

①

$$E(X) = \frac{1}{\beta} \int_0^{\infty} x e^{-\beta/x} dx = \beta$$

$$\Rightarrow E(\bar{X}) = E\left(\frac{1}{n} \sum X_i\right) = \frac{1}{n} \sum E(X_i) = \beta$$

$$\Rightarrow \bar{X} \text{ is u.e. of } \beta$$

②

$$f_{X_{(n)}}(x) = \begin{cases} \frac{n}{\theta^n} x^{n-1} & 0 < x < \theta \\ 0 & \text{o/w} \end{cases}$$

$$E(X_{(n)}) = \frac{n}{\theta^n} \int_0^{\theta} x^n dx = \frac{n}{n+1} \theta$$

$$\Rightarrow E\left(\frac{n+1}{n} X_{(n)}\right) = \theta$$

$$\Rightarrow \frac{n+1}{n} X_{(n)} \text{ is u.e. of } \theta$$

Also

$$f_X(x) = \begin{cases} \frac{1}{\theta} & 0 < x < \theta \\ 0 & \text{o/w} \end{cases}$$

$$E(X) = \frac{\theta}{2}$$

$$\Rightarrow E(2\bar{X}) = E\left(\frac{2}{n} \sum X_i\right) = \frac{2}{n} \sum E(X_i) = \theta$$

$$\Rightarrow 2\bar{X} \text{ is u.e. for } \theta$$

③

$$E(X) = \beta \int_0^{\infty} x e^{-\beta x} dx = \frac{1}{\beta}$$

$$E(\bar{X}) = \beta \Rightarrow \bar{X} \text{ is u.e. of } \beta.$$

$$\left(E(\bar{X}) = E\left(\frac{1}{n} \sum X_i\right) = \frac{1}{n} \sum E(X_i) \right)$$

$$(4) \quad T_1 = \sum X_i, \quad T_2 = \sum X_i^2$$

$$E(T_1^2) = V(T_1) + E^2(T_1) \\ = n\theta^2 + n^2\theta^2 = \theta^2 n(n+1)$$

$$\Rightarrow E\left(\frac{T_1^2}{n(n+1)}\right) = \theta^2$$

$$\Rightarrow \frac{T_1^2}{n(n+1)} \text{ is u.e. of } \theta^2$$

$$E(T_2) = E\left(\sum X_i^2\right) = \sum E(X_i^2) \\ = \sum (V(X_i) + E^2(X_i)) \\ = \sum (\theta^2 + \theta^2) = 2n\theta^2$$

$$\Rightarrow E\left(\frac{T_2}{2n}\right) = \theta^2$$

$$\Rightarrow \frac{T_2}{2n} \text{ is u.e. of } \theta^2$$

$$(5) \quad g(\theta) = \theta e^{-2\theta}$$

$$\delta_0(x) = \begin{cases} 1 & \text{if } x_1=0, x_2=1 \\ 0 & \text{o/w} \end{cases}$$

$$E(\delta_0(x)) = 1 \cdot P(x_1=0, x_2=1) \\ = P(x_1=0) P(x_2=1) \\ = e^{-\theta} \cdot \frac{e^{-\theta}\theta}{1!} = \theta e^{-2\theta}$$

$$\Rightarrow \delta_0(x) \text{ is u.e. of } \theta e^{-2\theta}$$

$$(6) \quad x_1, \dots, x_n \text{ i.i.d. } B(1, \theta)$$

$$\sum X_i \sim B(n, \theta)$$

$$E(T(x)) = \frac{\frac{1}{2}\sqrt{n} + E(\sum X_i)}{n + \sqrt{n}} = \frac{\frac{1}{2}\sqrt{n} + n\theta}{n + \sqrt{n}} \neq \theta$$

$$\Rightarrow T(x) \text{ is not u.e. of } \theta$$

$$\lim_{n \rightarrow \infty} E(T(\underline{x})) = \lim_{n \rightarrow \infty} \frac{\frac{\sqrt{n}}{2} + n\theta}{n + \sqrt{n}} = \theta$$

$\Rightarrow T(\underline{x})$ is unbiased in the limit for θ

⑦ X_1, \dots, X_n r.s. from $N(\mu, \sigma^2)$

$$\Rightarrow \bar{X} \sim N(\mu, \sigma^2/n)$$

$$Y = \frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1} \quad \text{indep}$$

If $Z \sim \chi^2_m$, then

$$\begin{aligned} E\left(\frac{1}{Z}\right) &= \frac{1}{2^{m/2} \Gamma(m/2)} \int_0^\infty z^{-1} e^{-z/2} z^{m/2-1} dz \\ &= \frac{1}{2^{m/2} \Gamma(m/2)} \int_0^\infty e^{-z/2} z^{\frac{m}{2}-1-1} dz \\ &= \frac{\sqrt{\frac{m}{2}-1}}{2^{m/2} \Gamma(m/2)} = \frac{1}{m-2} \end{aligned}$$

$$\begin{aligned} \& E\left(\frac{1}{\sqrt{Z}}\right) &= \frac{1}{2^{m/2} \Gamma(m/2)} \int_0^\infty e^{-z/2} z^{\frac{m}{2}-\frac{1}{2}-1} dz \\ &= \frac{\sqrt{\frac{m-1}{2}}}{2^{m/2} \Gamma(m/2)} = \frac{\sqrt{\frac{m-1}{2}}}{\sqrt{2} \Gamma(\frac{m}{2})} \end{aligned}$$

$$\Rightarrow E\left(\frac{1}{Y}\right) = E\left(\frac{\sigma^2}{(n-1)S^2}\right) = \frac{1}{(n-1)-2} = \frac{1}{n-3}$$

$$\Rightarrow E\left(\frac{1}{S^2}\right) = \frac{n-1}{n-3} \cdot \frac{1}{\sigma^2}$$

(4)

$$2 \ E \left(\frac{1}{\sqrt{y}} \right) = E \left(\frac{\sigma}{\sqrt{n-1} \ s} \right) = \frac{\sqrt{\frac{n-2}{2}}}{\sqrt{2} \ \sqrt{\frac{n-1}{2}}}$$

$$\Rightarrow E \left(\frac{1}{s} \right) = \frac{\sqrt{n-1} \ \sqrt{\frac{n-2}{2}}}{\sqrt{2} \ \sqrt{\frac{n-1}{2}}} \cdot \frac{1}{\sigma}$$

Since \bar{X} & s^2 are indep.

$$E \left(\frac{\bar{X}}{s^2} \right) = E(\bar{X}) \cdot E \left(\frac{1}{s^2} \right)$$

$$= \mu \cdot \frac{n-1}{n-3} \cdot \frac{1}{\sigma^2}$$

$$\Rightarrow E \left(\frac{n-3}{n-1} \cdot \frac{\bar{X}}{s^2} \right) = \frac{\mu}{\sigma^2}$$

$\Rightarrow \frac{n-3}{n-1} \cdot \frac{\bar{X}}{s^2}$ is an unbiased estimator of $\frac{\mu}{\sigma^2}$.

Further

$$E \left(\frac{\bar{X}}{s} \right) = E(\bar{X}) \cdot E \left(\frac{1}{s} \right)$$

$$= \mu \cdot \frac{\sqrt{n-1} \ \sqrt{\frac{n-2}{2}}}{\sqrt{2} \ \sqrt{\frac{n-1}{2}}} \cdot \frac{1}{\sigma}$$

$$\Rightarrow E \left(\sqrt{\frac{2}{n-1}} \cdot \frac{\sqrt{\frac{n-1}{2}}}{\sqrt{\frac{n-2}{2}}} \cdot \frac{\bar{X}}{s} \right) = \frac{\mu}{\sigma}$$

$\Rightarrow \sqrt{\frac{2}{n-1}} \cdot \frac{\sqrt{\frac{n-1}{2}}}{\sqrt{\frac{n-2}{2}}} \cdot \frac{\bar{X}}{s}$ is an unbiased estimator

of $\frac{\mu}{\sigma}$.

(5)

(8) x_1, \dots, x_n are i.i.d $B(1, \theta)$

$$g(\theta) = \theta^2 (1-\theta)$$

Define
$$g(\underline{x}) = \begin{cases} 1 & \text{if } x_1=1, x_2=1, x_3=0 \\ 0 & \text{o/w} \end{cases}$$

$$\begin{aligned} E_{\theta} g(\underline{x}) &= P(x_1=1, x_2=1, x_3=0) \\ &= P(x_1=1) P(x_2=1) P(x_3=0) \\ &= \theta^2 (1-\theta) \end{aligned}$$

$\Rightarrow g(\underline{x})$ is an u.e. of $g(\theta) = \theta^2 (1-\theta)$.

(9) (a) $f(x|\alpha) = \frac{1}{\alpha} e^{-\frac{x}{\alpha}} ; x > 0$

It p.d.f

$$f(\underline{x}|\alpha) = \frac{1}{\alpha^n} e^{-\frac{1}{\alpha} \sum x_i} \quad x_1, \dots, x_n > 0$$

$$= \left(\frac{1}{\alpha^n} e^{-\frac{1}{\alpha} \sum x_i} \right) \cdot 1.$$

$$\text{i.e. } f(\underline{x}|\alpha) = g\left(\alpha, \sum_{i=1}^n x_i\right) \cdot h(\underline{x}) \quad (h(\underline{x})=1).$$

By NFFT, $T(\underline{x}) = \sum_{i=1}^n x_i$ is suff for α .

$$9(b) \quad f(x|\beta) = e^{-(x-\beta)} \quad x > \beta$$

$$f(\underline{x}|\beta) = \begin{cases} e^{-\sum (x_i - \beta)} & , \quad x_1, \dots, x_n > \beta \\ 0 & \text{o/w} \end{cases}$$

$$\text{i.e., } f(\underline{x}|\beta) = \begin{cases} e^{-\sum x_i + n\beta} & , \quad x_{(n)} > \beta \\ 0 & \text{o/w} \end{cases}$$

$$\begin{aligned} \text{i.e., } f(\underline{x}|\beta) &= e^{n\beta - \sum x_i} I(\beta, x_{(n)}) \quad \left[I(a,b) = \begin{cases} 1 & a < b \\ 0 & \text{o/w} \end{cases} \right] \\ &= (e^{-\sum x_i}) (e^{n\beta} I(\beta, x_{(n)})) \\ &= h(\underline{x}) \quad g(\beta, x_{(n)}) \end{aligned}$$

By NFFT, $T(\underline{x}) = x_{(n)}$ is a suff statistic

$$\begin{aligned} 9(c) \quad f(\underline{x}|\alpha, \beta) &= \frac{1}{\alpha^n} \exp\left(-\frac{\sum x_i}{\alpha} + \frac{n\beta}{\alpha}\right) I(\beta, x_{(n)}) \\ &= \left(\frac{1}{\alpha^n} \exp\left(-\frac{\sum x_i}{\alpha} + \frac{n\beta}{\alpha}\right) \cdot I(\beta, x_{(n)}) \right) \cdot 1 \\ &= g(\alpha, \beta; (\sum x_i, x_{(n)})) \cdot h(\underline{x}) \end{aligned}$$

By NFFT, $T(\underline{x}) = (\sum x_i, x_{(n)})$ is jointly sufficient for (α, β) .

9 (d) $f(\underline{x} | \mu, \sigma) = \left(\frac{1}{\sigma \sqrt{2\pi}} \right)^n \left[\prod_{i=1}^n \left(\frac{1}{x_i} \right) \right] \cdot \exp \left(-\frac{1}{2\sigma^2} \sum (\log x_i - \mu)^2 \right)$ (7)

$$= \left(\frac{1}{\sigma^n} \exp \left(-\frac{n\mu^2}{2\sigma^2} - \frac{1}{2\sigma^2} \sum (\log x_i)^2 + \frac{\mu}{\sigma^2} \sum \log x_i \right) \right)$$

$$\times \left(\left(\frac{1}{\sqrt{2\pi}} \right)^n \prod_{i=1}^n x_i^{-1} \right)$$

$$= g(\mu, \sigma; \left(\sum \log x_i, \sum (\log x_i)^2 \right)) \cdot h(\underline{x})$$

" $\left(\left(\frac{1}{\sqrt{2\pi}} \right)^n \prod_{i=1}^n x_i^{-1} \right)$

By NFFT $(\sum \log x_i, \sum (\log x_i)^2)$ is jointly sufficient for (μ, σ)

(10) It p.d.f of x_1 & x_2

$$f(x_1, x_2) = \theta e^{-\theta x_1} \cdot 2\theta e^{-2\theta x_2}$$

$$f(x_1, x_2) = 2\theta^2 e^{-\theta(x_1 + 2x_2)}$$

$$f(x_1, x_2) = (\theta^2 e^{-\theta(x_1 + 2x_2)}) (2)$$

By NFFT $T(x_1, x_2) = x_1 + 2x_2$ is suff for θ

9 (e) $f(\underline{x} | \theta) = \begin{cases} \frac{1}{\theta^n} & ; -\theta/2 < x_1, \dots, x_n < \theta/2 \\ 0 & \text{o/w} \end{cases}$

$$= \begin{cases} \frac{1}{\theta^n} & ; |x_i| < \theta/2 ; i=1, 2, \dots, n \\ 0 & \text{o/w} \end{cases}$$

$$\text{i.e. } f(\underline{x}|\theta) = \begin{cases} \frac{1}{\theta^n} & ; \quad \text{Max}_i |x_i| < \theta/2 \\ 0 & \text{o/w} \end{cases} \quad (8)$$

$$\Rightarrow f(\underline{x}|\theta) = \frac{1}{\theta^n} I(\text{Max}_i |x_i|, \theta/2)$$

$$\Rightarrow \text{By NFFT } T(\underline{x}) = \text{Max}_i |x_i| \text{ is suff for } \theta.$$

(ii) jt p.d.f

$$f(\underline{x}|\theta) = \begin{cases} \prod_{i=1}^n \exp(i\theta - x_i) & \text{if } \frac{x_1}{1}, \frac{x_2}{2}, \dots, \frac{x_n}{n} \geq \theta \\ 0 & \text{o/w} \end{cases}$$

$$= \begin{cases} e^{-\sum_{i=1}^n x_i} e^{\theta \frac{n(n+1)}{2}} & \text{if } \text{Min}_i \frac{x_i}{i} \geq \theta \\ 0 & \text{o/w} \end{cases}$$

$$\text{i.e. } f(\underline{x}|\theta) = \left(e^{\theta \frac{n(n+1)}{2}} I\left(\theta, \text{Min}_i \frac{x_i}{i}\right) \right) \left(e^{-\sum_{i=1}^n x_i} \right)$$

$$= \underbrace{g\left(\theta, \text{Min}_i \frac{x_i}{i}\right)}_{\downarrow} \times \underbrace{h(\underline{x})}_{\downarrow}$$

$$\Rightarrow T(\underline{x}) = \text{Min}_i \frac{x_i}{i} \text{ is suff.}$$

(12) X_1, \dots, X_n i.i.d Beta(α, β)

(9)

(a) β is known - α is the unknown parameter

$$f(\underline{x}|\alpha) = \begin{cases} \left[\left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \right)^n (\prod x_i)^{\alpha-1} \right] \underbrace{\left[(\prod (1-x_i))^{\beta-1} \left(\frac{1}{\Gamma(\beta)} \right)^n \right]}_{h(\underline{x})} & 0 < x_1, \dots, x_n < 1 \\ 0 & \text{o/w} \end{cases}$$

By NFFT $\left(\prod_{i=1}^n x_i \right)$ is suff for α

(b) α is known - β is the unknown parameter

$$f(\underline{x}|\beta) = \begin{cases} \left[\left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)} \right)^n (\prod (1-x_i))^{\beta-1} \right] \underbrace{\left[(\prod x_i)^{\alpha-1} \left(\frac{1}{\Gamma(\alpha)} \right)^n \right]}_{h(\underline{x})}; & 0 < x_1, \dots, x_n < 1 \\ 0 & \text{o/w} \end{cases}$$

By NFFT $\prod_{i=1}^n (1-x_i)$ is suff for β

(c) α, β both unknown

$$f(\underline{x}|\alpha, \beta) = \begin{cases} \left[\left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \right)^n (\prod x_i)^{\alpha-1} (\prod (1-x_i))^{\beta-1} \right] & 0 < x_1, \dots, x_n < 1 \\ 0 & \text{o/w} \end{cases}$$

By NFFT $(\prod x_i, \prod (1-x_i))$ is jointly sufficient for (α, β) .

(13) T is suff for $\theta \in \Theta$ & $T = \psi(T^*)$

(10)

By NFFT, T is suff for θ iff.

$$f(\underline{x}|\theta) = g(\theta, t(\underline{x})) \cdot h(\underline{x})$$

$$\text{i.e. } f(\underline{x}|\theta) = g(\theta, \psi(t^*(\underline{x}))) \cdot h(\underline{x})$$

$$f(\underline{x}|\theta) = g'(\theta, t^*(\underline{x})) \cdot h(\underline{x})$$

$\Rightarrow T^*(\underline{x})$ is suff for θ .

(14)
$$f(\underline{x}|\theta) = \begin{cases} 1, & \theta - \frac{1}{2} < x_{(1)}, \dots, x_{(n)} < \theta + \frac{1}{2} \\ 0, & \text{o.w.} \end{cases}$$

$$\text{i.e. } f(\underline{x}|\theta) = I_{(\theta - \frac{1}{2}, x_{(1)})} I_{(x_{(n)}, \theta + \frac{1}{2})} \\ = g(\theta, (x_{(1)}, x_{(n)})) \cdot h(\underline{x})$$

By NFFT, $T(\underline{x}) = (x_{(1)}, x_{(n)})$ is jointly suff for θ

(15)
$$f(\underline{x}|\theta) = (\theta e^{-\theta x_1}) (2\theta e^{-2\theta x_2}) \dots (n\theta e^{-n\theta x_n})$$

$$\text{i.e. } f(\underline{x}|\theta) = \theta^n \left(\prod_{i=1}^n i \right) e^{-\theta \sum_{i=1}^n i x_i}$$

$$= \left(\prod_{i=1}^n i \right) \left(\theta^n e^{-\theta \sum_{i=1}^n i x_i} \right)$$

$$= h(\underline{x}) g(\theta, \sum_{i=1}^n i x_i)$$

By NFFT, $T(\underline{x}) = \sum_{i=1}^n i x_i$ is sufficient for θ .