Monte Carlo Simulation

Objective:

Approximate value of pi using monte carlo simulation.

In statistics, we often run into integrals of the form :-

$$I = \int_a^b h(x)g(x)dx$$

These are hard to compute using traditional methods. In such cases, we can use Monte Carlo Simulations to obtain an estimate for the value of the integral by transforming the integration problem into a procedure of sampling values from a probability distribution and calculating a statistic over those samples.

In this case, if the function g(x) is positivea on the interval (a, b) and the integral of the function is finite with $C = \int_b^a g(x)$

then, we can define a new probability distribution on the interval (a, b):

$$p(x) = \frac{g(x)}{C}$$

We can now rewrite the original integration as

$$I = C\int_a^b h(x)p(x)dx = CE_{p(x)}[h(x)]$$

The integral is same as a constant times the expected value of the integrand function. The expected value is approximated by sampling N points and computing the sample mean

$$E_{p(x)}[h(x)] \approx \frac{1}{N}\sum_i^N h(x_i)$$

Part A: Circle Approximation

As we know, area of a circle can be written as

$$I = \int_{-r}^{r} \int_{-r}^{r} \mathbb{1}(x^2 + y^2 \leq r^2) dx dy$$

where $\mathbb{1}$ represents the Indicator Random Variable, which takes the value 1 when the condition evaluates to true and 0 when it evaluates to false.
If we obtain this integral, we can calculate pi by using pi=I/r^2

$$I = (2r)(2r) \int \int \mathbb{1}(x^2 + y^2 \leq r^2) p(x) p(y) dx dy$$

$$= 4r^2 \int \int \mathbb{1}(x^2 + y^2 \leq r^2) p(x) p(y) dx dy$$

$$I \approx 4r^2 \frac{1}{S} \sum_{s=1}^{S} \mathbb{1}(x_s^2 + y_s^2 \leq r^2)$$

where S is the number of points sampled.

Start off with 10 sample points. For each successive run, increase the number of samples to 10 times to go on till 10^7 points.

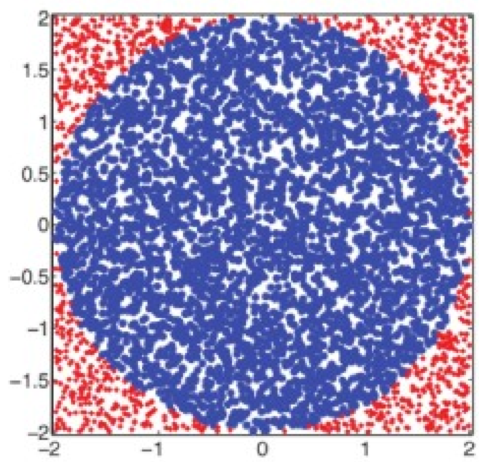

Fig 2.19 from Kevin Murphy - Machine Learning, A Probabilistic Perspective

For each run, a plot is drawn like the one given above.

Part B: Sphere Approximation

Except from the fact that points sampled out are in 3-Dimension, the approach remains same as circle case.