Sets, relations, functions
Version 1.0

1 Basic definitions and notation

We start by assuming that a set is a primitive notion and is a collection of elements that constitute it. Set theory has an axiomatic basis. Actually, multiple axiomatizations are available. The most widely accepted is the Zermelo-Fraenkel axiomatization. This axiomatization has to be carefully done otherwise it leads to paradoxes (see section 3.2).

We also assume we can build or represent sets using the standard set builder notation as: $S = \{s \in A \,|\, P(s)\}$. S is the set of all elements $s \in A$ such that $P(s)$ holds, P is a boolean predicate. For example, $Even = \{s \in \mathbb{N} \,|\, \text{isEven}(s)\}$ defines the set of positive even numbers. `isEven` is `True` when s is an even number. Here $\mathbb{N}$ is the set of natural numbers that is: $\{1, 2, 3, 4, \ldots\}$. We will use $\mathbb{N}_0$ to denote the natural numbers including 0 - i.e. $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$.

1.1 Relations, functions

In this subsection we define relations (both binary and n-ary), function, injection, surjection and bijection.

Definition 1.1 (Cartesian product)**.** Given sets $S_1, S_2, \ldots, S_n$ a **Cartesian product**

$$S1 \times S_2 \times \ldots \times S_n = \{(s_1, s_2, \ldots, s_n) \,|\, s_1 \in S_1, s_2 \in S_2, \ \ldots, s_n \in S_n\}$$

$\lhd$

Definition 1.2 (Relation)**.** Given sets S_1, S_2 a **binary relation** R is a subset of $S_1 \times S_2$. That is $R \subseteq S_1 \times S_2$.

More generally an **nary relation** R is a subset of $S_1 \times S_2 \times \ldots \times S_n$ or $R \subseteq S1 \times S_2 \times \ldots \times S_n$.
$\lhd$

Definition 1.3 (Function, domain, co-domain, range)**.** A **function** $f : A \to B$, where A, B are sets, associates at most one element $b \in B$ (often called the image) with an element $a \in A$ (often called the pre-image). The image b of a is often written as $b = f(a)$.

A is called the **domain** and B the **co-domain** of function f.

If all elements of A have an image in B then the function is **total** else it is **partial**. We will be concerned largely with total functions. By default all our functions are total unless explicitly indicated otherwise.

The set $range(A) = \{b \in B \,|\, b = f(a), a \in A\}$ is called the **range** of $f(A)$. It is the set of all image points in B for domain A. We denote the range of A by $range(A)$. Clearly, $range(A) \subseteq B$.
$\lhd$

Definition 1.4 (Injection or 1-1 function)**.** A function $f : A \to B$ denoted $f : A \rightarrowtail B$ is an **injection or 1-1 function** if $\forall a_1, a_2 \in A, \ f(a_1) = f(a_2) \implies a_1 = a_2$ or distinct points in A map to distinct points in B.

If f is an injection we can define $f^{-1} : B \to A$ or the inverse function from $range(A) \subseteq B$ to A. Note that f^{-1} can be a **partial** function.
$\lhd$

Definition 1.5 (Surjection or onto function)**.** A function $f : A \to B$, written $f : A \twoheadrightarrow B$, is a **surjection or onto function** if $\exists S \subseteq A, \ni B = range(S)$. That is every element in B has at least one pre-image in A.
$\lhd$

Definition 1.6 (Bijection or 1-1 onto function)**.** A function $f : A \to B$, written $A \rightleftarrows B$ is a **bijection or 1-1, onto function** if f is both an injection and a surjection.

If f is a bijection then $f^{-1} : B \to A$ is also a 1-1, onto function.
$\lhd$

Lemma 1.1. *If $f : A \twoheadrightarrow B$ then $\exists g \ni g : B \rightarrowtail A$. If there is a surjection f from A to B then there exists an injection g from B to A.*

Proof. The proof follows directly from the definitions. Since f is a surjection every element $b \in B$ has at least one pre-image in A which we can associate with b. If b has more than one pre-image then choose one arbitrarily and associate it with b. This gives us the injection g. □

1.2 Cardinality or size

The size or cardinality of a finite set is the number of elements in the set and is a non-negative integer.

The size or cardinality of a set S (whether finite or not) will be written as $card(S)$ or $|S|$.

Two finite sets have the same size exactly when both have the same number of elements.

$S_1 \subset S_2 \implies card(S_1) < card(S_2)$. The cardinality of a proper subset of a finite set is always smaller than the parent set.

When sets are infinite in size (e.g. $\mathbb{N}$ or $\mathbb{R}$) then our intuitive notions of size do not hold. In particular, subsets are not always smaller than the parent set. So, we need a different idea of size.

Definition 1.7 (Equipollent)**.** Set A is **equipollent** to B, written $A \sim B$ if $\exists f$, $f : A \rightleftarrows B$. That is there is a bijection between A and B.
$\lhd$

Loosely if $A \sim B$ they have the same size. Finding actual bijections between infinite sets can be difficult. This is made easier by the Cantor-Schröder-Bernstein (or CSB) theorem.

Theorem 1.1 (Cantor-Schröder-Bernstein)**.** *Let A, B be infinite sets. If there exist injections f, $f : A \rightarrowtail B$ and g, $g : B \rightarrowtail A$ then there exists a bijection h, $h : A \rightleftarrows B$.*

Proof. This is based on Julius König's proof [1].

Consider any $a \in A$. We can then construct the following sequence of elements from A and B starting from a - shown in bold below.

$$\ldots g^{-1}f^{-1}g^{-1}(a), f^{-1}g^{-1}(a), g^{-1}(a), \mathbf{a}, f(a), gf(a), fgf(a), \ldots$$

Note that the sequence can continue infinitely to the right since f, g, are injections. However, to the left there are 3 possibilities:

a) It may continue infinitely to the left.

b) It may terminate in the set A because g^{-1} is not defined.

c) It may terminate in set B because f^{-1} is not defined.

Since f and g are injections $range(A)$ can be a proper subset of B so f^{-1} is not defined for all elements of B. A similar argument holds for g^{-1}.

We will get a similar sequence for any element $b \in B$. In this case the sequence is:

$$\ldots, f^{-1}g^{-1}f^{-1}(b), g^{-1}f^{-1}(b), f^{-1}(b), \mathbf{b}, g(b), fg(b), gfg(b), \ldots$$

We claim that any element of A or B can occur in exactly one such sequence. This follows directly because f and g are injections. But let us examine it in more detail. Suppose some a_1 occurs in two distinct sequences. It is clear that to the right of a_1 both sequences are identical - $f(a_1), gf(a_1), fgf(a_1), \ldots$. The elements preceding a_1 in both sequences have to be $g^{-1}(a_1)$ and they are identical because g is an injection. This argument is repeated for $f^{-1}g^{-1}(a_1)$ and so on. So, the subsequences to the left of a_1 (whether finite or infinite) in both sequences will be exactly identical. Consequently, the two sequences in which a_1 occurs cannot be different and must be identical. An identical argument works if we assume that some b is in two different sequences.

Since we are building sequences for every element of A and B the set of sequences partition $A \cup B$. That is, if the sequences are $S_1, S_2, \ldots$ then $\bigcup S_i = A \cup B$ and $S_i \cap S_j = \varnothing$, $i \neq j$ (mutually disjoint and collectively exhaustive).

A bijection $h : A \rightleftarrows B$ can now be defined as follows:

$$h = \begin{cases} f \text{ if sequence terminates in } A. \\ g^{-1} \text{ if sequence terminates in } B. \\ f \text{ or } g^{-1} \text{ if sequence continues infinitely to the left, conventionally } f. \end{cases}$$

Since each $a \in A$ and $b \in B$ is in exactly one sequence h is clearly a bijection. Another way to see this: f, g are injections so h is also an injection as is h^{-1}. h is also a bijection since for each $a \in A$ it maps to a $b \in B$ via the sequence and each $b \in B$ is associated with an $a \in A$ again via the corresponding sequence. □

2 Countability

In this section we study countable sets and their properties. Look at chapter 7 in [3] and for historical information, set theoretic paradoxes (also called antinomies) and the ZFC axiomatization see [2].

Definition 2.1 (Countable set - defn. 1)**.** A set S is countable if there exists an injection $f : S \rightarrowtail \mathbb{N}$. This definition works for both finite and countably infinite sets. ◁

An alternative definition is:

Definition 2.2 (Countable set - defn 2)**.** A set S is countable if it is equipollent with some subset of $\mathbb{N}$(note it includes $\mathbb{N}$ itself). This definition also works for both finite and infinite sets. ◁

Exercise 2.1. Argue that the two definions are equivalent.

2.1 Properties of countable sets

We now discuss some useful properties of countable sets.

Theorem 2.1. *Any infinite subset S of $\mathbb{N}$ is countably infinite.*

Proof. Follows immediately from the definition since the identity function I is an injection from S to $\mathbb{N}$. □

An **enumeration** of a set S is a bijection between S and $\mathbb{N}$. One way an enumeration is constructed is by showing how all pairs of $S \times \mathbb{N}$ can be systematically listed. For example, consider the set of all primes P. This set can be enumerated as follows:

$$\{(2,1),(3,2),(5,3),(7,4),(11,5),(13,6),\ldots\}$$

So, the n^{th} prime is associated with n. Since there are infinitely many primes we know this pair construction can be continued ad infinitum Therefore, this is a bijection and so $\mathbb{N}$ and P are equipollent and P is countable. Enumeration is one standard way to show that a set is countable.

Theorem 2.2. *If S is countable and $S1 \subseteq S$ then $S1$ is countable.*

Proof. We show this only when S is countably infinite. The proof for the finite case is similar and easy. S is countable implies there is an injection $f : S \rightarrowtail \mathbb{N}$. The same f when restricted to $S1 \subseteq S$ is an injection from $S1$ to $\mathbb{N}$ and by theorem 2.2 $S1$ is countable. □

Theorem 2.3. *Every infinite set contains a countable subset.*

Proof. We construct an injection to $\mathbb{N}$. Let S be the infinite set. For every non-empty subset $T \subseteq S$ select a definite element as follows: Let s_1 be chosen arbitrarily from S. Choose s_k from the k^{th} subset of S such that it is different from $s_1, s_2, \ldots, s_{k-1}$. This process can be continued and we have an enumeration:

$$\{s_1, s_2, s_3, \ldots, s_n, \ldots\}$$

The subscript associates it with $\mathbb{N}$ giving us an injection - actually a bijection. So, we have the countable set $\{s_1, s_2, s_3, \ldots\}$ which is a subset of S. □

Theorem 2.4. *If X is a countably infinite set and $f : X \twoheadrightarrow Y$ is a surjection from X to Y then Y is countable.*

Proof. Since f is a surjection we can find an injection $g : Y \rightarrowtail X$. The $range(Y) \subseteq X$. X is countable so Y is countable. □

Theorem 2.5. *If S is a countable set then the Cartesian product $S \times S$ is countable.*

Proof. Since S is countable we have an injection from S to $\mathbb{N}$. So, it is enough to show that $\mathbb{N} \times \mathbb{N}$ is countable.

Consider the function $f((i,j)) \to 2^i 3^j$. It is enough to show that f is injective. But this follows directly from the prime number theorem which states that a composite number can be uniquely broken down into primes. So, if $n = 2^{i_1}3^{j_1}$ and $n = 2^{i_2}3^{j_2}$ then $i_1 = i_2,\ j_1 = j_2$ and therefore we have an injection. □

Theorem 2.6. *If $S_1, S_2, \ldots, S_n, \ldots$ is a countable sequence of countable sets then $S = \bigcup_i S_i$ is countable.*

Proof. Since each S_i is countable we can enumerate the elements of S_1, S_2 and more generally any S_i as follows:

$$\begin{aligned} S_1 &= \{s_{11}, s_{12}, s_{13}, \ldots,\} \\ S_2 &= \{s_{21}, s_{22}, s_{23}, \ldots,\} \\ &\ldots \\ S_i &= \{s_{i1}, s_{i2}, s_{i3}, \ldots,\} \\ &\ldots \end{aligned}$$

We observe that $S = \bigcup_i S_i = \bigcup_{i,j \in \mathbb{N}} s_{ij}$. Now define the following function:
$f((i,j)) = s_{ij}$ where $f : \mathbb{N} \times \mathbb{N} \twoheadrightarrow S$. This function is surjective since every element in S has a pre-image (i,j) in $\mathbb{N} \times \mathbb{N}$ according to the definition of f. So, S is countable using theorems 2.5 and 2.4. □

2.2 Examples of countable sets

In this section we look at examples of sets that can be shown to be countable. Consider the the set of integers $\mathbb{Z}$. We show that it is countable in two different ways.

The first way is to enumerate $\mathbb{Z}$ as follows:

$$\{0, 1, -1, 2, -2, 3, -3, \ldots\}$$

The above is an enumeration since every element of the $\mathbb{Z}$ occurs in the sequence and we can pair the elements with $\{1, 2, 3, 4, 5, \ldots\}$ - that is $\mathbb{N}$. So, $\mathbb{Z}$ is countable.

The second is make use of the CSB theorem and construct injections from $\mathbb{N}$ to $\mathbb{Z}$ and in the reverse direction. The injection from $\mathbb{N}$ to $\mathbb{Z}$ is just the identity function. From $\mathbb{Z}$ to $\mathbb{N}$ we define the function g:

$$\forall z \in \mathbb{Z},\ g(z) = \begin{cases} 2z - 1 & z > 0 \\ -2z & z \leq 0 \end{cases}$$

Suppose $g(z_1) = g(z_2)$ then this implies either $2z_1 - 1 = 2z_2 - 1 \implies z_1 = z_2$ or $-2z_1 = -2z_2 \implies z_1 = z_2$ or $2z_1 - 1 = -2z_2 \implies z_1 = \frac{1}{2} - z_2 \implies z_1 \notin \mathbb{Z}$. So, we conclude that g is an injection.

Now consider the set $\mathbb{Q}^+$ (set of positive rationals). We argue that this set is countable. Note that it is enough to consider only the positive rationals since negative rationals can be trivially put into a 1-1 correspondence with the positive rationals. The proof was originally given by Cantor and is called the dovetailing procedure.

	1	2	3	4	5	...
1	$\frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	...
2	$\frac{2}{1}$	$\frac{2}{2}$	$\frac{2}{3}$	$\frac{2}{4}$	$\frac{2}{5}$	...
3	$\frac{3}{1}$	$\frac{3}{2}$	$\frac{3}{3}$	$\frac{3}{4}$	$\frac{3}{5}$	...
4	$\frac{4}{1}$	$\frac{4}{2}$	$\frac{4}{3}$	$\frac{4}{4}$	$\frac{4}{5}$	...
5	...					
6	...					

In the above table it is clear that all positive rationals occur since each element of $\mathbb{N}$ occurs in the numerator and the denominator. If we traverse the table in a zig-zag manner as follows: $(1,1), (2,1),$ diagonally up to $(1,2)$, $(1,3),$ diagonally down to $(3,1)$, $(4,1),$ diagonally up to $(1,4), \ldots$.

The above is an enumeration of $\mathbb{Q}^+$ where every positive rational will occur. Duplicates are removed e.g. $\frac{2}{2}$, $\frac{2}{4}$, $\frac{3}{6}$ etc. We thus have an enumeration of $\mathbb{Q}^+$ and so it is a countable set.

Another proof finds an injection into $\mathbb{N}$ using the technique that was used for $\mathbb{N} \times \mathbb{N}$. Let $q = \frac{m}{n}$ be any rational number and $q = \frac{m'}{n'}$ be its reduced version (that is no common divisor, except 1, for m and n). The reduced fraction can be mapped to a unique number in $\mathbb{N}$ by $f : \mathbb{Q}^+(\text{reduced}) \rightarrowtail \mathbb{N}$ with $f(q) = p_1^{m'} p_2^{n'}$ where $p_1,\ p_2$ are primes.

3 Uncountable Sets and Incomputability

In this section we discuss the hierarchy of infinities that exist and Cantor's powerful diagnonalization proof method that is widely used. We also look at some semantic and set theoretic paradoxes and use the diagonalization method in a not so obvious way to show that incomputable or non-computable functions exist.

3.1 Uncountable sets

We first argue that the set $\Omega = \{0,1\}^\omega$, the set of all infinite strings of 0s and 1s is not countable. The proof is by contradiction. Assume that Ω is countable. Then we can enumerate the set. Let the following be one possible enumeration:

$$\begin{aligned} s_1 &= 11000010\ldots \\ s_2 &= 01010101\ldots \\ s_3 &= 00010101\ldots \\ s_4 &= 10110101\ldots \\ &\ldots \\ &\ldots \end{aligned}$$

Now consider the string obtained by picking all the bits on the diagonal. In the example above we get: $1101\ldots$ now complement each bit to get $0010\ldots$. We claim that the complemented string which is certainly a valid string and is in Ω cannot be part of the enumeration. This is true because by construction the string differs from s_1 in the first bit, with s_2 in the second bit and so on. So, it differs from every string in the enumeration in the diagonal bit and therefore cannot be part of the enumeration. So, our assumption that Ω can be enumerated is false and Ω is an infinite set that is not countable. So, we have the theorem:

Theorem 3.1. *The set $\Omega = \{0,1\}^\omega$ is not a countable set.*

We have seen a simple application of the diagonalization argument. We can easily extend the above argument to the open interval $(0,1) \subset \mathbb{R}$. Similar to Ω consider the set $\{0,1,2,3,4,5,6,7,8,9\}^\omega$ that is the set of all infinite strings over the digits 0 to 9. If we put a decimal point before the first digit then we have exactly the interval $(0,1)$. By an argument similar to the one above we can show that $(0,1)$ is not countable. Consequently, the set $\mathbb{R}$, the set of real numbers, is uncountable.

Theorem 3.2. $\mathbb{R}$ *is uncountable.*

Exercise 3.1. Show that $\mathbb{R}$ is uncountable by finding a bijection between $(0,1)$ and $\mathbb{R}$.

Cantor used a more subtle form of the diagonalization argument to show that we have an 'infinity of infinities'. He showed that a set and its power set can never be equipollent.

Theorem 3.3 (Cantor's theorem)**.** *For any set S, $S \nsim \mathcal{P}(S)$*

Proof. If S is finite then the proof is easy since $\mathcal{P}(S)$ contains $2^{card(S)}$ elements we can never find a bijection between S and $\mathcal{P}(S)$. So, let S be an infinite set. Consider a function $f : S \to \mathcal{P}(S)$. If for all f the function fails to be surjective then S is stricly smaller than $\mathcal{P}(S)$.

The proof is a diagonalization. Assume f is surjective. Note that f maps $s \in S$ to a subset $A \subseteq S$ since the co-domain is the set of all subsets of S. Define: $\tilde{S} = \{s \in S \mid s \notin f(s)\}$, it is the set of all s such that s itself is not present in $f(s)$ which is a subset of S. Now $\tilde{S}$ will be either non-empty or empty. Let it be non-empty. Now since f is a surjection there will exist some $\tilde{s} \in S$ such that $f(\tilde{s}) = \tilde{S}$. By the definition of $\tilde{S}$ we have: $\forall s \in S\ s \in \tilde{S}$ iff $s \notin f(s)$. Now choose $s = \tilde{s}$ and instantiate the above giving $\tilde{s} \in \tilde{S}$ iff $s \notin f(\tilde{s}) = \tilde{S}$, which is a contradiction. So, f cannot be surjective and $S \nsim \mathcal{P}(S)$.

We still have to consider the case when $\tilde{S} = \varnothing$. This means that for every $s \in S$, $s \in f(s)$ which implies that $\varnothing$ which is an element of $\mathcal{P}(S)$ does not have a pre-image in S. So, once again we have a contradiction and f fails to be surjective. So, we have proved that $S \nsim \mathcal{P}(S)$. □

For infinite sets that are not equipollent and where one can be considered to have a higher cardinality than another we will use the standard $<$ symbol. For example, we will write $cardinality(\mathbb{N}) < cardinality(\mathbb{R})$.

Cantor's proof allows us to create an infinite progression of infinities where no bijection can exist between successive infinite sets. The next infinity in the progression is the power set of the previous infinite set. These infinities are called cardinals and are represented by: $\aleph_0, \aleph_1, \aleph_2$ etc. $\aleph_0$ corresponds to countably infinite sets and is the 'smallest' infinity. $\aleph_1$ corresponds to sets whose cardinality is the same as $\mathbb{R}$. $\aleph_2 = cardinality(\mathcal{P}(\aleph_1))$ and so on.

A natural question that arises is: are there infinite sets with cardinalities between $\aleph_0$ and $\aleph_1$, that is does there exist an infinite set S such that $\mathbb{N} \nsim S \nsim \mathbb{R}$ and $cardinality(\mathbb{N}) < cardinality(S) < cardinality(\mathbb{R})$. Cantor conjectured that such an S does not exist. This is called the continuum hypothesis (CH). A more general question called the generalized continuum hypothesis (GCH) claims that there is no infinite set with cardinality between $\aleph_i$ and $\aleph_{i+1}$.

In 1939 Gödel proved that the GCH is consistent with Zermelo-Fraenkel set theory (minus the Axiom of Choice). That is one cannot derive the negation of the GCH from ZF set theory- see [4], [5]. In 1963 Cohen [6] proved that the CH cannot be proved from ZF set theory. So, since neither CH nor its negation can be proved from ZF it is independent of ZF set theory. For some more recent discussion and a different view which says that CH (or GCH) is not a proper mathematical or logical problem see Feferman [7].

3.2 Un/Incomputability

We now shift attention to how diagonalization is used in computation to obtain a very important result on the existence of non-computable functions. We start with some semantic and set theoretic paradoxes extend it to computation and see how a non-computable function emerges. The semantic and set theoretic paradoxes are discussed in [2], see [8] for incomputability.

Let us call predicates that apply to themselves **autological**. Examples of autological predicates are: English, short, polysyllabic, word. A predicate that is not autological we will call **heterological**. Examples are: Hindi, long, monosyllabic, red, etc. Most predicates are heterological. Note that both autological and heterological are themselves predicates. So, we can ask the question: is heterological autological or heterological. Let us assume heterological is autological. Then it applies to itself that is it is heterological - a contradiction. Now let heterological be heterological, then clearly it applies to itself and is therefore autological, again a contradiction. So, it is neither. But you expect a predicate to be one or the other and we have a paradoxical situation. There are other such semantic paradoxes (e.g. 'I am lying' or 'The barber who shaves all those who do not shave themselves'. Does the barber shave himself?). We still do not have a simple intuitive way to resolve such semantic paradoxes.

A more serious paradox from the mathematical standpoint is Russell's paradox. Let us define a set as **normal** if the set is not an element of itself and otherwise as **abnormal**. Let S be the set of all normal sets. The question is whether S is normal or abnormal. If S is normal then since it contains all normal sets it must contain itself and is therefore abnormal. On the other hand if it is abnormal and contains itself then since it is in S which is the set of all normal sets it must be normal. So we have a paradoxical situation for S.

Russell's paradox was a huge blow to set theory which was being developed as a foundation for mathematics. The key problem seems to be the unrestricted way in which a set is defined - called the axiom of unrestricted comprehension. Essentially, a set was being defined by: $\{x \mid P(x)\}$ where there is no constraint on x whatsoever. Multiple efforts were made to remove the paradoxes. One of which was Russell and Whitehead's theory of types. Zermelo and Fraenkel rescued set theory by constraining the axiom of comprehension so that it can only define subsets of a know/given set. So, the axiom now becomes: $\{x \in S \mid P(x)\}$, so x is now constrained to be a member of some set S. No, inconsistency has yet been found in ZF set theory and this along with the axiom of choice is the most widely accepted foundation for mathematics today.

We now take the semantic paradox and try to give it a computational flavour. Suppose we want to define a function `hetero(p)` which for a predicate (or program) `p` returns `True` if `p` is not true of itself and false otherwise. That is it implements heterological:

```
hetero(p) = not p(p)
```

If we apply hetero to itself then we get a non-terminating computation as follows:

```
hetero(p)  =not hetero(hetero)
           =not (not hetero(hetero))
           =not (not(not(hetero(hetero)))
           =...
```

There are other examples of functions that do not terminate for certain inputs. For example the following factorial function does not terminate for negative integers.

```
fac(n) = if (n==0) then 1 else n*fac(n-1)
```

So, let us assume that we can define a function `halts?(p, i)` that always terminates and returns `True` if `p(i)` terminates and `False` otherwise. Using this we rewrite the definition of `hetero` as follows:

```
hetero(p) = if (halts?(p, p)) then not p(p) else True
```

The meaning of `hetero` is: if `hetero(p)` does not terminate it returns `True` otherwise it returns `not p(p)`. Now notice that `hetero` always halts since `halts?` always halts and `p(p)` is executed exactly when `p` halts on `p` otherwise it just returns `True`. Let us again calculate `hetero(hetero)`.

```
hetero(hetero)  =if (halts?(hetero hetero)) then not hetero(hetero) else True
                =not hetero(hetero) Remark:  hetero terminates so executes then part
```

We get a contradiction hetero(hetero)=not hetero(hetero). If we try to restrict the power of the functions we can write and put the constraint that we cannot pass programs or predicates to a function only some kind of data then we can do the following. Assume that `r(p)` is some representation of the program - for example the textual representation. We can define the new hetero function `hetero*` that takes a representation of the program `r(p)` as input and uses `halt*?` instead of `halt?` as follows:

```
hetero*(r(p)) = if (halts*?(r(p),r(p))) then not interpret(r(p),r(p))
                else True
```

Here `interpret(r(p),i)` is a function with two arguments. It takes the representation of the program `r(p)` and then simulates the running of `p` on input `i`. It returns whatever `p(i)` returns and does not terminate if `p(i)` does not terminate. As a consequence we can say that $\forall p \forall i \; interpret(r(p), i) = p(i)$. With this observation let us evaluate `hetero*` with a representation of `hetero*` as input.

```
hetero*(r(hetero*))  =if (halts*?(r(hetero*),r(hetero*))) then not interpret(r(hetero*), r(hetero*))
                      else True
                     =not interpret(r(hetero*), r(hetero*)) Remark:  hetero* always terminates
                     =not hetero*(r(hetero*)) Remark:  uses semantics of interpret
```

Once again we have a contradiction. Since we can easily write the `interpret` function in all widely used languages like: C, C++, Java, Python, Lisp, Haskell etc. we must conclude that a function like `halts?` or `halts*?` cannot be programmed in those languages. Consequently, we have the following important result.

Theorem 3.4. *In any programming language L that is powerful enough to write an interpreter for a representation of programs in L it is not possible to program the halts? or halts*? function in L.*

The theorem asserts the non-computability of the halting functon.

References

[1] http://en.wikipedia.org/wiki/Cantor-Bernstein-Schroeder_theorem

[2] AA Fraenkel, Y Bar-Hillel, A Levy, Foundations of Set Theory, 2nd Ed., Elsevier, 1973.

[3] E Lehman, FT Leighton, AR Meyer, Mathematics for Computer Science, OCW courseware, 2013.

[4] K Gödel, The Consistency of the Continuum Hypothesis, Princeton Univ. Press, 1940.

[5] K Gödel, What is Cantor's Continuum Problem, Am. Math Monthly, 54, 515-525, 1947.

[6] P Cohen, Set Theory and the Continuum Hypothesis, WA Benjamin Inc., 1966.

[7] S Feferman, The Continuum Hypothesis is Neither a Definite Mathematical Problem nor a Definite Logical Problem, http://math.stanford.edu/ feferman/papers/CH_is_Indefinite.pdf.

[8] CAR Hoare, DCS Allison, Incomputability, Computing Surveys, 4(3),169-178, Sep. 1972.