CS203 ASSIGNMENT 5 SEMESTER I, 2017-18

Due Date: 16th September, 2017 **Maximum Marks: 30**

Instructions

- Submit the assignment in KD213 before 4 pm on 16th September .
- Yours answers should be precise and clearly written on A4 sheets.
- Cheating/plagiarizing in any form will be heavily penalized.
- Late submissions will receive a mark of zero.

An important aspect of study of curves is to understand when can they be parameterized. We explore this question below. Consider curves on a plane with rational coefficients; these are specified by elements of $\mathbb{Q}[x, y]$. A *parametrization* of curve $C(x, y) = 0$, $C(x, y) \in \mathbb{Q}[x, y]$, is given by equations $x = f(t)$ and $y = g(t)$ such that $C(f(t), g(t)) = 0$ for rational functions f and g. For example, parameterization of circle $x^2 + y^2 = 1$ is $x = \frac{2t}{t^2+1}$ and $y = \frac{t^2-1}{t^2+1}$.

Question 1. (10 marks) Given a parameterization of curve $C(x, y) = 0$, $C(x, y) \in \mathbb{Q}[x, y]$, via rational functions f, $g \in \mathbb{Q}(t)$, prove that it gives rise to a ring homomorphism from ring $\mathbb{Q}[x, y]$ to field $\mathbb{Q}(t)$, $A(x, y) \mapsto A(f(t), g(t))$, with kernel containing ideal $(C(x, y))$.

Question 2. (5 marks) Conversely, given any ring homomorphism $\phi : \mathbb{Q}[x, y] \mapsto \mathbb{Q}(t)$, show that its kernel is a prime ideal. It can be shown that the kernel is also a principle ideal and hence equals $(C(x, y))$ for some $C(x, y) \in \mathbb{Q}[x, y]$.

Therefore, ring homomorphisms from $\mathbb{Q}[x, y]$ to $\mathbb{Q}(t)$ capture parameterization of curves. Let ϕ be such a homomorphism with kernel $I = (C(x, y))$. Let $R = \mathbb{Q}[x, y]/I$. Since I is prime, R is an integral domain. Let F be its field of fractions. We can view ϕ as a homomorphism from F to $\mathbb{Q}(t)$ by extending it as:

$$\phi(\frac{A(x, y) + I}{B(x, y) + I}) = \frac{\phi(A(x, y))}{\phi(B(x, y))} = \frac{A(\phi(x), \phi(y))}{B(\phi(x), \phi(y))}.$$

As we have seen in the class, as a homomorphism of fields, ϕ must be 1-1.
A special case occurs when ϕ is an isomorphism.

Question 3. (15 marks) Prove that the map given by $\phi(x) = \frac{2t}{t^2+1}$ and $\phi(y) = \frac{t^2-1}{t^2+1}$ is an isomorphism from F to $\mathbb{Q}(t)$ where F is the field of fractions of integral domain $\mathbb{Q}[x, y]/(x^2 + y^2 - 1)$.