Math for CS I/Discrete Mathematics
Assignment 6 Solutions

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I Problem 1 Solution

1.1 Part (a)

In this part of the question we have 2 players A and B and they take turns rolling a die, they each need different value to win.
A wins with probability α and B wins with probability β in a certain rolling attempt.
Let A be the event that A wins and B be the event that B wins. I have to find the probability of A being the winner for 2 cases:
A rolls first:

Solution. The series of events possible in this case are:

$$A, \overline{AB}A, \overline{ABAB}A, \overline{ABABAB}A, \dots$$

The probability of A winning i.e. $P(A)$ is:

$$P(A) = \alpha + \alpha(1-\alpha)(1-\beta) + \alpha(1-\alpha)^2(1-\beta)^2 + \dots$$
$$P(A) = \frac{\alpha}{1-(1-\alpha)(1-\beta)}$$
$$P(A) = \frac{\alpha}{\alpha+\beta-\alpha\beta}$$

□

A rolls second:

Solution. The series of events possible in this case are:

$$\overline{B}A, \overline{BAB}A, \overline{BABAB}A, \dots$$

The probability of A winning i.e. $P(A)$ is:

$$P(A) = \alpha(1-\beta) + \alpha(1-\beta)^2(1-\alpha) + \alpha(1-\beta)^3(1-\alpha)^2 + \dots$$
$$P(A) = \frac{\alpha(1-\beta)}{1-(1-\alpha)(1-\beta)}$$
$$P(A) = \frac{\alpha(1-\beta)}{\alpha+\beta-\alpha\beta}$$

□

1.2 Part (b)

Solution. 2 coins A and B show heads with respective probabilities α and β.
They are tossed alternately such that A is tossed first and I have to find the probability such that A is first to show a head.
This case is same as

$$A, \overline{AB}A, \overline{ABAB}A, \overline{ABABAB}A, \dots$$

here, $\overline{A}$ denotes A not showing head and same applies to $\overline{B}$.

$$P(A) = \alpha + \alpha(1-\alpha)(1-\beta) + \alpha(1-\alpha)^2(1-\beta)^2 + \dots$$
$$P(A) = \frac{\alpha}{1-(1-\alpha)(1-\beta)}$$
$$P(A) = \frac{\alpha}{\alpha+\beta-\alpha\beta}$$

□

1.3 Part (c)

Solution. Consider the 2 events as below:
E_1: Out of 3 coin tosses, first 2 coins result in same outcome.
E_2: Outcomes of all 3 coin tosses are alike.
In the argument of the problem, it is true that out of 3 coin toss outcomes, atleast 2 must be same and the probability that the third is a head or a tail is $\frac{1}{2}$ which is independent of other 2.
The mistake in the argument is assuming that the 2 same outcomes are of the first 2 coins.
The probability given by the above argument is $\frac{1}{2}$ but this probability is $P(E_2 \mid E_1)$ and not $P(E_2)$.

$$P(E_2 \mid E_1) = \frac{P(E_2 \cap E_1)}{P(E_1)}$$
$$P(E_2 \mid E_1) = \frac{P(E_1 \mid E_2) \cdot P(E_2)}{P(E_1)}$$

$P(E_1) = \frac{1}{2}$, it can be trivially calculated.
$P(E_1 \mid E_2) = 1$ as event E_1 lies within event E_2.

$$P(E_2) = P(E_2 \mid E_1) \cdot P(E_1)$$
$$P(E_2) = \frac{1}{2} \cdot \frac{1}{2}$$
$$P(E_2) = \frac{1}{4}$$

The above value for the event E_2 is the one which it should be i.e. $\frac{1}{4}$, so it is consistent with the argument I made. □

1.4 Part (d)

Solution. There are 2 players A and B, A flips $n+1$ fair coins and B flips n fair coins.
I have to find the probability of A having more heads than B, consider the following events:
E: A gets more number of heads than B.
F: A gets more number of tails than B.
The events E and F are the only possible events as it is not possible for both A and B to have equal number of heads since they have unequal number of coins.
If $A(heads) > B(heads)$, it will be event E and if $A(heads) \leq B(heads)$, it will be event F and thus, these 2 events are exhaustive.
Also, we will have $P(E) = P(F)$ as heads and tails can be interchanged.

$$P(E) + P(F) = 1$$
$$2 \cdot P(E) = 1$$
$$P(E) = \frac{1}{2}$$

□

II Problem 2 Solution

2.1 Part (a)

Solution. We have a s-sided fair die, it is rolled r times, so the possible number of outcomes are s^r.
Consider the following events:
E_1: Side 1 did not appear in the rolls.
E_2: Side 2 did not appear in the rolls.
$\vdots$
E_s: Side s did not appear in the rolls.
Probability that atleast 1 side of the side does not show in the r rolls is $P(E_1 \cup E_2 \cup \cdots \cup E_s)$.

$$P(E_1 \cup E_2 \cup \cdots \cup E_s) = \frac{\binom{s}{1}(s-1)^r - \binom{s}{2}(s-2)^r \ldots (-1)^{s-2}\binom{s}{s-1}(s-s+1)^r}{s^r}$$
$$P(E_1 \cup E_2 \cup \cdots \cup E_s) = \frac{\sum_{i=1}^{s-1}(-1)^{i-1}\binom{s}{i}(s-i)^r}{s^r}$$

The probability p that each side has turned up atleast once if it was rolled r times.

$$p = 1 - P(E_1 \cup E_2 \cup \cdots \cup E_s)$$
$$p = \frac{s^r - \sum_{i=1}^{s-1}(-1)^{i-1}\binom{s}{i}(s-i)^r}{s^r}$$
$$p = \frac{\sum_{i=0}^{s}(-1)^{i}\binom{s}{i}(s-i)^r}{s^r}$$

□

2.2 Part (b)

Solution. Assuming there are 365 days in the year of people's birthdays for calculation, even if it is 366 the final answer will not change.
Let n be the number of students who have announced their birthdays before you, we have:

$$P(n=0) = 0$$
$$P(n=1) = \frac{365}{365} \cdot \frac{1}{365}$$
$$P(n=2) = \frac{365}{365} \cdot \frac{364}{365} \cdot \frac{2}{365}$$
$$P(n) = \frac{\binom{365}{n} n! \binom{n}{1}}{365^{n+1}}$$

I have to find out max value of $P(n)$, it will increase with increasing n till a specific value and then it will keep on decreasing.

$$P(n+1) < P(n)$$
$$\frac{\binom{365}{n+1}(n+1)!\binom{n+1}{1}}{365^{n+2}} < \frac{\binom{365}{n} n! \binom{n}{1}}{365^{n+1}}$$
$$(365-n)(n+1) < 365n$$
$$n^2 + n - 365 > 0$$

The above equation is valid for $n > 18.611$, so using the first integer value, the **maxima** will occur at $n = 19$.
So, the max probability of winning the prize will be when 19 students before you have announced their birthday.
So, state your birthday at 20^{th} place and you will have best probability of winning. □

III Problem 3 Solution

X has a mass function $f(x)$, i.e. $f(x) = P(X = x)$.

$$\Omega = \{\omega_1, \omega_2, \ldots, \omega_n\}$$
$$X = \{x_1, x_2, \ldots, x_m\}$$
$$P(X = x_i) = P(\omega_j \mid \omega_j \in \Omega, X(\omega_j) = x_i)$$

Using the above definitions of the mass function and probability of X, I will find the mass function for the given sets of X' in the following parts.

3.1 Part (a)

Solution. Consider the mass function for $-X$ to be $g(x)$.
We will have $g(x) = P(-X = x)$ or $g(x) = P(X = -x)$.
Thus, the function $g(x)$ is:

$$g(x) = f(-x)$$

□

3.2 Part (b)

Solution. Consider the mass function for $X^+ = max(0, X)$ to be $g(x)$.
X^+ will have only non-negative values of x and all the negative values will result in zero, so the final function will be as below.

$$g(x) = \begin{cases} f(x), & x > 0 \\ \sum_{x \le 0} f(x), & x = 0 \\ 0, & x < 0 \end{cases}$$

□

3.3 Part (c)

Solution. Consider the mass function for $X^- = max(0, -X)$ to be $g(x)$.
X^- will have only non-negative values of x and all the positive values will result in zero as it is $-X$, so the final function will be as below.

$$g(x) = \begin{cases} f(-x), & x > 0 \\ \sum_{x \le 0} f(-x), & x = 0 \\ 0, & x < 0 \end{cases}$$

□

3.4 Part (d)

Solution. Consider the mass function for $\mid X \mid = X^+ + X^-$ to be $g(x)$.
$\mid X \mid$ will have only non-negative values of x and all the values will be result of X and $-X$, so the final function will be as below.

$$g(x) = \begin{cases} f(-x) + f(x), & x > 0 \\ f(0), & x = 0 \\ 0, & x < 0 \end{cases}$$

□

3.5 Part (e)

Solution. Consider the mass function to be $g(x)$ for:

$$sgn(X) = \begin{cases} \frac{X}{|X|}, & X \neq 0 \\ 0, & X = 0 \end{cases}$$

$sgn(X)$ will have only 3 values of $x = \{-1, 0, 1\}$ i.e. all negative values will result in -1 and all positive values will result in 1, so the final function will be as below.

$$g(x) = \begin{cases} \sum_{x<0} f(x), & x = -1 \\ f(0), & x = 0 \\ \sum_{x>0} f(x), & x = 1 \\ 0, & \text{otherwise.} \end{cases}$$

□

IV Problem 4 Solution

4.1 Part (a)

The probability mass function is $f(x) = P(X = x)$ by definition and x here is the number on the card, since the box contains n cards numbered from 1 to n and one is picked at random.
So the following will be the probability mass function:

$$f(x) = \begin{cases} \frac{1}{n}, & \text{if } x = \{1, 2, \dots, n\}; \\ 0, & \text{otherwise.} \end{cases}$$

Solution. I have to find the value of $E[X]$ which is also called the mean expectation value.

$$E[X] = \sum_{x=1}^{n} x \cdot f(x)$$
$$E[X] = \frac{1}{n} \sum_{x=1}^{n} x$$
$$E[X] = \frac{1}{n} \cdot \frac{n(n+1)}{2}$$
$$E[X] = \frac{n+1}{2}$$

□

Solution. I have to find the value of $E[X^2]$ which is also called the second moment.

$$E[X^2] = \sum_{x=1}^{n} x^2 \cdot f(x)$$
$$E[X^2] = \frac{1}{n} \sum_{x=1}^{n} x^2$$
$$E[X^2] = \frac{1}{n} \cdot \frac{n(n+1)(2n+1)}{6}$$
$$E[X^2] = \frac{(n+1)(2n+1)}{6}$$

□

Solution. I have to find the value of second central moment which is also called variance.
The second central moment is $E[(X - E[X])^2] = E[X^2] - (E[X])^2$.

$$E[(X - E[X])^2] = E[X^2] - (E[X])^2$$
$$E[(X - E[X])^2] = \frac{(n+1)(2n+1)}{6} - \left(\frac{n+1}{2}\right)^2$$
$$E[(X - E[X])^2] = \frac{n^2 - 1}{12}$$

□

4.2 Part (b)

Assuming $E[X]$ exists, I have to prove that:

$$(E[X])^2 \le (E[|X|])^2 \le E[X^2]$$

Proof. We know that $x \le |x|$.
Multiplying the above equation by a non-negative number will not change the validity of the inequality and we know that $p(x) \ge 0$, so multiply by it.
We get $x \cdot p(x) \le |x| p(x)$, since it is valid for every value of x, we can do summation of it and still the inequality will be valid.

$$\begin{gathered} x \cdot p(x) \le |x| \cdot p(x) \\ \sum x \cdot p(x) \le \sum |x| \cdot p(x) \\ \left(\sum x \cdot p(x)\right)^2 \le \left(\sum |x| \cdot p(x)\right)^2 \\ (E[X])^2 \le (E[|X|])^2 \end{gathered}$$

By definition of variance for absolute x which is:

$$Var[|X|] = \sum (|x| - E[|X|])^2 \cdot p(|x|)$$

It is clear from above that $Var[|X|] \ge 0$.

$$\begin{gathered} E[X^2] - (E[|X|])^2 \ge 0 \\ E[X^2] \ge (E[|X|])^2 \end{gathered}$$

By above equations, I have proved the required result for a discrete random variable.

$$(E[X])^2 \le (E[|X|])^2 \le E[X^2]$$

For proving the result for a continuous random variable we can combine the above arguments with **Riemann Integral Hypothesis** i.e. breaking integral into summation of small strips. □