CS656A: Mini Project
Convergence to Equilibria in Plurality Voting

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1 Introduction

In this project, we studied the paper *Convergence to Equilibria in Plurality Voting*[1] by *Reshef Meir, Maria Polukarov, Jeffrey Rosenschein, Nicholas Jennings*

Often voting is quite complicated and is more than just electing whoever wins the most votes. There are several methods for determining a winner and some method maybe preferred over other depending on the type of election. Plurality voting is one such method wherein the candidate who gets the most number of votes wins. The voters may have same or different weights. This paper focuses on conditions under which the iterative game with plurality voting rule is guaranteed to converge to a Nash Equilibrium.

In this report, we begin by introducing by formally defining the game form for plurality voting and then proceed to the results presented in the paper.

2 Preliminaries

2.1 The Game Form

- Set of m candidates $C = \{c_1, ..., c_m\}$
- Set V of n voters
- candidates select an action from a set A ($A = C$ in Plurality Voting)
- Voting rule $f : A^n \longrightarrow 2^C \setminus \{0\}$
- Outcome of **joint action** is $f(a_1, ..., a_n)$
- **Extension of the game form:** only k out of n voters play strategically, ie, $\{a_1, ..., a_k\} = K \subseteq V$, while the rest are additional voters who have already casted their vote $B = V \setminus K = \{b_{k+1}, ..., b_n\}$
- Each voter has **weight** $w_i \in \mathbb{N}$
- Initial score of candidate c: $\hat{s}(c) = \sum_{j \in B: b_j = c} w_j$
- Final Score of candidate c: for joint action $\vec{a} \in A^k$, $s(c, \vec{a}) = \hat{s}(c) + \sum_{i \in K: a_i = c} w_i$
- $s(c) >_p s(c')$ if either $s(c) > s(c')$ or $s(c) = s(c')$ and c has higher priority (lower index)
- Types of Plurality tie-breaking rule:

– PL_D -Plurality rule with deterministic tie-breaking is defined as (note that it is a singleton set)

$$PL_D(\vec{\hat{s}}, \vec{w}, \vec{a}) = \{c : \forall c' \neq c, s(c, \vec{a}) >_p s(c', \vec{a})\}$$

– PL_R -Plurality rule with deterministic tie-breaking is defined as

$$PL_R(\vec{\hat{s}}, \vec{w}, \vec{a}) = argmax_{c \in C} s(c, \vec{a})$$

- **Outcome vector:** $\vec{s(a)} = (s(c_1, \vec{a}), ..., s(c_m, \vec{a}))$
- For a tie-breaking scheme T (T=D,R), **Game Form** $GF_T = \langle C, K, \vec{w}, \vec{\hat{s}} \rangle$ gives winner for the joint strategy $\vec{a}$ as $GF_T(\vec{a}) = PL_T(\vec{\hat{s}}, \vec{w}, \vec{a})$

2.2 Incentives

- R is a set of all strict orders over C
- $>_i \in R$ is a preference ordering of voter i over candidates
- **profile:** $\vec{r} = (>_1, ..., >_k)$ = preferences of all k agents
- **Normal Form Game:** $G_T = \langle GF_T, \vec{r} \rangle$
- We associate a voter's preferences with cardinal utilities. We extend the utilities to multiple winners (set W) as $u(W) = \frac{1}{|W|} \sum_{c \in W} u_i(c)$
- A utility function u is said to be consistent with preference relation $>_i$ if $u(c) > u(c') \Leftrightarrow c >_i c'$
- **Lemma1**[**1**]: For any utility function u which is consistent with preference order $>_i$, the following holds:
  1. $a >_i b \implies \forall W \subseteq C \setminus \{a, b\}, u(\{a\} \cup W) > u(\{b\} \cup W)$
  2. $\forall b \in W, a >_i b \implies u(a) > u(\{a\} \cup W) > u(W)$

  This lemma is quite easy to prove and hence the proof is skipped in this report.

2.3 Manipulation and Stability

Now that we have defined normal form game for plurality voting, *Improvement Step* and *Nash Equilibrium* are defined in the standard way. We define a few more terms which will be used often in the paper:

- **truthful vote:** $a^*(\vec{r}) = (a^*_1, ..., a^*_k)$ such that forall $c \neq a^*_i$, $a^*_i >_i c$. It is also called **truthful state** of G_T. $GF_T(a^*(\vec{r}))$ is called **truthful outcome** of the game.
- in the truthful state, if i has an improvement step, then it is called **manipulation**. Hence, if the game is in NE state, $\vec{r}$ can't be manipulated.

2.4 Game Dynamics

The game dynamics in plurality voting begin with each voter anouncing his initial vote. They are then informed of the winner accounrding to the initial votes. The voters proceed to change their votes and keep doing so until NE is reached, ie, no voter wants to change his vote. We make a few assumptions about the dynamics of change of votes. Firstly, the changes are improvement steps. Secondly, often the game begins with truthful votes.

Proposition 2[1]**:** If agents are allowed to re-vote simultaneously, the improvement process may never converge.
Proof: We prove the proposition by giving a simple counterexample:

- $C = \{a, b, c\}$ with initial score = (0,0,2)
- $V = \{1, 2\}$ with weights $w_1 = w_2 = 1$
- Preference ordering: $a >_1 b >_1 c$, $b >_2 a >_2 c$
- It can be seen that we observe a cycle if players simultaneously keep swapping between states (a,b) and (b,a)

Hence, the authors restrict the study in this paper to itertive change of votes, ie, only one voter changes his vote in an iteration. In the subsequent sections of the paper, we state the results shown in the paper and prove the most essential ones. Some of the results have just been stated in the paper due to space restriction. We prove few such results in this report which we found challenging and non-intuititve.

2.5 Truth Biased agents

Truth biased agents is a variation of the setting considered so far. These agents although cast vote to the candidate with higher preference, will switch to their true preference if their vote can't change the winner. Given two outcomes W and Z with vote to candidates a and a', if they don't have a preference among these, they cast their vote among a and a' to the one with higher preference

3 Results

3.1 Few more notation

Suppose at time t, voter i switches their vote from candidate a to candidate b

- the winner after the switch is further denoted by o_t and o_{t-1} for the one before the switch.
- $s(o_t)$ and $s(o_{t-1})$ denote their score when they are the winner
- For any candidate c, $s_t(c)$ represents their score excluding the vote of the current switching voter. For all voters, it follows that $s_{t-1}(c) = s_t(c)$

3.2 Best response or otherwise

Movement of any voter a can be classified into one of the following:

1. **type 1** $a_{i,t-1} \in o_{t-1}$ and $a_{i,t} \notin o_t$
2. **type 2** $a_{i,t-1} \notin o_{t-1}$ and $a_{i,t} \in o_t$
3. **type 3** $a_{i,t-1} \in o_{t-1}$ and $a_{i,t} \in o_t$

(Replace $\in$, $\notin$ with $=$, $\neq$ for deterministic tie-breaking)

These are the possible better replies that any player can make. Among these, the one which results in the best score for the voter a is their best reply. Clearly, there can't be any best reply of the type

1, as not switching the vote will be a better response than switching to a candidate who will lose even after counting their vote.

Please note that the notion of best reply has been defined as the immediate best response. All voters switch their vote to maximize their score immediately after the move. One can argue and show examples that the best response in the general notion may not align with these.

3.3 Partitions

The result space is broken into 16 partitions. These are made using following 4 criterias

1. Tie Breaking: Deterministic vs Randomized
2. Weights: Non-Weighted voting (Equal weight to every vote) vs Weighted voting (Different weights)
3. Response: Best response strategy or otherwise
4. Initial State: start from Truth value (most preferable) or Anywhere

Tie Breaking	Dynamics	Best response		Any better reply		Truth biased
	Initial State	Truth	Anywhere	Truth	Anywhere	
Deterministic	Non-weighted	V	V(3)	X(4b)	X	X
	Weighted ($k = 2$)	V	V(6a)	V(6b)	X(4a)	X
	Weighted ($k > 2$)	X(5)	X	X	X	X
Randomized	Non-weighted	V(8)	X(9)	X(10)	X(10)	X(11b)
	Weighted	X(7)	X	X	X	X

Table 1: The cells marked with V converge to Equilibria state, X suggests otherwise. The numbers in bracket identify the theorem/preposition (in paper)(not all present in report) which provides the proof/counterexample for the corresponding mark

3.4 Theorems and Counter Examples

Here we present some of the proofs/counterexamples to the above results

1. **Theorem 3** *Let G_D be a plurality game with deterministic tie-breaking. If all agents have weigh 1 and use best replies, then the game will converge to a NE from any state*

   For the moves of type 3, the voter must shift their vote to a candidate with preference higher than their current candidate. Any candidate can make only $m - 1$ such moves. So, there can't be more than $(m - 1) * k$ consecutive type 3 moves.

   For type 2, let's say the voter i changes their vote from a to b at time t. It can be claimed that at time $t' \geq t$, following two in-variants hold. 1. The score of candidate a doesn't increase. 2. There are at-least two candidates with score larger or equal to $s(o_{t-1})$. On looking at the step at $t' = t$, we have

   $$s_t(b) + 1 = s(o_t) >_p s(o_{t-1}) >_p s_{t-1}(a) + 1$$

   The proof of the two claims is through induction. The above observation helps with the base case. The score of candidate a decreased at $t' = t$. Also, two candidates b and $o_{t-1} \neq b$ have scores at-least $s(o_{t-1})$

   Now assume that the induction holds till t'. Looking at the step at time $t' + 1$. It holds that $s_{t'}(a) \leq_p s_{t-1}(a) <_p s(o_{t-1}) - 1$. Also, we have $s(o_{t'}) >_p s(o_{t-1})$. So, at-least 2 more votes are needed to make a a winning candidate, which implies, a receives no votes. i.e. score of candidate a doesn't increase. Further, voter j switches their vote at step $t'+1$ to some candidate c, which implies $s_{t'}(c) + 1 >_p s(o_{t'}) >_p s(o_{t-1})$. Here again, we have c and $o_{t'} \neq c$ with required scores larger than $s(o_{t-1})$. Thus, induction holds and the claims are true.

So, for candidate a, there can be at-most k moves of type 2, $(m-1)*k$ for all candidates. This implies the voting procedure converges in at-most m^2k^2 steps

2. Best response strategies guarantee convergence for equally weighted voters, but we don't get this guarantee when weights are different.

   Proposition 5[1]: There is a counterexample with 3 weighted agents that start from the truthful state and use best replies.

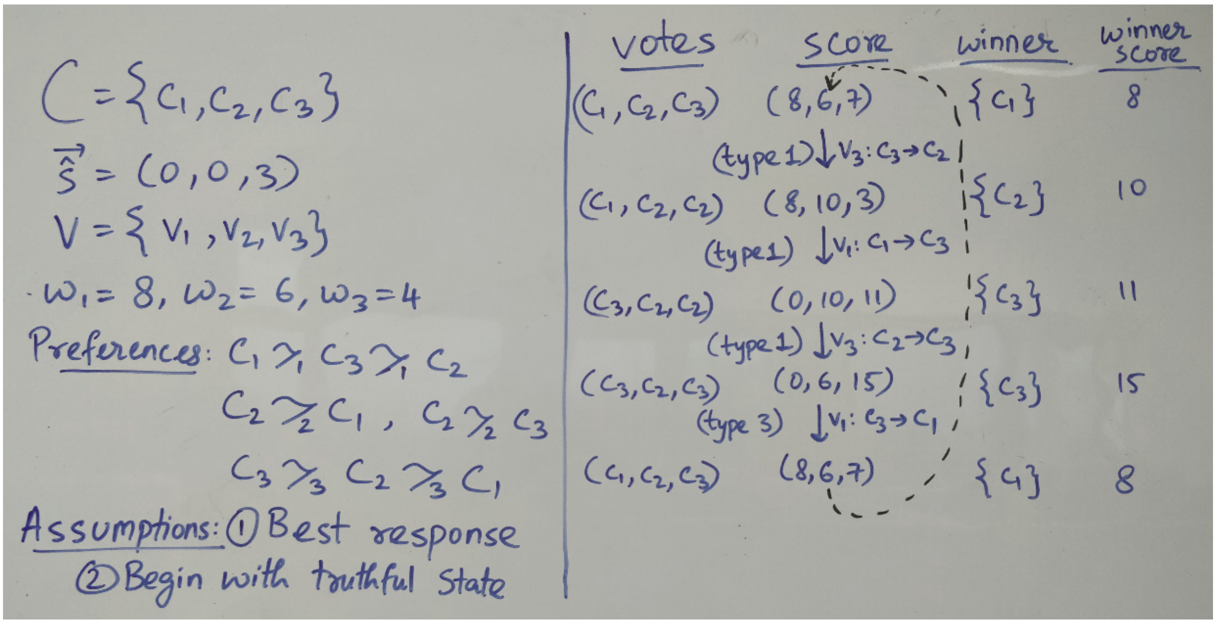

Figure 1: Counterexample: It is sufficient to show that a loop can exist in the state transformations. Because if it does, it is possible that voters keep following this loop indefinitely and never reach an equilibrium.

3. **Proposition 10(a)[1]** *If agents use arbitrary better replies, then there is a strong counterexample with 3 agents of weight 1*

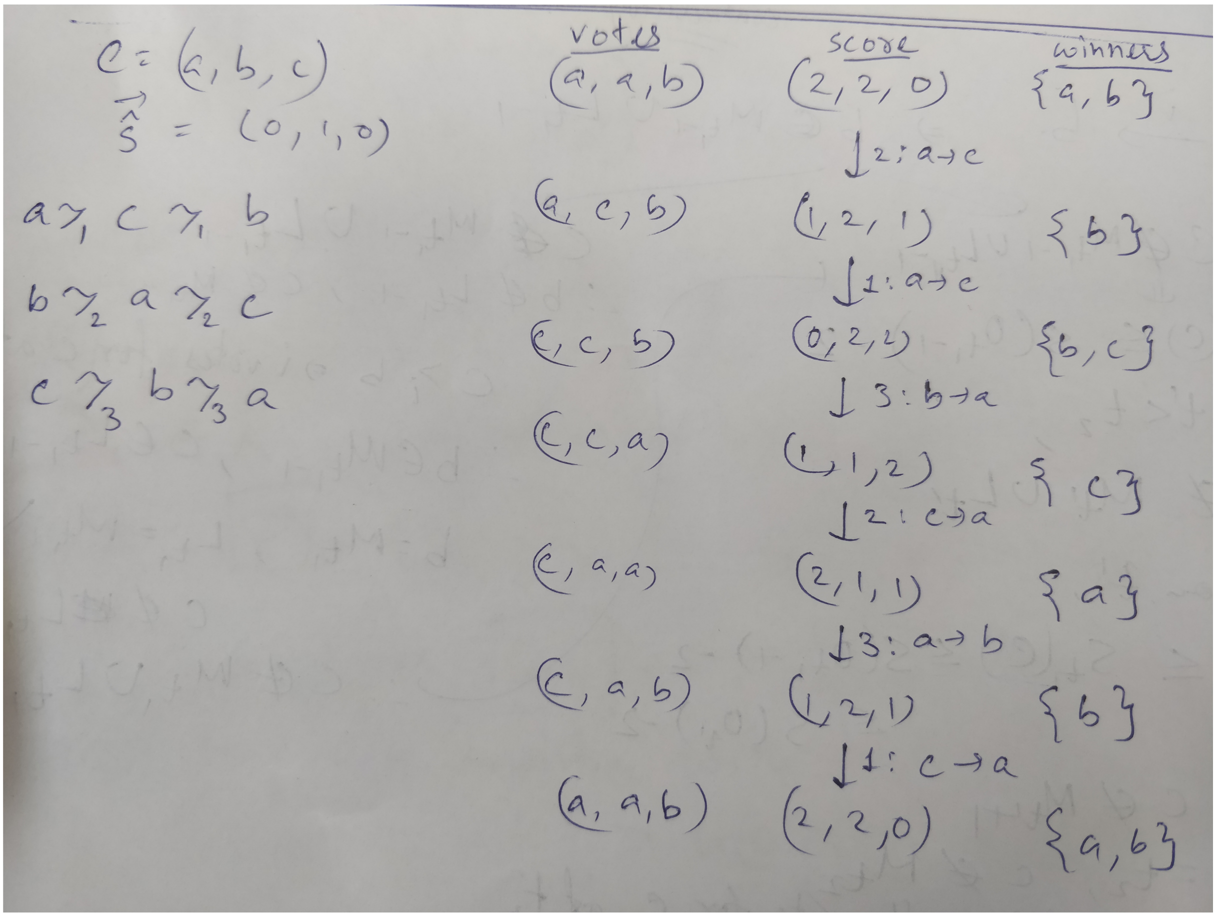

Figure 2: Counterexample: It is sufficient to show that a loop can exist in the state transformations. Because if it does, it is possible that voters keep following this loop indefinitely and never reach an equilibrium.

4. More Prepositions and proofs are present in Paper, but have been skipped to avoid repetition

References

[1] R. Meir, M. Polukarov, J. S. Rosenschein, and N. R. Jennings. Convergence to equilibria in plurality voting. In *AAAI*, volume 10, pages 823–828, 2010.