Supplementary Material

Formation control of differential-drive robots with input saturation and constraint on turning radius

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Abstract—This piece of work is a supplementary material to the original work submitted to IROS 2022. This supplementary material discusses the proof of those arguments/expressions/equations for which this material was cited in the original work. Note that the proof here is continuation of the proof given in the original work. Here, the tags/labels of sections, subsections and equations are kept the same as given in the original work.

II. TRAJECTORY TRACKING CONTROLLER WITH INPUT CONSTRAINTS

B. Upper bound of $\left|\dot{\hat{\theta}}_d\right|$

We begin by obtaining an upper bound on $\dot{\theta}_d$ by starting with (6). We substitute expressions of e_x, e_y, $\dot{e}_x$, $\dot{e}_y$ in (6) from (15), (16).

$$
\begin{aligned}
\left|\dot{\theta}_d\right| &= \left|\frac{e_x\dot{e}_y - e_y\dot{e}_x}{D^2}\right| \\
&= \left|\frac{e_x v\sin(\theta) - e_y v\cos(\theta) + e_y\dot{x}_d - e_x\dot{y}_d}{D^2}\right| \\
&= \left|\frac{-Dv_{max}\sin(2e_\theta)\frac{\eta_v D}{2(1+\eta_v D)} - D(v_{traj}^{\perp L})}{D^2}\right| \\
&= \left|-v_{max}\sin(2e_\theta)\frac{\eta_v}{2(1+\eta_v D)} - \frac{v_{traj}^{\perp L}}{D}\right| \\
&\leq \left|v_{max}\frac{\eta_v}{2(1+\eta_v D)}\right| + \frac{\left|v_{traj}^{\perp L}\right|}{D}
\end{aligned}
\tag{29}
$$

In the region outside the ultimate bound, $D(t) \geq b_D$. Also,

$$
b_D > \mu_D > \frac{\left|v_{traj}^{sup}\right|}{\eta_v\left(v_{max}\delta_\theta^2 - \left|v_{traj}^{sup}\right|\right)} \geq \frac{\left|v_{traj}^{sup}\right|}{\eta_v\left(v_{max} - \left|v_{traj}^{sup}\right|\right)}
\tag{30}
$$

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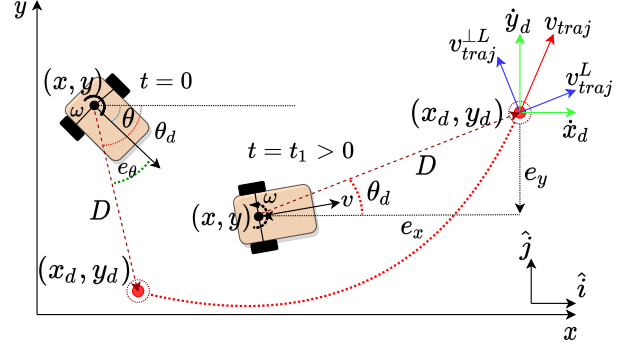
Fig. 1: Cartesian components of v_{traj} (Green): $(\dot{x}_d, \dot{y}_d)$, Components of v_{traj} along and $\perp$ to L (Blue): $(v_{traj}^L, v_{traj}^{\perp L})$. D denotes the distance between (x, y) and (x_d, y_d)

Continuing with the result of (29), (30)

$$
\begin{aligned}
\left|\dot{\theta}_d\right| &\leq \left|v_{max}\frac{\eta_v}{2(1+\eta_v b_D)}\right| + \frac{\left|v_{traj}^{\perp L}\right|}{b_D} \\
&< \left|\frac{\eta_v(v_{max}\delta_\theta^2 - \left|v_{traj}^{sup}\right|)}{2\delta_\theta^2}\right| + \eta_v(v_{max}\delta_\theta^2 - \left|v_{traj}^{sup}\right|) \\
\left|\dot{\theta}_d\right| &< \frac{3\eta_v v_{max}}{2}
\end{aligned}
\tag{31}
$$

Now, using (8) and upper bound on $\left|\dot{\theta}_d\right|$ in (31), we get

$$
\left|\dot{\hat{\theta}}_d\right| \leq \left|\dot{\theta}_d\right| + \varepsilon_\theta \implies \left|\dot{\hat{\theta}}_d\right| < \frac{3\eta_v v_{max}}{2} + \varepsilon_\theta
\tag{32}
$$

If we choose η_v as per (13), then choosing $\Omega_{max} = \frac{3\eta_v v_{max}}{2} + \varepsilon_\theta$ will satisfy (14).

Even if $D(t)$ goes arbitrarily close to zero, $\left|\dot{\hat{\theta}}_d\right|$ will remain upper bounded because

$$
\lim_{D\to 0} \frac{\left|v_{traj}^{\perp L}\right|}{D} = 0
\tag{33}
$$

The proof of (33) is as follows.

By definition, $D \to 0 \implies e_x \to 0$, $e_y \to 0$. Therefore, we can apply L'Hospital's rule to $\lim_{D\to 0} \frac{e_y}{e_x}$. From (16) it is clear that $e_x, e_y \to 0 \implies \dot{e}_x \to \dot{x}_d$, $\dot{e}_y \to \dot{y}_d$. We also define slope of reference trajectory as

$$
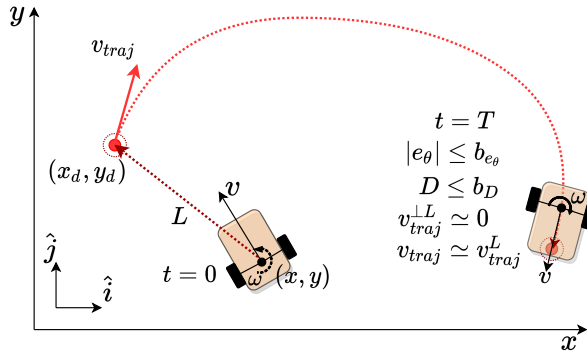\tan\left(\theta_{traj}^{slope}\right) = \frac{\dot{y}_d}{\dot{x}_d}
\tag{34}
$$

Fig. 2: Stable tracking achieved by differential drive robot using controller

Now, consider the following limit

$$\lim_{D \to 0} \tan(\theta_d) = \lim_{D \to 0} \frac{e_y}{e_x} = \lim_{D \to 0} \frac{\dot{e}_y}{\dot{e}_x} = \frac{\dot{y}_d}{\dot{x}_d} = \tan\left(\theta_{traj}^{slope}\right) \quad (35)$$

The result of (35) implies

$$\lim_{D \to 0} (\tan(\theta_d) - \tan\left(\theta_{traj}^{slope}\right)) = 0 \implies \lim_{D \to 0} (\theta_d - \theta_{traj}^{slope}) = 0 \quad (36)$$

By *Assumption 1*, the $\dot{\theta}_{traj}^{slope}$ is uniformly continuous, and $\dot{\theta}_d$ is also uniformly continuous as defined in (6). Therefore by *Lemma 8.2: Barbalat's Lemma* of [1], we get

$$\lim_{D \to 0} (\dot{\theta}_d - \dot{\theta}_{traj}^{slope}) = 0 \quad (37)$$

We can also write $v_{traj}^L = v_{traj} \cos\left(\theta_{traj}^{slope} - \theta_d\right)$ and $v_{traj}^{\perp L} = v_{traj} \sin\left(\theta_{traj}^{slope} - \theta_d\right)$. A magnified view of Fig.1 is given here for reference. Therefore, (37) implies $\lim_{D \to 0} \left|v_{traj}^{\perp L}\right| = 0$.

Applying L'Hospital's rule to $\lim_{D \to 0} \frac{\left|v_{traj}^{\perp L}\right|}{D}$ we get

$$\lim_{D \to 0} \frac{\left|v_{traj}^{\perp L}\right|}{D} \leq \lim_{D \to 0} \frac{\left|v_{traj}^{sup}\right|\left|\sin\left(\theta_{traj}^{slope} - \theta_d\right)\right|}{D}$$
$$\leq \lim_{D \to 0} \frac{\left|v_{traj}^{sup}\right|\left|\cos\left(\theta_{traj}^{slope} - \theta_d\right)\right|(\dot{\theta}_d - \dot{\theta}_{traj}^{slope})}{\dot{D}} \quad (38)$$

We rewrite (18) from the main paper here

$$\dot{V}_D = D\dot{D} = e_x \dot{e}_x + e_y \dot{e}_y$$
$$\dot{V}_D = -v_{max} \frac{\eta_v}{1 + \eta_v D} \cos^2(e_\theta)(e_x^2 + e_y^2) - \dot{x}_d e_x - \dot{y}_d e_y$$
$$\dot{D} = -v_{max} \frac{\eta_v D}{1 + \eta_v D} \cos^2(e_\theta) - (\dot{x}_d \cos(e_\theta) + \dot{y}_d \sin(e_\theta))$$
$$\dot{D} = -v_{max} \frac{\eta_v D}{1 + \eta_v D} \cos^2(e_\theta) - v_{traj}^L \quad (39)$$

We know $D \to 0 \implies e_x \to 0$, $e_y \to 0$. Following (38) we get

$$\lim_{D \to 0} \dot{D} \to \left|v_{traj}^L\right| \quad (40)$$

and $\left|v_{traj}^L\right| = v_{traj} \cos\left(\theta_{traj}^{slope} - \theta_d\right)$. Therefore, proceeding further with (38)

$$\lim_{D \to 0} \frac{\left|v_{traj}^{\perp L}\right|}{D} \leq \lim_{D \to 0} \frac{\left|v_{traj}^{sup}\right|\left|\cos\left(\theta_{traj}^{slope} - \theta_d\right)\right|(\dot{\theta}_d - \dot{\theta}_{traj}^{slope})}{\left|v_{traj}\right|\left|\cos\left(\theta_{traj}^{slope} - \theta_d\right)\right|}$$
$$\leq \lim_{D \to 0} \frac{\left|v_{traj}^{sup}\right|(\dot{\theta}_d - \dot{\theta}_{traj}^{slope})}{\left|v_{traj}\right|} = 0 \quad (41)$$

The above analysis proves that $\left|\hat{\dot{\theta}}_d\right|$ is upper bounded for all $D \in [0, \infty)$. From the results of II-A, II-B, we conclude that stable tracking, as defined in (9), is guaranteed without exceeding actuator limits.

III. Discussion

A. Selecting values of v_{max}, ω_{max}

In the controller proposed in (11), (12) of the main paper, $v(t)$ is a function of D, e_θ, and $\omega(t)$ is a function of e_θ, θ_d.

$$v = -v_{max} \cos(e_\theta) \frac{\eta_v D}{1 + \eta_v D} \quad (42)$$

$$\omega = -(\omega_{max} - \Omega_{max}) \tanh(\eta_\omega e_\theta) + \hat{\dot{\theta}}_d \quad (43)$$

Now consider a possible scenario where $D = 20m$, $e_\theta = \pi$, $\hat{\dot{\theta}}_d \to -\Omega_{max}$ and $\eta_v = 10$, $\eta_\omega = 20$. Then in this case both $v(t) \to v_{max}$, $\omega(t) \to -\omega_{max}$. If we choose v_{max}, ω_{max} as

$$v_{max} = \frac{R\omega_{r,\,l}^{max}}{2}, \quad \omega_{max} = \frac{R\omega_{r,\,l}^{max}}{L} \quad (44)$$

Then as per (1) of main paper, either ω_r or ω_l will reach $\omega_{r,\,l}^{max}$. Hence, the limits on motor angular velocities will be obeyed at all time, $t \geq 0$.

B. Alternate proposal for linear velocity control $v(t)$

The following linear velocity control can also be used in place of controller proposed in (11) of main paper.

$$v(t) = -v_{max} \cos(e_\theta) \tanh(\eta_v D) \quad (45)$$

The stability analysis for this controller can be done in the same way as done in section II-A of the main paper. Bounds of η_v, $\hat{\dot{\theta}}_d$ can be obtained for this controller as well..

References

[1] Hassan K. Khalil, 2002, *Nonlinear Systems*, Prentice Hall, 3rd ed.