Supplementary Material

Formation control of differential-drive robots with input saturation and constraint on formation size

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***Abstract*— This piece of work is a supplementary material to the original work in [1] submitted to ACC 2023. This supplementary material discusses the proof of those arguments/expressions/equations for which this material was cited in [1]. Note that the proofs here are continuation of the proofs given in [1]. Here, the tags/labels of sections, subsections and equations are kept the same as given in the original work. Section III contains discussion to provide additional clarity to the arguments presented in [1] .**

II. Trajectory Tracking Controller with Input Constraints

II-B. Upper bound of $\left|\hat{\dot{\theta}}_d\right|$

We begin by obtaining an upper bound on $\dot{\theta}_d$ by starting with (6) of [1]. We substitute expressions of $e_x,\ e_y,\ \dot{e}_x,\ \dot{e}_y$ in (6) from (15), (16) of [1].

$$
\begin{aligned}
\left|\dot{\theta}_d\right| &= \left|\frac{e_x\dot{e}_y - e_y\dot{e}_x}{D^2}\right| \\
&= \left|\frac{e_x v\sin(\theta) - e_y v\cos(\theta) + e_y\dot{x}_d - e_x\dot{y}_d}{D^2}\right| \\
&= \left|\frac{-Dv_{max}\sin(2e_\theta)\frac{\eta_v D}{2(1+\eta_v D)} - D(v^{\perp L}_{traj})}{D^2}\right| \\
&= \left|-v_{max}\sin(2e_\theta)\frac{\eta_v}{2(1+\eta_v D)} - \frac{v^{\perp L}_{traj}}{D}\right| \\
&\leq \left|v_{max}\frac{\eta_v}{2(1+\eta_v D)}\right| + \frac{\left|v^{\perp L}_{traj}\right|}{D}
\end{aligned} \tag{29}
$$

In the region outside the ultimate bound: $D(t) \geq b_D$. Also,

$$
b_D > \mu_D > \frac{\left|v^{sup}_{traj}\right|}{\eta_v\left(v_{max}\delta^2_\theta - \left|v^{sup}_{traj}\right|\right)} \geq \frac{\left|v^{sup}_{traj}\right|}{\eta_v\left(v_{max} - \left|v^{sup}_{traj}\right|\right)} \tag{30}
$$

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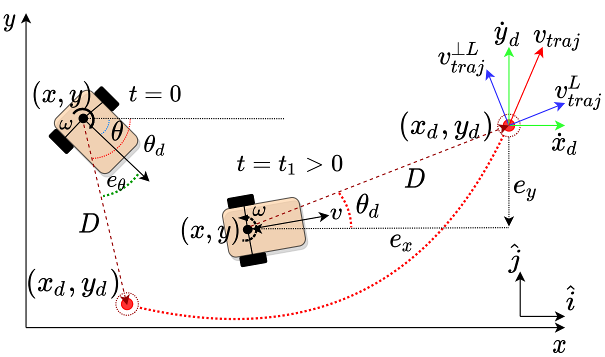

Fig. 1: Cartesian components of v_{traj} (Green): $(\dot{x}_d, \dot{y}_d)$, Components of v_{traj} along and $\perp$ to L (Blue): $(v^L_{traj}, v^{\perp L}_{traj})$. D denotes the distance between $(x,\ y)$ and $(x_d,\ y_d)$

Continuing with the result of (29), (30)

$$
\begin{aligned}
\left|\dot{\theta}_d\right| &\leq \left|v_{max}\frac{\eta_v}{2(1+\eta_v b_D)}\right| + \frac{\left|v^{\perp L}_{traj}\right|}{b_D} \\
&< \left|\frac{\eta_v\left(v_{max}\delta^2_\theta - \left|v^{sup}_{traj}\right|\right)}{2\delta^2_\theta}\right| + \eta_v\left(v_{max}\delta^2_\theta - \left|v^{sup}_{traj}\right|\right) \\
\left|\dot{\theta}_d\right| &< \frac{3\eta_v v_{max}}{2}
\end{aligned} \tag{31}
$$

Now, using (8) and upper bound on $\left|\dot{\theta}_d\right|$ in (31), we get

$$
\left|\hat{\dot{\theta}}_d\right| \leq \left|\dot{\theta}_d\right| + \varepsilon_\theta \implies \left|\hat{\dot{\theta}}_d\right| < \frac{3\eta_v v_{max}}{2} + \varepsilon_\theta \tag{32}
$$

If we choose η_v as per (13), then choosing $\Omega_{max} = \frac{3\eta_v v_{max}}{2} + \varepsilon_\theta$ will satisfy (14).

Even if $D(t)$ goes arbitrarily close to zero, $\left|\hat{\dot{\theta}}_d\right|$ will remain upper bounded because

$$
\lim_{D\to 0}\frac{\left|v^{\perp L}_{traj}\right|}{D} = 0 \tag{33}
$$

The proof of (33) is as follows.

By definition, $D \to 0 \implies e_x \to 0,\ e_y \to 0$. Therefore, we can apply L'Hospital's rule to $\lim_{D\to 0}\frac{e_y}{e_x}$. From (16) it is clear that $e_x, e_y \to 0 \implies \dot{e}_x \to \dot{x}_d,\ \dot{e}_y \to \dot{y}_d$. We also define slope of reference trajectory as

$$
\tan\left(\theta^{slope}_{traj}\right) = \frac{\dot{y}_d}{\dot{x}_d} \tag{34}
$$

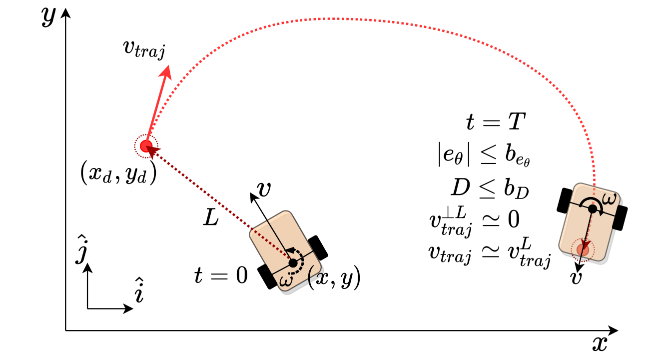

Fig. 2: Stable tracking achieved by differential drive robot using controller

From (16) of the main paper, we can write the following limits

$$D \to 0 \implies e_x \to 0,\ e_y \to 0$$

$$\lim_{D\to 0} \dot{e}_x = -\dot{x}_d, \quad \lim_{D\to 0} \dot{e}_y = -\dot{y}_d$$

With the above limits, we can simplify the limit in (34)

$$\lim_{D\to 0} \tan(\theta_d) = \lim_{D\to 0} \frac{e_y}{e_x} = \lim_{D\to 0} \frac{\dot{e}_y}{\dot{e}_x} = \frac{\dot{y}_d}{\dot{x}_d} = \tan\left(\theta_{traj}^{slope}\right) \tag{35}$$

The result of (35) implies

$$\lim_{D\to 0} \left(\tan(\theta_d) - \tan\left(\theta_{traj}^{slope}\right)\right) = 0 \implies \lim_{D\to 0} \left(\theta_d - \theta_{traj}^{slope}\right) = 0 \tag{36}$$

By *Assumption 1*, the $\dot{\theta}_{traj}^{slope}$ is uniformly continuous, and $\dot{\theta}_d$ is also uniformly continuous as defined in (6). Therefore by *Lemma 8.2: Barbalat's Lemma* of [1], we get

$$\lim_{D\to 0} \left(\dot{\theta}_d - \dot{\theta}_{traj}^{slope}\right) = 0 \tag{37}$$

We can also write $v_{traj}^{L} = v_{traj} \cos\left(\theta_{traj}^{slope} - \theta_d\right)$ and $v_{traj}^{\perp L} = v_{traj} \sin\left(\theta_{traj}^{slope} - \theta_d\right)$. A magnified view of Fig.1 is given here for reference. Therefore, (37) implies $\lim_{D\to 0} \left|v_{traj}^{\perp L}\right| = 0$. Applying L'Hospital's rule to $\lim_{D\to 0} \frac{\left|v_{traj}^{\perp L}\right|}{D}$ we get

$$\begin{aligned}
\lim_{D\to 0} \frac{\left|v_{traj}^{\perp L}\right|}{D} &\leq \lim_{D\to 0} \frac{\left|v_{traj}^{sup}\right| \left|\sin\left(\theta_{traj}^{slope} - \theta_d\right)\right|}{D} \\
&\leq \lim_{D\to 0} \frac{\left|v_{traj}^{sup}\right| \left|\cos\left(\theta_{traj}^{slope} - \theta_d\right)\right| \left(\dot{\theta}_d - \dot{\theta}_{traj}^{slope}\right)}{\dot{D}}
\end{aligned} \tag{38}$$

We rewrite (18) from the main paper here

$$\begin{aligned}
\dot{V}_D &= D\dot{D} = e_x \dot{e}_x + e_y \dot{e}_y \\
\dot{V}_D &= -v_{max} \frac{\eta_v}{1+\eta_v D} \cos^2(e_\theta)(e_x^2 + e_y^2) - \dot{x}_d e_x - \dot{y}_d e_y \\
\dot{D} &= -v_{max} \frac{\eta_v D}{1+\eta_v D} \cos^2(e_\theta) - (\dot{x}_d \cos(e_\theta) + \dot{y}_d \sin(e_\theta)) \\
\dot{D} &= -v_{max} \frac{\eta_v D}{1+\eta_v D} \cos^2(e_\theta) - v_{traj}^{L}
\end{aligned} \tag{39}$$

We know $D \to 0 \implies e_x \to 0,\ e_y \to 0$. Following (39) we get

$$\lim_{D\to 0} \dot{D} \to \left|v_{traj}^{L}\right| \tag{40}$$

and $\left|v_{traj}^{L}\right| = v_{traj} \cos\left(\theta_{traj}^{slope} - \theta_d\right)$. Therefore, proceeding further with (38)

$$\begin{aligned}
\lim_{D\to 0} \frac{\left|v_{traj}^{\perp L}\right|}{D} &\leq \lim_{D\to 0} \frac{\left|v_{traj}^{sup}\right| \left|\cos\left(\theta_{traj}^{slope} - \theta_d\right)\right| \left(\dot{\theta}_d - \dot{\theta}_{traj}^{slope}\right)}{\left|v_{traj}\right| \left|\cos\left(\theta_{traj}^{slope} - \theta_d\right)\right|} \\
&\leq \lim_{D\to 0} \frac{\left|v_{traj}^{sup}\right| \left(\dot{\theta}_d - \dot{\theta}_{traj}^{slope}\right)}{\left|v_{traj}\right|} = 0
\end{aligned} \tag{41}$$

The above analysis proves that $\left|\hat{\dot{\theta}}_d\right|$ is upper bounded for all $D \in [0, \infty)$. From the results of II-A, II-B, we conclude that stable tracking, as defined in (9), is guaranteed without exceeding actuator limits.

III. Discussion

III-A. Selecting values of v_{max}, ω_{max}

In the controller proposed (11), (12) of the main paper, $v(t)$ is a function of D, e_θ, and $\omega(t)$ is a function of e_θ, θ_d.

$$v = -v_{max} \cos(e_\theta) \frac{\eta_v D}{1+\eta_v D} \tag{42}$$

$$\omega = -(\omega_{max} - \Omega_{max}) \tanh(\eta_\omega e_\theta) + \hat{\dot{\theta}}_d \tag{43}$$

Now consider a possible scenario where $D = 20m$, $e_\theta = \pi$, $\hat{\dot{\theta}}_d \to -\Omega_{max}$ and $\eta_v = 10$, $\eta_\omega = 20$. Then in this case both $v(t) \to v_{max}$, $\omega(t) \to -\omega_{max}$. If we choose v_{max}, ω_{max} as

$$v_{max} = \frac{R\omega_{r,\ l}^{max}}{2}, \quad \omega_{max} = \frac{R\omega_{r,\ l}^{max}}{L} \tag{44}$$

Then as per (1) of main paper, either ω_r or ω_l will reach $\omega_{r,\ l}^{max}$. Hence, the limits on motor angular velocities will be obeyed at all time, $t \geq 0$.

III-B. Error in estimation of $\hat{\dot{\theta}}_d$

To estimate the error in $\hat{\dot{\theta}}_d$ relative to $\dot{\theta}_d$, we expand $\theta_d(t-\tau)$ using Taylor's series expansion about current time instance, $t > 0$,

$$\begin{aligned}
&\implies \theta_d(t-\tau) = \theta_d(t) - \dot{\theta}_d(t)\tau + \frac{\ddot{\theta}_d \tau^2}{2!} - \frac{\dddot{\theta}_d \tau^3}{3!} \dots \\
&\implies \frac{\theta_d(t) - \theta_d(t-\tau)}{\tau} - \dot{\theta}_d(t) = -\frac{\ddot{\theta}_d \tau}{2!} + \frac{\dddot{\theta}_d \tau^2}{3!} \dots \\
&\implies |\hat{\dot{\theta}}_d - \dot{\theta}_d| \leq M\left|\tau + \tau^2 + \tau^3 + \dots\right| \\
&\implies |\hat{\dot{\theta}}_d - \dot{\theta}_d| \leq M\left|\frac{\tau}{1-\tau}\right|
\end{aligned}$$

Here, $M = \max\left\{\frac{\ddot{\theta}_d \tau}{2!}, \frac{\dddot{\theta}_d \tau^2}{3!}, \ldots\right\}$. Hence, for a small time step or sampling time (e.g. $\tau = 0.01s$), the measurement error of $\dot{\theta}_d$ is upper bounded by

$$|\hat{\dot{\theta}}_d - \dot{\theta}_d| \leq M\tau$$

Now, lets choose a small positive number $\varepsilon_\theta > 0$ such that

$$|\hat{\dot{\theta}}_d - \dot{\theta}_d| \leq M\tau \leq \varepsilon_\theta \tag{45}$$

where $\varepsilon_\theta = \Omega(\tau) > 0$. Here, Ω is the Big-Ω notation respectively which adhere to convention mentioned in cited reference [2].

Note: The value of M defined above depends on the sensor resolution, robot's specification, the reference trajectory, etc. In the simulation shown in section III.C of [1], we have chosen $\varepsilon_\theta = 0.1$ for $\tau = 0.01s$ based on above discussion. Even if a higher value for ε_θ is chosen, the ultimate bound on e_θ would not increase much in magnitude due to the presence of natural logarithm, 'ln', in the expression of b_{e_θ}.

III-C. Significance of adjustable scalar ζ

In section II.A of the main paper, we have defined ζ such that $0 < \zeta \ll 1$. ζ was used in defining upper and lower bounds on the Lyapunov function V_D, V_{e_θ} defined in (17), (23) of the main paper. At the end of the stability analysis, we have derived ultimate bounds on D, e_θ.

$$\begin{aligned} \mu_D \leq D = \left\|e_x, e_y\right\|_2 \leq b_D \quad & \forall t \geq T = \max\{T_D, T_{e_\theta}\} \\ \mu_{e_\theta} \leq |e_\theta| \leq b_{e_\theta} \quad & \forall t \geq T = \max\{T_D, T_{e_\theta}\} \end{aligned} \tag{46}$$

The ultimate bounds depend on various parameters other than ζ. The expressions for these ultimate bounds are as follows.

$$\begin{aligned} \mu_D &= (1+\zeta)\frac{|v_{traj}^{sup}|}{\eta_v\left(v_{max}\delta_\theta^2 - |v_{traj}^{sup}|\right)} \\ b_D &= \frac{\sqrt{2+\zeta}}{\sqrt{2-\zeta}}\mu_D \\ \mu_{e_\theta} &= \frac{(1+\zeta)}{2\eta_\omega}\ln\left(\frac{\omega_{max} - \Omega_{max} + \varepsilon_\theta}{\omega_{max} - \Omega_{max} - \varepsilon_\theta}\right) \\ b_{e_\theta} &= \frac{\sqrt{2+\zeta}}{\sqrt{2-\zeta}}\mu_{e_\theta} \end{aligned} \tag{47}$$

These parameters and their value are listed below.

Controller parameters			
Parameter	Value	Parameter	Value
R	$0.02m$	η_v	10
L	$0.09m$	η_ω	10
$\omega_{r,\,l}^{max}$	$52 rad/s$	τ	0.01
v_{max}	$0.52\ m/s$	ε_θ	$0.1 rad/s$
ω_{max}	$11.5 rad/s$	Ω_{max}	$7.9 rad/s$

TABLE I

From (47), it is apparent that as $\zeta \to 0$, the upper bound will approach lower bound, the lower bound will approach a constant value depending on the parameters listed above.

$$\begin{aligned} \lim_{\zeta\to 0} b_D &= \lim_{\zeta\to 0}\mu_D = \frac{|v_{traj}^{sup}|}{\eta_v\left(v_{max}\delta_\theta^2 - |v_{traj}^{sup}|\right)} = 0.1524 \\ \lim_{\zeta\to 0} b_{e_\theta} &= \lim_{\zeta\to 0}\mu_{e_\theta} = \frac{1}{2\eta_\omega}\ln\left(\frac{\omega_{max}-\Omega_{max}+\varepsilon_\theta}{\omega_{max}-\Omega_{max}-\varepsilon_\theta}\right) = 2.7\times 10^{-3} \end{aligned} \tag{48}$$

From above discussion, it is clear that the primary role of ζ was to show the existence of ultimate bounds on error terms. More importantly, adjusting the value of ζ closer to 0 showed us that upper and lower bounds on error terms decrease in value, approach each other, and eventually converge to a constant value that depends on parameters listed in TABLE I. Fig.3, 4 illustrate this graphically.

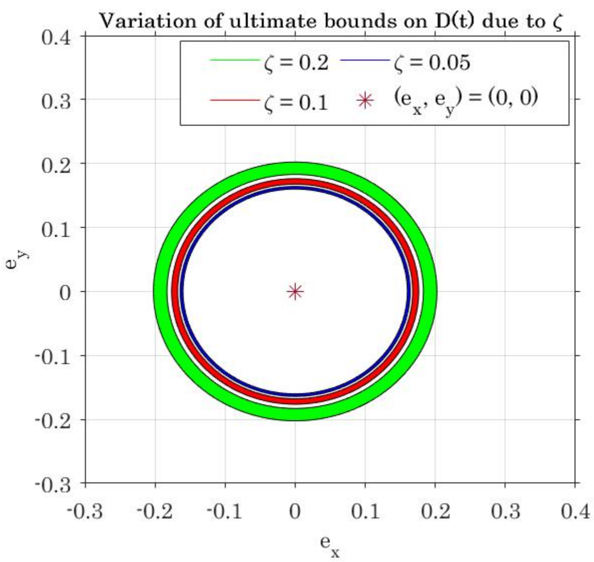

Fig. 3: Error bands for $D(t)$ shrinks and approach a radius of 0.1524 as the values of ζ is reduced

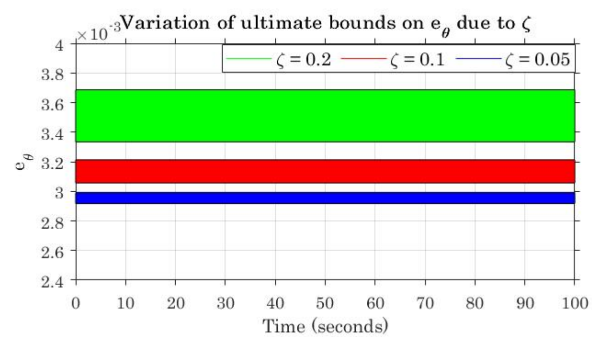

Fig. 4: Error bands for $e_\theta(t)$ shrinks and approach 2.7×10^{-3} as the values of ζ is reduced

III-D. Alternate proposal for linear velocity control $v(t)$

The following linear velocity control can also be used in place of controller proposed in (11) of main paper.

$$v(t) = -v_{max}\cos(e_\theta)\tanh(\eta_v D) \tag{49}$$

The stability analysis for this controller can be done in the same way as done in section II-A of the main paper. Bounds of η_v, $\hat{\dot{\theta}}_d$ can be obtained for this controller as well.

REFERENCES

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