# MSO 201 A : Homework 11

**[1]** Let $X_1, X_2, \ldots X_n$ be a random sample from an exponential distribution with p.d.f.

$$f_X(x) = \frac{1}{\beta}\exp\left(-\frac{x}{\beta}\right); x > 0$$

Show that $\bar{X} = \sum_{i=1}^{n} X_i \Big/ n$ is an unbiased estimator of $\beta$.

**[2]** Let $X_1, X_2, \ldots X_n$ be a random sample from $U(0, \theta)$; $\theta > 0$. Show that $\frac{n+1}{n} X_{(n)}$ and $2\,\bar{X}$ are both unbiased estimators of $\theta$.

**[3]** Let $X_1, X_2, \ldots X_n$ be a random sample from an exponential distribution with p.d.f.

$$f(x) = \beta \exp(-\beta\, x); x > 0$$

Show that $\bar{X}$ is an unbiased estimator of $1/\beta$.

**[4]** Let $X_1, X_2, \ldots X_n$ be a random sample from $N(\theta, \theta^2)$, $\theta > 0$. Show that $\left(\sum_{i=1}^{n} X_i\right)^2 \Big/ n(n+1)$ and $\sum_{i=1}^{n} X_i^2 \Big/ 2n$ are both unbiased estimators of $\theta^2$.

**[5]** Let $X_1, X_2, \ldots X_n$ be a random sample from $P(\theta)$; $\theta > 0$. Find an unbiased estimator of $\theta e^{-2\theta}$.

**[6]** Let $X_1, X_2, \ldots X_n$ be a random sample from $B(1, \theta); 0 \le \theta \le 1$.

**(a)** Show that the estimator $T(\underset{\sim}{X}) = \dfrac{\frac{1}{2}\sqrt{n} + \sum_{i=1}^{n} X_i}{n + \sqrt{n}}$ is not unbiased $\theta$?

**(b)** Show that $\lim_{n \to \infty} E\left(T(\underset{\sim}{X})\right) = \theta$.

(An estimator satisfying the condition in (b) is said to be unbiased in the limit)

**[7]** $X_1, \ldots, X_n$ be a random sample from $N(\mu, \sigma^2), \mu \in \Re, \sigma \in \Re^+$. Find unbiased estimators of $\mu/\sigma^2$ and $\mu/\sigma$.

**[8]** Let $X_1, X_2, \ldots X_n$ be a random sample from $B(1, \theta); 0 \le \theta \le 1$. Find an unbiased estimator of $\theta^2(1-\theta)$.

**[9]** Using Neyman Fisher Factorization Theorem, find a sufficient based on a random sample $X_1, X_2, \ldots X_n$ from each of the following distributions

(a) $f_\alpha(x) = \begin{cases} \dfrac{1}{\alpha}\exp\left(-\dfrac{x}{\alpha}\right) & \text{if } x > 0 \\ 0 & \text{otherwise.} \end{cases}$

(b) $f_\beta(x) = \begin{cases} \exp\left(-\left(x-\beta\right)\right) & \text{if } x > \beta \\ 0 & \text{otherwise.} \end{cases}$

(c) $f_{\alpha,\beta}(x) = \begin{cases} \dfrac{1}{\alpha}\exp\left(-\dfrac{(x-\beta)}{\alpha}\right) & \text{if } x > \beta \\ 0 & \text{otherwise.} \end{cases}$

(d) $f_{\mu,\sigma}(x) = \begin{cases} \dfrac{1}{x\,\sigma\,\sqrt{2\pi}}\ \exp\left(-\dfrac{(\log x_i - \mu)^2}{2\sigma^2}\right) & \text{if } x > 0 \\ 0 & \text{otherwise.} \end{cases}$

(e) $f_\theta(x) = \begin{cases} \dfrac{1}{\theta} & -\theta/2 \le x \le \theta/2 \\ 0 & \text{otherwise} \end{cases}$

**[10]** Let $X_1$ and $X_2$ be independent random samples with densities $f_1(x_1) = \theta e^{-\theta x_1}$ and $f_2(x_2) = 2\theta e^{-2\theta x_2}$ as the respective p.d.f.s where $\theta > 0$ is an unknown parameter and $0 < x_1, x_2 < \infty$. Using Neyman Fisher Factorization Theorem find a sufficient statistic for $\theta$.

**[11]** Let $X_1, \ldots, X_n$ be a random sample with densities

$$f_{X_i}(x) = \begin{cases} \exp(i\theta - x) & \text{if } x \ge i\theta \\ 0 & \text{otherwise.} \end{cases}$$

Using Neyman Fisher Factorization Theorem find a sufficient statistic for $\theta$.

**[12]** Let $X_1, X_2, \ldots X_n$ be a random sample from a $Beta\left(\alpha, \beta\right)$ distribution $\left(\alpha > 0, \beta > 0\right)$ with p.d.f.

$$f(x) = \begin{cases} \dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\, x^{\alpha-1}\left(1-x\right)^{\beta-1} & 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Show that

(a) $\prod_{i=1}^{n} X_i$ is sufficient for $\alpha$ if $\beta$ is known to be a given constant.

(b) $\prod_{i=1}^{n}\left(1 - X_i\right)$ is sufficient for $\beta$ if $\alpha$ is known to be a given constant.

(c) $\left(\prod_{i=1}^{n} X_i, \prod_{i=1}^{n}\left(1 - X_i\right)\right)$ is jointly sufficient for $\left(\alpha, \beta\right)$ if both the parameters are unknown.

**[13]** Let $T$ and $T^*$ be two statistic such that $T=\psi\left(T^*\right)$. Show that if $T$ is sufficient then $T^*$ is also sufficient.

**[14]** $X_1,...,X_n$ be a random sample from $U\left(\theta-1/2,\theta+1/2\right), \theta\in\Re$. Find a sufficient statistic for $\theta$.

**[15]** Let $X_1,...,X_n$ be independent random variables with $X_i\left(i=1,2,...,n\right)$ having the probability density function

$$f_i\left(x_i\right)=\begin{cases} i\,\theta\, e^{-i\,\theta\, x_i} & x_i>0 \\ 0 & \text{otherwise} \end{cases}$$

Find a sufficient statistic for $\theta$.