Problem Set #6

(1) $X \sim U(0,1)$ $\quad F_X(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le 1 \\ 1 & x > 1 \end{cases}$

Solutions using d.f. Method

(a) $Y = \sqrt{X}$

d.f. of Y : $F_Y(y) = P(\sqrt{X} \le y)$

$$= P(X \le y^2) = \begin{cases} 0 & y < 0 \\ y^2 & 0 \le y \le 1 \\ 1 & y > 1 \end{cases}$$

p.d.f. of Y is $f_Y(y) = \begin{cases} 0, & y < 0 \\ 2y, & 0 \le y \le 1 \\ 1, & y > 1 \end{cases} = \begin{cases} 2y, & 0 \le y \le 1 \\ 0, & o/w. \end{cases}$

(b) $Y = X^2$

d.f. of Y : $F_Y(y) = P(X^2 \le y)$

$$= \begin{cases} 0, & y < 0 \\ P(-\sqrt{y} \le X \le \sqrt{y}), & 0 \le y \le 1 \\ 1, & y > 1 \end{cases}$$

for $0 \le y \le 1$

$$P(-\sqrt{y} \le X \le \sqrt{y}) = P(0 \le X \le \sqrt{y}) = F_X(\sqrt{y}) = \sqrt{y}$$

$$\Rightarrow \quad F_Y(y) = \begin{cases} 0, & y < 0 \\ \sqrt{y}, & 0 \le y \le 1 \\ 1, & y > 1 \end{cases}$$

p.d.f. of Y : $f_Y(y) = \begin{cases} \dfrac{1}{2\sqrt{y}} & 0 \le y \le 1 \\ 0 & o/w. \end{cases}$

(c) $Y = 2X + 3 \rightarrow (3,5)$

d.f. of Y : $F_Y(y) = P(2X+3 \le y)$

$$= P\left(X \le \frac{y-3}{2}\right) = \begin{cases} 0, & y < 3 \\ \frac{y-3}{2}, & 3 \le y \le 5 \\ 1 & y > 5 \end{cases}$$

p.d.f. $f_Y(y) = \begin{cases} \frac{1}{2}, & 3 \le y \le 5 \\ 0 & o/w \end{cases} \Rightarrow Y \sim U(3,5)$

(d) $Y = -\lambda \log X \rightarrow (0, \infty)$

$$F_Y(y) = P(Y \le y) = P(-\lambda \log X \le y) = P(X > e^{-y/\lambda})$$

$$= 1 - P(X \le e^{-y/\lambda})$$

i.e. $F_Y(y) = \begin{cases} 0 & y < 0 \\ 1 - e^{-y/\lambda} & y \geq 0 \end{cases}$

p.d.f of Y : $f_Y(y) = \begin{cases} \frac{1}{\lambda} e^{-y/\lambda}, & y \geq 0 \\ 0 & o/w \end{cases}$

i.e $Y \sim Exp(\lambda)$ (scale λ)

(2) $X \sim U(0, \theta)$

$Y = \min(X, \theta/2) \rightarrow (0, \theta/2) \leftarrow$ range sp of Y

$$F_Y(y) = P(Y \le y) = P(\min(X, \theta/2) \le y)$$

$$= 1 - P(\min(X, \theta/2) > y)$$

$$= 1 - P(X > y \cap \theta/2 > y)$$

Now $P(X > y, \theta/2 > y) = 1$ If $y < 0$

$= 0$ If $y \geq \theta/2$

for $0 \leq y < \theta/2$; $P(X > y, \theta/2 > y) = P(X > y)$

$$= \frac{1}{\theta} \int_y^\theta dx = \frac{\theta - y}{\theta}$$

$$\Rightarrow F_Y(y) = \begin{cases} 0, & y < 0 \\ y/\theta, & 0 \leq y < \theta/2 \\ 1, & y \geq \theta/2 \end{cases}$$

Note: $F_Y(y)$ has a jump discontinuity at $\frac{\theta}{2}$

(3) $$f_X(x) = \begin{cases} \frac{1}{2}, & -\frac{1}{2} \leq x \leq 3/2 \\ 0, & o/w \end{cases} \qquad X \sim U\left(-\frac{1}{2}, \frac{3}{2}\right)$$

$Y = X^2 \rightarrow y \in (0, 9/4)$

$F_Y(y) = P(X^2 \leq y) = P(-\sqrt{y} \leq X \leq \sqrt{y})$

For $y < 0$; $F_Y(y) = 0$

& $y > \frac{9}{4}$; $F_Y(y) = 1$

For, $0 \leq y \leq \frac{1}{4}$; $F_Y(y) = \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{2} dx = \sqrt{y}$

for, $\frac{1}{4} < y \leq \frac{9}{4}$; $F_Y(y) = \int_{-\sqrt{y}}^{-1/2} 0 \cdot dx + \int_{-1/2}^{\sqrt{y}} \frac{1}{2} dx = \frac{1}{2}\left(\sqrt{y} + \frac{1}{2}\right)$

$$= \left(\frac{1}{4} + \frac{\sqrt{y}}{2}\right)$$

$$\Rightarrow \quad F_Y(y) = \begin{cases} 0 & y < 0 \\ \sqrt{y} & 0 \le y \le \frac{1}{4} \\ \frac{1}{2}\left(\sqrt{y} + \frac{1}{2}\right), & \frac{1}{4} \le y \le \frac{9}{4} \\ 1, & y \geq 9/4 \end{cases}$$

p.d.f. $$f_Y(y) = \begin{cases} \dfrac{1}{2\sqrt{y}}, & 0 \le y \le \frac{1}{4} \\ \dfrac{1}{4\sqrt{y}}, & \frac{1}{4} < y \le \frac{9}{4}. \end{cases}$$

(4) $$f_X(x) = \begin{cases} k \dfrac{x^{p-1}}{(1+x)^{p+q}}, & x > 0 \\ 0 & o/w \end{cases}$$

$$Y = \frac{1}{1+X} \Rightarrow X = \frac{1-Y}{Y} = g^{-1}(Y) \quad ; \qquad Y \in (0,1)$$

$$J = \frac{dx}{dy} = -\frac{1}{y^2}$$

$$f_Y(y) = \begin{cases} f_X(g^{-1}(y)) \; |J|, & 0 \le y \le 1 \\ 0 & o/w \end{cases}$$

$$= \begin{cases} k \cdot \left(\dfrac{1-y}{y}\right)^{p-1} \left(\dfrac{1}{y^{-1}}\right)^{p+q} \cdot \dfrac{1}{y^2}, & 0 \le y \le 1 \\ 0, & o/w \end{cases}$$

i.e $$f_Y(y) = \begin{cases} k \quad y^{q-1}(1-y)^{p-1}, & 0 \le x \le 1 \\ 0 & o/w. \end{cases}$$

$$K = \left(\text{Beta}(q, p)\right)^{-1}$$

$$\Rightarrow \quad Y \sim \text{Beta}(q, p)$$

(5) $$f_X(x) = \begin{cases} K \; x^{\beta-1} \, e^{-\alpha x^{\beta}}, & x > 0 \\ 0, & o/w \end{cases}$$

$Y = X^{\beta}$ $\quad J = \dfrac{dx}{dy} = \dfrac{1}{\beta} \, y^{1/\beta - 1} \quad \left(x = y^{1/\beta} = g^{-1}(y) \right)$

$\underline{y > 0}$

$$f_Y(y) = \begin{cases} f_X(g^{-1}(y)) \, |J|, & y > 0 \\ 0, & o/w \end{cases}$$

$$= \begin{cases} K \cdot \left(y^{1/\beta}\right)^{\beta-1} e^{-\alpha y} \left(\frac{1}{\beta} \cdot y^{1/\beta - 1} \right), & y > 0 \\ 0 & o/w \end{cases}$$

$$= \begin{cases} K \; y^{1-1/\beta} \, e^{-\alpha y} \; \frac{1}{\beta} \cdot y^{1/\beta - 1}, & y > 0 \\ 0 & o/w \end{cases}$$

$$= \begin{cases} K \, \dfrac{e^{-\alpha y}}{\beta}, & y > 0 \\ 0, & o/w. \end{cases}$$

$K = \alpha \beta \quad \Rightarrow \quad Y \sim \text{Exp}\left(\frac{1}{\alpha}\right).$

(6) $$f_V(v) = \begin{cases} K\, v^2 e^{-\beta v^2}, & v > 0 \\ 0, & o/w \end{cases}$$

$$E = \frac{1}{2} m v^2 \quad ; \quad V^2 = \frac{2E}{m}.$$

$$\frac{\partial e}{\partial v} = m v$$

$$J = \frac{\partial v}{\partial e} = \frac{1}{\sqrt{2me}}.$$

$$f_E(e) = \begin{cases} K \cdot \left(\frac{2e}{m}\right) e^{-\beta\left(\frac{2e}{m}\right)} \cdot \frac{1}{\sqrt{2me}} \quad ; & e > 0 \\ 0 & o/w \end{cases}$$

$$= \begin{cases} c \cdot e^{1/2} \exp\left(-\frac{2\beta}{m} e\right), & e > 0 \\ 0, & o/w \end{cases}$$

c is $\ni$ $$\int_0^\infty c\, e^{1/2} \exp\left(-\frac{2\beta}{m} e\right) de = 1$$

i.e. $$c \cdot \frac{\Gamma 3/2}{\left(\frac{2\beta}{m}\right)^{3/2}} = 1 \Rightarrow c = \frac{(2\beta/m)^{3/2}}{\Gamma 3/2}$$

$\Rightarrow E \sim$ Gamma (. . .)

(7) $f_X(x) = \begin{cases} \frac{3}{8}(x+1)^2, & -1 < x < 1 \\ 0, & o/w. \end{cases}$

$Y = 1 - X^2, \quad y \in (0,1)$

$x^2 = 1 - y \Rightarrow x = \pm\sqrt{1-y}$

$x \in (-1,0) \rightarrow x = -\sqrt{1-y} = g_1^{-1}(y) \rightarrow \left|\frac{dx}{dy}\right| = \frac{1}{2\sqrt{1-y}}$

$x \in (0,1) \rightarrow x = \sqrt{1-y} = g_2^{-1}(y) \rightarrow \left|\frac{dx}{dy}\right| = \frac{1}{2\sqrt{1-y}}.$

$$f_Y(y) = f_X(\underset{(-1,\,0)}{g_1^{-1}(y)}) \underset{(-1,0)}{\left|\frac{dx}{dy}\right|} + f_X(\underset{(0,\,1)}{g_2^{-1}(y)}) \underset{(-1,0)}{\left|\frac{dx}{dy}\right|}. \quad 0 < y < 1$$

$$= \frac{3}{8}\left(1-\sqrt{1-y}\right)^2 \cdot \frac{1}{2\sqrt{1-y}} + \frac{3}{8}\left(1+\sqrt{1-y}\right)^2 \cdot \frac{1}{2\sqrt{1-y}}$$

$$= \frac{3}{16\sqrt{1-y}}\left(\left(1-\sqrt{1-y}\right)^2 + \left(1+\sqrt{1-y}\right)^2\right)$$

$$= \frac{3}{16\sqrt{1-y}}\left(2\left(1+(1-y)\right)\right)$$

i.e. $f_Y(y) = \begin{cases} \frac{3}{8}\left((1-y)^{-1/2} + (1-y)^{1/2}\right), & 0 < y < 1 \\ 0, & o/w. \end{cases}$

(8) $Y = \min(X, 1-X) \rightarrow$ range of y $(0, \frac{1}{2})$

$F_Y(y) = P(Y \le y) = P(\min(X, 1-X) \le y) = 1 - P(\min(X, 1-X) > y)$

$= 1 - P(X > y, 1-X > y)$

$= 1 - P(X > y, 1-y > X)$

$= 1 - P(y < X < 1-y)$

$$P(y < X < 1-y) = \begin{cases} 1 & y \le 0 \\ \int_y^{1-y} dx & \text{if } 0 < y < \frac{1}{2} \\ 0 & y \geq \frac{1}{2} \end{cases}$$

$$\Rightarrow F_Y(y) = \begin{cases} 0 & y \le 0 \\ 2y & 0 < y < \frac{1}{2} \\ 1 & y \geq \frac{1}{2} \end{cases} \Rightarrow \text{p.d.f } f_Y(y) = \begin{cases} 2, & 0<y<\frac{1}{2} \\ 0, & o/w \end{cases}$$

~~p.d.f.~~ $Z = \frac{1-Y}{Y} = \frac{1}{Y} - 1 \rightarrow$ range of z is $(1, \infty)$

$F_Z(z) = P(Z \le z)$ -

If $z \le 1$, then $F_Z(z) = 0$

If $z > 1$, then $P(Z \le z) = P\left(\frac{1}{Y} - 1 \le z\right) = P\left(\frac{1}{Y} \le z+1\right)$

$$= P\left(Y \geq \frac{1}{z+1}\right) = 1 - P\left(Y < \frac{1}{z+1}\right)$$

$$= 1 - \frac{2}{z+1} \quad (\text{using d.f of } Y)$$

$$\Rightarrow F_Z(z) = \begin{cases} 0, & \text{if } z \le 1 \\ 1 - \frac{2}{z+1}, & \text{if } z > 1 \end{cases}$$

$$\Rightarrow \text{p.d.f. of } Z \text{ is } f_Z(z) = \begin{cases} \frac{2}{(z+1)^2}, & z > 1 \\ 0 & o/w \end{cases}$$

(9) $X \sim N(\mu, \sigma^2)$

$Y = 2X - 6$; $\quad y \in (-\infty, \infty)$

$$F_Y(y) = P(Y \le y) = P(2X - 6 \le y)$$
$$= P\left(X \le \frac{y+6}{2}\right)$$
$$= P\left(\frac{X-\mu}{\sigma} \le \left(\frac{y+6}{2} - \mu\right)/\sigma\right)$$
$$= \Phi\left(\frac{y+6-2\mu}{2\sigma}\right)$$

$$f_Y(y) = \phi\left(\frac{y+6-2\mu}{2\sigma}\right) \cdot \frac{1}{2\sigma} \qquad y \in (-\infty, \infty)$$
$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y-(2\mu-6)}{2\sigma}\right)^2} \cdot \frac{1}{2\sigma}$$
$$= \frac{1}{\sqrt{2\pi}(2\sigma)} \exp\left(-\frac{1}{2(4\sigma^2)}(y-(2\mu-6))^2\right)$$

$\Rightarrow Y \sim N(2\mu - 6, 4\sigma^2)$.

(10) $X \sim f_X(x)$

$Z = -\log F(X)$; $\quad z \in (0, \infty)$

$z = -\log F(x) \Rightarrow x = F^{-1}(e^{-z})$

$$\left|\frac{\partial z}{\partial x}\right| = \left(\frac{f(x)}{F(x)}\right) \Rightarrow |J| = \frac{F(x)}{f(x)}$$

$$\text{p.d.f. of } Z = f\left(F^{-1}(e^{-z})\right) \cdot \frac{F\left(F^{-1}(e^{-z})\right)}{f\left(F^{-1}(e^{-z})\right)}$$
$$= F\left(F^{-1}(e^{-z})\right) = e^{-z} ;$$

$$\Rightarrow f_Z(z) = \begin{cases} e^{-z}, & z > 0 \\ 0 & \text{o/w}. \end{cases}$$

(11) $f_X(x) = \begin{cases} 6x(1-x), & 0 \le x \le 1 \\ 0 & o/w \end{cases}$

$$F_X(x) = P(X \le x) = \begin{cases} 0, & x < 0 \\ \int_0^x (6y - 6y^2)\,dx, & 0 \le x \le 1 \\ 1, & x > 1 \end{cases}$$

$$= \begin{cases} 0, & x < 0 \\ x^2(3-2x), & 0 \le x \le 1 \\ 1, & x > 1 \end{cases}$$

$$F_X(X) = X^2(3-2X)$$

If $X \sim F_X(x)$ dist^n f^n, then $Y = F(X) \sim U(0,1)$

$\left[\begin{array}{l} X \sim f_X(x) \text{ p.d.f. \& d.f. } F \\ \text{Cont r.v.} \end{array} \right.$

$Y = F(X) \rightarrow y \in (0,1)$ [General Result as in prob 14]

$x = F^{-1}(y)$

$\frac{dy}{dx} = f(x) \qquad \left|\frac{dx}{dy}\right| = \left|\frac{1}{f(x)}\right|$

$\Rightarrow$ p.d.f. of Y : $f_Y(y) = \begin{cases} \frac{f_X(F^{-1}(y))}{f_X(F^{-1}(y))} = 1 & \text{if } 0 < y < 1 \\ 0 & o/w \end{cases}$

$\Rightarrow Y \sim U(0,1)$

$\Rightarrow$ For the given dist^n $X^2(3-2X) = F(X) \sim U(0,1)$

(12) $X \sim$ Double exponential

$$f_X(x) = \frac{1}{2} e^{-|x|} \; ; \qquad -\infty < x < \infty$$

$Y = |X|$ range of Y : $(0, \infty)$

$x \in (-\infty, 0) \rightarrow x = -y \rightarrow \left|\frac{dx}{dy}\right| = 1$

$x \in (0, \infty) \rightarrow x = y \rightarrow \left|\frac{dx}{dy}\right| = 1$

$$\Rightarrow f_Y(y) = f_X(\underbrace{g_1^{-1}(y)}^{(-\infty,0)}) \, |J| + f_X(g_2^{-1}(y)) \, |J|$$

$\uparrow$ in $(-\infty, 0)$

i.e. $f_Y(y) = \begin{cases} \frac{1}{2} e^{-y} + \frac{1}{2} e^{-y} , & 0 < y < \infty \\ 0 , & o/w \end{cases}$

$\therefore$ p.d.f. of Y : $f_Y(y) = \begin{cases} e^{-y}, & 0 < y < \infty \\ 0 & o/w. \end{cases}$

(13) $X_i = 0, 1, 2, 3$ for $i = 1, 2, 3$

$N = 1, 2, 3$

Possible configurations with 3 boxes and 3 balls

B_1	B_2	B_3
3	0	0
0	3	0
0	0	3
2	1	0
2	0	1
1	2	0
0	2	1
1	0	2
0	1	2
1	1	1

$\rightarrow$ each with prob $= \dfrac{1}{\binom{3+3-1}{3}} = \dfrac{1}{10}$

$\rightarrow$

$\downarrow N$	X_1	X_2	X_3
1	3	0	0
1	0	3	0
1	0	0	3
2	2	1	0
2	2	0	1
2	1	2	0
2	0	2	1
2	1	0	2
2	0	1	2
3	1	1	1

jt p.m.f. of (N, X_1)

$N \backslash X_1$	0	1	2	3	Marginal of N
1	2/10	0	0	1/10	3/10
2	2/10	2/10	2/10	0	6/10
3	0	1/10	0	0	1/10
Marginal of X_1	4/10	3/10	2/10	1/10	

jt. p.m.f. of (X_1, X_2)

$X_1 \backslash X_2$	0	1	2	3	Marg of X_1
0	1/10	1/10	1/10	1/10	4/10
1	1/10	1/10	1/10	0	3/10
2	1/10	1/10	0 ~~1/10~~	0	2/10
3	1/10	0	0	0	1/10
Marg of X_2	4/10	3/10	2/10	1/10	

(14) $\sum_{(x,y)} p(x,y) = c \sum_{(x,y)} (xy) = 1$

$\Rightarrow c\left[(1.1)+(2.1)+(2.2)+(3.1)\right] = 1$

$\Rightarrow c = \frac{1}{10}$

jt p.m.f.

X \ Y	1	2	
1	1/10	0	1/10
2	2/10	4/10	6/10
3	3/10	0	3/10
	6/10	4/10	1

marginal of X

marginal of Y

Conditional p.m.f. of X given Y=2, $\frac{p(x,2)}{p_Y(2)} = 1$ if $x = 2$
$= 0$ o/w

(15) jt p.m.f.

X \ Y	1	2	
1	3/18	5/18	8/18
2	4/18	6/18	10/18
	7/18	11/18	

marg of X

marg of Y

(b) $P(X=1, Y=1) = \frac{3}{18} \neq P(X=1) \cdot P(Y=1) = \frac{8}{18} \cdot \frac{7}{18}$
$\Rightarrow$ X & Y not ind.

(c) $P(X < Y) = P(X=1, Y=2) = \frac{5}{18}$

$P(X+Y>2) = P(X=1, Y=2) + P(X=2, Y=1) + P(X=2, Y=2)$

$= \frac{15}{18}$

(d) marg of X: $p_X(x) = P(X=x) = \frac{x+3}{9}$; $x = 1, 2$

$p_{Y|X=x} = \frac{\frac{1}{18}(x+2y)}{\frac{1}{18}(2x+6)} = \frac{x+2y}{2x+6}$; $y = 1, 2$

(16) jt p.m.f. of X_1, X_2, X_3

$$p_{X_1,X_2,X_3}(x_1,x_2,x_3) = \frac{\binom{13}{x_1}\binom{13}{x_2}\binom{13}{x_3}\binom{13}{5-x_1-x_2-x_3}}{\binom{52}{5}} \quad ; \quad x_i \geq 0 \ \& \ \sum_1^3 x_i \leq 5$$

$$p_{X_i}(x) = \frac{\binom{13}{x}\binom{39}{5-x}}{\binom{52}{5}} \qquad x = 0,1,2,3,4,5$$

$$p_{X_1}(x_1)\, p_{X_2}(x_2)\, p_{X_3}(x_3) \neq p_{X_1,X_2,X_3}(x_1,x_2,x_3)$$

$\Rightarrow (X_1, X_3, X_3)$ are not indep.

(17). X_1 : # of white balls

X_2 : # of black balls.

3 W, 2 B, 1 R $-\ \underline{\underline{7}}$

$$p_{X_1,X_2}(x_1,x_2) = \frac{3!}{x_1!\, x_2!\, (3-x_1-x_2)!}\left(\frac{3}{8}\right)^{x_1}\left(\frac{2}{8}\right)^{x_2}\left(\frac{3}{8}\right)^{3-x_1-x_2};$$

$(X_1, X_2) \sim \text{Mult}\left(3, \frac{3}{8}, \frac{2}{8}\right)$ $\qquad x_i \geq 0,\ x_1 + x_2 \leq 3$

$$p_{X_1}(x_1) = \binom{3}{x_1}\left(\frac{3}{8}\right)^{x_1}\left(\frac{5}{8}\right)^{3-x_1}; \quad x_1 = 0,1,2,3$$

$$p_{X_2}(x_2) = \binom{3}{x_2}\left(\frac{2}{8}\right)^{x_1}\left(\frac{6}{8}\right)^{3-x_2}; \quad x_2 = 0,1,2,3$$

i.e. $X_1 \sim B\left(3, \frac{3}{8}\right)$; $X_2 \sim B\left(3, \frac{2}{8}\right)$

$$p_{X_1}(x_1)\, p_{X_2}(x_2) \neq p_{X_1,X_2}(x_1,x_2)$$

$\Rightarrow X_1$ & X_2 are not indep.

(18) From the jt p.m.f. of X_1, X_2

$$P(X_1=0, X_2=0) = P(X_1=0, X_2=1) = P(X_1=1, X_2=0) = P(X_1=1, X_2=1) = \frac{1}{4}$$

Further $(X_1, X_2) \stackrel{d}{\equiv} (X_1, X_3) \stackrel{d}{\equiv} (X_2, X_3)$

& $P(X_i = 0) = \frac{1}{2} = P(X_i = 1)$; $i = 1, 2, 3$

$\Rightarrow X_1, X_2, X_3$ are pairwise indep

But $P(X_1=0, X_2=0, X_3=0) = \frac{1}{4} \neq P(X_1=0)\, P(X_2=0)\, P(X_3=0) = \frac{1}{8}$

$\Rightarrow X_1, X_2, X_3$ are not indep.