

Supplementary Material

Formation control of differential-drive robots with input saturation and constraint on turning radius

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Abstract—This piece of work is a supplementary material to the original work submitted to IROS 2022. This supplementary material discusses the proof of those arguments/expressions/equations for which this material was cited in the original work. Note that the proof here is continuation of the proof given in the original work. Here, the tags/labels of sections, subsections and equations are kept the same as given in the original work.

II. TRAJECTORY TRACKING CONTROLLER WITH INPUT CONSTRAINTS

B. Upper bound of $|\hat{\theta}_d|$

We begin by obtaining an upper bound on $\dot{\theta}_d$ by starting with (6). We substitute expressions of e_x , e_y , \dot{e}_x , \dot{e}_y in (6) from (15), (16).

$$\begin{aligned} |\dot{\theta}_d| &= \left| \frac{e_x \dot{e}_y - e_y \dot{e}_x}{D^2} \right| \\ &= \left| \frac{e_x v \sin(\theta) - e_y v \cos(\theta) + e_y \dot{x}_d - e_x \dot{y}_d}{D^2} \right| \\ &= \left| \frac{-D v_{\max} \sin(2e\theta) \frac{\eta_v D}{2(1+\eta_v D)} - D(v_{\text{traj}}^{\perp L})}{D^2} \right| \\ &= \left| -v_{\max} \sin(2e\theta) \frac{\eta_v}{2(1+\eta_v D)} - \frac{v_{\text{traj}}^{\perp L}}{D} \right| \\ &\leq \left| v_{\max} \frac{\eta_v}{2(1+\eta_v D)} \right| + \frac{|v_{\text{traj}}^{\perp L}|}{D} \end{aligned} \quad (29)$$

In the region outside the ultimate bound, $D(t) \geq b_D$. Also,

$$b_D > \mu_D > \frac{|v_{\text{traj}}^{\text{sup}}|}{\eta_v (v_{\max} \delta_\theta^2 - |v_{\text{traj}}^{\text{sup}}|)} \geq \frac{|v_{\text{traj}}^{\text{sup}}|}{\eta_v (v_{\max} - |v_{\text{traj}}^{\text{sup}}|)} \quad (30)$$

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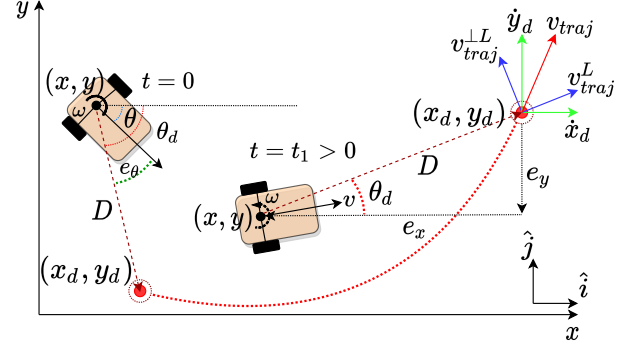


Fig. 1: Cartesian components of v_{traj} (Green): (\dot{x}_d, \dot{y}_d) , Components of v_{traj} along and \perp to L (Blue): $(v_{\text{traj}}^L, v_{\text{traj}}^{\perp L})$. D denotes the distance between (x, y) and (x_d, y_d)

Continuing with the result of (29), (30)

$$\begin{aligned} |\dot{\theta}_d| &\leq \left| v_{\max} \frac{\eta_v}{2(1+\eta_v b_D)} \right| + \frac{|v_{\text{traj}}^{\perp L}|}{b_D} \\ &< \left| \frac{\eta_v (v_{\max} \delta_\theta^2 - |v_{\text{traj}}^{\text{sup}}|)}{2 \delta_\theta^2} \right| + \eta_v (v_{\max} \delta_\theta^2 - |v_{\text{traj}}^{\text{sup}}|) \\ |\dot{\theta}_d| &< \frac{3 \eta_v v_{\max}}{2} \end{aligned} \quad (31)$$

Now, using (8) and upper bound on $|\dot{\theta}_d|$ in (31), we get

$$|\hat{\theta}_d| \leq |\dot{\theta}_d| + \varepsilon_\theta \implies |\hat{\theta}_d| < \frac{3 \eta_v v_{\max}}{2} + \varepsilon_\theta \quad (32)$$

If we choose η_v as per (13), then choosing $\Omega_{\max} = \frac{3 \eta_v v_{\max}}{2} + \varepsilon_\theta$ will satisfy (14).

Even if $D(t)$ goes arbitrarily close to zero, $|\hat{\theta}_d|$ will remain upper bounded because

$$\lim_{D \rightarrow 0} \frac{|v_{\text{traj}}^{\perp L}|}{D} = 0 \quad (33)$$

The proof of (33) is as follows.

By definition, $D \rightarrow 0 \implies e_x \rightarrow 0, e_y \rightarrow 0$. Therefore, we can apply L'Hospital's rule to $\lim_{D \rightarrow 0} \frac{e_y}{e_x}$. From (16) it is clear that $e_x, e_y \rightarrow 0 \implies \dot{e}_x \rightarrow \dot{x}_d, \dot{e}_y \rightarrow \dot{y}_d$. We also define slope of reference trajectory as

$$\tan(\theta_{\text{traj}}^{\text{slope}}) = \frac{\dot{y}_d}{\dot{x}_d} \quad (34)$$

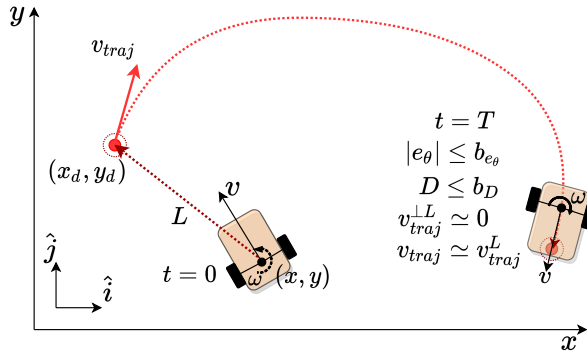


Fig. 2: Stable tracking achieved by differential drive robot using controller

Now, consider the following limit

$$\lim_{D \rightarrow 0} \tan(\theta_d) = \lim_{D \rightarrow 0} \frac{e_y}{e_x} = \lim_{D \rightarrow 0} \frac{\dot{e}_y}{\dot{e}_x} = \frac{\dot{y}_d}{\dot{x}_d} = \tan(\theta_{traj}^{slope}) \quad (35)$$

The result of (35) implies

$$\lim_{D \rightarrow 0} (\tan(\theta_d) - \tan(\theta_{traj}^{slope})) = 0 \implies \lim_{D \rightarrow 0} (\theta_d - \theta_{traj}^{slope}) = 0 \quad (36)$$

By *Assumption 1*, the $\dot{\theta}_{traj}^{slope}$ is uniformly continuous, and $\dot{\theta}_d$ is also uniformly continuous as defined in (6). Therefore by *Lemma 8.2: Barbalat's Lemma* of [1], we get

$$\lim_{D \rightarrow 0} (\dot{\theta}_d - \dot{\theta}_{traj}^{slope}) = 0 \quad (37)$$

We can also write $v_{traj}^L = v_{traj} \cos(\theta_{traj}^{slope} - \theta_d)$ and $v_{traj}^{\perp L} = v_{traj} \sin(\theta_{traj}^{slope} - \theta_d)$. A magnified view of Fig.1 is given here for reference. Therefore, (37) implies $\lim_{D \rightarrow 0} |v_{traj}^{\perp L}| = 0$.

Applying L'Hospital's rule to $\lim_{D \rightarrow 0} \frac{|v_{traj}^{\perp L}|}{D}$ we get

$$\begin{aligned} \lim_{D \rightarrow 0} \frac{|v_{traj}^{\perp L}|}{D} &\leq \lim_{D \rightarrow 0} \frac{|v_{traj}^{sup}| |\sin(\theta_{traj}^{slope} - \theta_d)|}{D} \\ &\leq \lim_{D \rightarrow 0} \frac{|v_{traj}^{sup}| |\cos(\theta_{traj}^{slope} - \theta_d)| (\dot{\theta}_d - \dot{\theta}_{traj}^{slope})}{\dot{D}} \end{aligned} \quad (38)$$

We rewrite (18) from the main paper here

$$\begin{aligned} \dot{V}_D &= D\dot{D} = e_x \dot{e}_x + e_y \dot{e}_y \\ \dot{V}_D &= -v_{max} \frac{\eta_v}{1 + \eta_v D} \cos^2(e_\theta) (e_x^2 + e_y^2) - \dot{x}_d e_x - \dot{y}_d e_y \\ \dot{D} &= -v_{max} \frac{\eta_v D}{1 + \eta_v D} \cos^2(e_\theta) - (\dot{x}_d \cos(e_\theta) + \dot{y}_d \sin(e_\theta)) \\ \dot{D} &= -v_{max} \frac{\eta_v D}{1 + \eta_v D} \cos^2(e_\theta) - v_{traj}^L \end{aligned} \quad (39)$$

We know $D \rightarrow 0 \implies e_x \rightarrow 0, e_y \rightarrow 0$. Following (39) we get

$$\lim_{D \rightarrow 0} \dot{D} \rightarrow |v_{traj}^L| \quad (40)$$

and $|v_{traj}^L| = v_{traj} \cos(\theta_{traj}^{slope} - \theta_d)$. Therefore, proceeding further with (38)

$$\begin{aligned} \lim_{D \rightarrow 0} \frac{|v_{traj}^{\perp L}|}{D} &\leq \lim_{D \rightarrow 0} \frac{|v_{traj}^{sup}| |\cos(\theta_{traj}^{slope} - \theta_d)| (\dot{\theta}_d - \dot{\theta}_{traj}^{slope})}{|v_{traj}| |\cos(\theta_{traj}^{slope} - \theta_d)|} \\ &\leq \lim_{D \rightarrow 0} \frac{|v_{traj}^{sup}| (\dot{\theta}_d - \dot{\theta}_{traj}^{slope})}{|v_{traj}|} = 0 \end{aligned} \quad (41)$$

The above analysis proves that $|\hat{\theta}_d|$ is upper bounded for all $D \in [0, \infty)$. From the results of II-A, II-B, we conclude that stable tracking, as defined in (9), is guaranteed without exceeding actuator limits.

III. DISCUSSION

A. Selecting values of v_{max} , ω_{max}

In the controller proposed in (11), (12) of the main paper, $v(t)$ is a function of D , e_θ , and $\omega(t)$ is a function of e_θ , θ_d .

$$v = -v_{max} \cos(e_\theta) \frac{\eta_v D}{1 + \eta_v D} \quad (42)$$

$$\omega = -(\omega_{max} - \Omega_{max}) \tanh(\eta_\omega e_\theta) + \hat{\theta}_d \quad (43)$$

Now consider a possible scenario where $D = 20m$, $e_\theta = \pi$, $\hat{\theta}_d \rightarrow -\Omega_{max}$ and $\eta_v = 10$, $\eta_\omega = 20$. Then in this case both $v(t) \rightarrow v_{max}$, $\omega(t) \rightarrow -\omega_{max}$. If we choose v_{max} , ω_{max} as

$$v_{max} = \frac{R\omega_{r,l}^{max}}{2}, \quad \omega_{max} = \frac{R\omega_{r,l}^{max}}{L} \quad (44)$$

Then as per (1) of main paper, either ω_r or ω_l will reach $\omega_{r,l}^{max}$. Hence, the limits on motor angular velocities will be obeyed at all time, $t \geq 0$.

B. Alternate proposal for linear velocity control $v(t)$

The following linear velocity control can also be used in place of controller proposed in (11) of main paper.

$$v(t) = -v_{max} \cos(e_\theta) \tanh(\eta_v D) \quad (45)$$

The stability analysis for this controller can be done in the same way as done in section II-A of the main paper. Bounds of η_v , $\hat{\theta}_d$ can be obtained for this controller as well..

REFERENCES

- [1] Hassan K. Khalil, 2002, *Nonlinear Systems*, Prentice Hall, 3rd ed.