

CALIBRATING THE NELSON-SIEGEL-SVENSSON MODEL BY GENETIC ALGORITHM

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ABSTRACT

Accurately fitting the term structure of interest rates is critical to central banks and other market participants. The Nelson-Siegel and Nelson-Siegel-Svensson models are probably the best-known models for this purpose due to their intuitive appeal and simple representation. However, this simplicity comes at a price. The difficulty in calibrating these models is twofold. Firstly, the objective function being minimized during the calibration procedure is nonlinear and has multiple local optima. Secondly, there is strong co-dependence among the model parameters. As a result, their estimated values behave erratically over time. To avoid these problems, we apply a heuristic optimization method, specifically the Genetic Algorithm approach, and show that it is able to construct reliable interest rate curves and stable model parameters over time, regardless of the shape of the curves.

Keywords [Yield Curve Modelling · Nelson-Siegel Model · Svensson Model]

1 Introduction

An accurate representation of the term structure of interest rates plays a key role in the area of financial risk management and financial economics. For example, investors often use the yield curve as a key representation of future economic activity and inflation. As pointed out in [Nielsen, 2015], its slope indicates the short term interest rates and is closely followed by investors. As such, estimation and representation of the term structure of interest rates has been an active area of research for several years.

Smoothed bootstrap is one widely used method of approximating the term structure. Practitioners use market data to bootstrap discrete spot rates and then use various spline methods to fit a smooth and continuous curve from the set of discrete rates [McCulloch, 1971, 1975, Vasicek and Fong, 1982, Shea, 1984, Steeley, 1991, Adams and Deventer, 1994, Fisher et al., 1994, Waggoner, 1997]. Because the smoothed bootstrap method has been criticized for having undesirable economic properties or being a “black box” model [Seber and Wild, 2003], Nelson and Siegel [1987] proposed another parametric method in which the entire forward curve is characterised by a single set of parameters. These parameters represent the long-run level of interest rates, the slope of the curve, and humps in the curve. This is a parsimonious model [Nelson and Siegel, 1987], and is consistent with the findings of Litterman and Scheinkman [1991] that most of the variation in returns on all fixed-income securities represented by the yield curve can be explained in terms of three attributes of the yield curve: level, steepness and curvature. Svensson [1994, 1995] later extend Nelson and Siegel [1987] by including a second hump/trough factor to allow a more complex range of term structure shapes.

Diebold and Rudebusch [2013] claim that the Nelson-Siegel (NS) and Nelson-Siegel-Svensson (NSS) models are probably the best-known yield curve fitting tools among central bankers, in industry, and in research. There are three main reasons:

- i This model is able to construct yields for all maturities without using interpolation,

- ii Factors obtained from the NS model have meaningful economic interpretations,
- iii Empirical test results show the calibrated yield curves fit the market data reasonably well, especially in the long end of the term structure.

First of all, NS and NSS models are extensively used by central banks and monetary policy makers, due to their intuitive appeal and ease of implementation [Anderson and Sleath, 2001, Monetary & Economic Department, 2005, Coroneo et al., 2011]. Meanwhile, fixed-income portfolio managers employ them to immunize their portfolios [Barrett et al., 1995]. The NS and NSS models also have recently experienced growing popularity in academic research. Anderson and Sleath [2001], Diebold and Li [2006], Lorenčič [2016] use the NS model as the benchmark to judge other models' performance in estimating and forecasting interest rate curves. They find that the NS model appears to be much more accurate at long horizon predictions than various standard benchmark forecasts. Martellini and Meyfredi [2007] employ the NS model to extract time-varying parameters used as proxies for factors affecting the shape of the yield curve and observe that the NS model achieves a high quality of fit with an average adjusted R^2 higher than 99%, and never below 99%, in their estimation data. Furthermore, when these curve-affecting factors are combined with a Student copula, Martellini and Meyfredi [2007] are able to generate Value-at-Risk (VaR) estimates that strongly dominate standard VaR estimates in the out-of-sample tests. The NS model is also used to bootstrap the riskless spot rate curves when calculating the credit risk spread for the U.S. market [Xie et al., 2008]. Finally, Coroneo et al. [2011] conclude that the NS yield curve model is compatible with the no-arbitrage constraints on the U.S. market.

Although the NS and NSS models have many appealing features, the estimation of model parameters is quite difficult. The estimates are usually obtained by minimising the difference between the zero rates and the corresponding observed market rates. This objective function is highly non-linear. One common approach uses nonlinear regression techniques to obtain all model parameters simultaneously. Cairns and Pritchard [2001] use the maximum likelihood technique to fit the model and find that the time series of model parameter estimates are very unstable. From one day to the next, the results can jump from one location to another. They refer to this phenomenon as a catastrophic jump: the size of the jump will typically be much larger than would be consistent with the corresponding changes in price. As a result, the search starting value is very important and the probability of getting local optima for the next day is very high even if one uses today's global optimum.

Another approach used in practice is to linearize the nonlinear objective function by fixing the shape parameter in the model that causes the nonlinearity, and then to solve a much simpler problem. There are two common ways to conduct the linearization. Nelson and Siegel [1987] propose the "grid search with OLS regression" method which is still one of the most popular market practices. More specifically, they conduct the Ordinary Least Squares (OLS) regression to estimate the remaining model parameters, conditional upon a grid of the fixed shape parameter. The second branch of research focuses on identifying a good value for the shape parameter and fixing it over the whole time series of term structures. For example, Barrett et al. [1995], Fabozzi et al. [2005] fix the shape parameter at 3 for annualized returns. Diebold and Li [2006], Coroneo et al. [2011] select the fixed shape parameter where the maturity at which the maximized loading on the medium-term is exactly 30 months to avoid hundreds of potentially changing numerical optimizations under the grid search methodology. In what follows, we will show that our methodology does not depend on any such ad-hoc procedures.

Although the linearized version of the model has a simpler objective function, it is typically not convex and has many local optima. Therefore, classical techniques, based on derivatives of the objective function, are not appropriate, and can even generate negative long-term rates [Gilli et al., 2010, Gilli and Schumann, 2010]. Regardless, researchers use such techniques. Anderson and Sleath [2001], Gurkaynak et al. [2007] and many more document that the estimated parameters demonstrate instability, in that they sometimes jump from one day to the next, often with little actual movement in underlying bond prices. The instability of model parameter estimates also comes from the selection of the fixed shape parameter. De Pooter [2007] concludes that the calibrated model parameters can take extreme values under different fixed shape parameters. Therefore, it is critical to carefully determine the shape parameter for each sample data and estimation window. Under the grid search method, although no shape parameter is pre-defined, trying hundreds or thousands of different starting points in the optimization is computationally expensive. However, the grid search method does not guarantee the final solution is globally optimal. Additionally, the NS and NSS models are well known for high multicollinearity between the slope and hump factor which can inflate the variance of the estimators. Consequently, they are more difficult to calibrate [Cabrera et al., 2014, Annaert et al., 2013].

Because of these limitations in the traditional linear estimation of NS and NSS model parameters, scholars are trying different approaches to deal with the ill-conditioned optimization objective function. Annaert et al. [2013] propose to switch to ridge regression whenever they detect high multicollinearity under a normal grid search OLS regression procedure. Although ridge regression is able to improve the out-of-sample predictability at the short and long ends of the term structure with lower mean absolute prediction errors and to obtain a lower volatility for some NS model parameters than traditional grid search, the shape parameter estimation still presents erratic time series behaviour.

Meanwhile, the grid search method has to be conducted for every observation date during the estimation period. This is computationally expensive, and the accuracy is limited by the size of the grid [Lakhany and Mausser, 2000].

In order to deal with the multiple local minima in the objective function, the high-correlation problem among regressors, potential catastrophic jump phenomena, and long computation time associated with the calibration of the NS and NSS models, we apply a heuristic optimization method, or specifically a Genetic Algorithm (GA) approach. We believe that GA can handle almost all of the above issues, due to the following features [Goldberg, 1989]:

- i Ability to handle non-differentiable, nonlinear and multimodal objective functions.
- ii Highly parallel implementation to cope with the computationally expensive objective functions.
- iii Ease of use; few control variables to steer the minimization. These variables are robust and easy to choose.
- iv Good convergence properties; ability to escape from local optima.
- v Good stability properties; ability to use today's solution as a good starting value for the next observation day.

By using the GA method, this paper finds that all model parameters estimated on each observation date are consistent with their corresponding economic meaning.

Our test results show that the NS yield curve has difficulty fitting the entire term structure, especially the long-term maturities of twenty years or more. The reason for this is that the convexity of the yield curve tends to pull down the yields on longer-term securities, giving the yield curve a concave shape at longer maturities. The NS specification, while fitting shorter maturities quite well, tends to have the forward rates asymptote too quickly to be able to capture the convexity effects at longer maturities. We find, however, that the NSS specification is sufficiently rich to capture the double humps in the curve. On the other hand, when the term structure of the calibrated yield curve shortens as the estimation window moves forward, the NS model is preferable.

When calibrating NS and NSS model parameters for a group of yield curve scenarios on a given date, we find that the two hump position parameters (λ, κ) are very similar across all scenarios. Therefore, we only need to calibrate these two parameters once, and can re-use them to estimate the remaining 3 or 4 factors for all remaining scenarios. By representing the scenarios in terms of the NSS parameters, we are able to compress the risk factor space and represent the yield curve sensitivities as a linear combination of the sensitivities on the NSS parameters.

The estimation of model parameters no longer shows erratic time series behaviours under the GA approach. This demonstrates that the GA method makes the estimates more stable. Furthermore, because today's solution is a good starting point for the next day's search, one can use considerably fewer iterations to obtain suitable results. This, combined with the fact that GA is well-suited to parallel computing, makes it possible to obtain useful results in a reasonable amount of time. This is especially true for a long estimation window, and when compared with the grid search method.

This paper makes several contributions to the interest rate calibration literature. First of all, although the GA method has become more standard in several fields of science, its application in estimation and modelling in finance appears to still be limited. Our paper makes up for this gap. Additionally, we successfully solve the multiple-optima and multi-collinearity problem in the calibration of the NS and NSS models and obtain all model parameters simultaneously. Considering that the NS and NSS models are one of the most popular interest rate models among market practitioners and regulators, our paper paves the way for them to fully understand and predict the interest rates and related derivatives.

The remainder of the paper proceeds as follows. Section 2 introduces the NS and NSS models. Section 3 explains the GA technique used in this paper. Section 4 presents the data and highlights relevant points. Section 5 illustrates the results and compares the model results with the behaviour of market data. It also examines through-time behaviour of the approach. Section 6 concludes.

2 Nelson-Siegel-Svensson Model

In this section we introduce the model. Our development closely follows Annaert et al. [2013]. In the Nelson-Siegel framework, the forward rate, $f(\tau)$ is modelled by the following expansion:

$$f(\tau) = \beta_0 f_0(\tau) + \beta_1 f_1(\tau) + \beta_2 f_2(\tau) \quad (1)$$

where $\{\beta_i\}_{i=1}^3$ are the coefficients of expansion and the basis functions $f_i(\tau)$ are given by:

$$\left. \begin{aligned} f_0(\tau) &= 1 \\ f_1(\tau) &= \exp(-\tau/\lambda) \\ f_2(\tau) &= (\tau/\lambda)\exp(-\tau/\lambda) \end{aligned} \right\} \quad (2)$$

The basis functions are not arbitrary, but represent the basic components of an interest rate term structure. The first basis function represents the long-term level. The second function represents an exponential decay and allows the term structure to slope upwards (with $\beta_1 < 0$) or downwards (with $\beta_1 > 0$). The third function produces a butterfly effect; viz., $\beta_2 > 0$ will produce a hump and $\beta_2 < 0$ will produce a trough. Finally, the parameter λ determines the location of the hump or the trough, as well as its steepness.

The interest rate term structure is obtained by averaging the forward rate curve (1) over different maturities. We will denote this curve by $R(\tau) = (1/\tau) \int_0^\tau f(s)ds$. It is easy to check that

$$R(\tau) = \beta_0 F_0(\tau) + \beta_1 F_1(\tau) + \beta_2 F_2(\tau) \quad (3)$$

where

$$\left. \begin{aligned} F_0(\tau) &= 1 \\ F_1(\tau) &= \frac{1 - \exp(-\tau/\lambda)}{\tau/\lambda} \\ F_2(\tau) &= \frac{1 - \exp(-\tau/\lambda)}{\tau/\lambda} - \exp(-\tau/\lambda). \end{aligned} \right\} \quad (4)$$

Equations (3) and (4) sufficiently describe term structures with a single hump. But in practical cases, market data may contain a second hump, generally at a longer maturity, in form of a downward bend. In order to model these cases, Svensson [1995] suggested using a second hump/trough term. The Nelson-Siegel-Svensson model is described by:

$$R(\tau) = \beta_0 F_0(\tau) + \beta_1 F_1(\tau) + \beta_2 F_2(\tau) + \beta_3 F_3(\tau) \quad (5)$$

where

$$\left. \begin{aligned} F_0(\tau) &= 1 \\ F_1(\tau) &= \frac{1 - \exp(-\tau/\lambda)}{\tau/\lambda} \\ F_2(\tau) &= \frac{1 - \exp(-\tau/\lambda)}{\tau/\lambda} - \exp(-\tau/\lambda) \\ F_3(\tau) &= \frac{1 - \exp(-\tau/\kappa)}{\tau/\kappa} - \exp(-\tau/\kappa). \end{aligned} \right\} \quad (6)$$

Our main focus in this paper is to propose a robust and stable algorithm for calibrating the coefficients $\{\beta_i\}_{i=1}^3 \cup \{\lambda\}$ for the Nelson-Siegel model given by (3) and (4) or the set $\{\beta_i\}_{i=1}^4 \cup \{\lambda, \kappa\}$ for the Nelson-Siegel-Svensson model described by equations (5) and (6). In what follows, we refer to this as the calibration problem.

3 Calibrating the Nelson-Siegel(-Svensson) Model

In Section 2 we described the Nelson-Siegel and Nelson-Siegel-Svensson models. We now propose methodologies for solving the calibration problem. As mentioned before, each term in the NSS expansion represents a feature of the term structure, and, as such, the parameters cannot take an arbitrary range. This means that fitting to the market data is a bound constrained optimization problem. The method we propose due to its flexibility is based on genetics and natural selection. In this section we shall briefly touch upon the features of the Genetic Algorithm (GA) approach. Interested readers are referred to the comprehensive monograph by Haupt and Haupt [2004]. In short, GA comes with several features that are very suitable to the present application. These include the ability to escape local minima, parallel implementation, and the ability to provide multiple solutions instead of a single local minimum.

We have implemented GA with the following features: we start with a randomly generated population using Quasi Random sequence generator [BRODA, 2016]. In the case of NSS, this sequence is generated in six dimensional space $(\{\beta_i\}_{i=1}^4, \lambda, \kappa)$. The first parameter, N , that we choose is the number of genes in the initial population. We impose this number to be a multiple of 4, which facilitates partitioning the population into winners and losers (half each), and parental gender (half again). In case a repeated evaluation is desired under changing market data we take $N = 1,024$; for all other cases, $N = 512$ should suffice. We next define the fitness function which, simply put, is the negative distance between the market data and the model values. The algorithm goes through the fundamental steps of natural selection, crossover and blending/mutation. With this fitness function we are able to categorize the population as elite, winners and losers. The number of elite members is taken to be three for these experiments. The tournament to decide the mother and father of a newborn is also based on three randomly selected winners of each gender. Then we apply the crossover and blending techniques to obtain the newborn. We use Eshelman-Shaffer Blending as described in Haupt and Haupt [2004]. In our implementation, the random weight used for blending is generated so that the new

point is never outside the bounds imposed by the problem. This obviates the need for trial and error. Finally, we apply mutations. The mutation rate for these experiments has been kept adaptive to the history of evolution. This idea is subject to criticism. From a purist's approach, the need to adaptively modify the mutation rate may suggest that the GA is not working. However, since our stopping criterion includes a limit on the number of generations, we are unable to consistently obtain optimal results without incorporating this feature. We claim that, with a sufficiently large number of generations, a constant mutation rate should produce equivalent results.

From the practitioner's point of view, our GA implementation is equipped with the following three non-standard features. First, we improve the blending operation in such a manner that the resulting operation yields parameter values within the original bounds constraints. For example, consider the following blending of a parameter p_i from the parents $p_i^{[MA]}$ and $p_i^{[PA]}$ (where the meaning of $PAter$ and $MAter$ should be clear):

$$\left. \begin{aligned} \hat{p}_i^1 &= \mu p_i^{[MA]} + (1 - \mu) p_i^{[PA]} \\ \hat{p}_i^2 &= \hat{\mu} p_i^{[MA]} + (1 - \hat{\mu}) p_i^{[PA]} \end{aligned} \right\} \quad (7)$$

where $\hat{p}_i^j, j \in \{1, 2\}$ is the new pair of blended values. The numbers μ and $\hat{\mu}$ in formula (7) are generated from a uniform distribution $U(a, b)$. In our implementation of this blending, we choose the parameters a and b in such a manner that any number generated from this distribution will respect the bounds on the parameter p_i . The next feature is the adaptive mutation feature discussed above. Another feature, that we empirically justify, is skewing the initial population with a selected group of genes. We use this feature to address stability of the NS(S) parameters. The keen observer will note that the hypothesis of the convergence proofs that rely on uniform distribution of the starting population is violated [Haupt and Haupt, 2004]. Practically, the induced set of genes has cardinality much smaller (about 2%-5%) than the population size. For example, in our tests we use 16 returning winners out of a population size of 1,024 or 512.

4 Data for testing the calibration fit

The set of data that we have used to test our proposed algorithm are OIS rates for the EUR currency taken from two different periods. By OIS rates we mean rates that were bootstrapped from Overnight Index Swap (OIS) pricing data. An OIS is an interest rate swap that exchanges a fixed rate payment with a floating rate payment on a fixed notional amount for the life of the swap. In this case, the floating rate is based on a specific published index of daily overnight rates in the currency of the swap. The term for the OIS typically ranges from 1 week to 2 years, however longer-termed maturities on the order of 10 years are becoming more common [Hull and White, 2012]. At maturity, the parties agree to exchange an amount, based on the agreed notional, as a difference of accruals from the fixed rate and the geometric average of the floating index rate. Since there is no exchange of notional, these swaps are virtually devoid of credit risk. The index rates used are computed differently for different markets. For the U.S. market it is the daily effective federal funds rate.

In Table (1)¹ we provide data for the dates September {22, 23, 27, 28, 29, 30}, 2011 (note the weekend is excluded), for the EUR OIS curve. The first analysis is done on the data corresponding to September 22. We selected this curve due to the existence of two humps. It will allow us to demonstrate the applicability of the NSS model. It was also chosen because of its complicated shape. In fact, careful observation suggests the presence of a third hump at the end of the term structure. In the settings of the algorithm we run, we shall use bounds that will avoid capturing this third hump. Strictly speaking, the NSS model itself is only meant to capture two such features.

Simply producing a good fit to complicated data is not the only objective of this paper. As mentioned in the introduction, one of the biggest challenges associated with the NSS model is the stability of the parameters under daily changing market conditions. To test how well our algorithm handles the stability issue, we iteratively fit to a full week of data, provided in Table (1).

The OIS data described above has clearly been processed. In order to show that our methodology works equally well for raw data, we will apply our proposed algorithm to the data used to build USD yield curves.² The following groups of instruments are chosen:

- i The first group of instruments is used to build the short end of the curve. These are typically the cash/deposit securities available for up to one year tenor. However, they are commonly used for curve construction for up to

¹This data is sourced from Reuters, and is bootstrapped in conjunction with cubic spline interpolation of the continuously compounded rate.

²This data is sourced from Bloomberg

Table 1: EUR OIS Curve : September 22-30, 2011

	2011-09-22	2011-09-23	2011-09-26	2011-09-27	2011-09-28	2011-09-29	2011-09-30
Term	Rate %	Rate %	Rate %	Rate %	Rate %	Rate %	Rate %
30	0.9231	0.8910	0.8928	0.9187	0.9470	0.8896	0.8909
60	0.7720	0.7328	0.7461	0.8125	0.8405	0.7959	0.8104
90	0.7105	0.6726	0.6842	0.7552	0.7843	0.7624	0.7655
180	0.5995	0.5793	0.6073	0.7012	0.7265	0.7005	0.6656
270	0.5710	0.5481	0.5807	0.6859	0.7193	0.7033	0.6652
365	0.5685	0.5417	0.5783	0.6815	0.7219	0.7135	0.6662
455	0.5743	0.5465	0.5864	0.6852	0.7338	0.7288	0.6784
545	0.5879	0.5612	0.6049	0.6969	0.7481	0.7477	0.6953
635	0.6127	0.5860	0.6342	0.7209	0.7756	0.7768	0.7192
730	0.6446	0.6211	0.6734	0.7550	0.8114	0.8110	0.7513
820	0.6840	0.6665	0.7149	0.7945	0.8538	0.8501	0.7883
910	0.7277	0.7158	0.7622	0.8412	0.8985	0.8953	0.8296
1,000	0.7786	0.7710	0.8187	0.8941	0.9492	0.9478	0.8790
1,095	0.8341	0.8313	0.8769	0.9513	1.0061	1.0062	0.9353
1,185	0.8944	0.8927	0.9371	1.0114	1.0686	1.0703	0.9940
1,275	0.9519	0.9534	0.9983	1.0732	1.1326	1.1352	1.0557
1,365	1.0138	1.0195	1.0606	1.1364	1.1995	1.1982	1.1181
1,460	1.0765	1.0832	1.1232	1.2005	1.2670	1.2660	1.1815
1,550	1.1358	1.1458	1.1852	1.2623	1.3330	1.3304	1.2443
1,640	1.1947	1.2080	1.2473	1.3251	1.3969	1.3936	1.3064
1,730	1.2505	1.2669	1.3070	1.3857	1.4585	1.4547	1.3665
1,825	1.3052	1.3252	1.3652	1.4440	1.5182	1.5134	1.4250
1,915	1.3586	1.3804	1.4202	1.4989	1.5743	1.5701	1.4806
2,005	1.4085	1.4325	1.4718	1.5505	1.6274	1.6228	1.5322
2,095	1.4574	1.4839	1.5228	1.6015	1.6787	1.6743	1.5826
2,190	1.5045	1.5330	1.5712	1.6503	1.7274	1.7242	1.6312
2,280	1.5489	1.5791	1.6167	1.6964	1.7737	1.7716	1.6774
2,370	1.5892	1.6220	1.6591	1.7396	1.8168	1.8155	1.7202
2,460	1.6294	1.6640	1.7011	1.7822	1.8594	1.8585	1.7620
2,555	1.6676	1.7038	1.7411	1.8222	1.8997	1.8999	1.8023
2,645	1.7037	1.7407	1.7801	1.8612	1.9391	1.9393	1.8404
2,735	1.7367	1.7749	1.8145	1.8955	1.9740	1.9756	1.8756
2,825	1.7697	1.8086	1.8488	1.9298	2.0090	2.0111	1.9099
2,920	1.8011	1.8407	1.8822	1.9636	2.0436	2.0452	1.9428
3,100	1.8591	1.9002	1.9415	2.0251	2.1065	2.1082	2.0045
3,285	1.9143	1.9559	1.9966	2.0831	2.1647	2.1670	2.0606
3,465	1.9646	2.0066	2.0493	2.1371	2.2182	2.2213	2.1135
3,650	2.0153	2.0558	2.1003	2.1893	2.2700	2.2731	2.1642
4,380	2.1992	2.2377	2.2837	2.3744	2.4636	2.4678	2.3539
5,475	2.3927	2.4252	2.4626	2.5574	2.6534	2.6602	2.5420
7,300	2.4628	2.4788	2.5164	2.6236	2.7240	2.7330	2.6156
9,125	2.4241	2.4419	2.4725	2.5820	2.6753	2.6799	2.5558
10,950	2.3637	2.3873	2.4207	2.5314	2.6193	2.6182	2.4905
14,600	2.3514	2.3672	2.4080	2.5220	2.6163	2.6110	2.4864
18,250	2.4092	2.4278	2.4633	2.5743	2.6732	2.6646	2.5443

Table 2: USD Bonds

Cusip	Coupon	Maturity Date	BidYield	MidYield	Issue Date
912796XG9	0	2020-08-27	0.099132786	0.090235681	2020-02-27
9127962H1	0	2020-09-24	0.106476035	0.100137247	2020-03-26
9127962T5	0	2020-10-29	0.111559138	0.106486972	2020-04-30
912796UC1	0	2021-01-28	0.124619268	0.119529857	2020-01-30
9127962Q1	0	2021-04-22	0.131905229	0.130635968	2020-04-23
9127963S6	0	2021-07-15	0.13447065	0.129391733	2020-07-16
912828Z60	1.375	2022-01-31	0.1561949	0.1459276	2020-01-31
9128282P4	1.875	2022-07-31	0.1594827	0.14802215	2017-07-31
9128283U2	2.375	2023-01-31	0.1649988	0.15592475	2018-01-31
912828S92	1.25	2023-07-31	0.1706244	0.1603927	2016-08-01
9128285Z9	2.5	2024-01-31	0.1954981	0.18908325	2019-01-31
912828Y87	1.75	2024-07-31	0.2134859	0.20969775	2019-07-31
912828Z52	1.375	2025-01-31	0.2410912	0.23515535	2020-01-31
912828Y79	2.875	2025-07-31	0.272484	0.2680819	2018-07-31
9128286A3	2.625	2026-01-31	0.3144243	0.309736	2019-01-31
912828Y95	1.875	2026-07-31	0.358822	0.3550592	2019-07-31
912828Z78	1.5	2027-01-31	0.3991395	0.3956219	2020-01-31
91282CAD3	0.375	2027-07-31	0.4339846	0.43171135	2020-07-31
9128283W8	2.75	2028-02-15	0.4596749	0.45677725	2018-02-15
9128284N7	2.875	2028-05-15	0.4735112	0.4697927	2018-05-15
9128284V9	2.875	2028-08-15	0.4898128	0.4862126	2018-08-15
9128286B1	2.625	2029-02-15	0.5144641	0.51104645	2019-02-15
912828YB0	1.625	2029-08-15	0.5303851	0.52870125	2019-08-15
912828Z94	1.5	2030-02-15	0.5543428	0.5527359	2020-02-18
912828ZQ6	0.625	2030-05-15	0.5773463	0.57652695	2020-05-15
912810FM5	6.25	2030-05-15	0.5428465	0.52192415	2000-02-15
912810FP8	5.375	2031-02-15	0.5607927	0.5487083	2001-02-15
912810FT0	4.5	2036-02-15	0.7299671	0.7275026	2006-02-15
912810QK7	3.875	2040-08-15	0.9737774	0.9697724	2010-08-16
912810RN0	2.875	2045-08-15	1.1736387	1.1715444	2015-08-17
912810SN9	1.25	2050-08-15	1.2167632	1.2161395	2020-05-15

three or six months. We recommend including only a single cash instrument with tenor equal to that of the curve being built.

- ii The second group of instruments is used to build the curve beyond the term points covered by the first group of instruments. This group contains USD coupon bonds.

Our data is based on all available USD (zero-)coupon bond yields trading on July 28, 2020 – a total of 357 instruments. Based on a typical use case, we selected 1, 2, 3, 6, 9-month, and 1, 1.5, 2, 2.5, 3, 3.5, 4, 4.5, 5, 5.5, 6, 6.5, 7, 7.5, 8, 8.5, 9, 9.5, 10, 12, 15, 20, 25, 30-year term nodes. From these 357 bonds, we finally choose 29 instruments whose maturities are closest to our selected term nodes. The list of bonds appear in Table (2).

5 Calibration Results

A fundamental reason for choosing GA is due to the ease of doing bound-constrained optimization. As mentioned in Section (2), the parameters of the NS(S) model are not arbitrary. They represent different features of the yield curve that are observed in the market. We can leverage this fact to provide consistent bounds to the parameters of the NS(S) model during a GA run. For example, based on economic intuition, β_0 is close to the empirical long-term spot rate and, as such, cannot be negative for the data we are analyzing. We also leverage the fact that λ, κ are the range parameters. Based on this, we select the bounds that appear in Table (3).

Next we discuss the parameters of the algorithm. Any algorithm that is affected by a large number of parameters is often criticized as being difficult to use. Practitioners may be indifferent to the selection of parameters: how they affect the solution and what the optimal values are. GA is no different in this regard. As such, our policy was to fix the parameters based on the recommendations made in Haupt and Haupt [2004]. In Table (4) we present those parameters.

Table 3: Bounds Constraints for the NSS Model - OIS Curve

Parameter	Lower Bound	Upper Bound
β_0	0.0	0.10
β_1	-0.10	1.0
β_2	-2.0	2.0
β_3	0.0	2.0
λ	0.0	4.0
κ	4.0	30.0

Table 4: Parameters for GA Experiments

Parameter	Value
Random Number Generator	Sobol Quasi Random Generator
Minumum Population Size	512
Maximum Population Size	1,024
Number of Elite Genes	3
Tournament Size	3
Minimum Mutation Rate	0.2
Maximum Mutation Rate	0.5
Returning Genes	64
Stopping Criterion	Maximum Iterations

It should be noted that even though our implementation of the algorithm allows for more complex stopping criteria than shown in Table (4), for the purpose of this study we imposed a fixed number of generations. A fixed number of generations makes it convenient to compare different runs of the algorithm. In practice, however, one would choose a more suitable stopping criteria such as those discussed in Haupt and Haupt [2004]. Table (4) indicates that the number of returning genes is set to 64. A smaller value could also suffice; these are the genes that move from one time step to another.

Before we discuss the experiments to fit the OIS Curve presented in Table (1), we need to demonstrate and empirically prove an earlier claim. Our choice of data was driven by the fact that it exhibits three turning points - one of which at the rear end will be ignored. Due to this, the NS model will be limited in its ability to fit the data. Our first result in Figure (1) shows exactly this. Note that the bound for λ was taken to be the union of the bounds for λ and κ in Table (3).

Now that we have confirmed the inadequacy of the NS model to handle the September 22, 2011 data, we examine how well the NSS model handles it. In Figure (2) we show the fit using the initial population of 512 genes. We let the algorithm run for a total of 5,000 generations.

We then repeat the experiment with a population size of 1,024. We immediately notice a significant improvement in the results for the same number of fixed iterations. However, we also see evidence of the difficulty with the NSS model mentioned in Section (1) and reported by several researchers and practitioners. The change in the error is significant but not drastic; however, the difference in the set of parameters produced by these two runs is substantial. This tells us that the objective function admits several local minima. Furthermore, it also shows why practitioners have had difficulty in obtaining a stable set of parameters over time. The key requirement here is that, as data changes from day to day in a smooth manner, the parameters of the NSS model should change proportionately. This is the basic requirement of a stable algorithm. Drastic changes in the calibrated parameter set with only a small changes in the data make the fitting technique unreliable.

The data we presented in Section (4) will allow us to demonstrate the effectiveness of our algorithm to handle the stability issues. In Figure (4) we show the daily absolute movement of the term structure of yields. These movements are not very small. In some cases they seem to be reducing the size of the first acute hump. Without any stability measure, these data may generate completely different solutions.

Our approach is to modify the GA by allowing one to selectively infuse a set of genes in the initial population. We choose these to be the winners of a previous GA run. From Table (4) we see that the number of such genes returning from one run to another is 64 for our experiments. The movement of yield rates is highly variable from day to day, as seen in Figure (4). As such, we cannot test for any pattern in the movement of the parameters. The best we can do is to check that they do not move drastically. For the stability test, we introduce another feature to our algorithm. We run the first experiment again for a minimum number of 10,000 generations. We then compensate for this extra time by

Figure 1: Fitting OIS Curve using Nelson Siegel Model

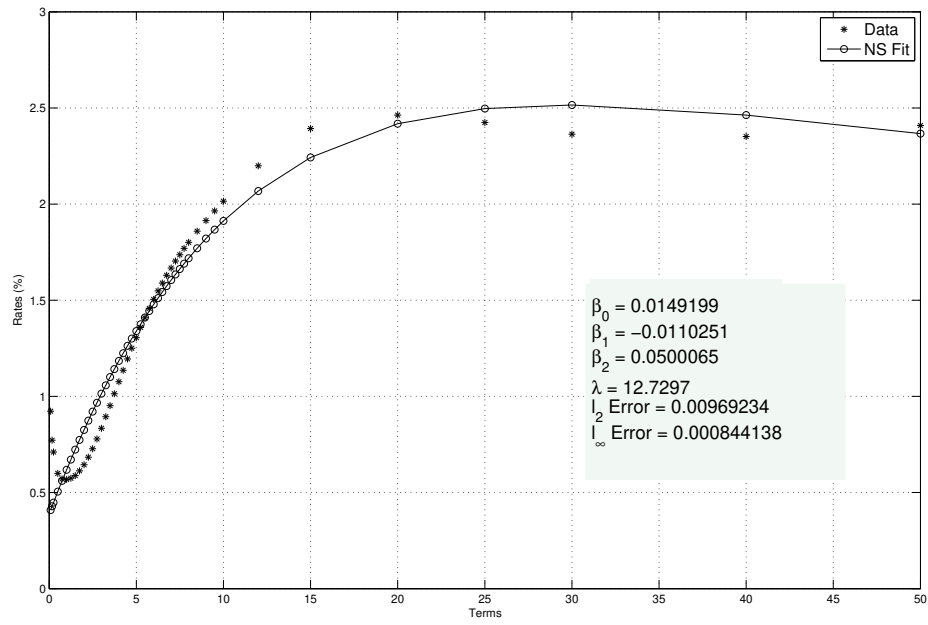


Figure 2: Fitting OIS Curve using Nelson Siegel Svensson Model, Population Size = 512

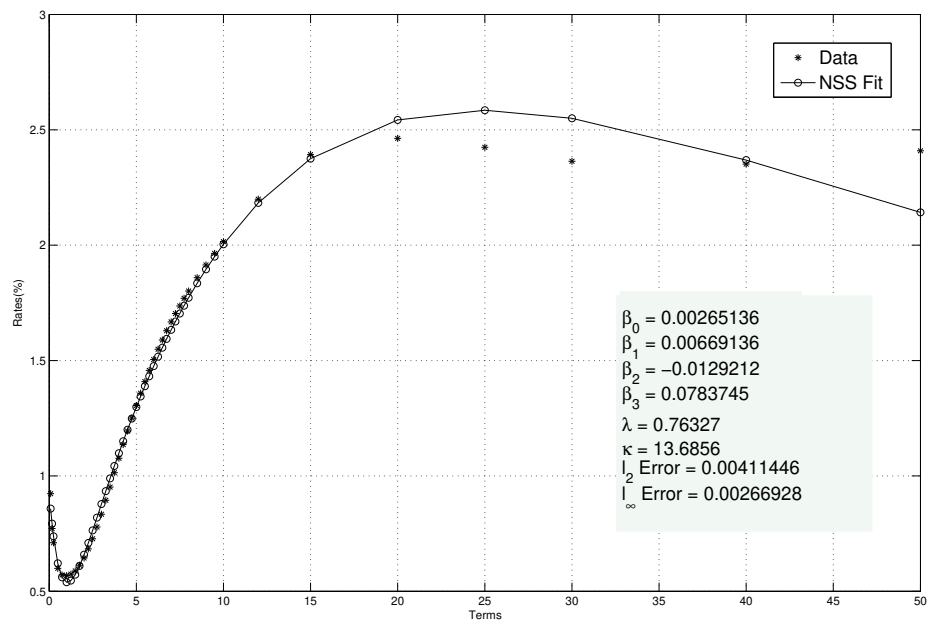


Figure 3: Fitting OIS Curve using Nelson Siegel Svensson Model, Population Size = 1,024

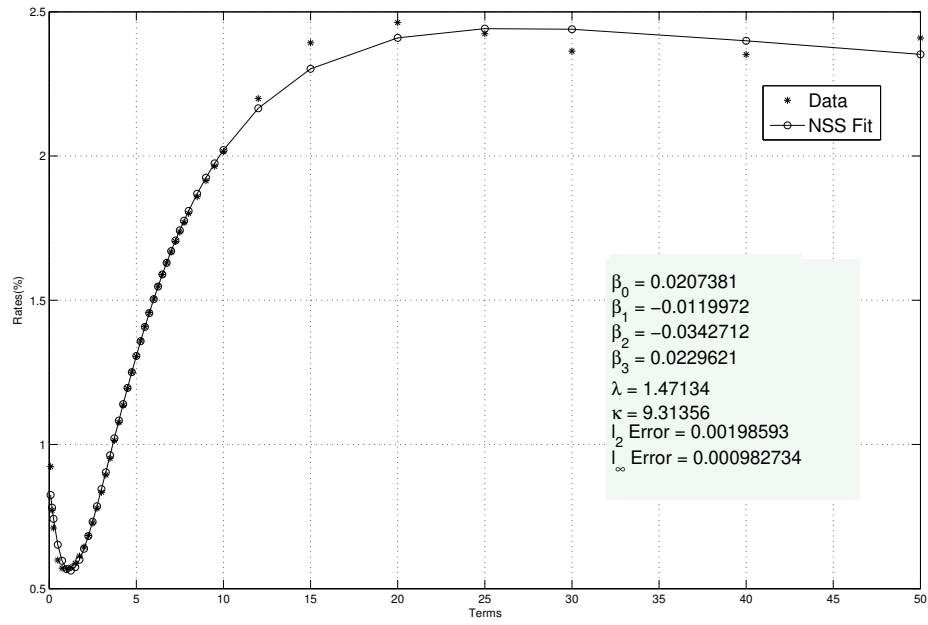


Figure 4: Daily Movements in OIS Curve During the Period September 22-30, 2011

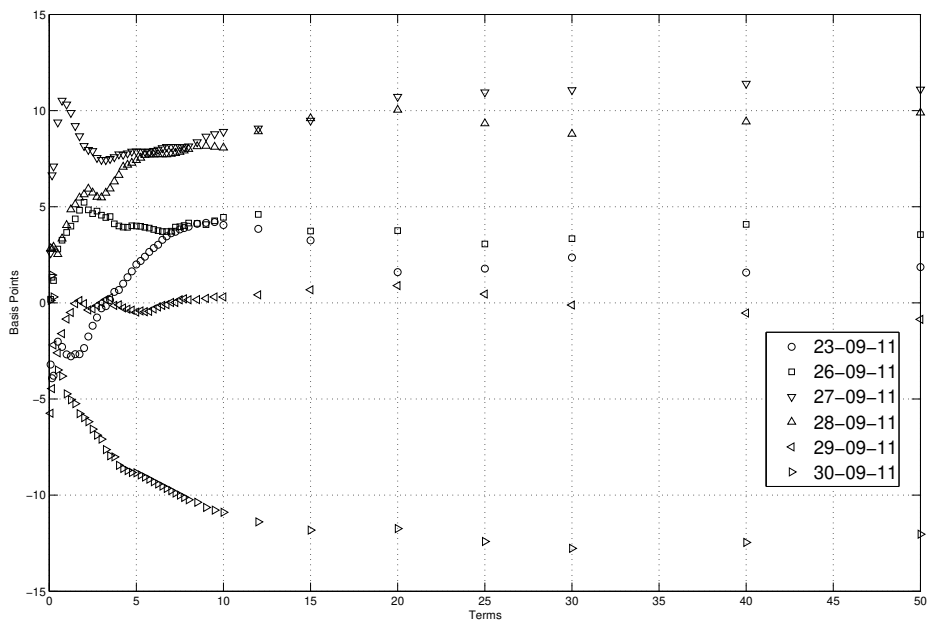


Table 5: Daily Changes in NSS Model Parameters for the Period September 22-30, 2011

β_0	β_1	β_2	β_3	λ	κ	l_2 Error	l_∞ Error
0.020780	-0.011995	-0.034771	0.023232	1.484620	9.050420	0.001950	0.000942
0.020662	-0.012027	-0.035353	0.023943	1.405530	9.186470	0.001921	0.000908
0.021055	-0.012224	-0.035449	0.024367	1.430270	8.800990	0.001910	0.000852
0.022008	-0.012384	-0.035328	0.024750	1.454020	9.090340	0.001906	0.000824
0.022405	-0.012521	-0.034934	0.026274	1.430930	9.459170	0.002099	0.000872
0.022416	-0.013029	-0.034438	0.027155	1.497920	9.110210	0.002057	0.000844
0.021804	-0.012981	-0.034442	0.027327	1.654630	8.129460	0.001763	0.000833

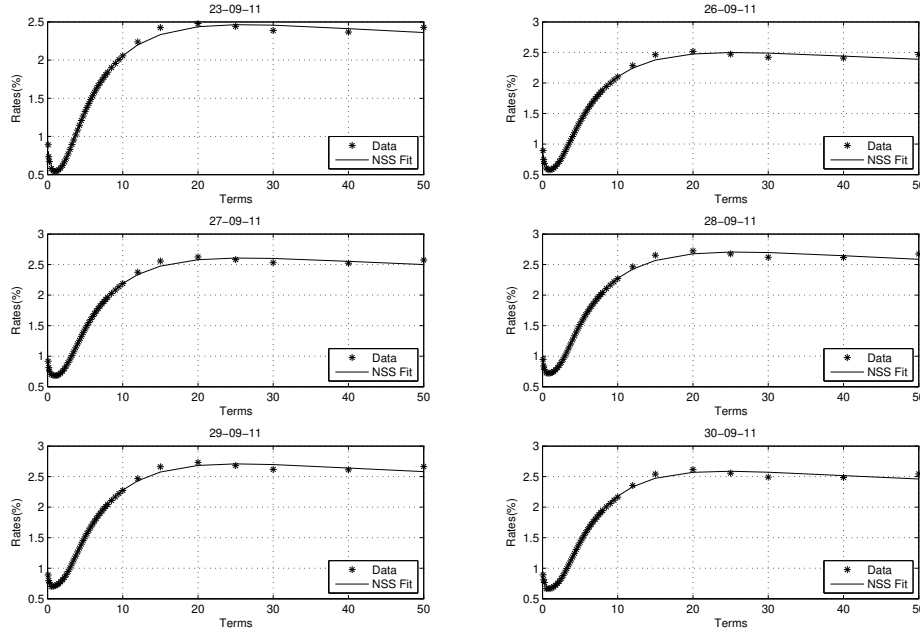


Figure 5: Fitting the OIS Curves from the Period September 22-30, 2011

running the subsequent data for only 1,000 generations. The parameters and the errors for these runs are presented in Table (5).

Table (5) also displays the tracking error in the 2-norm and the ∞ -norm in columns 7 and 8 respectively. It is gratifying to note that the errors in the subsequent runs are much less even with only 1,000 generations. Not only that, the last row shows a very good fit considering the fact that the movements are greatest for the last day in the period under study. In practice, one could incorporate a logic where the number of generations depends on some way on the deviation of the term structure from one day to another. One could simply run a full 10,000 generations every week or every other week. We do not investigate this idea here as the results show that tracking error is second lowest on the day of maximum deviation. We shall revisit these results in Section (6). In Figure (5) we show how well the NSS model fits over time. The results are visually impressive.

We conclude this Section by reporting the results of our fit on the raw data described in Table (2). β_1 , β_2 , and β_3 control the slope of the term structure and the degree of curvature. Compared with the OIS data, which has been smoothed by the cubic spline interpolation method, the USD yield curve is harder to capture. As such, we employ a wider range of parameter boundaries in Table (6) for this test.

The NSS fit to raw data appears in Figure (6). We see that the raw data is fitted quite well and observe no overfitting.

Table 6: Bounds Constraints for the NSS Model - USD Yield Curve

Parameter	Lower Bound	Upper Bound
β_0	0.0	0.10
β_1	-1.0	4.0
β_2	-2.0	4.0
β_3	-2.0	8.0
λ	0.0	6.0
κ	6.0	30.0

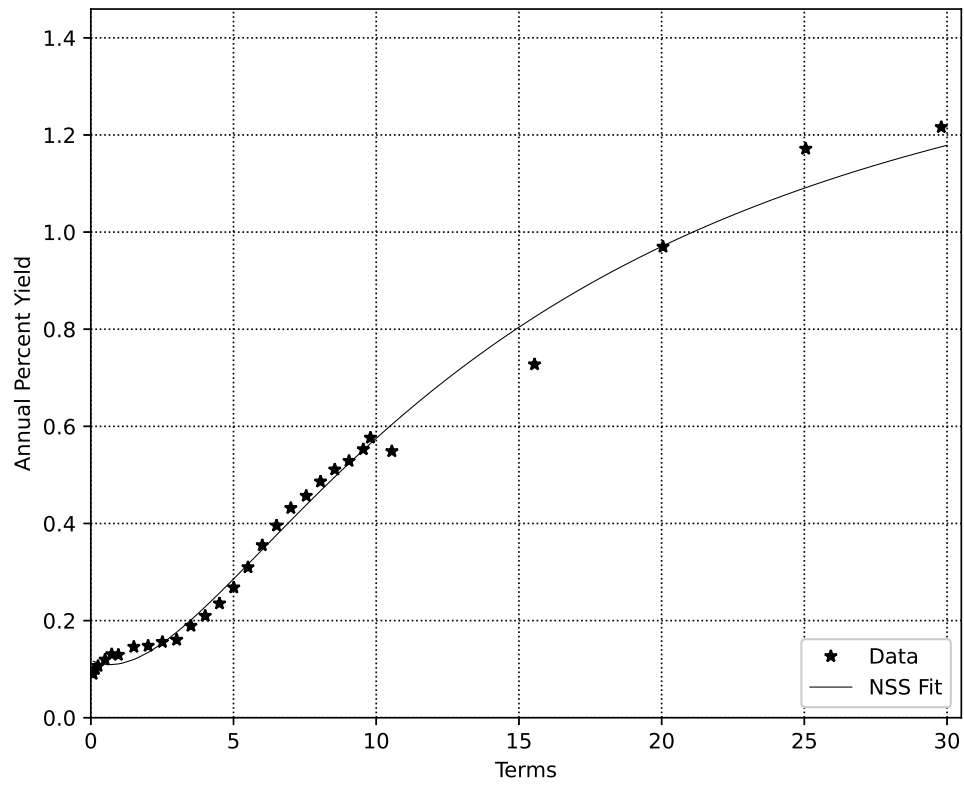


Figure 6: Fitting the USD Bond Data

6 Conclusions and Future Research

We devoted the whole of Section (5) to demonstrating the flexibility and the power of GA in dealing with the problems associated with the calibration of the NSS model. However, some objectivity in this regard is warranted. One is able to argue that the maximum error of about 10bp is rather high, considering the fact that daily movements are generally smaller than 10bp. One can see in Figures (3) and (5) that the third turning point is not well-captured. The algorithm tries to fit it in the least squares sense, resulting in prominent (point to point) tracking errors. Nevertheless, we do not believe that any alternate algorithm will be able to capture this region (along with the whole curve) significantly better. This is a result of the limitation of the NSS model itself. Secondly, fitting to a set of 45 term points is challenging. The fitness function is a simple squared distance metric based on a set of 45 points. Loosely speaking, the algorithm tries to capture data spread over 45 term nodes with a parametric representation with only 6 degrees of freedom. Every algorithm is susceptible to a tracking error of some magnitude. Our approach seems to have handled the raw data effectively, without overfitting.

Having said the above, there is always a room for improvement. Possible avenues of future research to improving the basic GA include:

- **Fitness Function :** One can try a different fitness function. For example, if one needs to capture the turning points accurately, one can use a weighted squared distance metric, allowing greater weights around the turning points.
- **Hybrid Approach :** In practice, one cannot run the GA algorithm indefinitely, hence convergence is an issue. In such cases using a hybrid approach can be rewarding. In this case a GA run can be subsequently be followed by, for example, a non-heuristic algorithm to finely pinpoint the accurate solution.

These are just a few suggestions. This does not necessarily mean that one would obtain a considerably better solution than the one proposed in this paper. By *considerably better*, we mean one that is able to reduce the tracking error to be insignificant when compared to the daily moves in yields. Clearly, the model itself may not allow this possibility.

All in all, our work demonstrates the potential of GA to solve calibration issues mentioned in Section (1). We back our claims by a series of experiments that check the quality of fit as well as the stability of the calibrated parameter set.

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