

CS201A: Math for CS I/Discrete Mathematics

#1

Max marks:140

Due on/before:23.00, 26-Aug-2017.

17-Aug-2017

The first question defines two types of binary relations that are important in CS - partial orders and equivalence relations. It introduces the basic properties of these two kinds of relations.

1. Let $R \subseteq S \times S$ be a binary relation. For elements $a, b \in S$ let us write aRb if $(a, b) \in R$. We say R is a *partial order* if R satisfies:

1. $\forall a \in S, aRa$. R is *reflexive*.
2. $\forall a, b \in S, aRb \wedge bRa \implies a = b$. R is *anti-symmetric*.
3. $\forall a, b, c \in S, aRb \wedge bRc \implies aRc$. R is *transitive*.

If S has a partial order defined on it it is often called a partially ordered set (or poset for short).

We say R is an *equivalence relation* if R satisfies:

1. $\forall a \in S, aRa$. R is *reflexive*.
2. $\forall a, b \in S, aRb \implies bRa$. R is *symmetric*.
3. $\forall a, b, c \in S, aRb \wedge bRc \implies aRc$. R is *transitive*.

$a, b \in S$ are *comparable* if aRb or bRa else they are *incomparable*. They are *compatible* if $\exists c$ such that cRz and cRb else they are *incompatible*.

Subset $T \subset S$ is called a *chain* iff any pair $a, b \in T$ are comparable. It is an *anti-chain* iff for no pair $a, b \in T$ is compatible. T is said to be *linked* iff any pair $a, b \in T$ is compatible. S is said to be *linear* or *linearly* ordered iff S is a chain under R . R is said to be a *linear order*.

Let S be a poset then $m \in S$ is a *minimal* element iff $\forall a \in S, aRm \implies a = m$; m is a *minimum* element iff $\forall a \in S, mRa$. Similarly, $M \in S$ is a *maximal* element if $\forall a \in S, MRa \implies a = M$; M is a *maximum* element if $\forall a \in S, aRM$.

- (a) Show that the divides relation where $S = \mathbb{N}$ and $a|b$ means a divides b is a partial order but not an equivalence relation.
- (b) Give an example of an equivalence relation from Euclidean geometry. Show that your relation satisfies all the properties.
- (c) Can a relation be both a partial order and an equivalence relation? Justify your answer.
- (d) Prove maximum elements are maximal.
- (e) Prove minimum elements are minimal.
- (f) Prove: If S has a minimum element then every subset is linked.
- (g) Prove: There can be at most one maximum element and at most one minimum element.
- (h) Prove: A maximal element in a linear order is a maximum and minimal element is a minimum.
- (i) Give an example of a poset where a unique minimal element need not be a minimum and a unique maximal element need not be a maximum.
- (j) A *partition* of set S is a collection of subsets of S , say S_1, S_2 , till S_n such that every element of S belongs to exactly one of S_i . It is clear from the definition that $i \neq j \implies S_i \cup S_j = \emptyset$ and $\bigcup_i S_i = S$. Prove that if R is an equivalence relation on S then it partitions S .

[5×10=50]

2. Recall we had two definitions for a countable set. S is *countable* if $\exists f$ such that $f : S \rightarrow \mathbb{N} \setminus \mathbb{N}_0$ (that is if there exists an injection f from S to \mathbb{N} or \mathbb{N}_0) or equivalently if $S \sim T$ where $T \subseteq \mathbb{N} \setminus \mathbb{N}_0$ (that is S is equipollent to a subset of \mathbb{N} or \mathbb{N}_0). We said in class that the two definitions are equivalent. Prove this. [10]

3. In class we claimed that the set of *algebraic numbers* (roots of polynomials with rational coefficients) is countable. Prove the claim. First show that $\mathbb{Q}[x]$ the set of polynomials with rational coefficients is countable. Then argue that the roots of such polynomials is also countable.

4. (a) A useful version of the CSB theorem is the following:

Theorem: If A, B are infinite sets and $f : A \rightarrow B, g : B \rightarrow A$ are surjections then there exists a bijection between A and B .

Prove the above theorem.

- (b) The set $S_b = \{0, 1\}^*$ is the set of all finite sequences of strings of 0s and 1s. Argue that S_b is countable.

- (c) Show that if A is an infinite set and S is countable then there is a bijection $f : A \rightleftarrows A \cup S$. [10,5,10=25]

5. This problem looks at another proof of the CSB theorem in multiple steps. Assume A and B are infinite sets and $f : A \rightarrow B$ and $g : B \rightarrow A$ are injections.

- (a) Consider the two sequences below:

$$A = A_0, A_1 = g(B_0), \dots, A_n = g(B_{n-1}), \dots$$

$$B = B_0, B_1 = f(A_0), \dots, B_n = f(A_{n-1}), \dots$$

What cardinality relation holds between the sets of the sequence: $A_0, B_1, A_2, B_3, \dots$. Similarly, between the sets of the sequence: $B_0, A_1, B_2, A_3, \dots$?

- (b) What subset relation holds between the sets of the sequences in part (a) above?

- (c) Show that if X_i , with i ranging over some index set is a collection of mutually disjoint sets and similarly Y_i (same index set for i) are also mutually disjoint and $X_i \sim Y_i$ then $\bigcup_i X_i \sim \bigcup_i Y_i$. Also, argue this does not hold if the mutually disjoint condition does not hold.

- (d) Define suitable sets A_i^*, B_i^* (based on A_i and B_i) such that the result in (c) can be applied to: $\bigcup_i A_i^* \sim \bigcup_i B_i^*$

- (e) Show that $A = (\bigcup_i A_i^*) \cup (\bigcap_i A_i)$ and $B = (\bigcup_i B_i^*) \cup (\bigcap_i B_i)$.

- (f) Argue that $f(\bigcap_i A_i) = \bigcap_i B_i$ and $g(\bigcap_i B_i) = \bigcap_i A_i$.

- (g) Put everything together to prove CSB. [5×7=35]

6. Consider the subset $F_n = \{s \in \{0, 1\}^\omega \mid s \text{ has only 0s after } n^{\text{th}} \text{ bit}\}$. For example, 011010... where there are only 0s after the last 1 is in F_5 (and also in $F_n, n > 5$).

In each case below justify your answer.

- (a) What is the cardinality of $F_n, n \in \mathbb{N}_0$? Justify.

- (b) Let $F \subset \{0, 1\}^\omega$ be the set of binary strings with only finitely many 1s. What is the cardinality of F ? Justify.

- (c) What is the cardinality of the set of binary strings that have infinitely many 1s? Justify. [5,5,10=20]