Assignment # 2 Solⁿ.

1. (i) $P(A\cup B|C) = \dfrac{P((A\cup B)\cap C)}{P(C)} = \dfrac{P(AC\cap BC)}{P(C)}$

$= P(A|C) + P(B|C) - P(AB|C)$

(ii) $P(A^c|C) = \dfrac{P(A^cC)}{P(C)} = \dfrac{P(C)-P(AC)}{P(C)} = 1 - P(A|C)$

2. (a) true

(b) $P(A|B) = \dfrac{P(AB)}{P(B)}$; $P(A|B^c) = \dfrac{P(AB^c)}{P(B^c)} = \dfrac{P(A)-P(AB)}{1-P(B)}$

Take $A\subset B$, $P(A)>0$, $P(B-A)>0$

$$P(A|B) + P(A|B^c) = \frac{P(AB)}{P(B)} + \frac{P(AB^c)}{P(B^c)}$$

$$= \frac{P(A)}{P(B)} < 1$$ False.

(c). Take $A\subset B$, i.e. $B^c\subset A^c$

$P(A|B) + P(A^c|B^c)$

$$= \frac{P(AB)}{P(B)} + \frac{P(A^cB^c)}{P(B^c)}.$$

$$= \frac{P(A)}{P(B)} + \frac{P(B^c)}{P(B^c)} > 1$$ false.

$P(A)>0$.

3. (a). $P(A|B) = \frac{1}{4} \Rightarrow P(AB) \neq 0$ False

(b). $A\subset B \Rightarrow P(AB) = P(A)$

given $P(B|A) = \dfrac{P(AB)}{P(A)} = \dfrac{1}{2}$

$\Rightarrow P(AB) \neq P(A)$ False.

(c) $P(A^c \mid B^c) = \dfrac{P(A^c B^c)}{P(B^c)} = \dfrac{1 - P(A) - P(B) + P(AB)}{1 - P(B)}$ — (*)

$P(B|A) = \frac{1}{2}$

& $P(A) = \frac{1}{4}$, $\Rightarrow P(AB) = \frac{1}{8}$

& $P(A|B) = \dfrac{P(AB)}{P(B)} = \dfrac{1/8}{P(B)} = \dfrac{1}{4} \Rightarrow P(B) = \frac{1}{2}$

$(*) \Rightarrow P(A^c \mid B^c) = \dfrac{1 - \frac{1}{4} - \frac{1}{2} + \frac{1}{8}}{1/2} = \dfrac{3}{4}$

4 (a): P(exactly 3 white balls, out of 4).

$= \binom{4}{3} \left(\frac{1}{2}\right)^3 \cdot \frac{1}{2} = \cdots$

(b) A : first ball placed is white

$P(A) = \frac{1}{2}$

B : urn contains exactly 3 white balls

$P(B|A) = \dfrac{P(AB)}{P(A)} = \dfrac{\frac{1}{2} \cdot \binom{3}{2} \left(\frac{1}{2}\right)^3}{1/2} = \dfrac{3}{8}$

5. D : A occurs before B

$P(D) = p_A + p_A p_c + p_A p_c^2 + \cdots$

$= \dfrac{p_A}{1 - p_c} = \dfrac{p_A}{p_A + p_B}$

6. A_i : component i functions

$P(\text{system functions}) = 1 - P(\cap A_i^c)$

$= 1 - (1-p)^n$.

7. A_i : question i is among the 90 questions that the student can answer correctly

reqd prob $P(A_1 A_2 A_3) = P(A_1) P(A_2 | A_1) P(A_3 | A_1 A_2)$

$$= \frac{90}{100} \cdot \frac{89}{99} \cdot \frac{88}{98} \ldots$$

8. Appl^n Bayes th^m

$$\text{reqd prob} = \frac{\frac{1}{2} \times \frac{2}{3}}{\frac{1}{2} \times \frac{2}{3} + \frac{2}{3} \times \frac{1}{3}} = \ldots$$

9. A^+, B^+, C^+, D^+ – events that A, B, C, D passes the paper with the (+) sign

$$\text{reqd prob} = P(A^+ | D^+) = \frac{P(A_+) P(D_+ | A_+)}{P(D^+)}$$

$$P(D^+ | A^+) = \left(\tfrac{1}{3}\right)^3 + \binom{3}{2}\left(\tfrac{2}{3}\right)^2 \cdot \tfrac{1}{3} = \frac{13}{27}$$

Also $P(D^+) = P(D^+ | A^+) P(A^+) + P(D^+ | A^c_+) P(A^c_+)$

$P(D^+ | A^c_+) = P(D \text{ passes with } + | A \text{ passes } -)$

$$= \binom{3}{1} \frac{2}{3} \left(\frac{1}{3}\right)^2 + \binom{3}{3} \left(\frac{2}{3}\right)^3 \left(\frac{1}{3}\right)^0 = \frac{14}{27}$$

$$P(D^+) = \frac{13}{27} \cdot \frac{1}{3} + \frac{14}{27} \cdot \frac{2}{3} = \frac{41}{81}$$

$$\Rightarrow P(A^+ | D^+) = \frac{13/27 \cdot 1/3}{41/81} = \frac{13}{41}$$

(10) $P\left(\begin{array}{l}\text{Silver coin in other} \\ \end{array} \middle| \text{Gold coin in one drawer}\right)$.

$$= \frac{1/3 \times 1/2}{1/3 \times 1 + \frac{1}{3} \times \frac{1}{2} + \frac{1}{3} \times 0} = \frac{1}{3}.$$

(11) (a) $P(c_1 c_2 c_3 c_4) = \prod_1^4 P(c_i) = - - -$.

(b) $P(c_1^c \cap c_2^c \cap c_3^c \cap c_4^c) = \prod_1^4 P(c_i^c)$ [explain why it is so]

$= - - -$

(c) $P(c_1 c_2^c c_3^c c_4^c) + P(c_1^c c_2 c_3^c c_4^c)$

$+ P(c_1^c c_2^c c_3 c_4^c) + P(c_1^c c_2^c c_3^c c_4)$.

$= P(c_1) \prod_2^4 P(c_i^c) + - - -$.

(d). $P(\text{at least one hits}) = 1 - P(\text{no one hits})$

$= 1 - P(c_1^c c_2^c c_3^c c_4^c)$.

$= - - -$

(12) $P(\bigcap_1^n A_i^c) = \prod_1^n P(A_i^c)$

$= \prod_1^n (1 - P(A_i)) \leq \prod_1^n \exp(-P(A_i))$

$\left[\begin{array}{l} 0 < x < 1 \\ 1 - x < e^{-x} \end{array}\right]$

i.e. $P(\bigcap_1^n A_i^c) \leq \exp(-\sum_1^n P(A_i))$.

13 $\Omega = \{1, 2, 3, 4\}$ $\mathcal{F}$: power set

$P(\{i\}) = \frac{1}{4}$ $i = 1, 2, 3, 4$

$A = \{1, 4\}$, $B = \{2, 4\}$, $C = \{3, 4\}$.

$P(A) = P(B) = P(C) = \frac{1}{2}$

$P(AB) = P(AC) = P(BC) = \frac{1}{4}$; $P(ABC) = \frac{1}{4}$

$\Rightarrow$ $P(AB) = P(A)\ P(B)$, $P(AC) = P(A)\ P(C)$

& $P(BC) = P(B)\ P(C)$.

i.e A, B, C are pair wise indep

but $P(ABC) = \frac{1}{4} \neq P(A) . P(B)\ P(C) = \frac{1}{8}$

$\Rightarrow$ A, B, C not mutually indep.

14 In prev prob setup take

$A = \{1, 2\}$, $B = \{3, 4\}$, $C = \{1\}$.

$P(A \mid B) < P(A)$

$P(B \mid C) < P(B)$ but $P(A \mid C) > P(A)$.

(15)

A_i : i girls are in the list
$i = 0, 1, 2, 3$

B : 1st student is girl

C : 2nd student is boy :- to obtain $P(C|B)$

$$P(B) = P(A_1)\,P(B|A_1) + P(A_2)\,P(B|A_2) + P(A_3)\,P(B|A_3)$$

$$= \frac{\binom{3}{1}\binom{5}{3}}{\binom{8}{4}} \times \frac{1}{4} + \frac{\binom{3}{2}\binom{5}{2}}{\binom{8}{4}} \times \frac{2}{4} + \frac{\binom{3}{3}\binom{5}{1}}{\binom{8}{4}} \times \frac{3}{4}$$

$$P(B) = \frac{105}{4 \cdot \times \binom{8}{4}}.$$

$$P(C|B) = \frac{P(CB)}{P(B)} = \frac{P\left(C \cap \left(\bigcup_1^3 A_i B\right)\right)}{P(B)} = \frac{P\left(\bigcup_1^3 C A_i B\right)}{P(B)}$$

$$= \sum_{i=1}^{3} \frac{P(CA_iB)}{P(B)} = \sum_1^3 P(C|A_iB)\,\frac{P(A_iB)}{P(B)}$$

$$= \sum_1^3 P(C|A_iB)\;P(A_i|B)$$

$$= 1 \times P(A_1|B) + \frac{2}{3} \times P(A_2|B) + \frac{1}{3} \times P(A_3|B)$$

$$\left[\begin{array}{l} P(A_1|B) = \dfrac{P(A_1)\,P(B|A_1)}{P(B)} = \dfrac{2}{7} \\ P(A_2|B) = \dfrac{P(A_2)\,P(B|A_2)}{P(B)} = \dfrac{4}{7} \quad \& \; P(A_3|B) = \dfrac{1}{7} \end{array}\right]$$

$$\Rightarrow P(C|B) = 1 \times \frac{2}{7} + \frac{2}{3} \times \frac{4}{7} + \frac{1}{3} \times \frac{1}{7}$$

$$= - - -$$

(16) A & B are in series

C & D in parallel

(a) P(the system works)

$= P(A \cap B \cap (C \cup D))$

$= P(A \cap B)\ P(C \cup D)$.

$= P(A)\ P(B)\ \big(P(C) + P(D) - P(C)\ P(D)\big)$

$= 0.9 \times 0.9\ (0.8 + 0.8 - 0.8 \times 0.8)$

$= \ldots$

(b) P(C is not working | system is working)

$$= \frac{P(\text{C is not working} \cap \text{system is working})}{P(\text{system is working})}$$

$$= \frac{P(A \cap B \cap C^c \cap D)}{P(\text{system is working})} \leftarrow \text{from (a)}$$

$$= \frac{P(A)\ P(B)\ P(C^c)\ P(D)}{P(\text{system is working})}$$

(contd)

(17) A_i : event that a fly survives i^{th} application

$i = 1, 2, 3, 4.$

Note that $A_4 \subset A_3 \subset A_2 \subset A_1$

$$\Rightarrow A_4 = A_1 \cap A_2 \cap A_3 \cap A_4$$

(a) reqd prob = P(a fly survives 4 applications)

$$= P(A_1 A_2 A_3 A_4)$$

$$= P(A_4)$$

$$= P(A_1)\, P(A_2 | A_1)\, P(A_3 | A_1 A_2)\, P(A_4 | A_1 A_2 A_3)$$

$$= (1-0.8)\ (1-0.4)\ (1-0.2)\ (1-0.1)$$

(from the given conditions)

$$= 0.2 \times 0.6 \times 0.8 \times 0.9$$

(b) $$P(A_4 | A_1) = \frac{P(A_4 \cap A_1)}{P(A_1)} = \frac{P(A_4))}{P(A_1)}$$

$$= 0.6 \times 0.8 \times 0.9.$$

(18) B_i : event that i of the paintings are forgeries

$i = 0(1)5$

$P(B_0) = 0.76$, $P(B_1) = 0.09$, $P(B_2) = 0.02$, $P(B_3) = 0.01$

$P(B_4) = 0.02$ & $P(B_5) = 0.1$ (given condⁿs)

A : event that the painting sent for authentication turns out to be a forgery.

$$\text{reqd prob} = P(B_5 \mid A) \underset{\substack{\uparrow \\ \text{Bayes th}^m}}{=} \frac{P(B_5)\,P(A \mid B_5)}{\sum_{i=0}^{5} P(B_i)\,P(A \mid B_i)}.$$

$$P(A) = \sum_{i=0}^{5} P(B_i)\,P(A \mid B_i)$$

$$= 0.76 \times 0 + 0.09 \times \frac{1}{5} + 0.02 \times \frac{2}{5}$$

$$+ 0.01 \times \frac{3}{5} + 0.02 \times \frac{4}{5} + 0.10 \times 1$$

$$= \ldots$$

$$P(B_5 \mid A) = \frac{0.10 \times 1}{P(A)} = \ldots$$

(19) A_n : event that family has n children

$$P(A_n) = \alpha p^n$$

B_K : event that family has K boys.

(a)
$$P(B_K) = \sum_{n=0}^{\infty} P(A_n)\, P(B_K | A_n)$$

$$= \sum_{n=K}^{\infty} P(A_n)\, P(B_K | A_n)$$

$$= \sum_{n=K}^{\infty} \alpha p^n \binom{n}{K} \left(\frac{1}{2}\right)^n$$

$$= \alpha p^K \binom{K}{K} \left(\frac{1}{2}\right)^K + \alpha p^{K+1} \binom{K+1}{K} \left(\frac{1}{2}\right)^{K+1} + \alpha p^{K+2} \binom{K+2}{K} \left(\frac{1}{2}\right)^{K+2} + \cdots$$

$$= \alpha p^K \left(\frac{1}{2}\right)^K \left(1 + (K+1)\frac{p}{2} + \frac{(K+1)(K+2)}{2!}\left(\frac{p}{2}\right)^2 + \cdots\right)$$

$$= \alpha p^K \left(\frac{1}{2}\right)^K \left(1 - \frac{p}{2}\right)^{-(K+1)}$$

$$= 2\alpha p^K (2-p)^{-(K+1)}$$

(b) req^d cond^{n}al prob $= P\left(\bigcup_{K=2}^{\infty} B_K \,\middle|\, \bigcup_{K=1}^{\infty} B_K\right)$

$$= \frac{P\left(\left(\bigcup_{K=2}^{\infty} B_K\right) \cap \left(\bigcup_{K=1}^{\infty} B_K\right)\right)}{P\left(\bigcup_{1}^{\infty} B_K\right)}$$

$$= P\left(\bigcup_{2}^{\infty} B_K\right) \Big/ P\left(\bigcup_{1}^{\infty} B_K\right)$$

$$= \sum_{k=2}^{\infty} P(B_k) \Big/ \sum_{k=1}^{\infty} P(B_k)$$

$$\sum_{2}^{\infty} P(B_k) = \frac{2\alpha p^2}{(2-p)^3}\left(1 + \frac{p}{2-p} + \left(\frac{p}{2-p}\right)^2 + \cdots\right)$$

$$\sum_{1}^{\infty} P(B_k) = \frac{2\alpha p}{(2-p)^2}\left(1 + \frac{p}{2-p} + \left(\frac{p}{2-p}\right)^2 + \cdots\right)$$

$\Rightarrow$ req. prob $= \dfrac{p}{2-p}$.