CS 580
ALGORITHM DESIGN AND ANALYSIS

NP and NP-completeness:
Part 2

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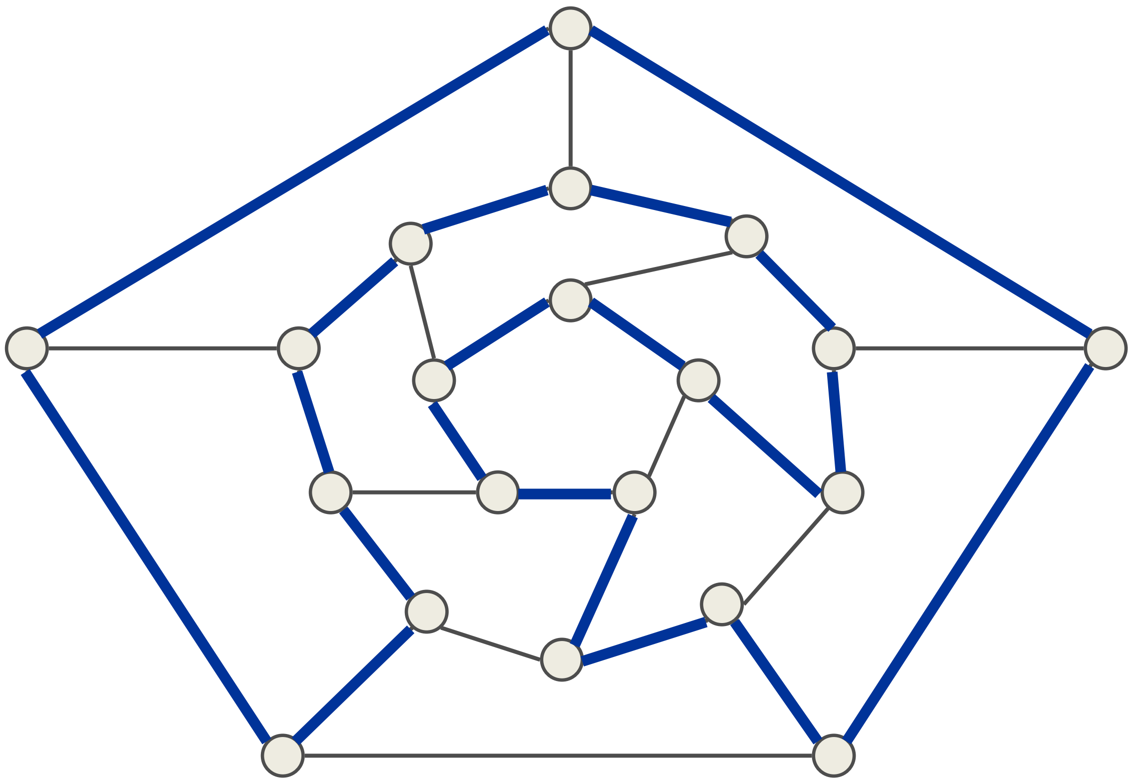
SO FAR

- So far:
  - Definition on NP and co-NP
  - NP-completeness
  - CIRCUIT-SAT and 3-SAT (and more) are NP-complete
- Today:
  - More NP-completeness proofs

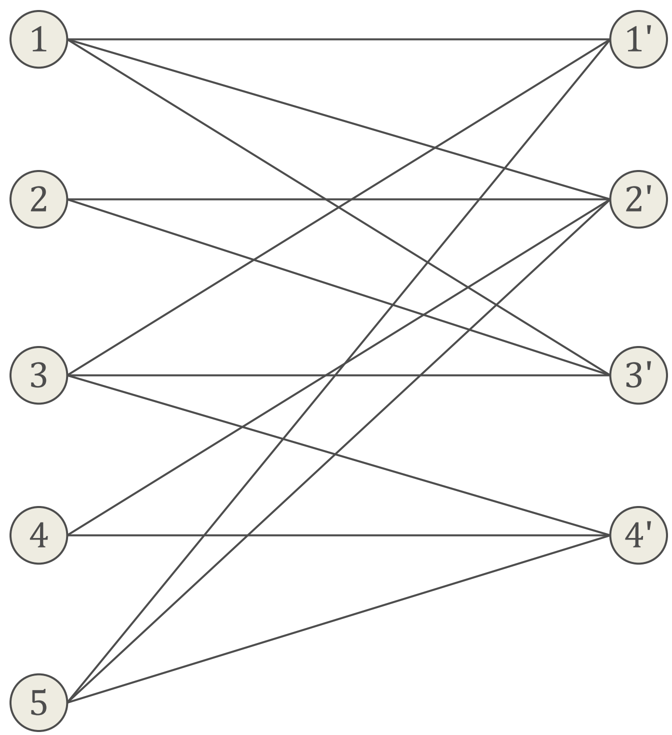
SEQUENCING PROBLEMS (8.5 IN KT)

HAMILTONIAN CYCLE

- HAM-CYCLE: Given an undirected graph $G = (V, E)$ does it contain a simple cycle C that has every node in G?



HAMILTONIAN CYCLE



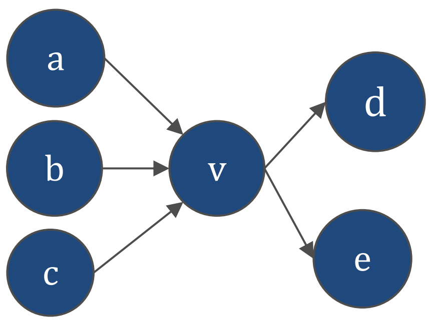
NO: bipartite graph with odd number of nodes.

DIRECTED HAMILTONIAN CYCLE

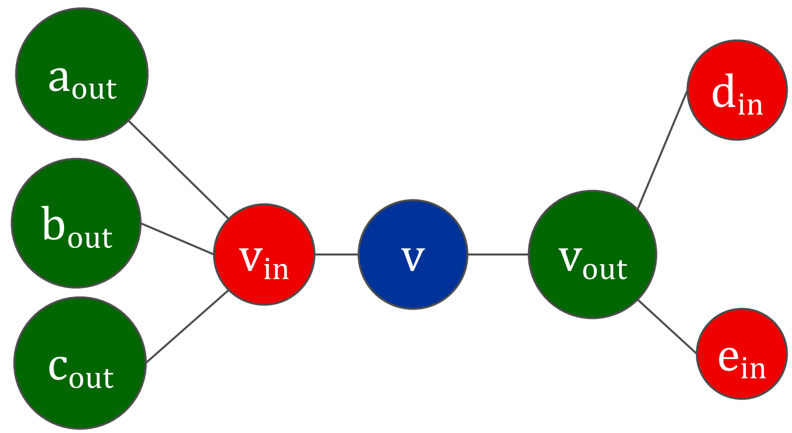
- DIR-HAM-CYCLE: Given a directed graph $G = (V, E)$, does there exist a Hamiltonian cycle (a simple directed cycle that contains all nodes)?
- Claim: DIR-HAM-CYCLE $\leq_P$ HAM-CYCLE
- Proof: Given a directed graph G we construct an undirected graph G' by replacing every node u with 3 nodes: u_{in}, u and u_{out}
  - Add edges (u_{in}, u) and (u, u_{out}) in G'
  - For every directed (u, v) edge in G we have an (undirected) edge (u_{out}, v_{in})

DIRECTED HAMILTONIAN CYCLE

G

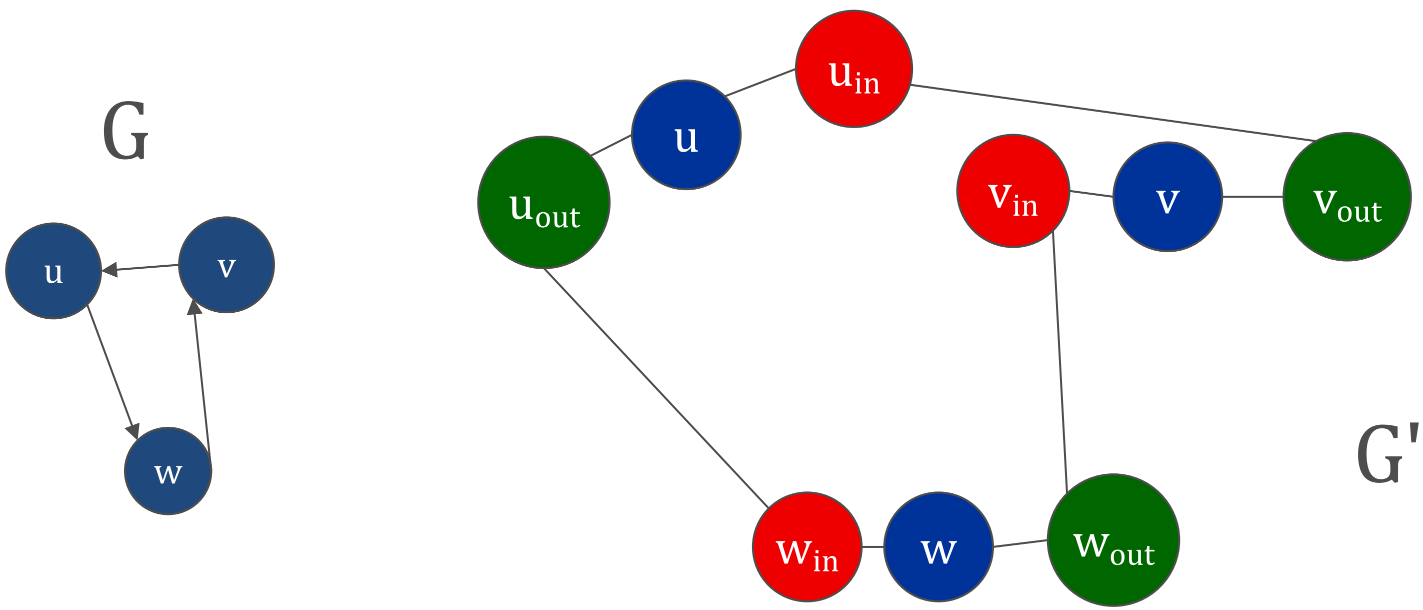


G'



DIRECTED HAMILTONIAN CYCLE

- Claim: G' has a Hamiltonian cycle if and only if G has one.
- Pf. $\Rightarrow$
  - Suppose G has a directed Hamiltonian cycle Γ (e.g., (u, w, v)).
  - Then G' has an undirected Hamiltonian cycle (same order): for each node z in the directed cycle replace z with (z_{in}, z, z_{out})



DIRECTED HAMILTONIAN CYCLE

- Claim: G' has a Hamiltonian cycle if and only if G has one.
- Pf. $\Leftarrow$
  - Suppose G' has an undirected Hamiltonian cycle Γ'.
  - Γ' must visit nodes in G' using one of following two orders:

    $\dots, z_{in}^{1}, z^{1}, z_{out}^{1}, z_{in}^{2}, z^{2}, z_{out}^{2}, \dots$

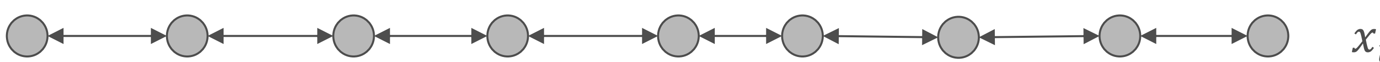
    $\dots, z_{out}^{2}, z^{2}, z_{in}^{2}, z_{out}^{1}, z^{1}, z_{in}^{1}, \dots$
  - The first corresponds to a directed Hamiltonian cycle in G. If the second is a Hamiltonian cycle, then its inverse is also an undirected Hamiltonian cycle (in G'), so again we're done.
- Therefore, DIR-HAM-CYCLE $\leq_P$ HAM-CYCLE

3-SAT REDUCES TO DIRECTED HAMILTONIAN CYCLE

- Claim: 3-SAT $\leq_P$ DIR-HAM-CYCLE
- Proof:
  - Given an instance ϕ of 3-SAT, we will construct an instance of DIR-HAM-CYCLE that has a Hamiltonian cycle if and only if ϕ is satisfiable.
  - Construction:
    - Create a graph G that has 2^n Hamiltonian cycles corresponding to the 2^n possible truth assignments of n variables

3-SAT REDUCES TO DIRECTED HAMILTONIAN CYCLE

- Given a 3-SAT instance ϕ with n variables x_i, and k clauses
  - Construction:
    - A (bidirectional) line for each variable x_i with $3k + 3$ nodes
    - Traversing from left to right = set $x_i = 1$
    - Traversing from right to left = set $x_i = 0$



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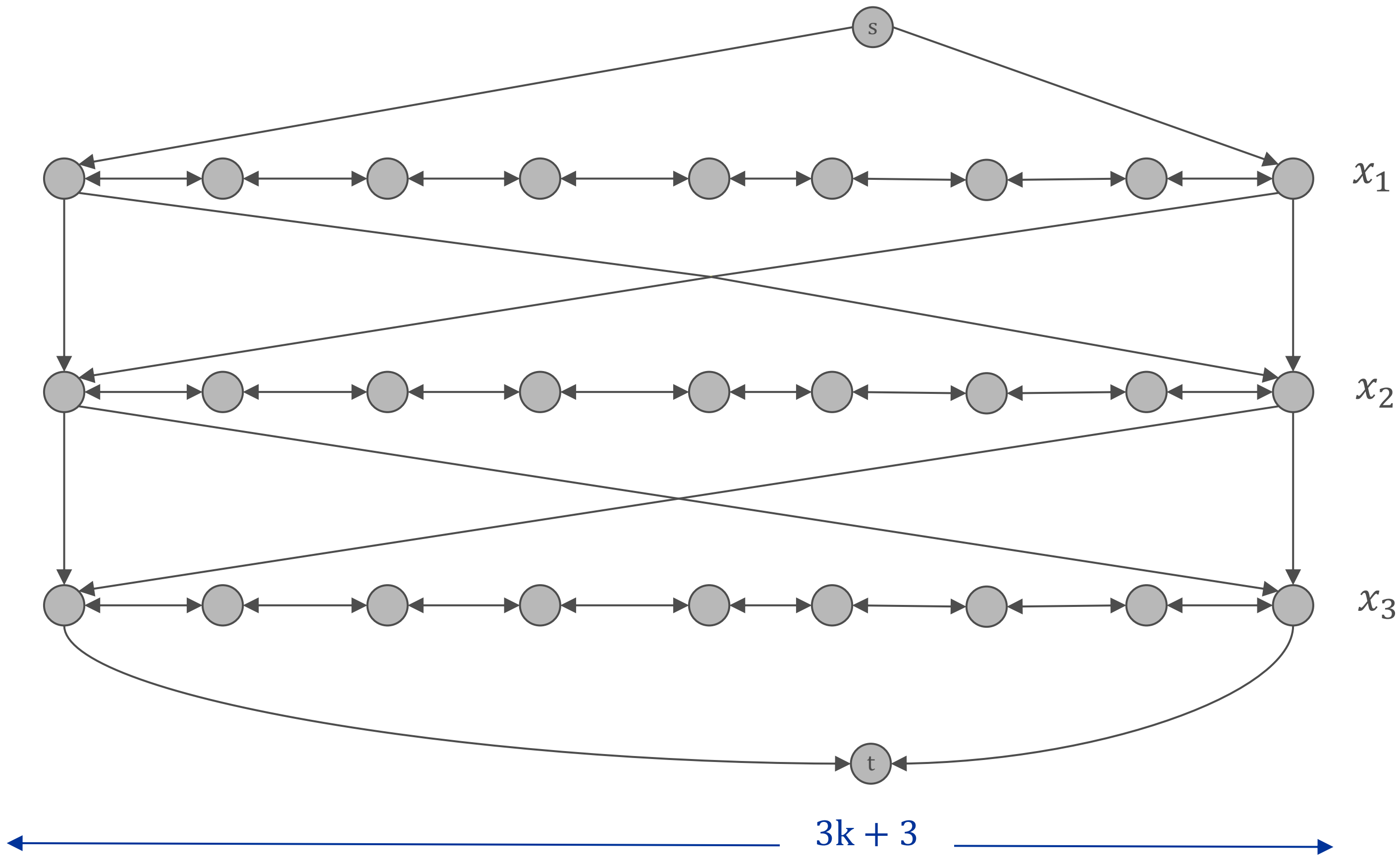

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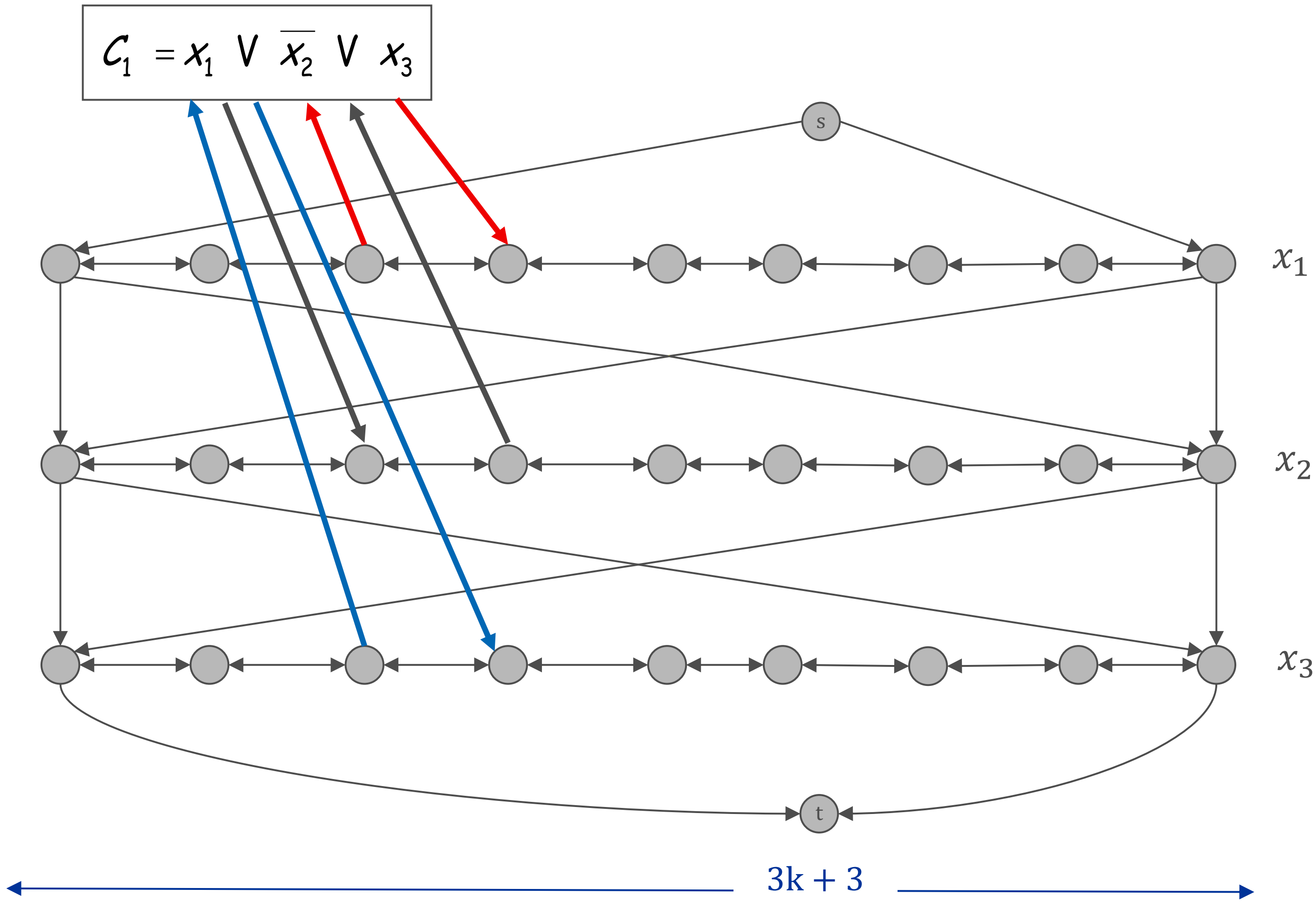

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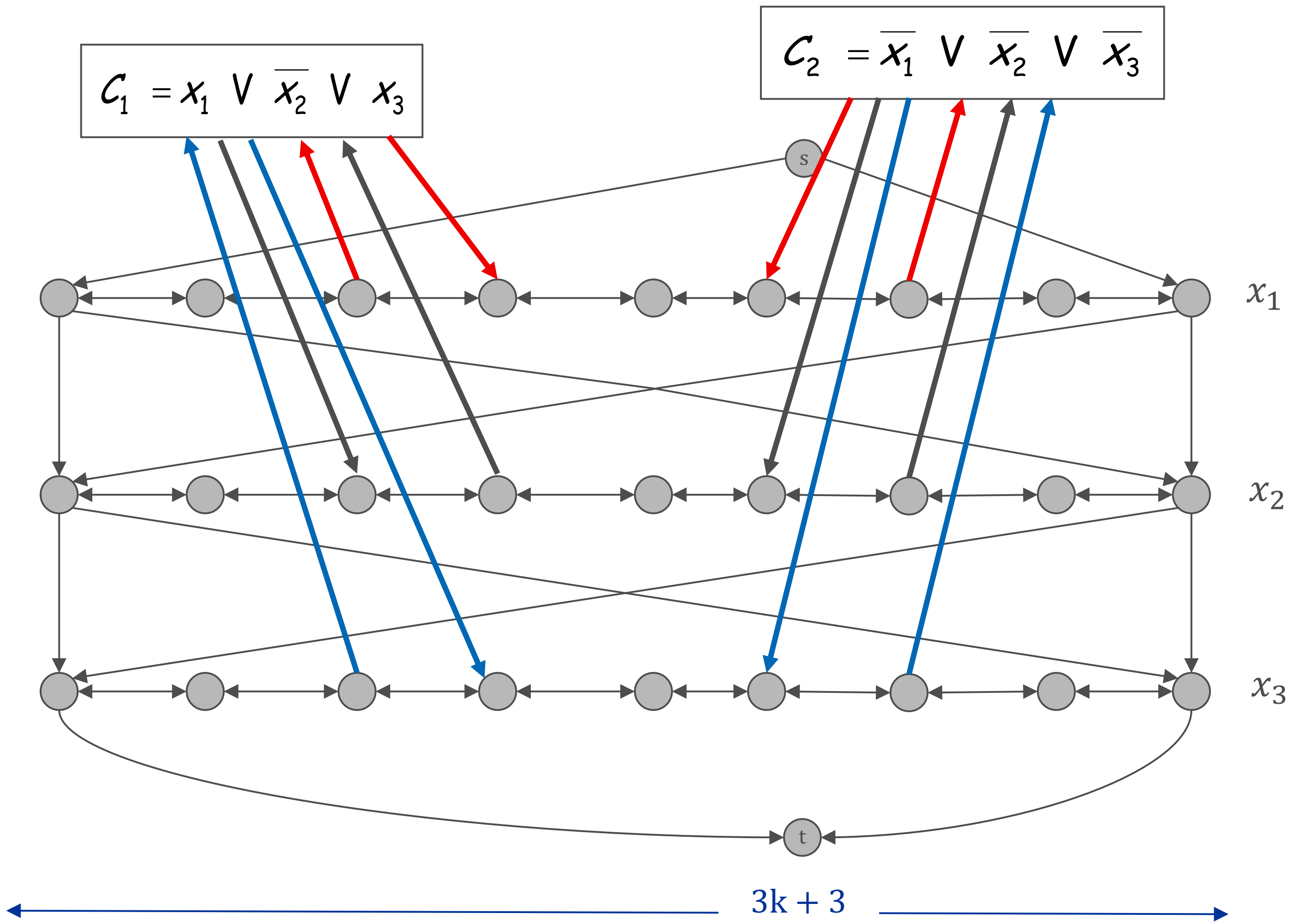

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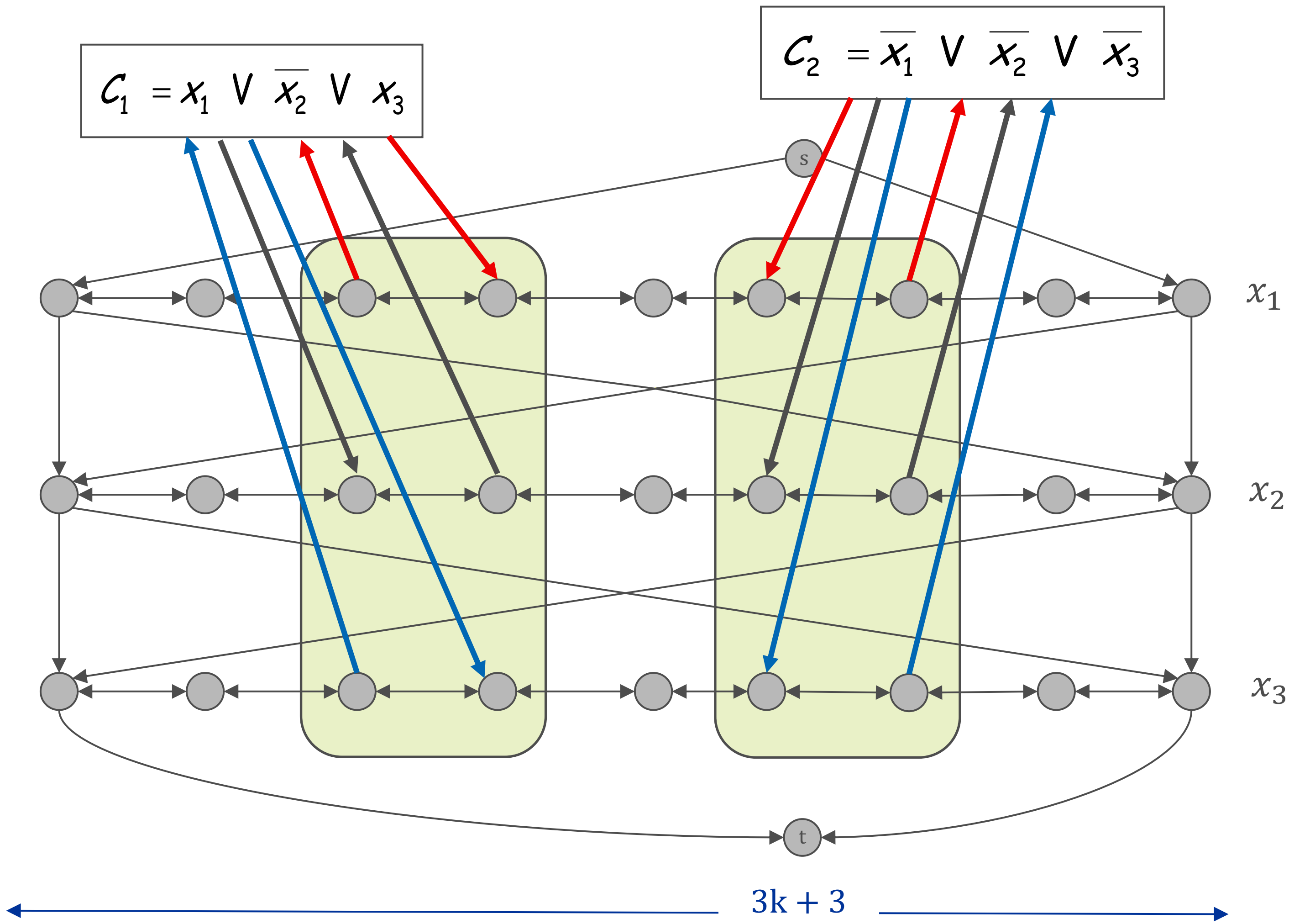


3-SAT REDUCES TO DIRECTED HAMILTONIAN CYCLE

- Claim: ϕ has a satisfying iff G has a Hamiltonian cycle
- Proof ($\Rightarrow$)
  - Suppose 3-SAT has a satisfying assignment x^*
  - Define the Hamiltonian cycle as follows:
    - If $x_i^* = 1$ traverse the i-th row from left to right, and otherwise traverse from right to left
    - Use the left/right most nodes to go from row to row
    - For each clause/node C_i there will be at least one row in which we can "go in and out" in the correct way, thus we include the node in the tour

3-SAT REDUCES TO DIRECTED HAMILTONIAN CYCLE

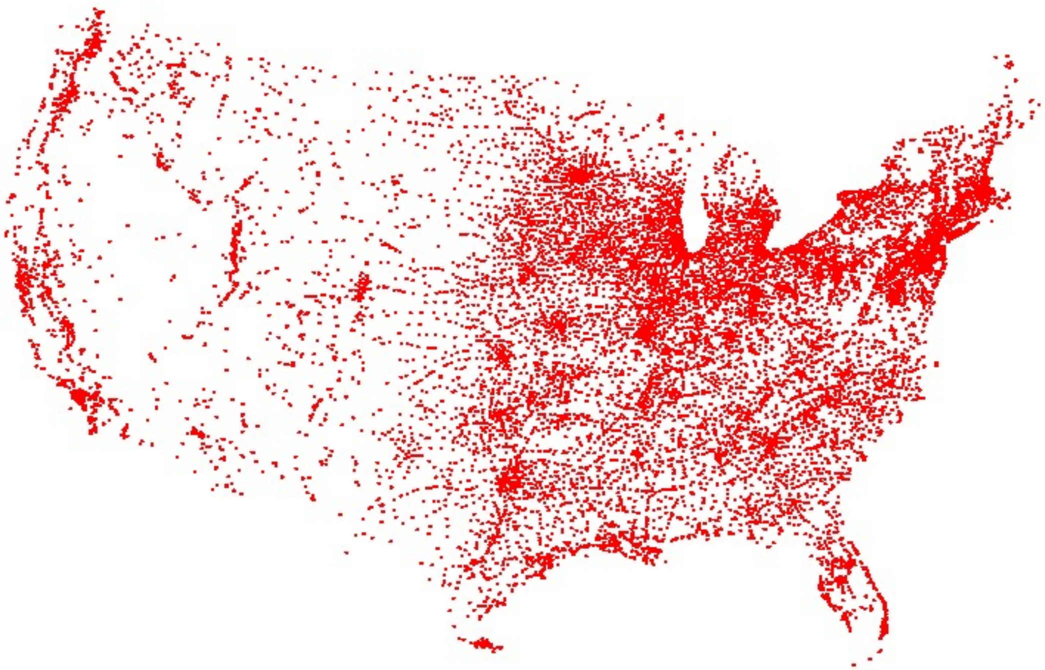
- Claim: ϕ has a satisfying iff G has a Hamiltonian cycle
- Proof ($\Leftarrow$)
  - Suppose G has a Hamiltonian cycle Γ
  - When Γ enters node/clause C_i using edge (x_j^k, C_i) it must leave using the mate edge
    - Otherwise there is no way to visit both nodes next to x_j^k
  - Thus, every clause C_i the nodes immediately before and after in the tour have an edge e_i between them
  - Remove all C_is and use e_i to get a tour Γ' in the new graph
  - In every Hamiltonian cycle of this graph, every row is either left to right or right to left
  - Set $x_j = 1$ if Γ' is left to right, and $x_j = 0$ otherwise
  - This is a valid assignment
  - Since Γ visited every node (including the C_i nodes) at least one path was in the "correct" direction relative to C_i, and thus the clause is satisfied.

LONGEST PATH

- SHORTEST-PATH: Given a directed graph G and two vertices s, t does there exist a simple path from s to t using at most k edges?
- LONGEST-PATH: Given a directed graph G and two vertices s, t does there exist a simple path from s to t using at least k edges?
- Claim: 3-SAT $\leq_P$ LONGEST-PATH
- Proof 1: Re-do the reduction to DIR-HAM-CYCLE, ignoring edge from t to s, for $k = \#vertices - 1$
- Proof 2: Show HAM-CYCLE $\leq_P$ LONGEST-PATH

TRAVELING SALESPERSON PROBLEM

- TSP: Given a set of n cities and a pairwise distance function $d(u, v)$ is there a tour of size $\leq D$?



All 13,509 cities in US with a population of at least 500
Reference: http://www.tsp.gatech.edu

TRAVELING SALESPERSON PROBLEM

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TRAVELING SALESPERSON PROBLEM

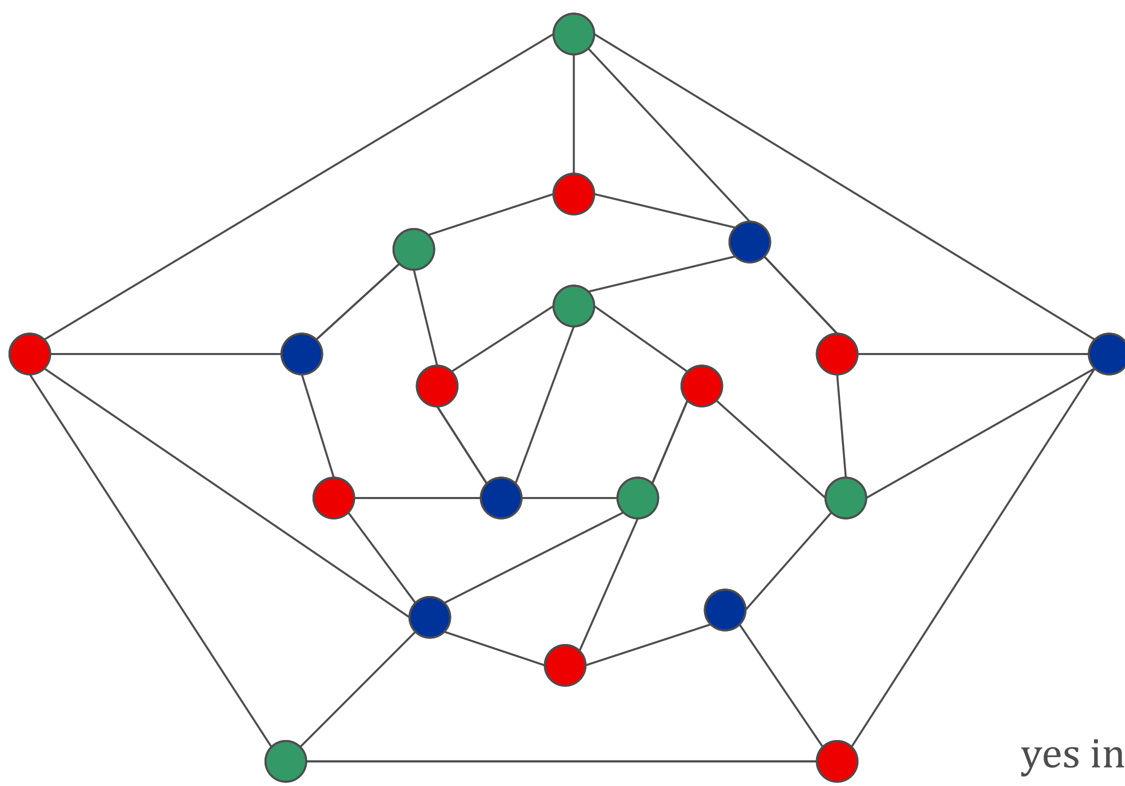
- Claim: HAM-CYCLE $\leq_P$ TSP
- Proof:
  - Given an instance G of HAM-CYCLE create an instance G' of TSP by adding the following distance function

$$d(u, v) = \begin{cases} 1 & \text{if } (u, v) \in E \\ 2 & \text{if } (u, v) \notin E \end{cases}$$

  - TSP has a tour of size $\leq n$, i.e. can use distance 1 edges, iff G is Hamiltonian
- Note: TSP instance above satisfies triangle inequality!

GRAPH COLORING

- 3-COLOR: Given an undirected graph G, does there exist a way to color the vertices red, green and blue so that adjacent nodes have different colors?



yes instance

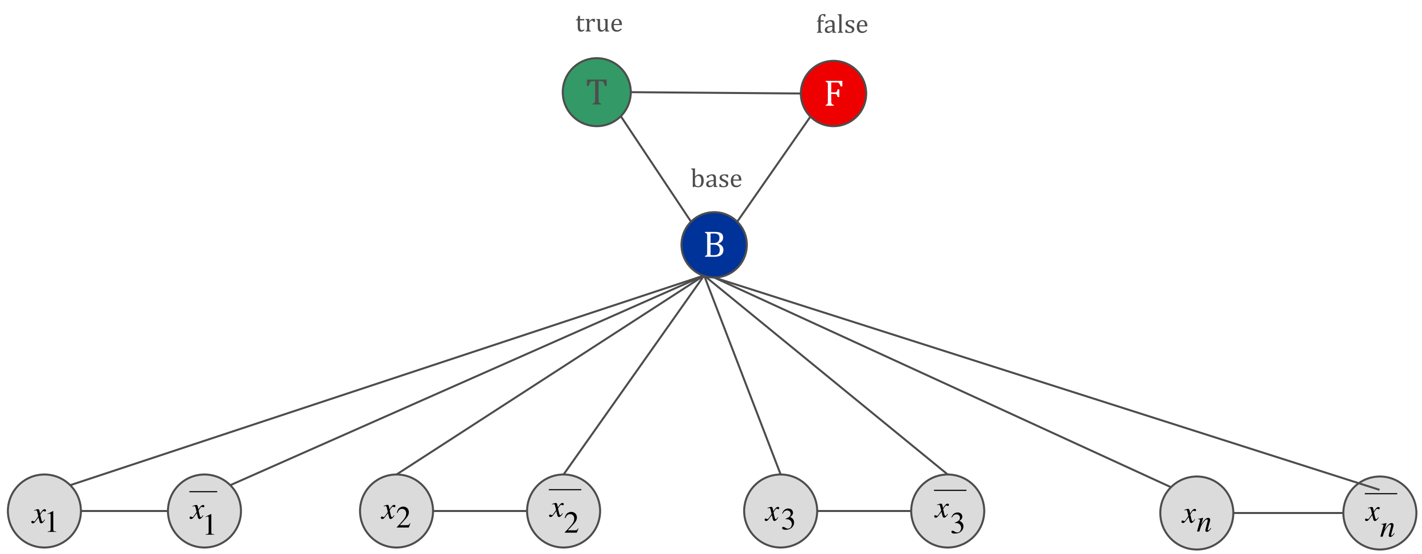
3-COLOR

- Claim: 3-SAT $\leq_P$ 3-COLOR
- Proof:
  - Given a 3-SAT instance ϕ we construct an instance of 3-COLOR (a graph G) that is 3 colorable iff ϕ is satisfiable
  - Construction:
    1. For each literal create a node
    2. Create 3 new nodes T, F, B. Connect them in a triangle. Connect every literal to B
    3. Connect each literal to its negation
    4. For each clause add a gadget (TODO) of 6 nodes and 13 edges

3-COLORABILITY

- Claim: Graph is 3-colorable iff ϕ is satisfiable.

- Pf. $\Rightarrow$ Suppose graph is 3-colorable.
  - T, F and B get different colors
  - (2) ensures each literal is (the color of) T or F.
  - (3) ensures a literal and its negation are opposites.



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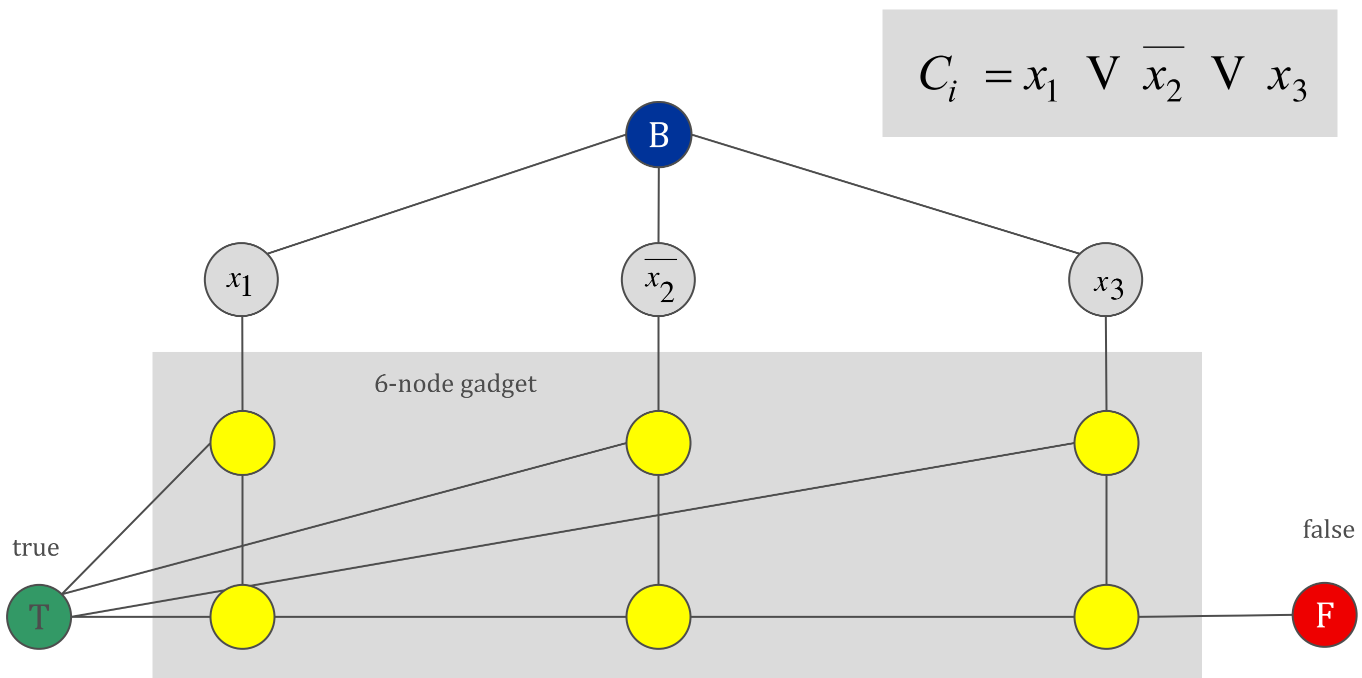
  - (4) ensures at least one literal in each clause is T.



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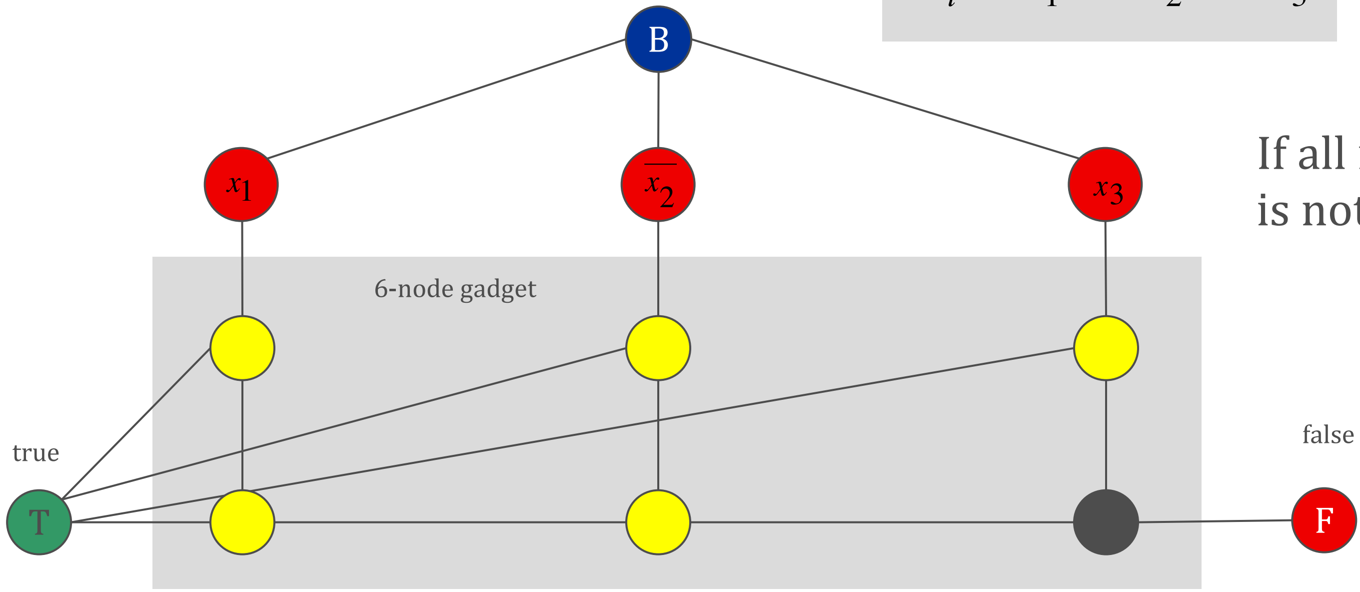
$$C_i = x_1 \vee \overline{x_2} \vee x_3$$



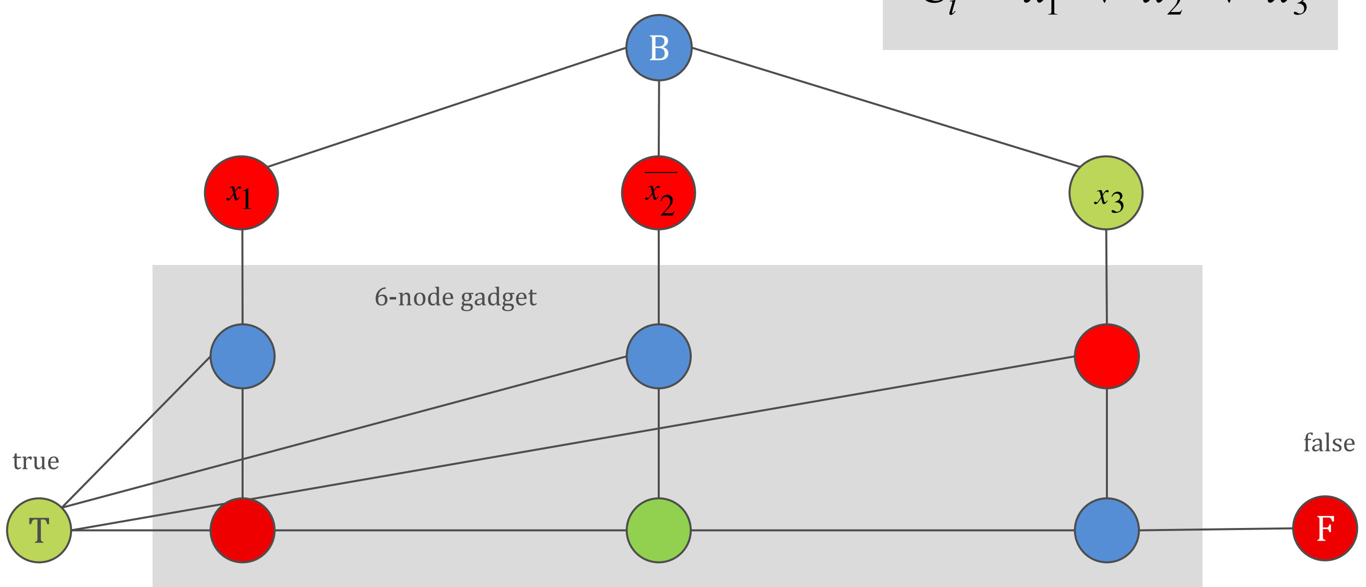
If all red, then the graph is not 3-colorable

3-COLORABILITY

- Claim. Graph is 3-colorable iff ϕ is satisfiable.

- Pf. $\Leftarrow$ Suppose 3-SAT formula satisfiable
  - Color all true literals T
  - Color node below green node F, and node below that B.
  - Color remaining middle row nodes B.
  - Color remaining bottom nodes T or F as forced.

$$C_i = x_1 \ \text{V} \ \overline{x_2} \ \text{V} \ x_3$$



COLORABILITY

- Planar graphs?
  - Recall definition of planar: can be “drawn” on the plane in a way that the edges don’t intersect
- PLANAR 3-COLOR: Still NP-complete!
- PLANAR 4-COLOR: $O(1)$!
  - Appel and Haken [1976]

NUMERICAL PROBLEMS

- SUBSET-SUM: Given natural numbers $w_1, \ldots, w_n$ and a target number W, is there a subset of $\{w_1, \ldots, w_n\}$ that adds up to W?
- Claim: SUBSET-SUM is NP-complete.
- PARTITION: Given natural numbers $w_1, \ldots, w_n$, can they be partitioned into two subsets that add up to the same value?
- Claim: SUBSET-SUM $\leq_P$ PARTITION
  - Intuition: pad the instance of SUBSET-SUM with two new numbers: $W + \sum_i w_i$ and $2\sum_i w_i - W$
- Claim: SUBSET-SUM is a special case of KNAPSACK

SUMMARY

- More reductions!
- Sequencing problems (8.5 in KT)
- Graph coloring (8.7 in KT)
- Numerical problems (8.8 in KT)