MSO 201 A : Homework 9

[1] Let $\{X_n\}$ be a sequence of random variables with $E(X_n) \to c$ and $V(X_n) \to 0$ as $n \to \infty$. Show that $X_n \xrightarrow{p} c$.

[2] Let $\{X_n\}$ be a sequence of random variables with $E(X_n) = \mu_n$ and finite variance such that $\frac{1}{n^2} V\left(\sum_{i=1}^{n} X_i\right) \to 0$ as $n \to \infty$. Show that WLLN holds and $\bar{X}_n \xrightarrow{p} \bar{\mu}_n = \frac{1}{n}\sum_{i=1}^{n} \mu_i$.

[3] Let $X_1, X_2, \ldots X_n$ be a random sample from $U(0,1)$. Let $Y_n = \min(X_1, \ldots, X_n)$ and $Z_n = \max(X_1, \ldots, X_n)$. Show that (a) $\sqrt{Y_n} \xrightarrow{p} 0$, (b) $Z_n^2 \xrightarrow{p} 1$ and (c) $Y_n^2 Z_n^2 \xrightarrow{p} 0$.

[4] Let $X_1, X_2, \ldots X_n$ be i.i.d. $N(0,1)$. Show that $\bar{X}_n / S_n \xrightarrow{p} 0$, where $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $S_n^2 = \frac{1}{n}\sum_{i=1}^{n}\left(X_i - \bar{X}_n\right)^2$.

[5] Suppose $Y_n \sim Bin(n, p)$, show that $(1 - Y_n/n) \xrightarrow{p} 1 - p$.

[6] Let $\{X_n\}$ be a sequence of independent random variables with

$$P(X_n = x) = \begin{cases} 1/2, & x = -n^{1/4},\ n^{1/4} \\ 0, & \text{otherwise.} \end{cases}$$

Show that $\bar{X}_n \xrightarrow{p} 0$.

[7] Let $\{X_n\}$ be a sequence of i.i.d. random variables with mean μ and finite variance. Show that

(a) $\frac{2}{n(n+1)}\sum_{i=1}^{n} i\, X_i \xrightarrow{p} \mu$

(b) $\frac{6}{n(n+1)(2n+1)}\sum_{i=1}^{n} i^2\, X_i \xrightarrow{p} \mu$

[8] Let $\{X_n\}$ be a sequence of i.i.d. random variables with $U(0,1)$ distribution and $Z_n = \left(\prod_{i=1}^{n} X_i\right)^{1/n}$. Show that $Z_n \xrightarrow{p} e^{-1}$.

[9] Let $\{X_n\}$ be a sequence of uncorrelated random variables with $E(X_n) = \mu_n$ and $V(X_n) = \sigma_n^2$. Show that if $\sum_{i=1}^{n} \sigma_i^2 \to \infty$ as $n \to \infty$, then WLLN holds for $\{X_n\}$.

[10] Let $\{X_n\}$ be a sequence of i.i.d. random variables with $U(0,1)$ distribution. Find c such that $\bar{X}_n \xrightarrow{p} c$.

[11] Let $\{X_n\}$ be a sequence of $N\left(\frac{1}{n}, 1-\frac{1}{n}\right)$. Show that $X_n \xrightarrow{\mathcal{L}} Z$, where $Z \sim N(0,1)$.

[12] Let $\{X_n\}$ be a sequence of i.i.d. random variables with $E(X_i)=\mu$, $Var(X_i)=\sigma^2$ and $E(X_i-\mu)^4=\sigma^4+1$. Find $\lim_{n\to\infty} P\left[\sigma^2-\frac{1}{\sqrt{n}} \le \frac{(X_1-\mu)^2+\ldots+(X_n-\mu)^2}{n} \le \sigma^2+\frac{1}{\sqrt{n}}\right]$.

[13] Let $X_1, X_2, \ldots X_n$ be i.i.d. $B(1,p)$, $S_n=\sum_{i=1}^{n} X_i$. Find n which would guarantee $P\left(\left|\frac{S_n}{n}-p\right| \ge 0.01\right) \le 0.01$, no matter whatever the unknown p may be.

[14] Let $X_1,\ldots,X_n$ be i.i.d. from a distribution with mean μ and finite variance σ^2. Prove that $\frac{\sqrt{n}\left(\bar{X}_n-\mu\right)}{S_n} \xrightarrow{\mathcal{L}} Z$, where $Z \sim N(0,1)$.

[15] The p.d.f. of a random variable X is

$$f(x)=\begin{cases} 1/x^2 & x \ge 1 \\ 0 & \text{otherwise} \end{cases}$$

Consider a random sample of size 72 from the distribution having the above p.d.f. Compute, approximately, the probability that more than 50 of these observations are less than 3.

[16] Let $X_1,\ldots,X_{100}$ be i.i.d. from Poisson(3) distribution and let $Y=\sum_{i=1}^{100} X_i$. Using CLT, find an approximate value of $P(100 \le Y \le 200)$.

[17] Let $X \sim Bin(100, 0.6)$. Find an approximate value of $P(10 \le X \le 16)$.

[18] The p.d.f. of X_n is given by

$$f_n(x)=\begin{cases} \frac{1}{\Gamma n} e^{-x} x^{n-1} & x>0 \\ 0 & \text{otherwise} \end{cases}$$

Find the limiting distribution of $Y_n = X_n / n$.

[19] Let $\bar{X}$ denote the mean of a random sample of size 64 from the Gamma distribution with density

$$f_n(x)=\begin{cases} \frac{1}{\Gamma p \alpha^p} e^{-x/\alpha} x^{p-1} & x>0 \\ 0 & \text{otherwise.} \end{cases}$$

With $\alpha=2, p=4$. Compute the approximate value of $P(7<\bar{X}<9)$.

[20] $X_1,\ldots.,X_n$ is a random sample from $U(0,2)$. Let $Y_n=\bar{X}_n$, show that $\sqrt{n}(Y_n-1) \xrightarrow{\mathcal{L}} N(0,1/3)$.