CS201: Math for CS I/Discreet Mathematics
Assignment 3 Solutions

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I Problem 1 Solution

1.1 Part (a)

We are given a **simple** graph $G = (10, 28)$.

Claim 1.1.1. *G has a cycle of length 4.*

Proof. Since the number of vertices and edges in the graph are 10 and 28 respectively, the average degree $d(G) = \frac{2|E|}{|V|}$ of the graph is 5.6 and thus, we can observe that there has to be atleast 2 vertices with degree sum atleast 12 i.e. $deg(v) + deg(u) \geq 12$.
If u and v are not adjacent to each other then, they are connected to 12 other vertices but there are only 8 vertices left so there will be atleast 4 common vertices between the 2 which gives us a cycle of 4.
If u and v are adjacent to each other then their degree sum is lowered to 10 and they are connected to 10 other vertices but again there are only 8 vertices left so there will be atleast 2 common vertices between the 2 which again gives us a cycle of 4 □

1.2 Part (b)

We are given a **simple** graph $G = (10, 38)$. I first make the following claim:

Claim 1.2.1. *There are atleast 3 vertices such that $deg(v) \geq 8$.*

Proof. Since the graph is simple, the maximum degree a vertex can have is $|V| - 1$ i.e. 9 in this case, also the sum of degrees of all vertices will be $2|E| = 76$.
Assume 8 vertices have $deg(v) \leq 7$ which means the sum of their degrees is ≤ 56 but that would mean the sum of degrees of other 2 vertices is ≥ 20 which is not possible, but now consider 7 vertices have $deg(v) \leq 7$ which means the sum of their degrees is ≤ 49 and this is possible if other 3 vertices have $deg(v) = 9$.
The above arguments prove that there are atleast 3 vertices such that $deg(v) \geq 8$. □

Now I will use the result of the above claim to prove the following claim.

Claim 1.2.2. *G contains K_4 as an induced sub-graph.*

Proof. Consider the 3 vertices v_1, v_2, v_3 such that their degree is 8, now since v_1 has degree 8, it will connected to 8 vertices out of which atleast 1 will be one out of v_2, v_3, same goes for v_2 and v_3, so 1 vertex out of 3 is connected to other 2.
So finally we have edges $(v_1, v_2), (v_2, v_3)$ and there will a vertex v_4 such that it is connected to all 3 earlier vertices, now we have 2 cases:
Case 1. There is an edge between v_1, v_3
    If there is an edge (v_1, v_3), then we are done and our **induced subgraph** K_4 is v_1, v_2, v_3, v_4.
Case 2. There is no edge between v_1, v_3
    Since the remaining degree of v_1 is 6, it will be connected to every other node of the graph and since we know that degree of any node can't be ≤ 2, there will be a node v_5 such that it is connected to v_4, v_1, v_2 which gives us an **induced graph** K_4 i.e. graph of nodes v_1, v_2, v_4, v_5. □

1.3 Part (c)

We will find the number of **automorphisms** for each graph in the following cases:

1.3.1 Part i

We are given a **complete** graph K_n.

Solution. Every node of this graph has degree $n-1$ and it is connected to every other node of the graph.
Since all nodes have degree $n-1$, thus we can consider each node as equivalent while finding the number of automorphisms, thus the final number will be:

$$^nC_1(n-1)! = n!$$

Thus, the final answer will be $n!$. □

1.3.2 Part ii

We are given a **Cycle** graph C_n.

Solution. Every node of this graph has degree 2 and thus degree of every node afterwards must also remain the same.
Since all nodes have degree 2, thus we can consider each node as equivalent while finding the number of automorphisms, thus the final answer will be:

$$^nC_1 2! = 2n$$

Thus, the final answer will be $2n$. □

1.3.3 Part iii

We are given a **Path** graph P_n.

Solution. The graph is considered to be a straight line with n nodes i.e. 2 nodes with degree 1 and other nodes with degree 2.
Thus the possible **number of automorphisms** of this graph is **2** as any other automorphism will either changes the degree of the vertices or changes the relative order of the vertices. □

1.3.4 Part iv

We are given a **Star** graph S_n.

Solution. In this graph there is 1 nodes of degree $n-1$ and all other nodes of degree 1 which means the node with degree $n-1$ lies at the centre and all other nodes are connected to it.
Now counting the possible number of automorphisms, the node at the centre will remain the same and other nodes can be considered equivalent which means the **number of automorphisms** is **(n-1)!**. □

1.4 Part (d)

We are given a **simple** graph G such that $\delta(G) = k$ where $\delta(G)$ is the minimum degree of the graph.

Claim 1.4.1. *G has a cycle of length atleast $k+1$.*

Proof. Since the degree of each vertex is k, there has to be atleast $k+1$ vertices in the graph since it is a simple graph.
Consider $x_0, \dots x_n$ to be a longest path in G, then all the neighbours of x_n lie on this path, thus we can say that $n \geqslant d(x_n) \geqslant k$.
If $i < n$ is minimal with $x_i x_n \in E(G)$, then $x_i \dots x_n x_i$ is a cycle of length atleast $k+1$. □

II Problem 2 Solution

2.1 Part (a)

We are given a **simple** graph $G = (V,E)$ with n nodes such that for distinct $u,v \in V$, we have $deg(u) + deg(v) \geq n$.

2.1.1 Part i

I have to prove the following claim

Claim 2.1.1. *G is connected.*

Proof. I will prove the claim by **contradiction**.
Assume the graph G is disconnected.
Since the graph is disconnected, there will be atleast 2 connected different sub-graphs of the graph which have no edge between them, let these components be A and B.
$\therefore$ there will be non-adjacent vertices u and v in A and B respectively, let A and B have total number of a and b vertices, so $a+b \leq n$.
The graph is simple thus a node in A can have a max degree of $|A|-1$ which means $deg(u) \leq a-1$ and $deg(v) \leq b-1$.

$$\begin{aligned} deg(u) + deg(v) &\leq a + b - 2 \\ deg(u) + deg(v) &< a + b \\ deg(u) + deg(v) &< n \end{aligned}$$

The above equation is clearly a **contradiction** as $deg(u)+deg(v) \geq n$ and thus graph G is connected. □

2.1.2 Part ii

I have to prove the following claim.

Claim 2.1.2. *G has a **H-cycle**.*

Proof. I will prove the claim by **contradiction**.
Assume the graph has no **H-cycle**.
Since we know graph is connected as we proved above as **Claim 2.1.1**, we consider a graph G with the most possible edges such that the graph G is not Hamiltonian.
Since the graph G has the most possible edges without an $H-cycle$, adding 1 edge must make it Hamiltonian, thus there must be a **Hamiltonian Path** - $\{v_1, v_2 \ldots v_n\}$ in the graph.
Since we know, G has no $H-cycle$, nodes v_1 and v_n do not have an edge between them, now consider v_i is neighbour of v_1, then v_{i-1} can't be the neighbour of v_n, otherwise we will get a $H-cycle$ which is $\{v_1, v_2 \ldots, v_{i-1}, v_n, v_{n-1}, \ldots, v_i, v_1\}$.
Since, if v_i is neighbour of v_1 then v_{i-1} is not the neighbour of v_n which gives us

$$deg(v_1) + deg(v_n) \leq n - 1 \tag{2.1}$$

Equation (2.1) is clearly a **contradiction** as $deg(v_1) + deg(v_n) \geq n$. □

2.1.3 Part iii

Now consider **simple** graph $G = (V, E)$ such that $|V| = n$ and $\forall u, v \in V$ we have $deg(u) + deg(v) \geq n - 1$.

Claim 2.1.3. ***Claim 2.1.2*** *does not hold for the above mentioned graph.*

Proof. In this case it is possible that the graph has no $H - cycle$ as by equation (2.1) we can see that it is possible to have $deg(v_1) + deg(v_n) = n - 1$ and not be a **Hamiltonian** graph.
Taking an example graph $G = (4, 4)$ such that $V = \{v_1, v_2, v_3, v_4\}$ and $E = \{(v_1, v_2), (v_1, v_3), (v_2, v_3), (v_3, v_4)\}$, this graph does not have a $H - cycle$ but satisfies the condition $deg(v_i) + deg(v_j) \geq 3$. □

2.2 Part (b)

We are given a **simple** graph $G = (V, E)$ with $|V| = n$ and $\forall v \in V, deg(v) \geq \frac{n}{2}$.

Claim 2.2.1. *G has a* ***H-cycle****.*

Proof. I will prove the claim by **contradiction**, so lets assume G doesn't have a $H - cycle$.
Since $deg(v) \geq \frac{n}{2}$, consider 2 nodes in V, $deg(u) + deg(v) \geq n$, thus by **Claim 2.1.1**, graph G is connected.
Now consider a path P and let this path be the longest path of the graph G, $P = \{v_1, v_2 \ldots v_k\}$, and since it is the longest path v_1 and v_k cannot have an edge with some other vertex u which is not in path P as it will yield a longer path.
Since $deg(v_1) \geq \frac{n}{2}$, and v_1 can have edges only with vertices in path P, we have $k \geq \frac{n}{2} + 1$.
Consider a vertex v_j in this path, if it is adjacent to v_k, then v_{j+1} can't be adjacent to v_1 otherwise the path P will form a cycle $\{v_1, v_{j+1}, \ldots, v_{k-1}, v_k, v_j \ldots, v_2, v_1\}$ and since G is connected this path P will cover all the nodes of the graph otherwise if there are other nodes, G will become disconnected.
Since, if v_j is adjacent to v_k, v_{j+1} can't be adjacent to v_1 which gives us

$$k \geq deg(v_1) + deg(v_k) + 1$$
$$k \geq n + 1$$

But the above equation is clearly a **contradiction** as there can be only n nodes. □

III Problem 3 Solution

3.1 Part (a)

We are given a complete graph K_n and I have to find the number of different **Hamiltonian Cycles** in the graph.

Solution. Since the graph is complete, every vertex is adjacent to every other vertex and thus we can take vertices in any order.
Since there are n vertices, $n!$ ways are possible of ordering the vertices in a cycle and since for having different H-cycle, the edge set must be different, so we need to eliminate the similar cases. We need to divide the answer by ${}^nC_1$ since we can start from any vertex, also since a cycle is same backwards and forwards, we need to eliminate those cases, thus dividing by 2, the final answer will be

$$\frac{n!}{2\,{}^nC_1} = \frac{(n-1)!}{2}$$

□

3.2 Part (b)

We are given a bi-partite complete graph $K_{m,n}$ where $|V_1| = m$ and $|V_2| = n$ and mn edges connect nodes in V_1 to nodes in V_2.

Claim 3.2.1. *$K_{m,n}$ has no **H-cycle** when $m \neq n$.*

Proof. Since the graph is complete bi-partite, there is an edge between each node of V_1 and V_2, so we can consider nodes of V_1 and V_2 equivalent among themselves.
For traversing between the 2 sets of vertices, we will move alternately as the graph is bi-partite.
Since $m \neq n$, let $m > n$, assume that G has a $H - cycle$, let cycle start at some node $u \in V_2$, so after traversing $2n$ edges, we will arrive back at u and since $m > n$, there will $m - n$ vertices left in V_1 which will not be visited.
By above arguments, it is clear $K_{m,n}$ will not have a **H-cycle** when $m \neq n$. □

Now, we consider $m = n$ and find the number of **H-cycles**.

Solution. Consider a vertex $v \in V_1$ and let's start the cycle from this point, since all nodes are equivalent, we can begin from any vertex and finally reach at that.
Now begin from v, we can choose 1 of n vertices from V_2, then 1 of $n-1$ vertices left of V_1 and continuing this we get $n!(n-1)!$ and since a cycle is same forwards and backwards, the final answer will be to divide it by 2.
Thus the answer when $m = n$ is $\frac{n!(n-1)!}{2}$. □