MSiA 400 Lab 2

Markov Chains

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Some Updates

- Office Hours: Mondays 10-11am MEC Conference room 431 (Or by appointment/Slack)
- Assignment1 due in 3 weeks (Can start working on Problem1)

Markov Chains

- Stochastic process
- Discrete time
- Countable or finite state space S

$$\{X_t \in S \mid t = 0, 1, 2, \cdots\}$$

- Markov property (memory-less)

$$P(X_{t+1} = i_{t+1} \mid X_t = i_t, \cdots, X_0 = i_0) = P(X_{t+1} = i_{t+1} \mid X_t = i_t)$$

Example

- Random Walk: $X_t = X_{t-1} \pm 1$ with equal probability
  - State Space?
  - Markov property?

Variations

- Time-homogenous (or stationary)

$$P(X_{t+1} = i \mid X_t = j) = P(X_t = i \mid X_{t-1} = j)$$

- Markov chain with memory (order m)

$$\begin{aligned} &P(X_t = i_t \mid X_{t-1} = i_{t-1}, \cdots, X_0 = i_0) \\ &\qquad = P(X_t = i_t \mid X_{t-1} = i_{t-1}, \cdots, X_{t-m} = i_{t-m}) \text{ for } t > m \end{aligned}$$

Applications

- Typing word prediction
- Credit risk measurement
- Predict market trends
- Predict weather
- ...

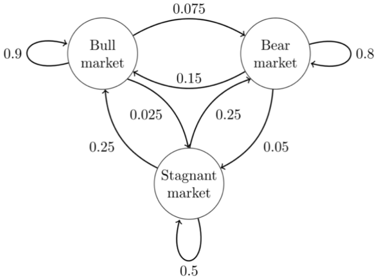
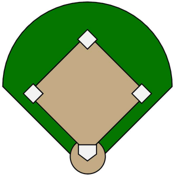


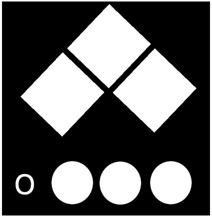
Figure 1: State diagram of a hypothetical stock market with 3 states: {bull, bear, stagnant}

Example: Baseball

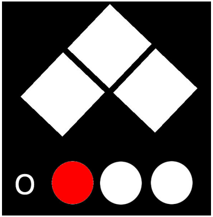
- Half of an inning
  - $3(2^3) + 1 = 25$ states
    - 0, 1, or 2 outs
    - 3 bases (w/ or w/o runners)
    - Inning over
  - Plate appearances $t \in \{0, 1, 2, \cdots\}$



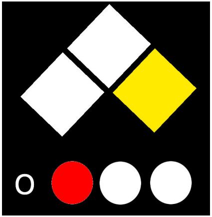
Example Path



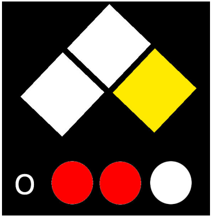
(a) 1st PA



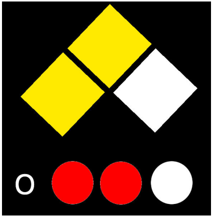
(b) 2nd PA



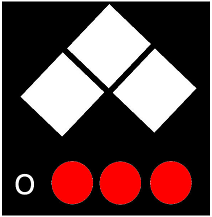
(c) 3rd PA



(d) 4th PA



(e) 5th PA



(f) Inning over

Transition Matrix

- Transition
  - System moves from state i to state j
- Transition probabilities
  - Probability of transition from state i to j: $P(X_{t+1} = j \mid X_t = i) = p_{ij}$
- Transition matrix

$$P = \begin{pmatrix} p_{00} & p_{01} & \cdots \\ p_{10} & p_{11} & \\ \vdots & & \ddots \end{pmatrix}$$

n-step Transition Probababilities

- Given the Markov chain starts in state i (at $t = 0$), what is the probability that the Markov chain is in state j at time n?
  $P(X_n = j \mid X_0 = i) = p_{ij}^{(n)}$
- $p_{ij}^{(n)}$ is called the ***n*-step transition probability** from state i to j
- If time-homogenous, $p_{ij}^{(n)}$ is the ij^{th} element of P^n
- Initial distribution a (probability distribution over states at time 0)

n-step Transition Probabability Example in R

- (An inning starts with no outs & no baserunners.) What is the probability that the inning is over within 5 batters?

```
# Import transition matrix
P = as.matrix(read.csv("baseball.csv", header=T, row.names=1))
P[1:4,1:4]
```

```
##             X0          X1         X2         X3
## 0 0.02941560 0.242163552 0.05214279 0.00000000
## 1 0.02477029 0.000350335 0.07733650 0.18433205
## 2 0.02005076 0.044111675 0.04715736 0.10124366
## 3 0.02628186 0.000066100 0.01276075 0.03686072
```

```
a = c(1,rep(0,24)) #initial state
a[1:20]
```

```
##  [1] 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
```

n-step Transition Probabability Example in R

- (An inning starts with no outs & no baserunners.) What is the probability that the inning is over within 5 batters?

```
# matrix multiplication uses %*%
prob5 = a %*% P %*% P %*% P %*% P %*% P
prob5[25]
```

```
## [1] 0.792758
```

```
#alternatively, use expm
library(expm)
```

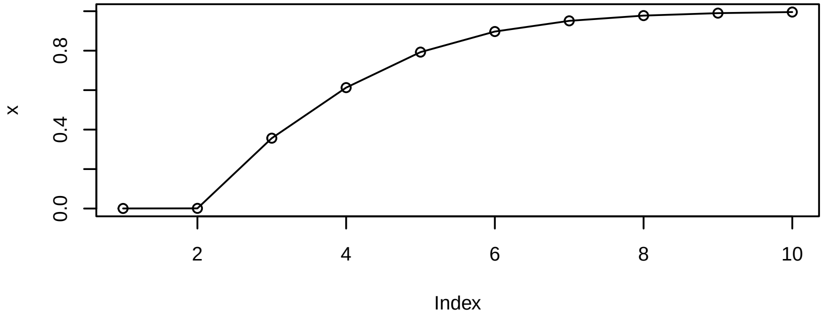
```
prob5 = a %*% (P %^% 5)
prob5[25]
```

```
## [1] 0.792758
```

n-step Transition Probabability Example in R

- Compute the probability that the inning is over within n batters

```
x=rep(0,10); at=a
for (i in 1:10){
    at = at %*% P
    x[i] = at[25] }
par(cex=0.7); plot(x, type="o")
```



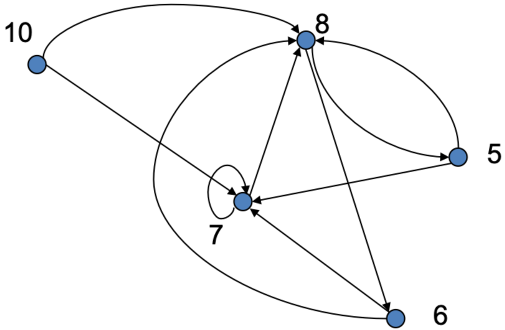
Classification of States

- State j is **accessible** from state i if $p_{ij}^{(n)} > 0$ for some n, denoted $i \to j$
- Two states i and j **communicate** if they are accessible from each other, denoted $i \leftrightarrow j$
- A Markov chain is **irreducible** if all states communicate with each other

Classification of States

- A state i has **period** k if any path starting at i and returning to it must take a number of steps divisible by k
  - If $k = 1$, state i is **aperiodic**
  - If $k > 1$, state i is **periodic**
  - A Markov chain is aperiodic if all its states are.
- An **absorbing** state is one where $p_{ii} = 1$
- $f_{ii} = P(X_n = i, \text{ for some } n \mid X_0 = i)$
  - State i is **recurrent** if $f_{ii} = 1$
  - State i is **transient** if $f_{ii} < 1$
  - Positive recurrent: Expected to comeback finite time
- A state is **ergodic** if it is positive recurrent and aperiodic. A Markov chain is ergodic if all its states are ergodic and it is irreducible.

Classification of States Example



- Is this Markov chain irreducible?
- What is the period of state 7?
- What is the period of state 8?
- Are there any absorbing states?

Classification of States Example

- For the baseball Markov chain
  - Is the Markov chain irreducible?
  - Is the Markov chain aperiodic?
  - Are there any absorbing states?
  - Are states recurrent or transient?

Steady-State

- A **stationary distribution** $\pi = \left(\pi_1, \cdots, \pi_{|S|}\right)^T$ is one that is invariant by the transition matrix P, i.e. $\pi = \pi P$
- If P is the transition matrix for an irreducible, aperiodic, positive recurrent, and time-homogenous Markov chain, then $\exists$ a unique stationary distribution π st

$$\lim_{n\to\infty} P^n = \begin{pmatrix} \pi_1 & \cdots & \pi_{|S|} \\ \vdots & \ddots & \vdots \\ \pi_1 & \cdots & \pi_{|S|} \end{pmatrix}$$

Equilibrium Distribution

- For equilibrium (or steady-state) probabilities

$$\begin{cases} \pi_j = \sum_{i \in S} \pi_i p_{ij}, \quad \forall j \in S \\ \sum_{j \in S} \pi_j = 1 \end{cases}$$

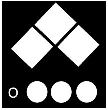
- We can compute π. Let Q be the matrix obtained by replacing the last row of $P^T - I$ with $\mathbf{1}$. Then $Q\pi = (0, \cdots, 0, 1)^T$. (Note: you can replace any row.)

Computing the Equilibrium Distribution in R

```r
P[25,1]=1; P[25,25]=0 #make the network irreducible
Q=t(P)-diag(25)
Q[25,]=rep(1,25)
rhs=c(rep(0,24),1)
Pi=solve(Q,rhs) #solves Q %*% Pi = rhs for Pi
Pi[1:4]
```

```
##          0          1          2          3
## 0.19198298 0.04768498 0.01495921 0.01148478
```

Path Probability



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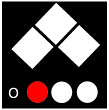
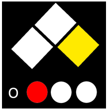
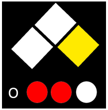


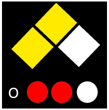
(b) 2nd PA



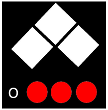
(c) 3rd PA



(d) 4th PA



(e) 5th PA



(f) Inning over

- What is the probability this path happens?

```
pathprob = P[1,9]*P[9,10]*P[10,18]*P[18,23]*P[23,25]
pathprob
```

```
## [1] 0.000752231
```

First Passage Time

- Time for a state to transition from state i to j for the first time is called the **first passage time**
- What is the expected number of steps to get to state j given that we started at state i?

$$E[\# \text{ of steps to } j \mid i] = \sum_{\text{all paths to } j} \text{length(path)} P(\text{path})$$

Mean First Passage Time

- Mean (or expected) first passage time

$$
\begin{aligned}
m_{ij} &= p_{ij} + \sum_{k \neq j} p_{ik} \left(1 + m_{kj}\right) \\
&= 1 + \sum_{k \neq j} p_{ik} m_{kj}
\end{aligned}
$$

- Matrix form
  - Define B as the submatrix of P obtained by removing the row & column corresponding to state j
  - Define m as a vector of mean first passage times over all $i \neq j$
  - Then $m = (I - B)^{-1}\mathbf{1}$

Mean First Passage Time in R

```
B=P[1:24,1:24]
Q=diag(24)-B
rhs=rep(1,24)
m=solve(Q,rhs)
m[1]
```

```
##        X0
## 4.428299
```