a subset of parameters are of interest, then the robustness metric can be computed only for these parameters.

We present the result assuming parametric uncertainties in A and in the initial conditions $x(0$ for simplicity because this is the most common case that would appear in estimating dynamics of physical systems—the external inputs and output measurements are usually parameter independent. However, if the matrices B and C are also parameter dependent, the result can be extended without loss of generality, as it will add more terms that quantify the parametric robustness contributions of the B and C matrices that are parameter dependent.

A. Parameter dependent A

Theorem 1. *For a given estimation (as in equation (2)) of a parametrized linear dynamical system with control inputs (as in equation (1)), the parametric robustness of the error in estimation for a given parameter θ_i is bounded above using a computable contribution of three terms: (1) the simple derivative of $\bar{A}$ with θ_i, (2) the worst-case norm of external inputs, and (3) the maximum time for which the inputs are applied N:*

$$\left\| \frac{\partial \bar{y}}{\partial \theta_i} \right\|^2 \leq K_1 \left\| \frac{\partial \bar{A}}{\partial \theta_i} \right\|^2 + K_2 \left\| \frac{\partial \bar{A}}{\partial \theta_i} \right\|^3 \left\| \bar{B} u \right\|_\infty + K_3 N^2 \left\| \frac{\partial \bar{A}}{\partial \theta_i} \right\|^2 \left\| \bar{B} u \right\|_\infty, \tag{16}$$

where $K_1, K_2,$ and K_3 are positive constants dependent on the initial condition, the log-norm of $\bar{A}$, and the augmented output matrix $\bar{C}$.

Proof. The main proof idea is to describe the parametric robustness of the estimation error in the form of parametric robustness of the matrix exponential, which enables the use of the lemmas developed earlier. We start by using the convolution equation [25], write $\bar{y}(t)$ for the augmented system as:

$$\bar{y}(t) = \bar{C} e^{\bar{A}t} \bar{x}(0) + \int_0^t \bar{C} e^{\bar{A}(t-\tau)} \bar{B} u(\tau) d\tau.$$

Then, using the definition of the Euclidean norm and the equation above, we can write the LHS of the theorem (the square of 2-norm of $\partial \bar{y}/\partial \theta_i$) as

$$\int_0^\infty \frac{\partial}{\partial \theta_i} \left[\int_0^t u^T(\tau) \bar{B}^T e^{\bar{A}^T(t-\tau)} \bar{C}^T d\tau + \bar{x}^T(0) e^{\bar{A}^T t} \bar{C}^T \right] \frac{\partial}{\partial \theta_i} \left[\bar{C} e^{\bar{A}t} \bar{x}(0) + \int_0^t \bar{C} e^{\bar{A}(t-\tau)} \bar{B} u(\tau) d\tau \right] dt$$

Expanding further, we obtain:

$$= \int_0^\infty \underbrace{\left[\int_0^t u^T(\tau) \bar{B}^T \frac{\partial e^{\bar{A}^T(t-\tau)}}{\partial \theta_i} \bar{C}^T d\tau + \bar{x}^T(0) \frac{\partial e^{\bar{A}^T t}}{\partial \theta_i} \bar{C}^T \right]}_{(a)} \tag{17}$$

$$\underbrace{\left[\bar{C} \frac{\partial e^{\bar{A}t}}{\partial \theta} \bar{x}(0) + \int_0^t \bar{C} \frac{\partial e^{\bar{A}(t-\tau)}}{\partial \theta} \bar{B} u(\tau) d\tau \right]}_{(b)} dt \tag{18}$$

We break down the above integral into four parts that result from the multiplication of the two additive terms in each of (a) and (b) above to get the following:

$$a_1 b_1 = \int_0^\infty \left(\int_0^t u^T(\tau) \bar{B}^T \frac{\partial e^{\bar{A}^T(t-\tau)}}{\partial \theta_i} \bar{C}^T d\tau \right) \left(\bar{C} \frac{\partial e^{\bar{A}^T t}}{\partial \theta_i} \bar{x}(0) \right) dt$$

$$a_1 b_2 = \int_0^\infty \left(\int_0^t u^T(\tau) \bar{B}^T \frac{\partial e^{\bar{A}^T(t-\tau)}}{\partial \theta_i} \bar{C}^T d\tau \right) \left(\int_0^t \bar{C} \frac{\partial e^{\bar{A}(t-\tau)}}{\partial \theta_i} \bar{B} u(\tau) d\tau \right) dt$$

which is equal to the 2-norm by definition,

$$a_1 b_2 = \left\| \int_0^t \bar{C} \frac{\partial e^{\bar{A}(t-\tau)}}{\partial \theta_i} \bar{B} u(\tau) d\tau \right\|_2^2.$$

For this term, we expand the partial derivative of the matrix exponential such that we can use Lemma 3, to obtain

$$a_1 b_2 \leq \|\bar{C}\| \left\| \frac{\partial \bar{A}}{\partial \theta_i} \right\|^2 \|\bar{B} u\|_\infty \left[\frac{t^2 |\mu| + t e^{-|\mu| t} - t}{|\mu|^2} \right]$$

by solving the integral of $te^{|\mu|t}$ (the residual term in Lemma 3) from 0 to ∞. Note that for all values of t, since $te^{-|\mu|t}$ is always less than t, and we assume that we integrate until a maximum time of N, we can conservatively upper bound the above by,

$$a_1 b_2 \leq \frac{N^2}{|\mu|} \|\bar{C}\| \left\| \frac{\partial \bar{A}}{\partial \theta_i} \right\|^2 \|\bar{B}u\|_\infty. \tag{19}$$

Then, for the third term, we write

$$a_2 b_1 = \int_0^\infty \bar{x}(0)^T \frac{\partial e^{\bar{A}^T t}}{\partial \theta_i} \bar{C}^T C \frac{\partial e^{\bar{A}t}}{\partial \theta_i} \bar{x}(0) dt.$$

For this term, taking the norm and using the sub-multiplicative property of the 2-norm to get an upper bound, we get that it is

$$\leq \int_0^\infty \left\| \frac{\partial e^{\bar{A}t}}{\partial \theta_i} \right\|^2 \|\bar{C}^T \bar{C}\| \, \|\bar{x}(0)\|^2 \, dt,$$

then, using Lemma 2 and evaluating the integral

$$\int_0^\infty t^2 e^{-2|\mu|t} dt = \frac{1}{4\,|\mu|^3}$$

we finally get that

$$a_2 b_1 \leq \frac{1}{4\mu^3} \left\| \frac{\partial \bar{A}}{\partial \theta_i} \right\|_2^2 \left\| \bar{C}^T \bar{C} \right\|_2 \|\bar{x}(0)\|_2^2 \tag{20}$$

where $|\mu|$ is the absolute value of the log-norm of the matrix $\bar{A}$. Then, the last term is

$$a_2 b_2 = \int_0^\infty \bar{x}^T(0) \frac{\partial e^{\bar{A}^T t}}{\partial \theta_i} \bar{C}^T \left(\int_0^t \bar{C} \frac{\partial e^{\bar{A}(t-\tau)}}{\partial \theta_i} \bar{B}u(\tau) d\tau \right) dt.$$

Note that the norm for this term is the same as $a_1 b_1$. The full expansion and bound for this term is shown in the Appendix. We give the final bound for these two terms together:

$$a_2 b_2 + a_1 b_1 \leq \frac{2}{|\mu|^5} \left(\|\bar{x}(0)\| \|\bar{C}^T \bar{C}\| \left\| \frac{\partial \bar{A}}{\partial \theta_i} \right\|^3 \|\bar{B}u\|_\infty \right) \tag{21}$$

Adding the bounds for $a_1 b_1$, $a_1 b_2$, $a_2 b_1$, and $a_2 b_2$ from equations (19, 20, 21) we obtain the desired result with the following values for the constants K_1, K_2 and K_3

$$K_1 = \frac{1}{4\mu^3} \left\| \bar{C}^T \bar{C} \right\|_2 \|\bar{x}(0)\|_2^2$$
$$K_2 = \frac{2}{|\mu|^5} \|\bar{x}(0)\| \|\bar{C}^T \bar{C}\|$$
$$K_3 = \frac{1}{|\mu|} \|\bar{C}\|.$$

□

B. Parameter dependent initial conditions

Here, we assume that initial conditions are dependent on parameters and the dynamics are parameter independent. Then, we can obtain a simpler bound on the parametric robustness of the estimation error as

Theorem 2. *For a given estimation of a parameter independent, observable, linear dynamical system with control inputs and parameter dependent initial conditions, the parametric robustness of the error in estimation for a parameter θ_i is bounded above by*

$$\left\| \frac{\partial y}{\partial \theta_i} \right\|^2 \leq \lambda_{\max}(P) \left\| \frac{\partial \bar{x}(0)}{\partial \theta_i} \right\|^2, \tag{22}$$

where P is the Lyapunov matrix that satisfies the observability equation $\bar{A}^T P + P\bar{A} = -\bar{C}^T \bar{C}$.

Proof. On expanding similar to the proof of the previous theorem, we obtain that all terms except the one dependent on the sensitivity of the initial condition will go to 0. This is because the system dynamics are parameter independent and on integrating the matrix exponential factors from 0 to ∞, bounded above with Lemma 2, will asymptotically go to zero, leaving only the following term

$$\left\| \frac{\partial \bar{y}}{\partial \theta_i} \right\|^2 \leq \int_0^\infty \frac{\partial \bar{x}(0)^T}{\partial \theta_i} e^{\bar{A}^T t} \bar{C}^T \bar{C} e^{\bar{A}t} \frac{\partial \bar{x}(0)}{\partial \theta_i} dt.$$

The rest of the proof follows the proof method in [22], where it is applied to the model reduction robustness problem. Specifically, we have from [26, Ch.5], that for an asymptotically stable system the unique matrix P in the theorem statement satisfies the observability Gramian:

$$P = \lim_{N\to\infty} W_o(N) = \lim_{N\to\infty} \int_0^N e^{\bar{A}^T t} \bar{C}^T \bar{C} e^{\bar{A}t} dt,$$

where $W_o(N)$ is the observability Gramian. Substituting this above and bounding by the maximum eigen value of P gives us the desired result. □

C. Special Cases

To validate our results, we study the following special cases
Case 1: $u = 0$ (No forced input)

$$a_1 b_1 = a_1 b_2 = a_2 b_2 = 0, \quad \text{thus,} \tag{23}$$

$$a_2 b_1 \leq \frac{1}{4\,|\mu|^3} \left\| \frac{\partial \bar{A}}{\partial \theta_i} \right\|^2 \|\bar{C}^T \bar{C}\|_2 \|\bar{x}(0)\|_2^2, \tag{24}$$

which is the same result as obtained in a previously published article [27] on the parametric robustness of model reduction. Note that the main result of this paper is to extend these proof techniques to systems with forced inputs, and for data-driven system estimation problems.
Case 2: $\bar{x}(0) = 0$ (Only forced response)

$$a_2 b_1 = a_1 b_1 = a_2 b_2 = 0, \quad \text{thus, we are left with} \tag{25}$$

$$\left\| \frac{\partial \bar{y}}{\partial \theta_i} \right\|^2 \leq K_3 N^2 \left\| \frac{\partial \bar{A}}{\partial \theta_i} \right\|^2 \left\| \bar{B}u \right\|_\infty. \tag{26}$$

Thus, the equivalence of the results in this paper with other findings validate our proposed bounds.

V. Applications

Now we will apply the methodology outlined above to compute the robustness upper bound for estimation of parametric linear dynamical systems. By computing the upper bound, we aim to show the effectiveness of the metric in capturing the robustness level for a given parametric uncertainty on parameters of interest. The robustness metric gives the worst case guarantee on the changes to the estimation error, as opposed to running the estimation algorithm multiple times for different parameter samples.

A. Additive parametric uncertainty

Consider the following parametrized system

$$A = \begin{bmatrix} -1.1+\theta & 0.0 \\ 0.0 & -0.5 \end{bmatrix}, \quad B = \begin{bmatrix} 1.0 \\ 0.0 \end{bmatrix}, \quad C = \mathbb{I}_2.$$

For a simple case study, we estimate the dynamics of this system using both a linear least squares method and Lasso regression while preserving the additive structure in an affine representation. In particular, we assume that the true system may be written as

$$A(\theta) = A_0 + \theta A_1,$$

with

$$A_0 = \begin{bmatrix} -1.1 & 0.0 \\ 0.0 & -0.5 \end{bmatrix} \quad \text{and} \quad A_1 = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 0.0 \end{bmatrix}.$$

Then, we collect state and control snapshots over a fixed time interval, and then construct a regression library. The estimation yields corresponding matrices $\tilde{A}_0$ and $\tilde{A}_1$ so that the estimated model can be written as

$$\tilde{A}(\theta) = \tilde{A}_0 + \theta\,\tilde{A}_1.$$

Subsequently, we compute the robustness metric by evaluating the sensitivity of the observed output $y = Cx$ with respect to θ and comparing it to the derived theoretical bound. For both methods, we observed conservativeness in the computed theoretical bound. This affirms the absolute worst-case nature of the bound derived in the paper. In future work, we will explore approximations to obtain less conservative bounds. This example shows the applicability of this method in studying the robustness of parametric uncertainties in data-driven estimation of system dynamics.

B. Towards choosing structure-preserving system estimates

Consider a system described by the following differential equation

$$\begin{aligned} \frac{dx_1}{dt} &= k_1 - d_1 x_1 \\ \frac{dx_2}{dt} &= k_2 \frac{K}{x_1 + K} - d_2 x_2. \end{aligned} \tag{27}$$

This model uses a Hill function in the second equation, which is commonly used to describe ultrasensitivity in biological system modeling, where $x_1 = K$ is the half-activation state and the x_2 state asymptotically converges to zero as x_1 increases. We propose two structure-preserving estimated models for this system:

(1) Assuming that the rate of change of x_1 is 0 we get, $x_1 = \frac{k_1}{d_1}$. The estimated model is,

$$\begin{aligned} \frac{dx_1}{dt} &= 0 \\ \frac{dx_2}{dt} &= k_2 \frac{K}{\frac{k_1}{d_1} + K} - d_2 x_2. \end{aligned}$$

(2) Alternatively, if we assume that K is very large relative to x_1, then the Hill function can be approximated as

$$\frac{K}{x_1 + K} \approx 1.$$

In this case, the second equation simplifies to

$$\frac{dx_2}{dt} \approx k_2 - d_2 x_2,$$

while the first equation remains unchanged. This model effectively decouples the dynamics, treating the effect of K as saturating. Now, with the results given in this paper, we can compare the robustness of the estimation to the different parameters k_1, d_1, k_2, K, and d_2 with time. The higher value of robustness metric will indicate the choice of the preferred estimated model. We linearize both estimated models and applied the robustness metric developed in this paper to get that $R_1 > R_2$, which is expected since the second estimated model removes the parameter with high value. In future work, we plan to comprehensively evaluate both case studies and extend the complexity of the examples to gain deeper insights into the robustness of estimation as quantified by our metric. Additionally, we intend to analyze the conservativeness of the proposed robustness metric across various data-driven estimation methods.

C. Code

All results can be reproduced by running the code associated with this paper[1].

VI. Conclusions and Future Work

In this paper, we consider the problem of parametric dependence in linear dynamical systems and its impact on data-driven system estimation pipelines. Specifically, we introduce a new method to compute the robustness of estimation error for linear dynamical systems with control inputs. This robustness metric is motivated by a sensitivity analysis perspective, quantifying the rate of change of error with respect to model or initial condition parameters. The main contribution of our paper is a closed-form upper bound for the robustness metric that can be selectively computed for parameters of interest. This approach enables systematic comparisons of various data-driven estimation methods in parametric dynamical systems, aiding researchers in selecting effective estimation strategies or combinations thereof. Moreover, our metric facilitates the evaluation of whether the

[1] `https://github.com/ayush9pandey/robest`

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estimated dynamics preserve the original system's parametric dependencies, which is desirable in many applications. Future research directions include an extensive exploration of application case studies to further assess the performance of the proposed metric across different estimation methods.

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Appendix A

For the a_2b_2 term in the proof of Theorem 1, we can obtain the bound as follows.

$$
\begin{aligned}
a_2b_2 = &\int_0^\infty \bar{x}^T(0)\frac{\partial e^{\bar{A}^T t}}{\partial\theta}\bar{C}^T \\
&\left(\int_0^t \bar{C}\frac{\partial e^{\bar{A}(t-\tau)}}{\partial\theta}\bar{B}u(\tau)d\tau\right)dt \\
= &\int_0^\infty \bar{x}^T(0)\frac{\partial e^{\bar{A}^T t}}{\partial\theta}\bar{C}^T\bar{C}\frac{\partial e^{\bar{A}^T t}}{\partial\theta} \\
&\left(\int_0^t e^{-\bar{A}\tau}\frac{\partial\bar{C}}{\partial\theta}\bar{B}u(\tau)d\tau\right)dt \\
\le &\int_0^\infty \|\bar{x}(0)\|\left\|\frac{\partial e^{\bar{A}^T t}}{\partial\theta}\right\|^2\|\bar{C}^T\bar{C}\| \\
&\left\|\int_0^t e^{-\bar{A}\tau}\frac{\partial\bar{C}}{\partial\theta}\bar{B}u(\tau)d\tau\right\|dt \\
\le &\|\bar{x}(0)\|\|\bar{C}^T\bar{C}\|\left\|\frac{\partial\bar{A}}{\partial\theta}\right\|^2 \\
&\int_0^\infty t^2e^{-2|\mu|t}\left\|\int_0^t e^{-\bar{A}\tau}\frac{\partial\bar{C}}{\partial\theta}\bar{B}u(\tau)d\tau\right\|dt
\end{aligned}
$$

Aside: To bound the inner integral, consider:

$$
\begin{aligned}
\left\|\int_0^t \frac{\partial e^{-\bar{A}\tau}}{\partial\theta}\bar{B}u(\tau)d\tau\right\| &\le \|\bar{B}u\|_\infty\int_0^t\left\|\frac{\partial e^{-\bar{A}\tau}}{\partial\theta_i}\right\|d\tau \\
&\le \|\bar{B}u\|_\infty\int_0^t\left\|\frac{\partial\bar{A}}{\partial\theta_i}\right\|\tau e^{|\mu|\tau}d\tau \\
&\le \|\bar{B}u\|_\infty\left\|\frac{\partial\bar{A}}{\partial\theta_i}\right\|\frac{[e^{|\mu|t}(|\mu|t-1)+1]}{|\mu|^2}
\end{aligned}
$$

This result can now be substituted back into the above integral to complete the bound on $a_2 b_2$:

$$a_2 b_2 \leq \|\bar{x}(0)\| \|\bar{C}^T \bar{C}\| \left\| \frac{\partial \bar{A}}{\partial \theta_i} \right\|^2 \times$$

$$\int_0^\infty t^2 e^{-2|\mu|t} \left\| \int_0^t \frac{\partial e^{-\bar{A}\tau}}{\partial \theta} \bar{B} u(\tau) d\tau \right\| dt$$

$$\leq \|\bar{x}(0)\| \|\bar{C}^T \bar{C}\| \left\| \frac{\partial \bar{A}}{\partial \theta} \right\|^2 \|\bar{B}u\|_\infty \left\| \frac{\partial \bar{A}}{\partial \theta_i} \right\| \frac{1}{|\mu|^2} I_1$$

where

$$I = \int_0^\infty t^2 e^{-2|\mu|t} \left[e^{|\mu|t}(|\mu|t - 1) + 1 \right] dt$$

Breaking the integral down into three separate integrals, we have:

$$= \frac{\|\bar{x}(0)\| \|\bar{C}^T \bar{C}\| \left\| \frac{\partial \bar{A}}{\partial \theta} \right\|^3 \|\bar{B}u\|_\infty}{|\mu|} \int_0^\infty t^3 e^{-|\mu|t} dt$$

$$- \frac{\|\bar{x}(0)\| \|\bar{C}^T \bar{C}\| \left\| \frac{\partial \bar{A}}{\partial \theta} \right\|^3 \|\bar{B}u\|_\infty}{|\mu|^2} \int_0^\infty t^2 e^{-|\mu|t} dt$$

$$+ \frac{\|\bar{x}(0)\| \|\bar{C}^T \bar{C}\| \left\| \frac{\partial \bar{A}}{\partial \theta} \right\|^3 \|\bar{B}u\|_\infty}{|\mu|^2} \int_0^\infty t^2 e^{-2|\mu|t} dt$$

These three integrals can then be individually evaluated as:

- $\displaystyle\int_0^\infty t^3 e^{-|\mu|t}\, dt = \frac{6}{|\mu|^4}$
- $\displaystyle\int_0^\infty t^2 e^{-|\mu|t}\, dt = \frac{2}{|\mu|^3}$
- $\displaystyle\int_0^\infty t^2 e^{-2|\mu|t}\, dt = \frac{1}{4|\mu|^3}$.

Substituting these integrals, we get the desired result used in the proof above.