Robust Controller Design Using H_∞-Loop Shaping Approach

Mid-sem evaluation report submitted in partial fulfillment to the requirements for the degree of

BACHELOR OF TECHNOLOGY

by

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Certificate

This is to certify that the report entitled **Robust Controller Design Using H_∞ Loop Shaping Approach**, submitted by **Ayush Pandey**, an undergraduate student, in the *Department of Electrical Engineering, Indian Institute of Technology, Kharagpur, India,* for the award of the degree of Bachelor of Technology, is a record of the work carried out by him under our supervision and guidance. Neither this report nor any part of it has been submitted for any degree or academic award elsewhere.

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Contents

1 Introduction

This project deals with the analysis and design of a H_∞ loop shaping controller using convex optimization based methods. A brief introduction to the project was given in [1]. This document reports the subseqest work done. The reader is referred to [1] for detailed results and discussions on preliminaries to robust control theory, H_∞ control, linear fractional transformation, small gain theorem for robust stability analysis and a few examples on these topics.

The H_∞ loop shaping design technique incorporates classical loop shaping method to obtain performance and robust stability tradeoffs along with H_∞ control optimization technique to gurantee a level of closed loop robust stability. This method ensures robustness against unstructured uncertainty which is described as perturbations to normalized coprime factors of the shaped plant.

The first two sections of this report give a brief discussion of solution of H_∞ control problem by finding out the H_∞ norm using a convex optimization technique. The preliminaries for the same have been covered and related proofs have been presented. The second section ends with a few examples on the same. The next section covers coprime factor uncertanity description to a plant, its robust stabilization and H_∞ loop shaping design procedure. The report ends with a justification on how this technique leads to a good and acceptable robust controller design.

2 Preliminaries

Various robust control theory preliminaries such as matrix norms, performance specifications, H_∞ and H_2 norms, coprime factorization, linear fractional transformation etc. have been covered in [1]. This section gives an introduction to convex optimization based technique to solve the H_∞ control optimization problem. Before 1995, two algebraic riccati equation based methods were used to solve the H_∞ control optimization problem, the disadvantage with these methods is that they require various assumptions to be satisfied which are usually not satisfied for many plant models. A convex optimization based approach was proposed in the mid 90s which was based on solving a few linear matrix inequalities (LMIs) inorder to result in a stabilizing controller for the H_∞ control problem. The LMI based approach is easily applicable to a range of systems and is covered in detail in Section(3).

2.1 Linear Matrix Inequality

An optimization problem with convex constraints can be formulated in the LMI framework. An LMI has the following form.

$$F(x) = F_0 + \sum_{i=1}^{m} x_i F_i > 0 \tag{1}$$

where, $x = [x_1 x_2 ... x_m]$ is the design variable and F_i, $i = 0, 1...m$ are symmetric matrices. The constraint shown in Eq.(1) indicates the positive definiteness of the matrix $F(x)$. Formulation of an optimization problem in the LMI framework helps the designer to make use of the available LMI solvers in softwares such as MATLAB to efficiently solve the optimization problem.

2.2 Bounded Real Lemma

As previously seen in [1], closed loop performance and robust stability requirements for a system expressed in LFT framework can be expressed as H_∞ norm minimization problem of certain stable closed loop transfer function matrix of the form $\|T_{zw}\|_\infty < \gamma$. The bounded real lemma gives equivalent conditions to this minimization in LMI framework which can be solved to effectively solve the H_∞ control problem. Consider a system in state space representation as follows.

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}, x(0) = 0 \tag{2}$$

$$\mathbf{y} = \mathbf{Cx} + \mathbf{Du} \tag{3}$$

In compact matrix notation, we can write (denoting the system TFM as G(s)):

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

The bounded real lemma states that $\|G(s)\|_\infty < \gamma \Leftrightarrow \exists\, Y = Y^T > 0$ such that:

$$\begin{bmatrix} YA + A^T Y & YB & C^T \\ B^T Y & -\gamma I & D^T \\ C & D & -\gamma I \end{bmatrix} < 0 \tag{4}$$

The above LMI can be solved for Y and γ using convex optimization techniques. The value of γ corresponds to the H_∞ norm of the TFM $G(s)$ which can be used to design a stabilizing controller.

Proof We can write

$$\|G(s)\|_\infty = \sup_{u\neq 0} \frac{\|y\|_2^2}{\|u\|_2^2} \tag{5}$$

We have, $\|G(s)\|_\infty < \gamma$ and we need to derive the relation in Eq.(4). We define a supply rate function as $(\gamma^2 \|u\|_2^2 - \|y\|_2^2)$. From Eq.(5), we have that the supply rate function is positive. Physically, this is the energy supply rate to a system whose input is u and output, y. Let, $V(x)$ be the energy storage function for this system. Then, we directly have that

$$\dot{V}(x) < (\gamma^2 \|u\|_2^2 - \|y\|_2^2) \tag{6}$$

That is, the rate of supply of energy to the system would always be more than the rate of storage of energy in the system. For a linear time invariant systems with minimal realization, this function $V(x)$ can be considered to be the Lyapunov function of the system. Now, let $V(x)$ be a quadratic Lyapunov function $x^T Px$. For the system in Eq.(2), we evaluate Eq.(6) and on simple manipulation we get,

$$\Leftrightarrow \begin{bmatrix} x^T & u^T \end{bmatrix} \begin{bmatrix} A^T P + PA + C^T C & PB + C^T D \\ B^T P + D^T C & -\gamma^2 I + D^T D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} < 0$$

$$\Leftrightarrow \begin{bmatrix} A^T P + PA + C^T C & PB + C^T D \\ B^T P + D^T C & -\gamma^2 I + D^T D \end{bmatrix} < 0$$

Now, we know that for a function to be a Lyapunov function candidate, the following properties are satisfied:

1. $V(0) = 0$
2. $V(x) > 0 \, \forall \, x \neq 0$

Using this and the fact that for an LTI system stability doesn't depend on the input, we can see from Eq.(6) that the system is stable as $\dot{V}(x) < 0$. On integrating Eq.(6) for time 0 to time t, we obtain

$$\Leftrightarrow V(x(t)) - V(x(0)) < \int_0^t (\gamma^2 \|u_2\|^2 - \|y_2\|^2) dt \tag{7}$$

Since $x(0) = 0$, We have $V(x(0)) = 0$ and $V(x(t))$ is always positive. Hence,

$$(\gamma^2 \|u\|_2^2 - \|y\|_2^2) > 0 \tag{8}$$

$$\Leftrightarrow \frac{\|y\|_2^2}{\|u\|_2^2} < \gamma^2 \tag{9}$$

$$\Leftrightarrow \|G(s)\|_\infty < \gamma \tag{10}$$

We have proved that $\|G(s)\|_\infty < \gamma$ is equivalent to, when $\exists Y = Y^T > 0$ such that

$$\begin{bmatrix} A^T P + PA + C^T C & PB + C^T D \\ B^T P + D^T C & -\gamma^2 I + D^T D \end{bmatrix} < 0$$

Now, using the Schur's complement lemma, we will prove the statement of the bounded real lemma. The Schur's complement lemma for a symmetric matrix states that

$$\begin{bmatrix} A & B \\ B^T & D \end{bmatrix} < 0 \Leftrightarrow A - BD^{-1}B^T < 0\,;\ D < 0 \Leftrightarrow D - B^T A^{-1} B < 0\,;\ A < 0$$

For the LMI in the bounded real lemma statement, using Schur's complement taking

$$A = \begin{bmatrix} YA + A^T Y & YB \\ B^T Y & -\gamma I \end{bmatrix},\ B = \begin{bmatrix} C^T \\ D^T \end{bmatrix},\ D = \begin{bmatrix} -\gamma I \end{bmatrix}$$

Using the above matrices in the Schur's complement's first form, we can prove the equivalence relation in the bounded real lemma, which completes the proof.

3 LMI Based Approach to H_∞ Control

As mentioned in the Introduction section, there are three methods to solve a H_∞ control problem. The first two methods, viz. the DGKF method and the Doyle-Glover method are based on finding out a stabilizing solution to certain algebraic riccati equation. The design procedure in these methods works only when the system satisfies a set of assumptions which are usually difficult to satisfy for a given plant model. On the other hand, the LMI based convex optimization approach only requires two assumptions mentioned in the next section, which are satisfied easily for a range of different systems. Also, it has been proved that the controller structure found in the two methods can be similarly derived in the LMI framework. Hence, the LMI based approach to H_∞ control is widely used due to its ease of design on the part of the designer. Using the bounded real lemma stated and proved in the previous section, the H_∞ norm of a

TFM can be found out by solving the LMI. This can be used to design a stabilizing controller $K(s)$ for a system by minimizing the H_∞ norm of the closed loop TFM. The LMI approach to H_∞ control requires two assumptions to be satisfied for the generalized plant P. Let a generalized plant be expressed in state space representation as follows.

$$P = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right]$$

The following two assumptions should be satisfied to apply LMI based approach to solve the H_∞ control problem:

1. (A,B_1,C_1) stabilizable and detectable.
2. $D_{22} = 0$

The controller design to minimize the H_∞ norm of the generalized plant using LMI approach has been covered in detail in [2]. A simple numerical example is shown in the next section which demonstrates the use of LMI toolbox and YALMIP toolbox to calculate the H_∞ norm of a given system in MATLAB using bounded real lemma.

3.1 Examples and Simulations

In [1], we took a simple stable system and calculated its infinity norm.

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}, D = 0_{2\times 2}$$

Using MATLAB's Robust Control Toolbox *hinfnorm* function, the value obtained was 2.2836. The same result will be shown in the following sections obtained by solving the LMI that the bounded real lemma gives using the LMI tooblox and the YALMIP toolbox [3].

3.1.1 LMI Toolbox

The *mincx* function in MATLAB helps in minimizing linear objective under LMI constraint. To use this, first an LMI needs to be defined using *setlmis*, *lmivar* and *lmiterm* commands. The details for these commands are available in MATLAB documentation. After an LMI has been defined in MATLAB, the linear objective function which needs to be minimized is defined using *defcx* function. Once this is done, the *mincx* command yields the minimized objective function value as well as the value at which this minimum occurs. This completes the H_∞ norm calculation. The MATLAB code written for the above system is avaliable at [4].

3.1.2 YALMIP Toolbox

YALMIP is a modelling language for advanced modeling and solution of convex and nonconvex optimization problems. Its function *optimize* can be used to solve a convex optimization problem. The YALMIP toolbox code works on symbolic decision variables which are defined by the user. These decision variables can be then used to define the LMI in a very user friendly manner. YALMIP toolbox can be used to solve both strict and non-strict constraint LMIs while MATLAB's LMI toolbox only handles strict constraints. For the system shown above, the MATLAB program which solves the LMI given by the bounded real lemma is available at [4]. As expected, the result for the H_∞ norm is same using all the three methods.

4 H_∞ Loop Shaping Control

The H_∞ controller design optimization methods face a few disadvantages with respect to model perturbations being limited by number of poles in the RHP and the possibility of undesirable pole-zero cancellation between the nominal plant and the H_∞ controller. In H_∞ loop shaping design technique, the perturbations are directly described on the coprime factors of the nominal model. This helps in relaxing the restriction on the number of RHP poles and also doesn't produce any pole-zero cancellation. More importantly, this design technique is computationally more efficient as it doesn't require an iterative procedure for finding γ. Finally, since this design technique inherits concepts of loop shaping which is a classical controller design technique, hence it is much more user friendly method to design a robust controller.

4.1 Coprime Factor Uncertainty Description

A brief introduction to coprime factorization was given in [1]. We consider left coprime facotrization of a plant P given by $\tilde{M}^{-1}\tilde{N}$, where $\tilde{M}$ and $\tilde{N} \in \mathbb{RH}_\infty$. Unstructured uncertainties can be described in a more general sense by perturbations on the coprime factors of a plant ($\tilde{\Delta}_m$ and $\tilde{\Delta}_n \in \mathbb{RH}_\infty$). The Figure(1) shows perturbations on the left coprime factors of the plant P considered above.

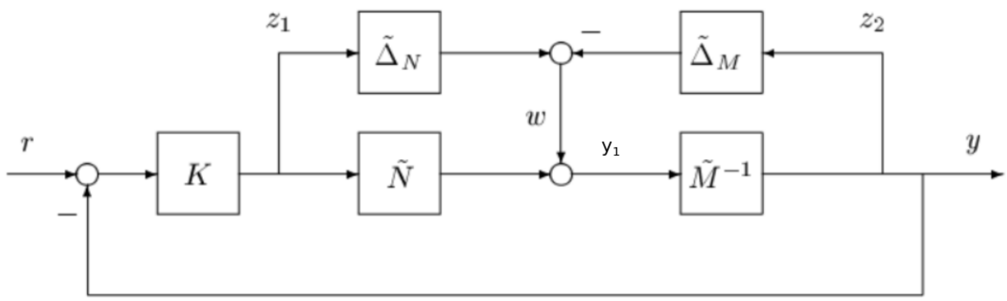

Figure 1: Perturbations to Coprime Factors : Block Diagram

From the block diagram we have,

$$y_1 = \tilde{M}y \tag{11}$$
$$y_1 = \tilde{N}u + \tilde{\Delta}_n u - \tilde{\Delta}_m y \tag{12}$$

We have from these two equations, eliminating y_1

$$y = (\tilde{M} + \tilde{\Delta}_m)^{-1}(\tilde{N} + \tilde{\Delta}_n)u \tag{13}$$

Hence, the perturbed plant in this case is given by

$$P = (\tilde{M} + \tilde{\Delta}_m)^{-1}(\tilde{N} + \tilde{\Delta}_n) \tag{14}$$

Now with this coprime factor perturbed plant, we will study the robust stabilization and the H_∞ loop shaping controller design technique.

4.2 Robust Stabilization of Coprime Factors

In Eq.(14), we assume that $\left\|[\tilde{\Delta}_m \ \ \tilde{\Delta}_n]\right\|_\infty < \epsilon$. We assume that in Fig.(1) the controller K internally stabilizes the nominal plant. The robust stability analysis for this system can be done by reducing the block diagram in Fig.(1) to LFT framework and applying the small gain theorem. For the equivalent $M - \Delta$ structure, we can obtain M as follows.

$$
\begin{aligned}
z_2 &= y\ ;\ z_1 = -ky \\
y &= \tilde{M}^{-1}(\tilde{N}u + w) \\
\Rightarrow z_1 &= -KPu - K\tilde{M}^{-1}w \\
\Rightarrow z_2 &= Gu + \tilde{M}^{-1}w \\
y &= -PKy + \tilde{M}^{-1}w \\
\Rightarrow y &= (I + PK)^{-1}\tilde{M}^{-1}w \\
\Rightarrow M &= \begin{bmatrix} K \\ I \end{bmatrix} (I + PK)^{-1}\tilde{M}^{-1}
\end{aligned}
$$

Hence, on using small gain theorem, we obtain the following condition for robust stability of the system.

$$
\left\| \begin{bmatrix} K \\ I \end{bmatrix} (I + PK)^{-1}\tilde{M}^{-1} \right\|_\infty \leq \frac{1}{\epsilon} \tag{15}
$$

Now, the following theorem from coprime factorization theory could be used to derive the generalized plant expression for the system shown in Fig.(1). The theorem gives the left and right coprime factorizations for a proper real-rational TFM. Let a system P have the following stabilizable and detectable minimal state space realization.

$$
P = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] \tag{16}
$$

Now, let B and L be matrices such that $A + LC$ and $A + BF$ are both stable. Then we can define:

$$
\begin{bmatrix} M & -Y_l \\ N & X_l \end{bmatrix} = \left[\begin{array}{c|cc} A + BF & B & -L \\ \hline F & I & 0 \\ C + DF & D & I \end{array} \right] \tag{17}
$$

$$
\begin{bmatrix} X_r & Y_r \\ -\tilde{N} & \tilde{M} \end{bmatrix} = \left[\begin{array}{c|cc} A + LC & -(B + LD) & L \\ \hline F & I & 0 \\ C & -D & I \end{array} \right] \tag{18}
$$

Then $P = NM^{-1} = \tilde{M}^{-1}\tilde{N}$ are rcf and lcf respectively. The above theorem can be proved by verifying the Bezout's identity for double coprime facotrization of plant P. Now, using Eq.(18), we get the following result for lcf of P.

$$
\begin{bmatrix} \tilde{N} & \tilde{M} \end{bmatrix} = \left[\begin{array}{c|cc} A + LC & B + LD & L \\ \hline C & D & I \end{array} \right] \tag{19}
$$

Using Eq.(19) and denoting the controller $\hat{K} = -K$ in Fig.(1), we have for $z = [z_1\ z_2]^T$

$$\begin{aligned} z_1 &= u \\ z_2 &= \tilde{M}^{-1}w + \tilde{M}^{-1}\tilde{N}u \\ y &= \tilde{M}^{-1}w + \tilde{M}^{-1}\tilde{N}u \end{aligned}$$

Hence, the generalized plant in the equivalent closed loop LFT structure (as shown in Fig.(2)) is as follows (in the transfer function form).

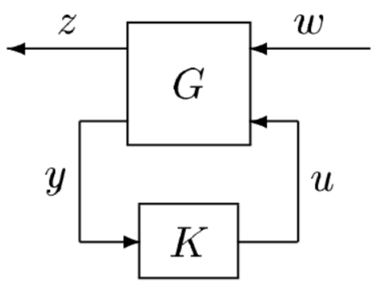

Figure 2: Lower LFT Representation

$$G(s) = \begin{bmatrix} \begin{bmatrix} 0 \\ \tilde{M}^{-1} \end{bmatrix} & \begin{bmatrix} I \\ P \end{bmatrix} \\ \tilde{M}^{-1} & P \end{bmatrix} \tag{20}$$

And in state space form is given by

$$G = \left[\begin{array}{c|cc} A & -L & B \\ \hline \begin{bmatrix} 0 \\ C \end{bmatrix} & \begin{bmatrix} 0 \\ I \end{bmatrix} & \begin{bmatrix} I \\ D \end{bmatrix} \\ C & I & D \end{array}\right] \tag{21}$$

Using $G(s) = C(sI - A)^{-1}B + D$ we can verify that Eq.(21) is indeed the state space representation of the TF representation given in Eq.(20).

$$C = \begin{bmatrix} \begin{bmatrix} 0 \\ C \end{bmatrix} \\ C \end{bmatrix}; B = \begin{bmatrix} -L & B \end{bmatrix}; D = \begin{bmatrix} \begin{bmatrix} 0 \\ I \end{bmatrix} & \begin{bmatrix} I \\ D \end{bmatrix} \\ I & D \end{bmatrix}$$

This is an important result which would help in the design of H_∞ loop-shaping design based controller using LMI approach.

4.3 Design Steps

H_∞ loop shaping controller design steps were studied as presented in [5]. Using pre-compensator and/or post-compensator the singular values of the nominal plant are shaped first to get a desired open loop shape. This shaped plant is represented as G_s, which is equal to W_2GW_1. Next, the maximum robust stability margin is calculated. It is given by

$$\epsilon_{max} = \sqrt{1 - \left\| \begin{bmatrix} \tilde{N}_s & \tilde{M}_s \end{bmatrix} \right\|_H^2} < 1 \tag{22}$$

where $G_s = \tilde{M}_s^{-1}\tilde{N}_s$ and $\tilde{M}_s\tilde{M}_s^* + \tilde{N}_s\tilde{N}_s^* = I$, i.e. $\tilde{M}_s$ and $\tilde{N}_s$ are normalized left coprime factors of the shaped plant G_s.
Now, selecting $\epsilon < \epsilon_{max}$, we go ahead with the controller design based on H_∞ optimization as follows.

$$\left\| \begin{bmatrix} I \\ K_\infty \end{bmatrix} (I + G_sK_\infty)^{-1}\tilde{M}_s^{-1} \right\|_\infty \leq \epsilon^{-1} \tag{23}$$

The final feedback controller is given by $K = W_1K_\infty W_2$. The calculation of ϵ_{max}, ϵ and the weights W_1 and W_2 are the choices that the designer has to make judiciously.

4.4 Solvability Conditions

The H_∞ loop-shaping controller design problem using LMI approach will be described in detail in this and the next section. First, in this section, conditions for the exsitence of a suboptimal controller for the generalized plant obtained for coprime factor uncertainity description in previous section (Eq.(21)) would be presented as LMIs. These existence conditions are LMIs which can be easily solved using convex optimization techniques.
The key idea in deriving these existence conditions is to use the LMI based existence conditions presented in [2] for H_∞ control problem by simply replacing the matrices A,B_1,B_2,C_1,C_2,D_{11},D_{12},D_{21} and D_{22} with those obtained for coprime factor uncertainity description (See Eq.(21))used in the design of H_∞ loop-shaping controller. The following subsections give a detailed insight into the Theorems that would be used to derive the existence conditions. The theorems statements are as described in [2].

4.4.1 Generalized Plant Description

For the LFT structure shown in Fig.(3), it has been shown in [1] that $T_{zw} = \mathscr{F}_l(P, K)$, where $\mathbb{F}_l(P, K) = p_{11} + p_{12}K(I - p_{22}K)^{-1}p_{21}$ for

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$

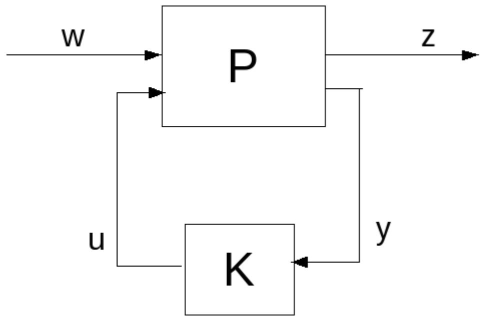

Figure 3: Lower LFT Representation of Plant P and Controller K

The above can be represented in state space form. Let $x \in \mathbb{R}^{n\times 1}$ be the state vector. For outputs $z \in \mathbb{R}^{p_1\times 1}$ and $y \in \mathbb{R}^{p_2\times 1}$ and inputs $w \in \mathbb{R}^{m_1\times 1}$ and $u \in \mathbb{R}^{m_2\times 1}$, we can write the following equations:

$$\dot{x} = Ax + B_1 w + B_2 u \tag{24}$$

$$z = C_1 x + D_{11} w + D_{12} u \tag{25}$$

$$y = C_2 x + D_{21} w + D_{22} u \tag{26}$$

In packed matrix notation, the above can be written as follows.

$$\left[\begin{array}{c} \dot{x} \\ \hline z \\ y \end{array}\right] = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] \left[\begin{array}{c} x \\ \hline w \\ u \end{array}\right]$$

It is straightforward from the above discussion that $A \in \mathbb{R}^{n\times n}$, $B_1 \in \mathbb{R}^{n\times m_1}$, $B_2 \in \mathbb{R}^{n\times m_2}$, $C_1 \in \mathbb{R}^{p_1\times n}$, $C_2 \in \mathbb{R}^{p_2\times n}$, $D_{11} \in \mathbb{R}^{p_1\times m_1}$, $D_{12} \in \mathbb{R}^{p_1\times m_2}$, $D_{21} \in \mathbb{R}^{p_2\times m_1}$ and $D_{22} \in \mathbb{R}^{p_2\times m_2}$. Next, the controller can be written as follows:

$$K = \left[\begin{array}{c|c} A_k & B_k \\ \hline C_k & D_k \end{array}\right]$$

where $A_k \in \mathbb{R}^{k\times k}$, $B_k \in \mathbb{R}^{k\times p_2}$, $C_k \in \mathbb{R}^{m_2\times k}$ and $D_k \in \mathbb{R}^{m_2\times p_2}$ are the state matrices for the controller K, where $u = K(s)y$. We can also write,

$$\dot{x}_k = A_k x_k + B_k y \tag{27}$$

$$u = C_k x_k + D_k y \tag{28}$$

We begin by writing the closed loop structure shown in Fig. (3) in state space form. This can be done by absorbing the controller K into the state space representation to obtain the following:

$$\left[\begin{array}{c} \dot{x} \\ \dot{x}_k \\ \hline z \end{array}\right] = \left[\begin{array}{c|c} A_{cl} & B_{cl} \\ \hline C_{cl} & D_{cl} \end{array}\right] \left[\begin{array}{c} x \\ x_k \\ \hline w \end{array}\right] \tag{29}$$

where the closed loop transfer function from w to z is given by $\mathscr{F}(P,K)(s) = C_{cl}(sI - A_{cl})^{-1}B_{cl} + D_{cl}$. We will next show the derivation of these closed loop matrices.

$$u = Ky$$

From Eq.(26) and (28), we have

$$\begin{aligned}
&y = C_2x + D_{21}w + D_{22}C_kx_k + D_{22}D_ky \\
\Rightarrow\ &y = (I - D_{22}D_k)^{-1}C_2x + (I - D_{22}D_k)^{-1}D_{21}w + (I - D_{22}D_k)^{-1}D_{22}C_kx_k \\
&z = C_1x + D_{11}w + D_{12}C_kx_k + D_{12}D_ky \\
\Rightarrow\ &z = (C_1 + D_{12}D_k(I - D_{22}D_k)^{-1}C_2)x + (D_{12}C_k + D_{12}D_k(I - D_{22}D_k)^{-1}D_{22}C_k)x_k + \\
&\quad (D_{11}w + D_{12}D_k(I - D_{22}D_k)^{-1}D_{21})w \\
&\dot{x} = Ax + B_1w + B_2C_kx_k + B_2D_k(I - D_{22}D_k)^{-1}C_2x + B_2D_k(I - D_{22}D_k)^{-1}D_{21}w + \\
&\quad B_2D_k(I - D_{22}D_k)^{-1}D_{22}C_kx_k \\
\Rightarrow\ &\dot{x} = (A + B_2D_k(I - D_{22}D_k)^{-1}C_2)x + (B_2C_k + B_2D_k(I - D_{22}D_k)^{-1}D_{22}C_k)x_k + \\
&\quad (B_1 + B_2D_k(I - D_{22}D_k)^{-1}D_{21})w \\
&\dot{x}_k = A_kx_k + B_k(I - D_{22}D_k)^{-1}C_2x + B_k(I - D_{22}D_k)^{-1}D_{12}C_kx_k + \\
&\quad B_k(I - D_{22}D_k)^{-1}D_{21}w
\end{aligned}$$

Using the above equations, we can find out A_{cl}, B_{cl}, C_{cl} and D_{cl} to satisfy Eq.(29).

$$A_{cl} = \begin{bmatrix} A + B_2D_k(I - D_{22}D_k)^{-1}C_2 & B_2C_k + B_2D_k(I - D_{22}D_k)^{-1}D_{22}C_k \\ B_k(I - D_{22}D_k)^{-1}C_2 & A_k + B_k(I - D_{22}D_k)^{-1}D_{22}C_k \end{bmatrix}$$

$$B_{cl} = \begin{bmatrix} B_1 + B_2D_k(I - D_{22}D_k)^{-1}D_{21} \\ B_k(I - D_{22}D_k)^{-1}D_{21} \end{bmatrix}$$

$$C_{cl} = \begin{bmatrix} C_1 + D_{12}D_k(I - D_{22}D_k)^{-1}C_2 & D_{12}C_k + D_{12}D_k(I - D_{22}D_k)^{-1}D_{22}C_k \end{bmatrix}$$

$$D_{cl} = D_{11} + D_{12}D_k(I - D_{22}D_k)^{-1}D_{21}$$

Also, as it is clear that if D_{22} is chosen as a null matrix the above equations are simplified to a great extent. So, taking $D_{22} = 0$, we get:

$$A_{cl} = \begin{bmatrix} A + B_2D_kC_2 & B_2C_k \\ B_kC_2 & A_k \end{bmatrix}, B_{cl} = \begin{bmatrix} B_1 + B_2D_kD_{21} \\ B_kD_{21} \end{bmatrix}$$

$$C_{cl} = \begin{bmatrix} C_1 + D_{12}D_kC_2 & D_{12}C_k \end{bmatrix}, D_{cl} = D_{11} + D_{12}D_kD_{21}$$

Hence, $D_{22} = 0$ is one of the assumptions in the LMI approach to H_∞ controller design. Also, we can simplify the above matrices by using the following notation:

$$A_0(\in \mathbb{R}^{(n+k)\times(n+k)}) = \begin{bmatrix} A & 0 \\ 0 & 0_{k\times k} \end{bmatrix}; B_0(\in \mathbb{R}^{(n+k)\times(m_1)}) = \begin{bmatrix} B_1 \\ 0_{k\times m_1} \end{bmatrix} \tag{30}$$

$$C_0(\in \mathbb{R}^{(p_1)\times(n+k)}) = \begin{bmatrix} C_1 & 0_{p_1\times k} \end{bmatrix}; \mathscr{B}(\in \mathbb{R}^{(n+k)\times(m_2+k)}) = \begin{bmatrix} 0_{n\times k} & B_2 \\ I_k & 0_{k\times m_2} \end{bmatrix} \tag{31}$$

$$\mathscr{C}(\in \mathbb{R}^{(p_2+k)\times(n+k)}) = \begin{bmatrix} 0_{k\times n} & I_k \\ C_2 & 0_{p_2\times k} \end{bmatrix}; \mathscr{D}_{12}(\in \mathbb{R}^{(p_1)\times(m_2+k)}) = \begin{bmatrix} 0_{p_1\times k} & D_{12} \end{bmatrix} \tag{32}$$

$$\mathscr{D}_{21}(\in \mathbb{R}^{(p_2+k)\times(m_1)}) = \begin{bmatrix} 0_{k\times m_1} \\ D_{21} \end{bmatrix} \tag{33}$$

This notation is helpful because it only depends on plant data and as we will see in the following expression the closed loop matrices will depend affinely on controller data if the above notation is used.

$$A_{cl}(\in \mathbb{R}^{(n+k)\times(n+k)}) = A_0 + \mathscr{B}\theta\mathscr{C}; B_{cl}(\in \mathbb{R}^{(n+k)\times m_1}) = B_0 + \mathscr{B}\theta\mathscr{D}_{21}$$

$$C_{cl}(\in \mathbb{R}^{p_1\times(n+k)}) = C_0 + \mathscr{D}_{12}\theta\mathscr{C}; D_{cl}(\in \mathbb{R}^{p_1\times m_1}) = D_{11} + \mathscr{D}_{12}\theta\mathscr{D}_{21};$$

where

$$\theta(\in \mathbb{R}^{(m_2+k)\times(p_2+k)}) = \begin{bmatrix} A_k & B_k \\ C_k & D_k \end{bmatrix}$$

The above can be verified by simple substitution.

Other than the assumption that $D_{22} = 0$, we also assume that the plant is stabilizable and detectable. Hence, using LMI approach to design H_∞ controller only these two assumptions are needed. This is one of the advantages of the LMI approach compared to the DGKF method as mentioned in previous sections. Now, we present the solvability conditions for the existence of a suboptimal controller for the generalized plant described above.

4.4.2 Controller Existence Conditions

Theorem 1 Consider the proper plant $P(s)$ and assume that the two assumptions mentioned above hold true. Define

$$\mathscr{P} := \begin{bmatrix} \mathscr{B}^T & 0_{(k+m_2)\times m_1} & \mathscr{D}_{12}^T \end{bmatrix} \tag{34}$$

$$\mathscr{D} := \begin{bmatrix} \mathscr{C} & \mathscr{D}_{21} & 0_{(k+p_2)\times p_1} \end{bmatrix} \tag{35}$$

and let $W_{\mathscr{P}}$ and $W_{\mathscr{D}}$ be the basis of null spaces of $\mathscr{P}$ and $\mathscr{D}$, respectively. Then the set of $\gamma-$suboptimal controllers of order k is nonempty iff there exists some $(n+k)\times(n+k)$ positive definite matrix X_{cl} such that:

$$W_{\mathscr{P}}^T \Phi_{X_{cl}} W_{\mathscr{P}} < 0 \tag{36}$$

$$W_{\mathscr{D}}^T \Psi_{X_{cl}} W_{\mathscr{D}} < 0 \tag{37}$$

where

$$\Phi_{X_{cl}} := \begin{bmatrix} A_0 X_{cl}^{-1} + X_{cl}^{-1} A_0^T & B_0 & X_{cl}^{-1} C_0^T \\ B_0^T & -\gamma I & D_{11}^T \\ C_0 X_{cl}^{-1} & D_{11} & -\gamma I \end{bmatrix} \tag{38}$$

$$\Psi_{X_{cl}} := \begin{bmatrix} A_0^T X_{cl} + X_{cl} A_0 & X_{cl} B_0 & C_0^T \\ B_0^T X_{cl} & -\gamma I & D_{11}^T \\ C_0 & D_{11} & -\gamma I \end{bmatrix} \tag{39}$$

Proof:
From the Bounded Real Lemma described in Section(2.2), for some matrix $X_{cl} = X_{cl}^T > 0 \in \mathbb{R}^{(n+k)\times(n+k)}$, we can write that $\|G_{cl}\|_\infty < \gamma$ is equivalent to

$$\begin{bmatrix} A_{cl}^T X_{cl} + X_{cl} A_{cl} & X_{cl} B_{cl} & C_{cl}^T \\ B_{cl}^T X_{cl} & -\gamma I & D_{cl}^T \\ C_{cl} & D_{cl} & -\gamma I \end{bmatrix} < 0 \tag{40}$$

where G_{cl} in packed matrix notation can be written as,

$$G_{cl} = \left[\begin{array}{c|c} A_{cl} & B_{cl} \\ \hline C_{cl} & D_{cl} \end{array}\right].$$

Now, using Eq.(39) and notation given in (30), we can write Eq.(40) as follows:

$$\begin{bmatrix} A_0^T X_{cl} + \mathscr{C}^T \theta^T \mathscr{B}^T X_{cl} + X_{cl} A_0 + X_{cl} \mathscr{B} \theta \mathscr{C} & X_{cl} B_0 + X_{cl} \mathscr{B} \theta \mathscr{D}_{21} & C_0^T + \mathscr{C}^T \theta^T \mathscr{D}_{12}^T \\ B_0^T X_{cl} + \mathscr{D}_{21}^T \theta^T \mathscr{B}^T X_{cl} & -\gamma I & D_{11}^T + \mathscr{D}_{21}^T \theta^T \mathscr{D}_{12}^T \\ C_0 + \mathscr{D}_{12} \theta \mathscr{C} & D_{11} + \mathscr{D}_{12} \theta \mathscr{D}_{21} & -\gamma I \end{bmatrix} < 0$$

Now observing that,

$$\mathscr{D}^T \theta^T \mathscr{P}_{X_{cl}} = \begin{bmatrix} \mathscr{C}^T \theta^T \\ \mathscr{D}_{21} \theta^T \\ 0 \end{bmatrix} \begin{bmatrix} \mathscr{B}^T X_{cl} & 0 & \mathscr{D}_{12}^T \end{bmatrix} = \begin{bmatrix} \mathscr{C}^T \theta^T \mathscr{B}^T X_{cl} & 0 & \mathscr{C}^T \theta^T \mathscr{D}_{12}^T \\ \mathscr{D}_{21} \theta^T \mathscr{B}^T X_{cl} & 0 & \mathscr{D}_{21} \theta^T \mathscr{D}_{12}^T \\ 0 & 0 & 0 \end{bmatrix}$$

We have,

$$\Psi_{X_{cl}} + \mathscr{D}^T \theta^T \mathscr{P}_{X_{cl}} + \mathscr{P}_{X_{cl}}^T \theta \mathscr{D} < 0 \tag{41}$$

This equation is solvable if and only if,

$$W_{\mathscr{P}_{X_{cl}}}^T \Psi_{X_{cl}} W_{\mathscr{P}_{X_{cl}}} < 0 W_{\mathscr{D}}^T \Psi_{X_{cl}} W_{\mathscr{D}} < 0 \tag{42}$$

where $W_{\mathscr{P}_{X_{cl}}}$ and $W_{\mathscr{D}}$ are the basis of the null spaces of $\mathscr{P}_{X_{cl}}$ and $\mathscr{D}$ respectively. For the proof of this lemma, the reader is referred to Section 3 of [2]. Now, we will reduce the Eq. (42) as follows to prove the statement of the theorem.

$$I = \begin{bmatrix} X_{cl} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} X_{cl}^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

So, we can write,

$$\mathscr{P} = \mathscr{P}_{X_{cl}} \begin{bmatrix} X_{cl}^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

Using the above and finding basis of the null spaces, we have,

$$W_{\mathscr{P}_{X_{cl}}} = \begin{bmatrix} X_{cl}^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} W_{\mathscr{P}}$$

where $W_{\mathscr{P}_{X_{cl}}}$ and $W_{\mathscr{P}}$ are the basis of the null space of $\mathscr{P}_{X_{cl}}$ and $\mathscr{P}$ respectively. Now, rewriting Eq.(42), we get

$$W_{\mathscr{P}}^T \left(\begin{bmatrix} X_{cl}^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \Psi_{X_{cl}} \begin{bmatrix} X_{cl}^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \right) W_{\mathscr{P}} < 0$$

Simplifying, we finally get,

$$W_{\mathscr{P}}^T \Phi_{X_{cl}} W_{\mathscr{P}} < 0$$

Hence, the theorem is proved. The next theorem eradicates the need to compute X_{cl}^{-1} and also shows the role of plant parameters in the solvability conditions.

Theorem 2 Consider a continuous time plant $P(s)$ of order n and minimal realization given in Eq.(16). Let $\mathscr{N}_R$ and $\mathscr{N}_S$ be bases of null space of (B_2^T, D_{12}^T) and (C_2, D_{21}) respectively. The suboptimal H_∞ problem of parameter γ is solvable if and only if the following three conditions (formulated in LMI) are satisfied.

$$\begin{bmatrix} \mathscr{N}_R & 0 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} AR + RA^T & RC_1^T & B_1 \\ C_1 R & -\gamma I & D_{11} \\ B_1^T & D_{11}^T & -\gamma I \end{bmatrix} \begin{bmatrix} \mathscr{N}_R & 0 \\ 0 & I \end{bmatrix} < 0 \quad (43)$$

$$\begin{bmatrix} \mathscr{N}_S & 0 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} A^T S + SA & SB_1 & C_1^T \\ B_1^T S & -\gamma I & D_{11}^T \\ C_1 & D_{11} & -\gamma I \end{bmatrix} \begin{bmatrix} \mathscr{N}_S & 0 \\ 0 & I \end{bmatrix} < 0 \quad (44)$$

$$\begin{bmatrix} R & I \\ I & S \end{bmatrix} \geq 0 \quad (45)$$

Proof: Our main aim with this Theorem is to express the existence conditions derived in Theorem 1 in terms of plant parameters. We express X_{cl} and X_{cl}^{-1} as follows

$$X_{cl} := \begin{bmatrix} S & N \\ N^T & * \end{bmatrix}; \; X_{cl}^{-1} := \begin{bmatrix} R & M \\ M^T & * \end{bmatrix} \quad (46)$$

where $R, S \in \mathbb{R}^{n\times n}$ and $M, N \in \mathbb{R}^{n\times k}$. Consider Eq.(36) and substitute X_{cl} from Eq.(46). We have,

$$\Phi_{X_{cl}} = \begin{bmatrix} AR + RA^T & AM & B_1 & RC_1^T \\ M^T A^T & 0 & 0 & M^T C_1^T \\ B_1^T & 0 & -\gamma I & D_{11}^T \\ C_1 R & C_1 M & D_{11} & -\gamma I \end{bmatrix} \tag{47}$$

From Eq.(34), we can find out the basis of null space of $\mathscr{P}$ in the following form

$$W_{\mathscr{P}} = \begin{bmatrix} W_1 & 0 \\ 0 & 0 \\ 0 & I_{m_1} \\ W_2 & 0 \end{bmatrix}$$

where

$$\mathscr{N}_R := \begin{bmatrix} W_1 \\ W_2 \end{bmatrix}$$

is any basis of null space of $\begin{bmatrix} B_2^T & D_{12}^T \end{bmatrix}$.
Above can be found out be finding indiviudal elements of $W_{\mathscr{P}}$ that satisfy the following which make $W_{\mathscr{P}}$ the basis of null space of $\mathscr{P}$.

$$\mathscr{P} W_{\mathscr{P}} = 0 \tag{48}$$

Now, the Eq.(36) can be reduced using the above and Eq.(47) to read as given below:

$$\begin{bmatrix} W_1 & 0 \\ 0 & I_{m_1} \\ W_2 & 0 \end{bmatrix}^T \begin{bmatrix} AR + RA^T & B_1 & RC_1^T \\ B_1^T & -\gamma I & D_{11}^T \\ C_1 R & D_{11} & -\gamma I \end{bmatrix} \begin{bmatrix} W_1 & 0 \\ 0 & I_{m_1} \\ W_2 & 0 \end{bmatrix} < 0 \tag{49}$$

Substituting for W_1 and W_2 from above, we can write

$$\begin{bmatrix} \mathscr{N}_R & 0 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} AR + RA^T & RC_1^T & B_1 \\ C_1 R & -\gamma I & D_{11} \\ B_1^T & D_{11}^T & -\gamma I \end{bmatrix} \begin{bmatrix} \mathscr{N}_R & 0 \\ 0 & I \end{bmatrix} < 0 \tag{50}$$

which proves the first statement of the Theorem given in Eq.(43).
Using a similar approach we will next prove the second statement of the Theorem in Eq.(44). Consider Eq.(37), i.e. $W_{\mathscr{Q}} \Psi_{X_{cl}} W_{\mathscr{Q}} < 0$. We will reduce this equation of Theorem 1 in the terms of plant parameters. From Eq.(46), we can write $\Psi_{X_{cl}}$ as follows:

$$\Psi_{X_{cl}} = \begin{bmatrix} A^T S + SA & A^T N & SB_1 & C_1^T \\ N^T A & 0 & N^T B_1 & 0 \\ B_1^T S & B_1^T N & -\gamma I & D_{11}^T \\ C_1 & 0 & D_{11} & -\gamma I \end{bmatrix} \tag{51}$$

From Eq.(35), we have

$$\mathscr{D} = \begin{bmatrix} 0_{k\times n} & I_{k\times k} & 0_{k\times m_1} & 0_{k\times p_1} \\ C_{2_{p_2\times n}} & 0_{p_2\times k} & D_{21_{p_2\times m_1}} & 0_{p_2\times p_1} \end{bmatrix} \tag{52}$$

Using the above, and using the proper dimensions as mentioned above to solve $\mathscr{D}W_{\mathscr{D}} = 0$ to get $\mathscr{W}_{\mathscr{D}}$, the basis of null space of $\mathscr{D}$. We get

$$W_{\mathscr{D}} = \begin{bmatrix} W_3 & 0 \\ 0 & 0 \\ W_4 & 0 \\ 0 & I \end{bmatrix} \tag{53}$$

where

$$\mathscr{N}_S = \begin{bmatrix} W_3 \\ W_4 \end{bmatrix} \tag{54}$$

is the basis of null space of $\begin{bmatrix} C_2 & D_{21} \end{bmatrix}$. Substituting, we get,

$$\begin{bmatrix} \mathscr{N}_S & 0 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} A^TS + SA & SB_1 & C_1^T \\ B_1^TS & -\gamma I & D_{11}^T \\ C_1 & D_{11} & -\gamma I \end{bmatrix} \begin{bmatrix} \mathscr{N}_S & 0 \\ 0 & I \end{bmatrix} < 0 \tag{55}$$

The third statement in Eq.(45) is clearly seen as $X_{cl} > 0$ and from Eq.(46), we get R and S satisfying Eq.(45). Hence, we have proved the Theorem to obtain the existence conditions for a γ-suboptimal H_∞ controller formulated in LMI. Next, we will use these conditions to derive the conditions for existence for H_∞ loop-shaping design.

4.4.3 Conditions for H_∞ loop-shaping design

For the generalized plant expression obtained for coprime factor uncertainity description to design an H_∞ loop-shaping controller, we can obtain the solvability conditions for existence of a γ-suboptimal controller as follows. From the generalized plant expression in Eq.(21) we have.

$$B_2 = B_{n\times p}; D_{12} = \begin{bmatrix} I_p \\ 0_{m\times p} \end{bmatrix} \tag{56}$$

To find $\mathscr{N}_R$, i.e. the basis of null space of $\begin{bmatrix} B_2^T & D_{12}^T \end{bmatrix}$ we have the rank of null space $= (n+m)$ and $\mathscr{N}_R \in \mathbb{R}^{(n+p+m)\times(n+m)}$. We get

$$\mathscr{N}_R = \begin{bmatrix} I_n & 0_{n\times m} \\ -B_{p\times n}^T & 0_{p\times m} \\ 0_{m\times n} & I_m \end{bmatrix} \tag{57}$$

Similarly, we see that for $\mathscr{N}_S$, the basis of null space of $\begin{bmatrix} C_2 & D_{21} \end{bmatrix}$, we have for H_∞ loop-shaping design from Eq.(21)

$$C_2 = C_{m\times n} \; ; \; D_{21} = I_m \tag{58}$$

The rank of $\mathscr{N}_S \in \mathbb{R}^{(m+n)\times n}$ hence will be equal to n. We can then write

$$\mathscr{N}_S = \begin{bmatrix} I_n \\ -C_{m\times n} \end{bmatrix} \tag{59}$$

4.4.4 LMI Formulation

From $\mathscr{N}_R$ and $\mathscr{N}_S$ obtained above, we can reduce the solvability conditions for the existence of a suboptimal H_∞ loop-shaping controller as three LMIs given below. These three LMIs can be solved to find out the value of γ, R and S which can then be used to design the controller as shown in the following section. The three LMIs which need to be solved for R, S and γ are:

$$\begin{bmatrix} AR + RA^T - \gamma BB^T & RC^T & -L \\ CR & -\gamma I & I \\ -L^T & I & -\gamma I \end{bmatrix} < 0 \tag{60}$$

$$A^T S + SA + C^T L^T S + SLC - \gamma C^T C < 0 \tag{61}$$

$$\begin{bmatrix} R & I \\ I & S \end{bmatrix} \geq 0 \tag{62}$$

Other than the three convex conditions mentioned above, if we want to ensure that the obtained controller is of order k where $k < n$, then another conditions should be satisfied.

$$\text{Rank}(I - RS) \leq k$$

Some examples simulating a few physical systems on MATLAB to solve the above equation are presented in the next section.

4.5 Examples and Simulations

The controller solvability conditions LMIs for a few numerical examples and physical systems have been solved and presented in this section.

4.5.1 A BLDC Hub Motor Driving a Three Wheeled Mobile Robot

Using system identification technique in MATLAB with the data of the input voltage to the motor and the output velocity measurements, the following estimated TF was obtained. We will show that an H_∞ loop-shaping controller design can be done by showing the existence of the γ-suboptimal controller by solving the LMIs developed in previous section. The plant model is given by,

$$G(s) = \frac{0.001011s^2 + 0.0006967s + 1.801 \times 10^{-6}}{s^3 + 1.16s^2 + 0.3351s + 0.007859} \tag{63}$$

To obtain the shaped plant, we choose a first order weight W_1 as follows and $W_2 = 1$ is chosen

$$W_1(s) = \frac{10s + 3}{s(s + 8)} \tag{64}$$

Using the above, we find the following minimal state space representation of the shaped plant. Using this, we solve the LMIs formulated in the previous section

$$A = \begin{bmatrix} -9.1600 & -2.4038 & -0.6722 & -0.0157 & 0 \\ 4.0000 & 0 & 0 & 0 & 0 \\ 0 & 1.0000 & 0 & 0 & 0 \\ 0 & 0 & 1.0000 & 0 & 0 \\ 0 & 0 & 0 & 1.0000 & 0 \end{bmatrix};\ B = \begin{bmatrix} 0.0625 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{65}$$

$$C = \begin{bmatrix} 0 & 0.0404 & 0.0400 & 0.0084 & 0.0000 \end{bmatrix};\ D = 0 \tag{66}$$

The valus of γ obtained on solving the LMIs is 1.3937. We take another similar example next.

4.5.2 A Wheeled Mobile Robot System

A second order SISO model of a wheeled mobile robot has been derived in [6]. We use the TF model given to check if a γ-suboptimal H_∞ loop-shaping controller exists for the system or not.

$$G(s) = \frac{0.3961}{0.2256s^2 + 0.3645s + 1.469} \tag{67}$$

With the same weights as in previous example, we get the minimal state space realization of the shaped plant as follows

$$A = \begin{bmatrix} -1.6157 & -3.2558 & 2.5000 & 0.7500 \\ 2.0000 & 0 & 0 & 0 \\ 0 & 0 & -8.0000 & 0 \\ 0 & 0 & 1.0000 & 0 \end{bmatrix};\ B = \begin{bmatrix} 0 \\ 0 \\ 4 \\ 0 \end{bmatrix} \tag{68}$$

$$C = \begin{bmatrix} 0 & 0.8779 & 0 & 0 \end{bmatrix};\ D = 0 \tag{69}$$

Solving the LMIs we get the value of $\gamma = 1.2761$.

4.5.3 F-16 Fighter Aircraft Model

Finally, we show a MIMO system example. The longitudinal model of a F-16 fighter aircraft is shown below. The state space representation as shown below has been taken from [7] Pages 370-371. For details on definition of states, the equilibrium point about which linearization is done and other details the reader is referred to [7].

$$A = \begin{bmatrix} -0.3220 & 0.0640 & 0.0364 & -0.9917 & 0.0003 & 0.0008 & 0 \\ 0 & 0 & 1 & -0.0037 & 0 & 0 & 0 \\ -30.6492 & 0 & -3.6784 & 0.6646 & -0.7333 & 0.1315 & 0 \\ 8.5396 & 0 & -0.0254 & -0.4764 & -0.0319 & -0.0620 & 0 \\ 0 & 0 & 0 & 0 & -20.2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -20.2 & 0 \\ 0 & 0 & 0 & 57.2958 & 0 & 0 & -1 \end{bmatrix} \tag{70}$$

$$B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 20.2 & 0 \\ 0 & 20.2 \\ 0 & 0 \end{bmatrix} \tag{71}$$

$$C = \begin{bmatrix} 0 & 0 & 0 & 57.2958 & 0 & 0 & -1 \\ 0 & 0 & 57.2958 & 0 & 0 & 0 & 0 \\ 57.2958 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 57.2958 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}; \; D = [0]_{4\times 2} \tag{72}$$

Solving the LMIs we obtain $\gamma = 3.1587$. With this we move on to controller design at the end of which we will show H_∞ controller design for these physical systems.

4.6 Controller Design

Using the Bounded Real Lemma developed in Section 2.2 and solving the corresponding LMI, we can find out the controller by using the values of R and S obtained by solving the controller existence conditions. The steps to reconstruct the controller are given below.

1. Compute two full column rank matrices M and N such that

$$MN^T = I - RS \tag{73}$$

2. From the matrices M and N calculated in the first step, X_{cl} can be found out as follows. Recall that, as defined previously, X_{cl} is a positive definite

matrix $in\mathbb{R}^{(n+k)\times(n+k)}$ which satisfies Eq. (46). Solving the following equation, X_{cl} can be computed.

$$\begin{bmatrix} S & I \\ N^T & 0 \end{bmatrix} = \begin{bmatrix} I & R \\ 0 & M^T \end{bmatrix} \tag{74}$$

Note that Eq.(74) is always solvable when $S > 0$ and M has full column rank.

3. As seen in Theorem 1, from Eq.(40) we can solve the following Bounded Real Lemma inequality:

$$\begin{bmatrix} A_{cl}^T X_{cl} + X_{cl} A_{cl} & X_{cl} B_{cl} & C_{cl}^T \\ B_{cl}^T X_{cl} & -\gamma I & D_{cl}^T \\ C_{cl} & D_{cl} & -\gamma I \end{bmatrix} = \Psi_{X_{cl}} + \mathscr{D}^T \theta^T \mathscr{P}_{X_{cl}} + \mathscr{P}_{X_{cl}}^T \theta \mathscr{D} < 0 \tag{75}$$

4. The above LMI can be solved for θ to obtain the controller parameters.

$$\theta = \begin{bmatrix} A_k & B_k \\ C_k & B_k \end{bmatrix}. \tag{76}$$

4.7 Controller Design Examples

5 Conclusions and Future Work

In this project we covered the basics of robust control theory and studied the theory towards general H_∞ control framework. Using which, H_∞ loop-shaping design procedure was studied. The project initially dealt with robustness analysis of uncertain systems. The same was demonstrated by simulating various physical systems on MATLAB. After the analysis, controller design was considered using the LMI approach. Advantages of this approach over the conventional iterative approaches was made clear through simulations and examples. Finally, an H_∞ loop-shaping controller design procedure using LMI approach was demonstrated. This approach was simulated on MATLAB for various physical systems. The future work in this project would be to apply the controller design theory developed to a mobile robotic system to achieve robust stability and performance. The controller developed up till now is a linear robust controller. Hence, to successfully apply this controller to a non-linear system various challenges would need to be overcome.

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