MSO 201 A : Homework 6

[1] The probability density function of the random variable

$$f_X(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

i.e. $X \sim U(0,1)$. Find the distribution of the following functions of X

(a) $Y = \sqrt{X}$; (b) $Y = X^2$; (c) $Y = 2X + 3$; (d) $Y = -\lambda \log X; \lambda > 0$.

[2] Let X be a random variable with $U(0,\theta)$, $\theta > 0$ distribution. Find the distribution of $Y = \min(X, \theta/2)$.

[3] The probability density function of X is given by

$$f_X(x) = \begin{cases} 1/2 & -1/2 \le x \le 3/2 \\ 0 & \text{otherwise.} \end{cases}$$

Find the distribution of $Y = X^2$.

[4] The probability density function of X is given by

$$f_X(x) = \begin{cases} k \dfrac{x^{p-1}}{(1+x)^{p+q}} & x > 0 \\ 0 & \text{otherwise,} \end{cases}$$

$p, q > 0$. Derive the distribution of $Y = (1+X)^{-1}$.

[5] The probability density function of X is given by

$$f_X(x) = \begin{cases} k\, x^{\beta-1} \exp\left(-\alpha x^{\beta}\right) & x > 0 \\ 0 & \text{otherwise,} \end{cases}$$

$\alpha, \beta > 0$. Derive the distribution of $Y = X^{\beta}$

[6] According to the Maxwell-Boltzmann law of theoretical physics, the probability density function of V, the velocity of a gas molecule, is

$$f_V(v) = \begin{cases} k\, v^2 \exp\left(-\beta v^2\right) & v > 0 \\ 0 & \text{otherwise,} \end{cases}$$

where, $\beta > 0$ is a constant which depends on the mass and absolute temperature of the molecule and $k > 0$ is a normalizing constant. Derive the distribution of the kinetic energy $E = mV^2/2$.

[7] The probability density function of the random variable X is

$$f_X(x) = \begin{cases} \frac{3}{8}(x+1)^2 & -1 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Find the distribution of the following functions of $Y = 1 - X^2$.

[8] Let X be a random variable with $U(0,1)$ distribution. Find the distribution function of $Y = \min(X, 1-X)$ and the probability density function of $Z = (1-Y)/Y$.

[9] Suppose $X \sim N(\mu, \sigma^2)$, $\mu \in \Re, \sigma \in \Re^+$. Find the distribution of $2X - 6$.

[10] Let X be a continuous random variable on (a,b) with p.d.f f and c.d.f. F. Find the p.d.f. of $Z = -\log(F(X))$.

[11] Let X be a continuous r.v. having the following p.d.f.

$$f(x) = \begin{cases} 6x(1-x) & \text{if } 0 \le x \le 1 \\ 0 & \text{if otherwise} \end{cases}.$$

Derive the distribution function of X and hence find the p.d.f. of $Y = X^2(3-2X)$.

[12] Let X be distributed as double exponential with p.d.f. $f(x) = \frac{1}{2}e^{-|x|}; x \in \Re$. Find the p.d.f. of $Y = |X|$

[13] 3 balls are placed randomly in 3 boxes B_1, B_2 and B_3. Let N be the total number of boxes which are occupied and X_i be the total number of balls in the box B_i, $i = 1, 2, 3$. Find the joint p.m.f. of (N, X_1) and (X_1, X_2). Obtain the marginal distributions of N, X_1 and X_2 from the joint p.m.f.s.

[14] The joint p.m.f. of X and Y is given by

$$p(x,y) = \begin{cases} c\,xy & \text{if } (x,y) \in \{(1,1),(2,1),(2,2),(3,1)\} \\ 0 & \text{otherwise.} \end{cases}$$

Find the constant c, the marginal p.m.f. of X and Y and the conditional p.m.f. of X given $Y = 2$.

[15] The joint p.m.f. of X and Y is given by

$$p(x,y) = \begin{cases} (x+2y)/18 & \text{if } (x,y) \in \{(1,1),(1,2),(2,1),(2,2)\} \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the marginal distributions.

(b) Verify whether X and Y are independent random variables.

(c) Find $P(X < Y), P(X + Y > 2)$.

(d) Find the conditional p.m.f. of Y given $X = x, x = 1, 2$.

[16] 5 cards are drawn at random without replacement from a deck of 52 playing cards. Let the random variables X_1, X_2, X_3 denote the number of spades, the number of hearts, the number of diamonds, respectively, that appear among the five cards. Find the joint p.m.f. of X_1, X_2, X_3. Also determine whether the 3 random variables are independent.

[17] Consider a sample of size 3 drawn with replacement from an urn containing 3 white, 2 black and 3 red balls. Let the random variables X_1 and X_2 denote the number of white balls and number of black balls in the sample, respectively. Determine whether the two random variables are independent.

[18] Let $\underset{\sim}{X} = \left(X_1, X_2, X_3\right)^T$ be a random vector with joint p.m.f.

$$f_{X_1,X_2,X_3}\left(x_1,x_2,x_3\right)=\begin{cases}1/4 & \left(x_1,x_2,x_3\right)\in\mathcal{X}\\ 0 & \text{otherwise.}\end{cases}$$

$\mathcal{X}=\left\{(1,0,0),(0,1,0),(0,0,1),(1,1,1)\right\}$. Show that X_1, X_2, X_3 are pairwise independent but are not mutually independent.

Note: The above problem shows that random variables may be pairwise independent without being mutually independent.