Task 1:

Calculate the Hamilton’s equations for the above system.

$$H(x,p) = \frac{1}{2}kx^2 + \frac{p^2}{2m}$$

Hamilton's equations of the system are given by

$$\frac{-dp}{dt} = \frac{\partial H}{\partial x} = kx$$
$$\frac{dx}{dt} = \frac{\partial H}{\partial p} = \frac{p}{m}$$

Task 2:

Solve the Hamilton’s equations computationally using the finite difference method and study the time evolution of the position and momentum of the oscillator.

On solving the hamilton's equations, we get

$$\frac{d^2x}{dt^2} = -w_0^2 x$$

On converting into finite difference, we get

$$\frac{x(t+\Delta t) - 2x(t) + x(t-\Delta t)}{(\Delta t)^2} = -w_0^2 x(t)$$
$$\implies x(t+\Delta t) = (2 - w_0^2(\Delta t)^2)x(t) - x(t-\Delta t)$$

Initially, $x(0) = x_0$ and $x(\Delta t) = x_0 + v_0\Delta t$ (where v_0 is the initial velocity, $w_0^2 = \frac{k}{m}$)

Similarly for momentum, we get

$$\frac{d^2p}{dt^2} = \frac{-kp}{m}$$

On converting into finite difference, we get

$$\frac{p(t+\Delta t) - 2p(t) + p(t-\Delta t)}{(\Delta t)^2} = \frac{-kp(t)}{m}$$
$$\implies p(t+\Delta t) = (2 - \frac{k(\Delta t)^2}{m})p(t) - p(t-\Delta t)$$

Initially, $p(0) = p_0 = mv_0$, and $p(\Delta t) = p_0 - kx_0\Delta t$