CS 580 Algorithm Design, Analysis, And Implementation
Fall 2021 Vassilis Zikas HW 5

Due Wednesday Nov. 17 at 11:59 p.m.

1. (50 points, 10 + 5 + 5 + 15 + 15) Consider the *shortest s-t path problem* in which the input is an undirected graph $G = (V,E)$ with nonnegative edge weights and vertices $s,t \in V$. The goal is to find the shortest path (minimum cost) *s-t* path in G.

   Consider the following primal and dual LP for the shortest *s-t* path problem. Here $S' = \{S \subseteq V : s \in S, t \notin S\}$ i.e. S' is the set of all *s-t* cuts in G. $\delta(S)$ represents edges in the cut S i.e. $\delta(S) = |\{(u,v) \in E : |\{u,v\} \cap S| = 1\}|$. The LPs are given below:

minimize	$\sum_{e\in E} c_e x_e$			maximize	$\sum_{S\in S'} y_S$	
subject to	$\sum_{e\in\delta(S)} x_e \geq 1$	$\forall S \in S'$		subject to	$\sum_{S\in S': e\in\delta(S)} y_S \leq c_e$	$\forall e \in E$
	$x_e \geq 0$	$\forall e \in E.$			$y_S \geq 0$	$\forall S \in S'$

   Let $G : (V,E)$ be the input graph. Consider the following algorithm based on these LPs.

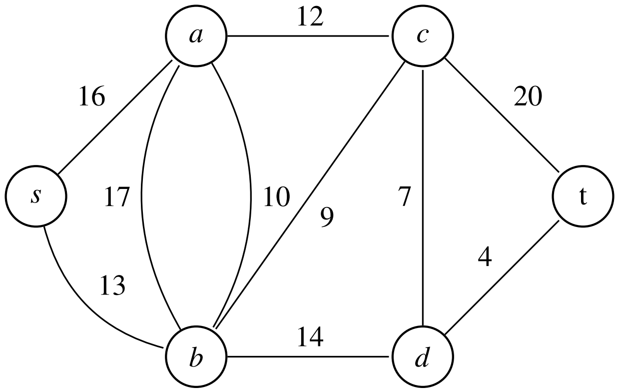
   - Initialize $y_S = 0,\ x_e = 0$ for all edges $e \in E$ and all cuts $S \in S'$.
   - **While** there is no *s-t* path formed in G using edges in $\{e : x_e > 0\}$ **do**
     Let $C \subseteq V$ be the connected component formed by edges whose $x_e = 1$ (C always contains s).[1]
     Increase y_C until there is an edge $e' \in \delta(C)$ such that $\sum_{S\in S': e'\in\delta(S)} y_S = c_{e'}$
     Set $x_{e'} = 1$.
   - Return an *s-t* path in G using edges in $\{e : x_e > 0\}$.

   Let the path returned be denoted by P.

   (a) What is an interpretation of the primal and dual LP – explain in sentences, as was done in class for the Chicago vs. Detroit Pizza example? In particular, explain what each variable, the objective and each constraint means.

   (b) Consider the following graph. We execute the above algorithm on this graph. In the first iteration, some set C_1 is chosen, and some edge e_1 has $x_{e_1} = 1$ at the end of the first iteration. Similarly we choose sets C_2, C_3 and edges e_2, e_3 in the second and third iteration, respectively. What are $C_1, C_2, C_3, e_1, e_2, e_3$, as well as $y_{C_1}, y_{C_2}, y_{C_3}$?

[1] In the first step, when $x_e = 0$ for all e, C is just the node s.



(c) Prove that at any point of the algorithm the set F of edges with $x_e = 1$, i.e. $F_t = \{e \in E : x_e = 1\}$ at the end of iteration t of the algorithm, forms a tree .

(d) Prove that for any $S \in S'$, if $y_S > 0$ at the end of the algorithm, then $|P \cap \delta(S)| = 1$ (where P is the path returned by the algorithm).

(e) Prove that the given algorithm returns a shortest s-t path in G. You can use parts (b) and (c) as stated (even if you didn't provide a solution for them).

2. (50 points, 10 + 10 + 10 + 20 respectively) For each of the following problems either design and analyze an efficient algorithm (i.e. atleast polynomial time), or prove that they are NP-hard.

   (a) Given an undirected graph G, with positive edge weights, does G have a spanning tree of weight atmost 42?

   (b) Given an undirected graph G, does G have a spanning tree with exactly 2 leaves?

   (c) Given an undirected graph G, does G have a spanning tree with maximum degree 2?

   (d) Given an undirected graph G, does G have a spanning tree with at most 42 leaves?

   Any vertex with degree one in a tree is a referred to as a leaf.