Problem Set # 7 : solution

(1) jt p.d.f. of (X, Y)

$$f_{X,Y}(x,y) = \begin{cases} 4xy, & 0<x<1,\ 0<y<1 \\ 0, & o/w \end{cases}$$

Marginal of X:

$$f_X(x) = \int_0^1 4xy\, dy = 2x \qquad 0<x<1$$
$$= 0 \qquad o/w.$$

Sly $$f_Y(y) = \begin{cases} 2y, & 0<y<1 \\ 0, & o/w. \end{cases}$$

observe that $f_{X,Y}(x,y) = f_X(x)\, f_Y(y)$

$\Rightarrow X \,\&\, Y$ are indep.

$$P(0<X<\tfrac{1}{2}, \tfrac{1}{4}<Y<1) = P(0<X<\tfrac{1}{2})\, P(\tfrac{1}{4}<Y<1)$$
$$= \left(\int_0^{1/2} 2x\,dx\right)\left(\int_{1/4}^{1} 2y\,dy\right) = \ldots$$

$$P(X+Y<1) = \int_0^1 P(X<1-y)\, f_Y(y)\, dy \rightarrow X \,\&\, Y \text{ are indep.}$$
$$= \int_0^1 \left[\int_0^{1-y} 2x\,dx\right] 2y\,dy$$
$$= \ldots$$

(2) $f_X(x) = \int_0^{\infty} e^{-x} e^{-y} \, dy = e^{-x} \qquad x > 0$

$\qquad\qquad\qquad\qquad\qquad\qquad = 0 \qquad o/w$

$f_Y(y) = e^{-y} \qquad y > 0$

$\qquad\quad 0 \qquad o/w$

$f_{X,Y}(x,y) = f_X(x) \, f_Y(y)$

$\Rightarrow$ X & Y are indep.

(3) $f_X(x) = \int_x^{\infty} 2 e^{-x} e^{-y} \, dy$

$= 2e^{-x} e^{-x} = 2e^{-2x} \qquad x > 0$

$\qquad\qquad = 0 \qquad o/w$

$\text{ly } f_Y(y) = 2\int_0^y e^{-y} e^{-x} \, dx = 2e^{-y}(1-e^{-y}) \qquad y > 0$

$\qquad\qquad\qquad\qquad = 0 \qquad o/w$

$f(x,y) \neq f(x) \, f(y)$

$\Rightarrow$ X & Y are not indep.

(4) $f_X(x) = 12x \int_0^1 (y - y^2) \, dy = 12x \left[\frac{y^2}{2} - \frac{y^3}{3} \right]_0^1$

$= 12x \cdot \frac{1}{6} = 2x$

$\Rightarrow f_X(x) = \begin{cases} 2x & 0 < x < 1 \\ 0 & o/w \end{cases}$

$f_Y(y) = 12y(1-y) \int_0^1 x \, dx = 6y(1-y) \qquad 0 < y < 1$

$\qquad\qquad\qquad\qquad = 0 \qquad o/w$

$f(x,y) = f(x) \, f(y)$

$\Rightarrow$ X & Y are indep.

(5) $\int_0^1 \int_x^1 f(x,y) \, dy \, dx = 1$, i.e. $c\int_0^1 x^2 \int_x^1 y \, dy \, dx = 1$

$\Rightarrow c \int_0^1 x^2 \, \frac{1}{2}(1-x^2) \, dx = 1$

$\Rightarrow \frac{c}{2} \left[\frac{x^3}{3} - \frac{x^5}{5} \right]_0^1 = 1 \quad \Rightarrow c = 15$

(b) $f_X(x) = 15x^2 \int_x^1 y\,dy = \begin{cases} 15/2\, x^2(1-x^2), & 0 < x < 1 \\ 0, & o/w \end{cases}$

$f_Y(y) = 15\,y \int_0^y x^2\,dx = \begin{cases} 5y^4, & 0 < y < 1 \\ 0, & o/w. \end{cases}$

(c) $P(X+Y \le 1) = \iint\limits_{\substack{x+y\le 1 \\ x<y}} 15\,x^2 y\,dy\,dx$

$= 15\int_0^{1/2} x^2 \int_x^{1-x} y\,dy\,dx = 15\int_0^{1/2} x^2 \left.\frac{y^2}{2}\right|_x^{1-x} dx$

$= \ldots\ldots = \frac{15}{192}.$

<u>Alt</u>

$P(X+Y \le 1) = \iint\limits_{\substack{x+y\le 1 \\ x<y}} 15\,x^2 y\,dx\,dy$

$= 15\int_0^1 y \int_0^{\min(y,1-y)} x^2\,dx\,dy$

$= 15\int_0^{1/2} y \int_0^y x^2\,dx\,dy + 15\int_{1/2}^1 y \int_0^{1-y} x^2\,dx\,dy$

$= \ldots\ldots = \frac{15}{15\times 32} + \frac{15}{10\times 32} = \frac{15}{192}$

(6) $f_X(x) = \int_0^{1-x} f_{X,Y}(x,y)\,dy = 6\int_0^{1-x}(1-x-y)\,dy = 6\left[(1-x)y - \frac{y^2}{2}\right|_0^{1-x}$

$= \begin{cases} 3(1-x)^2, & 0 < x < 1 \\ 0, & o/w \end{cases}$

Sly (by symmetry)

$f_Y(y) = \begin{cases} 3(1-y)^2, & 0 < y < 1 \\ 0, & o/w. \end{cases}$

$$P(2X+3Y<1) = 6\int_0^{1/2}\int_0^{\frac{1-2x}{3}} (1-x-y)\,dy\,dx$$

$$= 6\int_0^{1/2}\left[(1-x)y - \frac{y^2}{2}\right]_0^{\frac{1-2x}{3}} dx$$

$$= 6\int_0^{1/2}\left\{(1-x)\left(\frac{1-2x}{3}\right) - \frac{1}{2}\left(\frac{1-2x}{3}\right)^2\right\} dx$$

$$= 6\int_0^{1/2}\left(\frac{1+2x^2-3x}{3} - \frac{1+4x^2-4x}{18}\right) dx$$

$$= 6\int_0^{1/2} \frac{8x^2-14x+5}{18}\,dx$$

$$= \frac{6}{18}\left(8\frac{x^3}{3} - 14\frac{x^2}{2} + 5x\right|_0^{1/2}$$

$$= \frac{6}{18}\left(\frac{8}{3}\cdot\frac{1}{8} - 7\cdot\frac{1}{4} + \frac{5}{2}\right) = \frac{13}{36}$$

Alt $$P(2X+3Y<1) = 6\int_0^{1/3}\int_0^{\frac{1-3y}{2}} (1-y-x)\,dx\,dy$$

$$= 6\int_0^{1/3}\left((1-y)x - \frac{x^2}{2}\right|_0^{\frac{1-3y}{2}} dy$$

$$= 6\int_0^{1/3}\left[(1-y)\left(\frac{1-3y}{2}\right) - \frac{1}{2}\left(\frac{1-3y}{2}\right)^2\right] dy$$

$$\cdots = \frac{13}{36}$$

(7) $f_X(x) = \int_0^1 f(x,y)\,dy = \int_0^1 (x+y)\,dy$

$$= \begin{cases} x + \frac{1}{2} & 0 < x < 1 \\ 0 & \text{o/w} \end{cases}$$

$$f_{Y|X} = \frac{f(x,y)}{f_X(x)} = \frac{x+y}{\frac{1}{2}(2x+1)} = \frac{2(x+y)}{(2x+1)} \quad 0<y<1$$

$$E(Y|X) = \int_0^1 y \frac{2(x+y)}{2x+1}\,dy = \frac{2}{2x+1}\int_0^1 (xy+y^2)\,dy$$

$$= \frac{2}{2x+1}\left(\frac{x}{2} + \frac{1}{3}\right) = \frac{2(3x+2)}{6(2x+1)} = \frac{3x+2}{6x+3}$$

$$E(Y^2|X) = \int_0^1 y^2 \frac{2(x+y)}{2x+1}\,dy = \frac{2}{2x+1}\int_0^1 (y^2x + y^3)\,dy$$

$$= \frac{2}{2x+1}\left(\frac{x}{3} + \frac{1}{4}\right) = \frac{2(4x+3)}{12(2x+1)} = \frac{4x+3}{6(2x+1)}$$

$$V(Y|X) = E(Y^2|X) - E^2(Y|X)$$

$$= \frac{4x+3}{6(2x+1)} - \left(\frac{3x+2}{3(2x+1)}\right)^2 = \ldots$$

(8) $f(x,y) = f(x|y)\,g(y) = c\,d\,x\,y^2 \; ; \; 0<x<y, \; 0<y<1$

$$\int_0^1 g(y)\,dy = 1 \Rightarrow d\int_0^1 y^4\,dy = 1 \Rightarrow d = 5$$

$$\Rightarrow f(x,y) = 5c\,x\,y^2 \quad ; \quad 0<x<y<1$$

$$\Rightarrow 5c\int_0^1 y^2 \int_0^y x\,dx\,dy = 1 \Rightarrow \frac{5c}{2}\int_0^1 y^4\,dy = 1$$

$$\Rightarrow c = 2$$

$$\Rightarrow f(x,y) = 10xy^2 \;; \quad 0 < x < y < 1$$
$$= 0 \qquad o/w$$

$$f_X(x) = 10x \int_x^1 y^2\,dy \qquad 0 < x < 1$$

$$f_X(x) = \begin{cases} \frac{10}{3}x(1-x^3) & 0 < x < 1 \\ 0 & o/w. \end{cases}$$

$$P(0.25 < X < 0.5) = \frac{10}{3}\int_{1/4}^{1/2} (x - x^4)\,dx = \ldots$$

$$P\left(\frac{1}{4} < X < \frac{1}{2} \,\middle|\, Y = 0.625\right) = \int_{1/4}^{1/2} f_{X|Y=y}\,dx$$

$$= 2\int_{1/4}^{1/2} \frac{x}{(.625)^2}\,dx = \frac{\cancel{2}}{(.625)^2}\frac{1}{\cancel{2}}\left(\left(\frac{1}{2}\right)^2 - \left(\frac{1}{4}\right)^2\right)$$
$$= \ldots$$

(9) Marg of X from $h(x,y)$

$$f_X(x) = \int_{-\infty}^{\infty} h(x,y)\,dy = \int_{-\infty}^{\infty} f(x)g(y)\{1 + \alpha[2F(x)-1][2G(y)-1]\}\,dy$$

$$= f(x)\int_{-\infty}^{\infty} g(y)\,dy + f(x)\alpha[2F(x)-1]\int_{-\infty}^{\infty} g(y)(2G(y)-1)\,dy$$

$$= f(x) \times 1 + f(x)\alpha[2F(x)-1]\int_{-\infty}^{\infty} g(y)(2G(y)-1)\,dy$$

$$\int_{-\infty}^{\infty} g(y)(2G(y)-1)\,dy \overset{G(y)=u}{=\!=} \int_0^1 (2u-1)\,du = \cancel{2}\frac{u^2}{\cancel{2}} - u \Big|_0^1 = 0$$

$$\Rightarrow f_X(x) = f(x) + 0$$

Sly $f_Y(y) = \int_{-\infty}^{\infty} h(x,y)\,dx = g(y)$

$$h(x,y) = f_X(x) \cdot f_Y(y) = f(x)\,g(y) \quad \text{iff} \quad \alpha = 0.$$

(10) $$f_{X,Y}(x,y) = f_{Y|X=x}(y|x)\, f_X(x)$$

$$= \begin{cases} 8xy, & 0 < x < y < 1 \\ 0 & o/w \end{cases}$$

Marginal p.d.f. of Y

$$f_Y(y) = \begin{cases} 8y \int_0^y x\,dx = 4y^3, & 0 < y < 1 \\ 0, & o/w. \end{cases}$$

Conditional p.d.f. of X given y

$$f_{X|Y=y}(x|y) = \begin{cases} \dfrac{8xy}{4y^3} = \dfrac{2x}{y^3}, & 0 < x < y;\ 0 < y < 1 \\ 0, & o/w. \end{cases}$$

$$E(X|Y=y) = \frac{2}{y^2}\int_0^y x^2\,dx = \frac{2}{y^2}\cdot\frac{y^3}{3} = \frac{2y}{3}$$

$$\Rightarrow E\left(X|Y=\tfrac{1}{2}\right) = \frac{1}{3}$$

$$E(X^2|Y=y) = \frac{2}{y^2}\int_0^y x^3\,dx = \frac{2}{y^2}\cdot\frac{y^4}{4} = \frac{y^2}{2}$$

$$\Rightarrow E\left(X^2|Y=\tfrac{1}{2}\right) = \frac{1}{8}$$

$$V(X|Y=y) = E(X^2|Y=y) - E^2(X|Y=y)$$

$$= \frac{1}{8} - \frac{1}{9} = \frac{1}{72}.$$

(11) jt. m.g.f.

$$M_{X_1,X_2}(t_1,t_2) = E(e^{t_1X_1+t_2X_2})$$

$$= \int_0^\infty \int_0^\infty e^{t_1x_1+t_2x_2}\, e^{-(x_1+x_2)}\, dx_2\, dx_1$$

$$= \int_0^\infty e^{-x_2(1-t_1)}\, dx_1 \int_0^\infty e^{-x_2(1-t_2)}\, dt_2$$

$$= (1-t_1)^{-1}(1-t_2)^{-1} \quad \text{If } t_1, t_2 < 1$$

Note : Since X_1 & X_2 are indep, we can write

$$M_{X_1,X_2}(t_1,t_2) = M_{X_1}(t_1)\, M_{X_2}(t_2)$$

m.g.f. of $Z = X_1 + X_2$

$$M_Z(t) = E\left(e^{t(X_1+X_2)}\right) = (1-t)^{-2}, \quad t<1$$

$$E(Z) = \frac{\partial M_Z(t)}{dt}\Big|_{t=0} = 2(1-t)^{-3}\Big|_{t=0} = 2$$

$$E(Z^2) = \frac{\partial^2 M_Z(t)}{dt^2}\Big|_{t=0} = 6(1-t)^{-4}\Big|_{t=0} = 6 \quad \Rightarrow V(Z) = 2$$

(12) $$M_{X_1,X_2}(t_1,t_2) = E\left(e^{t_1X_1+t_2X_2}\right)$$

$$= EE\left(e^{t_1X_1+t_2X_2}\,|\,X_1\right)$$

$$= E\left(e^{t_1X_1}\, E\left(e^{t_2X_2}\,|\,X_1\right)\right)$$

Since $X_2|X_1 \sim N\left(\mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x_1-\mu_1),\ \sigma_2^2(1-\rho^2)\right)$

$E(e^{t_2X_2}|X_1) \rightarrow$ cond'nl m.g.f of X_2 given X_1

$$M_{X_1,X_2}(t_1,t_2) = E\left(e^{t_1X_1}\left(e^{t_2\left(\mu_2+\rho\frac{\sigma_2}{\sigma_1}(X_1-\mu_1)\right)+\frac{t_2^2}{2}\sigma_2^2(1-\rho^2)}\right)\right)$$

$$= e^{t_2\mu_2+\frac{t_2^2}{2}\sigma_2^2(1-\rho^2)}\, E\left(e^{t_1X_1+t_2\rho\frac{\sigma_2}{\sigma_1}X_1}\right) e^{-t_2\rho\frac{\sigma_2}{\sigma_1}\mu_1}$$

$$= e^{t_2\mu_2+\frac{t_2^2}{2}\sigma_2^2(1-\rho^2)-t_2\rho\frac{\sigma_2}{\sigma_1}\mu_1} \times E\left(e^{\left(t_1+t_2\rho\frac{\sigma_2}{\sigma_1}\right)X_1}\right)$$

$$= e^{t_2\mu_2+\frac{t_2^2}{2}\sigma_2^2(1-\rho^2)-t_2\rho\frac{\sigma_2}{\sigma_1}\mu_1}\, e^{\left(t_1+t_2\rho\frac{\sigma_2}{\sigma_1}\right)\mu_1+\frac{\sigma_1^2}{2}\left(t_1+t_2\rho\frac{\sigma_2}{\sigma_1}\right)}$$

$$= \exp\left(t_2\mu_2+\frac{t_2^2}{2}\sigma_2^2(1-\cancel{\rho^2})-\cancel{t_2\rho\frac{\sigma_2}{\sigma_1}\mu_1}+t_1\mu_1+\cancel{t_2\rho\frac{\sigma_2}{\sigma_1}\mu_1}\right.$$
$$\left.+\frac{\sigma_1^2}{2}\left(t_1^2+\cancel{t_2^2\rho^2\frac{\sigma_2^2}{\sigma_1^2}}+2t_1t_2\rho\frac{\sigma_2}{\sigma_1}\right)\right)$$

$$= \exp\left(t_2\mu_2+\frac{t_2^2}{2}\sigma_2^2+t_1\mu_1+\frac{t_1^2}{2}\sigma_1^2+t_1t_2\rho\sigma_1\sigma_2\right)$$

$$= \exp\left(t_1\mu_1+t_2\mu_2+\frac{1}{2}\left(t_1^2\sigma_1^2+t_2^2\sigma_2^2+2t_1t_2\sigma_1\sigma_2\rho\right)\right)$$

$$\left.\frac{\partial M_{X_1X_2}(t_1,t_2)}{\partial t_1}\right|_{t_1=0,t_2=0} = \mu_1 \quad \text{\&ly} \quad \left.\frac{\partial M_{X_1,X_2}}{\partial t_2}\right|_{t_1=0,t_2=0} = \mu_2$$

& $V(X_1)=\sigma_1^2$, $V(X_2)=\sigma_2^2$

~~Extra~~ $E(X_1X_2) = \left.\frac{\partial^2 M_{X_1,X_2}(t_1,t_2)}{\partial t_1\,\partial t_2}\right|_{t_1=0,t_2=0} = \rho\sigma_1\sigma_2+\mu_1\mu_2$

$\Rightarrow \mathrm{Cov}(X_1,X_2) = (\rho\sigma_1\sigma_2+\mu_1\mu_2)-\mu_1\mu_2 = \rho\sigma_1\sigma_2$

$\Rightarrow \mathrm{Corr}(X_1,X_2) = \rho.$

(13) $f(x,y) = \begin{cases} 2, & 0 < x < y < 1 \\ 0, & \text{o/w} \end{cases}$

$$f_X(x) = \int_x^1 2\,dy = 2(1-x), \quad 0 < x < 1$$

$$= 0 \quad \text{o/w}$$

$$f_Y(y) = \int_0^y 2\,dx = 2y, \quad 0 < y < 1$$

$$= 0 \quad \text{o/w}$$

$$f_{Y|X=x} = \begin{cases} \frac{2}{2(1-x)}, & x < y < 1 \\ 0 & \text{o/w} \end{cases}$$

$$f_{X|Y=y} = \begin{cases} \frac{2}{2y}, & 0 < x < y \\ 0 & \text{o/w.} \end{cases}$$

$$E(Y|X) = \int_x^1 y \cdot \frac{1}{1-x}\,dy = \frac{1-x^2}{2(1-x)} = \frac{1+x}{2}$$

$$E(Y^2|X) = \int_x^1 y^2 \frac{1}{1-x}\,dy = \frac{1-x^3}{3(1-x)}.$$

$$\Rightarrow V(Y|X) = E(Y^2|X) - E^2(Y|X) = \frac{1-x^3}{3(1-x)} - \frac{1+x}{2}$$

$$\text{Sly } E(X|Y), \; E(X^2|Y) \text{ and hence } V(X|Y).$$

(14) (a) $\text{Cov}(X, b) = E(X - E(X))(b - E(b)) = 0 = \text{Cov}(Y, b) = \text{Cov}(Z, b)$

(b) $\text{Cov}(X, aY+b) = E(X - E(X))(aY+b - E(aY+b))$

$= E(X - E(X))(aY + \cancel{b} - aE(Y) - \cancel{b})$

$= a\,\text{Cov}(X, Y)$

(c) $\text{Cov}(X, Y+Z) = E(X - E(X))(Y+Z - E(Y) - E(Z))$

$= E(X - E(X))[(Y - E(Y)) + (Z - E(Z))]$

$= \text{Cov}(X, Y) + \text{Cov}(X, Z)$

(d) $\text{Cov}(X, aY+b) = a\,\text{Cov}(X, Y)$

$$\text{Corr}^n(X, aY+b) = \frac{\text{Cov}(X, aY+b)}{[V(X)\ V(aY+b)]^{1/2}} = \frac{a\,\text{Cov}(X,Y)}{[V(X)\ a^2 V(Y)]^{1/2}}$$

$= \text{Corr}(X, Y)$

(15) $\text{Cov}(W_1, W_2) = \text{Cov}\left(X_1, \frac{\sqrt{3}-1}{2} X_1 + \frac{3-\sqrt{3}}{2} X_2\right)$

$= \frac{\sqrt{3}-1}{2} V(X_1) + \frac{3-\sqrt{3}}{2}\text{Cov}(X_1, X_2) = \frac{\sqrt{3}-1}{2}\sigma^2$

$V(W_1) = \sigma^2$ & $V(W_2) = \left(\frac{\sqrt{3}-1}{2}\right)^2 \sigma^2 + \left(\frac{3-\sqrt{3}}{2}\right)^2 \sigma^2 = (\sqrt{3}-1)^2\sigma^2$

$\Rightarrow \rho_{W_1, W_2} = \frac{1}{2}$

Sly ρ_{W_1, W_3} & ρ_{W_2, W_3}

$\downarrow$

$\text{Cov}(W_1, W_3) = \text{Cov}(X_1, (\sqrt{2}-1)X_2 + (2-\sqrt{2})X_3) = 0$

$\Rightarrow \rho_{W_1, W_3} = 0$

(16) (a) $P(3 < Y < 8) \qquad Y \sim N(1, 25)$

$$= P\left(\frac{3-1}{5} < \frac{Y-1}{5} < \frac{8-1}{5}\right) = \Phi\left(\frac{7}{5}\right) - \Phi\left(\frac{2}{5}\right)$$

$= \ldots$ (from table)

(b) $P(3 < Y < 8 \mid X = 7) \qquad \left[Y|X \sim N\left(1 + \rho \frac{5}{4}(x-3), 25(1-\rho^2)\right)\right.$

i.e. $\left. Y|X=7 \sim N(4, 16) \right]$

$$= P\left(\frac{3-4}{4} < \frac{Y-4}{4} < \frac{8-4}{4} \middle| X=7\right)$$

$$= \Phi(1) - \Phi(-0.25) = \ldots$$

(c) $P(-3 < X < 3) \qquad X \sim N(3, 16)$

$$= P\left(\frac{-3-3}{4} < \frac{X-3}{4} < \frac{3-3}{4}\right) = \Phi(0) - \Phi\left(-\frac{6}{4}\right) = \ldots$$

(d) $P(-3 < X < 3 \mid Y = 4) \qquad \left[X|Y \sim N\left(3 + \rho \frac{4}{5}(y-1), 16(1-\rho^2)\right)\right]$

i.e $X|Y=4 \sim N(4.44, (3.2)^2)$

$$= P\left(\frac{-3-4.44}{3.2} < \frac{X-4.44}{3.2} < \frac{3-4.44}{3.2} \middle| Y=4\right)$$

$$= \Phi\left(-\frac{1.44}{3.2}\right) - \Phi\left(-\frac{7.44}{3.2}\right) = \ldots$$

(17) $(X, Y) \sim N_2(5, 10, 1, 25, \rho) \quad ; \rho > 0$

$Y|X=5 \sim N_2(10, 25(1-\rho^2))$

$$P(4 < Y < 16 \mid X = 5) = P\left(\frac{4-10}{5\sqrt{1-\rho^2}} < \frac{Y-10}{5\sqrt{1-\rho^2}} < \frac{16-10}{5\sqrt{1-\rho^2}} \middle| X=5\right)$$

$$= \Phi\left(\frac{6}{5\sqrt{1-\rho^2}}\right) - \Phi\left(-\frac{6}{5\sqrt{1-\rho^2}}\right)$$

$$= 2\,\Phi\left(\frac{6}{5\sqrt{1-\rho^2}}\right) - 1 = 0.954 \text{ (given condition)}$$

$$\Rightarrow \Phi\left(\frac{6}{5\sqrt{1-\rho^2}}\right) = 0.977 = \Phi(2)$$

$$\Rightarrow \frac{6}{5\sqrt{1-\rho^2}} = 2 \Rightarrow 1-\rho^2 = 0.36 \Rightarrow \rho = 0.8 \text{ (as } \rho > 0).$$

(18) $E(Y) = \sum_1^{15} E(X_i) = 30$

$V(Y) = 15\, V(X_i) = 45$; $V(Z) = 10 \times 3 = 30$

$$\text{Cov}(Y, Z) = \text{Cov}\left(\sum_1^{15} X_i, \sum_{11}^{20} X_i\right) = 5\,V(X_i) = 15$$

$$\rho_{Y,Z} = \frac{15}{[45 \times 30]^{1/2}}$$

(19) $U = X - Y$; $V = 2X - 3Y$

$E(U) = -5$ $\qquad$ $E(V) = 2 \times 15 - 3 \times 20 = -30$

$V(U) = V(X) + V(Y) - 2\,\text{Cov}(X,Y)$ $\qquad$ $V(V) = 4\,V(X) + 9\,V(Y) - 12\,\text{Cov}(X,Y)$

Now,

$$\rho_{X,Y} = -0.6 = \frac{\text{Cov}(X,Y)}{\sqrt{25 \times 100}} = \frac{\text{Cov}(X,Y)}{50} \Rightarrow \text{Cov}(X,Y) = -30$$

$\Rightarrow V(U) = 185$ & $V(V) = 100 + 900 + 360 = 1360$

$\Rightarrow \text{Cov}(U, V) = \text{Cov}(X - Y, 2X - 3Y) = \longrightarrow$

$$= 2V(X) - 3\,Cov(X,Y) - 2\,Cov(Y,X) + 3V(Y)$$

$$= 2 \times 25 - 5(-30) + 3 \times 100 = 500$$

$$\Rightarrow \rho_{U,V} = \frac{500}{\sqrt{185 \times 1360}}.$$

(20) X : r.v. denoting life time

$X \sim Exp(50)$ p.d.f. $f(x) = \begin{cases} \frac{1}{50} e^{-x/50}, & x > 0 \\ 0 & o/w. \end{cases}$

$$F_X(x) = 1 - e^{-x/50}, \quad x > 0$$

Y_1 : # of bulbs out of 8 to have lifetime < 40

Y_2 : . . . . ≥ 40 & < 60

Y_3 : . . . . ≥ 60 & ≤ 80

Y_4 : . . . . > 80

$$P(X < 40) = F_X(40) = 1 - e^{-40/50} = p_1, \text{ say}$$

$$P(40 \leq X < 60) = F_X(60) - F_X(40) = e^{-\frac{40}{50}} - e^{-60/50} = p_2, \text{ say}$$

$$P(60 \leq X \leq 80) = F_X(80) - F_X(60) = e^{-60/50} - e^{-80/50} = p_3, \text{ say}$$

$$P(X > 80) = 1 - p_1 - p_2 - p_3 = e^{-80/50}.$$

jt. p.m.f. of Y_1, X_2, Y_3 is multinomial. $(8, p_1, p_2, p_3)$.

$$\Rightarrow P(Y_1 = 2, Y_2 = 3, Y_3 = 2) = \frac{8!}{2!\,3!\,2!\,1!} p_1^2\, p_2^3\, p_3^2 (1 - p_1 - p_2 - p_3).$$

Marginal dist^n $Y_3 \sim Bin\left(8, p_3 = e^{-60/50} - e^{-80/50}\right)$

$$E(Y_3) = 8\left(e^{-60/50} - e^{-80/50}\right)$$

Cond^n al $\quad Y_3 \mid Y_2 = y_2 \sim Bin\left(8 - y_2, \dfrac{p_3}{1-p_2}\right)$

$$\Rightarrow E(Y_3 \mid Y_2 = 1) = (8-1)\left(\frac{e^{-60/50} - e^{-80/50}}{1 - e^{-40/50} + e^{-60/50}}\right).$$

(21) (a)

x \ y	0	1	2	
0	1/3	0	0	1/3
1	0	1/3	0	1/3
2	0	0	1/3	1/3
	1/3	1/3	1/3	

$E(X) = 1 = E(Y)$

$V(X) = E(X^2) - 1$

$= \frac{5}{3} - 1 = \frac{2}{3} = V(Y)$

$E(XY) = (0\times 0)\frac{1}{3} + (1\times 1)\frac{1}{3} + (2\times 2)\times\frac{1}{3}$

$= 5/3$

$Cov(X,Y) = \frac{5}{3} - 1 = \frac{2}{3}$

$\rho_{X,Y} = 1$

(b)

x \ y	0	1	2
0	0	0	1/3
1	0	1/3	0
2	1/3	0	0

Sly

$\Rightarrow \rho_{X,Y} = -1$

(c)

x \ y	0	1	2
0	1/3	0	0
1	0	1/3	0
2	1/3	0	0

$\rho_{X,Y} = 0.$

(22)

$x \backslash y$	1	2	
1	$1/10$	0	$1/10$
2	$2/10$	$4/10$	$6/10$
3	$3/10$	0	$3/10$
	$6/10$	$4/10$	

$$E(X) = \frac{1}{10} + 2\,\frac{6}{10} + 3\,\frac{3}{10} = \frac{22}{10}$$

$$E(Y) = \frac{6}{10} + 2\,\frac{4}{10} = \frac{14}{10}$$

$$E(X^2) = \frac{1}{10} + 4\,\frac{6}{10} + 9\,\frac{3}{10} = \frac{52}{10}$$

$$E(Y^2) = \frac{6}{10} + 4\,\frac{4}{10} = \frac{22}{10}$$

$$V(X) = \frac{52}{10} - \left(\frac{22}{10}\right)^2 = \ldots$$

$$V(Y) = \left(\frac{22}{10}\right)^2 - \left(\frac{14}{10}\right)^2 = \ldots$$

$$E(XY) = (1\times1)\,\frac{1}{10} + (2\times1)\,\frac{2}{10} + (2\times2)\,\frac{4}{10} + (3\times1)\,\frac{3}{10}$$

$$= \frac{30}{10} = 3$$

$$\operatorname{Cov}(X,Y) = E(XY) - E(X)E(Y) = 3 - \frac{22}{10}\cdot\frac{14}{10} = \ldots$$

$$\operatorname{Corr}^n(X,Y) = \frac{\operatorname{Cov}(X,Y)}{[V(X)\,V(Y)]^{1/2}} = \ldots$$

jt. m.g.f. of (X,Y)

$$M_{X,Y}(t_1,t_2) = \sum_{x,y} e^{t_1 x + t_2 y}\, P(X=x, Y=y)$$

$$= e^{t_1+t_2} \times \frac{1}{10} + e^{2t_1+t_2}\,\frac{2}{10} + e^{2(t_1+t_2)}\,\frac{4}{10}$$

$$+ e^{3t_1+t_2}\cdot\frac{3}{10}$$

(23) $M_{X,Y}(u,v) = E\left(e^{uX+vY}\right)$

$$\Psi(u,v) = \log M_{X,Y}(u,v)$$

$$\frac{\partial \Psi(u,v)}{\partial u} = \frac{1}{M_{X,Y}(u,v)}\cdot\frac{\partial M_{X,Y}(u,v)}{\partial u}$$

$$\left.\frac{\partial \Psi(u,v)}{\partial u}\right|_{u=0,\,v=0} = \frac{1}{M(0,0)}\cdot E(X) = E(X)$$

Sly $\dfrac{\partial \Psi(0,0)}{\partial v} = \dfrac{\partial \Psi(u,v)}{\partial v}\Big|_{u=0,v=0} = E(Y)$

$$\frac{\partial^2 \Psi(u,v)}{\partial u^2} = \frac{1}{M(u,v)} \frac{\partial^2 M(u,v)}{\partial u^2} + \left[\frac{-1}{(M(u,v))^2} \frac{\partial M(u,v)}{\partial u}\right]\left(\frac{\partial M(u,v)}{\partial u}\right)$$

$$= \frac{1}{M(u,v)} \cdot \frac{\partial^2 M(u,v)}{\partial u^2} - \left(\frac{\partial M(u,v)}{\partial u} \cdot \frac{1}{M(u,v)}\right)^2$$

$$\frac{\partial^2 \Psi(u,v)}{\partial u^2}\Big|_{u=0,v=0} = E(X^2) - E^2(X) = V(X) = \frac{\partial^2 \Psi(0,0)}{\partial u^2}$$

Sly $\dfrac{\partial^2 \Psi(u,v)}{\partial v^2}\Big|_{u=0,v=0} = V(Y)$

$$\frac{\partial^2 \Psi(u,v)}{\partial v \partial u} = \frac{1}{M(u,v)} \cdot \frac{\partial^2 M(u,v)}{\partial v \partial u} - \frac{1}{(M(u,v))^2} \cdot \frac{\partial M(u,v)}{\partial v} \cdot \frac{\partial M(u,v)}{\partial u}$$

$$\frac{\partial^2 \Psi(u,v)}{\partial v \partial u}\Big|_{u=0,v=0} = E(XY) - E(X)E(Y)$$

i.e. $\dfrac{\partial^2 \Psi(u,v)}{\partial v \partial u}\Big|_{u=0,v=0} = Cov(X,Y)$

(24) Marginal p.d.f. of X

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} f_\rho(x,y)\,dy + \frac{1}{2}\int_{-\infty}^{\infty} f_{-\rho}(x,y)\,dy$$

$$= \frac{1}{2}\phi(x) + \frac{1}{2}\phi(x) \qquad [\phi(x) \text{ p.d.f. of } N(0,1)]$$

$$= \phi(x) \Rightarrow X \sim N(0,1)$$

Sly $f_Y(y) = \phi(y) \Rightarrow Y \sim N(0,1)$

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{X,Y}(x,y)\, dx\, dy$$

$$= \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{\rho}(x,y)\, dx\, dy + \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{-\rho}(x,y)\, dx\, dy$$

$$= \frac{1}{2}(\rho) + \frac{1}{2}(-\rho) = 0$$

$$\text{Cov}(X,Y) = E(XY) - E(X)E(Y)$$

$$= 0 - 0.0 = 0$$

$\rho(X,Y) = 0 \Rightarrow X$ & Y are uncorrelated

Since, $f_{X,Y}(x,y) \neq f_X(x)\, f_Y(y)$.

X & Y are not independent

(25) $\int_0^1\int_{-x}^{x} K\, dy\, dx = 1 \Rightarrow K\int_0^1 2x\, dx = 1 \Rightarrow K = 1$

Marginal of X, $f_X(x) = \int_{-x}^{x} K\, dy = \begin{cases} 2x, & 0 < x < 1 \\ 0 & o/w \end{cases}$

Marginal of Y, $f_Y(y) = \int_{|y|}^{1} dx = \begin{cases} 1-|y|, & -1 < y < 1 \\ 0 & o/w \end{cases}$

Conditional distⁿ of $Y|X=x$; $f_{Y|X=x} = \begin{cases} \frac{1}{2x}, & -x < y < x \\ 0, & o/w \end{cases}$

$$E(Y|X=x) = \int_{-x}^{x} y\frac{1}{2x}\, dy = 0$$

Sly $E(X|Y=y) = \int_{|y|}^{1} x\frac{1}{1-|y|}\, dx = \frac{1-y^2}{2(1-|y|)}$.

$\left(f_{X|Y=y} = \begin{cases} (1-|y|)^{-1}, & |y| < x < 1 \\ 0 & o/w \end{cases} \right)$

$$f_{X,Y}(x,y) = 1 \neq f_X(x)\, f_Y(y)$$

$\Rightarrow X$ & Y are not indep.

$$E(XY) = \int_0^1 \int_{-x}^{x} xy \, dy \, dx = 0$$

$$E(Y) = E\, E(Y \mid X) = 0.$$

$$\Rightarrow \operatorname{Cov}(X,Y) = \rho_{X,Y} = 0$$

$\Rightarrow X$ & Y are uncorrelated

(26) $M_{X,Y}(s,t) = \left\{ a\left(e^{s+t}+1\right) + b\left(e^s + e^t\right) \right\}$, $\left(a, b > 0,\ a+b = \frac{1}{2}\right)$

$$E(X) = \frac{\partial}{\partial s}\left(a\left(e^{s+t}+1\right) + b\left(e^s + e^t\right)\right)\Big|_{s=t=0}$$

$$= a e^t e^s + b e^s \Big|_{t=s=0} = a+b = \frac{1}{2} = E(Y)$$

$$E(X^2) = \frac{\partial^2}{\partial s^2}\left(a\left(e^{s+t}+1\right) + b\left(e^s + e^t\right)\right)\Big|_{s=t=0}$$

$$= a e^t e^s + b e^s \Big|_{s=t=0} = a+b = \frac{1}{2} = E(Y^2)$$

$$V(X) = V(Y) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

$$E(XY) = \frac{\partial^2}{\partial t \partial s}\left(a\left(e^{s+t}+1\right) + b\left(e^s + e^t\right)\right)\Big|_{s=t=0}$$

$$= a e^t e^s \Big|_{s=t=0} = a$$

$$\therefore \operatorname{Cov}(X,Y) = a - \frac{1}{4} \Rightarrow \rho_{X,Y} = \frac{a - \frac{1}{4}}{\frac{1}{4}} = \underline{\underline{4a-1}}$$

(27) $Var\left(\frac{X}{3}+\frac{2Y}{3}\right)\left(= Var\left(\frac{2X}{3}+\frac{Y}{3}\right) \because V(X)=V(Y)\right)$

$= \frac{1}{9}V(X) + \frac{4}{9}V(Y) + 2\,Cov\left(\frac{X}{3}, \frac{2Y}{3}\right)$

$= \frac{2}{9} + \frac{8}{9} + \frac{4}{9} \times \frac{2}{3} = \frac{2}{9} + \frac{8}{9} + \frac{8}{27} = \frac{38}{27}$

$Cov\left(\frac{X}{3}+\frac{2Y}{3}, \frac{2X}{3}+\frac{Y}{3}\right)$

$= \frac{2}{9}V(X) + \frac{1}{9}Cov(X,Y) + \frac{4}{9}Cov(X,Y) + \frac{2}{9}V(Y)$

$= \frac{4}{9} + \frac{2}{27} + \frac{8}{27} + \frac{4}{9} = \frac{34}{27}$

$Corr^2\left(\frac{X}{3}+\frac{2Y}{3}, \frac{2X}{3}+\frac{Y}{3}\right) = \frac{34/27}{38/27} = \underline{\underline{\frac{34}{38}}}$