CS 580
ALGORITHM DESIGN AND ANALYSIS

Linear Programming 2:
Duality

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SO FAR

- Linear programming
  - Basic definitions
  - Some problem formulations
  - Simplex
- Today:
  - Duality
  - Linear programs as games

DUALITY

- Example due to Tim Roughgarden
- $max\, x_1 + x_2$
- Subject to:
  - $4x_1 + x_2 \leq 2$
  - $x_1 + 2x_2 \leq 1$
  - $x_1 \geq 0$
  - $x_2 \geq 0$
- Claim: The optimal solution is $x_1 = \frac{3}{7}, x_2 = \frac{2}{7}$ for an objective of $\frac{5}{7}$

DUALITY

- $max\; x_1 + x_2$, subject to:
  - $4x_1 + x_2 \leq 2, x_1 + 2x_2 \leq 1, x_1, x_2 \geq 0$
- Claim: The optimal solution is $x_1 = \frac{3}{7}, x_2 = \frac{2}{7}$ for an objective of $\frac{5}{7}$
- How can we confirm this??

DUALITY

- $max\, x_1 + x_2$, subject to:
  - $4x_1 + x_2 \leq 2, x_1 + 2x_2 \leq 1, x_1, x_2 \geq 0$
- Claim: The optimal solution is $x_1 = \frac{3}{7}, x_2 = \frac{2}{7}$ for an objective of $\frac{5}{7}$
- How can we confirm this??
- Objective $= x_1 + x_2 \leq 4x_1 + x_2 \leq 2!$
  - So, optimal can be at most 2

DUALITY

- $max\, x_1 + x_2$, subject to:
  - $4x_1 + x_2 \leq 2, x_1 + 2x_2 \leq 1, x_1, x_2 \geq 0$
- Claim: The optimal solution is $x_1 = \frac{3}{7}, x_2 = \frac{2}{7}$ for an objective of $\frac{5}{7}$
- How can we confirm this??
- Objective $= x_1 + x_2 \leq x_1 + 2x_2 \leq 1$

DUALITY

- $x_1 + x_2 = \frac{4}{7}x_1 + \frac{3}{7}x_1 + \frac{1}{7}x_2 + \frac{6}{7}x_2$
- $= \frac{1}{7}(4x_1 + x_2) + \frac{3}{7}(x_1 + 2x_2)$
- $\leq \frac{1}{7}2 + \frac{3}{7}1$
- $= \frac{5}{7}$
- Cool!
- This is a proof that we have an optimal solution!!

DUALITY

- Arbitrary LP
  - We'll call this the Primal (P) LP
- $\max \sum_{i=1}^{n} c_i x_i$, s.t.
  - $\sum_{i=1}^{n} a_{1i} x_i \leq b_1$
  - $\sum_{i=1}^{n} a_{2i} x_i \leq b_2$
  - …
  - $\sum_{i=1}^{n} a_{mi} x_i \leq b_m$
  - $x_i \geq 0$, for all $i = 1, \ldots, n$

DUALITY

- Arbitrary LP
  - We'll call this the Primal (P) LP
- $\max \vec{c}^T \vec{x}$, s.t.
  - $A \cdot \vec{x} \leq \vec{b}$
  - $\vec{x} \geq 0$
- Where $A_{j,i} = a_{j,i}$
- $\vec{c}$ and $\vec{x}$ are vectors in n dimensions
- $\vec{b}$ is a vector in m dimensions
- A is an m by n matrix

DUALITY

- In order to get our upper bound on the objective, we were trying to express the objective by combining constraints
- Multiply constraint j by a number $y_j \geq 0$
- We want the coefficient of x_i in the objective, i.e. c_i, to be at most the coefficient in the combo of constraints
  - $c_i \leq \sum_{j=1}^{m} y_j a_{j,i}$
- In matrix notation:
  - $A^T \vec{y} \geq \vec{c}$

WEAK DUALITY

- Objective $= \sum_{i=1}^{n} c_i x_i$
- $\leq \sum_{i=1}^{n} \left(\sum_{j=1}^{m} y_j a_{j,i} \right) x_i$
- $= \sum_{j=1}^{m} y_j \sum_{i=1}^{n} a_{j,i} x_i$
- $\leq \sum_{j=1}^{m} y_j b_j$
- Matrix way:
    - $\vec{c}^T \, \vec{x} \leq (A^T \vec{y})^T \, \vec{x} = (\vec{y})^T A \vec{x} \leq (\vec{y})^T \vec{b}$
- Overall, OPT at most $\sum_{j=1}^{m} y_j b_j$ for all $\vec{y}$ such that $A^T \vec{y} \geq \vec{c}$

WEAK DUALITY

- This is a whole other LP!
- min $\vec{y}^T \vec{b}$
- Subject to: $A^T \vec{y} \geq \vec{c}\,, \vec{y} \geq 0$
- We call this LP the Dual (D)
- **Theorem (Weak Duality)**: OPT of P at most OPT of D
  - Remember max flow and min cut?

WEAK DUALITY

- Primal:
  - $max\, x_1 + x_2$
  - Subject to:
    - $4x_1 + x_2 \leq 2$
    - $x_1 + 2x_2 \leq 1$
    - $x_1, x_2 \geq 0$
- Dual:
  - $\min 2y_1 + y_2$
  - Subject to:
    - $4y_1 + y_2 \geq 1$
    - $y_1 + 2y_2 \geq 1$
    - $y_1, y_2 \geq 0$

DUALITY

- Recipe for taking duals:

Primal	Dual
variables $x_1, \ldots, x_n$	n constraints
m constraints	variables $y_1, \ldots, y_m$
objective function $\mathbf{c}$	right-hand side $\mathbf{c}$
right-hand side $\mathbf{b}$	objective function $\mathbf{b}$
$\max \mathbf{c}^T\mathbf{x}$	$\min \mathbf{b}^T\mathbf{y}$
constraint matrix $\mathbf{A}$	constraint matrix $\mathbf{A}^T$
ith constraint is "$\leq$"	$y_i \geq 0$
ith constraint is "$\geq$"	$y_i \leq 0$
ith constraint is "$=$"	$y_i \in \mathbb{R}$
$x_j \geq 0$	jth constraint is "$\geq$"
$x_j \leq 0$	jth constraint is "$\leq$"
$x_j \in \mathbb{R}$	jth constraint is "$=$"

WEAK DUALITY COROLLARIES

- OPT P $\leq$ OPT D
  - Therefore, if P is unbounded, D is infeasible!
  - If P is infeasible, then D is unbounded!
  - If x, y are two feasible solutions for the primal and dual, and $c^T x = y^T b$, then x and y are both optimal!

COMPLEMENTARY SLACKNESS

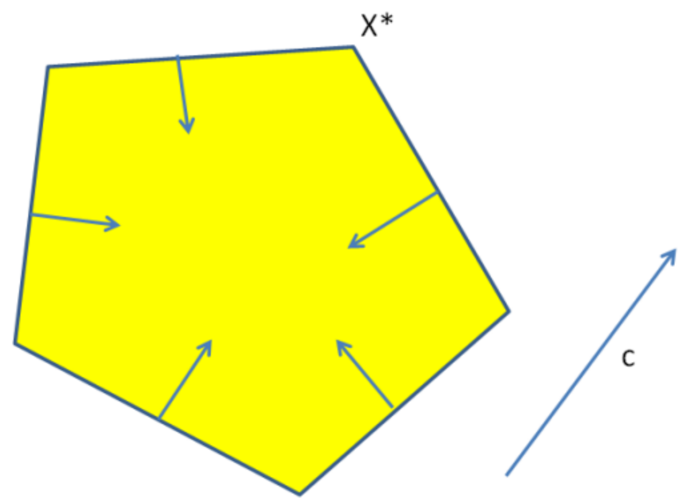
- **Complementary Slackness**: If both of the following conditions hold, then $\vec{x}$ and $\vec{y}$ are optimal
  - When $x_i \neq 0$, $\vec{y}$ satisfies the i-th constraint of D with equality
  - When $y_j \neq 0$, $\vec{x}$ satisfies the j-th constraint of P with equality

COMPLEMENTARY SLACKNESS

- Proof:
  - What's the i-th constraint of D?
    - $c_i \leq \sum_{j=1}^{m} y_j a_{j,i}$
  - Thus, $c_i x_i = \left(\sum_{j=1}^{m} y_j a_{j,i}\right) x_i$
  - What's the j-th constraint of P?
    - $\sum_{i=1}^{n} a_{ji} x_i \leq b_j$
  - Thus, $y_j \left(\sum_{i=1}^{n} a_{j,i} x_i\right) = y_j b_j$
  - Overall, $\vec{c}^T \vec{x} = (A^T \vec{y})^T \vec{x} = (\vec{y})^T A \vec{x} = (\vec{y})^T \vec{b}$

COMPLEMENTARY SLACKNESS

- A particle is pushed in direction c until it rests at x^*
- Total "force" on the particle is 0
  - "Force" from constraint i is $-A_i$, the i-th row of the constraint matrix
  - Dual variable y_i: magnitude of force of constraint i
- Complementary slackness: A "wall" can exert force only if you touch it

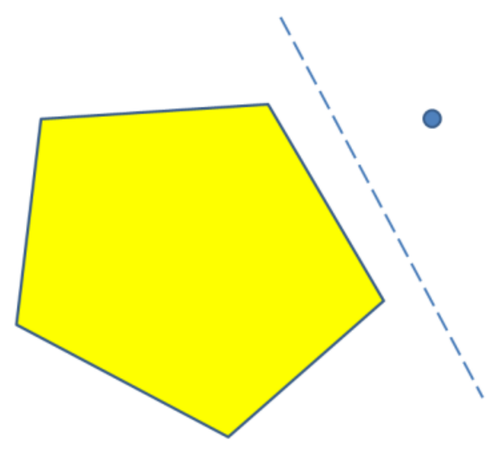


STRONG DUALITY

- **Theorem (Strong Duality)**: OPT of P is equal to OPT of D
- Sketch:
- Separating hyperplane → Farka's Lemma
- Farka's Lemma → strong LP duality

STRONG DUALITY

- **Separating Hyperplane Theorem:**
  - Let C be a closed and convex region of $\mathbb{R}^n$ and $z \notin C$ a point. Then, there exists a hyperplane that separates z from C



STRONG DUALITY

- Farka's lemma:
  - Given a matrix $A \in \mathbb{R}^{m \times n}$ and a right-hand side $b \in \mathbb{R}^m$ exactly one of the following is true:
    - There exists $x \in \mathbb{R}^n$: $Ax \leq b$
    - There exists $y \in \mathbb{R}^m$: $A^T y = 0, y \geq 0$ and $y^T b < 0$

STRONG DUALITY

- Farka's lemma to strong duality (sketch):
  - Assume that the optimal value of the dual was γ and the primal's optimal value was strictly less
  - Add the $-c^T x \leq -\gamma$ constraint to the primal
  - Use Farka's lemma to argue that $[\mathrm{A} \quad -c]^{\mathrm{T}}[y\ z] = 0, [y\ z]^T[b - \gamma] < 0$ and $y \geq 0, z \geq 0$
  - It must be that $z > 0$ (why?)
  - $yA - zc = 0 \rightarrow \left(\frac{y}{z}\right) A = c$
  - $yb - z\gamma < 0$ and $\frac{y}{z}$ is feasible, so γ is not optimal

BREAK

- Weak duality
  - Primal value smaller or equal to Dual value
- Complementary slackness
  - In an optimal pair of solutions, positive variables in one program correspond to tight constraints in the other
- Strong duality
  - Primal value is equal to Dual value

2 PLAYER ZERO SUM GAMES

- Rock-Paper-Scissors

Alice/Bob	Rock	Paper	Scissors
Rock	(0,0)	(-1,1)	(1,-1)
Paper	(1,-1)	(0,0)	(-1,1)
Scissors	(-1,1)	(1,-1)	(0,0)

- Alice wins if Bob loses, and vice versa
- Minimizing the opponents reward is the same as maximizing your reward
- Minimax value: The highest value a player can guarantee without knowing the actions of the other player
  - Equivalently, “if I go first, and the other player plays after they’ve seen my strategy, what’s the best I can do?”

2 PLAYER ZERO SUM GAMES

Alice/Bob	Rock	Paper	Scissors
Rock	(0,0)	(-1,1)	(1,-1)
Paper	(1,-1)	(0,0)	(-1,1)
Scissors	(-1,1)	(1,-1)	(0,0)

- Clearly, if we focus on deterministic strategies, Alice (or Bob) cannot guarantee a minimax value better than -1
  - If Alice's strategy is to play "Rock", then Bob will play "Paper"

2 PLAYER ZERO SUM GAMES

Alice/Bob	Rock	Paper	Scissors
Rock	0	-1	1
Paper	1	0	-1
Scissors	-1	1	0

- What about randomized strategies?
- Say, with probability ½ Alice plays "Rock" and with probability ½ Alice plays "Paper".
- What would Bob do, knowing this strategy?
  - If he plays "Rock", with probability ½ it's a tie, and with probability ½ he loses
  - If he plays "Paper", with probability ½ he wins, and with probability ½ it's a tie
  - If he plays "Scissors", with probability ½ he loses, and with probability ½ he wins

2 PLAYER ZERO SUM GAMES

Alice/Bob	Rock	Paper	Scissors
Rock	0	-1	1
Paper	1	0	-1
Scissors	-1	1	0

- So, Bob responds with “Paper” to Alice’s strategy
- Alice’s value: $\frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot 0 = -\frac{1}{2}$
- Better than every deterministic strategy
- Can she do even better?
  - Yes! Playing (1/3,1/3,1/3) gives value zero!
  - Is this optimal? How would you show this?

2 PLAYER ZERO SUM GAMES

- Input: An n by m matrix A
  - $A_{i,j}$ is the reward of the “row” player, when she plays action i and the “column” player plays action j
- Problem: Compute minimax value

2 PLAYER ZERO SUM GAMES

- Let p_i be the probability that the row player assigns to action i
- We have a strategy $\vec{p} = (p_1, p_2, \dots, p_n)$
- The column player will see this strategy and then try to maximize his reward, i.e. minimize the reward of the row player
  - The column player picks a <u>single</u> action $j \in [m]$
  - Why???
- Reward $\min_{j \in [m]} \sum_{i \in [n]} p_i A_{i,j}$
- Best strategy for row player:

$$\max_{feasible\ \vec{p}} \min_{j \in [m]} \sum_{i \in [n]} p_i A_{i,j}$$

2 PLAYER ZERO SUM GAMES

- Write an LP!
- Variables: $p_1, \ldots, p_n$ and v (the minimax value)
- Objective: $\max v$
- Subject to:
  - $1 \leq \sum_{i \in [n]} p_i \leq 1$ (feasibility)
  - $\sum_{i \in [n]} p_i A_{i,j} \geq v, \forall j \in [m]$ (minimax)
  - $\forall i \; p_i \geq 0$

2 PLAYER ZERO SUM GAMES

- Theorem: The LP described computes the minimax value
- Proof:
- We will show that
  - (1) Every valid strategy corresponds to a feasible solution for the LP, and the corresponding minimax value is equal to the LPs objective
  - (2) Every feasible LP solution corresponds to a valid strategy, whose minimax value is at least the LPs objective

2 PLAYER ZERO SUM GAMES

- Proof of (1)
- A valid strategy $\vec{x}$ is a distribution over actions, i.e. non-negative numbers $x_1, \ldots, x_n$ that add up to 1
  - Setting $p_i = x_i$ satisfies the feasibility constraints
- Given a valid strategy $\vec{x}$ the column player best responds, resulting in value $v^* = \min_{j \in [m]} \sum_{i \in [n]} x_i A_{i,j}$
  - Setting $v = v^*$ we have that the minimax constraints are satisfied and that the LP objective is precisely the value v^*

2 PLAYER ZERO SUM GAMES

- Proof of (2)
- Let $p_1, \dots, p_n, v$ be a feasible solution of the LP
- Then, setting $x_i = p_i$ we get a valid strategy for the row player
- The column player will best respond and give value $v^* = \min_{j\in[m]} \sum_{i\in[n]} x_i A_{i,j}$
  - v is smaller than $\sum_{i\in[n]} x_i A_{i,j}$ for all j, thus $v \le v^*$
  - Note that in an optimal solution, v will be as large as possible, i.e. at least one of the minimax constraints will be tight. Therefore $v = v^*$

2 PLAYER ZERO SUM GAMES

- What if the column player goes first?
- Write an LP!
- Objective: min c
- Subject to:
  - $1 \leq \sum_{j\in[m]} q_j \leq 1$ (feasibility)
  - $\sum_{j\in[m]} q_j A_{i,j} \leq c, \forall i \in [n]$ (minimax)
  - $\forall j\ q_j \geq 0$

2 PLAYER ZERO SUM GAMES

Objective: max v
Subject to:

$$\sum_{i\in[n]} p_i = 1$$
$$\text{v} - \sum_{i\in[n]} p_i A_{i,j} \leq 0, \forall j \in [m]$$
$$\forall i \in [n], p_i \geq 0$$

Objective: min c
Subject to:

$$\sum_{j\in[m]} q_j = 1 \qquad \text{(Variable } v\text{)}$$
$$\text{c} - \sum_{j\in[m]} q_j A_{i,j} \geq 0, \forall i \in [n] \qquad \text{(Variable } p_i\text{)}$$
$$\forall j \in, q_j \geq 0$$

2 PLAYER ZERO SUM GAMES

- Minimax Theorem:

$$\max_{feasible\ \vec{p}} \min_{j \in [m]} \sum_{i \in [n]} p_i A_{i,j} = \min_{feasible\ \vec{q}} \max_{i \in [n]} \sum_{j \in [m]} q_j A_{i,j}$$

- Different version:

$$\max_{\vec{p}} \min_{\vec{q}} p^T A q = \min_{\vec{q}} \max_{\vec{p}} p^T A q$$

- Proof: Strong Duality!
  - Other direction is true as well!!

SUMMARY

- Linear programs:
  - Weak and Strong Duality
  - Complementary Slackness
  - 2 player zero sum games