Information-Performance Tradeoffs in Control

Final Report by
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INTRODUCTION

Over the past few decades the fields of communication and control have been developed extensively to support the emerging needs of the society such as higher speed internet, better telephone communication networks, wireless access to devices over Internet of Things (IoT), reliable long distance satellite communication, home automation and many other related fields. While communication theory is mainly concerned with the transmission of information from one point to another reliably it does not deal with what is done with the information once it is transmitted. Control theory, on the other hand, mainly deals with using the information it receives as feedback in order to achieve better performing systems, which are not prone to disturbances, noises and other process variations. Although there is substantial individual research on control and information theory, the research on networked control systems (control systems with communication between its components) is still in a nascent stage.

Networked control systems are increasingly finding wide applications but they also pose newer challenges which are neither addressed in control theory nor in information theory. The components of a networked control system are physically distributed with communication links between the plant, controller, observer and/or the actuators. These links connecting the different components are often noisy and pose various other constraints (See [1] for details). Hence, the control design and analysis needs to be done keeping in mind these information bottlenecks. In control system design, it is usually desired to minimize a performance index. Because of the information bottlenecks, there exists a tradeoff between the best achievable performance and the observation accuracy (it is intuitive that worse the observation accuracy, worse would be the performance of the system).

We dealt with two important settings of networked control systems in this project:

1. Noiseless rate-limited communication of data between observer and controller
2. Communication over Additive White Gaussian Noise (AWGN) channel between the observer and the controller.

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BACKGROUND

Throughout this paper, unless otherwise mentioned, we deal with a Linear Quadratic Gaussian (LQG) control system setting with a communication link between the observer and the controller. This section covers LQG control theory very briefly and introduces the two communication constraints — rate-limited channel and AWGN channel in the feedback path.

Classical Partially Observed LQG Problem

For a stochastic system, the LQG control scheme estimates the states from the available outputs. Using this estimation of states a controller is designed such that a quadratic performance index is minimized. Consider a discrete time stochastic system, the state equation can be written as

$$x_{t+1} = A_t x_t + B_t u_t + w_t \tag{2.1}$$

The output equation is

$$y_t = C_t x_t + v_t \tag{2.2}$$

where w_t and v_t are assumed to be zero mean Gaussian noises with covariances W_t and V_t in process and measurement respectively. For states $x_t \in \mathbb{R}^{n\times 1}$, inputs $u_t \in \mathbb{R}^{k\times 1}$ and outputs $y_t \in \mathbb{R}^{p\times 1}$ we have, the stochastic state transition matrix $A_t \in \mathbb{R}^{n\times n}$, the stochastic control multipliers, $B_t \in \mathbb{R}^{n\times k}$ and the stochastic output multipliers, $C_t \in \mathbb{R}^{p\times n}$. We also assume that the distribution of the initial value of the state x, x_0 is known and that w, v and x_0 are all independent of each other. The system is stationary which ensures that the mean and covariances will not change with time. The following block diagram representation gives a clearer picture of LQG control.

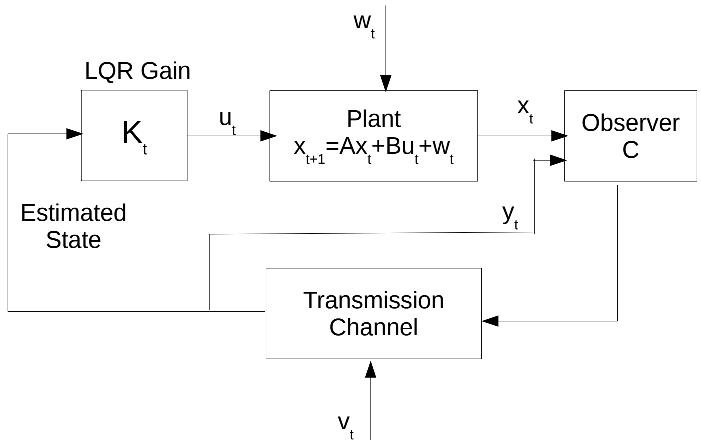

Figure 2.1: Representative Block Diagram of Classical Partially Observed LQG Control

The aim is to find a control law u_t so that the following performance index is minimized, where $Q \geq 0$ is the state weighting matrix and $R > 0$ is the control weight matrix.

$$b(\cdot) = E\left(\sum_{t=0}^{N-1}\left(x_t^T Q x_t + u_t^T R u_t\right) + x_N^T Q x_N\right) \tag{2.3}$$

The control law is given by

$$u_t = -L_t E[x_t | y_1, y_2, ..., y_t] \tag{2.4}$$

where L_t is the LQR gain and $E[x_t|y_1, y_2, ..., y_t]$ is the minimum mean square error estimate of x_t given that all past output measurements are available. It can be proved that the calculation of the LQR gain L_t and the MMSE estimate of the state using Kalman Filter can be performed independent of each other [2].

Rate-Limited Communication Channel

If in Fig.(2.1) the observer data is transmitted via a digital communication link to the controller then the system performance would also be dependent on the data rate at which this communication is done. Hence, there exists a tradeoff between communication rate and optimal performance (faster the rate, better the performance). We studied the tightness of the bounds on the performance cost for the rate-limited noiseless communication channel using different quantization schemes. The information-theoretic lower and upper bounds have been given in [3]. We also considered a system where the system matrix A (see Eq.(2.1)) is uncertain but has a known probability distribution.

Additive White Gaussian Channel

We studied a system where the observations are corrupted by additive white Gaussian noise during the transmission from the observer to the controller, which is often the case when analog communication is used to transmit information. Continuing on the work in the recent paper [4] where the system disturbances w (See Eq.(2.1)) have been assumed to have Gaussian distribution, we instead dealt with other probability distributions of w. We tried to show that the information-theoretic bounds (as given in [3]) are tight even when w is not Gaussian.

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AWGN COMMUNICATION CONSTRAINT

Consider the scalar system in Eq.(2.1) but with communication over AWGN channel between observer and the controller. Using controller and estimator (Kalman Filter) Algebraic Riccati Equations (AREs) (see [4] and [5]) we have the expressions for the controller gain $L(t)$ and Kalman gain $P(t)$ for the control over AWGN channel case.

$$P(t+1) = A^2 P(t)\left(1 - \left(\frac{P(t)}{P(t)+1}\right)\left(\frac{SNR}{SNR+1}\right)\right) + W \tag{3.1}$$

$$S(t) = \frac{A^2 R S(t+1)}{S(t+1)+R} + Q \tag{3.2}$$

$$L(t) = \frac{A S(t+1)}{S(t+1)+R} \tag{3.3}$$

Using the above, the expression for optimal cost(b^*) achieved vs SNR of the channel was derived as follows.

$$b^*(\cdot) = \sum_{t=0}^{T}\left[Q\Sigma(t) + \left(S(t)\left(A^2\Sigma(t-1) + W - \Sigma(t)\right)\right)\right] \tag{3.4}$$

where $\Sigma(t)$ can be obtained using Kalman filter gains $P(t)$ as follows

$$\Sigma(t) = \frac{P(t) - W}{A^2} \tag{3.5}$$

To tackle the problem of non-Gaussian system disturbance w, we used the result derived in [3]. The following equation taken from [3] gives the lower bound to the rate distortion function in terms of optimal cost b.

$$\mathbb{R}(b) \geq \log(|det(A)|) + \frac{n}{2}\log\left(1 + \frac{N(w)|det(M)|^{\frac{1}{n}}}{(b - b_{min})/n}\right) \tag{3.6}$$

where b_{min} is the minimum cost as calculated for the classical LQG case (without communication), $N(w)$ is the entropy power given by

$$N(w) = \frac{1}{2\pi e}exp\left(\frac{2}{n}h(w)\right) \tag{3.7}$$

where $h(w)$ is the differential entropy of w and M can be obtained by solving the following Algebraic Riccati Equation (ARE):

$$S = Q + A^T(S - M)A \tag{3.8}$$

$$M = SB(R + B^T SB)^{-1} B^T S \tag{3.9}$$

For the scalar case with $n = 1$, we have

$$\mathbb{R}(b) \geq \log|A| + \frac{1}{2}\log\left(1 + \frac{N(w)|M|}{b - b_{min}}\right) \tag{3.10}$$

Using Shannon's channel capacity we used the above bound for the AWGN channel case that we are concerned with here. The lower bound given in Eq.(3.6) is valid for any distribution of system disturbance w. The channel capacity for noise corrupted channel is given by

$$\mathbb{C} = \frac{n}{2}\log(1 + SNR) \tag{3.11}$$

where SNR is the signal to noise ratio of the AWGN channel.
To study the tightness of the lower bound given in Eq.(3.6) we simulated the system and compared it with the lower bound plot. The result is shown in Fig.(3.1). We used a Laplace distribution for w (with zero mean and $\sigma_w = 1$) to simulate the system and to calculate the lower bound.

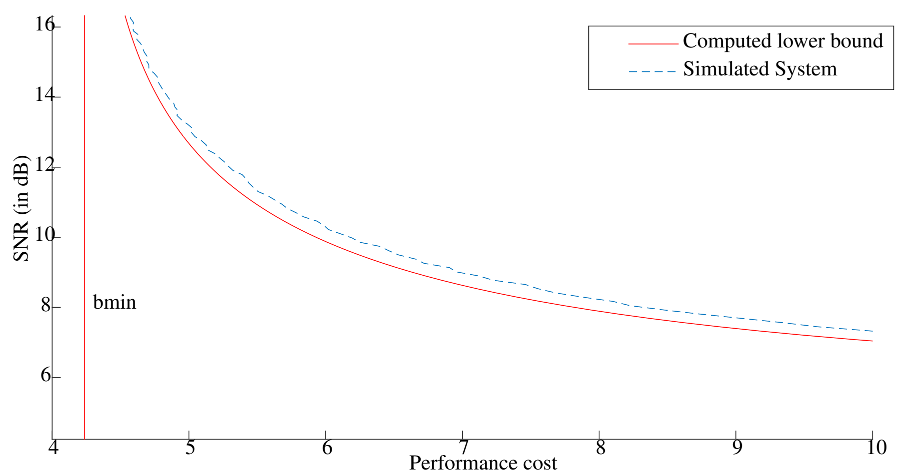

Figure 3.1: Comparing the computed optimal cost with the simulated optimal cost for system with AWGN communication channel. The fully observable system has the parameter $A = 2$ and the system disturbance w is zero mean Laplacian with $\sigma_w = 1$.

We also studied the partially observable case where (i.e. $y = x + v$) in a similar fashion. The results for partially observable system and for different system parameters and distributions of w are available at the web link given in [6].

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RATE-LIMITED COMMUNICATION CHANNEL CONSTRAINT

Similar to the AWGN channel case, our approach to studying the tradeoff in the rate-limited channel case involves using the lower bound from [3], Eq.(3.6) and Shannon's channel capacity concept. We demonstrated the tightness of the bound by using a simple uniform quantization scheme to implement the rate-limited communication. We observed that the lower bound is closely followed with this scheme. Towards the end of this section, we also present some initial results on a similar tradeoff study for a system with uncertain parameter A, which has a known probability distribution in the same setting of rate-limited communication between the observer and the controller.

We approximated the rate-distortion function ($R(b)$ in the bound, See Eq.(3.6) using the quantizer entropy. To calculate the output entropy of the quantizer in the simulated system, we estimated the probabilities of each sample falling in the different quantization bins. Using the expression of entropy $H(x) = \sum_i P_i(x) \log\left(\frac{1}{P_i(x)}\right)$, we calculated the entropy of the quantizer output for each Δ, the quantizer step size. The simulated system was then compared with the bound, the result is shown in the Fig.(4.1).

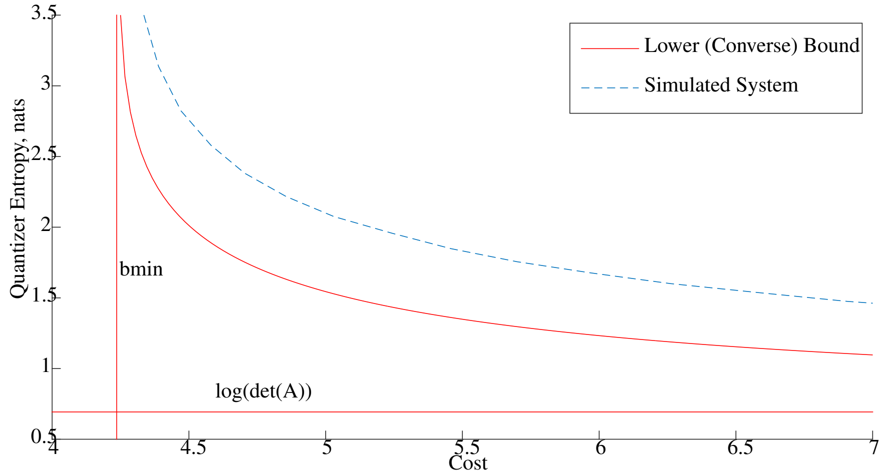

Figure 4.1: Comparing the lower bound to the computed optimal cost with the simulated optimal cost for system with rate-limited channel. The fully observable system has the parameter $A = 2$, and the system disturbance w is zero mean Gaussian with $\sigma_w = 1$

The result for partially observable case is shown in Fig.(4.2).

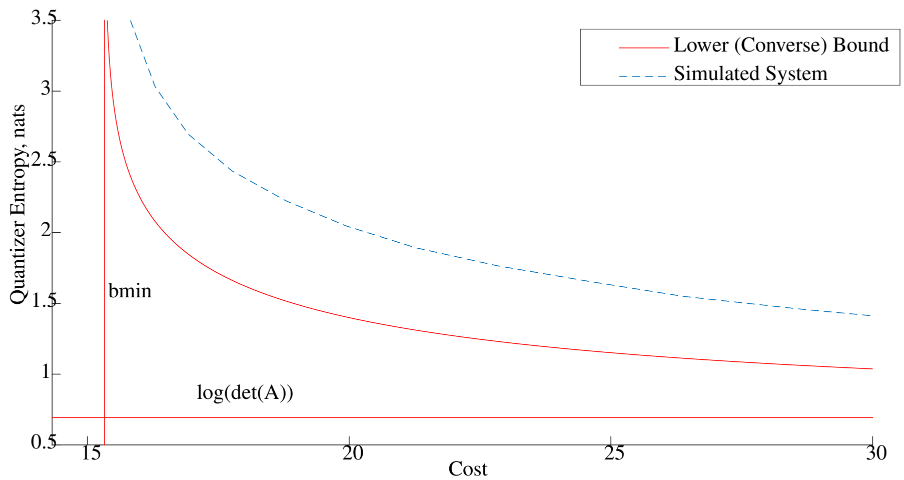

Figure 4.2: Comparing the lower bound to the computed optimal cost with the simulated optimal cost for system with rate-limited channel. The partially observable system has the parameter $A = 2$, the system disturbance w is zero mean Gaussian with $\sigma_w = 1$ and observation noise is assumed to be zero mean Gaussian with $\sigma_v = 1$

Further results showing the upper bound as well along with lower bound and simulated system are available on the web link [6]. Other results also include simulations for different system parameters and different probablity distributions for the system disturbance w.

4.1 Multiplicative Parameter Uncertainty in the Plant

The system parameter A might itself be uncertain having a known probability distribution. This is often the case in digital control systems which inherently have white parameters occuring due to sampling period or in some controller parameters. There are various known examples of economic systems as well which exhibit uncertain system parameters. The issue of stabilizability of such systems has been a topic of research since the 1970s (see [7]). Because of the wide applications of these kind of systems, there have been many studies focused on general LQG control design for such systems. However the most general case where the system is partially observable and the system parameter A has a probability distribution has been an unsolved problem. As described in [8] and other related research papers, the separation theorem which plays a major role in LQG control design is not valid for the system with uncertain parameter. Although the partially observable case with random system parameter(s) remains unsolved, the effect of communication between the observer and the controller has been a focus of recent research in this area. In [9], stabilizability conditions have been derived for such a system and a rate-distortion bound similar to Eq.(3.6) has been given. To study the tradeoff between the rate of communication and the optimal achievable LQG cost, we compared the uniform quantizer entropy with the bound (similar to the rate-limited case). The result is shown in Fig.(4.3).

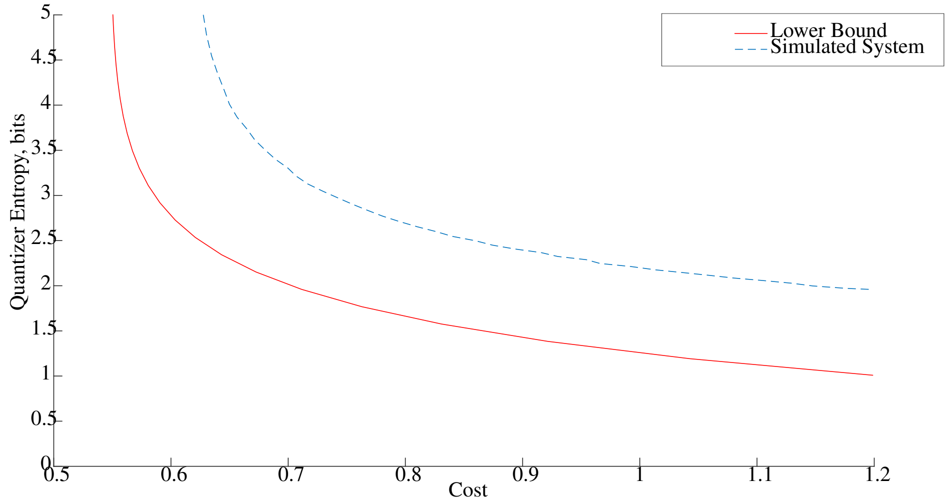

Figure 4.3: Comparing the lower bound to the optimal cost in the rate-limited channel case with uncertain system parameter A using a uniform quantization scheme. The fully observable system has the parameter $A = 2$ and the system disturbance w is zero mean Gaussian with $\sigma_w = 1$

As we can see that the uniform quantizer in this case does not follow the bound as closely as we observed when A was fixed. In order to improve the performance we improved the quantizer design by using the Lloyd-Max quantizer for a fixed number of points. Some results with this quantizer are available in [6].

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CONCLUSIONS AND FUTURE WORK

5.1 For AWGN channel

As expected, for high SNR values, the optimal cost achieved was lower. This was exhibited in all of the results as the cost tends to the minimum value in the high SNR regime. The important inference that can be drawn from the comparison between the simulations and the lower bound given in [3] is that the lower bound is tight for non Gaussian system disturbance w as well. We demonstrated this result for both partially observable and fully observable cases. Hence, we can say that the lower bound given in [3] is tight for non-Gaussian w's as well.

We only considered a scalar system with only a single channel of communication between the observer and the controller. It would be interesting to study the trade-off for MIMO (multi-input and multi-output) systems as well as when there are more than one channels available to transmit the information. The optimal power allocation over multiple channels would be an interesting problem to solve.

5.2 Rate-Limited Channel

Similar to the AWGN case, it was expected that for higher entropy of the output of the quantizer (i.e. smaller quantization bins), the optimal cost would tend to the minimum. The results obtained are in line with this expectation, as we observed that the optimal cost tends to the minimum cost (b_{min}) for higher values of quantizer entropy. The major conclusion is that even on using the simplest of schemes for quantization (a uniform quantizer), we observed that the performance cost values were quite close to the lower bounds given in [3] both for partially observable and fully observable case.

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