Problem Set # 9

(1) $P(|X_n - c| \geq \epsilon) \leq \dfrac{E(X_n-c)^2}{\epsilon^2}$ by chebyshev's ineq

$$= E(X_n - EX_n + EX_n - c)^2 / \epsilon^2$$

$$= \frac{E(X_n - EX_n)^2 + (EX_n - c)^2}{\epsilon^2}$$

Since $EX_n \to c$ and $VX_n \to 0$ as $n \to \infty$

$$\frac{E(X_n - EX_n)^2 + (EX_n - c)^2}{\epsilon^2} = \frac{V(X_n) + (EX_n - c)^2}{\epsilon^2} \longrightarrow 0 \text{ as } n \to \infty \ \forall \epsilon > 0.$$

$\Rightarrow P(|X_n - c| \geq \epsilon) \to 0$ as $n \to \infty$ $\forall \epsilon > 0$

$\Rightarrow X_n \xrightarrow{p} c.$

(2) $S_n = \sum_{i=1}^{n} X_i$; Take $a_n = \sum \mu_i$ and $b_n = n$

$$\frac{S_n - a_n}{b_n} = \frac{\sum(X_i) - \sum \mu_i}{n}$$

$$P\left(\left|\frac{S_n - a_n}{b_n}\right| \geq \epsilon\right) = P\left(\left|\frac{\sum X_i - \sum \mu_i}{n}\right| \geq \epsilon\right)$$

$$\leq \frac{E(\sum X_i - \sum \mu_i)^2}{n^2 \epsilon^2}$$

$$= \frac{E(\sum X_i - E(\sum X_i))^2}{n^2 \epsilon^2}$$

$$= \frac{V(\sum X_i)}{n^2 \epsilon^2} \longrightarrow 0 \text{ as } n \to \infty \ \forall \epsilon > 0.$$

from the given condition

$$\Rightarrow \frac{S_n - a_n}{b_n} \xrightarrow{p} 0$$

$\Rightarrow$ WLLN holds for $\{X_n\}$.

Furthermore,
$$\frac{S_n - a_n}{b_n} = \frac{\sum X_i}{n} - \frac{\sum \mu_i}{n}$$
$$= \overline{X}_n - \bar{\mu}_n \xrightarrow{p} 0$$
i.e. $\overline{X}_n \xrightarrow{p} \overline{\mu}_n$.

3. $X_1, \ldots X_n$ i.i.d $U(0,1)$

$Y_n = \min(X_1, \ldots X_n)$; $Z_n = \max(X_1, \ldots X_n)$

d.f.
$$F_{Y_n}(y) = P(\min(X_1, \ldots X_n) \le y)$$
$$= 1 - P(\min(X_1, \ldots X_n) > y)$$
$$= 1 - (1 - F_X(y))^n$$

p.d.f.
$$f_{Y_n}(y) = n(1 - F_X(y))^{n-1} f_X(y) \quad ; \quad 0 < y < 1$$

i.e.
$$f_{Y_n}(y) = \begin{cases} n(1-y)^{n-1} & ; \quad 0 < y < 1 \\ 0 & \text{o/w.} \end{cases}$$

$$P(|Y_n| > \epsilon) \le \frac{E Y_n^2}{\epsilon^2}$$

Now
$$E Y_n^2 = n \int_0^1 y^2 (1-y)^{n-1}\, dy = n \int_0^1 (1-x)^2 x^{n-1}\, dx$$
$$= n \int_0^1 (1 + x^2 - 2x) x^{n-1} dx = n\left(\frac{1}{n} + \frac{1}{n+2} - \frac{2}{n+1}\right)$$
$$\longrightarrow 0 \quad \text{as } n \to \infty$$

$$\Rightarrow \frac{E Y_n^2}{\epsilon^2} \longrightarrow 0 \quad \text{as } n \to \infty \quad \forall \epsilon > 0. \text{ however small.}$$

$\Rightarrow \quad P(|Y_n| > \epsilon) \longrightarrow 0 \quad$ as $n \to \infty$

$\Rightarrow \quad Y_n \xrightarrow{p} 0$

p.d.f. of Z_n : $f_{Z_n}(z) = \begin{cases} n z^{n-1}, & 0 < z < 1 \\ 0, & \text{o/w} \end{cases}$

$$P(|Z_n - 1| > \epsilon) \le \frac{E(Z_n-1)^2}{\epsilon^2} = \frac{E Z_n^2 + 1 - 2E(Z_n)}{\epsilon^2}$$

$$E Z_n = n \int_0^1 z^n \, dz = \frac{n}{n+1}$$

$$E Z_n^2 = n \int_0^1 z^{n+1} \, dz = \frac{n}{n+2}$$

$\Rightarrow \quad \dfrac{E(Z_n-1)^2}{\epsilon^2} = \dfrac{1}{\epsilon^2}\left(\dfrac{n}{n+2} + 1 - 2\dfrac{n}{n+1}\right) \longrightarrow 0 \quad$ as $n \to \infty$ $\forall \epsilon > 0$ fixed.

$\Rightarrow \quad P(|Z_n - 1| > \epsilon) \longrightarrow 0 \quad$ as $n \to \infty$

$\Rightarrow \quad Z_n - 1 \xrightarrow{p} 0 \quad$ i.e. $Z_n \xrightarrow{p} 1$

$\Rightarrow \quad Z_n^2 \xrightarrow{p} 1 \quad$ (If $X_n \xrightarrow{p} X$ for g cont $g(X_n) \xrightarrow{p} g(X)$).

Since $Y_n \xrightarrow{p} 0$ & $Z_n^2 \xrightarrow{p} 1$

$Y_n \cdot Z_n^2 \xrightarrow{p} 0 \quad$ (If $X_n \xrightarrow{p} X$ & $Y_n \xrightarrow{p} Y$ then $X_n Y_n \xrightarrow{p} XY$).

(4) $X_1, \ldots X_n$ i.i.d. $N(0,1)$

WLLN (Khintchine's WLLN) $\Rightarrow$ $\frac{1}{n}\sum X_i \xrightarrow{p} EX_1 = 0$

i.e. $\bar{X}_n \xrightarrow{p} 0$

$$S_n^2 = \frac{1}{n}\sum X_i^2 - \bar{X}_n^2$$

Note that $X_1^2, X_2^2, \ldots X_n^2$ are i.i.d with $E(X_1^2) = 1$

WLLN (Khintchine's WLLN) $\Rightarrow$ $\frac{1}{n}\sum_1^n X_i^2 \xrightarrow{p} EX_1^2 = 1$

Since $\bar{X}_n \xrightarrow{p} 0 \Rightarrow \bar{X}_n^2 \xrightarrow{p} 0.$

$$\Rightarrow S_n^2 = \frac{1}{n}\sum X_i^2 - \bar{X}_n^2 \xrightarrow{p} 1 - 0 = 1$$

$$\Rightarrow S_n^{-1} \xrightarrow{p} 1$$

Since $\bar{X}_n \xrightarrow{p} 0$ & $S_n^{-1} \xrightarrow{p} 1$

$$\bar{X}_n S_n^{-1} \xrightarrow{p} 0 \left(= 0/1\right)$$

(5) $Y_n \sim Bin(n, p)$

$$P\left(\left|\frac{Y_n}{n} - p\right| > \epsilon\right) \le \frac{E\left(\frac{Y_n}{n} - p\right)^2}{\epsilon^2} = \frac{E(Y_n - np)^2}{n^2\epsilon^2}$$

$$= \frac{npq}{n^2\epsilon^2} = \frac{pq}{n\epsilon^2} \longrightarrow 0 \quad \text{as } n \to \infty$$

$$\Rightarrow \frac{Y_n}{n} \xrightarrow{p} p$$

$$\Rightarrow \left(1 - \frac{Y_n}{n}\right) \xrightarrow{p} p.$$

(6) $E(X_n) = 0$; $V(X_n) = E X_n^2 = \frac{\sqrt{n}}{2} + \frac{\sqrt{n}}{2} = \sqrt{n}$

$E\bar{X}_n = 0$; $V(\bar{X}_n) = \frac{1}{n^2} \sum_{i=1}^{n} \sqrt{i} \le \frac{n\sqrt{n}}{n^2} \to 0$ as $n \to \infty$

$\Rightarrow P(|\bar{X}_n - 0| > \epsilon) \le \frac{E\bar{X}_n^2}{\epsilon^2} = \frac{V\bar{X}_n}{\epsilon^2} \to 0$ as $n \to \infty \ \forall \epsilon > 0$

$\Rightarrow \bar{X}_n \xrightarrow{p} 0$.

(7) (a) $Y_n = \frac{2}{n(n+1)} \sum_{i=1}^{n} i X_i$

$E Y_n = \frac{2}{n(n+1)} \sum_{1}^{n} i \mu = \mu$

$V Y_n = \frac{4}{n^2(n+1)^2} \sum_{1}^{n} i^2 \sigma^2 = \frac{4\sigma^2}{n^2(n+1)^2} \cdot \frac{n(n+1)(2n+1)}{6} \to 0$ as $n \to \infty$

$\Rightarrow P(|Y_n - \mu| > \epsilon) \le \frac{E(Y_n - \mu)^2}{\epsilon^2} = \frac{V(Y_n)}{\epsilon^2} \to 0$ as $n \to \infty$

$\Rightarrow Y_n \xrightarrow{p} \mu$.

(b) Sly as in (a)

(8) $X_1, \ldots X_n$ i.i.d $U(0,1)$

Let $T_i = -\log X_i \quad i = 1(1)n$

$T_i \sim Exp(0,1) \qquad f_{T_i}(t) = \begin{cases} e^{-t} & t > 0 \\ 0, & o/w. \end{cases}$

$E T_i = 1$

$T_1, \ldots T_n$ are i.i.d. $Exp(0,1)$ with $E T_i = 1$

$Z_n = \left(\prod_{1}^{n} X_i\right)^{1/n}$

$-\log Z_n = \frac{1}{n} \sum_{1}^{n} (-\log X_i) = \frac{1}{n} \sum_{1}^{n} T_i$

WLLN $\Rightarrow \frac{1}{n} \sum_{1}^{n} T_i \xrightarrow{p} E(T_1) = 1$

i.e. $-\log Z_n \xrightarrow{p} 1$

$\Rightarrow Z_n \xrightarrow{p} e^{-1}$.

9 $E X_i = \mu_i \quad , V(X_i) = \sigma_i^2$

If $\sum_1^n \sigma_i^2 \to \infty$, take $a_n = \sum_1^n \mu_i$

& $b_n = \sum_1^n \sigma_i^2$

then
$$P\left(\left| \frac{S_n - a_n}{b_n} \right| > \epsilon \right) = P\left(\left| \frac{1}{\sum \sigma_i^2} \sum (X_i - \mu_i) \right| > \epsilon \right)$$
$$\le \frac{E\left(\sum (X_i - \mu_i)\right)^2}{\left(\sum \sigma_i^2\right)^2 \epsilon^2}$$
$$= \frac{\sum \sigma_i^2}{\epsilon^2 \left(\sum \sigma_i^2\right)^2} = \frac{1}{\epsilon^2 \sum \sigma_i^2} \to 0 \text{ as } n \to \infty$$

$$\Rightarrow \quad \frac{S_n - a_n}{b_n} \xrightarrow{p} 0$$

i.e. WLLN holds for $\{X_n\}$.

10 $X_1 \cdots X_n$ i.i.d. $U(0,1)$ with $E X_i = \frac{1}{2}$

W.LLN $\Rightarrow \frac{1}{n} \sum X_i \xrightarrow{p} E X_1 = \frac{1}{2}$

i.e. $\bar{X}_n \xrightarrow{p} \frac{1}{2}$

(11) $F_{X_n}(x) = P(X_n \le x)$

$$= P\left(\frac{X_n - 1/n}{\sqrt{1-\frac{1}{n}}} \le \frac{x - \frac{1}{n}}{\sqrt{(1-\frac{1}{n})}}\right)$$

$$= \Phi\left(\frac{x - 1/n}{\sqrt{1-\frac{1}{n}}}\right) \to \Phi(x) \quad \text{as } n \to \infty$$

$$\Rightarrow X_n \xrightarrow{L} X \sim N(0,1)$$

Alt m.g.f of X_n

$$M_{X_n}(t) = \exp\left(t/n + \frac{t^2}{2}\left(1-\frac{1}{n}\right)\right)$$

$$\longrightarrow e^{t^2/2} \leftarrow \text{m.g.f of } N(0,1).$$

$$\Rightarrow X_n \xrightarrow{L} X \sim N(0,1).$$

(12) $Y_i = (X_i - \mu)^2$

$E(Y_i) = E(X_i - \mu)^2 = \sigma^2$

$V(Y_i) = E\left((X_i-\mu)^2 - \sigma^2\right)^2$

$$= E\left(X_i - \mu\right)^4 + \sigma^4 - 2\sigma^2 E(X_i - \mu)^2$$

$$= (\sigma^4 + 1) + \sigma^4 - 2\sigma^4 = 1$$

i.e. $E(Y_i) = \sigma^2$; $V(Y_i) = 1$ $\forall i$ & $Y_1 \cdots Y_n$ i.i.d

$S_n = \Sigma Y_i$ $\quad ES_n = n\sigma^2$

$\quad VS_n = n$

CLT $\Rightarrow$ $\dfrac{S_n - ES_n}{\sqrt{V(S_n)}} \xrightarrow{L} N(0,1)$

i.e. $\dfrac{(X_1-\mu)^2 + \cdots + (X_n-\mu)^2 - n\sigma^2}{\sqrt{n}} \xrightarrow{L} X \sim N(0,1)$

$$\lim_{n\to\infty} P\left(\sigma^2 - \frac{1}{\sqrt{n}} \le \frac{(X_1-\mu)^2 + \cdots + (X_n-\mu)^2}{n} \le \sigma^2 + \frac{1}{\sqrt{n}}\right)$$

$$= \lim_{n\to\infty} P\left(-\frac{1}{\sqrt{n}} \le \frac{(X_1-\mu)^2 + \cdots + (X_n-\mu)^2 - n\sigma^2}{n} \le \frac{1}{\sqrt{n}}\right)$$

$$= \lim_{n\to\infty} P\left(-1 \le \frac{(X_1-\mu)^2 + \cdots + (X_n-\mu)^2 - n\sigma^2}{\sqrt{n}} \le 1\right)$$

$$= \Phi(1) - \Phi(-1) = 2\Phi(1) - 1 = \ldots$$

(13) $E S_n = np$; $V S_n = npq$

$$P\left(\left|\frac{S_n}{n} - p\right| \geq t\right) \le \frac{E(S_n - np)^2}{t^2 n^2} = \frac{np(1-p)}{n^2 t^2} \le \frac{1}{4n t^2} \le 0.01 \text{ (given)}$$

$$\Rightarrow n \geq \frac{1}{0.04\, t^2} \quad \text{for } t = 0.01$$

$$n \geq \ldots$$

(14) CLT $\Rightarrow$ $\dfrac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \xrightarrow{L} Z \sim N(0,1)$

Also $S_n^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2 \xrightarrow{p} \sigma^2$

$$\left(S_n^2 = \underbrace{\frac{1}{n}\sum X_i^2}_{\xrightarrow{p}\ \sigma^2+\mu^2} - \underbrace{\bar{X}^2}_{\xrightarrow{p}\ \mu^2} \xrightarrow{p} \sigma^2 \right)$$

$\Rightarrow S_n \xrightarrow{p} \sigma$,

using Slutsky's lemma

$\left(X_n \xrightarrow{L} X \; ; \; Y_n \xrightarrow{p} c \quad X_n/Y_n \xrightarrow{L} X/c \right)$.

$$\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \Big/ \left(S_n/\sigma\right) \xrightarrow{L} X \sim N(0,1)$$

i.e. $\dfrac{\sqrt{n}(\bar{X}_n - \mu)}{S_n} \xrightarrow{L} X \sim N(0,1)$.

(15) $X_1 \dots X_{72}$ r.s. from $f(x) = \begin{cases} \frac{1}{x^2} & x > 1 \\ 0 & \text{o/w} \end{cases}$

Define $Y_i = \begin{cases} 1 & \text{if } X_i < 3 \\ 0 & \text{o/w} \end{cases}$

$$P(Y_i = 1) = P(X_i < 3) = \int_1^3 \frac{1}{x^2}\,dx = \frac{2}{3} = \theta \text{ say}$$

$Y_1, \dots Y_{72}$ are i.i.d $B(1, \theta)$

$$Y = \sum_{i=1}^{72} Y_i \sim B(72, \theta = 2/3)$$

CLT $\Rightarrow$ $\dfrac{Y - 72 \times 2/3}{\sqrt{72 \times \frac{2}{3} \times \frac{1}{3}}} \xrightarrow{L} Z \sim N(0,1)$

i.e. $\dfrac{Y - 48}{4} \xrightarrow{L} Z \sim N(0,1)$

$$P(Y > 50) = 1 - P(Y \le 50) = 1 - P(Y \le 50.5)$$ Continuity correction.

$$= 1 - P\left(\frac{Y-48}{4} \le \frac{50.5-48}{4}\right)$$

$$\approx 1 - \Phi\left(\frac{2.5}{4}\right) = \dots$$

(16) $X_1 \dots X_{100}$ i.i.d $P(3)$

$E(X_1) = 3$; $V(X_1) = 3$; $Y = \sum_{i=1}^{100} X_i \sim P(300)$ $\Rightarrow E(Y) = 300$, $V(Y) = 300$

CLT $\Rightarrow$ $\dfrac{Y - 300}{10\sqrt{3}} \left(= \dfrac{S_n - ES_n}{\sqrt{\operatorname{Var} S_n}}\right) \xrightarrow{L} N(0,1)$

$$P(100 \le Y \le 200) = P(99.5 \le Y \le 200.5)$$ Cont correction

$$= P\left(\frac{99.5 - 300}{10\sqrt{3}} \le \frac{Y - 300}{10\sqrt{3}} \le \frac{200.5 - 300}{10\sqrt{3}}\right)$$

$$\approx \Phi\left(\frac{200.5 - 300}{10\sqrt{3}}\right) - \Phi\left(\frac{99.5 - 300}{10\sqrt{3}}\right).$$

$= \dots$

17 $X \sim \text{bin}(100, 0.6)$

CLT $\Rightarrow$ $$\frac{X - 100 \times 0.6}{\sqrt{100 \times 0.6 \times 0.4}} = \frac{X-60}{\sqrt{24}} \xrightarrow{L} Z \sim N(0,1)$$

$$\Rightarrow P(10 \le X \le 16) = P(9.5 \le X \le 16.5)$$

$$= P\left(\frac{9.5-60}{\sqrt{24}} \le \frac{X-60}{\sqrt{24}} \le \frac{16.5-60}{\sqrt{24}}\right)$$

$$\approx \Phi\left(\frac{16.5-60}{\sqrt{24}}\right) - \Phi\left(\frac{9.5-60}{\sqrt{24}}\right)$$

$$= \ldots\ldots$$

18 X_n has p.d.f

$$f_n(x) = \begin{cases} \frac{1}{\Gamma n} e^{-x} x^{n-1} & x > 0 \\ 0 & \text{o/w} \end{cases}$$

m.g.f of X_n

$$M_{X_n}(t) = \frac{1}{\Gamma n} \int_0^\infty e^{tx} e^{-x} x^{n-1}\, dx$$

$$= \frac{1}{\Gamma n} \int_0^\infty e^{-x(1-t)} x^{n-1}\, dx$$

$$= \frac{1}{(1-t)^n} = (1-t)^{-n}$$

m.g.f of $Y_n = \frac{X_n}{n}$ is

$$M_{Y_n}(t) = E\left(e^{t\frac{X_n}{n}}\right) = \left(1 - \frac{t}{n}\right)^{-n}$$

$$\longrightarrow e^t \text{ as } n \to \infty$$

m.g.f r.v. ~~degen~~ degen at $x=$

$$Y_n \xrightarrow{L} X \text{ (degen at 1)}$$

(19) $f_{X_n}(x) = \begin{cases} \frac{1}{\Gamma p\, \alpha^p} e^{-x/\alpha} x^{p-1} & x > 0 \\ 0 & o/w \end{cases}$ (10)

$\alpha = 2, \ p = 4$

$E(X_n) = \alpha p = 8$; $V(X_n) = \alpha^2 p = 16 = \sigma^2$

$E(\bar{X}) = 8$; $V(\bar{X}) = \frac{16}{n} = \frac{1}{4}$

By CLT $\frac{\sqrt{n}(\bar{X} - \alpha p)}{\sqrt{\alpha^2 p}} \xrightarrow{L} Z \sim N(0,1)$

i.e. $2(\bar{X} - 8) \xrightarrow{L} Z \sim N(0,1)$

$$P(7 < \bar{X} < 9) = P\Big(2(7-8) < 2(\bar{X} - 8) < 2(9-8)\Big)$$
$$\approx P(-2 < Z < 2)$$
$$= \Phi(2) - \Phi(-2) = 2\Phi(2) - 1$$

(20) $X_i \sim U(0,2)$ $E X_i = \frac{1}{2}\int_0^2 x\,dx = 1$

$E X_i^2 = \frac{1}{2}\int_0^2 x^2\,dx = \frac{4}{3}$; $V(X_i) = \frac{1}{3}$

$X_1, \ldots X_n$ i.i.d with $EX_1 = 1$ & $VX_1 = \frac{1}{3}$

By CLT $\sqrt{n}(\bar{X}_n - 1) \xrightarrow{L} N(0, \frac{1}{3})$.

i.e. $\sqrt{n}(Y_n - 1) \xrightarrow{L} N(0, \frac{1}{3})$.