Rate-cost tradeoffs in control

Victoria Kostina, Babak Hassibi

Caltech ISIT 2016 BARCELONA

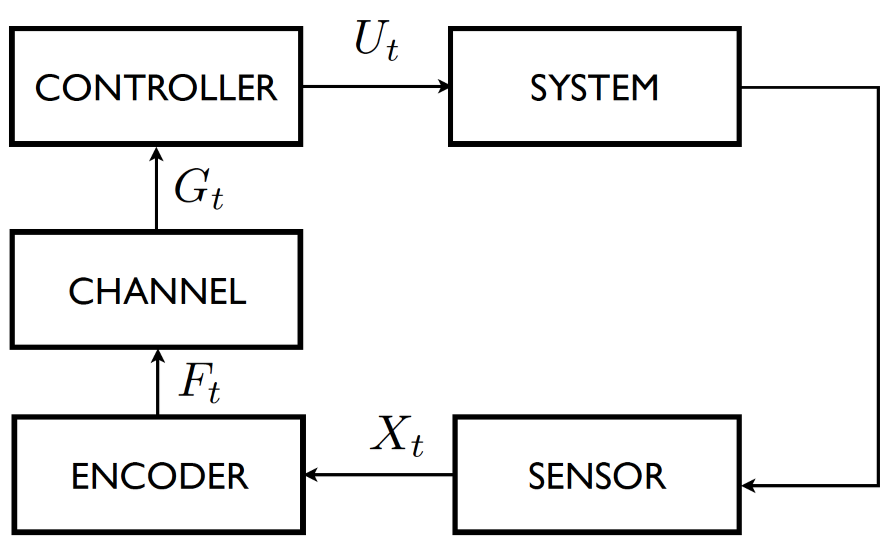
Control under communication constraints

Consider the following discrete time stochastic linear system:

$$X_{t+1} = \mathsf{A}X_t + \mathsf{B}U_t + V_t, \qquad (1)$$

where

- random vector $X_t \in \mathbb{R}^n$ is the system state;
- random vector $V_t \in \mathbb{R}^n$ is the process noise;
- $U_t \in \mathbb{R}^m$ are deterministic controls;
- A and B are fixed matrices of dimensions $n \times n$ and $n \times m$, respectively.



The efficacy of a given control law at time t is measured by the linear quadratic regulator (LQR) cost function:

$$\mathrm{LQR}(U_0, \ldots, U_t) \triangleq \frac{1}{t+1}\mathbb{E}\left[\sum_{i=0}^{t}\left(X_i^T \mathsf{Q} X_i + U_i^T \mathsf{R} U_i\right) + X_{t+1}^T \mathsf{S}_{t+1} X_{t+1}\right],$$

where $\mathsf{Q} \geq 0$, $\mathsf{R} > 0$ and $\mathsf{S}_{t+1} \geq 0$ are known matrices.

Goal

Design an encoder and a controller to minimize the LQR cost.

Minimum cost without rate constraints

Suppose that $V_1, V_2, \ldots$ are i.i.d. with common distribution P_V, and that (A, B) is controllable. Without rate constraints, the minimum attainable cost is given by

$$b_{\min} = \mathrm{tr}(\Sigma_V \mathsf{S}),$$

where Σ_V is the covariance matrix of V, and S is the solution to the algebraic Ricatti equation

$$\mathsf{S} = \mathsf{Q} + \mathsf{A}^T(\mathsf{S} - \mathsf{M})\,\mathsf{A}$$
$$\mathsf{M} \triangleq \mathsf{S}\mathsf{B}(\mathsf{R} + \mathsf{B}^T\mathsf{S}\mathsf{B})^{-1}\mathsf{B}^T\mathsf{S}.$$

Cost = mean-square deviation from 0

In this special case, $\mathsf{Q} = \mathsf{I}_n$, $\mathsf{R} = 0$.
If B is invertible, $\mathsf{S} = \mathsf{M} = \mathsf{I}_n$, $U_t^\star = -\mathsf{B}^{-1}\mathsf{A}X_t$, and

$$b_{\min} = \mathrm{Var}[V].$$

Rate-cost function

$$\mathbb{B}_t(r) \triangleq \min_{\substack{U_i, F_i, G_i,\ i=1,\ldots t-1:\\ U_i = f(G^{i-1}),\ I(F_i; G_i | G^{i-1}) \leq r,}} \mathrm{LQR}(U_0, \ldots, U_t)$$

Definition 1 (Information rate-cost function)**.** *The information rate-cost function is defined as*

$$\mathbb{R}(b) \triangleq \min\left\{ r\colon\ \limsup_{t\to\infty} \mathbb{B}_t(r) < b, \right\}$$

Minimum cost with rate constraints

Theorem 1. *Suppose that $V_1, V_2, \ldots$ are i.i.d. with common distribution P_V. Assume that X_0 and V have a density, and that (A, B) is controllable. For any LQR cost $b > b_{\min}$, the rate-cost function of the fully observed linear stochastic system (1) is bounded from below as follows.*

a) If $\mathrm{rank}\,\mathsf{B} = n$*, then*

$$\mathbb{R}(b) \geq \log|\det \mathsf{A}| + \frac{n}{2}\log\left(1 + \frac{N(V)|\det \mathsf{M}|^{\frac{1}{n}}}{(b - b_{\min})/n}\right), \qquad (2)$$

where $N(V)$ is the entropy power of V.

b) More generally, for all $b > b_{\min}$,

$$\mathbb{R}(b) \geq \sum_{i\colon |\lambda_i(\mathsf{A})| \geq 1} \log|\lambda_i(\mathsf{A})|. \qquad (3)$$

Cost = mean-square deviation from 0

In this special case, (2) reduces to

$$\mathbb{R}(b) \geq \log|\det \mathsf{A}| + \frac{n}{2}\log\left(1 + \frac{N(V)}{(b - \mathrm{Var}[V])/n}\right).$$

Achievability scheme

A simple lattice quantization scheme that only quantizes the *innovation*, that is, the difference between the controller's belief about the current state and the true state, achieves cost $b > b_{\min}$ at the entropy rate

$$H \leq \log|\det \mathsf{A}| + \frac{n}{2}\log\left(\frac{N(V)|\det \mathsf{M}|^{\frac{1}{n}}}{(b - b_{\min})/n}\right) + O_1(\log n) + O_2(b - b_{\min})$$

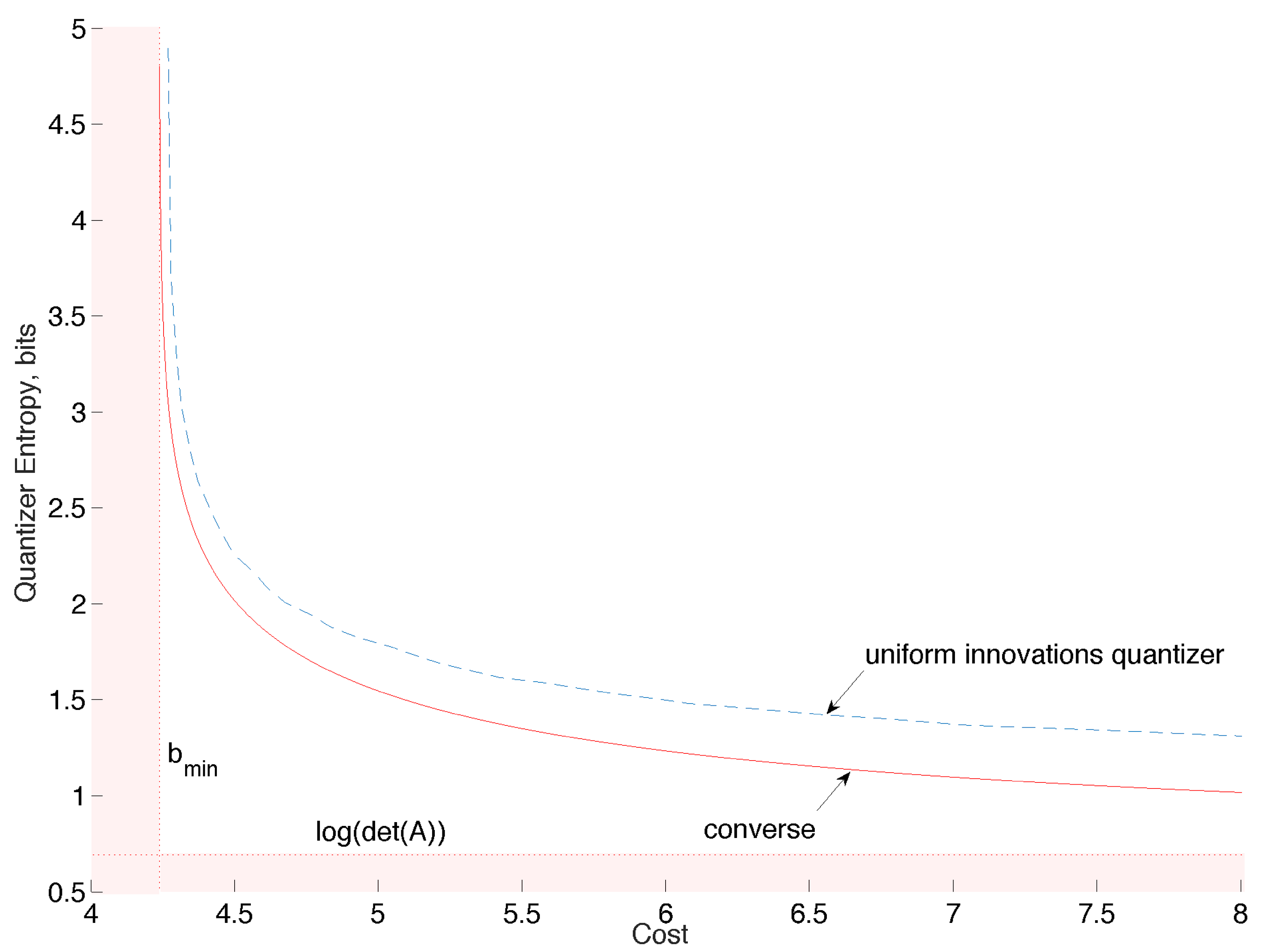


Figure: The minimum quantizer entropy compatible with cost b. Scalar system, $n = 1$, $\mathsf{A} = 2$, $\mathsf{B} = \mathsf{Q} = \mathsf{R} = 1$, $V \sim \mathcal{N}(0, 1)$. Courtesy of Ayush Pandey.

References

[1] S. Tatikonda, A. Sahai, and S. Mitter, "Stochastic linear control over a communication channel," *IEEE Transactions on Automatic Control*, vol. 49, no. 9, pp. 1549–1561, 2004.

[2] G. N. Nair and R. J. Evans, "Stabilizability of stochastic linear systems with finite feedback data rates," *SIAM Journal on Control and Optimization*, vol. 43, no. 2, pp. 413–436, 2004.