EM Algorithm

We begin by making a note of all the information provided to us.
Probability of chosing gold coin, $P(G) = \rho$
Probability of chosing silver coin, $P(S) = 1 - \rho$

$$P(H|G) = P(T|G) = 0.5$$
$$P(H|S) = 0.25 \, , P(T|S) = 0.75$$

Also, two random variables are defined as: $Z \in \{G, S\}$, and $X \in \{H, T\}$
The goal is to find out ρ by repeatedly chosing the coin and observing the outcomes of the coin tosses.

Part A

Given:

$$\gamma_\rho(z, x) = Pr(Z = z|X = x)$$

Now for finding $\gamma_\rho(z, x)$ as a function of ρ we consider all the possible pair of values for X and Z and calculate the probability.
Case 1: $Z = G$ and $X = H$

$$\gamma_\rho(z, x) = Pr(Z = G|X = H)$$
$$\Rightarrow \gamma_\rho(z, x) = \frac{Pr(G \cap H)}{Pr(H)} = \frac{Pr(H|G)Pr(G)}{Pr(H)}$$
$$\Rightarrow \gamma_\rho(z, x) = \frac{0.5 \times \rho}{0.5 \times \rho + 0.25 \times (1 - \rho)} = \frac{2\rho}{1 + \rho}$$

Case 2: $Z = G$ and $X = T$
By using the similar calculations as above we get,

$$\gamma_\rho(z, x) = Pr(Z = G|X = T)$$
$$\gamma_\rho(z, x) = \frac{0.5 \times \rho}{0.5 \times \rho + 0.75 \times (1 - \rho)} = \frac{2\rho}{3 - \rho}$$

Case 3: $Z = S$ and $X = H$
By using the similar calculations as above we get,

$$\gamma_\rho(z, x) = Pr(Z = S|X = H)$$
$$\gamma_\rho(z, x) = \frac{0.25 \times (1 - \rho)}{0.25 \times (1 - \rho) + 0.5 \times \rho} = \frac{1 - \rho}{1 + \rho}$$

Case 4: $Z = S$ and $X = T$
By using the similar calculations as above we get,

$$\gamma_\rho(z, x) = Pr(Z = S|X = T)$$
$$\gamma_\rho(z, x) = \frac{0.75 \times (1 - \rho)}{0.75 \times (1 - \rho) + 0.5 \times \rho} = \frac{3(1 - \rho)}{3 - \rho}$$

Thus the above cases gives us the required variation of $\gamma_\rho(z, x)$ as a function of ρ.

Part B

In the given part, ρ is estimated using Expectation Maximization(EM) algorithm, using a set of observations. Given:

$$\mathcal{L}_\rho(\rho') = \sum_{i=1}^{N} \sum_{z \in \{G,S\}} \gamma_\rho(z, x_i).logPr(z, x; \rho')$$

The above expression is used for updating the current estimation(ρ) to a new estimate ρ', by maximising the above expression.
As the solution of the problem, we need to find:

$$arg\ \max_{\rho'} \mathcal{L}_\rho(\rho')$$

Now consider $\mathcal{L}_\rho(\rho')$ as below:

$$\mathcal{L}_\rho(\rho') = \sum_{i=1}^{N} \big\{\gamma_\rho(G, x_i) logPr(G, x_i, \rho') + \gamma_\rho(S, x_i) logPr(S, x_i, \rho')\big\}$$

The condition can be written as:

$$arg\ \max_{\rho'} \sum_{i=1}^{N} \big\{\gamma_\rho(G, x_i) logPr(G, x_i; \rho') + \gamma_\rho(S, x_i) logPr(S, x_i; \rho')\big\}$$

$$= arg\ \max_{\rho'} \Big\{\big[N_H \gamma_\rho(G, H) logPr(G, H; \rho') + N_T \gamma_\rho(G, T) logPr(G, T; \rho')\big] + \big[N_H \gamma_\rho(S, H) logPr(S, H; \rho')$$

$$+ N_T \gamma_\rho(S, T) logPr(S, T; \rho')\big]\Big\}$$

Now the point of maxima for the above expression can be found by differentiating the above w.r.t. ρ'

$$\frac{d(expr)}{d\rho'} = 0$$

The expression is given by:

$$expr = N_H \frac{2\rho}{1+\rho} log(\frac{\rho'}{2}) + N_T \frac{2\rho}{3-\rho} log(\frac{\rho'}{2}) + N_H \frac{1-\rho}{1+\rho} log(\frac{1-\rho'}{4}) + N_T \frac{3(1-\rho)}{3-\rho} log(\frac{3(1-\rho')}{4})$$

On taking the derivative and then equating that to zero gives us:

$$\frac{2\rho}{\rho'} \left[\frac{N_H}{1+\rho} + \frac{N_T}{3-\rho} \right] = \frac{1-\rho}{1-\rho'} \left[\frac{N_H}{1+\rho} + \frac{3N_T}{3-\rho} \right]$$

$$\Rightarrow \frac{1}{\rho'} = \frac{1-\rho}{2\rho} \left[\frac{N_H}{1+\rho} + \frac{3N_T}{3-\rho} \right] \left[\frac{N_H}{1+\rho} + \frac{N_T}{3-\rho} \right]^{-1} + 1$$

The above equation describes the required solution.