INTERNATIONAL INSTITUTE OF INFORMATION TECHNOLOGY

HYDERABAD

REAL ANALYSIS (MA4.101) PROJECT

RIEMANN INTEGRATION

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Abstract

In this project, we aim to define Riemann integration and explain its mathematical construct by using the ideas of upper and lower riemann sums. We introduce certain criterias of riemann integrability and arrive at an important theorem that every continuous function is riemann integrable.

We also look at some of the deficiencies of Riemann integral and how it can be overcome using riemann stiljes integral and lebesgue integral. Finally we explore some of the important applications of Riemann integral.

Introduction

Integration is the very basic essence of calculus. Together with differentiation it forms the core of calculus which finds innumerable and indispensable applications in the field of mathematics, physics, astronomy etc. Integration can be viewed as a reverse process of differentiation. However, the Riemann Integral is different from an 'anti-derivative' in the sense that it is calculated for a bounded function within certain limits.

Riemann Integration provides us with the method of calculating the area under the curve by computing the 'Lower Riemann Sum' and the 'Upper Riemann Sum'. It approximates the area under the graph of function f(x) using rectangles and shows that the larger the number of rectangles we consider the more accurate and precise area we get. It also specifies the criteria for a function to be Riemann Integrable. There are also certain shortcomings of the Riemann integral such as only applicable to bounded functions which can be remedied by with the Riemann–Stieltjes integral, and most disappear with the Lebesgue integral.

The Riemann Integral

Before Calculating the Riemann Integral, we need a few concepts and definitions under our belt to understand it better.

Definition:

Suppose a, b ∈ R with a < b. A partition of [a, b] is a finite list of the form $x_0, x_1, \ldots, x_n$, where $a = x_0 < x_1 < \cdots < x_n = b$.

In simple terms, a Partition is used to denote the sub-intervals of [a,b] such that it is the union of those intervals. Moreover,

$$[a,b] = [x_0,x_1]U[x_1,x_2]U\ldots\ldots.U[x_{n-1},x_n]$$

Next we define the supremum and infimum of the function in the interval [a,b].

Inf f = inf{f(x):x ∈ A} and sup f = sup{ f(x) : x ∈ A}.

Suppose f : [a, b] → R is a bounded function and P is a partition $x_0, \ldots, x_n$ of [a, b].

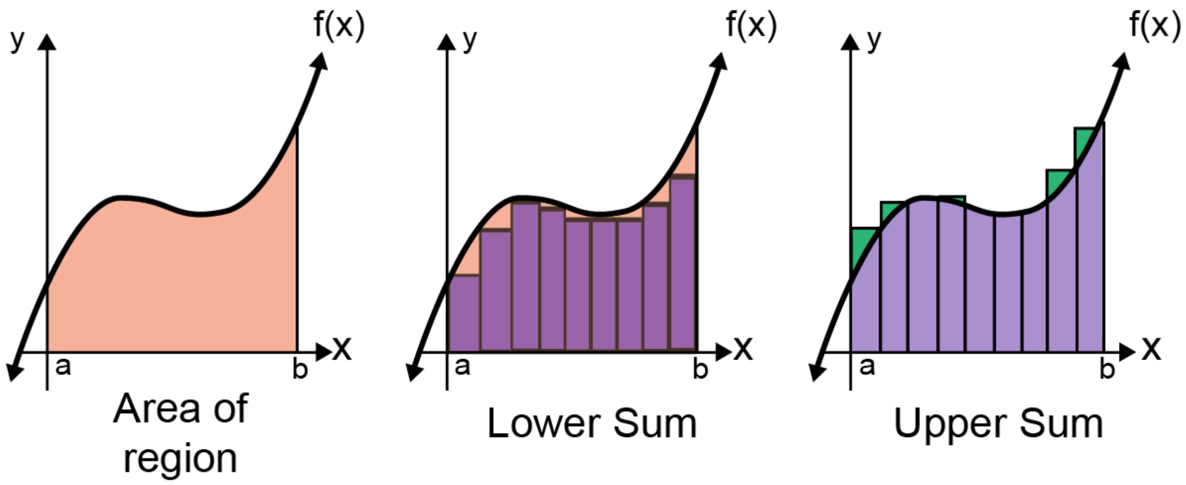
The lower Riemann sum L(f , P, [a, b]) and the upper Riemann sum U(f , P, [a, b]) are defined by

$$L(f,P,[a,b]) = \sum_{j=1}^{n} \left(x_j - x_{j-1}\right) \inf_{[x_{j-1},x_j]} ff,P,[a,b]) = \textstyle\sum\_(j$$

$$= 1)\hat{}\,n\,▒\ (x\_j - x\_(j-1)\ )\ \ (sup)┬[x\_(j-1), x\_j\ ]\ f$$

It should be noted that the lower Riemann sum is slightly less than the area under the graph and the upper Riemann sum is slightly more than the area under the graph.

The following figures help us to understand these ideas more clearly. The base of the j^{th} rectangle has length $x_j - x_{j-1}$ and has height $\inf_{[x_{j-1},x_j]} f$ r the lower Riemann Sum and height $\sup_{[x_{j-1},x_j]} f$ for the upper Riemann Sum.



As we can observe that the lower Riemann sums and the Upper Riemann sums are a fairly good approximation of the area under the curve provided the intervals into which [a,b] is divided becomes smaller and smaller. In other words, the more fine a partition is the more accurate area it gives. This is what inspires our next theorem.

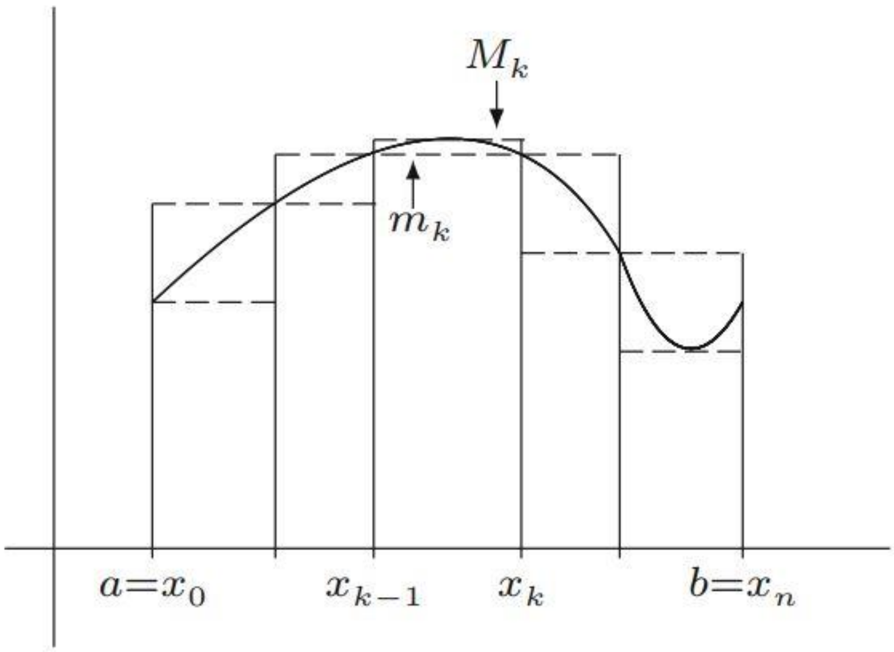
Theorem:

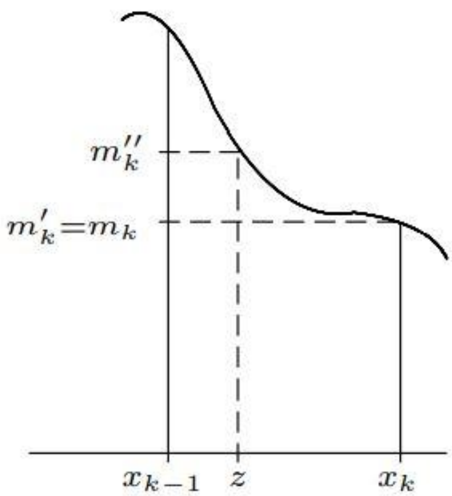
Suppose $f : [a, b] \to \mathbf{R}$ is a bounded function and P, P' are partitions of $[a, b]$ such that the list defining P is a sublist of the list defining P'. Then

$$L(f, P, [a, b]) \leq L\big(f, P', [a, b]\big) \leq U\big(f, P', [a, b]\big) \leq U(f, P, [a, b])$$

Proof:

Consider what happens when we refine P by adding a single point z to some subinterval $[x_{k-1}, x_k]$ of P.





Focusing on the lower sum, we have

$$\begin{aligned} m_k(x_k - x_{k-1}) &= m_k(x_k - z) + m_k(z - x_{k-1}) \\ &\leq m_k'(x_k - z) + m_k''(z - x_{k-1}) \end{aligned}$$

Where

$$m_k' = inf\{f(x): x \in [z, x_k]\} \text{ and } m_k'' = inf\{f(x): x \in [x_{k-1}, z]\}$$

are each necessarily as large or larger than m_k.

By induction, we have L(f,P,[a,b]) ≤ L(f,P',[a,b]), and an analogous argument holds for the upper sums.

Now we define the Lower and Upper Riemann Integrals as follows:

Let P be the collection of all possible partitions of the interval [a, b]. The upper integral of f is defined to be U(f) = inf{U(f,P_e,[a,b]) : $P_e \in P$}. In a similar way, define the lower integral of f by L(f) = sup{L(f,P_e,[a,b]) : $P_e \in P$}

Criteria for Riemann Integrability

A bounded function on a closed bounded interval is called Riemann integrable if its lower Riemann integral equals its upper Riemann integral.

The function f is Riemann Integrable if and only if U(f) = L(f), and when this is true its value is denoted by $\int_a^b f \;\; = U(f) = L(f)$

In this case the Integral is assigned the value equal to the Upper Riemann Integral which is equal to the Lower Riemann Integral.

So we can define the criteria of integrability as follows:

A bounded function f is integrable on [a, b] if and only if, for every ε > 0, there exists a partition P of [a, b] such that

$$U(f,P_e) - L(f,P_e) < \varepsilon$$

Proof:

Let ε> 0. If such a partition P exists, then

$U(f) - L(f) \le U(f,P) - L(f,P) < \varepsilon$. Because ε is arbitrary, it must be that U(f) = L(f), so f is integrable.

For the converse part, we use the triangle inequality theorem;

As U(f) is the greatest lower bound of the upper sums, we know that, given some > 0, there must exist a partition P_1 such that

$$U(f,P_1) < U(f) + \varepsilon/2 \; .$$

Likewise, there exists a partition P2 satisfying

$$L(f,P_2) > L(f) - \varepsilon/2 \; .$$

Now, let $P = P_1 \cup P_2$ be the common refinement. Keeping in mind that the integrability of f means U(f) = L(f),

we can write

$U(f,P) - L(f,P) \leq U(f,P1) - L(f,P2) < (U(f) + \varepsilon/2) - (L(f) - \varepsilon/2) = \varepsilon/2 + \varepsilon/2 = \varepsilon$

The next theorem is very important regarding continuity of a function and its Riemann integrability.

Theorem: If f is continuous on [a,b], then it is integrable.

Proof:

Assume a function f that is continuous in the interval [a,b]. Then f is uniformly continuous.

Hence, given any $\varepsilon > 0$, there exists $\delta > 0$ such that if $x, y \in [a, b]$ and $|x - y| < \delta$,

it follows that $|f(x) - f(y)| < \frac{\varepsilon}{b-a}$

let $P = \{x_0 = a, x_1, ..., x_{n-1}, x_n = b\}$ be a partition of [a, b] such that $\Delta xi < \delta$ for every i.

F attains its maximum and minimum on [a, b], and therefore for each subinterval created by the partition. More precisely, for any interval $[x_{i-1}, x_i]$ there exists points a_i , $z_i \in [x_{i-1}, xi]$ such that $m_i = f(a_i)$, $Mi = f(z_i)$.

But since $|a_i - z_i| < \delta$, it follows that $|f(z_i) - f(a_i)| = f(z_i) - f(a_i) < \frac{\varepsilon}{b-a}$.

Now we have

$U(f, P) - L(f, P) = \sum_{i=1}^{n} (Mi - mi))\Delta xi = \sum_{i=1}^{n} \big(f(zi) - f(ai)\big)\Delta xi$

$< \sum_{i=1}^{n} \frac{\varepsilon}{b-a}\Delta xi = \frac{\varepsilon}{b-a}.(b-a) = \varepsilon$

So, f is Riemann Integrable.

Properties of Riemann Integral

Let I be a bounded interval, and let f : I → R and g : I → R be Riemann integrable functions on I.

- The function f + g is Riemann integrable, and we have

  $\int (f + g) = \int f + \int g$
- the function cf is Riemann integrable, and we have $\int cf = c.\int f$ where c is a Real Number.
- f −g is Riemann integrable, and $\int (f - g) = \int f - \int g$
- If f(x) ≥ 0 for all x ∈ I, then $\int f \geq 0$
- if f(x) ≥ g(x) for all x ∈ I, then $\int f \geq \int g$
- Suppose that {J, K} is a partition of I into two intervals J and K. Then the functions $f|_J : J \to R$ and $f|_K : K \to R$ are Riemann integrable on J and K respectively, and we have $\int f = \int f_j + \int f_k$

Some Riemann Integrable functions

1. Piece wise Constant Functions

These are the the class of “simple” functions which we can integrate very easily.

Definition:

Let I be a bounded interval, let f : I → R be a function, and let P be a partition of I. We say that f is piece wise constant with respect to P if for every J ϵ P, f is constant on J.

Definition :

Let I be a bounded interval, and let f : I → R be a function. We say that f is piece wise constant on I if there exists a partition P of I such that f is piece wise constant with respect to P.

Piece wise Constant Integral :

Let I be a bounded interval, let P be a partition of I. Let f : I → R be a function which is piece wise constant with respect to P. Then we define the piece wise constant integral p.c$\int_{[P]} f$ of f with respect to the partition P by the formula:

$$p.c.\int_{[P]} f := \sum_{J \epsilon P} c_J |J|$$

The piece wise constant integral corresponds intuitively to one’s notion of area, given that the area of a rectangle ought to be the product of the lengths of the sides. (Of course, if f is negative somewhere, then the “area” c_j |J| would also be negative.)

2. Monotone Functions

In addition to piece wise continuous functions, another wide class of functions is Riemann integrable, namely the monotone functions.

Theorem:

Let [a, b] be a closed and bounded interval and let f : [a, b] → R be a monotone function. Then f is Riemann integrable on [a, b].

Proof. Without loss of generality we may take f to be monotone increasing (instead of monotone decreasing). we know that f is bounded. Now let N > 0 be an integer, and partition [a, b] into N half-open intervals [a + (b - a)/N*j, a + (b - a)/N*(j + 1)) : 0 ≤ j ≤ N − 1 of length (b - a)/N , together with the point b. Then we have

$$\int_{I}^{-} f \leq \sum_{j=0}^{N-1} (sup_{x\epsilon[a+(b-a)/N*j,a+(b-a)/N*(j+1))]} f(x))(b-a)/N$$

(the point b clearly giving only a zero contribution). Since f is monotone increasing, we thus have

$$\int_{I}^{-} f \leq \sum_{j=0}^{N-1} f(a+(b-a)/N*(j+1))(b-a)/N$$

Similarly we have

$$\int_{-I} f \geq \sum_{j=0}^{N-1} f(a+(b-a)/N*(j))b-a)/N$$

Thus we have

$$\int_{I}^{-} f - \int_{-I} f \leq (f(a+(b-a)/N*(j+1)) - f(a+(b-a)/N*(j)))(b-a)/N$$

Using telescoping series we thus have

$$= (f(b) - f(a))(b - a)/N$$

But N was arbitrary, so we can conclude as in the proof of Theorem of continuous functions that f is Riemann integrable.

The Riemann-Stieltjes Integral

Let I be a bounded interval, let $\alpha : I \to R$ be a monotone increasing function, and let $f : I \to R$ be a function. Then there is a generalization of the Riemann integral, known as the Riemann-Stieltjes integral. This integral is defined just like the Riemann integral, but with one twist: instead of taking the length $|J|$ of intervals J, we take the α-length $\alpha[J]$, defined as follows. If J is a point or the empty set, then $\alpha[J] := 0$. If J is an interval of the form [a, b], (a, b), (a, b], or [a, b), then $\alpha[J] := \alpha(b) - \alpha(a)$. Note that in the special case where α is the identity function $\alpha(x) := x$, then $\alpha[J]$ is just the same as $|J|$. However, for more general monotone functions α, the α-length $\alpha[J]$ is a different quantity from $|J|$. Nevertheless, it turns out one can still do much of the above theory, but replacing $|J|$ by $\alpha[J]$ throughout.

Definition (α-length)

Let I be a bounded interval, and let $\alpha : X \to R$ be a function defined on some domain X which contains I. Then we define the α-length $\alpha[I]$ of I as follows. If I is a point or the empty set, we set $\alpha[I] = 0$. If I is an interval of the form [a,b], [a,b), (a, b], or (a, b) for some $b > a$, then we set $\alpha[I] = \alpha(b) - \alpha(a)$.

Definition (Riemann-Stieltjes integral)

Let I be a bounded interval, and let P be a partition of I. Let $\alpha: X \to R$ be a function defined on some domain X which contains I, and let $f : I \to R$ be a function which is piece wise constant with respect to P. Then we define

$$p.c \int_{[P]} f d\alpha := \sum_{J \in P} c_J \alpha[J]$$

where c_J is the constant value of f on J.

The Two Fundemental Theorems of Calculus

The fundamental theorem of calculus states that differentiation and integration are inverse operations in an appropriately understood sense. The theorem has two parts: in one direction, it says roughly that the integral of the derivative is the original function; in the other direction, it says that the derivative of the integral is the original function.

Theorem (Fundamental theorem of calculus I).

If F : [a, b] → R is continuous on [a,b] and differentiable in (a,b) with F’ = f where f : [a,b] → R is Riemann integrable, then

$$\int_a^b f(x)dx = F(b) - F(a).$$

Let P = a = x_0, x_1, ...x_{n-1}, x_n = b be a partition of [a, b]. Then

$$F(b) - F(a) = \int_{k=1}^{n} [F(x_k) - F(x_{k-1})]$$

The function F is continuous on the closed interval [x_{k-1},x_k] and differentiable in the open interval (x_{k-1},x_k) with F’ = f . By the mean value theorem, there exists $x_{k-1} < c_k < x_k$ such that

$$F(x_k) - F(x_{k-1}) = f(c_k)(x_{k-1} - x_k)$$

Since f is Riemann integrable, it is bounded, and it follows that

$$m_k(x_{k-1} - x_k) \leq F(x_k) - F(x_{k-1}) \leq M_k(x_{k-1} - x_k)$$

where

$$M_k = sup_{[x_{k-1},x_k]} f, m_k = inf_{[x_{k-1},x_k]} f.$$

Hence, L(f,P) ≤ F(b) - F(a)≤ U(f,P) for every partition P of [a,b], which implies that L(f) ≤ F(b) - F(a)≤ U(f). Since f is integrable, L(f) = U(f) and

$$F(b) - F(a) = \int_a^b f$$

In above Theorem, we assume that F is continuous on the closed interval [a, b] and differentiable in the open interval (a, b) where its usual two-sided derivative is defined and is equal to f. It isn't necessary to assume the existence of the right derivative of F at a or the left derivative at b, so the values of f at the endpoints are arbitrary.

The above Theorem imposes the integrability of F as a hypothesis. Every function F that is continuously differentiable on the closed interval [a,b] satisfies this condition, but the theorem remains true even if F is a discontinuous, Riemann integrable function.

Theorem (Fundamental theorem of calculus II)

Suppose that f : [a, b] → R is integrable and F F : [a, b] → R is defined by

$$F(x) = \int_a^x f(t)dt.$$

Then F is continuous on [a, b]. Moreover, if f is continuous at a ≤ c ≤ b, then F is differentiable at c and F'(c) = f(c).

Proof:

Since f is Riemann integrable, it is bounded, and |f| ≤ M for some M ≥ 0. It follows that

$$|F(x+h) - F(x)| = |\int_x^{x+h} f(t)dt| \leq M|h|,$$

which shows that F is continuous on [a, b] (in fact, Lipschitz continuous). Moreover, we have

$$\frac{F(c+h)-F(c)}{h} = \frac{1}{h}\int_{c}^{c+h} f(t)dt.$$

if f is continuous at c, then F is differentiable at c with

$$F'(c) = lim_{h-0}[\frac{F(c+h)-F(c)}{h}] = lim_{h-0}\frac{1}{h}\int_{c}^{c+h} f(t)dt.$$

where we use the appropriate right or left limit at an endpoint. The assumption that f is continuous is needed to ensure that F is differentiable.

Applications of Riemann Integral

The Riemann integral finds its important application in various fields of mathematics and science. Some of them are:

- To Test the convergence of series: Let f : [0, ∞) → R be a monotone decreasing function which is non-negative (i.e., f(x) ≥ 0 for all x ≥ 0). Then the sum $\sum_{0}^{\infty} f(n)$ is convergent if and only if $\int_{0}^{\infty} f$ is finite. Thus it can be used to test the convergence of a series if the series satisfies the above criterion.

- It is used to calculate the area under the curve of various functions using the expression $\int_{a}^{b} f(x) - g(x)$ where f(x) and g(x) are continuous in [a,b].

- The integral finds its application in various field of physics like calculation velocity, acceleration etc. of non-uniformly moving bodies, calculating electric field at a point due to various non-uniform charge distributions etc.

- It is also used in solving differential equations and representation of functions using trigonometric series.

- It is also used to find the volume and surface area of more complicated 3D shapes accurately.

Deficiencies in Riemann Integral

Although Riemann integral defines how to calculate 'area under the curve', it is unsuitable in many scenarios. Three of them are:

- Riemann integration does not handle functions with many discontinuities;
- Riemann integration does not handle unbounded functions
- Riemann integration does not work well with limits.

For example, a function that is not Riemann integrable is

f:[0,1}→R by

$$f(x) = 1, \text{ if x is rational}$$
$$f(x) = 0, \text{ if x is irrational}$$

$$\text{If } [a, b] \subset [0, 1] \text{ with } a < b, \text{ then}$$

$$\inf_{[a,b]} f = 0 \text{ and } \sup_{[a,b]} f = 1$$

because [a, b] contains an irrational number and contains a rational number. Thus L(f , P, [0, 1]) = 0 and U(f , P, [0, 1]) = 1 for every partition P of [0, 1]. Hence L(f , [0, 1]) = 0 and U(f , [0, 1]) = 1. Because L(f , [0, 1]) ≠U(f , [0, 1]), we conclude that f is not Riemann integrable.

This example is disturbing because, there are far fewer rational numbers(countable) than irrational numbers(uncountable). Thus f should, in some sense, have integral 0. However, the Riemann integral of f is not defined.

However, f can be integrated using Lebesgue Integral which calculates area using horizontal strips contrary to Riemann Integral which calculates integral using vertical strips. Lebesgue Integral assigns the value 1 to the integral of f.

Conclusion

The main ideas of Riemann Integral were presented in this project. We saw the concept of integration as the sum of areas of rectangles and the larger the no of rectangles the more accurate the area. The upper and lower Riemann sums provide a fairly good approximation of the area enclosed by the curve. The Sums calculated over every partition give the Lower and Upper Riemann integral and if the two are equal the function is said to be Riemann integrable and the value of integral is the value of common integral.

The Integral follows many basic properties. It has various important applications in the field of maths and science. The integral is intimately connected with the Fundamental theorems of calculus. There are a few shortcomings of the Riemann integral like integrating the Dirchlet function which can be remedied by Riemann-Stieltjes integral and Lebesgue integral. In a nutshell, the Riemann integral can be considered as the baseline on which further modifications can be made to accommodate integration of various types of functions.

References

https://www.researchgate.net

https://www.courses.lumenlearning.com

https://www.wikipedia.com

https://www.mathworld.wolfram.com

https://www.byjus.com

Terence Tao- Analysis 1

Elementary Analysis – Kenneth Ross

Understanding Analysis – Stephen Abbott