CS 580
ALGORITHM DESIGN AND ANALYSIS

Randomized Algorithms 0:
Review

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BASIC PROBABILITY

- We have a random experiment, which defines an outcome space Ω
  - E.g., flip 2 fair coins
  - $\Omega = \{(H,H),(H,T),(T,H),(T,T)\}$
- Each outcome ω occurs with some probability $\Pr[\omega] \in [0,1]$
  - E.g. $\Pr[(H,H)] = 1/4$
- $\sum_{\omega \in \Omega} \Pr[\omega] = 1$
- An event $A \subseteq \Omega$ is a set outcomes
- $\Pr[A] = \sum_{\omega \in A} \Pr[\omega]$
  - E.g. A could be the event that the first coin is H
  - $\Pr[A] = 1/2$

BASIC PROBABILITY

- A random variable X is a function from outcomes to real numbers
  - It's not a variable
  - It's also not random
  - It's just a function
- E.g. X could be the number of H after flipping a fair coin twice
  - $\Pr[X]$?
  - No such thing
  - $\Pr[X = 1]$?
  - ½
- Expectation $E[X] = \sum_{\omega \in \Omega} \Pr[\omega] \cdot X(\omega)$
  - Equivalently $E[X] = \sum_{v \in \mathbb{R}} v \cdot \Pr[X = v]$

BASIC PROBABILITY

- Linearity of expectation
- Given n random variables $X_1, \ldots, X_n$ defined in the same probability space we have

$$E\left[\sum_i X_i\right] = \sum_i E[X_i]$$

- Note that the random variables don't have to be independent!

BASIC PROBABILITY

- Markov's inequality
  - "The probability of being above average is at most half"
- Formally, if X is a non-negative random variable and $\lambda > 0$, then $\Pr[X \geq \lambda] \leq \frac{E[X]}{\lambda}$

BASIC PROBABILITY

- Union Bound
  - “Probability that one of n things happened is at most the sum of probabilities of each thing”
- Let $A_1, \dots, A_n$ be events. Then

$$\Pr\left[\cup_{i\in[n]} A_i\right] \le \sum_{i\in[n]} \Pr[A_i]$$

BASIC PROBABILITY: CHERNOFF BOUNDS

- Chernoff Bounds
  - “Sums of independent things never go far from their expectation”
- Theorem: Let $X = \sum_{i \in [n]} X_i$ be the sum of mutually n independent random variables, such that $X_i = 1$ with probability p_i and $X_i = 0$ otherwise (w.p. $1 - p_i$). Let $\mu = E[X] = \sum_{i \in [n]} p_i$. Then, for $0 < \delta < 1$:

$$\Pr[X \geq (1+\delta)\mu] \leq \left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^{\mu}$$

BASIC PROBABILITY: CHERNOFF BOUNDS

- $\Pr[X \geq (1+\delta)\mu] \leq \left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^{\mu}$
- Pretty scary looking…
- Intuition:
  - Markov says $\Pr[X \geq a] \leq E[X]/a$
  - Apply Markov to $Y = e^{tX}$
  - $e^{\sum stuff} = \prod e^{stuff}$
  - Product of numbers smaller than 1 becomes small really really fast!

BASIC PROBABILITY: CHERNOFF BOUNDS

- Technical part: what is $E[e^{tX}]$?
  - $E[e^{tX_i}] = p_i e^t + (1 - p_i)e^0 \leq e^{p_i(e^t - 1)}$
  - Used that $1 + y \leq e^y$
- $E[e^{tX}] = E\left[e^{t\sum_i X_i}\right] = \prod_i E[e^{tX_i}]$
- $\leq \prod_i e^{p_i(e^t-1)} = e^{\sum_i p_i(e^t-1)} = e^{\mu(e^t-1)}$

BASIC PROBABILITY: CHERNOFF BOUNDS

- $\Pr[X \geq (1+\delta)\mu] = \Pr[e^{tX} \geq e^{t(1+\delta)\mu}]$
- $\leq E[e^{tX}]/e^{t(1+\delta)\mu}$
- $\leq \frac{e^{\mu(e^t-1)}}{e^{t(1+\delta)\mu}} = \left(\frac{e^{e^t-1}}{e^{t(1+\delta)}}\right)^{\mu}$
- Pick $t = \ln(1+\delta)$:
- $\Pr[X \geq (1+\delta)\mu] \leq \left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^{\mu}$

BASIC PROBABILITY: CHERNOFF BOUNDS

- Application: Flip a coin (H w.p. p) n times. Let X be the number of H.
- $X = \sum_{i\in[n]} X_i$, where X_i is the indicator for the event that the i-th coin was H
- For $n = 1000$ and $p = 1/2$, $E[X] = 500$
- $\Pr[X \geq 600]$?
  - Markov says at most $5/6$
  - Chernoff says at most $0.000083 \ldots$
  - $\Pr[X \geq (1+\delta)500] \leq \left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^{500}$ and plug in $\delta = 0.2$

BASIC PROBABILITY: CHERNOFF BOUNDS

- Many flavors of Chernoff:
  - $\Pr[X \geq (1+\delta)\mu] \leq e^{-\frac{\mu\delta^2}{3}}$
  - $\Pr[X \leq (1-\delta)\mu] \leq e^{-\frac{\mu\delta^2}{2}}$
  - Other bounds depend on the variance…