MSO 201 A : Homework 12

[1] $X_1,...,X_n$ be a random sample from $N(\mu, \sigma^2)$ distribution. Find the Cramer-Rao Lower Bounds (CRLB) on the variances of unbiased estimators of μ and σ^2. Can you find unbiased estimators μ and σ^2 whose variances attain the respective CRLB?

[2] $X_1,....,X_n$ is a random sample from $Gamma(\alpha,\beta)$

$$f(x|\alpha,\beta)=\begin{cases}\dfrac{1}{\Gamma\alpha\ \beta^{\alpha}}e^{-x/\beta}x^{\alpha-1} & \text{if } x>0\\ 0 & \text{otherwise}\end{cases}$$

α is assumed to be known. Find the Fisher Information $I(\beta)$ and the CRLB on the variances of unbiased estimators of β.

[3] $X_1,...,X_n$ be a random sample from $P(\theta), \theta\in(0,\infty)$. Find the CRLB on the variances of unbiased estimators of the following estimands: (a) $g(\theta)=\theta$, (b) $g(\theta)=\theta^2$ and (c) $g(\theta)=e^{-\theta}$.

[4] Suppose $X_1,...,X_n$ be a random sample from $B(1,\theta), \theta\in(0,1)$. Find the CRLB on the variances of unbiased estimators of the following estimands: (a) $g(\theta)=\theta^4$ (b) $g(\theta)=\theta(1-\theta)$.

[5] $X_1,...,X_n$ be a random sample from $U(0,\theta), \theta>0$. Show that (a) $\dfrac{n}{n+1}X_{(n)}$ is a consistent estimator of θ and (b) $e^{X_{(n)}}$ is consistent for e^{θ}, where $X_{(n)}=\max(X_1,...,X_n)$.

[6] $X_1,...,X_n$ be a random sample from $U(\theta-1/2,\theta+1/2), \theta\in\Re$. Show that $X_{(1)}+1/2$, $X_{(n)}-1/2$ and $\left(X_{(1)}+X_{(n)}\right)/2$ are all consistent estimators of θ, $X_{(n)}=\max(X_1,...,X_n)$ and $X_{(1)}=\min(X_1,...,X_n)$.

[7] $X_1,...,X_n$ be a random sample from

$$f(x)=\begin{cases}\dfrac{1}{2}(1+\theta x) & -1<x<1\\ 0 & \text{otherwise.}\end{cases}$$

Where, $\theta\in(-1,1)$. Find a consistent estimator for θ.

[8] $X_1,\ldots,X_n$ be a random sample from $P(\theta)$. Find a consistent estimator of $\theta^3\left(3\sqrt{\theta}+\theta+12\right)$.

[9] Let $X_1,\ldots,X_n$ be a random sample from $Gamma(\alpha,\beta)$ with density

$$f(x)=\begin{cases}\dfrac{1}{\Gamma\alpha\,\beta^{\alpha}}\,e^{-x/\beta}x^{\alpha-1} & x>0\\ 0 & \text{otherwise.}\end{cases}$$

Where, α is a known constant and β is an unknown parameter. Show that $\sum_{i=1}^{n}X_i\Big/n\alpha$ is a consistent estimator of β.

[10] Let $X_1,\ldots,X_n$ be a random sample from each of the following distributions having the following density or mass functions. Find the maximum likelihood estimator (MLE) of θ in each case.

(a) $f(x;\theta)=\begin{cases}\dfrac{e^{-\theta}\theta^{x}}{x!} & x=0,1,2,\ldots\\ 0 & \text{otherwise.}\end{cases}$

(b) $f(x;\theta)=\begin{cases}\theta x^{\theta-1} & 0<x<1\\ 0 & \text{otherwise.}\end{cases}$

(c) $f(x;\theta)=\begin{cases}\dfrac{1}{\theta}e^{-x/\theta} & x>0\\ 0 & \text{otherwise.}\end{cases}$

(d) $f(x;\theta)=\begin{cases}\dfrac{1}{2}e^{-|x-\theta|} & -\infty<x<\infty\\ 0 & \text{otherwise.}\end{cases}$

(e) $X\sim U(-\theta/2,\theta/2)$.

[11] Let $X_1,\ldots,X_n$ be a random sample from the distribution having p.d.f.

$$f(x;\theta_1,\theta_2)=\begin{cases}\dfrac{1}{\theta_2}e^{-(x-\theta_1)/\theta_2} & x\geq\theta_1\\ 0 & \text{otherwise.}\end{cases}$$

Find the MLEs of θ_1 and θ_2.

[12] Let $X_1,\ldots,X_n$ be a random sample from the distribution having p.d.f.

$$f(x;\alpha,\lambda)=\begin{cases}\dfrac{\lambda^{\alpha}}{\Gamma\alpha}e^{-\lambda x}x^{\alpha-1} & x\ge 0\\ 0 & \text{otherwise.}\end{cases}$$

Find the MLEs of α and λ.

[13] Let $X_1,\ldots,X_n$ be a random sample from the function having p.d.f.

$$f(x;\mu,\sigma)=\begin{cases}\dfrac{1}{2\sqrt{3}\sigma} & \mu-\sqrt{3}\sigma<x<\mu+\sqrt{3}\sigma\\ 0 & \text{otherwise.}\end{cases}$$

Find the MLEs of μ and σ.

[14] Let $X_1,\ldots,X_n$ be a random sample from $U(\theta-1/2,\theta+1/2),\theta\in\Re$. Show that any statistic $u(X_1,\ldots,X_n)$ such that it satisfies

$$X_{(n)}-\frac{1}{2}\le u(X_1,\ldots,X_n)\le X_{(1)}+\frac{1}{2}$$

is a maximum likelihood estimator of θ. In particular $\left(X_{(1)}+X_{(n)}\right)/2$ and $\frac{3}{4}\left(X_{(1)}+\frac{1}{2}\right)+\frac{1}{4}\left(X_{(n)}-\frac{1}{2}\right)$ are MLEs of θ.

[15] The lifetimes of a component are assumed to be exponential with parameter λ. Ten of these components were placed on a test independently. The only data recorded were the number of components that had failed (out of 10 put to test) in less than 100 hours, which was recorded to be 3. Find the maximum likelihood estimate of λ.

[16] A salesman of used cars is willing to assume that the number of sales he makes per day is a Poisson random variable with parameter μ. Over the past 30 days he made no sales on 20 days and one or more sales on each of the remaining 10 days. Find the maximum likelihood estimate of μ.

[17] Let $X_1,\ldots,X_n$ be a random sample from each of the following distributions. Find the method of moments estimator (MOME) of the corresponding unknown parameters in each of the situations.

(a) $X\sim P(\theta)$; **(b)** $X\sim U(-\theta/2,\theta/2)$;

(c) $X\sim Exp(0,\theta)$; **(d)** $X\sim Exp(\alpha,\beta)$;

(e) $X\sim G(\alpha,\beta)$ with p.d.f. $f(x;\alpha,\beta)=\begin{cases}\dfrac{1}{\beta^{\alpha}\Gamma\alpha}e^{-x/\beta}x^{\alpha-1} & x\ge 0\\ 0 & \text{otherwise.}\end{cases}$