CS 580 Algorithm Design, Analysis, And Implementation
Fall 2021 Vassilis Zikas HW 1

Problem 1

Collaborators: List students that you have discussed problem 1 with.

(a) (i) $\log n, \ln n, \log 4n^3$

(ii) $2n^2 + n$

(iii) n^4

(iv) $n!, 3^n, n^n$

Ranking of the function w.r.t n:
(i) < (ii) < (iii) < (iv)

(b) f(n) and g(n) are always greater than 1. (given)
For the upper bound:
$f(n) \leq \max(f(n), g(n))$ and g(n) $\leq \max(f(n), g(n))$
$\implies f(n) + g(n) \leq 2\max(f(n), g(n))$
$\implies f(n) + g(n) = O(\max(f(n), g(n)))$

For the lower bound:
For every n, we can say that either f(n) would be greater than g(n) or vice versa.
$\implies f(n) + g(n) \geq \max(f(n), g(n))$
$\implies f(n) + g(n) = \Omega(\max(f(n), g(n)))$

From these two, we can say that:
$\implies f(n) + g(n) = \Theta(\max(f(n), g(n)))$

(c)

$$H_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n}$$

For calculating the upper bound, assume k = $\lceil \log(n+1) \rceil$

$$\begin{aligned}
\implies H_n &\leq 1 + \frac{1}{2} + \ldots + \frac{1}{2^k - 1} \\
&\leq 1 + (\frac{1}{2} + \frac{1}{3}) + (\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}) + \ldots + (\frac{1}{2^{k-1}} + \ldots + \frac{1}{2^k - 1}) \\
&\leq 1 + (\frac{1}{2} + \frac{1}{2}) + (\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}) + \ldots + (\frac{1}{2^{k-1}} + \ldots + \frac{1}{2^{k-1}}) \\
&\leq 1 + 1 + 1 + 1 + 1 + \ldots. + 1 = k \\
\implies H_n &\leq \lceil \log(n+1) \rceil \sim \log n \qquad (1)
\end{aligned}$$

For calculating the lower bound, assume $k = \lfloor \log n \rfloor$

$$\implies H_n \geq 1 + \frac{1}{2} + ... + \frac{1}{2^k}$$
$$\geq 1 + \frac{1}{2} + (\frac{1}{3} + \frac{1}{4}) + (\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}) + ... + (\frac{1}{2^{k-1}+1} + ... + \frac{1}{2^k})$$
$$\geq 1 + \frac{1}{2} + (\frac{1}{4} + \frac{1}{4}) + (\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}) + ... + (\frac{1}{2^k} + ... + \frac{1}{2^k})$$
$$\geq 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + ... + \frac{1}{2} = 1 + + \frac{k}{2}$$
$$\implies H_n \geq 1 + \frac{1}{2}\lfloor \log n \rfloor \sim \frac{1}{2}\log n \tag{2}$$

From (1) and (2), we can conclude that $H_n = \Theta(\log n)$

Problem 2

Collaborators: List students that you have discussed problem 2 with.

Type solution here.

Problem 3

Collaborators: List students that you have discussed problem 3 with.

(a) No, Lil Omega should not accept the proof provided by Lil Delta.
The given proof makes incorrect use of mathematical induction to prove the statement that all students have same eye colour.
The flaw in the proof is that it assumes that the two subsets of n students out of n+1 students would share a common element in between them always. However, this is not true if we take n+1 = 2.
In that case, we can have one student with black eye colour and other with brown eye colour. So, both of them would satisfy the base condition of P(1) but they won't satisfy P(2) as now there would be no common element in between them.
Hence, the proof given here is not acceptable.

(b) P(n) : In a full m-ary tree with n nodes, L(T) = (m-1)I(T) + n-1
Now, let us prove this by mathematical induction:
P(1) : only root node is there in the tree, n=1
L(T) = 0 as well as I(T) = 0
So, this is true for P(1).

Now, let's assume that the given equation is true for P(n), i.e
L(T) = (m-1)I(T) + n-1

Now, we will prove for P(n+m) because in a full m-ary tree, each node has either m children or 0 children.
Let say, we added m children to a leaf node which was at depth d in the tree.

$$I^{'}(T) = I(T) + d$$

that leaf node now becomes internal node

$$L^{'}(T) = L(T) - d + m(d+1)$$

that leaf node now becomes internal node and m new leaf nodes are added at depth d+1

$$(m-1)I(T) + n - 1 - d$$

$$m * I(T) - I(T) + n - 1$$

$$m * (I(T) + d) - (I(T) +$$

$$(m-1) * I^{'}(T) + (n + m$$

$\implies$ P(n+m) is also true

Hence, proved.