

Problem set #5

(1)

Belt 1

$X \sim \text{Exp}$ with mean $\alpha \sim \frac{1}{\alpha} e^{-x/\alpha} ; x > 0$

Belt 2

$Y \sim \text{Exp}$ with mean $2\alpha \sim \frac{1}{2\alpha} e^{-x/2\alpha} ; x > 0$

$P(\text{system works beyond } \alpha)$

$$\begin{aligned} P(X > \alpha \cap Y > \alpha) &= P(X > \alpha) P(Y > \alpha) \\ &= \left(\int_{\alpha}^{\infty} \frac{1}{\alpha} e^{-x/\alpha} dx \right) \left(\int_{\alpha}^{\infty} \frac{1}{2\alpha} e^{-x/2\alpha} dx \right) \\ &= e^{-1} \times e^{-1/2} = e^{-3/2} \end{aligned}$$

$$\begin{aligned} (2) \quad (a) \quad P(X > 5) &= P\left(\frac{X-10}{6} > \frac{5-10}{6}\right) = P\left(Z > -\frac{5}{6}\right); Z \sim N(0,1) \\ &= 1 - \Phi\left(-\frac{5}{6}\right) \\ &= 1 - (1 - \Phi\left(\frac{5}{6}\right)) \\ &= \Phi\left(\frac{5}{6}\right) = .7967 \end{aligned}$$

$$\begin{aligned} (b) \quad P(4 < X < 16) &= P\left(\frac{4-10}{6} < Z < \frac{16-10}{6}\right) = P(-1 < Z < 1) \\ &= \Phi(1) - \Phi(-1) = 2\Phi(1) - 1 \end{aligned}$$

$$(c) \quad P(X < 8) = P\left(Z < \frac{8-10}{6}\right) = \Phi\left(-\frac{1}{3}\right) = 1 - \Phi\left(\frac{1}{3}\right) = \dots$$

$$(3) \quad P(X \leq 0) = \frac{1}{2} = P(X \geq 0) \Rightarrow \mu = 0$$

$$P(-1.96 \leq X \leq 1.96) = 0.95$$

$$P\left(-\frac{1.96}{\sigma} \leq \frac{X}{\sigma} \leq \frac{1.96}{\sigma}\right) = 0.95$$

$$P\left(-\frac{1.96}{\sigma} \leq Z \leq \frac{1.96}{\sigma}\right) = 0.95 ; Z \sim N(0,1)$$

$$2\Phi\left(\frac{1.96}{\sigma}\right) - 1 = 0.95$$

$$\Phi\left(\frac{1.96}{\sigma}\right) = 0.975$$

$$\Rightarrow \frac{1.96}{\sigma} = \Phi^{-1}(0.975) = 1.96$$

$$\Rightarrow \sigma = 1$$

(4) X : lifetime r.v.

$$X \sim N(\mu, \sigma^2)$$

$$\mu = 1.4 \times 10^6 \text{ hrs}$$

$$\sigma = 3 \times 10^5 \text{ hrs}$$

$$P(X < 1.8 \times 10^6)$$

$$= P\left(\frac{X - 1.4 \times 10^6}{3 \times 10^5} < \frac{0.4 \times 10^6}{3 \times 10^5}\right)$$

$$= \cancel{\Phi\left(\cancel{z} < \cancel{\frac{4}{3}}\right)} = P\left(z < \frac{4}{3}\right) \quad [z \sim N(0,1)]$$

$$= \Phi\left(\frac{4}{3}\right) = 0.918$$

Y : r.v. denoting # of chips that have lifetime
 $< 1.8 \times 10^6 \text{ hr}$

$$Y \sim \text{Bin}(10, 0.918)$$

$$\Rightarrow P(Y \geq 2) = 1 - P(Y < 2)$$

$$= 1 - P(Y=0) - P(Y=1)$$

$$= 1 - \binom{10}{0} (0.918)^0 (1-0.918)^{10} - \binom{10}{1} (0.918)^1 (1-0.918)^9$$

$$= \dots$$

$$(5) \quad X \sim N(0, 1)$$

$$\begin{aligned} \forall t > 0 \quad P(|X| \geq t) &= 1 - P(|X| < t) \\ &= 1 - P(-t < X < t) \\ &= 1 - [\Phi(t) - \Phi(-t)] \\ &\quad \swarrow \\ &= 1 - [2\Phi(t) - 1] \\ &= 1 - [2(1 - P(X > t)) - 1] \\ &= 2 - 2 + 2P(X > t) = 2P(X > t) \end{aligned}$$

$$\begin{aligned} P(X > t) &= \frac{1}{\sqrt{2\pi}} \int_t^{\infty} e^{-x^2/2} dx \\ &\leq \frac{1}{\sqrt{2\pi}} \int_t^{\infty} \frac{x}{t} e^{-x^2/2} dx \quad [t < x < \infty] \\ &\quad y = x^2/2 \end{aligned}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{t} \int_{t^2/2}^{\infty} e^{-y} dy = \frac{1}{\sqrt{2\pi}} \frac{e^{-t^2/2}}{t}$$

$$\Rightarrow P(|X| \geq t) \leq 2 \frac{1}{\sqrt{2\pi}} \frac{e^{-t^2/2}}{t} = \sqrt{\frac{2}{\pi}} \frac{e^{-t^2/2}}{t}$$

$$(6) \quad \begin{array}{ccccccc} X=x & 0 & 1 & 2 & - & - & \dots \\ P(X=x) & p_0 & p_1 & p_2 & - & - & \dots \end{array}$$

$$\sum_{k=0}^{\infty} (1 - F(k)) = \sum_{k=0}^{\infty} P(X > k) = P(X > 0) + P(X > 1) + P(X > 2) + \dots$$

$$\begin{aligned} &= (p_1 + p_2 + p_3 + \dots) \\ &\quad + (p_2 + p_3 + \dots) \\ &\quad + (p_3 + p_4 + \dots) \end{aligned}$$

$$= p_1 + 2p_2 + 3p_3 + \dots$$

$$= \sum_{i=1}^{\infty} i p_i = \sum_{i=0}^{\infty} i P(X=i) = E(X)$$

(7) d.f. $F(x) = \begin{cases} 0, & x < 0 \\ 1 - e^{-\beta x^2}, & x \geq 0. \end{cases}$

$\beta > 0$

p.d.f. $f(x) = \begin{cases} 2\beta x e^{-\beta x^2}, & x \geq 0 \\ 0 & \text{o.w.} \end{cases}$

$$E(X) = 2\beta \int_0^{\infty} x^2 e^{-\beta x^2} dx$$

$$= \beta \int_0^{\infty} y^{1/2} e^{-\beta y} dy = \beta \cdot \frac{\Gamma^{3/2}}{\beta^{3/2}} = \frac{1}{2} \frac{\sqrt{\pi}}{\sqrt{\beta}} = \mu.$$

$$E(X^2) = 2\beta \int_0^{\infty} x^3 e^{-\beta x^2} dx = \beta \int_0^{\infty} y e^{-\beta y} dy = \beta \frac{\Gamma_2}{\beta^2} = \frac{1}{\beta}$$

$$V(X) = E(X^2) - (E(X))^2 = \frac{1}{\beta} - \mu^2 = \frac{1}{\beta} - \frac{\pi}{4\beta}$$

median: m_0

$$m_0 \Rightarrow F(m_0) = \frac{1}{2} = 1 - F(m_0)$$

$$\text{i.e. } 2\beta \int_0^{m_0} x e^{-\beta x^2} dx = 2\beta \int_{m_0}^{\infty} x e^{-\beta x^2} dx = \frac{1}{2}$$

$$\text{i.e. } 1 - e^{-\beta m_0^2} = \frac{1}{2}$$

$$\Rightarrow m_0 = \dots$$

$$\begin{aligned}
 (8) \quad 1 - \Phi(x) &= \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-y^2/2} dy \\
 &= \frac{1}{\sqrt{2\pi}} \left[\int_x^{\infty} \frac{1}{y} (y e^{-y^2/2}) dy \right] \\
 &= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{y} \cdot (-e^{-y^2/2}) \Big|_x^{\infty} \right. \\
 &\quad \left. - \int_x^{\infty} \left(-\frac{1}{y^2}\right) (-e^{-y^2/2}) dy \right] \\
 &= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{x} e^{-x^2/2} - \int_x^{\infty} \frac{1}{y^2} e^{-y^2/2} dy \right] \\
 &\qquad\qquad\qquad \geq 0.
 \end{aligned}$$

$$\Rightarrow 1 - \Phi(x) \leq \frac{1}{x} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} = \frac{\phi(x)}{x}.$$

(9) p.d.f. of (7)
Mode - pt at which $f(x)$ is max^m.

$$f'(x) = 2\beta (x e^{-\beta x^2} (-2\beta x) + e^{-\beta x^2})$$

$$f'(x) = 0 \Rightarrow 2\beta x^2 = 1 \Rightarrow x = \frac{1}{\sqrt{2\beta}}$$

$$\begin{aligned} f''(x) &= 2\beta \frac{d}{dx} (e^{-\beta x^2} (1 - 2\beta x^2)) \\ &= 2\beta (e^{-\beta x^2} (-4\beta x) + (1 - 2\beta x^2) e^{-\beta x^2} (-2\beta x)) \end{aligned}$$

$$f''(x) \Big|_{x=\frac{1}{\sqrt{2\beta}}} = 2\beta \left(e^{-\frac{1}{2}} (-4\sqrt{\beta/2}) \right) < 0.$$

$\beta > 0$

$\Rightarrow m^*$, the mode of the distⁿ is at $\frac{1}{\sqrt{2\beta}}$.

$$m^* = \frac{1}{\sqrt{2\beta}}$$

$$\mu = E(x) = \frac{\sqrt{\pi}}{2} \cdot \frac{1}{\sqrt{\beta}} = \left(\frac{\sqrt{\pi}}{2} \right) \sqrt{2} m^*$$

$$\text{i.e. } \mu = \sqrt{\frac{\pi}{2}} m^*.$$

$$\Delta \quad 2 m^{*2} - \mu^2 = 2 \left(\frac{2}{\pi} \mu^2 \right) - \mu^2$$

$$= \frac{4}{\pi} \mu^2 - \mu^2 = \frac{4}{\pi} \cdot \frac{\pi}{4} \cdot \frac{1}{\beta} - \mu^2$$

i.e.

$$2 m^{*2} - \mu^2 = \sigma^2$$

$$\begin{aligned} &= \left(\frac{4}{\pi} = 1 \right) \mu^2 \Rightarrow \mu^2 = \frac{1}{\beta} - \mu^2 \\ &= E x^2 - \mu^2 \\ &= V(x). \end{aligned}$$

$$(10) \quad X \sim P(\lambda)$$

$$\text{p.m.f. } P(X=x) = \begin{cases} \frac{e^{-\lambda} \lambda^x}{x!}, & x=0, 1, 2, \dots \\ 0, & \text{o/w.} \end{cases}$$

$$Y = X^2 - 5 \Rightarrow \text{Range space of } Y = \{-5, -4, -1, 4, 11, \dots\} \\ = \mathcal{Y}$$

$$P(Y=y) = P(X^2 - 5 = y) = P(X^2 = y+5)$$

$$\text{p.m.f. of } Y : P(Y=y) = P(X = \sqrt{y+5}) = \begin{cases} \frac{e^{-\lambda} \lambda^{\sqrt{y+5}}}{(\sqrt{y+5})!}, & y \in \mathcal{Y} \\ 0, & \text{o/w.} \end{cases}$$

(11)

$$X \sim B(n, p)$$

$$\text{p.m.f. } P(X=x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x}, & x=0, 1, \dots, n \\ 0, & \text{o/w.} \end{cases}$$

$$Y = n - X \Rightarrow \mathcal{Y} = \{0, 1, \dots, n\}.$$

$$P(Y=y) = P(n - X = y) = P(X = n - y)$$

$$\Rightarrow \text{p.m.f. of } Y \quad P(Y=y) = \begin{cases} \binom{n}{n-y} p^{n-y} (1-p)^{n-(n-y)}; & y=0, 1, \dots, n \\ 0, & \text{o/w} \end{cases}$$

$$= \begin{cases} \binom{n}{y} (1-p)^y p^{n-y}, & y=0, 1, \dots, n \\ 0, & \text{o/w.} \end{cases}$$

$$\Rightarrow Y \sim B(n, 1-p).$$

$$(12) \quad Y = X^2 \rightarrow \text{range of } Y = \{0, 1, 4, 9\}$$

$$\text{p.m.f. } P(Y=y) = \begin{cases} P(X=0) & y=0 \\ P(X=-1)+P(X=1) & y=1 \\ P(X=-2)+P(X=2) & y=4 \\ P(X=3) & y=9 \end{cases}$$

$$= \begin{cases} \frac{1}{5}, & y=0 \\ \frac{1}{6} + \frac{1}{15}, & y=1 \\ \frac{1}{5} + \frac{1}{3}, & y=4 \\ \frac{1}{30}, & y=9 \end{cases}$$

$$(13) \quad P(X=x) = \begin{cases} \frac{1}{3} \left(\frac{2}{3}\right)^x, & x=0, 1, 2, \dots \\ 0, & \text{o/w} \end{cases}$$

$$Y = \frac{X}{X+1} \Rightarrow X = \frac{Y}{1-Y}$$

$$\text{range space of } Y = \left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots\right\}$$

$$\begin{aligned} P(Y=y) &= P\left(\frac{X}{X+1} = y\right) = P\left(X = \frac{y}{1-y}\right) \\ &= \begin{cases} \frac{1}{3} \left(\frac{2}{3}\right)^{\frac{y}{1-y}}, & y = 0, \frac{1}{2}, \frac{2}{3}, \dots \\ 0 & \text{o/w} \end{cases} \end{aligned}$$

(14) p.m.f. of X

$$P(X=x) = \begin{cases} e^{-1}, & x=0 \\ \frac{e^{-1}}{2(|x|)!}, & x \in \{\pm 1, \pm 2, \dots\} \\ 0, & \text{o/w.} \end{cases}$$

$$Y = |X| \quad Y = \{0, 1, 2, \dots\}$$

~~P(Y=0)~~ $P(Y=0) = P(X=0) = e^{-1}$

$$\begin{aligned} P(Y=1) &= P(X=-1) + P(X=1) \\ &= \frac{e^{-1}}{2} + \frac{e^{-1}}{2} = e^{-1} \end{aligned}$$

$$\begin{aligned} P(Y=2) &= P(X=-2) + P(X=2) \\ &= \frac{e^{-1}}{2 \cdot 2!} + \frac{e^{-1}}{2 \cdot 2!} = \frac{e^{-1}}{2!} \end{aligned}$$

slly for $k=1, 2, \dots$

$$\begin{aligned} P(Y=k) &= P(X=-k) + P(X=k) \\ &= \frac{e^{-1}}{2 \cdot k!} + \frac{e^{-1}}{2 \cdot k!} = \frac{e^{-1}}{k!} \end{aligned}$$

p.m.f. of Y

$$P(Y=y) = \begin{cases} \frac{e^{-1}}{y!}, & y = 0, 1, 2, \dots \\ 0, & \text{o/w} \end{cases}$$

$$Y \sim P(1)$$