CS 580
ALGORITHM DESIGN AND ANALYSIS

Greedy Algorithms 2:
Scheduling (cf. KT 4.1 & 4.2)

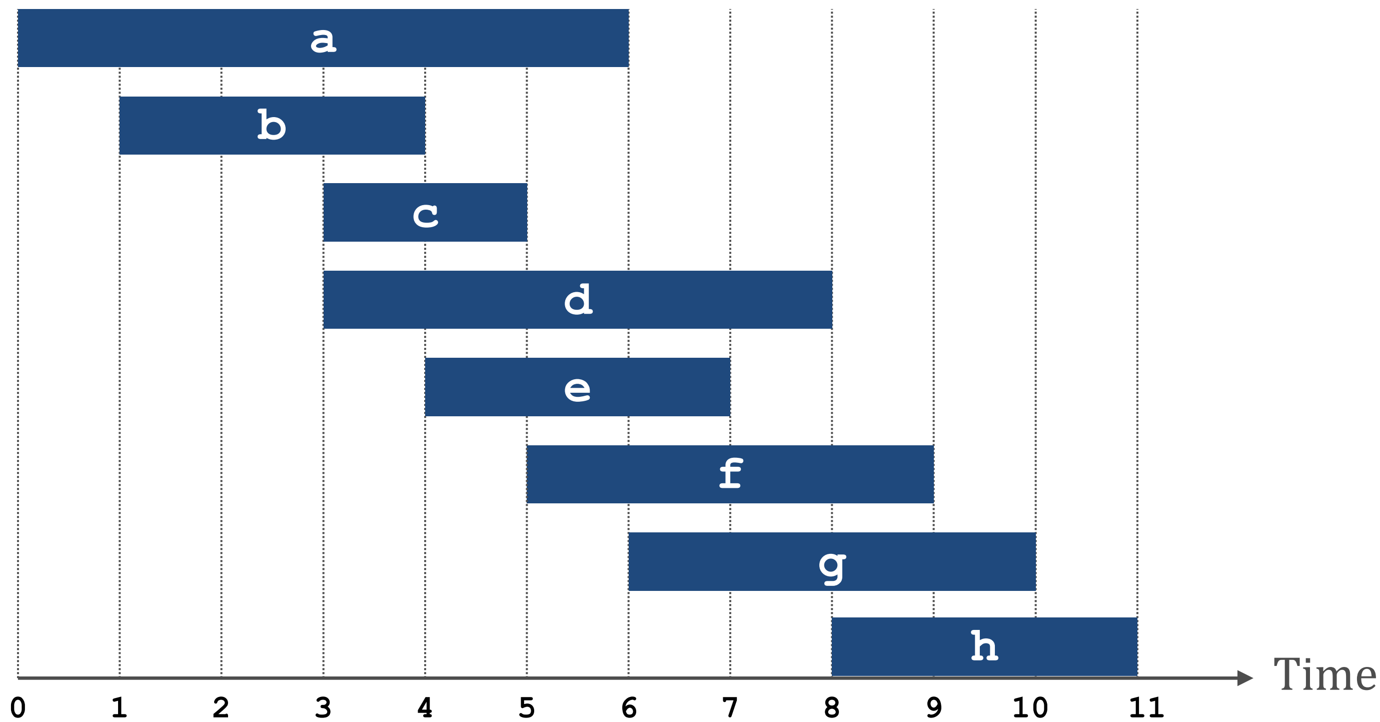
Vassilis Zikas

SO FAR

- Shortest paths
  - Dijkstra
  - Bellman-Ford
- Today:
  - Greedy algorithms that don’t have to do with graphs!
  - Interval scheduling (4.1 in KT)
  - Scheduling to minimize lateness (4.2 in KT)

INTERVAL SCHEDULING

- There's an incoming set of jobs $\{1, \ldots, n\}$
- The ith job corresponds to an interval $[s_i, t_i]$
- Two jobs are compatible if they don't overlap
- Goal: find maximum subset of compatible jobs



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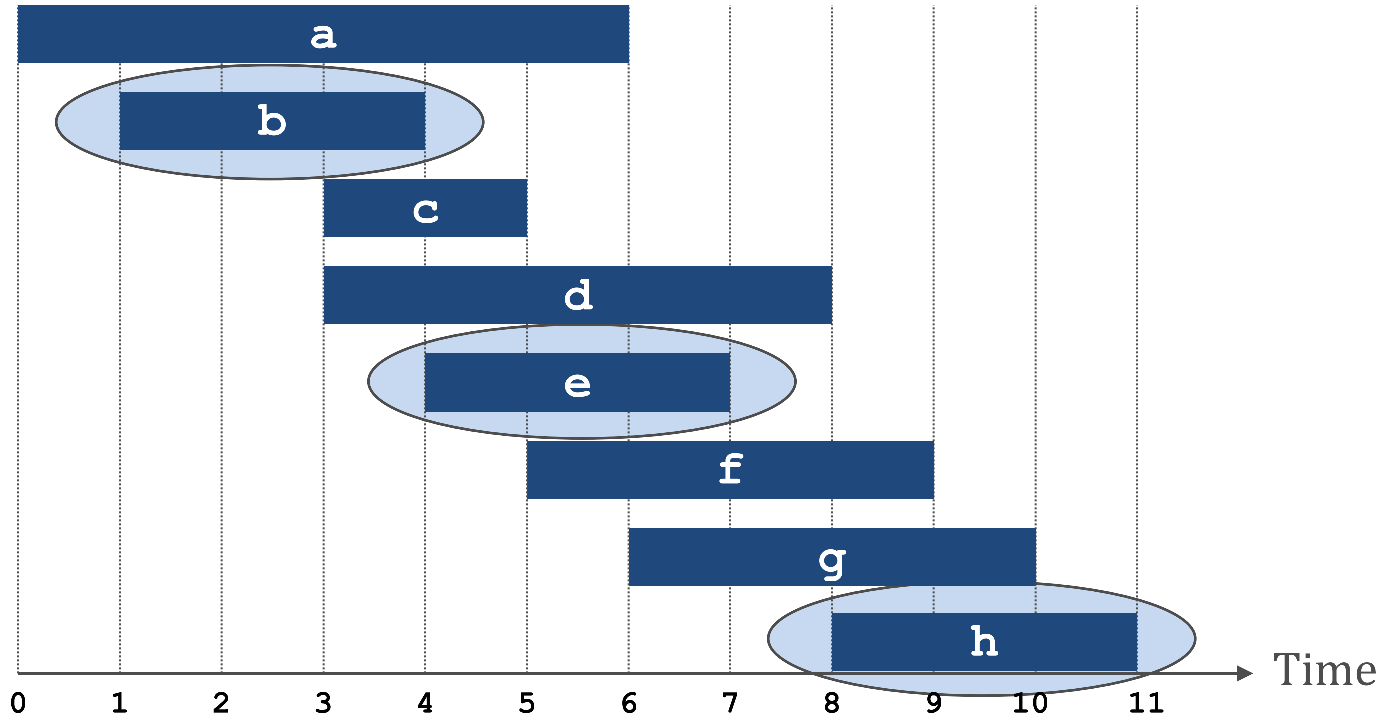

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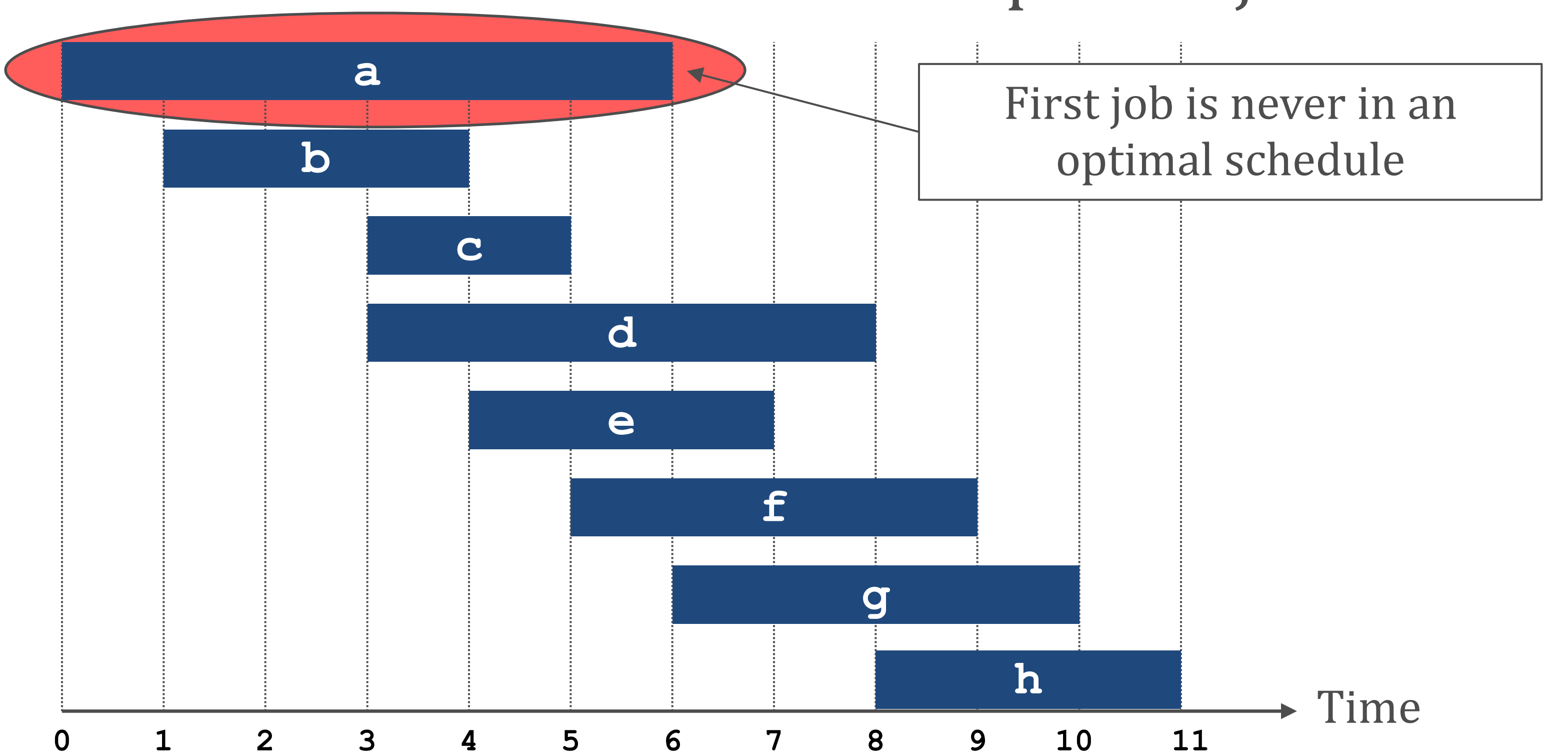

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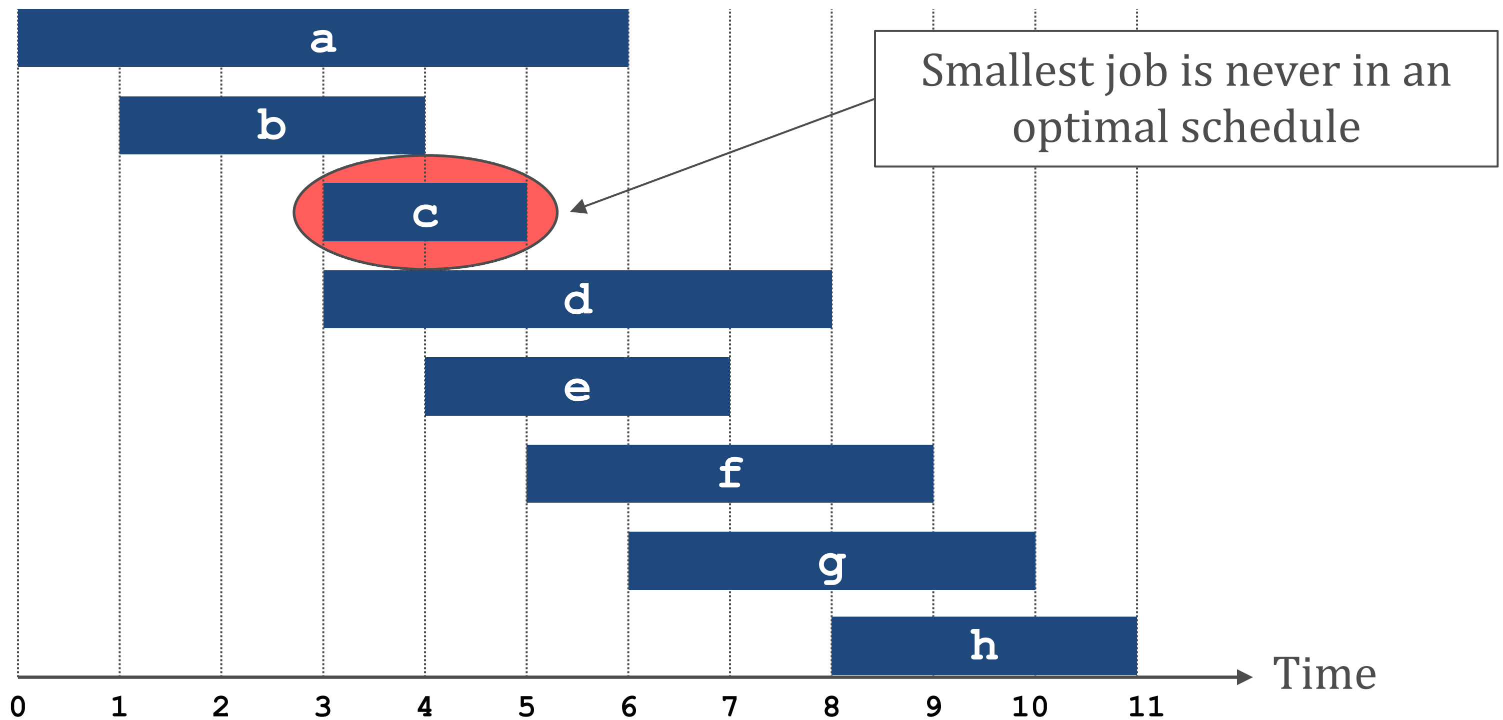

INTERVAL SCHEDULING GREEDY TEMPLATES

- Earliest start time?
- Earliest finish time?
- Pick smallest size first?
- Minimize number of overlaps?

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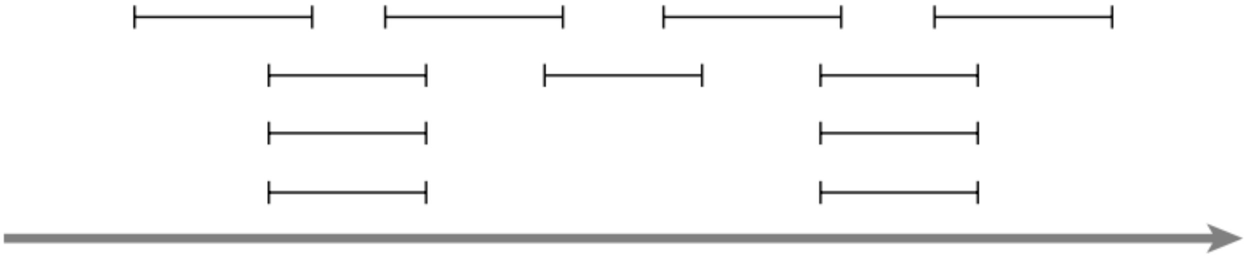

INTERVAL SCHEDULING: EARLIEST FINISH TIME?

- Consider algorithms in increasing order of finish time t_i
  - Take a job if compatible with the ones picked so far

1. Sort by finish time to get $t_1 \leq t_2 \leq \cdots \leq t_n$
2. $A \leftarrow \emptyset$.   //Set of jobs selected so far
3. For $j = 1, \ldots, n$:
   - If j compatible with A: $A = A \cup \{j\}$

INTERVAL SCHEDULING

- Running time $O(nlogn)$
  - Sorting takes $O(nlogn)$
  - Checking if a job is compatible equivalent to $s_i \geq f_{j^*}$, the last job added: $O(n)$

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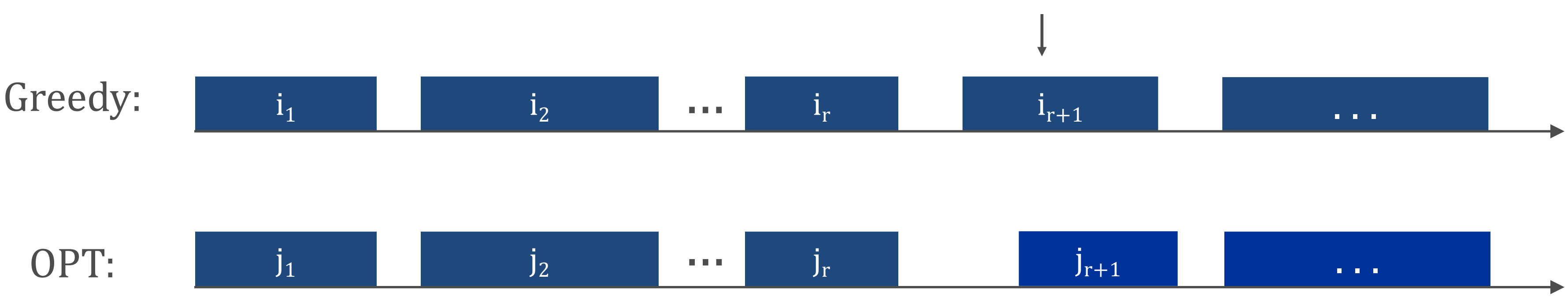
INTERVAL SCHEDULING

- Theorem: The greedy algorithm is optimal

Proof

- Assume greedy is not optimal
- Let $i_1, \dots, i_k$ be the jobs selected by greedy
- Let $j_1, \dots, j_m$ be the optimal set of jobs
    - $m > k$

By definition, i_{r+1} finishes before j_{r+1}



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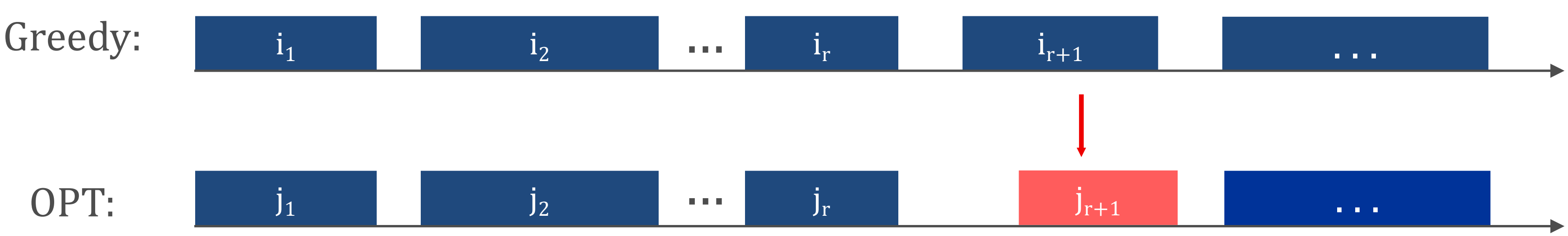
Replacing j_{r+1} with i_{r+1} won't break feasibility



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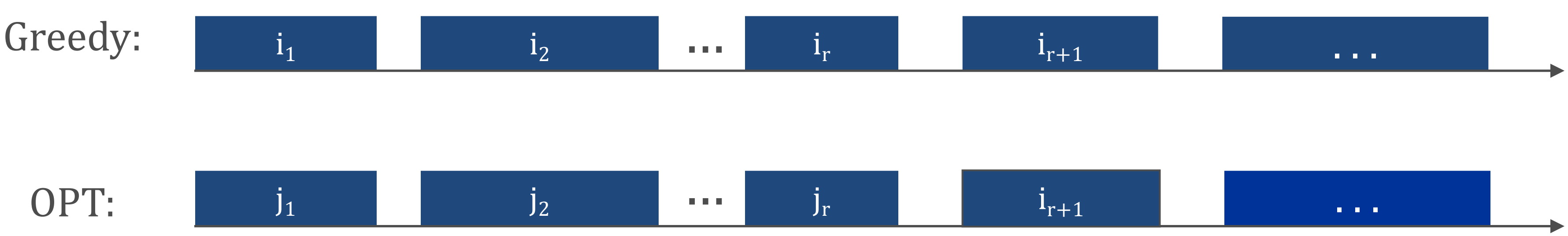
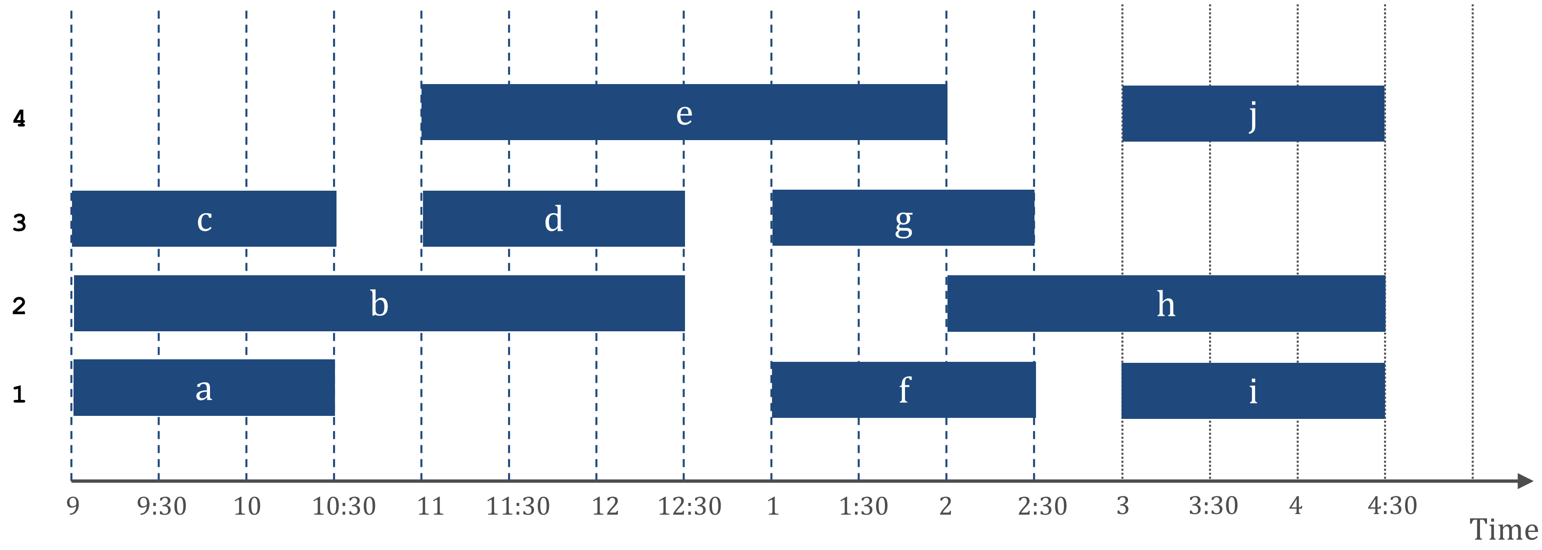
Size of OPT didn't change either
Contradiction: r was supposed to be maximal



INTERVAL PARTITIONING (SCHEDULING ALL INTERVALS)

- Lecture j starts at s_j and finishes at f_j
- Goal: find minimum number of classrooms to schedule all lectures so that no two occur at the same time in the same room

INTERVAL PARTITIONING (SCHEDULING ALL INTERVALS)



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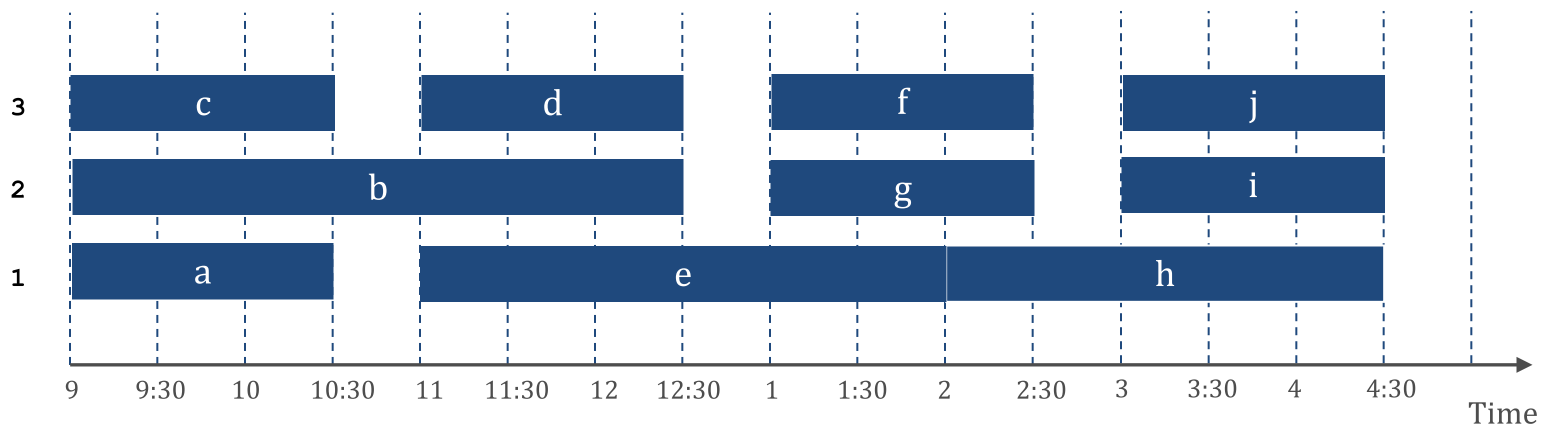
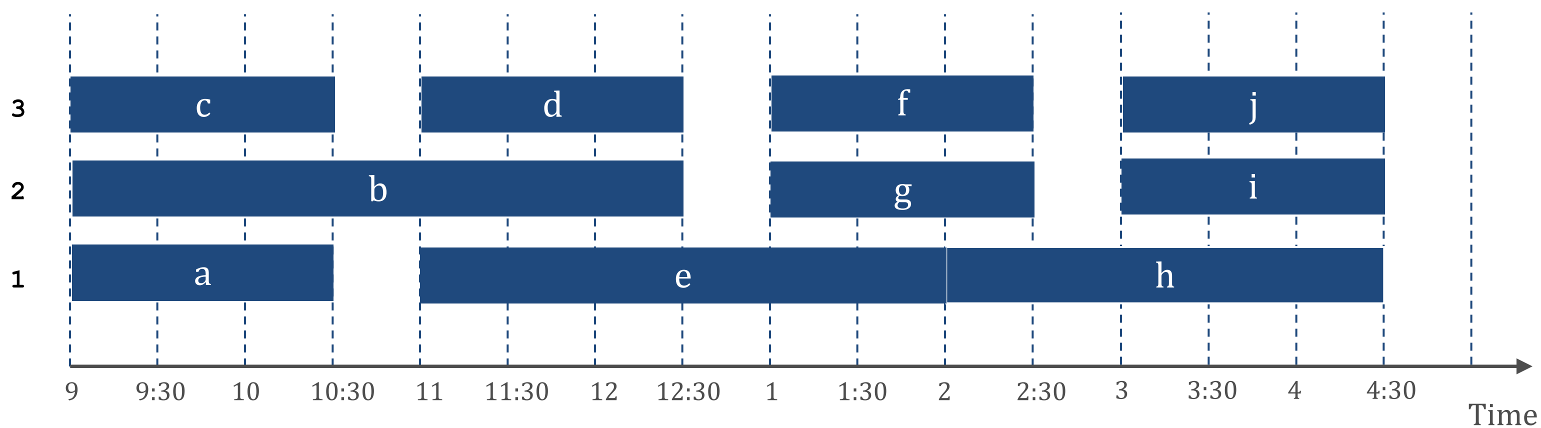


INTERVAL PARTITIONING (SCHEDULING ALL INTERVALS)

- Perhaps possible with two?
  - No: a, b and c overlap, so we need at least 3
- Definition: The depth of a set of intervals is the maximum number that pass over any single point in time



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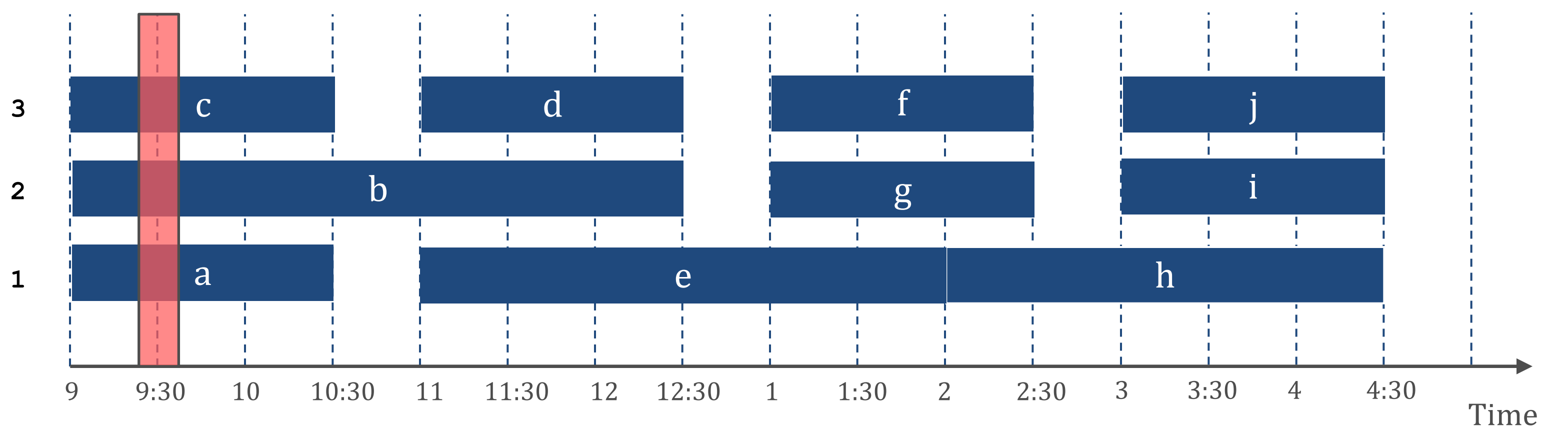


INTERVAL PARTITIONING

- Question: does there exist a schedule equal to the depth?
  - If the answer is yes, then this is clearly optimal

INTERVAL PARTITIONING

- Greedy algorithm: consider lectures in increasing starting time
  - assign to any compatible classroom (including a new one)

1. Sort intervals by start time to get $s_1 \leq \cdots \leq s_n$
2. For $j = 1$ to n:
   - If lecture j is compatible with some classroom k, schedule j in k
   - Otherwise, schedule j in a new classroom

INTERVAL PARTITIONING

- Implementation:
  - $O(n \log(n))$ to sort
  - At each time, need to figure out if a lecture is compatible
    - Use priority queue to remember end times of classrooms!

INTERVAL PARTITIONING

Theorem: Greedy is optimal

<u>Proof</u>

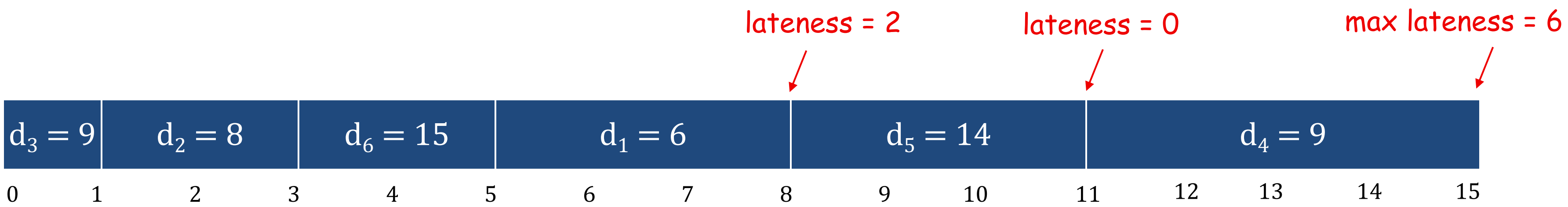
- Let d be the number of classrooms greedy needs
- Classroom d opened because we needed to schedule some job j that was incompatible with all $d - 1$ previous classrooms
- These d jobs (the $d - 1$ plus j) <u>all</u> **end** after s_j
- Since we sorted by start time, <u>all</u> these jobs also **start** before (or at) s_j
- Therefore, there exists a set of d jobs that overlap at s_j → need at least d classrooms

SCHEDULING TO MINIMIZE LATENESS

- Problem:
  - We have a single resource and need to process n jobs
  - We can process only one job at a time
  - Job j requires t_j units of time and expires at time d_j
    - If j starts at s_j it finishes at $f_j = s_j + t_j$
  - Lateness: $\ell_j = \max\{0, f_j - d_j\}$
  - Goal: minimize maximum latency $\mathrm{L} = \max_j \ell_j$

SCHEDULING TO MINIMIZE LATENESS

	1	2	3	4	5	6
t_j	3	2	1	4	3	2
d_j	6	8	9	9	14	15



SCHEDULING TO MINIMIZE LATENESS

- Greedy template: Consider jobs in *some* order
  - Shortest processing time first
    - Sort by t_j (ascending)
  - Smallest slack first
    - Sort by $d_j - t_j$ (ascending)

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	1	2
t_j	1	10
d_j	100	10

Processing time ≠ Urgency
Deadline can be very far in the future

SCHEDULING TO MINIMIZE LATENESS

- Greedy template: Consider jobs in *some* order
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    - Sort by t_j (ascending)
  - Smallest slack first
    - Sort by $d_j - t_j$ (ascending)

	1	2
t_j	1	10
d_j	2	10

Deadline - Processing time ≠ Urgency
Ignores small deadlines that take little time

SCHEDULING TO MINIMIZE LATENESS

- Greedy algorithm: Earliest deadline first
  - Initially seems like a bad idea to not take processing time into account…

1. Sort jobs so that $d_1 \leq d_2 \leq \cdots \leq d_n$
2. t=0
3. For $j = 1 \ldots n$:
   - Assign job j to interval $[t, t + t_j]$
   - $t = t + t_j$

SCHEDULING TO MINIMIZE LATENESS

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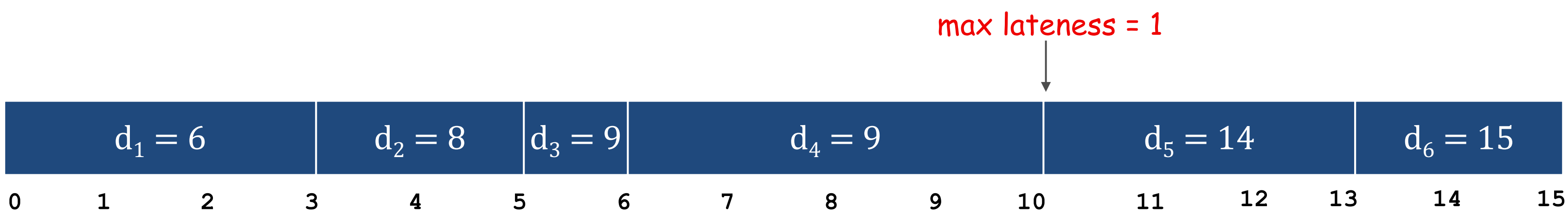
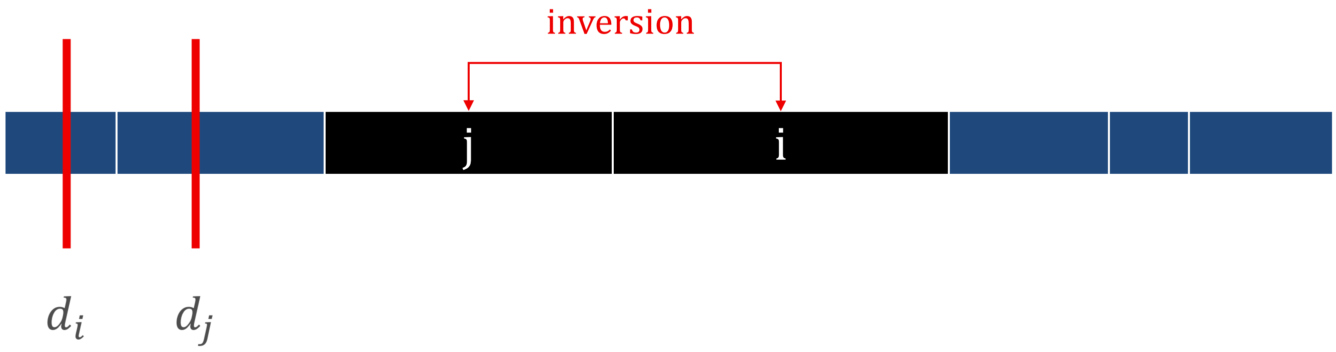


SCHEDULING TO MINIMIZE LATENESS

- **Observation**: There exists an optimal schedule with no idle time
  - Proof:
  - Assume the optimal schedule has some time interval $[t, t']$ where nothing is scheduled
  - For all jobs that start after t, move their start time earlier by $t' - t$
  - Still feasible, and latencies only decreased
- **Observation**: Greedy has no idle time

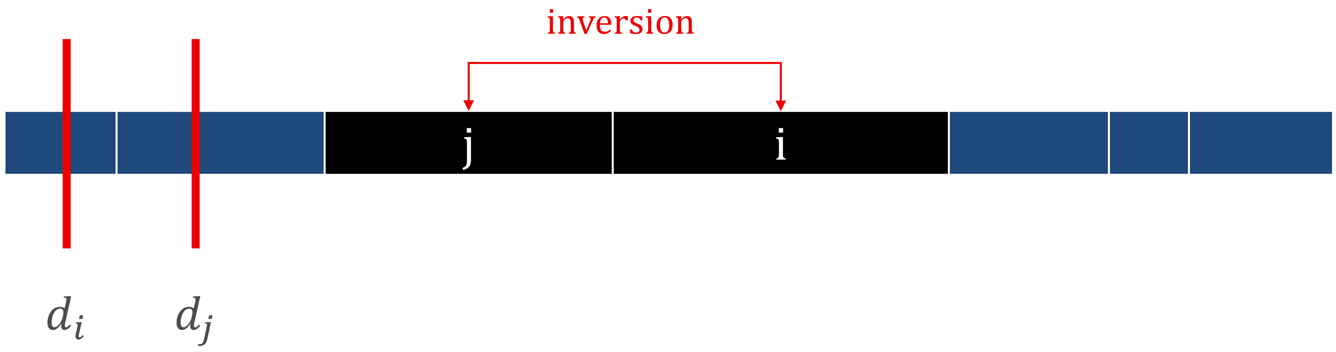
SCHEDULING TO MINIMIZE LATENESS

- **Definition**: Given a schedule S, an inversion is a pair of jobs i and j such that $d_i < d_j$ but j is scheduled before i
  - By definition, greedy has no inversions



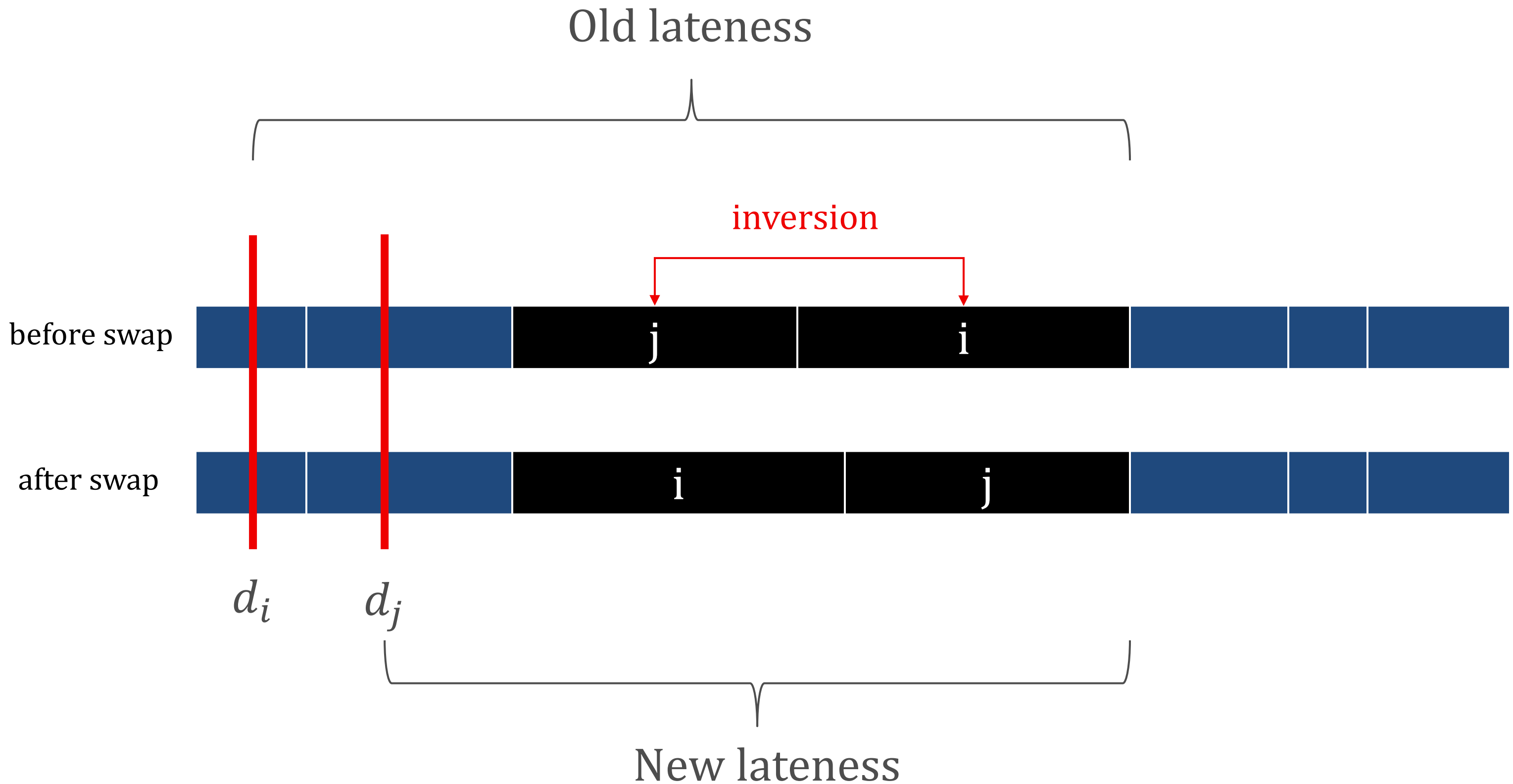
SCHEDULING TO MINIMIZE LATENESS

- Observation: If an inversion exists, then a "consecutive" inversion exists
  - Start from the left, until you find the first violation



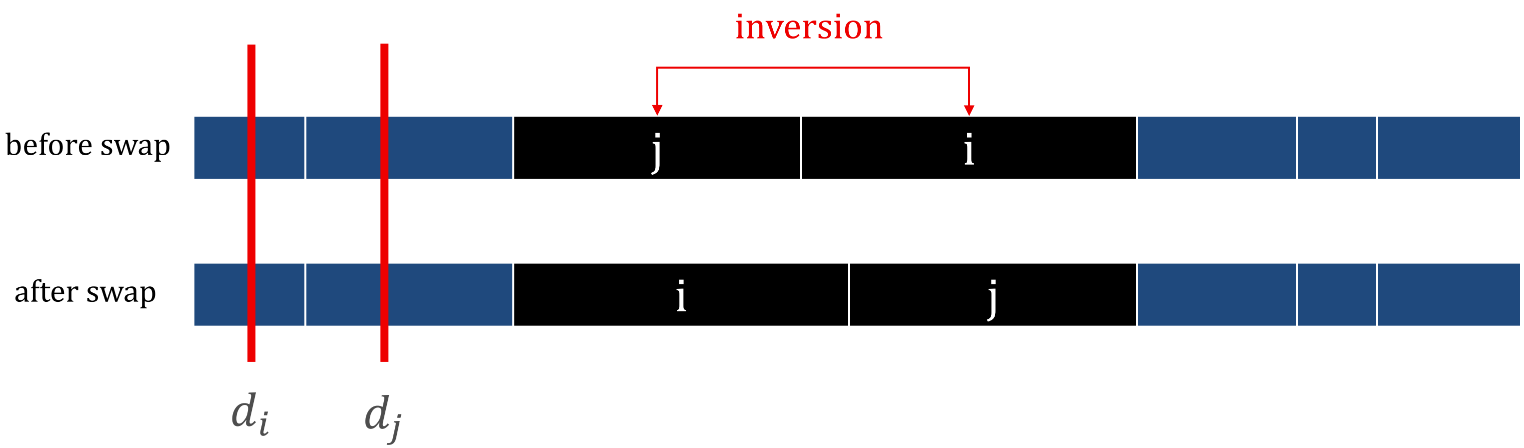
SCHEDULING TO MINIMIZE LATENESS

- What happens if we swap?
  - Everyone other than *i* and *j* remain unchanged



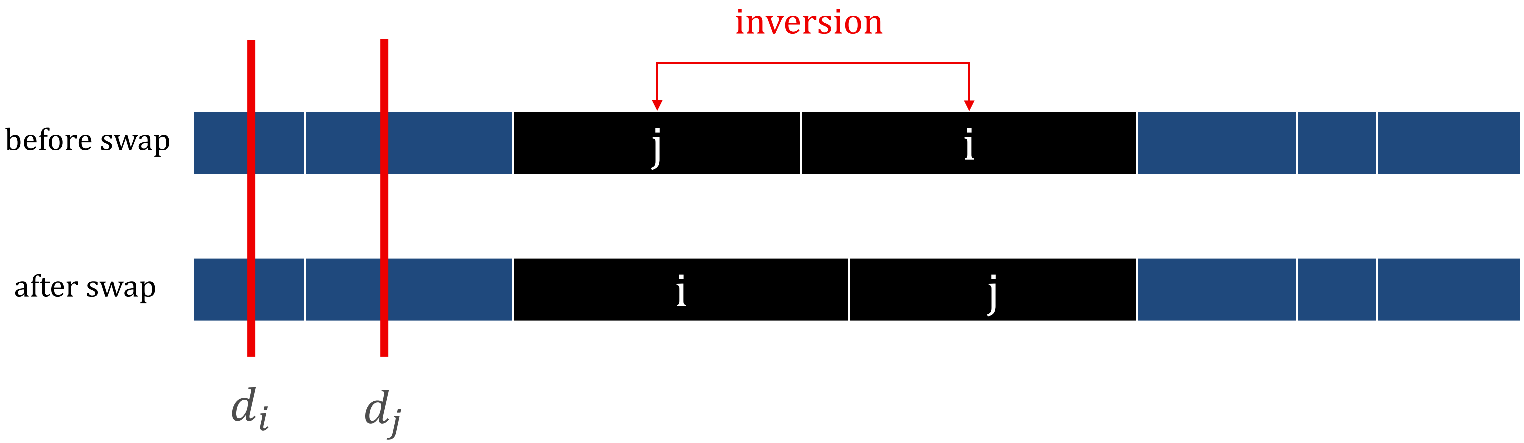
SCHEDULING TO MINIMIZE LATENESS

- What happens if we swap?
  - ℓ'_j



SCHEDULING TO MINIMIZE LATENESS

- What happens if we swap?
  - $\ell'_j = f'_j - d_j$



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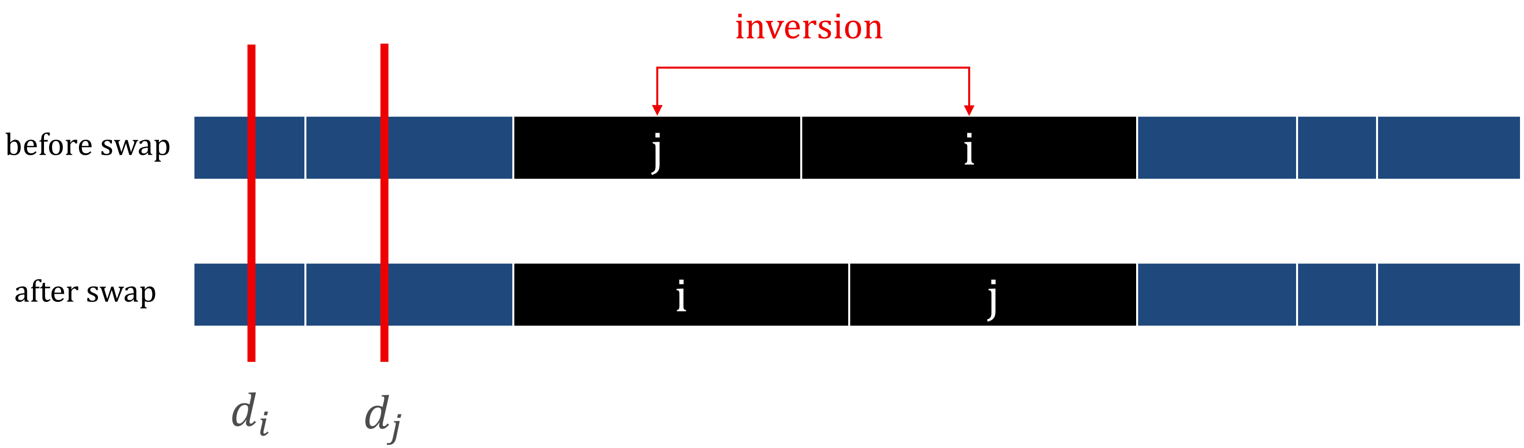

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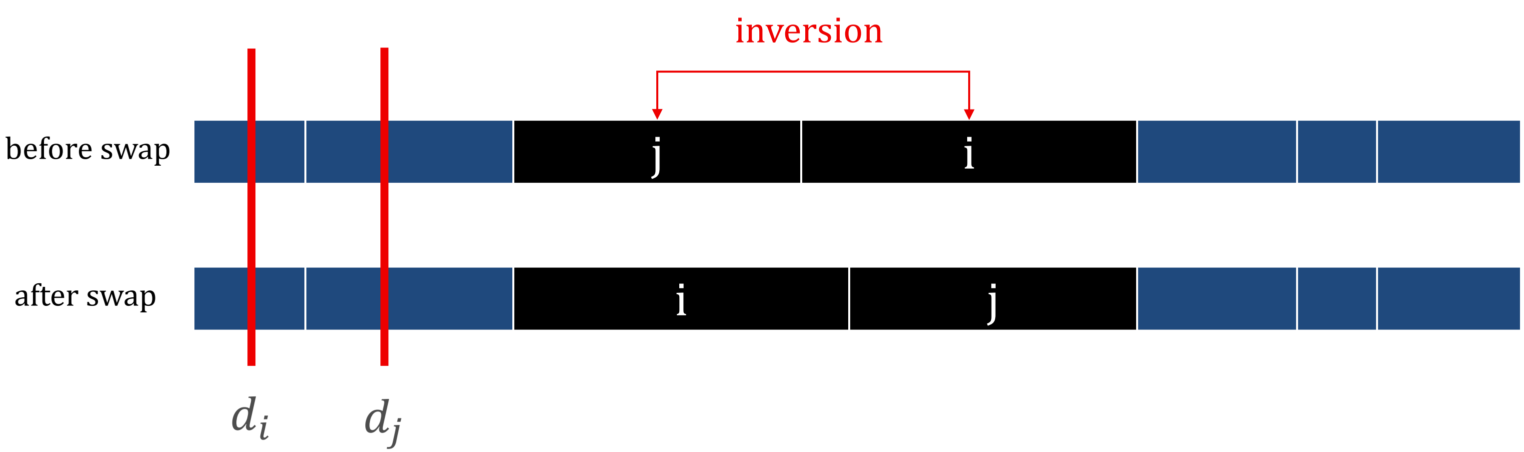

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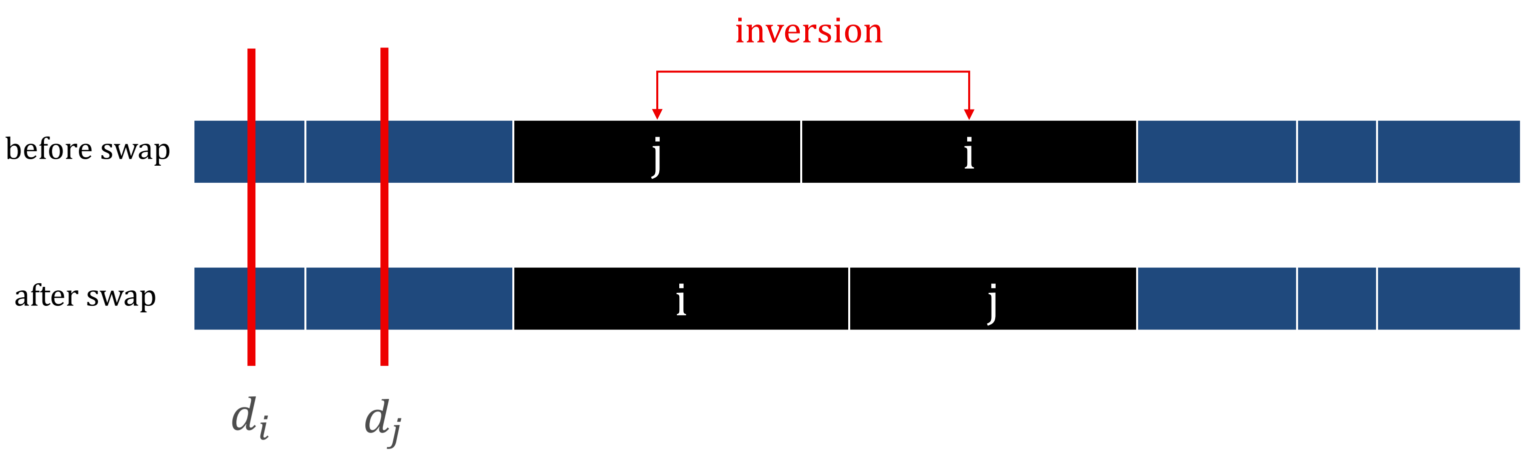

SCHEDULING TO MINIMIZE LATENESS

- Theorem: Greedy is optimal
- Proof:
- Let S^* be an optimal schedule (strictly) better than greedy, with the fewest number of inversions
- Wlog S^* has no idle times
- If S^* has no inversions, then $S^* = S$
- If S^* has an inversion, let (i, j) be an adjacent inversion: swap it!
  - Maximum latency didn't decrease (otherwise S^* is not optimal)
  - But, number of inversions decreased! Contradiction

SUMMARY

- Interval scheduling
- Interval Partioning
- Scheduling to minimize latency

- Meta lesson: similar (same?) analysis
    - Gradually transform the solution of any other algorithm to one of greedy, without changing the performance