Problem 4

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(a) Given : T_1 and T_2 are two spanning trees of graph G.
As T_1 is a spanning tree, it has n vertices and n-1 edges.
It is given in the question that edge e belongs to T_1 but not T_2. So, if we remove the edge e from T_1, we will get two disconnected components, let say M and N.
We know that T_2 is also a spanning tree of graph G, so it must have an edge which connects the components M and N. (Property of spanning tree)
Let say that edge is d.
So, we can say that T_1 - e + d is a tree, is connected and has n-1 edges.
$\implies$ T_1 - e + d is spanning tree of graph G.

Similarly, we can show this for T_2.
As T_2 is a spanning tree, it has n vertices and n-1 edges.
It is given in the question that edge d belongs to T_2 but not T_1. So, if we remove the edge d from T_2, we will get two disconnected components, let say M and N.
We know that T_1 is also a spanning tree of graph G, so it must have an edge which connects the components M and N. (Property of spanning tree)
From above, we know that edge was e.
So, we can say that T_2 - d + e is a tree, is connected and has n-1 edges.
$\implies$ T_2 - d + e is spanning tree of graph G.
Hence, proved.

(b) Let E_1 be the number of edges in T_1 and E_2 be the number of edges in T_2. We can say that $|E_1|$ = $|E_2|$ = |V|-1 because they are spanning trees.
Now assume that there exists an edge e = {a,b} $\in E_2 \setminus E_1$ but there does not exist a mapping for e, i.e edge e' $\in E_1 \setminus E_2$ such that T_1 - e' + e is a spanning tree.So, this means that for all e' $\in E_1 \setminus E_2$, T_1 - e' + e is disconnected.
Now consider edge set $E_{1'} = E_1 \cup$ e.
T_1 was a spanning tree and adding an edge e to a spanning tree creates a cycle, so this means that $E_{1'}$ has a cycle. We also know that E_2 does not have a cycle (by definition), this means there exists an edge p $\in E_{1'}$ such that p $\neq$ e and p $\notin E_2$. Let us denote the set of all edges in the cycle which are not in E_2 by P.
For every edge p in P, we can say that T_1 + e - p has |V| - 1 edges and is connected because e and p are part of same cycle. Hence T_1 + e - p is a spanning tree for each p $\in$ P.
Now consider $E_{2'} = E_2 \setminus$ e.
This creates two disconnected components, let say A and B. Now let us consider the cases for all edges p = {u,v} $\in$ P:

(i) u and v have no path between them in $E_{2'}$:

This means that u belongs to one of the disconnected components in $E_{2'}$ and v belongs to the other. So adding an edge p again connects the components A and B and makes T_2 + p - e a spanning tree. However, this is contradiction to our assumption that e' does not exist.

(ii) u and v have path between them in $E_{2'}$:
If case (i) holds for any edge in P, then we have proved that there exists e' = p. So, this case considers the possibility that none of the edge in P satisfy case(i).
Consider the vertices a and b which were connected by edge e. They are also connected in $E_{2'}$. For all edges in cycle in $E_{1'}$ that also belongs to $E_{2'}$, we take these edges to the path and for all edges in P, we know that there is some path between them in $E_{2'}$. Hence, this forms a path from a to b without using edge e. This path also exists in E_2 which implies that this path and edge e are two possible paths for a and b in E_2. This is a contradiction as E_2 should have unique path between each pair of vertices. This means that there exists an edge in set P which satisfies case (i) and hence can be used as e'.

Hence, this shows that there exists an e' for each e.