MSO 201 A : Homework 4

[1] Find the expected number of throws of a fair die required to obtain a 6.

[2] Consider a sequence of independent coin flips, each of which has a probability p of being heads. Define a random variable X as the length of the run (of either heads or tails) started by the first trial. Find $E(X)$.

[3] Verify whether $E(X)$ exists in the following cases:

(a) X has the p.m.f. $P(X=x)=\begin{cases}\left(x(x+1)\right)^{-1}, & \text{if } x=1,2,\ldots\\ 0, & \text{otherwise.}\end{cases}$

(b) X has the p.d.f. $f(x)=\begin{cases}\left(2x^2\right)^{-1}, & \text{if } |x|>1,\\ 0, & \text{otherwise.}\end{cases}$

(c) X (Cauchy r.v.) has the p.d.f. $f(x)=\dfrac{1}{\pi}\dfrac{1}{1+x^2}; -\infty<x<\infty.$

[4] Find the mean and variance of the distributions having the following p.d.f. / p.m.f.

(a) $f(x)=a\,x^{a-1}, 0<x<1, a>0.$

(b) $f(x)=1/n;\ x=1,2,\ldots,n;\ n>0$ is an integer

(c) $f(x)=\dfrac{3}{2}(x-1)^2;\ 0<x<2$

[5] Find the mean and variance of the Weibull random variable having the p.d.f.

$$f(x)=\begin{cases}\dfrac{c}{a}\left(\dfrac{x-\mu}{a}\right)^{c-1}\exp\left\{-\left(\dfrac{x-\mu}{a}\right)^{c}\right\} & \text{if } x>\mu\\ 0 & \text{otherwise.}\end{cases}$$

Where, $c>0$, $a>0$ and $\mu\in(-\infty,\infty)$.

[6] A median of a distribution is a value m such that $P(X\geq m)\geq 1/2$ and $P(X\leq m)\geq 1/2$, with equality for a continuous distribution. Find the median of the distribution with p.d.f. $f(x)=3x^2, 0<x<1; =0$, otherwise.

[7] Let X be a continuous, nonnegative random variable with d.f. $F(x)$. Show that $E(X)=\int_0^\infty\left(1-F(x)\right)dx.$

[8] A target is made of three concentric circles of radii $1/\sqrt{3}$, 1, $\sqrt{3}$ feet. Shots within the inner circle give 4 points, within the next ring 3 points and within the third ring 2 points. Shots outside the target give 0. Let X be the distance of the hit from the centre (in feet) and let the p.d.f. of X be

$$f(x)=\begin{cases}\dfrac{2}{\pi\left(1+x^2\right)} & x>0\\ 0 & \text{otherwise.}\end{cases}$$

What is the expected value of the score in a single shot?

[9] Find the moment generating function (m.g.f.) for the following distributions

(a) X is a (Binomial r.v.) discrete random variable with p.m.f.

$$P(X=x)=\begin{cases}\begin{pmatrix}n\\x\end{pmatrix}p^x(1-p)^{n-x} & x=0,1,2,\ldots,n\\ 0, & \text{otherwise.}\end{cases}$$

n is a positive integer.

(b) X is a (Poisson r.v.) discrete random variable with p.m.f.

$$P(X=x)=\begin{cases}\dfrac{e^{-\lambda}\lambda^x}{x!}, & x=0,1,2,\ldots\\ 0, & \text{otherwise.}\end{cases}$$

$\lambda>0.$

(c) X is a (Gamma r.v.) continuous random variable with p.d.f.

$$f_X(x)=\begin{cases}\dfrac{e^{-x/\beta}\,x^{\alpha-1}}{\Gamma(\alpha)\,\beta^\alpha}, & x>0\\ 0, & \text{otherwise.}\end{cases}$$

$\alpha,\beta>0.$

Find $E(X)$ and $V(X)$ from the m.g.f.s.

[10] The m.g.f. of a random variable X is given by

$$M_X(t)=\frac{1}{2}e^{-5t}+\frac{1}{6}e^{4t}+\frac{1}{8}e^{5t}+\frac{5}{24}e^{25t}$$

Find the distribution function of the random variable.

[11] Let X be a random variable with $P(X\le 0)=0$ and let $\mu=E(X)$ exists. Show that $P(X\ge 2\mu)\le 0.5.$

[12] Let X be a random variable with $E(X)=3$ and $E\left(X^2\right)=13$, determine a lower bound for $P(-2<X<8)$.

[13] Let X be a random variable with p.m.f.

$$P(X=x)=\begin{cases}1/8 & x=-1,1\\ 6/8 & x=0\\ 0 & \text{otherwise.}\end{cases}$$

Using the p.m.f., show that the bound for Chebychev's inequality cannot be improved.

[14] A communication system consists of n components, each of which will independently function with probability p. The system will be able to operate effectively if at least one-half of its components function.

(a) For what value of p a 5-component system is more likely to operate effectively than a 3-component system?

(b) In general, when is a $(2k+1)$-component system better than a $(2k-1)$-component system?

[15] An interviewer is given a list of 8 people whom he can attempt to interview. He is required to interview exactly 5 people. If each person (independently) agrees to be interviewed with probability $2/3$, what is the probability that his list will enable him to complete his task?

[16] A pipe-smoking mathematician carries at all times 2 match boxes, 1 in his left-hand pocket and 1 in his right-hand pocket. Each time he needs a match he is equally likely to take it from either pocket. Consider the moment when the mathematician first discovers that one of his matchboxes is empty. If it is assumed that both matchboxes initially contained N matches, what is the probability that there are exactly k matches in the other box, $k = 0, 1, \ldots, N$?