MSiA401: Predictive Analytics

Instructions:

- Put your name on the exam.
- Submit the exam questions with your answer sheets.
- If you don't have a fancy calculator (i.e., one that computes logs and exponents), you will receive full credit for showing work up until you need to look up the log/exponent.
- As always, I hope that my exam is a learning experience and that it helps you get a deeper understanding of the concepts.

1. (22 points) Which cable TV customers are likely to "cut the cord," where they drop cable TV services in favor of streaming? I have a data set with 200,000 customers. The `cordcut` variable equals 1 when a customer cuts the cord in a certain "target" period of time and 0 if not. We will consider three variable, all from the period of time prior to the "target" period: (1) `quality` (0, 1, 2, 3, 4, 5) indicates the quality of internet service (mostly speed), where 0 indicates that the customer does not have internet service, and higher numbers indicate better quality; (2) `timeshift` indicates if a customer is doing time-shifted viewing (e.g., using TiVo or a digital video recorder); (3) `loguse` is the log average amount of data (Gb) used by a customer in a month prior to the target period (those without internet service have a value of 0). Use the following notation: β_0 is the intercept, β_1 is for level 1 quality, ..., β_8 is for `timeshift:loguse`. I added a column `j` (for β_j) to the output below to help you keep track.

```
> fit=glm(cordcut ~ factor(quality) + timeshift*loguse, binomial, dat)
> fit
j                  Estimate Std. Error z value Pr(>|z|)
0 (Intercept)      -6.97542    0.14357 -48.585  < 2e-16 ***
1 factor(quality)1  1.60920    0.17584   9.151  < 2e-16 ***
2 factor(quality)2  2.00514    0.15399  13.021  < 2e-16 ***
3 factor(quality)3  2.40319    0.16321  14.725  < 2e-16 ***
4 factor(quality)4  2.57628    0.18066  14.260  < 2e-16 ***
5 factor(quality)5  2.41648    0.25600   9.439  < 2e-16 ***
6 timeshift         0.41101    0.10800   3.806 0.000141 ***
7 loguse            0.20940    0.01883  11.118  < 2e-16 ***
8 timeshift:loguse -0.11245    0.02592  -4.338 1.44e-05 ***

    Null deviance: 26062  on 199999  degrees of freedom
Residual deviance: 24743  on 199991  degrees of freedom
```

(a) (3 points) Suppose we wished to test the overall significance of the model, i.e., $H_0 : \beta_1 = \cdots = \beta_8 = 0$ against H_1 : at least one $\beta_j \neq 0 (j = 1, \ldots, 8)$ at the $\alpha = 5\%$ level. Perform a likelihood ratio test of this hypothesis showing work. Hint: the 95^{th} percentiles of a chi-squared distribution are as follows:

```
> rbind(df=1:10, p95=round(qchisq(.95, 1:10), 2))
    [,1] [,2] [,3] [,4]  [,5]  [,6]  [,7]  [,8]  [,9] [,10]
df  1.00 2.00 3.00 4.00  5.00  6.00  7.00  8.00  9.00 10.00
p95 3.84 5.99 7.81 9.49 11.07 12.59 14.07 15.51 16.92 18.31
```

Answer: LRT=$26062 - 24743 = 1319 > 15.51$*, so reject* H_0*.*

(b) (3 points) The `drop1` output is as follows:

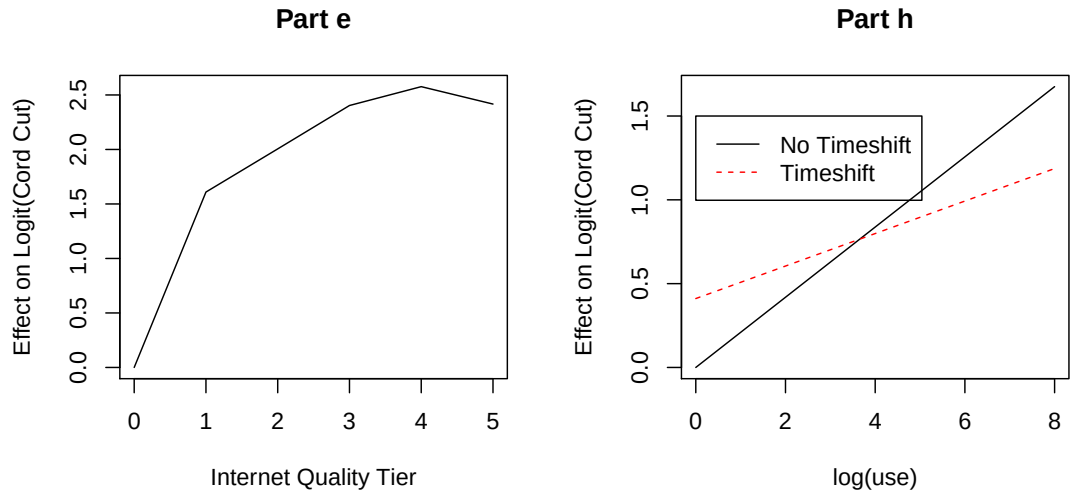
```
> drop1(fit, test="Chisq")
                 Df Deviance   AIC    LRT  Pr(>Chi)
<none>                 24743 24761
factor(quality)   5    25077 25085  ????? < 2.2e-16 ***
timeshift:loguse  1    24762 24778  ????? 1.598e-05 ***
```

What hypothesis does the `factor(quality)` line test? State the null and alternative. I deleted the value of the likelihood ratio test statistic; what is the correct value? *Answer:* $H_0 : \beta_1 = \cdots = \beta_5 = 0$ *versus* H_1 *: at least one* $\beta_j \neq 0$*,* $j = 1, \ldots, 5$*.* $LRT = 25077 - 24743 = 334$*.*

(c) (3 points) If we typed `step(fit)` to do a backward selection, what specifically would R do and why? *Answer: Terminate because the AIC of this model (24761) is smaller than the AICs of the other two models.*

(d) (2 points) What does the estimate 1.60920 of β_1 tell you? *Answer: The log odds of cord cutting of those with quality 1 internet service is 1.6092 great than those without internet service (quality 0).*

(e) (3 points) The `quality` variable was included as a factor (0 is the base category) in order to allow for nonlinearities. Is the relationship between `quality` and the logit of cutting the cord nonlinear? To answer this question, create a "partial plot" showing the effect on the logit against `quality` $\in [0, 5]$. Note that you are holding the values of `timeshift` and `loguse` constant and that the effect of these other variables is to shift the `quality` function up or down without changing its shape.



(f) (2 points) Write one sentence summarizing the relationship between `quality` and the logit of cutting. *Answer: Those without internet service are much less likely to cut the cord, and thereafter the logit of cutting increases roughly linearly with quality.*

(g) (3 points) Is there a significant (i.e., use $\alpha = .05$) interaction between time shifting and the log of use? Perform a hypothesis test to answer this question. State the null and alternative, P-value and decision. *Answer: $H_0 : \beta_8 = 0$ versus $H_1 : \beta_8 \neq 8$, $P = 0.0000144 < .05$, so reject H_0. There is an interaction. Full credit if they use the* `drop1` *output, with $P = .00001598$*

(h) (3 points) If we were to hold `quality` constant, what is the effect of `timeshift` and `loguse` on the logit if cutting? To answer this, create a graph with `loguse` on the horizontal axis and separate lines for `timeshift`. Label the slopes of the two lines and describe how the intercepts differ. Note that `loguse` ranges from 0 to about 8.2. *Answer: For* `timeshift=0` *the RHS of the equation is $a + 0.2094$*`loguse`*, but for* `timeshift=1` *it is $(a + 0.4110) + (0.2094 - 0.1125)$*`loguse`*. The line for those who time shift has an intercept that is 0.4110 greater, but its slope is less steep (0.0969).*

2. (18 points) Consider a classification problem with $K = 3$ classes, one predictor x, and a large sample of size n observations. Let p_{ik} be the probability that observation i comes from class k, where $p_{i1} + p_{i2} + p_{i3} = 1, \forall i$. The estimated coefficients are:

$$\log\left(\frac{p_{i1}}{p_{i3}}\right) = 1 - 2x_i \qquad \text{and} \qquad \log\left(\frac{p_{i2}}{p_{i3}}\right) = 1 - 1x_i.$$

(a) (3 pts) Interpret -1, the estimated x coefficient in $\log(p_2/p_3) = 1 - 1x$.

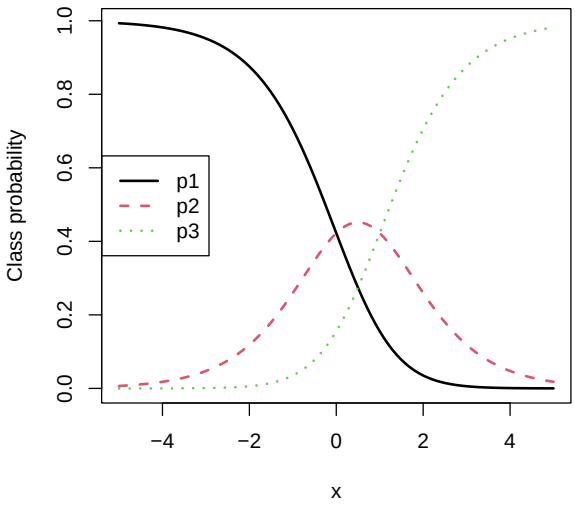
(b) (3 pts) Find $\log(p_{i2}/p_{i1})$.

(c) (3 pts) Find p_{i3}.

(d) (3 pts) Find p_{i2}.

(e) (3 pts) For which value of x does $p_2 = p_3$? Show work. This is a *decision boundary* for the maximum likelihood assignment rule.

(f) (3 pts) Use the graph to the right, which plots p_k against x, to describe how x affects p_2? How does this interpretation compare with part (a)?



Answer:

(a) A one-unit increase in x is associated with a change of -1 in the logit $\log(p_2/p_3)$, i.e., larger values of x are associated with smaller log odds of being in class 2 versus class 3.

(b) $\log\left(\frac{p_{i2}}{p_{i1}}\right) = (1 - x_i) - (1 - 2x_i) = x_i$

(c) $p_{i3} = 1/(1 + e^{1-2x_i} + e^{1-x_i})$

(d) $p_{i2} = p_{i3}e^{1-x_i}$

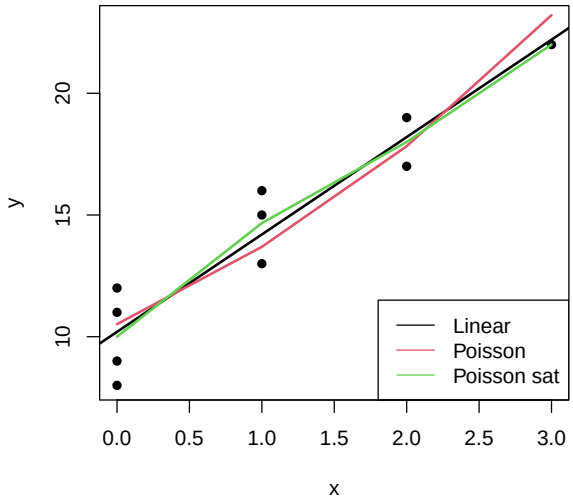
(e)

$$\begin{aligned} p_2 &= p_3 \\ \cancel{p_3}e^{1-x} &= \cancel{p_3} \\ \log(e^{1-x}) &= \log(1) \\ 1 - x &= 0 \\ x &= 1 \end{aligned}$$

(f) As x increases p_2 initially increases and then decreases. Students don't need to say this but the maximum value is achieved at $x = 0.5$ and the effect resembles a normal curve. In part (a) the effect on the logit was linear, but this part shows that effects on probabilities can be more complicated (i.e., not monotonic) in the multiclass problem.

3. (15 points) A substance used in medical research is shipped by airfreight in cartons of 1000 ampules (ampule=small glass container). The data below are from $n = 10$ shipments where x is the number of aircraft transfers and y is the number of broken ampules upon arrival. Let μ_i be the mean number of broken ampules for observation i. I have estimated three models: (1) OLS linear regression, (2) Poisson linear regression, (3) Poisson saturated model. The plot

to the right shows the observed points and fitted values from the three models.




```
> plot(y~x, dat, pch=16, ylim=c(8, 25))
> fit = lm(y~x, dat); abline(fit)                                  # Model 1
> fit2 = glm(y~x, poisson, dat); summary(fit2)                     # Model 2
Coefficients:
            Estimate Std. Error z value Pr(>|z|)
(Intercept)  2.35295    0.13174   17.86  < 2e-16 ***
x            0.26384    0.07924    3.33 0.000869 ***
    Null deviance: 12.5687  on 9  degrees of freedom
Residual deviance:  1.8132  on 8  degrees of freedom

> x = 0:3; lines(x, predict(fit2, data.frame(x=x), type="resp"), col=2)
> fit3 = glm(y ~ factor(x), poisson, dat); summary(fit3)          # Model 3
Coefficients:
            Estimate Std. Error z value Pr(>|z|)
(Intercept)   2.3026     0.1581  14.563  < 2e-16 ***
factor(x)1    0.3830     0.2185   1.753  0.07958 .
factor(x)2    0.5878     0.2297   2.559  0.01051 *
factor(x)3    0.7885     0.2654   2.970  0.00297 **
    Null deviance: 12.5687  on 9  degrees of freedom
Residual deviance:  1.4391  on 6  degrees of freedom
> lines(x, predict(fit3, data.frame(x=x), type="resp"), col=3)
> legend("bottomright",c("Linear","Poisson","Poisson sat"), col=1:3, lty=1)
```

(a) (3 pts) Using Model 2, state the estimated regression equation. *Answer:* $\log(\mu_i) = 2.35 + 0.264x_i$

(b) (3 pts) Another use of the saturated model when there are multiple observations for different x values is to test the *goodness of fit* (GOF). If the Poisson linear model (Model 2) holds then the saturated model 3, which estimates the mean of y for each unique value of x separately, will not improve the fit. Alternatively, if there are substantial nonlinearities in the relationship then the saturated model will have a much better fit, and therefore lower deviance. The null is that the linear model holds and the alternative is that it does not. Test at the 5% level whether the saturated model improves the deviance. Note that problem 1a gave 95th percentiles of the chi-squared distribution for different df. Show work, including the degrees of freedom. *Answer: Test stat* $= 1.8132 - 1.4391 = 0.374 < 5.99 = \chi^2_{0.95,2}$*, so do not reject* H_0*. Conclude the linear model is plausible.*

(c) (3 pts) Estimate the mean number of broken ampules when there are $x = 0$ transfers using Model 2. *Answer:* $\hat{\mu} = e^{2.35295+0.26384(0)} \approx 10.52$.

(d) (3 pts) Management wishes to estimate the probability that 10 or fewer ampules are broken when there are $x = 0$ transfers. Discuss how to do this with Model 2. Note that I am not asking you to do it, but say how you would do it. *Answer: Use* $\hat{\mu}$ *from the previous part to compute*

$$P(X \leq 10) = \sum_{x=0}^{10} \frac{\hat{\mu}^x e^{-\hat{\mu}}}{x!}$$

In R you could use `ppois(10, exp(2.35295)) =0.5186935`*. If students don't know the Poisson PMF then give full credit if they indicate summing the PMF with the estimated mean.*

(e) (3 pts) Which model, (1) or (2) , appears to be a better fit here? Discuss briefly. Note that we are <u>not</u> considering the saturated model in this part. *Answer: The linear model (1) is better for two reasons. First, the fit passes through the middle of the scatter, while the (red) Poisson model has some strange fitted values for* $x = 3$ *and also* $x = 1$*. This is due to the log link function and exponentiating the linear predictor. Second, the scatterplot does not show increasing variance (heteroscedasticity) we usually to see with count variables, further justifying the simple linear model.*

4. Suppose X is a discrete RV with the following PMF, with parameter $\theta \in [0, 1]$:

X	0	1	2	3
$P(X)$	$2\theta/3$	$\theta/3$	$2(1-\theta)/3$	$(1-\theta)/3$

The random sample $(3, 0, 2, 1, 3, 2, 1, 0, 2, 1)$ was drawn. Find the MLE of θ.

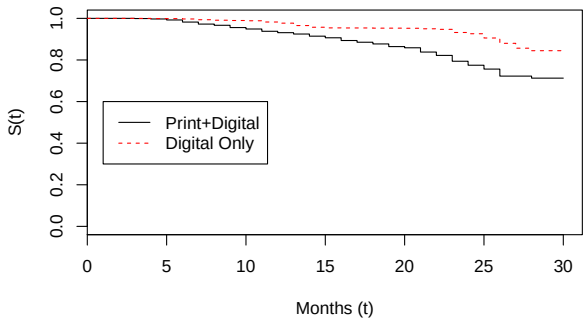
Answer: The likelihood is

$$L(\theta) = \left(\frac{2\theta}{3}\right)^2 \left(\frac{\theta}{3}\right)^3 \left(\frac{2(1-\theta)}{3}\right)^3 \left(\frac{1-\theta}{3}\right)^2$$

The log-likelihood is $l(\theta) = C + 5\log\theta + 5\log(1-\theta)$*, with derivative*

$$\frac{dl(\theta)}{d\theta} = \frac{5}{\theta} - \frac{5}{1-\theta} = 0 \quad \Longrightarrow \quad \hat{\theta} = \frac{1}{2}.$$

5. (10 points) The plot below is from a KM model of the newspaper data, stratifying by the publication type (print+digital versus digital only).



(a) (3 points) What does the plot tell you? *Answer: Digital only survive longer than print+digital.*

(b) (4 points) What can be concluded from the following output? State the appropriate null and alternative hypotheses, defining symbols that you use.

```
> survdiff(Surv(start, churn.x) ~ PubType, dat2)
Call: survdiff(formula = Surv(start, churn.x) ~ PubType, data = dat2)

                            N Observed Expected (O-E)^2/E (O-E)^2/V
PubType=Daily           47118      631      343       242       346
PubType=Daily Digital 112717      521      809       103       346
 Chisq= 346  on 1 degrees of freedom, p= <2e-16
```

Answer: Let $S_1(t)$ *be the survival function for daily and* $S_2(t)$ *be the survival function for daily digital. This output allows for us to test* $H_0 : S_1(t) = S_2(t)$ *for all* t *versus a two-sided alternative. The* P*-value is* $2 \times 10^{-16} \ll 0.05$ *and so we reject the null hypothesis and conclude that the survival curves are different. Another way to think about this is that the differences in curves shown in the plot from part (a) are not due to sampling variation.*

(c) (3 points) Here are the first few rows of KM output for the digital only package (for a sample of the original data). I deleted the survival function at time 4, $S(4)$. What is the correct value. Show work.

```
                PubType=Digital only
 time n.risk n.event survival    std.err lower 95% CI upper 95% CI
    3   6051       1  0.99983 0.00016525      0.99951      1.00000
    4   5943       4      ??? 0.00037471      0.99843      0.99990
    5   5823       2      ??? 0.00044629      0.99794      0.99969
  ...
```

Answer: $(1 - 1/6051)(1 - 4/5943) \approx 0.99983(1 - 4/5943) = 0.99916$

There are additional questions from other finals

1. (9 points) The Poisson distribution has the following PMF:

$$P(X = x) = \frac{\lambda^x e^{-\lambda}}{x!}, \quad (\lambda > 0; \quad x = 0, 1, 2, \ldots).$$

Suppose that we have a random sample of $n = 2$ observations, x_1 and x_2, from a Poisson distribution with parameter λ, where the observations are independent. This problem will derive the maximum likelihood estimate of λ.

(a) (2 points) Find $L(\lambda) = P(X_1 = x_1 \cap X_2 = x_2)$ *Answer:*

$$L(\lambda) = \frac{\lambda^{x_1} e^{-\lambda}}{x_1!} \cdot \frac{\lambda^{x_2} e^{-\lambda}}{x_2!} = \frac{\lambda^{x_1+x_2} e^{-2\lambda}}{x_1! x_2!}$$

(b) (2 points) Find $l(\lambda) = \log[L(\lambda)]$. *Answer:* $l(\lambda) = (x_1+x_2)\log\lambda - 2\lambda - \log(x_1!x_2!)$

(c) (3 points) Find the value of λ that maximizes $l(\lambda)$. *Answer:*

$$\frac{dl(\lambda)}{d\lambda} = \frac{x_1 + x_2}{\lambda} - 2 = 0$$

so we find $\lambda = (x_1 + x_2)/2$*, the sample mean.*

(d) (2 points) How do you know that your value of λ is a maximum? *Answer: The second derivative is*

$$\frac{d^2 l(\lambda)}{d\lambda^2} = -\frac{x_1 + x_2}{\lambda^2} < 0,$$

since both x_j *and* λ *are not negative.*

2. (30 points) Data from 37 patients receiving a non-depleted allogeneic bone marrow transplant were examined to see which variables were associated with the development of acute graft-versus-host disease (GvHD), which is a binary response variable. The predictor variables are the age of the recipient (`Rage`), the age of the donor (`Dage`), whether or not the donor had been pregnant (`preg`), an index of mixed epidermal cell-lymphocyte reactions (`index`) and the `type` of leukemia (there are 3 types, 1= acute myeloid leukemia, 2=acute lymphocytic leukemia and 3=chronic myeloid leukemia).

(a) (3 points) I first estimated a full logistic regression model, with the output below:

```
                  Estimate Std. Error z value Pr(>|z|)
(Intercept)      -5.343720   2.624293  -2.036   0.0417
Rage              0.003882   0.084106   0.046   0.9632
Dage              0.112187   0.081436   1.378   0.1683
preg              1.705127   1.199851   1.421   0.1553
log(index)        1.835935   0.892520   2.057   0.0397
as.factor(type)2 -0.405541   1.256857  -0.323   0.7470
as.factor(type)3  1.676694   1.297599   1.292   0.1963

    Null deviance: 51.049  on 36  degrees of freedom
Residual deviance: 26.288  on 30  degrees of freedom
```

Test whether the overall model is significant at the 5% level. State the null and alternative and your decision. Hint: the 95th percentile of a chi-square distribution with 6 degrees of freedom is 12.59. *Answer:* $H_0 : \beta_1 = \cdots = \beta_6 = 0$ *versus* H_1 : *at least one* $\beta_j \neq 0$. *The test statistic is* $51.049 - 26.288 = 24.761 > 12.59$ *so we reject* H_0.

(b) (3 points) Test whether the type of leukemia has an effect, i.e., do we need the two leukemia dummies? A model without the two leukemia dummies has a residual deviance of 29.27. Hint: the 95% percentile of a chi-square distribution with 2 df is 5.992. *Answer:* $H_0 : \beta_5 = \beta_6 = 0$ *versus* H_1 : *at least one of* β_5 *and* β_6 *is nonzero. The test statistics is* $29.27 - 26.288 = 2.984 < 5.992$ *so we cannot reject* H_0.

(c) (2 points) I created three dummies for the three types of leukemia (they equal 1 for that type and 0 otherwise): `aml`, `all` and `cml`). I entered all variables into a backward selection logistic regression model giving the following model:

```
             Estimate Std. Error z value Pr(>|z|)
(Intercept)  -2.5464     0.9485  -2.685  0.00726
log(index)    1.4877     0.7197   2.067  0.03872
cml           2.2506     1.1060   2.035  0.04187
preg          2.4955     1.1012   2.266  0.02344

    Null deviance: 51.049  on 36  degrees of freedom
Residual deviance: 28.848  on 33  degrees of freedom
```

State the estimated regression equation. *Answer:*

$$\log\left(\frac{\pi}{1-\pi}\right) = -2.5465 + 1.4877\log(\texttt{index}) + 2.2506\texttt{cml} + 2.4955\texttt{preg}$$

(d) (3 points) Use this model to estimate the *probability* if GvHD for someone with an index of 1.10, acute lymphocytic leukemia, and whose donor had been pregnant. *Answer:* $\hat{\eta} = -2.5464 + 1.4877 * \log(1.1) + 2.4955 = 0.0909$ *and* $\hat{\pi} = e^{0.0909} = 0.52$.

(e) (2 points) Using the model from the previous part, how many *times* greater are the *odds* (not probability or log odds) of GvHD for someone with chronic myeloid leukemia compared with one of the other two types of leukemia? *Answer:* $e^{2.2506} = 9.49$ *times more likely.*