Supplementary Material

Formation control of differential-drive robots with input saturation and constraint on turning radius

Ayush Agrawal[1], Mukunda Bharatheesha[2] and Shishir Kolathaya[3]

***Abstract*— This piece of work is a supplementary material to the original work submitted to IROS 2022. This supplementary material discusses the proof of those arguments/expressions/equations for which this material was cited in the original work. Note that the proof here is continuation of the proof given in the original work. Here, the tags/labels of sections, subsections and equations are kept the same as given in the original work.**

II. Trajectory Tracking Controller with Input Constraints

B. Upper bound of $\left|\hat{\dot{\theta}}_d\right|$

We begin by obtaining an upper bound on $\dot{\theta}_d$ by starting with (6). We substitute expressions of e_x, e_y, $\dot{e_x}$, $\dot{e_y}$ in (6) from (15), (16).

$$\begin{aligned}\left|\dot{\theta}_d\right| &= \left|\frac{e_x\dot{e_y} - e_y\dot{e_x}}{D^2}\right| \\ &= \left|\frac{e_x v\sin(\theta) - e_y v\cos(\theta) + e_y\dot{x_d} - e_x\dot{y_d}}{D^2}\right| \\ &= \left|\frac{-Dv_{max}\sin(2e_\theta)\frac{\eta_v D}{2(1+\eta_v D)} - D(v_{traj}^{\perp L})}{D^2}\right| \\ &= \left|-v_{max}\sin(2e_\theta)\frac{\eta_v}{2(1+\eta_v D)} - \frac{v_{traj}^{\perp L}}{D}\right| \\ &\leq \left|v_{max}\frac{\eta_v}{2(1+\eta_v D)}\right| + \frac{\left|v_{traj}^{\perp L}\right|}{D}\end{aligned} \tag{29}$$

In the region outside the ultimate bound, $D(t) \geq b_D$. Also,

$$b_D > \mu_D > \frac{\left|v_{traj}^{sup}\right|}{\eta_v\left(v_{max}\delta_\theta^2 - \left|v_{traj}^{sup}\right|\right)} \geq \frac{\left|v_{traj}^{sup}\right|}{\eta_v\left(v_{max} - \left|v_{traj}^{sup}\right|\right)} \tag{30}$$

[1] Ayush Agarwal has received his Bachelor's degree from the Department of Mechanical Engineering at the Indian Institute of Technology Bombay, India. `ayush.agrawal.c2021@iitbombay.org`, `aayush.agrawal149@gmail.com`

[2]Mukunda Bharatheesha is a guest faculty at Robert Bosch Centre for Cyber Physical Systems and a Senior Member of Techinal Staff at ARTPARK, Indian Institute of Science, Bengaluru, India `mukundab@iisc.ac.in`, `mukunda@artpark.in`

[3]Shishir Kolathaya is an Assistant Professor at Robert Bosch Centre for Cyber Physical Systems and the department of Computer Science and Automation (CSA), Indian Institute of Science, Bengaluru, India `shishirk@iisc.ac.in`

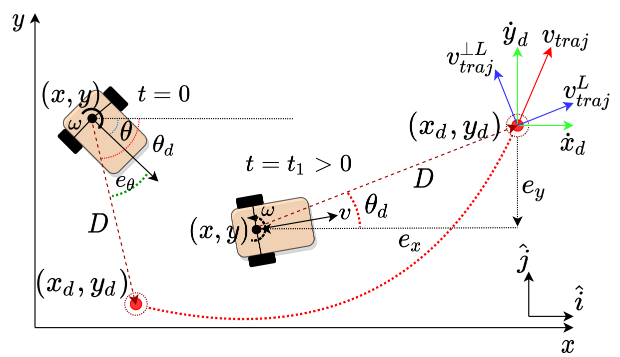

Fig. 1: Cartesian components of v_{traj} (Green): $(\dot{x_d}, \dot{y_d})$, Components of v_{traj} along and $\perp$ to L (Blue): $(v_{traj}^{L}, v_{traj}^{\perp L})$. D denotes the distance between $(x,\ y)$ and $(x_d,\ y_d)$

Continuing with the result of (29), (30)

$$\begin{aligned}\left|\dot{\theta}_d\right| &\leq \left|v_{max}\frac{\eta_v}{2(1+\eta_v b_D)}\right| + \frac{\left|v_{traj}^{\perp L}\right|}{b_D} \\ &< \left|\frac{\eta_v\left(v_{max}\delta_\theta^2 - \left|v_{traj}^{sup}\right|\right)}{2\delta_\theta^2}\right| + \eta_v\left(v_{max}\delta_\theta^2 - \left|v_{traj}^{sup}\right|\right) \\ \left|\dot{\theta}_d\right| &< \frac{3\eta_v v_{max}}{2}\end{aligned} \tag{31}$$

Now, using (8) and upper bound on $\left|\dot{\theta}_d\right|$ in (31), we get

$$\left|\hat{\dot{\theta}}_d\right| \leq \left|\dot{\theta}_d\right| + \varepsilon_\theta \implies \left|\hat{\dot{\theta}}_d\right| < \frac{3\eta_v v_{max}}{2} + \varepsilon_\theta \tag{32}$$

If we choose η_v as per (13), then choosing $\Omega_{max} = \frac{3\eta_v v_{max}}{2} + \varepsilon_\theta$ will satisfy (14).

Even if $D(t)$ goes arbitrarily close to zero, $\left|\hat{\dot{\theta}}_d\right|$ will remain upper bounded because

$$\lim_{D\to 0}\frac{\left|v_{traj}^{\perp L}\right|}{D} = 0 \tag{33}$$

The proof of (33) is as follows.

By definition, $D \to 0 \implies e_x \to 0,\ e_y \to 0$. Therefore, we can apply L'Hospital's rule to $\lim_{D\to 0}\frac{e_y}{e_x}$. From (16) it is clear that $e_x, e_y \to 0 \implies \dot{e_x} \to \dot{x_d},\ \dot{e_y} \to \dot{y_d}$. We also define slope of reference trajectory as

$$\tan\left(\theta_{traj}^{slope}\right) = \frac{\dot{y_d}}{\dot{x_d}} \tag{34}$$

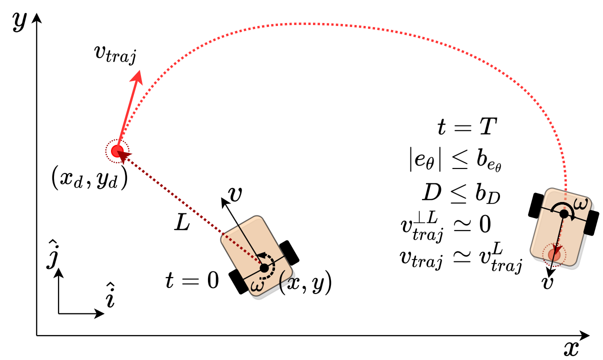

Fig. 2: Stable tracking achieved by differential drive robot using controller

Now, consider the following limit

$$\lim_{D\to 0} \tan(\theta_d) = \lim_{D\to 0} \frac{e_y}{e_x} = \lim_{D\to 0} \frac{\dot{e}_y}{\dot{e}_x} = \frac{\dot{y}_d}{\dot{x}_d} = \tan\left(\theta_{traj}^{slope}\right) \tag{35}$$

The result of (35) implies

$$\lim_{D\to 0} \left(\tan(\theta_d) - \tan\left(\theta_{traj}^{slope}\right)\right) = 0 \implies \lim_{D\to 0} \left(\theta_d - \theta_{traj}^{slope}\right) = 0 \tag{36}$$

By *Assumption 1*, the $\dot{\theta}_{traj}^{slope}$ is uniformly continuous, and $\dot{\theta}_d$ is also uniformly continuous as defined in (6). Therefore by *Lemma 8.2: Barbalat's Lemma* of [1], we get

$$\lim_{D\to 0} \left(\dot{\theta}_d - \dot{\theta}_{traj}^{slope}\right) = 0 \tag{37}$$

We can also write $v_{traj}^{L} = v_{traj} \cos\left(\theta_{traj}^{slope} - \theta_d\right)$ and $v_{traj}^{\perp L} = v_{traj} \sin\left(\theta_{traj}^{slope} - \theta_d\right)$. A magnified view of Fig.1 is given here for reference. Therefore, (37) implies $\lim_{D\to 0} \left|v_{traj}^{\perp L}\right| = 0$. Applying L'Hospital's rule to $\lim_{D\to 0} \frac{\left|v_{traj}^{\perp L}\right|}{D}$ we get

$$\begin{aligned} \lim_{D\to 0} \frac{\left|v_{traj}^{\perp L}\right|}{D} &\le \lim_{D\to 0} \frac{\left|v_{traj}^{sup}\right| \left|\sin\left(\theta_{traj}^{slope} - \theta_d\right)\right|}{D} \\ &\le \lim_{D\to 0} \frac{\left|v_{traj}^{sup}\right| \left|\cos\left(\theta_{traj}^{slope} - \theta_d\right)\right| \left(\dot{\theta}_d - \dot{\theta}_{traj}^{slope}\right)}{\dot{D}} \end{aligned} \tag{38}$$

We rewrite (18) from the main paper here

$$\begin{aligned} \dot{V}_D &= D\dot{D} = e_x \dot{e}_x + e_y \dot{e}_y \\ \dot{V}_D &= -v_{max} \frac{\eta_v}{1+\eta_v D} \cos^2(e_\theta)(e_x^2 + e_y^2) - \dot{x}_d e_x - \dot{y}_d e_y \\ \dot{D} &= -v_{max} \frac{\eta_v D}{1+\eta_v D} \cos^2(e_\theta) - (\dot{x}_d \cos(e_\theta) + \dot{y}_d \sin(e_\theta)) \\ \dot{D} &= -v_{max} \frac{\eta_v D}{1+\eta_v D} \cos^2(e_\theta) - v_{traj}^{L} \end{aligned} \tag{39}$$

We know $D \to 0 \implies e_x \to 0,\ e_y \to 0$. Following (38) we get

$$\lim_{D\to 0} \dot{D} \to \left|v_{traj}^{L}\right| \tag{40}$$

and $\left|v_{traj}^{L}\right| = v_{traj} \cos\left(\theta_{traj}^{slope} - \theta_d\right)$. Therefore, proceeding further with (38)

$$\begin{aligned} \lim_{D\to 0} \frac{\left|v_{traj}^{\perp L}\right|}{D} &\le \lim_{D\to 0} \frac{\left|v_{traj}^{sup}\right| \left|\cos\left(\theta_{traj}^{slope} - \theta_d\right)\right| \left(\dot{\theta}_d - \dot{\theta}_{traj}^{slope}\right)}{\left|v_{traj}\right| \left|\cos\left(\theta_{traj}^{slope} - \theta_d\right)\right|} \\ &\le \lim_{D\to 0} \frac{\left|v_{traj}^{sup}\right| \left(\dot{\theta}_d - \dot{\theta}_{traj}^{slope}\right)}{\left|v_{traj}\right|} = 0 \end{aligned} \tag{41}$$

The above analysis proves that $\left|\dot{\hat{\theta}}_d\right|$ is upper bounded for all $D \in [0, \infty)$. From the results of II-A, II-B, we conclude that stable tracking, as defined in (9), is guaranteed without exceeding actuator limits.

REFERENCES

[1] Hassan K. Khalil, 2002, *Nonlinear Systems*, Prentice Hall, 3*rd* ed.