CS201A : Mathematics for CS - I (Discrete Mathematics)
Assignment 2 Solution

Question 1

(a) Show that the divides relation where S = $\mathbb{N}$ and a|b means a divides b is a partial order but not an equivalence relation.

Reflexive: $\forall a \in S$, $\frac{a}{a} = 1 \in S \Rightarrow$ (a, a) $\in$ R as a|a. So R is Reflexive.
Anti-Symmetric:
Assume that for a, b, $\in$ S, b|a and a|b where a$\neq$b.
$\Rightarrow$ a = bx and b = ay, for some x, y $\in \mathbb{N}$
$\Rightarrow$ a = ayx
$\Rightarrow$ xy = 1
$\Rightarrow$ a = b
which is contradictory. Hence, R is Anti-Symmetric.
Transitive:
Let a, b, c $\in$ S and (a, b) , (b, c) $\in$ R.
$\Rightarrow$ a | b , b | c
$\Rightarrow$ b = ax , c = by , for some x, y $\in \mathbb{N}$
$\Rightarrow$ c = axy , where xy $\in \mathbb{N}$
$\Rightarrow$ a | c
$\Rightarrow$ R is Transitive.
Symmetric:
Let, a = 5, b = 10.
Here, a divides b but b does not divide a.
$\Rightarrow$ (a, b) $\in$ R but (b, c) $\notin$ R.
R is not Symmetric.

Since R is Reflexive, Anti-Symmetric and Transitive, R is patial order relation.
Since R is not symmetric, R is not equivalence relation.

(b) Give an example of an equivalence relation from Euclidean geometry.
Let A and B are two triangles.
Define relation R : (A, B) $\in$ R if A and B are congruent triangles.
It is easy to check that relation R is Reflexive, Symmetric and Transitive.
Hence, R is an equivalence relation.

(c) Can a relation be both a partial order and an equivalence relation? Justify your answer.
Yes.
Let's define a binary relation R on set $\mathbb{N}$ such that
$\forall$ a, b $\in \mathbb{N}$, (a, b) $\in$ R $\Leftrightarrow$ a = b.
It is easy to check that relation R is both partial order and equivalence relation.
Example : R = { (1, 1) , (2, 2) , (3, 3) }

(d) Prove maximum elements are maximal.
Let S be a poset with a relation R. If M is maximum element then $\forall$ a $\in$ S, aRM.
If MRa then we have
MRa, aRM (as M is maximum) and R anti-symmetric (as S is poset) $\Rightarrow$ a = M
If not MRa then MRa $\Rightarrow$ a = M
Hence, Since $\forall$ a $\in$ S, MRa $\Rightarrow$ a = M, M is maximal if M is maximum element.

(e) Prove minimum elements are minimal.
Let S be a poset with a relation R. If m is maximum element then $\forall$ a $\in$ S, mRa.
If aRm then we have
aRm, mRa (as m is minimum) and R anti-symmetric (as S is poset) $\Rightarrow$ a = m

If not aRm then aRm $\Rightarrow$ a = m
Hence, Since $\forall$ a $\in$ S, aRm $\Rightarrow$ a = m, m is minimal if m is minimum element.

(f) If S has a minimum element then every subset is linked.
Let S has a minimum element (say m) hence $\forall$ a $\in$ S, mRa
Consider a subset T of S.
Here, $\forall$ x, y $\in$ T, we have mRx, mRy (as m is a minimum element in S) $\Rightarrow$ x and y are compatible. (Which is true $\forall$ x, y $\in$ T)
$\Rightarrow$ T is linked.
Above will be true for all subset of S.
Hence, If S has a minimum element then every subset is linked.

(g) There can be at most one maximum element and at most one minimum element.
Let set S is a poset and on the contrary assume that it has two maximim element M_1 and M_2
$\Rightarrow$ $\forall$ a $\in$ S, aRM_1 and $\forall$ a $\in$ S, aRM_2
$\Rightarrow$ M_2RM_1 and M_1RM_2 (choosing 'a' from set S)
$\Rightarrow$ $M_1 = M_2$ (As S is a poset)
$\Rightarrow$ Set S has at most one maximum element.
Similarly, we can also prove that set S has at most one minimum element.

(h) A maximal element in a linear order is a maximum and minimal element is a minimum.
Let set S is linear order under relation R.
$\Rightarrow$ S is a chain under R.
$\Rightarrow$ $\forall$ a, b $\in$ S, aRb or bRa
Let maximal element is M.
$\Rightarrow$ aRM or MRa (as S is a chain)
If MRa then a=M
So, we have aRM $\forall$ a $\in$ S
Hence, A maximal element in a linear order is a maximum.
Similarly, we can prove that, A minimal element in a linear order is a minimum.

(i) Give an example of a poset where a unique minimal element need not be a minimum and a unique maximal element need not be a maximum.
Consider a poset S = $\mathbb{Z} \bigcup \{x\}$ with relation R defined as
$\forall$ a, b $\in \mathbb{Z}$, aRb $\Rightarrow$ a < b and xRx otherwise.
Here x is not related to any interger in $\mathbb{Z}$, hence x is minimal as well as maximal element.
Moreover, x is neither minimum nor maximum.

(j) A partition of set S is a collection of subsets of S, say S_1, S_2, till S_n such that every element of S belongs to exactly one of S_i. It is clear from the definition that i $\neq$ j $\Rightarrow S_i \cap S_j = \Phi$ and $\bigcup_i S_i = S$. Prove that if R is an equivalence relation on S then it partitions S.
Consider an equivalence relation R on set S.
Define an equivalence class of S as [s] = $\{a \in S | aRs\}$
Define a partition set of S as P = $\{[s] | s \in S\}$.
It is easy to check that R is reflexive, symmetric and transitive.
Also, they are disjoint as for any a, b, c $\in$ S if c $\in$ [a] and c $\in$ [b] then cRa and cRb.
By symmetric property, aRc and cRb.
By transitivity, aRb. Hence, [a] = [b]. So, each element in parition set P of S is pairwise disjoint.
Therefore, R partitions S if R is an equivalence relation on R.

(2) **Def^n 1 : countable set :-**

(i) Suppose that there is a set S such that $\boxed{f : S \to N/N_0}$ ie there is a injection from S to N/N_0

Let D be a set of images of $s \in S$

$$D = \{f(s) \mid s \in S\}$$

So, $D \subseteq N/N_0$.

Now there is a injection from S to D.
Also D being the set of images of set S, there is an injection from set D to S

$\therefore$ $D \sim S$ (D is equivalent to S)
($\because$ bijection)

(ii) **Def^n 2 :**

If $S \sim T$ where $T \subseteq N/N_0$
that is S is equipollent to a subset of N and
suppose $f : S \to T$ is that bijective function

Define $g : S \to N$ such that $g(a) = f(a)$ $\forall a \in S$

$\therefore$ given g is injective because f is injective

$\therefore$ $S \sim T$ where $T \subseteq N/N_0$ is equivalent to $f : S \to N/N_0$

Sol. 3.

Lemma 1: The set of polynomials $P(x) = a_0 + a_1 x + \dots + a_n x^n$ (of arbitrary degree) with rational coefficients is countable.

<u>Proof</u>: The set of all polynomials with rational coefficients is the union of countable collection of sets $A_0, A_1, \dots A_n, \dots$ where A_n denotes set of all polynomials of degree $\le n$ with rational coefficients.

We know that the union of a finite collection or countable collection of countable sets is a countable set. Therefore we now only need to show that each A_n is countable. For $n=0$, A_n is the set of all rational numbers which is countable.

Every element of A_{n+1} can be written in form

$$g(x) + a_{n+1} x^{n+1}$$

where $g(x)$ is polynomial of degree $\le n$ with rational coefficients (countable by Induction hypothesis) and set of all numbers a_{n+1} is countable.

Since every element of A_{n+1} can be assigned a pair $(g(x), a_{n+1})$ where a_{n+1} & $g(x)$ range over countable sets and set of all ordered pairs over two countable sets are countable. Therefore A_{n+1} is countable; Therefore Lemma 1 is proved.

Using Lemma 1 set of all polynomials with rational coefficients are countable, but each of these polynomials has only finite number of roots (Polynomial of degree n can have at max n roots), Therefore roots of all polynomial with rational coefficients are countable.

SOL 4.

(A) given $f: A \to B$ is a surjection, we can construct an injection from $B \to A$ as follows:

$$h(b) = a \in A \text{ s.t } f(a) = b \in B$$

If there are more than one such 'a', arbitrarily choose any one.

$\because f$ is a surjection, h is defined for all $b \in B$ mapping to unique $a \in A$

Similarly using $g: A \to B$ we construct an injection from $A \to B$, Lets call it J

$$J(a) = b \in B \text{ s.t } g(b) = a \in A$$

In case of more than one such 'b', arbitrarily choose any one.

Now we have $h: B \to A$ and $J: A \to B$ and both are injection hence using CSB Theorem we conclude $A \sim B$.

(B) $S_b = \{0,1\}^*$ is set of all finite sequences of strings of 0's and 1's

Let B_n be set of sequences of 0's and 1's of length n.

$$S_B = \bigcup_{n=0}^{\infty} B_n$$

Each B_n is finite as $|B_n| = 2^n$

Since union of countable collection of countable or finite sets is countable, Therefore S_b is countable.

(C) Consider most extreme case where S is Countably infinite and disjoint from ~~S~~ A. Then A has a Countably infinite subset $B = \{b_0, b_1, b_2, \ldots\}$, since If a set is infinite then it has a Countable infinite subset.

~~A~~ Let $S = \{s_0, s_1, s_2, \ldots\}$.

Define function $f : A \to A \cup S$

$$f(b_n) = b_{n/2}, \text{ if } n \text{ is even}$$

$$f(b_n) = s_{(n-1)/2}, \text{ if } n \text{ is odd}$$

$$f(x) = x \text{ if } x \in A \setminus \{b_0, b_1, \ldots\}$$

Then f maps ~~B~~ A one to one onto $A \cup S$.

Ques →

Given :- A and B are infinite sets
$f: A \rightarrowtail B$ $\quad g: B \rightarrowtail A$ are both injections.

a) Sequences given :-
$A = A_0$, $A_1 = g(B_0)$ . . . , $A_n = g(B_{n-1})$, ...
$B = B_0$, $B_1 = f(A_0)$, . . . , $B_n = f(A_{n-1})$, ...

We know that, composition of injections is also an injection. So, $\underline{A_0, B_1, A_2, B_3, \ldots}$ for this sequence,

note that, $B_1 = f(A_0)$ i.e. onto function (surjection).
↳ as B_1 is range of (A_0) on f.
↳ also, B_1, A_2, B_3 . . . all are having 1-1 mapping from A_0. (due to composition of injections)
so, $A_0 \sim B_1$ forms a bijection.
hence, A_0 & B_1 are equipotent.

Similarly, we can say for $B_1 \sim A_2$.
Hence, we can say that
$A_0 \sim B_1 \sim A_2 \sim B_3 \sim \ldots$
so, all have same cardinalities
$|A_0| = |B_1| = |A_2| \ldots$

Same logic can be Applied for the sequence,
$B_0, A_1, B_2, A_3, \ldots$
g is one-one func^n from B to A
and also $g(B_0) = A_1$ i.e. onto func^n
so, $|B_0| = |A_1| = |B_2| = \ldots$
as all has following relations -
$B_0 \sim A_1 \sim B_2 \sim A_3 \ldots$

(b) Here, we know g is a one-one function from B to A

So, we can see that

$A_1 \subseteq A_0$

because, $g(B_0) \subseteq A$

as range of a function is a subset to its co-domain

and $A = A_0$ .. (also)

So, $A_1 \subseteq A_0$. — (1)

by induction we can say,

$A_{i+1} \subseteq A_i$ so, $A_0 \supseteq A_1 \supseteq \ldots \supseteq A_n$

Similarly, f is a one-one function from A to B

So, we can see that,

$B_1 \subseteq B_0$

because, $f(A_0) \subseteq B$

as range of a function is a subset of codomain.

and $B = B_0$ (also)

So, $B_1 \subseteq B_0$ — (2)

by induction, we can say,

$B_{i+1} \subseteq B_i$, so, $B_0 \supseteq B_1 \supseteq B_2 \ldots \supseteq B_n$

(C) <u>Given</u>:- X_i, Y_i (where i ranging over some index set)
↳ collection of mutually disjoint sets.

and if $X_i \sim Y_i$ then $\bigcup_i X_i \sim \bigcup_i Y_i$.

<u>Soln</u>:-
as $X_i \sim Y_i$ so, X_i is equipollent to Y_i.

so, $\exists f_i : X_i \rightleftarrows Y_i \quad \forall i$ (f_i is a bijection)

so, let us define a function f, such that

$f : \bigcup_i X_i \to \bigcup_i Y_i$ and $\forall x \; x \in \bigcup_i X_i$

there is a unique index i from where this x come from, i.e. $x \in X_i$.

and $f(x) = f_i(x) \in Y_i$. [as $X_i \sim Y_i$ and each element of Y_i has a pre-image in X_i)

so, $f(x)$ is onto and 1-1 as each function $f_i \; \forall f_i$ is a bijection.

and hence, f: is a bijection b/w $\bigcup_i X_i$ and $\bigcup_i Y_i$.

Now, we have to prove that, All X_i's need to be pairwise disjoint, same thing implies for All Y_i's.

⇒ let us suppose, All X_i's are not pairwise disjoint,

so, $\exists$ an x s.t. $x \in X_i$ as well as $x \in X_j$ and $i \neq j$.

suppose, X_i & X_j are not disjoint but Y_i & Y_j are disjoint.

so, $|X_i \cup X_j| = |X_i| + |X_j| - 1$ as x is common, but

$|Y_i \cup Y_j| = |Y_i| + |Y_j|$

as they are disjoint.

but $f_i : X_i \rightsquigarrow Y_i$ as $|X_i| = |Y_i|$

$f_j : X_j \rightsquigarrow Y_j$ ||ly, $|X_j| = |Y_j|$

as both are bijection.

So,

Hence $|X_i \cup X_j| \neq |Y_i \cup Y_j|$

But for ~~$f : \cup X_i$~~

$f : \cup_i X_i \to \cup_i Y_i$ to be

bijection, $|\cup_i X_i| = |\cup_i Y_i|$

but it is giving a

contradiction. Hence, disjoint

Condition is necessary to hold.

d) To define suitable sets A_i^*, B_i^* (based on A_i & B_i) s.t. result of c holds.

We assume $A_i^* = A_{i-1} - A_i \; \forall i \geq 1$

$A_1^* = A_0 - A_1$

$A_2^* = A_1 - A_2$

⋮

so, we can say $A_1^* \cap A_2^* = \{\}$

||ly, for $B_i^* = B_{i-1} - B_i \; \forall i \geq 1$

so, $B_1^* \cap B_2^* = \{\}$.

we know, $A_0 \sim B_1$ and $A_1 \sim B_2$.

$\Rightarrow A_0 - A_1 \sim B_1 \sim B_2 \Rightarrow A_1^* \sim B_2^*$

||ly, $A_2^* \sim B_1^*$.

all A_i^* are disjoint and all B_i^* are disjoint and hence, by (C), union of all A_i's and union of all B_i's are in bijection to eachother.

(e) Show that $A = (\cup_i A_i^*) \cup (\cap_i A_i)$ and

$B = (\cup_i B_i^*) \cup (\cap_i B_i)$.

Soln :- $A_i^* = A_{i-1} - A_i$ at some point at end we come across a condition that,

$A_{n-1} \supseteq A_n$.

$A_{n-1} = A_n$, and

$$\bigcup_i A_i^* = A - A_n$$

$\hookrightarrow A_1^* \cup A_2^* \cup \dots A_n^*$

$A_0 - \cancel{A_1} + \cancel{A_1} - \cancel{A_2} + \dots - \cancel{A_{n-1}} - A_n$

$A_0 - A_n$

and $\boxed{A_0 = A}$ so, $\underline{A - A_n = \bigcup_i A_i^*}$.

and $\boxed{\bigcap_i A_i = A_n}$

so, $\bigcup_i A_i \cup \bigcap_i A_i = (A - \cancel{A_n}) \cup \cancel{A_n}$

$= A$

so, $\boxed{\left(\bigcup_i A_i\right) \cup \left(\bigcap_i A_i\right) = A}$

similarly, $\underline{\left(\bigcup_i B_i\right) \cup \left(\bigcap_i B_i\right) = B}$.

(f) $f\left(\bigcap_i A_i\right) = \bigcap_i B_i$ and

$g\left(\bigcap_i B_i\right) = \bigcap_i A_i$

$\rightarrow$ as $\bigcap_i A_i = A_n$ from part (e) we have taken it in such a way

i.e. $A_{n-1} = A_n$ i.e. $A_{n-1} \supseteq A_n$.

$A_n \sim B_{n-1}$ & $A_{n-1} \sim B_n$ — ①

also, $A_{n-1} \sim A_n$ $[$as $A_{n-1} = A_n]$ — ②

so, $B_n = f(A_{n-1})$ so, $A_{n-1} \sim B_n$ as f is a bijection

$A_n = A_{n-1}$ from ②,

so, $A_n \sim B_n$

$\bigcap A_i \sim \bigcap B_i$

$f(\bigcap A_i) = \bigcap B_i$

|||ly $g(\bigcap B_i) = \bigcap_i A_i$

(g) we have observed many things in above parts that are sufficient to prove CSB.

→ $\exists f_1 : \bigcup_i A_i^* \to \bigcup_i B_i^*$ and f_1 is bijection.
(say f_1)

→ also, from above part we have proven,

$\exists f_2 : \bigcap_i A_i \to \bigcap_i B_i$ and f_2 is bijection
(say f_2)

→ also, $(\bigcup_i A_i^*) \cup (\bigcap_i A_i) = A$

$(\bigcup_i B_i^*) \cup (\bigcap_i B_i) = B$.

so, from all above points we see that

A and B are in bijection.

Say, $H : A \to B$ forms a bijection.

(6)

(a) $F_n = \{ s = \{0,1\}^{\omega} \mid s$ has only 0s after nth bit $\}$

if n be the first n bits of infinite binary string, So cardinality of $F_n = 2^n$

(b) Let positions of occurrence of 1's be $a_1, a_2, \dots a_k$

$\Rightarrow$ we map ~~a~~ ~~str~~ string s with function $f: F \to N$ such that

$f(s) = p_1^{a_1} \times p_2^{a_2} \times \dots p_k^{a_k}$ where k & a_i are finite $\forall P \in \{1, 2, \dots k\}$

Hence $|F| = \aleph_0$

(c) $S = \{0,1\}^{\omega}$ is uncountable

Proof: Consider the power set of N, this set is uncountable & we can have bijection from this set to set S by considering a subset of N & mapping the infinite string to this subset in such a way that if an element of N is absent in subset then put 0 at its place & 1 otherwise.

Now since power set of N is uncountable then set S is also uncountable.

Now we will find the cardinality of set T which is a set of binary strings that have infinitely many 1s.

Sol we can write set $T = \{0,1\}^\omega - F$ where set F is the set from part (b) the set of binary string with finitely many 1s.

$$T + F = \{0,1\}^\omega$$

Since set on RHS is uncountable F is countable (b) thus set T is uncountable.