CS203 ASSIGNMENT 4 SEMESTER I, 2017-18

Due Date: 10th September, 2017 **Maximum Marks: 50**

Instructions

- Submit the assignment in KD213 before 1 pm on 10th September .
- Yours answers should be precise and clearly written.
- Cheating/plagiarizing in any form will be heavily penalized.
- Late submissions will receive a mark of zero.
- Any doubts regarding the assignment can be raised in the discussion forum on moodle.
- Please look up on the wikipedia page for the definition of Partially Ordered Set and Zorn's Lemma.
- For all the question you can assume that R is a commutative ring with the multiplicative identity as 1.

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1. (30 points) An element x in Ring R is *nilpotent* if there exist a non-zero positive integer n such that $x^n = 0$.
   - Prove that set of all nilpotent elements of a ring R forms an ideal. This ideal is known as *nilradical* of R.
   - Let S be the intersection of all the prime ideals of the ring R. Then prove that S contains nilradical of R.
   - Assume that x is not a nilpotent element. Let

     $$\mathcal{Y} = \{I \mid I \text{ is an ideal of } R \text{ and } \forall n \in \mathbb{Z}_{>0}, x^n \notin I\}$$

     Then prove that $\mathcal{Y}$ forms a partially ordered set with respect to the set inclusion operation.
   - Prove that $\mathcal{Y}$ has a maximal element by using Zorn's Lemma.
   - Prove that every maximal element of $\mathcal{Y}$ is a prime ideal.
   - Prove that all the elements of S are nilpotent elements.

2. (10 points) Let I be an ideal of Ring R. Consider the following homomorphism $\phi : R \to R/I$ defined as $\forall x \in R$, $\phi(x) = x + I$. Let

   $$\mathcal{Y} = \{U \mid U \text{ is an ideal of } R \text{ and } I \subseteq U\}.$$

   Let $\mathcal{Y}'$ be set of all ideals of R/I. Prove that there exists a bijection between $\mathcal{Y}$ and $\mathcal{Y}'$.

3. (10 points) Let R be a Noetherian ring.
   1. Every non-unit element of ring R is contained in a maximal ideal.
   2. Let S be the intersection of all the maximal ideal of ring R. Then prove that $x \in S$ iff $1 - x$ is a unit in R.