MSO201A (Mid-Semester Examination)
Time: 2 hours.
Maximum Points = 30.

YOU ARE REQUESTED TO ANSWER SERIALLY IN MANNER

1. (a) Let X and Y be continuous random variables with the joint probability density function

$$f(x,y) = \begin{cases} 2, & \text{if } 0 \le x \le y \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

Find Covariance between X and Y. 3.5 points

(b) Let X be a continuous random variable with the probability density function

$$f(x) = \begin{cases} \frac{x^2}{18}, & \text{if } -3 < x < 3 \\ 0, & \text{otherwise.} \end{cases}$$

Find $P[|X| < 1]$ and $P[X^2 < 9]$. $3 + 1 = 4$ points

2. (a) Let (X, Y) be a random vector having the joint probability density function

$$f(x,y) = \phi(x)\phi(y)\left\{1 + 2\pi xy\phi(x)\phi(y)\right\},$$

where $x \in \mathbb{R}$, $y \in \mathbb{R}$, and $\phi(z) = \frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}$, $z \in \mathbb{R}$. Derive the probability density function of the random variable X. 3.5 points

(b) Let X be a discrete random variable, whose mass points are -1, 0 and 1, and $P[X = 0] = \frac{1}{4}$. Find $E(X^2)$. 4 points

3. (a) Suppose that (X_1, X_2) follows bivariate normal distribution with location parameter $= (0, 0)$, scale parameter $= (1, 1)$ and correlation coefficient $= \frac{1}{2}$. Suppose that $Y_1 = \frac{X_1 + X_2}{\sqrt{2}}$ and $Y_2 = \frac{X_1 - X_2}{\sqrt{2}}$. Does (Y_1, Y_2) jointly follow bivariate normal distribution? Justify your answer. 3.5 points

(b) Let X be a random variable having normal distribution with mean $= \mu$ and variance $= \sigma^2$. Find the distribution of $2X - 6$. 4 points

4. (a) Let 0.37 be a random number generated from a uniform distribution over $(0, 1)$. Using this random number, how will you generate a random number from the exponential distribution (the probability density function of the exponential distribution is $f(x) = e^{-x}$, $x \ge 0$)? What will be the random number from the exponential distribution?

$2 + 1.5 = 3.5$ points.

(b) The cumulative distribution function of a random variable X defined over $0 \le x < \infty$ is $F_X(x) = 1 - e^{-\beta x^2}$, $\beta > 0$. Find mean and variance of the random variable X. $1.5 + 2.5 = 4$ points.

Solutions :-

1.a) Note that

$$\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y)$$ — 0.5 mark (1st step) (i.e, if the student knows this formula).

Now,

$$E(XY) = \int_0^1 \left[\int_x^1 xy\, f(x,y)\, dy\right] dx$$

$$= \int_0^1 y \left[\int_0^y 2x\, dx\right] dy = \int_0^1 y^3\, dy = \frac{y^4}{4}\Big|_0^1 = \frac{1}{4}$$

→ 1.5 marks

Next,

$$E(X) = \int_0^1 \int_0^y x\, f(x,y)\, dx\, dy$$

$$= \int_0^1 \left[\int_0^y 2x\, dx\right] dy = \int_0^1 y^2\, dy = \frac{1}{3}$$

→ 0.5 mark

[Note that one can compute $E(X)$ from the marginal density of X. Here Marginal density of X :-

$$f_X(x) = \int_x^1 f(x,y)\, dy = \int_x^1 2\, dy = 2(1-x), \quad 0 \le x \le 1$$

Hence, $E(X) = \int_0^1 x f_X(x)\, dx = \int_0^1 x\, 2(1-x)\, dx = \frac{1}{3}$.].

Similarly,

$$E(Y) = \int_0^1 \int_0^y y\, f(x,y)\, dx\, dy = \int_0^1 \left[y \int_0^y 2\, dx\right] dy$$

$$= \int_0^1 2y^2\, dy = \frac{2}{3}.$$

→ 0.5 mark

[Similarly, here also, one can compute $E(Y)$ from the marginal density of Y].

Hence,

$$\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y) = \frac{1}{4} - \frac{1}{3} \times \frac{2}{3}$$

$$= \frac{1}{36}.$$

→ 0.5 mark.

b) Note that the CDF of X is

1st step $\underline{F_X(x)} = \begin{cases} 0 & \text{if } x < -3 \\ \int_{-3}^{x} f(y)\,dy = \frac{x^3+27}{54} & \text{if } -3 \le x \le 3 \\ 1 & \text{if } x > 3. \end{cases}$

$\hookrightarrow$ 2 marks.

2nd step

Now $\underline{P[|X| < 1]} = P[-1 < X < 1]$
$= F(1) - F(-1)$
$= \frac{28}{54} - \frac{26}{54} = \frac{2}{54} = \frac{1}{27}.$

1 marks

3rd step

$P[X^2 < 9] = P[-3 < X < 3] = F(3) - F(-3)$
$= 1 - 0 = 1.$

1 marks.

Alternative Method:-

$$\underbrace{P[|X| < 1] = \int_{-1}^{1} f(x)\,dx}_{1 \text{ mark}} = \underbrace{\int_{-1}^{1} \frac{x^2}{18}\,dx = \frac{1}{27}}_{2 \text{ marks.}}$$

$$\underbrace{P[X^2 < 9] = P[-3 < X < 3] = \int_{-3}^{3} \frac{x^2}{18}\,dx}_{1 \text{ mark}} = 1$$

2. a) It is given that

$$f(x,y) = \phi(x)\phi(y)\{1 + 2\pi xy\,\phi(x)\phi(y)\}.$$

f_X : Marginal density of X.

So, $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$

$$= \int_{-\infty}^{\infty} [\phi(x)\phi(y)\{1 + 2\pi xy\,\phi(x)\phi(y)\}\,dy$$

$$= \underbrace{\phi(x)\int_{-\infty}^{\infty}\phi(y)\,dy + 2\pi x\,\phi^2(x)\int_{-\infty}^{\infty} y\,\phi^2(y)\,dy}_{1'5 \text{ marks}}$$

$$= \underbrace{\phi(x)}_{\text{since } \int_{-\infty}^{\infty}\phi(y)dy = 1} + 2\pi x\,\phi^2(x)\int_{-\infty}^{\infty} \underbrace{y \times \frac{1}{2\pi} e^{-\frac{y^2}{2}}}_{\text{odd } f^n}\,dy$$

$$= \phi(x) \longrightarrow 2 \text{ marks}$$

Hence, $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}},\ x \in \mathbb{R}$ (the probability density function of $N(0,1)$).

b) $E(X^2) = (-1)^2 P[X=-1] + 0^2 P[X=0]$
$\qquad + 1^2 P[X=1]$

$$= P[X=-1] + P[X=1] \longrightarrow 2 \text{ marks.}$$

$$= 1 - P[X=0]$$

$$= 1 - \frac{1}{4} = \frac{3}{4} \longrightarrow 2 \text{ marks}$$

3. a) Given $\underset{\sim}{X} = (X_1, X_2) \sim N_2\left(0, 0, 1, 1, \frac{1}{2}\right)$.

$$Y_1 = \frac{X_1 + X_2}{\sqrt{2}} \quad \& \quad Y_2 = \frac{X_1 - X_2}{\sqrt{2}}$$

Consider $C_1 Y_1 + C_2 Y_2$ [C_1, C_2 arbitrary constants]

1 marks

(i.e, if a student realises that the linear ~~combinatio~~ combination has to be considered)

1st step

$$= C_1\left(\frac{X_1 + X_2}{\sqrt{2}}\right) + C_2\left(\frac{X_1 - X_2}{\sqrt{2}}\right).$$

$$= \left(\frac{C_1}{\sqrt{2}} + \frac{C_2}{\sqrt{2}}\right) X_1 + \left(\frac{C_1}{\sqrt{2}} - \frac{C_2}{\sqrt{2}}\right) X_2$$

$$:= K_1 X_1 + K_2 X_2 \quad (K_1, K_2 \text{ are arbitrary constant})$$

$\Downarrow$

2nd step

2marks

It follows ~~uni~~ univariate normal distribution since $(X_1, X_2) \sim N_2(\cdots)$.

Since ~~Hence~~, $C_1 Y_1 + C_2 Y_2 \sim N_1(\quad)$ for any C_1 & C_2,

3rd step

$\Rightarrow (Y_1, Y_2) \sim N_2(\cdots)$.

0.5 mark

[Note :- If a student writes 'yes' ~~dont~~ without any explanation, give 1 mark].

b) $X \sim N(\mu, \sigma^2)$

$Y = 2X - 6 \in (-\infty, \infty)$.

$$F_Y(y) = P[Y \leq y] = P[2X - 6 \leq y]$$

CDF of Y

$$= P\left[X \leq \frac{y+6}{2}\right]$$

$$= P\left[\frac{X - \mu}{\sigma} \leq \frac{\frac{y+6}{2} - \mu}{\sigma}\right]$$

$$= \Phi\left(\frac{y + 6 - 2\mu}{2\sigma}\right).$$

2 marks

Therefore,

$$f_Y(y) = \frac{d}{dy}\left[\Phi\left(\frac{y+6-2\mu}{2\sigma}\right)\right]$$

PDF of Y

$$= \phi\left(\frac{y+6-2\mu}{2\sigma}\right) \times \frac{1}{2\sigma}$$

$$= \underline{\frac{1}{\sqrt{2\pi}\,2\sigma}\, e^{-\frac{1}{2(4\sigma^2)}\left(y-(2\mu-6)\right)^2}}, \quad y \in \mathbb{R}.$$

Hence, $Y \sim N_1(2\mu - 6, 4\sigma^2)$. 2 marks.

<u>Alternative method:-</u>

If $X \sim N(\mu, \sigma^2)$, then

$$\underline{M_X(t) = e^{t\mu + \frac{1}{2}t^2\sigma^2}} \longrightarrow 1.5 \text{ marks}$$

MGF of X

(with derivation or without derivation)

If $Y = 2X - 6$, we have

$$M_Y(t) = E\left(e^{tY}\right) = E\left(e^{t(2X-6)}\right)$$

MGF of Y

$$= e^{-6t}\, E\left[e^{X(2t)}\right]$$

$$= e^{-6t} \times e^{\mu(2t) + \frac{1}{2}\sigma^2(2t)^2}$$

$$= \underline{e^{t(2\mu-6) + \frac{1}{2}t^2(4\sigma^2)}}.$$

Hence, $Y \sim N(2\mu - 6, 4\sigma^2)$. 2.5 marks.

4.

a) X: Random variable with distribution F (continuous).

U: Random variable with uniform distribution over (0,1).

Then $F^{-1}(U)$'s ~~follows F~~. CDF will be F.

Hence, based on the random number from Unif(0,1), we can generate the random number from exponential distribution, and the random number will be $F^{-1}(0.37)$, where F is the CDF of exponential distribution.

2 marks.

Here $\overset{\nearrow \text{p.d.f.}}{f(x)} = e^{-x}, \; x \geq 0$

so, $\underset{\swarrow \text{c.d.f.}}{F(x)} = \int_0^x e^{-y}\,dy = 1 - e^{-x}, \; x \geq 0.$

Hence $F^{-1}(0.37)$ will be the sol$^{\underline{n}}$ of x in

$$1 - e^{-x} = 0.37$$

$$\Leftrightarrow x = \log 1/0.63 \approx 0.46203546$$

<u>Answer</u>:- 0.46203546 will be the random number from exp. distribution

1.5 marks.

b) Given that

$$\underset{\text{CDF}}{F(x)} = \begin{cases} 0 & \text{if } x < 0 \\ 1 - e^{-\beta x^2} & \text{if } x \geq 0 \end{cases}$$

So, $$\underset{\text{PDF}}{f(x)} = \begin{cases} 2\beta x e^{-\beta x^2}, & x \geq 0 \\ 0, & \text{o.w.} \end{cases} \longrightarrow 1 \text{ mark}$$

Therefore,

$$E(X) = 2\beta \int_0^\infty x^2 e^{-\beta x^2} dx$$

$$= \beta \int_0^\infty y^{1/2} e^{-\beta y} dy = \beta \frac{\Gamma 3/2}{\beta^{3/2}} = \frac{1}{2} \frac{\sqrt{\pi}}{\sqrt{\beta}}.$$

$\longrightarrow$ 0.5 mark.

Note that $Var(X) = E(X^2) \{E(X)\}^2$. $\longrightarrow$ 0.5 mark

Now, $$E(x^2) = 2\beta \int x^3 e^{-\beta x^2} dx$$

$$= \beta \int_0^\infty y \cdot e^{-\beta y} dy = \beta \frac{\Gamma 2}{\beta^2} = \frac{1}{\beta}.$$

$\longrightarrow$ 1.5 mark.

So, $$Var(X) = E(x^2) - \{E(X)\}^2$$

$$= \frac{1}{\beta} - \frac{\pi}{4\beta} = \frac{4-\pi}{4\beta}$$

$\longrightarrow$ 0.5 mark.