(1) $x_1, \ldots, x_n$ i.i.d. $N(\mu, \sigma^2)$

$$f(x|\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$$

$$\log f = K - \frac{1}{2}\log\sigma^2 - \frac{1}{2\sigma^2}(x-\mu)^2$$

$$\frac{\partial \log f}{\partial \mu} = \frac{(x-\mu)}{\sigma^2} \quad ; \quad \frac{\partial^2 \log f}{\partial \mu^2} = -\frac{1}{\sigma^2}.$$

$$-E\left(\frac{\partial^2 \log f}{\partial \mu^2}\right) = \frac{1}{\sigma^2} = I(\mu)$$

CRLB for an u.e. for $\mu = \dfrac{\sigma^2}{n}$.

Since $V(\bar{X}) = \dfrac{\sigma^2}{n}$; $\bar{X}$ attains CRLB.

$$\frac{\partial \log f}{\partial \sigma^2} = -\frac{1}{2\sigma^2} + \frac{1}{2\sigma^4}(x-\mu)^2$$

$$\frac{\partial^2 \log f}{\partial (\sigma^2)^2} = \frac{1}{2\sigma^4} - \frac{(x-\mu)^2}{\sigma^6}.$$

$$I(\sigma^2) = -E\left(\frac{\partial^2 \log f}{\partial(\sigma^2)^2}\right) = -\frac{1}{2\sigma^4} + \frac{1}{\sigma^4} = \frac{1}{2\sigma^4}.$$

CRLB for an u.e. for $\sigma^2 = \dfrac{2\sigma^4}{n}$.

Now $s^2 = \dfrac{1}{n-1}\sum_1^n (X_i - \bar{X})^2$ is UMVUE for σ^2 with

$$V(s^2) = \frac{2\sigma^4}{n-1} > \text{CRLB}$$

Since UMVUE is the unbiased est with lowest variance in the class of all unbiased estimators, CRLB can't be attained by any unbiased estimator of σ^2.

(2) $f(x|\beta) = \dfrac{1}{\Gamma\alpha \;\; \beta^{\alpha}} \, e^{-x/\beta} \, x^{\alpha-1} \; ; \quad x > 0$

$$\log f = -\log \Gamma\alpha - \alpha \log \beta - \frac{x}{\beta} + (\alpha - 1)\log x$$

$$\frac{\partial \log f}{\partial \beta} = -\frac{\alpha}{\beta} + \frac{x}{\beta^2}$$

$$\frac{\partial^2 \log f}{\partial \beta^2} = \frac{\alpha}{\beta^2} - 2\,\frac{x}{\beta^3}.$$

$$I(\beta) = -E\left(\frac{\partial^2 \log f}{\partial \beta^2}\right) = -\frac{\alpha}{\beta^2} + 2\,\frac{\alpha\beta}{\beta^3} = \frac{\alpha}{\beta^2}.$$

$\Rightarrow$ CRLB for u.e. of β : $\dfrac{1}{n \cdot \frac{\alpha}{\beta^2}} = \dfrac{\beta^2}{n\alpha}$.

(3) $X_1, \ldots X_n$ i.i.d. $P(\theta)$

$$f(x|\theta) = \frac{e^{-\theta}\,\theta^x}{x!}$$

$$\log f(x|\theta) = -\theta + x\log\theta - \log x!$$

$$\frac{\partial \log f}{\partial \theta} = -1 + \frac{x}{\theta} \;;\quad \frac{\partial^2 \log f}{\partial \theta^2} = -\frac{x}{\theta^2}.$$

$$I(\theta) = -E\left(\frac{\partial^2 \log f}{\partial \theta^2}\right) = \frac{1}{\theta}$$

CRLB for any u.e. of θ : $\dfrac{1}{n\,\frac{1}{\theta}} = \dfrac{\theta}{n}$

CRLB for any u.e. of $g(\theta) = \theta^2$: $\dfrac{(2\theta)^2}{\frac{n}{\theta}} = \dfrac{4\theta^3}{n}$

CRLB for any u.e. of $g(\theta) = e^{-\theta}$: $\dfrac{(-e^{-\theta})^2}{\frac{n}{\theta}} = \dfrac{\theta\, e^{-2\theta}}{n}$.

(4) $X_1, \ldots X_n$ i.i.d $B(1,\theta)$

$$f(x|\theta) = \theta^x (1-\theta)^{1-x}$$

$$\log f(x|\theta) = x \log\theta + (1-x)\log(1-\theta)$$

$$\frac{\partial \log f}{\partial\theta} = \frac{x}{\theta} + \frac{(1-x)}{1-\theta}(-1) = \frac{x}{\theta(1-\theta)} - \frac{1}{1-\theta}.$$

$$I(\theta) = E\left(\frac{\partial \log f}{\partial\theta}\right)^2 = V\left(\frac{\partial \log f}{\partial\theta}\right) = \frac{\theta(1-\theta)}{(\theta(1-\theta))^2} = \frac{1}{\theta(1-\theta)}.$$

CRLB for u.e. of θ^4: $\dfrac{(4\theta^3)^2}{n \cdot \frac{1}{\theta(1-\theta)}} = \dfrac{16\,\theta^7(1-\theta)}{n}$

CRLB for u.e. of $\theta(1-\theta)$: $\dfrac{(1-2\theta)^2}{n \cdot \frac{1}{\theta(1-\theta)}} = \dfrac{(1-2\theta)^2\,\theta(1-\theta)}{n}.$

(5) $X_1, \ldots X_n$ i.i.d. $U(0,\theta)$

$$P[\,|X_{(n)} - \theta| \geq \epsilon] \leq \frac{E(X_{(n)}-\theta)^2}{\epsilon^2} = \frac{E X_{(n)}^2 + \theta^2 - 2\theta E X_{(n)}}{\epsilon^2}$$

$$f_{X_{(n)}}(x) = \begin{cases} \frac{n}{\theta^n} x^{n-1}, & 0 < x < \theta \\ 0, & o/w. \end{cases}$$

$$E X_{(n)} = \frac{n}{\theta^n}\int_0^\theta x^n\,dx = \frac{n}{n+1}\cdot\theta$$

$$E X_{(n)}^2 = \frac{n}{\theta^n}\int_0^\theta x^{n+1}\,dx = \frac{n}{n+2}\theta^2$$

$$\Rightarrow P[\,|X_{(n)}-\theta| \geq \epsilon] \leq \frac{1}{\epsilon^2}\left[\frac{n}{n+2}\theta^2 + \theta^2 - 2\theta\frac{n}{n+1}\theta\right] \to 0 \text{ as } n \to \infty$$

$$\Rightarrow X_{(n)} \xrightarrow{P} \theta$$

$$\Rightarrow \frac{n}{n+1} X_{(n)} \xrightarrow{p} \theta$$

$\Rightarrow \frac{n}{n+1} X_{(n)}$ is a consistent estimator for θ

Further since $X_{(n)} \xrightarrow{p} \theta$

$$e^{X_{(n)}} = g(X_{(n)}) \xrightarrow{p} g(\theta) = e^{\theta}.$$

$\Rightarrow e^{X_{(n)}}$ is a consistent estimator for e^{θ}.

(6) $X_1, \ldots X_n$ i.i.d $U(\theta - \frac{1}{2}, \theta + \frac{1}{2})$

$$F_X(x) = \int_{\theta-\frac{1}{2}}^{x} dx = (x - \theta + \tfrac{1}{2})$$

$$f_{X_{(1)}}(x) = n(1 - F_X(x))^{n-1} f(x)$$

i.e. $$f_{X_{(1)}}(x) = \begin{cases} n(\theta - x + \frac{1}{2})^{n-1}, & \theta - \frac{1}{2} \le x \le \theta + \frac{1}{2} \\ 0, & o/w. \end{cases}$$

$$E X_{(1)} = n \int_{\theta-\frac{1}{2}}^{\theta+\frac{1}{2}} x(\theta - x + \tfrac{1}{2})^{n-1} dx = \theta + \frac{1}{2} - \frac{n}{n+1}$$

$$E X_{(1)}^2 = n \int_{\theta-\frac{1}{2}}^{\theta+\frac{1}{2}} x^2(\theta - x + \tfrac{1}{2})^{n-1} dx$$

$$= (\theta + \tfrac{1}{2})^2 + \frac{n}{n+2} - \frac{n}{n+1}(2\theta + 1)$$

$$P\left[\left|X_{(1)}-\left(\theta-\tfrac{1}{2}\right)\right| \geq \epsilon\right] \leq \frac{E\left(X_{(1)}-\left(\theta-\frac{1}{2}\right)\right)^2}{\epsilon^2}$$

$$\text{r.h.s.} = \frac{1}{\epsilon^2}\left[E\left(X_{(1)}^2\right)+\left(\theta-\tfrac{1}{2}\right)^2 - 2\left(\theta-\tfrac{1}{2}\right)E\left(X_{(1)}\right)\right]$$

$$= \frac{1}{\epsilon^2}\left[\left\{\left(\theta+\tfrac{1}{2}\right)^2+\frac{n}{n+2}-\frac{n}{n+1}(2\theta+1)\right\}+\left(\theta-\tfrac{1}{2}\right)^2 - 2\left(\theta-\tfrac{1}{2}\right)\left(\theta+\tfrac{1}{2}-\frac{n}{n+1}\right)\right]$$

$$\to \frac{1}{\epsilon^2}\left[\left\{\left(\theta+\tfrac{1}{2}\right)^2+1-(2\theta+1)\right\}+\left(\theta-\tfrac{1}{2}\right)^2-2\left(\theta-\tfrac{1}{2}\right)\left(\theta-\tfrac{1}{2}\right)\right] \quad \text{as } n \to \infty$$

$$= 0$$

$$\Rightarrow P\left[\left|X_{(1)}-\left(\theta-\tfrac{1}{2}\right)\right| \geq \epsilon\right] \to 0 \quad \text{as } n \to \infty$$

$$\Rightarrow X_{(1)} \xrightarrow{p} \theta - \tfrac{1}{2}. \quad — (1)$$

We can similarly prove that

$$X_{(n)} \xrightarrow{p} \theta + \tfrac{1}{2} \quad — (2)$$

Combining (1) & (2), we get.

$$\frac{X_{(1)}+X_{(n)}}{2} \xrightarrow{p} \theta$$

$\Rightarrow \dfrac{X_{(1)}+X_{(n)}}{2}$ is a consistent estimator for θ

Also,

$X_{(1)}+\frac{1}{2}$ is a consistent estimator for θ (from (1))

& $X_{(n)}-\frac{1}{2}$ is a consistent estimator for θ (from (2)).

(7) $X_1, \ldots X_n$ i.i.d. $f_X(x) = \begin{cases} \frac{1}{2}(1+\theta x), & -1 < x < 1 \\ 0, & \text{o/w.} \end{cases}$

$$E(X) = \frac{1}{2}\int_{-1}^{1}(1+\theta x)\,dx = \frac{\theta}{3}$$

$\Rightarrow X_1, \ldots X_n$ are i.i.d. with $E(X_1) = \frac{\theta}{3}$

By Khintchine's WLLN

$$\frac{1}{n}\sum_{1}^{n} X_i \xrightarrow{p} E(X_1)$$

i.e. $\bar{X} \xrightarrow{p} \frac{\theta}{3} \Rightarrow 3\bar{X} \xrightarrow{p} \theta$

$\Rightarrow 3\bar{X}$ is a consistent estimator for θ.

(8) $X_1, \ldots X_n$ are i.i.d. $P(\theta)$

$E(X_i) = \theta \quad \forall i = 1(1)n$

By WLLN $\quad \bar{X}_n \xrightarrow{p} \theta$

$$\Rightarrow g(\bar{X}_n) \xrightarrow{p} g(\theta)$$

$$\Rightarrow \bar{X}_n^3\left(3\sqrt{\bar{X}_n} + \bar{X}_n + 12\right) \xrightarrow{p} \theta^3\left(3\sqrt{\theta} + \theta + 12\right)$$

$\Rightarrow \bar{X}_n^3\left(3\sqrt{\bar{X}_n} + \bar{X}_n + 12\right)$ is a consistent estimator for

$\theta^3\left(3\sqrt{\theta} + \theta + 12\right)$.

(9) $X_1, \dots X_n$ are i.i.d. Gamma(α, β)

α is known

$E(X) = \alpha\beta$ for $X \sim$ Gamma(α, β)

By WLLN

$$\frac{1}{n}\sum_1^n X_i \xrightarrow{p} E(X_1)$$

i.e. $$\frac{1}{n}\sum_1^n X_i \xrightarrow{p} \alpha\beta$$

$$\Rightarrow \frac{1}{n\alpha}\sum_1^n X_i \xrightarrow{p} \beta.$$

$\Rightarrow \frac{1}{n\alpha}\sum_1^n X_i$ is a consistent estimator for β.

[Note: $T = \sum_1^n X_i \sim$ Gamma$(n\alpha, \beta)$
can be proved using m.g.f. approach]

(10)

(a) $X_1, \dots X_n$ i.i.d $P(\theta)$

likelihood f^n $L(\theta|\underline{x}) = \theta^{\Sigma x_i} e^{-n\theta} (\Pi x_i!)^{-1}$

$$l(\theta|\underline{x}) = \log L(\theta|\underline{x}) = \Sigma x_i \log\theta - n\theta - \log(\Pi x_i!)$$

$$\frac{\partial l}{\partial\theta} = \frac{\Sigma x_i}{\theta} - n \;;\; \frac{\partial l}{\partial\theta} = 0 \Rightarrow \hat{\theta} = \bar{x}$$

$$\left.\frac{\partial^2 l}{\partial\theta^2}\right|_{\theta=\hat{\theta}} = -\frac{\Sigma x_i}{\hat{\theta}^2} < 0$$

$$\Rightarrow \hat{\theta}_{MLE} = \bar{X}$$

10(b) $X_1, \dots X_n$ i.i.d with p.d.f. $f_X(x) = \begin{cases} \theta x^{\theta-1}, & 0<x<1 \\ 0, & o/w \end{cases}$

$$L(\theta|\underline{x}) = \theta^n \left(\prod_1^n x_i\right)^{\theta-1}$$

$$l(\theta|\underline{x}) = \log L(\theta|\underline{x}) = n\log\theta + (\theta-1)\sum_1^n \log x_i$$

$$\frac{\partial l}{\partial \theta} = \frac{n}{\theta} + \sum \log x_i$$

$$\frac{\partial l}{\partial \theta} = 0 \Rightarrow \hat{\theta} = -\frac{n}{\sum_1^n \log x_i}.$$

$$\left.\frac{\partial^2 l}{\partial \theta^2}\right|_{\theta=\hat{\theta}} = -\frac{n}{\hat{\theta}^2} < 0 \Rightarrow \hat{\theta}_{MLE} = -\frac{n}{\sum_1^n \log X_i}.$$

1(c) $X_1, \dots X_n$ i.i.d with p.d.f. $f_X(x) = \begin{cases} \frac{1}{\theta} e^{-x/\theta}, & x>0 \\ 0, & o/w \end{cases}$

$$L(\theta|\underline{x}) = \frac{1}{\theta^n} e^{-\sum x_i/\theta}$$

$$l(\theta|\underline{x}) = \log L(\theta|\underline{x}) = -n\log\theta - \frac{1}{\theta}\sum x_i$$

$$\frac{\partial l}{\partial \theta} = -\frac{n}{\theta} + \frac{1}{\theta^2}\sum x_i$$

$$\frac{\partial l}{\partial \theta} = 0 \Rightarrow \hat{\theta} = \bar{x}$$

$$\left.\frac{\partial^2 l}{\partial \theta^2}\right|_{\theta=\hat{\theta}} = \left.\left(\frac{n}{\theta^2} - \frac{2n\bar{x}}{\theta^3}\right)\right|_{\theta=\hat{\theta}} = \frac{n}{\bar{x}^2} - \frac{2n\bar{x}}{\bar{x}^3} = -\frac{n}{\bar{x}^2} < 0$$

$$\Rightarrow \hat{\theta}_{MLE} = \bar{X}$$

10 (d) $X_1, \ldots, X_n$ i.i.d with p.d.f. $f_X(x) = \begin{cases} \frac{1}{2} e^{-|x-\theta|}, & -\infty < x < \infty \\ 0, & o/w \end{cases}$

$L(\theta|\underset{\sim}{x})$ is maximized if $\Sigma|x_i - \theta|$ is minimized

Realize that $\Sigma|x_i - \theta|$ is minimized w.r.t. θ at

$$\hat{\theta} = \text{median}(x_1, \ldots x_n)$$

$$\Rightarrow \hat{\theta}_{MLE} = \text{median}(X_1, \ldots X_n)$$

10 (e) $X_1, \ldots X_n$ i.i.d. with $U(-\frac{\theta}{2}, \frac{\theta}{2})$

likelihood f^n $L(\theta|\underset{\sim}{x}) = \begin{cases} \frac{1}{\theta^n}, & -\frac{\theta}{2} \le x_1, \ldots x_n \le \frac{\theta}{2} \\ 0, & o/w. \end{cases}$

$$L(\theta|\underset{\sim}{x}) = \begin{cases} \frac{1}{\theta^n}, & \text{if } |x_i| \le \frac{\theta}{2}; i = 1(1)n. \\ 0, & o/w \end{cases}$$

i.e. $L(\theta|\underset{\sim}{x}) = \begin{cases} \frac{1}{\theta^n}, & \text{if } \max\limits_i |x_i| \le \frac{\theta}{2} \\ 0, & o/w. \end{cases}$

$L(\theta|\underset{\sim}{x})$ is maximized at minimum value of θ given $\underset{\sim}{x}$

$$\Rightarrow \hat{\theta}_{MLE} = 2 \max_i |X_i|$$

(11) $X_1, \ldots X_n$ i.i.d. $\text{Exp}(\theta_1, \theta_2)$

$$\hat{\theta}_{1(MLE)} = X_{(1)}$$

$$\hat{\theta}_{2(MLE)} = \frac{1}{n}\sum_{1}^{n}(X_i - X_{(1)})$$

Done in class.

(12) $X_1, \ldots X_n$ i.i.d. $f_X(x) = \begin{cases} \dfrac{\lambda^\alpha}{\Gamma\alpha} e^{-\lambda x} x^{\alpha-1}, & x \geq 0 \\ 0, & o/w \end{cases}$

$$L(\underset{\sim}{\theta} \mid \underset{\sim}{x}) = \frac{\lambda^{n\alpha}}{(\Gamma\alpha)^n} e^{-\lambda \sum_1^n x_i} \prod_1^n x_i^{\alpha-1}$$

$$\ell(\underset{\sim}{\theta} \mid \underset{\sim}{x}) = \log L(\underset{\sim}{\theta} \mid \underset{\sim}{x}) = n\alpha \log\lambda - n\log\Gamma\alpha + (\alpha-1)\sum \log x_i - \lambda\sum_1^n x_i$$

likelihood eq^{n}s

$$\frac{\partial \log L}{\partial\lambda} = \frac{n\alpha}{\lambda} - \sum x_i = 0 \quad -(1)$$

$$\frac{\partial \log L}{\partial\alpha} = n\log\lambda - n\frac{\Gamma\alpha'}{\Gamma\alpha} + \sum\log x_i = 0 \quad -(2)$$

$$(1) \Rightarrow \lambda = \frac{\alpha}{\bar{x}}$$

From (2), we get

$$n\log\left(\frac{\alpha}{\bar{x}}\right) - n\frac{\Gamma\alpha'}{\Gamma\alpha} + \sum\log x_i = 0 \quad (*)$$

Solving (*) by numerical method gives $\hat{\alpha}_{MLE}$

& $\hat{\lambda}_{MLE} = \hat{\alpha}_{MLE}/\bar{X}$.

(13) $X_1, \ldots X_n$ i.i.d. with p.d.f.

$$f(x;\mu,\sigma) = \begin{cases} \frac{1}{2\sqrt{3}\sigma}, & \mu - \sqrt{3}\sigma < x < \mu + \sqrt{3}\sigma, \\ 0, & o/w \end{cases}$$

likelihood f^n

$$L((\mu,\sigma)|\underset{\sim}{x}) = \begin{cases} \left(\frac{1}{2\sqrt{3}\sigma}\right)^n, & \text{if } \underbrace{\mu - \sqrt{3}\sigma \le x_{(1)} \le \ldots \le x_{(n)} \le \mu + \sqrt{3}\sigma}_{(*)} \\ 0, & o/w. \end{cases}$$

using condition (*),

$$\mu - \sqrt{3}\sigma \le x_{(1)} \ \& \ x_{(n)} \le \mu + \sqrt{3}\sigma$$

$$\Rightarrow \mu \le x_{(1)} + \sqrt{3}\sigma \ \& \ x_{(n)} - \sqrt{3}\sigma \le \mu$$

$$\Rightarrow x_{(n)} - \sqrt{3}\sigma \le \mu \le x_{(1)} + \sqrt{3}\sigma$$

For a given σ, $L((\mu,\sigma)|\underset{\sim}{x})$ is maximized w.r.t. μ if

$$\mu \in (x_{(n)} - \sqrt{3}\sigma, \ x_{(1)} + \sqrt{3}\sigma) \quad \left(o/w \ L(\mu,\sigma|\underset{\sim}{x}) = 0\right)$$

$\Rightarrow$ Any value of μ in the above interval is an MLE of μ

In particular

$$\frac{(X_{(n)} - \sqrt{3}\sigma) + (X_{(1)} + \sqrt{3}\sigma)}{2} = \frac{X_{(n)} + X_{(1)}}{2} = \hat{\mu}^{(\sigma)}_{MLE}$$

Since the above MLE is indep of σ, it is MLE of μ $\forall \sigma$

$$\Rightarrow \hat{\mu}_{MLE} = \frac{X_{(n)} + X_{(1)}}{2}$$

Further, $L(\hat{\mu},\sigma)$ is maximized w.r.t. σ if σ is minimum.

Observe that

$$\sqrt{3}\,\sigma \geq \mu - x_{(1)} \quad \& \quad \sqrt{3}\,\sigma \geq x_{(n)} - \mu$$

at the MLE of μ; $\quad \sqrt{3}\,\sigma \geq \dfrac{x_{(n)} - x_{(1)}}{2}$

$$\Rightarrow \hat{\sigma}_{MLE} = \frac{X_{(n)} - X_{(1)}}{2\sqrt{3}}$$

(14) $X_1, \ldots, X_n$ i.i.d $U[\theta - 1/2, \theta + \frac{1}{2})$, $\theta \in \mathbb{R}$

$$L(\theta \mid \underset{\sim}{x}) = \begin{cases} 1, & \theta - \frac{1}{2} \leq x_{(1)} \leq \ldots \leq x_{(n)} \leq \theta + \frac{1}{2} \\ 0, & \text{o/w} \end{cases}$$

L is maximized w.r.t. θ iff

$$\theta - \frac{1}{2} \leq x_{(1)} \quad \& \quad x_{(n)} \leq \theta + \frac{1}{2} \quad \left(\max_{\theta} L = 1\right)$$

i.e., $x_{(n)} - \frac{1}{2} \leq \theta \leq x_{(1)} + \frac{1}{2}$.

$\Rightarrow$ Any statistic $U(\underset{\sim}{X}) \ni$

$x_{(n)} - \frac{1}{2} \leq u(x_1, \ldots x_n) \leq x_{(1)} + \frac{1}{2}$ is an MLE of θ.

In particular, $\dfrac{X_{(1)} + X_{(n)}}{2}$ is an MLE of θ

In general, $\alpha\left(X_{(1)} + \frac{1}{2}\right) + (1-\alpha)\left(X_{(n)} - \frac{1}{2}\right)$; $\forall\, 0 < \alpha < 1$ is an MLE of θ

With $\alpha = \frac{3}{4}$, we have the above estimator as

$\frac{3}{4}\left(X_{(1)} + \frac{1}{2}\right) + \frac{1}{4}\left(X_{(n)} - \frac{1}{2}\right)$ is an MLE of θ

(15)

X : r.v. denoting lifetime of the component

$X \sim$ Exp distⁿ with mean λ

$$f_X(x) = \begin{cases} \frac{1}{\lambda} e^{-x/\lambda}, & x > 0 \\ 0, & o/w \end{cases}$$

Define
$$Y_i = \begin{cases} 1, & \text{if } i^{\text{th}} \text{ component has life} < 100 \text{ hrs} \\ 0, & o/w. \end{cases}$$

$$P(Y_i = 1) = P(X < 100) = \frac{1}{\lambda}\int_0^{100} e^{-x/\lambda}\,dx = \left(1 - e^{-100/\lambda}\right)$$

$Y_1, \ldots Y_n$ i.i.d $B\left(1, \left(1 - e^{-100/\lambda}\right)\right) \equiv B(1, \theta)$
$(n = 10)$

with $\theta = 1 - e^{-100/\lambda}$.

$\Rightarrow \hat{\theta}_{MLE} = \bar{Y}$ (done in class)

Further, $\lambda = -\dfrac{100}{\log(1-\theta)} = g(\theta)$

$\Rightarrow$ MLE of $g(\theta)$ is $g(\hat{\theta}_{MLE})$.

$$\Rightarrow \hat{\lambda}_{MLE} = -\frac{100}{\log(1-\hat{\theta}_{MLE})} = -\frac{100}{\log(1-\bar{X})}.$$

From the given data $\bar{X} = \frac{3}{10}$

$\Rightarrow$ The maximum likelihood estimate of λ computed from the given data is $\left(-\dfrac{100}{\log\left(\frac{7}{10}\right)}\right)$

(16) X : r.v. denoting the no. of sales per day

$X \sim P(\mu)$ (from the assumptions)

Define $Y_i = \begin{cases} 1, & \text{if } 0 \text{ sales on day } i \\ 0, & \text{o/w} \end{cases}$

$P(Y_i = 1) = P(X = 0) = e^{-\mu}$.

$Y_1, \ldots, Y_{30}$ i.i.d. $B(1, e^{-\mu}) \equiv B(1, \theta)$ $(\theta = e^{-\mu})$

$$\hat{\theta}_{MLE} = \bar{Y}$$

Further, $\mu = -\log \theta$

$\Rightarrow \hat{\mu}_{MLE} = -\log \hat{\theta}_{MLE}$

$\Rightarrow$ ML estimate of μ from the given data is given by $-\log(20/30)$.

(17)

(a) $X_1, \dots X_n$ i.i.d. $P(\theta)$

$$\mu_1' = E(X) = \theta$$

MOME of θ; $\hat{\theta}_{MOME} = m_1' = \overline{X}$.

(b) $X_1 \dots X_n$ i.i.d $U(-\theta/2, \theta/2)$

done in class.

(c) $X_1, \dots X_n$ i.i.d $Exp(0, \theta)$

$$\mu_1' = E(X) = \frac{1}{\theta}\int_0^\infty x e^{-x/\theta}\,dx = \theta$$

$$\Rightarrow \hat{\theta}_{MOME} = m_1' = \overline{X}$$

(d) $X_1, \dots X_n$ i.i.d. $Exp(\alpha, \beta)$

done in class.

(e) $X_1 \dots X_n$ i.i.d. $G(\alpha, \beta)$ with p.d.f

$$f(x;\alpha,\beta) = \begin{cases} \frac{1}{\beta^\alpha \Gamma\alpha} e^{-x/\beta} x^{\alpha-1}, & x > 0 \\ 0, & \end{cases}$$

$$\mu_1' = E(X) = \frac{1}{\beta^\alpha \Gamma\alpha}\int_0^\infty x^{\alpha+1-1} e^{-x/\beta}\,dx = \frac{\Gamma\alpha+1\ \beta^{\alpha+1}}{\Gamma\alpha\ \beta^\alpha} = \alpha\beta$$

$$\mu_2' = E(X^2) = \frac{1}{\beta^\alpha \Gamma\alpha} \int_0^\infty x^{\alpha+2-1} e^{-x/\beta} dx$$

$$= \frac{\Gamma(\alpha+2)\, \beta^{\alpha+2}}{\Gamma\alpha\, \beta^\alpha} = (\alpha+1)\alpha\beta^2,$$

Equate
$$\left.\begin{aligned} \bar{X} &= m_1' = \alpha\beta \\ \frac{1}{n}\sum X_i^2 &= m_2' = \alpha(\alpha+1)\beta^2 \end{aligned}\right\}$$

$$\Rightarrow \quad \frac{m_2'}{m_1'} = \alpha\beta + \beta = m_1' + \beta$$

$$\Rightarrow \quad \hat{\beta}_{MOME} = \frac{m_2' - (m_1')^2}{m_1'} = \frac{\frac{1}{n}\sum X_i^2 - \bar{X}^2}{\bar{X}}$$

i.e. $$\hat{\beta}_{MOME} = \frac{\frac{1}{n}\sum(X_i - \bar{X})^2}{\bar{X}}$$

& $$\hat{\alpha}_{MOME} = \frac{\bar{X}}{\left(\frac{1}{n}\sum(X_i-\bar{X})^2 / \bar{X}\right)} = \frac{\bar{X}^2}{\frac{1}{n}\sum(X_i-\bar{X})^2}$$