Math for CS I/Discrete Mathematics
Assignment 5 Solutions

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I Problem 1 Solution

I have to find the number of ways a committee of n persons can be formed from a group of 7 women and 4 men.
Choosing women or men for the committee are independent of each other and I will use this fact in later parts of this problem.

1.1 Part (a)

Case of $n = 5$ and committee has 3 women and 2 men.

Solution. Number of ways of choosing 3 women out of 7 women for the committee is P_1.
Number of ways of choosing 2 men out of 4 men for the committee is P_2.
The total number of ways will be just the product of P_1 and P_2 as they are independent, so we have:

$$
\begin{aligned}
P_1 &= \binom{7}{3} \\
P_2 &= \binom{4}{2} \\
P &= P_1 * P_2 \\
P &= \binom{7}{3} \cdot \binom{4}{2} \\
P &= 210
\end{aligned}
$$

□

1.2 Part (b)

Committee must have equal number of men and women. ($n > 0$)

Solution. Consider choosing $i = \{1, 2, 3, 4\}$ women and men for the committee as maximum number of men are 4 and the number of women and men in the committee must be equal, also since the ways of choosing men or women are independent of each other, we have:

$$
\begin{aligned}
P_i &= \binom{7}{i} \cdot \binom{4}{i} \\
P &= \sum_{i=1}^{4} P_i \\
P &= 329
\end{aligned}
$$

□

1.3 Part (c)

The committee has $n = 4$ persons and one of them is Mr. Sharma (A man).

Solution. Since we have already chosen Mr. Sharma, we have to choose 3 more people for the committee and it doesn't matter whether we choose women or men.
So we have to choose 3 people out of 10, we have:

$$P = \binom{10}{3}$$
$$P = 120$$

□

1.4 Part (d)

The committee has $n = 4$ persons and atleast 2 are women.

Solution. The total number of ways of choosing $n = 4$ persons for the committee is P_0.
From this I have to choose only the cases in which atleast 2 women are selected, so I will remove the ways to select 0 women and 4 men, 1 women and 3 men for the committee, let the cases to remove be P_r, so we have:

$$P_0 = \binom{11}{4}$$
$$P_r = \binom{7}{0} \cdot \binom{4}{4} + \binom{7}{1} \cdot \binom{4}{3}$$
$$P = P_0 - P_r$$
$$P = 301$$

□

1.5 Part (e)

The committee has $n = 4$ persons, two of each gender and Mr. and Mrs. Sharma cannot both be in the committee.

Solution. The total number of ways of choosing $n = 4$ persons such that 2 are women and 2 are men is P_0.
From P_0, I have to remove the number of cases in which Mr. and Mrs. Sharma are both placed on the committee, let these cases be P_r.
For calculating P_r, we have to chose one man and one women as 1 of each is already chosen, so we have:

$$P_0 = \binom{7}{2} \cdot \binom{4}{2}$$
$$P_r = \binom{6}{1} \cdot \binom{3}{1}$$
$$P = P_0 - P_r$$
$$P = 108$$

□

II Problem 2 Solution

2.1 Part (a)

In this part of the question we are given k colors and we have to find in how many ways we can color a graph such that it is a proper coloring.
The answer to this question will be a polynomial in k which is called chromatic polynomial $P_k(G)$ of a graph G.

2.1.1 Part (i)

We are given the graph $G = K_5$ which is a complete graph made of 5 nodes.

Solution. Since every node of this graph has an edge with every other node of the graph, in order to have a proper coloring, all the nodes must be of different colors otherwise it will not be a proper coloring.
Since all nodes must be of different colors, the least possible value of k must be 5 to have a proper coloring and the polynomial $P_k(K_5)$ will be as follows:

$$P_k(K_5) = \binom{k}{5} \cdot 5!$$
$$P_k(K_5) = k(k-1)(k-2)(k-3)(k-4)$$

The above solution follows from selecting 5 different colors. □

2.1.2 Part (ii)

We are given the graph $G = C_4$ which is a cyclic graph made of 4 nodes (a square).

Solution. The value of k must be atleast 2 otherwise there will be no proper coloring.
I will break the problem in 3 cases and count the ways of the 3 cases separately and add them.
Case 1: Coloring the graph in only 2 colors.
The graph can be colored by 2 colors in 2 ways, so the no. of ways of coloring in this case will be

$$P_1 = \binom{k}{2} \cdot 2!$$

Case 2: Coloring the graph in 3 colors.
The graph can be colored by 3 colors in $3! \cdot 2$ ways, so the no. of ways of coloring in this case will be

$$P_2 = \binom{k}{3} \cdot 12$$

Case 3: Coloring the graph in 4 colors.
The graph can be colored by 4 colors in 4! ways, so the no. of ways of coloring in this case will be

$$P_3 = \binom{k}{4} \cdot 4!$$

So, the total ways of coloring the graph will be just the sum of the above 3 values.

$$
\begin{aligned}
P &= P_1 + P_2 + P_3 \\
P &= k(k-1) + 2k(k-1)(k-2) + k(k-1)(k-2)(k-3) \\
P &= k(k-1)(k^2 - 3k + 3) \\
P &= (k-1)^4 + (k-1)
\end{aligned}
$$

P is the required polynomial. □

2.2 Part (b)

We are given n types of objects and we have to select r objects from them with repetition.
Since there are n types of objects, suppose we select x_i objects of i^{th} type till the x_n, we have to select r objects, so the sum of all these will be r, i.e.

$$x_1 + x_2 + \cdots + x_n = r$$

Considering the above problem, we can form factor polynomial which is $1 + x + x^2 + x^3 \ldots$ for the value of certain x^i, as the value of $x^i \geq 0$.
So, the problem is to find out the coefficient of x^r in $P(x) = (1 + x + x^2 \ldots)^n$:

$$
\begin{aligned}
P(x) &= (1 + x + x^2 + x^3 \ldots)^n \\
P(x) &= (1 - x)^{-n} \\
P(x) &= 1 + \sum_{i=1}^{\infty} \binom{n+i-1}{i} x^i
\end{aligned}
$$

The coefficient of x^r in $P(x)$ is

$$P = \binom{n+r-1}{r}$$

P is the number of ways of selecting r objects from n types of objects with repetition.

III Problem 3 Solution

3.1 Part (a)

We are given the equation $x_1 + x_2 + x_3 + x_4 = 12$ and I have to find the number of possible 4-tuples (n_1, n_2, n_3, n_4) such that $n_i \geq 0$ and $x_i = n_i$.

3.1.1 Part (i)

Solution. The problem can be considered as distributing 12 balls in 4 containers such that the containers can also be empty.
Since $x_i \geq 0$, we can form factor polynomial like $1 + x + x^2 + x^3 \ldots$ for each container, so the problem is to find the coefficient of x^{12} in the following expression $P(x)$:

$$P(x) = (1 + x + x^2 + x^3 \ldots)^4 \tag{3.1}$$

$$P(x) = (1 - x)^{-4} \tag{3.2}$$

$$P(x) = 1 + \sum_{i=1}^{\infty} \binom{4+i-1}{i} x^i \tag{3.3}$$

The coefficient of x^{12} in $P(x)$ is P:

$$P = \binom{12+3}{3}$$

$$P = 455$$

□

3.1.2 Part (ii)

The problem is exactly the similar to the previous one, we just need to alter the number of balls.

Solution. Since we know $x_i \geq 1$, let $x_i' = x_i - 1$, thus we have $x_i' \geq 0$, so the equation will be:

$$x_1 + x_2 + x_3 + x_4 = 12$$

$$(x_1 - 1) + (x_2 - 1) + (x_3 - 1) + (x_4 - 1) = 12 - 4$$

$$x_1' + x_2' + x_3' + x_4' = 8$$

The coefficient of x^8 in equation (3.3) is P:

$$P = \binom{8+3}{3}$$

$$P = 165$$

□

3.1.3 Part (iii)

The problem is exactly the same to the previous one, we just need to alter the number of balls.

Solution. Since we know $x_1 \geq 2, x_2 \geq 2, x_3 \geq 4, x_4 \geq 0$, let $x'_1 = x_1 - 2, x'_2 = x_2 - 2, x'_3 = x_3 - 4, x'_4 = x_4$, thus we have $x'_i \geq 0$, so the equation will be:

$$x_1 + x_2 + x_3 + x_4 = 12$$
$$(x_1 - 2) + (x_2 - 2) + (x_3 - 4) + x_4 = 12 - 8$$
$$x'_1 + x'_2 + x'_3 + x'_4 = 4$$

The coefficient of x^4 in equation (3.3) is P:

$$P = \binom{4+3}{3}$$
$$P = 35$$

□

3.2 Part (b)

There are 7 friends say a, b, c, d, e, f, g and the problem is to find the number of ways to invite a different subset of 3 friends for dinner on 7 successive nights such that each pair of friends are together at just one dinner.

Solution. Since there are 7 friends and subsets of 3 are to be formed considering that 2 friends can be together only once, 1 friend can go to dinner atmost 3 times.
Consider a friend a, the no. of sets of 3 consisting a as one of the member will be:

$$\binom{6}{2} = 15$$

For moving further let's assume that 3 (maximum amount for a) pairs are selected as

$$(a, b, c); (a, d, e); (a, f, g)$$

Now there can be no pair containing a as one of the member and still consistent with our conditions.
Let's make remaining 2 pairs which contain friend b, there will be only 2 ways to do that depending on the way we choose for a which will be as follows

$$(b, d, g); (b, e, f) \mid (b, e, g); (b, d, f)$$

Now there can be no pair containing a or b as one of the member and still consistent with our conditions.
Let's make remaining 2 pairs which contain friend c, there will only 1 way to do that depending on the way we choose for b which will be as follows

$$(c, d, f); (c, e, g) \mid (c, e, f); (c, d, g)$$

Note the seven 3-tuples which we have formed, these contain all the 7 friends exactly 3 times, so we have considered all the possible tuples which can exist by this way of counting.
Thus, the number of ways of forming seven 3-tuples which satisfy our initial conditions are

$$15 * 2 * 1 = 30$$

Now, finally distributing there 7 tuples into 7 nights we get the final answer:

$$P = 15 * 2 * 1 * 7! = 151200$$

$\square$

IV Problem 4 Solution

4.1 Part (a)

We are given a series a_n which gives the number of partitions that add up to at most n.
Let's consider another series p_n which gives us the number of partitions that add up to exactly n, so we have

$$a_n = a_{n-1} + p_n$$

The above expression is valid for $n \geq 2$ and adds the partitions that add up to at most $n-1$ and the number of partitions that add up to exactly n.
So I will find the generating function of a_n by finding the generating function of p_n.

Solution. We will have following initial conditions for p_n and a_n as n is to be considered positive:

$$a_0 = p_0 = 0$$
$$a_1 = p_1 = 1$$

Consider A and P to be generating functions of a_n and p_n respectively.

$$\begin{aligned}
a_n &= a_{n-1} + p_n \\
\sum_{i=2}^{\infty} a_i x^i &= \sum_{i=2}^{\infty} a_{i-1} x^i + \sum_{i=2}^{\infty} p_i x^i \\
A - a_1 x - a_0 &= x(A - a_0) + P - p_1 x - p_0 \\
A(1-x) &= P
\end{aligned}$$

So the above equation finally gives us:

$$A(x) = \frac{P(x)}{1-x} \tag{4.1}$$

Now let's calculate $P(x)$, consider the following equation:

$$x_1 + 2x_2 + 3x_3 + \ldots = n$$
$$\sum_{i=1}^{\infty} i \cdot x_i = n$$

The value of x_i will give the number of partitions of size i in a certain arrangement.
So, by calculating the number of possible non-negative solutions of the above equation, we can find the number p_n, consider the product of factor polynomials as below

$$(1 + x + x^2 \ldots) \cdot (1 + x^2 + x^4 \ldots) \cdot (1 + x^3 + x^6 \ldots) \ldots$$

The coefficient of x^n in the above expression will be p_n, and coefficient of $x^0 = 1$ must be zero as $p_0 = 0$, so we have:

$$P(x) = -1 + (1 + x + x^2 \ldots) \cdot (1 + x^2 + x^4 \ldots) \cdot (1 + x^3 + x^6 \ldots) \ldots \tag{4.2}$$
$$P(x) = \prod_{i=1}^{\infty} (1 - x^i)^{-1} - 1 \tag{4.3}$$

Putting value of (4.3) in (4.1), we get:

$$A(x) = (1-x)^{-1} \left(\prod_{i=1}^{\infty} (1 - x^i)^{-1} - 1 \right)$$

□

4.2 Part (b)

This part is similar to **4.3 Part (c)** of this problem, I will use the generating function derived in that part in my answer.
Let the 3 parts be a,b,c such that $a \le b \le c$, also by the given condition $a+b \ge c$, since the sum of 3 parts is n, we have $a+b+c=n$.
The only difference in this part and the next one is the case of $a+b=c$, so I will calculate the generating function $P(x)$ for the case $a+b=c$ and add that to eq (4.6) to get the final generating function $A(x)$ corresponding to the sequence a_n.

Solution. Since $a+b=c$, putting this in $a+b+c=n$, we get $2(a+b)=n$.
Since we know that the parts are positive integers, we have $a,b \ge 1$, so consider the following substitutions:

$$\begin{aligned} a &= 1+x \\ b &= 1+x+y \\ x,y &\ge 0 \end{aligned}$$

The above equations are consistent with our initial conditions, also finding unique (x,y) will give us a unique (a,b,c), so the expression now becomes:

$$4x+2y=n-4$$

We will calculate non-negative integral solutions of the above equations and this will give us p_n (corresponding to $P(x)$) for a certain n. Calculating the coefficient of x^{n-4} in the below expression will give the value of p_n:

$$Q(x)=(1+x^2+x^4\dots)\cdot(1+x^4+x^8\dots)$$

So we will shift above generating function to the right by multiplying the function by x^4

$$P(x)=\frac{x^4}{(1-x^2)(1-x^4)} \tag{4.4}$$

Adding equation (4.6) to equation (4.4), we get the final answer equation (4.5).

$$A(x)=\frac{x^4}{(1-x^2)(1-x^4)}+\frac{x^3}{(1-x^2)(1-x^3)(1-x^4)} \tag{4.5}$$

Above function $A(x)$ is the required generating function for the sequence a_n. □

4.3 Part (c)

We are given a sequence a_n which gives the number of different triangles with integral sides that add upto n.
Let the sides be a,b,c such that $a \le b \le c$, also since they form a triangle, we have $a+b>c$, since the perimeter of the triangle is n, $a+b+c=n$.

Solution. Since we know the sides will be a positive integer $a,b,c \ge 1$, so consider the following substitutions:

$$\begin{aligned} a &= 1+x+z \\ b &= 1+x+y+z \\ c &= 1+x+y+2z \\ x,y,z &\ge 0 \end{aligned}$$

The above equations are consistent with our initial conditions, also finding unique (x, y, z) will give us a unique (a, b, c), so the expression now becomes:

$$3x + 2y + 4z = n - 3$$

We will calculate non-negative integral solution of the above equations and this will give us a_n for a certain n.
Calculating the coefficient of x^{n-3} in the below expression will give the value of a_n:

$$P(x) = (1 + x^2 + x^4 \dots) \cdot (1 + x^3 + x^6 \dots) \cdot (1 + x^4 + x^8 \dots)$$

So we will shift above generating function to the right by multiplying the function by x^3

$$A(x) = \frac{x^3}{(1-x^2)(1-x^3)(1-x^4)} \tag{4.6}$$

Above function $A(x)$ is the required generating function for the sequence a_n. $\square$

V Problem 5 Solution

We are given 3 different sequences which are a_n, b_n, c_n and the relation between them are:

$$a_n = a_{n-1} + b_{n-1} + c_{n-1} \tag{5.1}$$
$$b_n = 3^{n-1} - c_{n-1} \tag{5.2}$$
$$c_n = 3^{n-1} - b_{n-1} \tag{5.3}$$
$$a_1 = b_1 = c_1 = 1 \tag{5.4}$$

Solution. Due to similarity of equation (5.2) and (5.3), also the initial values b_1 and c_1 are same, so the series b_n and c_n will be exactly same.
Let the generating function of b_n be

$$B = \sum_{i=1}^{\infty} b_i x^i$$

Also by putting the value of $n = 2$ in (5.2), we will get $b_2 = 2$.
Let's combine equation (5.2) and (5.3) and solve to find the generating function B.

$$b_n = b_{n-2} + 2 \cdot 3^{n-2}$$
$$\sum_{i=3}^{\infty} b_i x^i = \sum_{i=3}^{\infty} (b_{i-2} x^i + 2 \cdot 3^{i-2} x^i)$$
$$B - b_1 x - b_2 x^2 = x^2 B + 2x^2 \left(\frac{1}{1-3x} - 1 \right)$$
$$B(1-x^2) = b_1 x + b_2 x^2 + \frac{6x^3}{1-3x}$$
$$B = \frac{x - x^2}{(1-3x)(1-x^2)}$$
$$B = \frac{x}{(1-3x)(1+x)}$$
$$B = \frac{1}{4} \left(\frac{1}{1-3x} - \frac{1}{1+x} \right)$$
$$B = \frac{1}{4} \left(\sum_{i=1}^{\infty} (3^i - (-1)^i) x^i \right)$$

Solving the above expression we will get:

$$b_n = \frac{1}{4} \left(3^n - (-1)^n\right) \tag{5.5}$$
$$c_n = \frac{1}{4} \left(3^n - (-1)^n\right) \tag{5.6}$$

Let the generating function of a_n be

$$A = \sum_{i=1}^{\infty} a_i x^i$$

Putting the values from equation (5.5) and (5.6) in (5.1), solving the equations we get:

$$\begin{aligned} a_n &= a_{n-1} + \frac{1}{2}\left(3^{n-1} - (-1)^{n-1}\right) \\ A - a_1 x &= xA + \frac{x}{2}\left(\sum_{i=1}^{\infty}(3^i - (-1)^i)x^i\right) \\ A(1-x) &= x\left[a_1 + \frac{1}{2}\left(\frac{1}{1-3x} - \frac{1}{1+x}\right)\right] \\ A &= \frac{x}{1-x} - \frac{x}{2(1-x^2)} + \frac{1}{4(1-3x)} - \frac{1}{4(1-x)} \end{aligned}$$

Solving the above expression of generating function we will get:

$$a_n = \frac{3^n}{4} + \frac{1}{2} + \frac{(-1)^n}{4} \tag{5.7}$$

□