positive value. So the value of $F$ may increase by at most $h(v) = h(u) - 1 \leq 2|V| - 1 - 1 = 2|V| - 2$. From Corollary 6.24 the number of saturated Push operations is at most $2|V|.|E|$. So $F$ can increase by at most $2|V|.|E|.(2|V| - 2)$ due to all the saturated Push.

Finally let us consider Relabel operation. Total contribution to $F$ due to all the Relabel operations on a vertex, $u$, can be at most equal to all the increments to $h(u)$. Since $h$ monotonically increases and its maximum value is $2|V| - 1$, total contribution to $F$ due to one vertex is $2|V| - 1$. Hence total increase in $F$ due to all the Relabel operations is at most $(|V| - 2)(2|V| - 1)$.

Putting all together $0 + 2|V|.|E|.(2|V| - 2) + (|V| - 2)(2|V| - 1) - K \geq 0$ or $K \leq 2|V|^2 + 4|V|.|E|(|V| - 1) = 4|V|(|E|.(|V| - 1) + |V|/2) \leq 4|V|.^2|E|$. $\square$

Push operation takes $O(1)$ time, Relabel takes $O(|V|)$ time, Insert/Extract takes $O(1)$, and *Insert_Edges* takes $O(|V|)$ time. After the initialization, each Push may be followed by an *Insert_Edges*. So each of these cost $O(|V|)$. Each Relabel is followed by an *Insert_Edges* so each Relabel costs $O(|V|)$. So the total cost is at $O(|V|.(4|V|^2.|E| + 2|V|.|E| + 2|V|^2)) = O(|V|^3.|E|)$.

THEOREM 6.26. *Algorithm 85 correctly computes a maximum flow in a flow network in $O(|V|^3.|E|)$ time.*

**6.4.4. The case of multiple Sources and Sinks.** Going back to our example of a coal mine and a steel mill let us consider a variant of the problem where there are multiple mines that produce coal and multiple mills that consume it. Further assume that each mine and each mill has a maximum limit for its production and consumption. Figure 6.4.4 shows a sample network. Label $b_i$ at vertex $i$ is to be interpreted as follows. If $b_i > 0$, then $i$ is a source (mine) and it can produce at the rate up to $b_i$; if $b_i < 0$, then $i$ is a sink (mill) and can consume at the rate at most $-b_i$; and if $b_i = 0$, then $i$ is an intermediate node that neither produces nor consumes. Formally, the flow must satisfy the condition $0 \leq \sum_j f(i,j) \leq b_i$ for positive $b_i$; $0 \leq \sum_j f(j,i) \leq -b_i$ for negative $b_i$; and $\sum_j f(i,j) = 0$ if $b_i = 0$.

This variant of the problem can be formulated using one-source - one-sink model discussed above. Add two special nodes, $s$ and $t$, and add one edge from $s$ to each source $i$ and assign capacity $b_i$ to it and add one edge from each sink $i$ to $t$ and assign capacity $-b_i$. See an example in figure 6.4.4.

There is a one to one correspondence between solution of the original network and the new network. A solution of the new network gives a solution, of equal flow magnitude, of the original multi-source multi-sink network with the same flow by simply removing the added edges. Similarly given a solution of the original network add flow to the new edge $(s, s_i)$ ( edge $(t_i, t)$) equal to the flow emerging (sinking) from $s_i$ (into $t_i$) we can construct a valid flow of the transformed network with same magnitude. Hence we see that single source and single sink model is general enough.

## 6.5. Minimum Weight Circulation

**6.5.1. Weighted Flow.** In this section we will discuss a more general flow problem and discuss a classic solution for it. We will also see that the maximum flow problem is a special case of this problem. In this chapter we will associate a cost/weight (or per-unit cost) for each edge in the network. If $u$ units are flowing along an edge $e$, then total cost of the flow through this edge will be $u.w(e)$ where

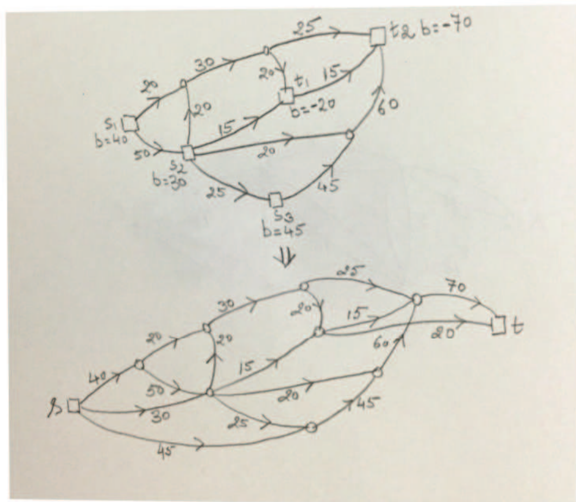


FIGURE 3. An example of a multi source/sink network and reduction to a single source/sink network

$w()$ will be called the weight function. To handle this situation in a general manner, we define a flow network as a *directed multi-graph* $(V, E, c, w)$ where $c$ is the capacity function and $w$ is the weight function. In this case there can be multiple edges between a pair of vertices. Each edge $e \in E$ will have its own capacity $c(e)$ and weight $w(e)$. The value of $c(e)$ is non-negative but the value of $w(e)$ can be any real number including negative. Since negative value is allowed we choose to call it a weight function rather than a cost function. We will denote the end-vertices of $e$ by $e^l$ and $e^r$ where edge is directed from $e^l$ to $e^r$. If two *parallel* edges, $e_1, e_2$ (i.e., $e_1^l = e_2^l$ and $e_1^r = e_2^r$) have same weight (i.e., $w(e_1) = w(e_2)$), then we may replace them by a single edge $e$ such that $e^l = e_1^l = e_2^l$, $e^r = e_1^r = e_2^r$, $w(e) = w(e_1) = w(e_2)$, and $c(e) = c(e_1) + c(e_2)$. This is why we did not assume multiple parallel edges in our previous discussion.

In the modified definition of a flow network we need to change the definition of a flow. Previously $f(u, v)$ was equal to $-f(v, u)$ and we had stated the capacity condition as $-c(v, u) \leq f(u, v) \leq c(u, v)$. Earlier if $a$ unit of flow was in the edge $(u, v)$ and $b$ unit of flow was in $(v, u)$, then we defined $f(u, v) = a - b$. But in the new definition $f$ assumes non-negative value. It is a function $f : E \to \mathbb{R}^+ \cup \{0\}$ because there may be multiple edges between the same pair of vertices. Given a

flow $f$ in a flow-network, its cost is defined by $\sum_{e \in E} w(e).f(e)$. In this more general network we will discuss a solution of the following problem.
*Minimum Cost Flow Problem*: Compute a maximum flow in the network that has minimum weight.

Before we start to discuss the solution of this problem we make a modification in the structure of the flow networks. The new structure is called a *circulation-network*.

**6.5.2. Circulation.** A circulation-network is similar to a flow network but it does not have any source-node or a sink-node.

Definition 6.27. A weighted circulation-network is a directed multi-graph $G = (V, E, c, w)$, where $c : E \to \mathbb{R}^+$ is a non-negative valued function called capacity-function and $w : E \to \mathbb{R}$ is a real-valued function called weight-function.

Homogeneous structure of the circulation networks (no special vertices as $s$ and $t$) make them simple to work with as we will see later.

Definition 6.28. A circulation in a circulation-network is a function $f : E \to \mathbb{R}^+$ satisfying following properties:

(1) $0 \leq f(e) \leq c(e)$ for all $e \in E$,
(2) $\sum_{e:e^l=x} f(e) = \sum_{e:e^r=x} f(e)$ for all $x \in V$.

The total cost/weight of a circulation is defined by $\sum_{e \in E} w(e).f(e)$. The minimum cost/weight circulation problem is defined as follows.
**Minimum Cost Circulation Problem** Given a circulation network $(V, E, c, w)$, compute a circulation for it with minimum $\sum_{e \in E} w(e).f(e)$.

In the remainder of this chapter we will develop an algorithm to solve this problem. But first, let us justify that this problem subsumes the maximum flow problem and the minimum cost flow problem.

Claim 6.29. *For a flow network $G = (V, E, c)$ we can define a circulation network $G'$ such that there is a 1 to 1 correspondence between the flows in $G$ and circulations in $G'$.*

Proof. Modify the given flow network into a circulation network by adding an edge $e_0$ from $t$ to $s$ with $\infty$ capacity. Retain the edges of $G$ and their capacities.

Let $f$ be any flow in $G$. Then define a circulation $f'$ in $G'$ by defining $f'(e) = f(e)$ for all $e \in E$ and define $f'(e_0) = |f|$. Observe that $f'$ is a valid circulation.

Conversely for any circulation $f'$ in $G'$ define $f$ for $G$ as $f(e) = f'(e)$ for all $e \in E$. Then observe that $f$ is a valid flow in $G$ and $|f| = f'(e_0)$. □

Using this construct we will show how to reduce the maximum flow problem and minimum cost flow problem into minimum cost circulation problem. Here is a trivial claim.

Claim 6.30. *Let $G = (V, E, c)$ be an unweighted flow network. Let $G' = (V, E \cup \{e_0\}, c, w')$ be the corresponding circulation network as defined above. Let $w'(e) = 0$ for all $e \in E$ and $w'(e_0) = -1$. Then $G$ has a flow of magnitude $m$ if and only if the corresponding circulation has cost $-m$.*

This claim shows that the flow in $G$ corresponding to the minimum cost circulation $f'$ is a maximum flow for $G$. So it shows that the maximum flow problem can be reduced the minimum cost circulation problem. Next consider the following claim which shows how to reduce the minimum cost flow problem to minimum cost circulation problem.

Claim 6.31. *Let $G = (V, E, c, w)$ be a weighted flow network. Let $G' = (V, E \cup \{e_0\}, c', w')$ be the corresponding circulation network as defined above. Let $w'(e) = w(e)$ for all $e \in E$ and $w'(e_0) = 0$. Then the cost any flow of $G$ is same as the cost of the corresponding circulation of $G'$.*

Definition 6.32. Given a circulation $f$ in a circulation network $G = (V, E, c, w)$ and a circulation $f$ in it. Then its *flow-graph* is the directed graph $G^f = (V, \{e \in E | f(e) > 0\})$. If $G^f$ is a simple cycle then $f$ is called a *cyclic circulation*.

For the remainder of this chapter we will express flow/circulation as a vector in $\mathbb{R}^{|E|}$. A flow $f$ will be denoted by a vector in which $e$-th component will have value $f(e)$. In addition any simple cycle $C$ will be denoted by a 0-1 vector in which $e$-th component is 1 if $e$ belongs to the cycle. Otherwise it is 0. In this notation a cyclic circulation can be denoted by $f = |f|.C$ where $G^f$ is the cycle $C$ because all edges of $C$ carry $|f|$ units of flow.

Suppose $C_1, C_2, \ldots, C_t$ are cycles in a circulation network and $\alpha_1, \ldots, \alpha_t$ are positive real numbers. Then $f = \alpha_1.C_1 + \alpha_2.C_2 + \cdots + \alpha_t.C_t$ defines a vector in which $f(e) = \sum_{j | e \in C_j} \alpha_j$ . Observe that $f$ satisfies the Kirchhoff's law at every vertex. So $f$ is a valid circulation if it also satisfies the capacity condition, i.e., $f(e) \leq c(e)$. Following result shows that every circulation can be decomposed into cyclic circulations.

Lemma 6.33. *Given any circulation network and any circulation $f$ in it, there exist cycles $C_i$ in the network graph such that $f = \alpha_1.C_1 + \cdots + \alpha_t.C_t$, where each $\alpha_i > 0$.*

Proof. The proof is by induction on the number of edges in $G^f$. The claim is trivial if $G^f$ is an empty graph. Suppose the claim holds if $G_f$ has $k$ or less edges. Now suppose $G^f$ has $k+1$ edges. In a circulation the magnitude of the inflow into a vertex is same as the magnitude of the outflow from that vertex. Hence in $G^f$, either a vertex is isolated (no incoming or no outgoing edges incident on them) or it has at least one incoming and at least one outgoing edge. Hence the graph must have at least one (directed) cycle. Let $C : x_0 x_1 \ldots, x_{r-1} x_0$ be one of the cycles in $G^f$. So $f(x_0, x_1) > 0, f(x_1, x_2) > 0, \ldots, f(x_{r-1}, x_0) > 0$. Let $\alpha = \min_i f(x_i, x_{i+1})$, here index summation is modulo $r$. Let $f'$ be defined as follows. $f'(y, z) = f(y, z)$ for all $(y, z)$ not in $C$ and $f'(y, z) = f(y, z) - \alpha$ for $(y, z)$ in $C$. So $f' = f - \alpha.C$. It is easy to verify that $f'$ is also a circulation and $G_{f'}$ has at least one less edge than in $G_f$. Hence from the induction hypothesis $f' = \alpha_1.C_1 + \cdots + \alpha_{t-1}.C_{t-1}$. Then $f = \alpha_1.C_1 + \cdots + \alpha_{t-1}.C_{t-1} + \alpha.C$. □

Definition 6.34. The residual graph of a circulation in a circulation network is defined in the similar fashion as it is defined in the flow network. Let $G = (V, E, c, w)$ be a circulation network and let $f$ be a circulation in it. $E^{rev}$ is set of new edges given by $\{e^{rev} | e \in E\}$. Observe that if $e_1$ is an edge from $x$ to $y$ and $e_2$ is an edge from $y$ to $x$, then $e_1^{rev}$ and $e_2^{rev}$ are **new edges** which are from $y$ to $x$ and $x$ to $y$ respectively. The residual graph of a circulation $f$ is the circulation network $G_f = (V, \{e \in E | f(e) < c(e)\} \cup \{e^{rev} \in E^{rev} | f(e) > 0\}, c_f, w_f)$ where (i) $c_f(e) = c(e) - f(e)$ for $e \in E$ because we can add $c(e) - f(e)$ flow to $e$ without violating capacity; (ii) $c_f(e^{rev}) = f(e)$ for all $e \in E$ because if $\alpha$ amount is flowing along $(x, y) \in E$, then we can decrease at most $\alpha$ flow from this edge or increase at most $\alpha$ along the virtual edge $(y, x)$; (iii) If $(x, y) \in E$, then by adding flow along $(y, x)$ we are effectively decreasing "real flow" leading to decrease in the cost at the rate of $w(x, y)$. So $w(y, x) = -w(x, y)$ for all $(y, x) \in E^{rev}$. More generally $w(e^{rev}) = -w(e)$ for all $e \in E \cup E^{rev}$. Thus $w_f$ is independent of $f$ and it is same as $w$ on original edges and $-w$ on the virtual edges. Note that we have used the superscript $rev$ for all edges of $E \cup E^{rev}$, where $(e^{rev})^{rev} = e$.

Following theorem gives a criterion for a minimum cost circulation in any circulation network. It is similar to the flow network case where we had a criterion for maximum flow (having no augmenting path) in the sense that it will lead to an algorithm for computing a minimum cost circulation.

THEOREM 6.35. *A circulation $f$, in a circulation network $G = (V, E, c, w)$ with finite costs and finite capacities, is a minimum cost circulation if and only if $G_f$ has no (directed) cycle with negative cost (cost of a cycle is the sum of the costs of its edges).*

PROOF. All capacities and costs in $G$ are finite so the cost of any circulation must be finite. Consider an arbitrary circulation $f$ in it. Suppose the residual graph $G_f$ has a cycle $C$ with negative cost $-\beta < 0$. Let $\alpha = \min_{e \in C} c_f(e)$. Hence $f' = f + \alpha.C$ is a valid circulation with cost $Cost(f') = Cost(f) + \alpha.(-\beta) < Cost(f)$. Hence $f$ is not a minimum cost circulation.

Conversely suppose $f$ is not a minimum cost circulation and let $f'$ be a minimum cost circulation in the network. Let us define a new circulation $f''$ in $G_f$ as follows: For all $e \in E$, if $f'(e) \geq f(e)$, then $f''(e) = f'(e) - f(e)$ and if $f'(e) < f(e)$, then $f''(e^{rev}) = f(e) - f'(e)$. To see that $f''$ satisfies the capacity constraints in $G_f$ we need to consider two cases: (i) If $f(e) < f'(e)$, then $f'(e) - f(e) \leq c(e) - f(e) = c_f(e)$; (ii) If $f'(e) < f(e)$, then $f(e) - f'(e) = c_f(e^{rev}) - f'(e) \leq c_f(e^{rev})$. Besides, it is easy to verify that as a vector $f'' = f' - f$. Since $f$ and $f'$ satisfy the Kirchhoff's law, $f''$ will also satisfy it. Hence $f''$ is also a circulation. We also have the relation $w(f'') = w(f') - w(f) < 0$ because $f'$ is a minimum cost circulation.

From Lemma 6.33 $f''$ can be decomposed into cyclic circulations, $f'' = \alpha_1 C_1 + \cdots + \alpha_r C_r$, where each $C_i$ is a cycle in $G_f$. So we have $w(f'') = \sum_i \alpha_i.w(C) < 0$. This implies that for at least one $C_i$, $w(C_i) < 0$. □

Note that the edges of $G_f$ constitute a subset of $E \cup E^{rev}$ hence the edges of cycles in $G_f$ also belong to $E \cup E^{rev}$. It may be observed that if $C$ is a clockwise cycle in $G_f$, then for any edge $e \in E$, if $e$ occurs in $C$, then it must be pointing in the clockwise direction and if $e^{rev}$ occurs in $C$, then $e$ must be pointing in the anti-clockwise direction.

This theorem gives a method to compute a minimum circulation, like Ford Fulkerson's method computes a maximum flow. Algorithm 86 describes Klein's negative cycle cancellation method. It iteratively selects a cycle having negative weight in the residual graph and augments maximum possible flow along it so that at least one of the edges gets saturated and gets removed from the residual graph, i.e., the cycle breaks.

Theorem 6.35 assures that if this algorithm terminates, then $f$ will be a minimum cost circulation. Klein's algorithm suffers from the same problem that Ford-Fulkerson algorithm did. How to guarantee that the algorithm will terminate? Surprisingly, as we saw in the Edmonds and Karp solution of termination problem

**for** $e \in E$ **do**
  $f(e) := 0;$
**end**
Compute residual graph $G_f$;
**while** $G_f$ *has a negative cost cycle* **do**
  Find a negative cost cycle $C$ in $G_f$;
  $\alpha := \min_{e \in C} c_f(e);$
  $f := f + \alpha.C;$
  Compute residual graph $G_f$;
**end**
Return $f$;

**Algorithm 80:** Klein's Cycle Cancellation Method

in Ford-Fulkerson algorithm, here too there exists a solution for termination problem based on a strategic way to select the negative weight cycles for cancellation. So we need a strategy for the selection of negative cost cycles in Klein's method to ensure termination.

**6.5.3. Minimum Mean Cost Strategy.** Goldberg and Tarjan showed that the Klein's method will terminate if in each iteration that cycle from $G_f$ is selected which has the minimum mean cost (average cost). If a cycle has length $L$ and its cost is $\beta$, then the mean-cost is $\beta/L$. If the circulation in not optimum, then from Theorem 6.35 there must exist at least one cycle with negative cost in $G_f$. The mean-cost of a cycle will be negative if and only if its total cost is negative. Therefore if a circulation is not optimal, then the minimum mean-cost will obviously be negative. In the remainder of this section we will derive an upper-bound on the number of iterations in the Klein's method under this strategy of cycle selection. Let $f$ be a circulation in a circulation network $G$. We will denote the mean-cost of the minimum mean-cost cycle in $G_f$ by $\mu_f$. Recall that in Chapter 1 we have discussed how to compute a minimum mean-weight cycle.

Let $|V| = n$ and $|E| = m$. Then Goldberg and Tarjan showed that in $2m$ iterations of Klein's method the magnitude of the minimum mean-cost will reduce by a factor of $1/n$, under minimum mean-weight strategy. That is, if the circulation after $r$ iterations is $f$ and that after $r + 2m$ iterations is $f'$, then $-\mu_{f'} \leq (1 - 2/n)(-\mu_f)$.

The minimum cost circulation problem is trivial if the costs of all the edges are non-negative because in that case $f = 0$ is the minimum cost circulation. Hence we will assume that the cost of at least one edge in $G$ is negative. Assume that $W_0(< 0)$ is the smallest edge cost in $E \cup E^{rev}$.

THEOREM 6.36. *In a graph with integral costs the Klein's Algorithm computes a minimum cost circulation in* $O(m^2 n \log^2(-W_0.n))$ *time if a minimum mean-cost cycle is cancelled (augmented) in every iteration, where* $W_0$ *is the smallest edge cost in* $E \cup E^{rev}$.

PROOF. If $f$ is a circulation such that $-\mu < 1/n$. Then the total cost of the minimum mean-cost cycle $C$ in $G_f$ will be $w(C) = |C|.\mu > (-1/n)|C| \geq (-1/n).n = -1$ where $|C|$ denotes the length of the cycle. The cost of the edges are given to be integral so $w(C) \geq 0$. Therefore there cannot be any negative cost cycle in $G_f$. Thus the Algorithm 86 must terminate as soon as $-\mu$ decreases below $1/n$.

Let $W_0$ be the smallest edge cost in $E \cup E^{rev}$. So initially $-\mu_i \leq -W_0$. After $k.(2m)$ iterations if the circulation is $f$, then from Goldberg Tarjan result $-\mu_f \leq \mu_i(1-2/n)^k \leq (-W_0)(1-2/n)^k$. From the previous paragraph we see that the algorithm will terminate after $k.(2m)$ iterations if $(-W_0)(1-2/n)^k < 1/n$ or $k.\log\,(1-2/n) < -\log(-W_0.n)$. Using $\log(1-x) = -x - x^2/2 - x^3/3 - \ldots$ we get $k > (\log(-W_0.n))/(2/n) = n.\log(-W_0.n)/2$. So we deduce that the algorithm will terminate after at most $(2m).n.\log(-W_0.n)+1$ iterations.

From Theorem 1.40 we know that $\mu_f$ can be computed in $O(n.m.\log_2\,(-W_0))$. By adding $\mu_f$ to all the edges of $G_f$ the cycle with minimum mean-wright cycle will reduce to a cycle of weight zero and all other cycles's weights will become positive. This zero weight cycle (minimum weight cycle) can be computed in $O(n^3)$ time, see Claim 1.37. The cost of computing $G_f$ is $O(|V|+|E|)$. Hence the total time cost of one iteration is $O(n+m+n^3+n.m.\log_2\,(-W_0)) = O(n^3+n.m.\log_2\,(-W_0))$. The total time complexity of the algorithm is $O(m^2.n^2(\log(-W_0.n).\log(-W_0) + n^4.m.\log(-W_0.n)))$ if the edge costs are integral. □

**6.5.4. Goldberg Tarjan Theorem.** In the last part of the chapter we will establish the Goldberg Tarjan result. We will show that in the successive iterations $-\mu_f$ never increase and then we will establish the main claim. In order to do this we define a new weight function called *reduced weight function* which is more "homogeneous". Let $p : V \to \mathbb{R}^+$ be a non-negative function, called a potential function. Then we define a reduced weight function $w_p$ given by $w_p(e) = w(e) + p(x) - p(y)$ for every edge $e = (x, y)$ in $E \cup E^{rev}$. For any edge $e \in E \cup E^{rev}$, $e^{rev}$ will denote the edge $(y, x)$ irrespective of whether $e$ belongs to $E$ or $E^{rev}$. For any $e = (x, y) \in E \cup E^{rev}$, $w_p(e^{rev}) = w(e^{rev}) + p(y) - p(x) = -w(e) + p(y) - p(x) = -w_p(e)$.

Given an $\epsilon > 0$, the circulation $f$ is said to be $\epsilon$-optimal with respect to a given potential function $p$ if $w_p(e) \geq -\epsilon$ for all $e$ in $G_f$. The smallest $\epsilon$, for which there exists a potential function such that $f$ is $\epsilon$-optimal, is denoted by $\epsilon_f$. We will first show that $\epsilon_f = -\mu_f$.

LEMMA 6.37. *Let $f$ be any circulation, then $\epsilon_f = -\mu_f$.*

PROOF. Let us first show that $\epsilon_f \leq -\mu_f$. Consider the modified weight function $w' = w - \mu_f$. As $\mu_f$ is the smallest mean-weight of any cycle in $G_f$ with respect to the weight function $w$, $G_f$ cannot have any negative weight cycle with respect to $w'$. Add a new vertex $s$ and edges $(s, x)$ with weight $w'(s, x) = 0$ for all $x \in V$. Define the potential function $p(x) = dist_{w'}(s, x)$ for all $x \in V$, where $dist_{w'}(s, x)$ is the weight of the minimum weight path from $s$ to $x$ with respect to the weight function $w'$.

Consider an arbitrary edge $e = (x, y)$ in $G_f$. Consider a path from $s$ to $y$ which is the shortest path to $x$ followed by the edge $e$. So $p(y) \leq p(x) + w'(e) = p(x) + w(e) - \mu_f$. So $\mu_f \leq w(e) + p(x) - p(y) = w_p(e)$. So $G_f$ is $(-\mu_f)$-optimal. Hence by the definition $\epsilon_f \leq -\mu_f$.

To prove the converse consider the cycle $C$ which has the minimum mean weight with respect to $w$. Suppose it has $L$ edges. Then $L.\mu_f = \sum_{e \in C} w(e)$. Let $p$ be the potential function with respect to which $f$ is $\epsilon_f$-optimal. For any cycle $C'$, $\sum_{e\in C'} w(e) = \sum_{e\in C'} w_p(e)$ because potential terms get cancelled. In particular, $\sum_{e\in C} w(e) = \sum_{e\in C} w_p(e) \geq \sum_{e\in C}(-\epsilon_f) = L.(-\epsilon_f)$. Therefore we have $\mu \geq -\epsilon_f$. $\square$

Next we will show that $-\mu_f$ is a monotonically decreasing function.

LEMMA 6.38. *$-\mu_{f_i} \geq -\mu_{f_{i+1}}$ for all $i$, where $f_i$ denotes the circulation after $i$ iterations.*

PROOF. Consider an arbitrary iteration $i$. Suppose $C$ is the minimum mean-weight cycle in $G_{f_i}$ which was cancelled in the $i+1$-st iteration. Let its length be $L$. Let $p$ be the potential function with respect to which $f_i$ is $\epsilon_{f_i}$-optimal. So $\sum_{e\in C} w_p(e) = \sum_{e\in C} w(e) = L.\mu_{f_i}$. By the definition of $\epsilon_{f_i}$-optimality $w_p(e) \geq -\epsilon_{f_i} = \mu_{f_i}$ for all $e$. So $w_p(e) = \mu_{f_i}$ for all $e \in C$. In this sense $w_p$ is a homogeneous weight function.

We will now show that $w_p(e) \geq \mu_{f_i}$ for every $e$ in $G_{f_{i+1}}$. This will imply that $\mu_{f_{i+1}} \geq \mu_{f_i}$ because there might a better choice of function $p$ giving a tighter lower bound for $w_p(e)$.

Observe that the edges of $E_{f_{i+1}} \setminus E_{f_i}$ are such edges which were not in $G_{f_i}$ (i.e., were saturated) and their reverse appeared in $C$. Let $p$ be the optimal potential function for $G_{f_i}$. If $e \in E_{f_{i+1}} \cap E_{f_i}$, then $w_p(e) \geq -\epsilon_f = \mu_{f_i}$.

If $e = (x, y) \in E_{f_{i+1}} \setminus E_{f_i}$ then $e^{rev} \in C$ and $w_p(e^{rev}) = \mu_{f_i}$. So $w_p(e) = w(e) + p(x) - p(y) = -w(e^{rev}) - p(y) + p(x) = -w_p(e^{rev}) = -\mu_{f_i}$. So $w_p(e) = -\mu_{f_i} > \mu_{f_i}$ because $\mu_{f_i}$ is negative. Therefore $w_p(e') \geq \mu_{f_i}$ for all $e' \in E_{f_{i+1}}$. Thus $\mu_{f_{i+1}} \geq \mu_{f_i}$. $\square$

Now we have necessary background to prove the main result.

LEMMA 6.39. *$-\mu_{f_{q+2m}} \leq (1 - 2/n)(-\mu_{f_q})$ for all $q$.*

PROOF. Our argument to prove this result is very similar to that in the proof of Lemma 6.13 and Theorem 6.15. Consider any $2m+1$ consecutive iterations. At least one edge in every augmenting cycle (the cycle to be cancelled) must be critical. Recall that an edge is critical if it gets saturated due to the current augmenting cycle and vanishes from the next residual graph. So in this span at least one edge must be critical twice. Consider arbitrary iteration $q$. Assume that an edge $(x, y)$ is critical in $G_{f_i}$ and again in $G_{f_j}$ for some $q \leq i < j \leq q + 2m$. Since $(x, y)$ was missing in $G_{f_{i+1}}$ it must have reappeared some time between iterations $i+2$ and $j$. For this to happen, edge $(y, x)$ must occur in the augmenting cycle (not necessarily as critical) in $G_{f_k}$ for some $i+1 \leq k \leq j-1$. So we have $i < k < j$ such that an edge $(x, y)$ occurs in the augmenting cycle $C_i$ of $G_{f_i}$ and $(y, x)$ occurs in the augmenting cycle $C_k$ of $G_{f_k}$.

Let $i \leq r \leq s \leq k$ such that $[r, s]$ is the shortest interval having the property that there is an edge present in augmenting cycle $C_r$ and its reverse is present in the augmenting cycle $C_s$. As shown above there exists at least one such interval $[i, k]$ so $[r, s]$ exists. Partition the edges of $C_s$ into two sets $A$ and $B$ where $A$ contains those edges which are either not present in $G_{f_r}$ or its reverse is present in the augmenting cycle $C_r$ in $G_{f_r}$. Thus by the choice of the pair $(r, s)$, $A$ is non-empty. $B$ is the set of the remaining $C_s$ edges. So every $B$-edge is present in $G_{f_r}$. Let $e$ be any arbitrary edge in $A$ such that $e^{rev}$ does not occur in $C_r$. So $e$ does not occur in $G_{f_r}$. For $e$ to be present in $G_{f_s}$, $e^{rev}$ must occur in $C_q$ for some $r < t < s$. Then we have a pair $(t, s)$ such that $e^{rev}$ occurs in $C_t$ and $e$ occurs in $C_s$ and this is a shorter interval than that of $(r, s)$. This contradicts the choice of the pair $(r, s)$. So we can deduce that reverse of every $A$-edge occurs in $C_r$. Also, by the definition of $(r, s)$, $A$ is non-empty.

Let $p$ be the potential function such that $w_p(e) \geq -\epsilon_{f_r}$ for all $e \in G_{f_r}$. So $\sum_{e\in C_s} w(e) = \sum_{e\in C_s} w_p(e) = \sum_{e\in A} w_p(e) + \sum_{e\in B} w_p(e) = -\sum_{e\in A} w_p(e^{rev}) + \sum_{e\in B} w_p(e)$. Every $e \in B$ is also present in $G_{f_r}$ so $w_p(e) \geq -\epsilon_{f_r}$. Every $e \in A$ is such that $e^{rev}$ belongs to $C_r$. Earlier we have seen that reduced weight of every edge of the augmenting cycle is $-\epsilon_f$. So $w_p(e^{rev}) = -\epsilon_{f_r}$ for all $e \in A$. Therefore $\sum_{e\in C_s} w(e) \geq \epsilon_{f_r}.|A| + (-\epsilon_{f_r}).|B|$. Let the length of $C_s$ be $L$. So $|A| + |B| = L$ and we have seen that $|A| \geq 1$. So $\sum_{e\in C_s} w(e) \geq (|L| - 1)(-\epsilon_{f_r}) + \epsilon_{f_r} = -(|L| - 2).\epsilon_{f_r}$.

We also have $\sum_{e\in C_s} w(e) = |L|.(\mu_{f_s}) = |L|.(-\epsilon_{f_s})$. Hence $|L|.(-\epsilon_{f_s}) \geq -(|L| - 2).\epsilon_{f_r}$ or $\epsilon_{f_s} \leq (1-2/|L|)\epsilon_{f_r} \leq (1-2/n)\epsilon_{f_r}$. From Lemma 6.38 $\epsilon_f$ is a non-increasing function. So $\epsilon_{f_{q+2m}} \leq \epsilon_{f_s} \leq (1 - 2/n)\epsilon_{f_r} \leq (1 - 2/n)\epsilon_{f_q}$. $\square$