CS 580 Algorithm Design, Analysis, And Implementation
Fall 2021 Vassilis Zikas HW 1

Due Wednesday Sept. 15 at 11:59 p.m.

1. **Asymptotic analysis.**

   (a) Rank the following functions (representing running times) from smallest to largest (in terms of growth with respect to n).
   $$n!, \log n, \ln n, 3^n, n^n, n^4, 2n^2+n, \log(4n^3)$$
   Group together functions in the same asymptotic class. You do not need to show any work/ explain your answers.

   (b) Suppose that $f(n)$ and $g(n)$ are positive function always greater than 1 for $n \geq 2^{200}$. Let $h(n) = f(n) + g(n)$. Prove or disprove: $h(n) \in \Theta(\max\{f(n), g(n)\})$.

   (c) The n^{th} Harmonic number is the sum of the reciprocal of the first n natural numbers i.e.
   $$H_n = \sum_{i=1}^{n} \frac{1}{i}$$
   Show that H_n is $\Theta(\log n)$.

2. **Graph Traversal.** Unlike countries nowadays which have complicated transportation systems, there is a country which is formed in a tree shape by railways. This means that in this country there are n cities and $n-1$ railways between cities. Now the president of this country would like to know the longest railway path in this country.

   (a) Design and analyze an algorithm to find the longest path in $O(n^2)$.

   (b) Prove the following statement: suppose that the longest railway path in this country starts from city a to city b, then for any given city x, the longest railway path starting from x is either from x to a or from x to b.

   (c) Design and analyze an algorithm to find the longest path in $O(n)$.

3. **Proof Techniques.**

   (a) Lil Omega shared a conjecture with Lil Delta. Lil Delta claimed to prove the conjecture and emailed the proof to Lil Omega. Lil Omega is not good at proofs and is asking for help from the experts. The content of the email from Lil Delta is given below. Should Lil Omega accept the proof? Provide reasoning for your answer.

   Theorem. *All students have the same eye color.*

   Proof. We prove the statement by induction.
   Since in the statement of the theorem there is no variable to induct on, we restate the theorem as follows.
   $P(n)$: In any set of n students, all students have the same eye color.

Base case: $P(1)$ is true. In any set of one student, the student has the same eye color.

Inductive step: We assume $P(n)$ is true to prove $P(n+1)$.

Let the $n+1$ students be $S_1, S_2, ..., S_n, S_{n+1}$.

By our inductive hypothesis, any set of n students have the same eye color.

Therefore, the first n students $S_1, S_2, ..., S_n$ have the same eye color.

Similarly, the last n students $S_2, ..., S_n, S_{n+1}$ have the same eye color.

From preceding arguments, the eye color of S_1 is same as the eye color of $S_2, ..., S_n$, and the eye color of S_{n+1} is same as the eye color of $S_2, ..., S_n$.

It follows that eye colors of S_1, S_{n+1}, and $S_2, ..., S_n$ are same, i.e., $P(n+1)$ is true. $\square$

(b) A full m-ary tree T ($m \geq 2$) is a rooted tree in which each node has either 0 or m children.

Let $d_T(x)$ denote the depth of a node x in T, which is the number of edges contained in the path from root of T to x.

Let $L(T)$ denote the sum of the depths of leaf nodes in T, i.e., $L(T) = \sum_{x \in Leaves(T)} d_T(x)$, and $I(T)$ denote the sum of the depths of internal nodes in T, i.e., $I(T) = \sum_{x \in NonLeaves(T)} d_T(x)$.

Using induction, prove that for a full m-ary tree T with n nodes, $L(T) = (m-1)I(T) + n - 1$.

4. **2 points** Have you assigned pages on Gradescope?