Abstract Algebra
Assignment 5 Solutions

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I Problem 1 Solution

We are given a curve $C(x,y)=0$, $C(x,y)\in\mathbb{Q}[x,y]$ and it is parameterized by rational functions $f,g\in\mathbb{Q}(t)$ where $\mathbb{Q}(t)$ is a field and $\mathbb{Q}[x,y]$ is a ring.

Claim 1.1. *There is a ring homomorphism from ring $\mathbb{Q}[x,y]$ to field $\mathbb{Q}(t)$, $A(x,y)\mapsto A(f(t),g(t))$*

Proof. Assume a mapping $\phi:\mathbb{Q}[x,y]\mapsto\mathbb{Q}(t)$ such that $\phi(x)=f(t)$ and $\phi(y)=g(t)$.
Since $x=f(t)$ and $y=g(t)$ which is the result of the parametrization done on $C(x,y)$ and $f(t),g(t)$ are rational functions, so

$$ax+by=af(t)+bg(t) \tag{1.1}$$
$$ax^n=a(f(t))^n \tag{1.2}$$
$$x^my^n=(f(t))^m(g(t))^n \tag{1.3}$$

Since $A(x,y)\in\mathbb{Q}[x,y]$ is a polynomial in x,y we know that it will have combination of integral powers of x and y with constants, so by (1.1), (1.2), (1.3) and mapping of ϕ we have the following equations.

$$\phi(A(x,y))=A(\phi(x),\phi(y)) \tag{1.4}$$

Since we know that sum or product of any polynomials in $\mathbb{Q}[x,y]$ will also be a polynomial as it is a ring and it has **closure** property.

$$\phi(A_1(x,y)+A_2(x,y))=A_1(\phi(x),\phi(y))+A_2(\phi(x),\phi(y)) \tag{1.5}$$
$$\phi(A_1(x,y)+A_2(x,y))=\phi(A_1)+\phi(A_2) \tag{1.6}$$
$$\phi(A_1(x,y)*A_2(x,y))=A_1(\phi(x),\phi(y))*A_2(\phi(x),\phi(y)) \tag{1.7}$$
$$\phi(A_1(x,y)*A_2(x,y))=\phi(A_1(x,y))*\phi(A_2(x,y)) \tag{1.8}$$

By above properties ring homomorphism is satisfied as $A(\phi(x),\phi(y))=A(f(t),g(t))$, where $A(f(t),g(t))$ is a rational function in $\mathbb{Q}(t)$ as all the coefficients of polynomial $A(x,y)$ are rational numbers and putting $x=f(t),y=g(t)$ where f,g are rational functions, we will finally get a rational function $A(f(t),g(t))$. □

Claim 1.2. *$Kernel(\phi)$ contains ideal $(C(x,y))$ i.e. principal ideal of $C(x,y)$.*

Proof. Now, we will find the elements of the set **kernel(ϕ)**, it is defined as.

$$kernel(\phi)=\{A(x,y)\mid A(x,y)\in\mathbb{Q}[x,y],\phi(A(x,y))=0\} \tag{1.9}$$
$$\phi(A(x,y))=A(f(t),g(t))=0 \tag{1.10}$$

By equation (1.10), we can see that the curve $C(x,y)=0$ where $x=f(t),y=g(t)$ belongs to $kernel(\phi)$. Also, $kernel(\phi)$ is an ideal we can verify that.
If $A(f(t),g(t))=0$ and $B(f(t),g(t))=0$, then $(A+B)(f(t),g(t))=0$ then $(A+B)(f(t),g(t))\in kernel(\phi)$, also if $A(f(t),g(t))=0$ then $A(f(t),g(t))*B(x,y)=0,B(x,y)\in\mathbb{Q}[x,y]$ then $A(f(t),g(t))*B(x,y)\in kernel(\phi)$.
Since $C(x,y)\in kernel(\phi)$ then $(C(x,y))\in kernel(\phi)$ where $(C(x,y))$ is the principal ideal of $C(x,y)$. □

II Problem 2 Solution

We are given a ring homomorphism $\phi : \mathbb{Q}[x,y] \mapsto \mathbb{Q}(t)$.

Claim 2.1. *$Kernel(\phi)$ is a **prime ideal**.*

Proof. By the definition of $kernel(\phi)$, we have

$$kernel(\phi) = \{A(x,y) \mid A(x,y) \in \mathbb{Q}[x,y], \phi(A(x,y)) = 0\} \tag{2.1}$$

By proof of **Claim 1.2**, $kernel(\phi)$ is an **ideal**.
Assume polynomial $A(x,y)*B(x,y) \in kernel(\phi)$ where $A(x,y), B(x,y) \in \mathbb{Q}[x,y]$ then we have $\phi(A(x,y)*B(x,y)) = 0$, so we have $\phi(A(x,y)) * \phi(B(x,y)) = 0$ which implies either $\phi(A(x,y)) = 0$ or $\phi(B(x,y)) = 0$ i.e. $A(x,y) \in kernel(\phi)$ or $B(x,y) \in kernel(\phi)$.
The above statement implies that $kernel(\phi)$ is a **prime ideal**. □

Claim 2.2. *An **algebraic set** V is irreducible if $I(V)$ is a prime ideal.*

Proof. I will prove the above claim by **contradiction**. Consider $V = V_1 \cup V_2$ is reducible.
Then, $V_i \subsetneq V$ implies $I(V_i) \supsetneq I(V)$ for each i.
Let $F_i \in I(V_i) \backslash I(V)$, then $F_i \notin I(V)$ for each i, but $F_1F_2 \in I(V)$ since for all $P \in V$, either $F_1(P) = 0$ or $F_2(P) = 0$. Thus, $I(V)$ is not prime which is a **contradiction**. □

Claim 2.3. *$Kernel(\phi)$ is **principal ideal** i.e. $kernel(\phi) = (C(x,y)), C(x,y) \in \mathbb{Q}[x,y]$.*

Proof. I have already proved that $kernel(\phi)$ is a **prime ideal**, also by the proof of **Claim 1.2**, we can see that ideal $kernel(\phi)$ contains principal ideal $(C(x,y))$ and also since the ideal $kernel(\phi)$ is **prime ideal**, $\exists A(x,y) \in \mathbb{Q}[x,y]$ such that $A(x,y)$ is an **irreducible curve** by **Claim 2.2** and $C(x,y)$ is either equal to $A(x,y)$ or it is a multiple of $A(x,y)$.
$A(x,y) \in \mathbb{Q}[x,y]$ is an irreducible curve as $kernel(\phi)$ is a **prime ideal** and it will map to an algebraic set V which is the algebraic set of $A(x,y) \in \mathbb{Q}[x,y]$.
So, by above arguments we can say that $kernel(\phi) = (A(x,y)), A(x,y) \in \mathbb{Q}[x,y]$ as the ideal $kernel(\phi)$ will be the ideal which will be an ideal of algebraic set which will be irreducible. □

III Problem 3 Solution

We are given a map $\phi(x) = \frac{2t}{t^2+1}$ and $\phi(y) = \frac{t^2-1}{t^2+1}$ and a field of fractions $F = \mathbb{Q}[x,y]/(x^2+y^2-1)$ and field $\mathbb{Q}(t)$.

Claim 3.1. *ϕ is an **isomorphism** from F to $\mathbb{Q}(t)$.*

Proof. We are given a map $\phi(x) = \frac{2t}{t^2+1}$ and $\phi(y) = \frac{t^2-1}{t^2+1}$, rearranging this we have.

$$t = \frac{2(1+\phi(y))}{\phi(x)} \tag{3.1}$$

From the proof of **Claim 1.1**, we have.

$$\phi\left(\frac{A(x,y)}{B(x,y)}\right) = \frac{\phi(A(x,y))}{\phi(B(x,y))} = \frac{A(\phi(x),\phi(y))}{B(\phi(x),\phi(y))} \tag{3.2}$$

Calculating the value of $\phi(x^2+y^2-1)$.

$$\phi(x^2+y^2-1) = (\phi(x))^2 + (\phi(y))^2 - 1 \tag{3.3}$$

$$\left(\frac{2t}{t^2+1}\right)^2 + \left(\frac{t^2-1}{t^2+1}\right)^2 - 1 = 0 \tag{3.4}$$

So $\forall C(x,y) \in I, \phi(C(x,y)) = 0$, as every polynomial in I will just be a multiple of x^2+y^2-1 as it is a principal ideal, and applying the homomorphism to field F.

$$\phi\left(\frac{A(x,y)+I}{B(x,y)+I}\right) = \frac{\phi(A(x,y))}{\phi(B(x,y))} = \frac{A(\phi(x),\phi(y))}{B(\phi(x),\phi(y))} \tag{3.5}$$

Considering equation (3.1), there is a value $(x,y)\forall t$ by the mapping $\phi(x) = \frac{2t}{t^2+1}$ and $\phi(y) = \frac{t^2-1}{t^2+1}$, thus homomorphism is **onto** as every t is mapped to $(\phi(x),\phi(y))$ and thus every possible function in the field $\mathbb{Q}(t)$ is mapped to a function in field F.
We know that a field has only 2 ideals which are $\{0\}$ and (1) i.e. **zero ideal** and **F-{0}**, and since $kernel(\phi) = \{0\}$.
If we assume $A \neq B$ where $A, B \in F$ but $\phi(A) = \phi(B)$, also if we have $\phi(X) = 0$, then $X = 0$ as $kernel(\phi) = \{0\}$.

$$\phi(A-B) = \phi(A) - \phi(B) = 0 \tag{3.6}$$

$$A - B = 0 \Rightarrow A = B \tag{3.7}$$

The above statement is a contradiction to our assumption, thus we have a unique mapping, also by equation (3.1), we can see that homomorphism of field F to $\mathbb{Q}(t)$ is **1-1**.
Now as we have shown the homomorphism is **1-1** and **onto**, the homomorphism is an **isomorphism**. $\square$