Solution : Homework 10

1. $$E(X) = \frac{1}{\beta}\int_0^{\infty} x\, e^{-\frac{\beta}{x}}\, dx = \beta$$

$$\Rightarrow \cancel{E\left(\frac{1}{n}\sum_{i=1}^{n}\right.}$$

Hence, $$E(\overline{X_n}) = \frac{1}{n}\sum_{i=1}^{n} E(X_i) = \frac{1}{n}\sum_{i=1}^{n}\beta = \beta.$$

2. Done in class.

3. Consider

$$T_n(X_1, \ldots, X_n) = \begin{cases} 1 & \text{if} \quad X_1 = 0,\ X_2 = 1 \\ 0 & \text{otherwise} \end{cases}$$

Now, $$\begin{aligned} E(T_n) &= 1 \cdot P[X_1 = 0,\ X_2 = 1] \\ &= P[X_1 = 0]\, P[X_2 = 1] \\ &= e^{-\theta} \times \frac{e^{-\theta}\theta^{1}}{1!} = \theta \cdot e^{-2\theta}. \end{aligned}$$

$\Rightarrow$ T_n is an unbiased estimator for $\theta \cdot e^{-2\theta}$.

4. Consider

$$T_n(x_1, \cdots, x_n) = \begin{cases} 1 & \text{if } x_1 = 1, x_2 = 1, x_3 = 0 \\ 0 & \text{otherwise} \end{cases}$$

Now,

$$E(T_n) = P[X_1 = 1, X_2 = 1, X_3 = 0]$$

$$= P[X_1 = 1]\, P[X_2 = 1]\, P[X_3 = 0]$$

$$= \theta^2 (1-\theta).$$

$\Rightarrow$ T_n is an unbiased estimator of $\theta^2(1-\theta)$.

5. Joint p.d.f. of X_1 and X_2 is

$$f_{X_1, X_2}(x_1, x_2) = \left(\theta \cdot e^{-\theta x_1}\right) \times \left(2\theta\, e^{-2\theta x_2}\right)$$

$$= 2\theta^2 e^{-\theta(x_1 + 2x_2)}$$

$$= \left\{\theta^2 e^{-\theta(x_1 + 2x_2)}\right\} \times (2).$$

Hence, by FT, $T(X_1, X_2) = X_1 + 2X_2$ is a sufficient statistic for θ.

6. Done in class.

7. $$f(x_1, \ldots, x_n \mid \theta) = \begin{cases} 1 & \text{if } \theta - \frac{1}{2} \leq x_{(1)} \cdots < x_{(n)} < \theta + \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

i.e.,
$$f(\underset{\sim}{x} \mid \theta) = 1 \times 1_{\{\theta - \frac{1}{2},\, x_{(1)}\}} \times 1_{\{x_{(n)},\, \theta + \frac{1}{2}\}}$$

[Here $1_{\{a,b\}} = 1$ if $a < b$
$= 0$ otherwise]

$$= g\left(\theta, \left(x_{(1)}, x_{(n)}\right)\right) h(\underset{\sim}{x})$$

$$g\left(\theta, \left(x_{(1)}, x_{(n)}\right)\right) = 1\{\theta - \tfrac{1}{2}, x_{(1)}\} \times 1\{x_{(n)}, \theta + \tfrac{1}{2}\}.$$

$$h(\underset{\sim}{x}) \to 1$$

Hence, by FT, $T_n(X_1, \ldots, X_n) = (X_{(1)}, X_{(n)})$ is jointly sufficient for θ.