Getting around No-Cloning Theorem

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Abstract

In any computation model, the ability of cloning information is very useful. Due to linearity of operations in quantum computing, one can not produce exact clones of states. Restriction on cloning of states is one important feature that sets Quantum computing apart from classical. Besides forcing us to make multiple queries to the oracle, it also acts as the backbone of post quantum cryptography. We explore the possibility of cloning quantum states using deterministic(but inaccurate) and probabilistic cloning systems.

1 Introduction

A perfect cloning method will take the input state and generate an identical copy of this while not affecting the input state. Also, once this process is finished, the two states which we receive(the input and the copy) should not be entangled. We have been through a brief tour about different methods of cloning these states.

These methods can be broken apart into two categories, deterministic and probabilistic. Deterministic cloning makes use of only unitary operations on the given system to generate a clone of the input state. From the restriction of no-cloning theorem, once can not achieve the goal through this method. We can design unitary operations such that the output states are *close* to ideal. Probabilistic cloning makes use of measurements along with unitary operators, and does a post-selection on measurement results. This can be used to make proper clones of the input state, but only probabilistically. There is a third category, hybrid. This lies somewhere in between of the other two methods. The output states are not same as ideal, but are closer ideal than those from deterministic with probability less than 1 but larger than that in probabilistic. We explore the first two in this report.

2 Deterministic Cloning Machines

2.1 Formal Definition

For the deterministic setting, one aims to construct an unitary U which does the following task. Here a is the input state and b is the target for storing the copy.

$$U\left|s\right\rangle_a\left|x\right\rangle_b \rightarrow \left|s\right\rangle_a\left|s\right\rangle_b$$

The discussion in this section is about cloning two dimensional states. That means the states to clone are only one qubit. The complete setup which helps us in generating a copy of the input state is coined as *Quantum Cloning Machine*. We introduce another state $\left|Q\right\rangle$ (not necessarily one qubit) which acts as the copying device. The previous definition can be altered to

$$U\left|s\right\rangle_a\left|x\right\rangle_b\left|Q\right\rangle_x \rightarrow \left|s\right\rangle_a\left|s\right\rangle_b\left|\tilde{Q}\right\rangle_x$$

By the no-cloning theorem, no such Unitary U, which can clone any arbitrary input state exists. We then ask if there exists QCMs which can do the task such that the output qubits are very close to the input.

For this, we need some notion of similarity of states or distance between two states. Two particular measures have been used to quantitatively evaluate the similarity/distance. These measures operate on density matrices, which are the mixed state representation of the qubits.

Given, two mixed states $\hat{\rho}_1$ and $\hat{\rho}_2$, their similarity can be measured using

- Hilbert Schmidt Norm:

  $$D = (\|\hat{\rho}_1 - \hat{\rho}_2\|_2)^2$$

  where

  $$\left\|\hat{A}\right\|_2 = [Tr(\hat{A}^\intercal\hat{A})]^{1/2}$$

  This norm can be interpreted as distance or the difference in two states. Ideally, we'd like this to be zero.

- Fidelity:

  $$F = Tr({\hat{\rho}_1}^{1/2}\hat{\rho}_2{\hat{\rho}_1}^{1/2})^{1/2}$$

  Fidelity can be interpreted as similarity of two states. The maximum value for this measure is 1

Besides ensuring that the obtained state of system is close to the desirable state, we need to ensure that the system obtained is not entangled. We may further need to perform operations or measurements on one of the qubit and if we do so on one of the qubits of an entangled system, the other will get affected, which is not desirable.

Three different deterministic QCMs have been analyzed

2.2 Wootters-Zurek QCM

This machine was discussed by Wootters and Zurek in their paper [1]. They define the unitary operation on the basis vectors. Now, they analyze the QCM's performance on a superposition state. The unitary is such that it copies the basis states ideally.

The copying process is defined by the following transformation on the basis vectors $|0\rangle$ and $|1\rangle$

$$\begin{aligned}|0\rangle_a |e\rangle_b |Q\rangle_x &\rightarrow |0\rangle_a |0\rangle_b |Q_0\rangle_x \\ |1\rangle_a |e\rangle_b |Q\rangle_x &\rightarrow |1\rangle_a |1\rangle_b |Q_1\rangle_x\end{aligned}$$

$|Q_0\rangle_x$ and $|Q_1\rangle_x$ are copying machine states. Given ${}_x\langle Q|Q\rangle_x = 1$, the copying machine states $|Q_0\rangle_x$ and $|Q_1\rangle_x$ are normalized to unity, which means ${}_x\langle Q_0|Q_0\rangle_x = {}_x\langle Q_1|Q_1\rangle_x = 1$

We also assume ${}_x\langle Q_0|Q_1\rangle_x = {}_x\langle Q_1|Q_0\rangle_x = 0$

For any superposition state

$$|s\rangle_a = \alpha |0\rangle + \beta |1\rangle$$

For simplicity, we assume α and β are real. Also, we have $\alpha^2 + \beta^2 = 1$. The authors in [2] have claimed that the results also hold for the complex numbers. This has not been verified by the student. Using the transformation on the superposition state $|s\rangle_a$

$$\begin{aligned}|s\rangle_a |e\rangle_b |Q\rangle_x &\rightarrow \alpha |0\rangle_a |0\rangle_b |Q_0\rangle_x + \beta |1\rangle_a |1\rangle_b |Q_1\rangle_x \\ &= |\Psi\rangle_{abx}^{out}\end{aligned}$$

Evaluate $\hat{\rho}_{abx}^{out}$ from this and take partial trace on the third state to get $\hat{\rho}_{ab}^{out}$

$$\begin{aligned}\hat{\rho}_{ab}^{out} &= Tr_x[\hat{\rho}_{abx}^{out}] \\ &= \alpha^2 |00\rangle\langle 00| Tr[|Q_0\rangle\langle Q_0|] + \beta^2 |11\rangle\langle 11| Tr[|Q_1\rangle\langle Q_1|] + \\ &\alpha\beta(|00\rangle\langle 11| Tr[|Q_0\rangle\langle Q_1|] + |11\rangle\langle 00| Tr[|Q_1\rangle\langle Q_0|]) \\ &= \alpha^2 |00\rangle\langle 00| + \beta^2 |11\rangle\langle 11|\end{aligned}$$

The density operator describing the input state and target state can be expressed as

$$\begin{aligned}\hat{\rho}_a^{out} &= Tr_b[\hat{\rho}_{ab}^{out}] = \alpha^2 |0\rangle\langle 0| + \beta^2 |1\rangle\langle 1| \\ \hat{\rho}_b^{out} &= Tr_b[\hat{\rho}_{ab}^{out}] = \alpha^2 |0\rangle\langle 0| + \beta^2 |1\rangle\langle 1|\end{aligned}$$

The ideal output state for both reads

$$\hat{\rho}_a^{id} = |s\rangle_{aa}\langle s| = \alpha^2 |0\rangle\langle 0| + \beta^2 |1\rangle\langle 1| + \alpha\beta(|0\rangle\langle 1| + |1\rangle\langle 0|)$$

To measure the distance between the ideal output and the output from our machine, we define difference measure D_a as

$$D_a = Tr[\hat{\rho}_a^{id} - \hat{\rho}_a^{out}]^2$$

Substituting values for $\hat{\rho}_a^{id}$ and $\hat{\rho}_b^{id}$, we get $D_a = 2\alpha^2\beta^2$ Clearly, this machine works ideally for basis states where α is either 0 or 1. But the performance is not similar for superposition states. On average, we expect the distance to be

$$\bar{D}_a = \int_0^1 d\alpha^2 D_a(\alpha^2) = \frac{1}{3}$$

Moreover, these qubits are entangled. For the ideal output with no entanglement, we can write the combined state as:

$$\hat{\rho}_{ab}^{id} = \hat{\rho}_a^{id} \otimes \hat{\rho}_b^{id}$$

A crude way to quantitatively express the entanglement of these state would be by the distance between the ideal and actual density operator. Solving which, we get

$$D_{ab}^{(1)} = Tr[\hat{\rho}_{ab}^{out} - \hat{\rho}_a^{out} \otimes \hat{\rho}_b^{out}]^2 = D_a D_b = 4\alpha^4\beta^4$$

We can also evaluate the Hilbert Schmidt norm for the density matrices $\hat{\rho}_{ab}^{out}$ and $\hat{\rho}_{ab}^{id}$. This is a measure of difference of the combined state of the output and ideal

$$D_{ab}^{(2)} = Tr[\hat{\rho}_{ab}^{out} - \hat{\rho}_{ab}^{id}]^2 = D_a + D_b$$

Another measure can be defined as difference between ideal output defined by $\hat{\rho}_{ab}^{id}$ and the product of the states of two single qubit systems

$$D_{ab}^{(3)} = Tr[\hat{\rho}_{ab}^{id} - \hat{\rho}_a^{out} \otimes \hat{\rho}_b^{out}]^2$$

The input state averaged value for the above norms is $\bar{D}_{ab}^{(1)} = \frac{2}{15}$, $\bar{D}_{ab}^{(2)} = \frac{2}{3}$ and $\bar{D}_{ab}^{(3)} = \frac{8}{15}$ Observe that, when we calculate $D_{ab}^{(2)}$, the diagonal elements of the two density operators match, but the off diagonal elements are absent in the $\hat{\rho}_{ab}^{out}$ operator. This gives us a sense that the off-diagonal elements get dropped.

2.3 Universal QCM

An universal QCM has the characteristic that the quality of the output of this machine doesn't depend on the input state(unlike Wootters-Zurek QCM where the quality was dependent on input state parameter α)

This machine is designed such that the previously defined measures D_a and D_{ab}^2 are independent of the input qubit state and take minimum values. The proposed transformation is

$$\begin{aligned}|0\rangle_a |e\rangle_b |Q\rangle_x &= |0\rangle_a |0\rangle_b |Q_0\rangle_x + [|0\rangle_a |1\rangle_b + |1\rangle_a |0\rangle_b] |Y_0\rangle_x \\ |1\rangle_a |e\rangle_b |Q\rangle_x &= |1\rangle_a |1\rangle_b |Q_1\rangle_x + [|0\rangle_a |1\rangle_b + |1\rangle_a |0\rangle_b] |Y_1\rangle_x\end{aligned}$$

this gives us following constraints

$$\begin{aligned}{}_x\langle Q_i|Q_i\rangle_x + 2{}_x\langle Y_i|Y_i\rangle_x &= 1 \qquad i = 0, 1 \\ {}_x\langle Y_0|Y_1\rangle_x = {}_x\langle X_0|X_1\rangle_x &= 0 \\ {}_x\langle Q_i|Y_i\rangle_x &= 0 \qquad i = 0, 1\end{aligned}$$

For other inner products, we assign values

$$\begin{aligned}{}_x\langle Q_i|Y_j\rangle_x &= \frac{\eta}{2} \qquad i \neq j \\ {}_x\langle Y_0|Y_0\rangle_x &= {}_x\langle Y_1|Y_1\rangle_x = \xi\end{aligned}$$

We have

$$\begin{aligned}|s\rangle_a |e\rangle_b |Q\rangle_x &\rightarrow \alpha |0\rangle_a |0\rangle_b |Q_0\rangle_x + \beta |1\rangle_a |1\rangle_b |Q_1\rangle_x \\ &+ [|0\rangle_a |1\rangle_b + |1\rangle_a |0\rangle_b](\alpha |Y_0\rangle_x + \beta |Y_1\rangle_x) \\ &= |\Psi\rangle_{abx}^{out}\end{aligned}$$

The density operator for output qubit evaluates to

$$\begin{aligned}\hat{\rho}_a^{out} &= |0\rangle_{aa}\langle 0| [\alpha^2 + (\beta^2 - \alpha^2)\xi] \\ &+ \eta[|0\rangle_{aa}\langle 1| + |1\rangle_{aa}\langle 0|]\alpha\beta \\ &+ |1\rangle_{aa}\langle 1| [\beta^2 + (\alpha^2 - \beta^2)\xi]\end{aligned}$$

We already have $\hat{\rho}_a^{id}$. The Hilbert Schmidt norm D_a evaluates to

$$D_a = 2\xi^2(4\alpha^4 - 4\alpha^2 + 1) + 2\alpha^2(1-\alpha^2)(\eta - 1)^2$$

Since, we want D_a to be independent of parameter α, we can find a relation between η and ξ by solving the following

$$\frac{\partial}{\partial \alpha^2} D_a = 0$$

The parameters η and ξ are related as

$$\eta = 1 - 2\xi$$

And the distance D_a can be rewritten as

$$D_a = 2\xi^2$$

Now, η can also be eliminated using $D_{ab}^{(2)} = Tr[\hat{\rho}_{ab}^{out} - \hat{\rho}_{ab}^{id}]$.

$$\begin{aligned}\hat{\rho}_{ab}^{out} =& (\alpha^2 \left|00\right\rangle \left\langle 00\right| + \beta^2 \left|11\right\rangle \left\langle 11\right|)(1-2\xi) \\ &+ [(\left|00\right\rangle + \left|11\right\rangle)(\left\langle 01\right| + \left\langle 10\right|) \\ &+ (\left|01\right\rangle + \left|10\right\rangle)(\left\langle 00\right| + \left\langle 11\right|)](1-2\xi)\alpha\beta \\ &+ \xi(\left|01\right\rangle + \left|10\right\rangle)(\left\langle 01\right| + \left\langle 10\right|)\end{aligned}$$

Since, this value has to be independent of parameter α^2, again it can be claimed

$$\frac{\partial}{\partial \alpha^2} D_{ab}^{(2)} = 0$$

Solving the above, we find $\xi = \frac{1}{6}$. For this value of ξ, the norm $D_{ab}^{(2)}$ evaluates to $\frac{2}{9}$ independent of parameter α^2

We can also calculate the fidelity for this values of ξ and η. Define $\left|\Phi_1\right\rangle_a$ and $\left|\Phi_2\right\rangle_a$ as an orthonormal basis

$$\begin{aligned}\left|\Phi_1\right\rangle_a &= \alpha\left|0\right\rangle_a + \beta\left|1\right\rangle_a \\ \left|\Phi_2\right\rangle_a &= \beta\left|0\right\rangle_a - \alpha\left|1\right\rangle_a\end{aligned}$$

then we have

$$\hat{\rho}_a^{out} = \frac{5}{6}\left|\Phi_1\right\rangle_{a\,a}\left\langle \Phi_1\right| + \frac{1}{6} + \left|\Phi_2\right\rangle_{a\,a}\left\langle \Phi_2\right|$$

and

$$\hat{\rho}_a^{id} = \left|\Phi_1\right\rangle_{a\,a}\left\langle \Phi_1\right|$$

Using $\hat{\rho}_a^{id}$ as $\hat{\rho}_1$ and $\hat{\rho}_a^{out}$ as $\hat{\rho}_2$, fidelity measure F is equal to a constant $\sqrt{\frac{5}{6}}$

Thus, we can claim that our QCM is Universal as all the input states are copied equally well.

We introduce two basis states $\left|\uparrow\right\rangle$ and $\left|\downarrow\right\rangle$ and after solving for $\left|Q_0\right\rangle_x$, $\left|Q_1\right\rangle_x$, $\left|Y_0\right\rangle_x$ and $\left|Y_1\right\rangle_x$, we find the following relations

$$\begin{aligned}\left|Y_0\right\rangle_x &= (1/\sqrt{6}, 0) & \left|Y_1\right\rangle_x &= (0, 1/\sqrt{6}) \\ \left|X_0\right\rangle_x &= (0, \sqrt{2/3}) & \left|X_1\right\rangle_x &= (0, \sqrt{2/3})\end{aligned}$$

We find that these four states are linearly dependent and can be related as

$$\left|Y_0\right\rangle_x = \frac{1}{2}\left|Q_1\right\rangle_x \quad \left|Y_1\right\rangle_x = \frac{1}{2}\left|Q_0\right\rangle_x$$

Thus, the transformation can be rewritten as

$$\begin{aligned}\left|0\right\rangle_a\left|e\right\rangle_b\left|Q\right\rangle_x &= \sqrt{\frac{2}{3}}\left|00\right\rangle\left|\uparrow\right\rangle + \sqrt{\frac{1}{6}}(\left|01\right\rangle + \left|10\right\rangle)\left|\downarrow\right\rangle \\ \left|1\right\rangle_a\left|e\right\rangle_b\left|Q\right\rangle_x &= \sqrt{\frac{2}{3}}\left|11\right\rangle\left|\downarrow\right\rangle + \sqrt{\frac{1}{6}}(\left|01\right\rangle + \left|10\right\rangle)\left|\uparrow\right\rangle\end{aligned}$$

The output states from this transformation has better input-averaged values for D_a or $D_{ab}^{(2)}$. This QCM copies all input states equally well, while the previous acted like a perfect QCM for basis states, but performed poorly for superposition states. One may prefer any of these over each other according to their requirement. Nevertheless, the output states received from this QCM is also entangled.

2.4 Measurement

2.5 Cloning states in a given neighborhood

References

[1] W.K. Wootters and W. H. Zurek, Nature (London) **299**, 802 (1982)

[2] V. Buzek and M. Hillery, Phys. Rev. A 54 (1996) 1844.