CS 580 Algorithm Design, Analysis, And Implementation
Fall 2021 Vassilis Zikas HW 4

Problem 1

Collaborators: List students that you have discussed problem 1 with: Vishnu Teja Narapareddy, Tulika Sureka

1. (a) Primal LP
   This representation stands for finding all the s-t paths in the given graph and then the objective function tries to minimise the cost along that path. Hence, it helps us to get the shortest s-t path in the given undirected graph.
   Variable c_e denotes the weight of that edge.
   Variable x_e denotes edges which are considered in the s-t paths.
   The first constraint denotes that the number of edges taken from each cut for finding the s-t path must be atleast 1.
   The second constraint denotes that for each edge, number of times that edge is chosen cannot be less than 0. It has to be atleast 0.

   (b) Dual LP

2.

3.

4.

5.

Problem 2

Collaborators: List students that you have discussed problem 2 with: Vishnu Teja Narapareddy, Tulika Sureka

1. This problem can be solved in polynomial time.
   We can use the Prim's algorithm to construct the minimum spanning tree and adding up the weights on the edges which are there in the tree. If the weight for the spanning tree comes out to be less than or equal to 42, then we can say that the graph G has a spanning tree of weight atmost 42 else we say that the graph cannot a spanning tree with weight less than or equal to 42.

2. We will try to show that the given problem is NP-hard problem.
   We can reduce the undirected hamiltonian path problem to the given 2 leaf spanning tree problem.

Construction:
From the given undirected graph G = (V, E), start vertex s and end vertex t, we can construct a graph G' = (V', E') such that V' = V $\cup\{s_1, t_1\}$ and E' = E $\cup\{(s_1, s), (t, t_1)\}$. Now, we will prove that there is a hamiltonian path from s to t in G if and only if there is a spanning tree in G' with exactly 2 leaves.

Proof:
Let's assume there is a hamiltonian path P from s to t in graph G. As P is a spanning tree in G, we can construct P' = P $\cup\{(s_1, s), (t, t_1)\}$, which will be a spanning tree in G'. Also, every vertex in the spanning tree P in G has degree 2 except vertex s and t, which have a degree of 1. Similarly analysing P', we can say that every vertex common in graph G and G' has degree 2, including s and t, because of the edge construction done by us earlier. Only the 2 new nodes added by us s_1, t_1 have a degree of 1. Thus, we can say that the spanning tree has exactly 2 leaves.
Now, let's assume there is a spanning tree H = (V', F) of G' with exactly 2 leaves. As the new vertices s_1, t_1 added by us have degree 1 in G', they will have the same degree in subgraph H too. This means that every other node in the subgraph H must have degree greater than 2 as we assumed that H has only 2 leaves. We can say that F must also contain a path from s to t because as we start from s, every node must be connected to some other node in the graph as the degree of each node is greater than or equal to 2 (shown above) and this other node cannot be in the set F as spanning trees are acyclic in nature. Hence, this gives a hamiltonian path from s to t in G.

3. For this, we show that the given problem is equivalent to the undirected hamiltoninan path problem.

   2-Spanning tree means a spanning tree in which each vertex has degree less than or equal to 2 in the path. This statement is equivalent to finding undirected hamiltonian path in the graph G as all the vertices in the path should also have a degree atmost 2.
   Hence, we can say that the given problem is NP-hard.

4. We can prove this problem as a NP-hard problem.
   We can reduce undirected Hamiltonian path problem to the given problem, which we already know that it is a NP-hard problem.

   Construction:
   From the given undirected graph G, we construct another graph H such that

   Construction:
   Construct a graph G' from G in the following manner:

   - Add a vertex z with edges to every other vertex in z
   - Now add 41 vertices,namely, l_1, l_2, ..., l_{41}, each with edges to t and nothing else.

   It can be clearly seen that H can be built from G in ploynomial time.

   Let say, we refer to a spanning tree of G' as almost-hamiltonian if it has atmost 42 leaves. We can say that G contains a Hamiltonian path if and only if H contains an almost-hamiltonian spanning tree.
   Proof:
   Let G has a hamiltoninan path P. Suppose P starts at vertex s and ends at vertex t. Let T be the subgraph of G' obtained by adding the edge tz and all possible zl_i. Then we can say that T is a spanning tree of H with exactly 42 leaves, which are s and the newly added 41 vertices l_i.

Suppose G' has almost-hamiltoninan spanning tree T. Each node l_i is a leaf in T, so T must consist of 41 edges zl_i and a simple path from z to some vertex s of G. Let t be the only neighbour of z in T that is not in leaf l_i, and let P be the unique path in T from s to t. This path visits every vertex of G, which means that P is a Hamiltonian path in G.