Project Report

Solving Infinite Plate Problem with general traction boundary condition in the circular hole using Fourier Series

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1 Introduction

The problem here is about obtaining the solution field i.e stress field, strain field and displacement field for a homogeneous-isotropic-elastic 2-D circular domain with infinite boundary and having a hole of radius R at which we are applying generalised traction boundary condition i.e both $\sigma_{rr}\ and\ \sigma_{r\theta}$.
The aim will be to solve the problem for different forms of loading i.e continuous loading, piecewise-continuous loading and point load. In order to solve the above problem, general solution of the **Biharmonic equation** in Fourier form in cylindrical coordinates will be used.

$$\nabla^4\phi = 0 \Rightarrow \phi\ is\ the\ Airy\ Stress\ Function \tag{1}$$

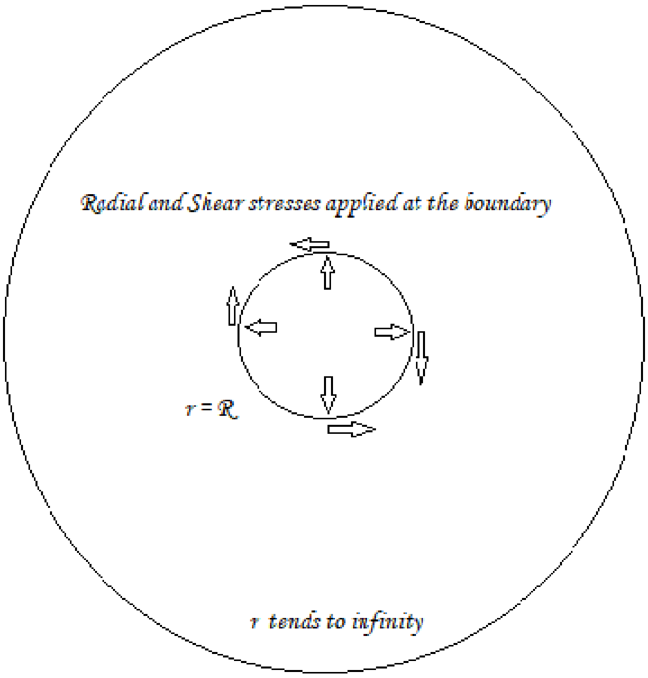


Figure 1: Schematic of the problem

The above problem has the following boundary conditions in terms of traction and displacements ⇒

$$\sigma_{rr}(R,\theta) = -P(\theta)$$
$$\sigma_{r\theta}(R,\theta) = -S(\theta)$$
$$\sigma_{r\theta} \to 0\ as\ r \to \infty \mid \sigma_{rr} \to 0\ as\ r \to \infty$$
$$u_\theta(r,\theta) \to 0\ as\ r \to \infty \mid u_r(r,\theta) \to 0\ as\ r \to \infty$$

Here $P(\theta)\ and\ S(\theta)$ are the radial and shear stresses as a function of θ applied at the inner boundary at r=R. We will solve for the following loading conditions ⇒

$$Continuous\ loading \to P(\theta) =\ sin(4\theta) \mid S(\theta)\ =\ cos(4\theta)$$

$$Piecewise\ continuous\ loading \to P(\theta) = \begin{cases} 0 & 0 < \theta < \frac{\pi}{4} \\ 1 & \frac{\pi}{4} \le \theta \le \frac{3\pi}{4} \\ 0 & \frac{3\pi}{4} < \theta < \frac{5\pi}{4} \\ 1 & \frac{5\pi}{4} \le \theta \le \frac{7\pi}{4} \\ 0 & \frac{7\pi}{4} < \theta < 2\pi \end{cases} \mid S(\theta) = 0$$

$$Point\ loading \to P(\theta) = Dirac(\theta - \frac{\pi}{2}) + Dirac(\theta - \frac{3\pi}{2}) \mid S(\theta) = 0$$

2 Theory

2.1 Plane Stress

We are solving our problem under plane stress condition. Under this condition,
$\sigma_{z\theta} = \sigma_{zr} = \sigma_{zz} = 0 \; and \; \epsilon_{zr} = \epsilon_{z\theta} = 0$. In matrix form,

$$\sigma = \begin{pmatrix} \sigma_{rr} & \sigma_{r\theta} & 0 \\ \sigma_{\theta r} & \sigma_{\theta\theta} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\epsilon = \begin{pmatrix} \epsilon_{rr} & \epsilon_{r\theta} & 0 \\ \epsilon_{\theta r} & \epsilon_{\theta\theta} & 0 \\ 0 & 0 & \epsilon_{zz} \end{pmatrix}$$

One can observe that $\epsilon_{zz} \neq 0$. This is because as per the Genralised Hooke's Law,

$$\epsilon_{zz} = \frac{-\upsilon(\sigma_{rr} + \sigma_{\theta\theta})}{E}$$

Since $\sigma_{rr} \; and \; \sigma_{\theta\theta} \; are \; non-zero \; functions$, $so \; \epsilon_{zz} \; has \; to \; be \; non-zero.$ Other equation of Generalised Hooke's Law that we will be using in our analysis,

$$\epsilon_{rr} = \frac{\sigma_{rr} - \upsilon\sigma_{\theta\theta}}{E}$$
$$\epsilon_{\theta\theta} = \frac{\sigma_{\theta\theta} - \upsilon\sigma_{rr}}{E}$$
$$\epsilon_{r\theta} = \frac{(1+\upsilon)\sigma_{r\theta}}{E}$$

2.2 Stress Function Formulation

For a static problem, the set of equations that are used to solve a problem are,
Equilibrium Equations $\Rightarrow$

$$\frac{\partial\sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + \frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta} = 0$$
$$\frac{\partial\sigma_{r\theta}}{\partial r} + \frac{2\sigma_{r\theta}}{r} + \frac{1}{r}\frac{\partial\sigma_{r\theta}}{\partial\theta} = 0$$

Strain Compatibility $\Rightarrow$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial\epsilon_{\theta\theta}}{\partial r}\right) - r\frac{\partial\epsilon_{rr}}{\partial r} + \frac{\partial^2\epsilon_{\theta\theta}}{\partial\theta^2} = 2\frac{\partial}{\partial r}\left(r\frac{\partial\epsilon_{r\theta}}{\partial\theta}\right)$$

Since we are dealing with elastic-isotropic-homogeneous material, we can use the equations for generalised Hooke's Law

$$\epsilon_{rr} = \frac{\sigma_{rr} - \upsilon\sigma_{\theta\theta}}{E}$$
$$\epsilon_{\theta\theta} = \frac{\sigma_{\theta\theta} - \upsilon\sigma_{rr}}{E}$$

$$\epsilon_{zz} = \frac{-\upsilon(\sigma_{rr} + \sigma_{\theta\theta})}{E}$$

$$\epsilon_{r\theta} = \frac{(1+\upsilon)\sigma_{r\theta}}{E}$$

We are solving a 2-D problem with elastic-isotropic-homogeneous material. As the first step of solution we will plug in the Hooke's Law equations in Strain Compatibility equation. After simplifying the expression obtained from the last step, we will get a new equation along with the equilibrium equations $\Rightarrow$

$$\nabla^2(\sigma_{rr} + \sigma_{\theta\theta}) = 0$$

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} = 0$$

$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{2\sigma_{r\theta}}{r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} = 0$$

It should be noted that the above set of equations is independent of any material property and also not involve any body forces. Hence we can say that for a 2-D problem having only traction boundary conditions (all the traction vectors should lie in the plane of the problem otherwise the plane stress condition will be violated) and no displacement boundary condition or body forces, the stress field that will be obtained is the same for all the elastic-isotropic-homogeneous materials i.e for the same in-plane loading obeying the discussed conditions, Steel and Aluminium will have the same stress field. But, the strain field and the displacement field will be different since they obtained by using the Hooke's law which involve material property. Let's define the Airy stress function $\phi \Rightarrow$

$$\sigma_{rr} = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}$$

$$\sigma_{\theta\theta} = \frac{\partial^2 \phi}{\partial r^2}$$

$$\tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right)$$

ϕ is defined such that it satisfies the last set of equations. Use of stress function automates the equilibrium and we get a **Biharmonic Equation** $\Rightarrow$

$$\nabla^4\phi = 0 \;\Rightarrow\; \left(\frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial r^2}\right)\left(\frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial r^2}\right) = 0$$

2.3 Solution of Biharmonic Equation

The stress and displacements must be single-valued and continuous and hence they must be periodic functions of θ, since $(r, \theta + 2m\pi)$ defines the same point as (r, θ), when m is an integer. It is therefore natural to seek the general solution of the Biharmonic equation in the form $\Rightarrow$

$$\phi = \sum_{n=0}^{+\infty} f_n(r)cos(n\theta) + \sum_{n=1}^{+\infty} g_n(r)sin(n\theta)$$

Substituting this expression into the Biharmonic equation, we find that the $f_n,\ g_n$ must satisfy the ordinary differential equation,

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{n^2}{r^2}\right)\left(\frac{\mathrm{d}^2 f_n}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}f_n}{\mathrm{d}r} - \frac{n^2 f_n}{r^2}\right) = 0$$

For $n \neq 0, 1$ the general solution for the above equation is

$$f_n(r) = A_{n1}r^{n+2} + A_{n2}r^{-n+2} + A_{n3}r^n + A_{n4}r^{-n}$$

$A_{n1}, ..A_{n4}$ are arbitrary constants.

When $n = 0, 1$ the solution develops repeated roots (*degenerate cases*) and so we get a different form of equation given as

$$f_0(r) = A_{01}r^2 + A_{02}r^2 ln(r) + A_{03}ln(r) + A_{04}$$
$$f_1(r) = A_{11}r^3 + A_{12}rln(r) + A_{13}r + A_{04}r^{-1}$$

Keeping in mind the degenerate cases for $n = 0, 1$ and the preceding results, we can write down a general solution of the elasticity problem in polar coordinates, such that the stress components form a Fourier series in θ. We have

$$\begin{aligned}\phi = {} & A_{01}r^2 + A_{02}r^2 ln(r) + A_{03}ln(r) + A_{04}\theta \\ & +(A_{11}r^3 + A_{12}rln(r) + A_{14}r^{-1})cos(\theta) + A_{13}r\theta sin(\theta) \\ & +(B_{11}r^3 + B_{12}rln(r) + B_{14}r^{-1})sin(\theta) + B_{13}r\theta cos(\theta) \\ & +\sum_{n=2}^{+\infty}(A_{n1}r^{n+2} + A_{n2}r^{-n+2} + A_{n3}r^n + A_{n4}r^{-n})cos(n\theta) \\ & +\sum_{n=2}^{+\infty}(B_{n1}r^{n+2} + B_{n2}r^{-n+2} + B_{n3}r^n + B_{n4}r^{-n})sin(n\theta)\end{aligned}$$

This is called the Michell solution. The stress components can be obtained by using the relation between Airy stress function and stresses.

$$\begin{aligned}\sigma_{rr} = {} & 2A_{01} + A_{02}(2ln(r) + 1) + \frac{A_{03}}{r^2} \\ & +\left(2A_{11}r + \frac{A_{12}}{r} + A_{13}r - \frac{2A_{14}}{r^3}\right)cos(\theta) \\ & +\left(2B_{11}r + \frac{B_{12}}{r} + B_{13}r - \frac{2B_{14}}{r^3}\right)sin(\theta) \\ & -\sum_{n=2}^{+\infty}\left(A_{n1}(n-2)(n+1)r^n + \frac{A_{n2}(n+2)(n-1)}{r^n} + A_{n3}n(n-1)r^{n-2} + \frac{A_{n4}n(n+1)}{r^{n+2}}\right)cos(n\theta) \\ & -\sum_{n=2}^{+\infty}\left(B_{n1}(n-2)(n+1)r^n + \frac{B_{n2}(n+2)(n-1)}{r^n} + B_{n3}n(n-1)r^{n-2} + \frac{B_{n4}n(n+1)}{r^{n+2}}\right)sin(n\theta)\end{aligned}$$

$$\begin{aligned}\sigma_{\theta\theta} = {} & 2A_{01} + A_{02}(2ln(r) + 3) - \frac{A_{03}}{r^2} \\ & +\left(6A_{11}r + \frac{A_{12}}{r} + \frac{2A_{14}}{r^3}\right)cos(\theta) \\ & +\left(6B_{11}r + \frac{B_{12}}{r} + \frac{2B_{14}}{r^3}\right)sin(\theta)\end{aligned}$$

$$+\sum_{n=2}^{+\infty}\left(A_{n1}(n+2)(n+1)r^n+\frac{A_{n2}(n-2)(n-1)}{r^n}+A_{n3}n(n-1)r^{n-2}+\frac{A_{n4}n(n+1)}{r^{n+2}}\right)cos(n\theta)$$
$$+\sum_{n=2}^{+\infty}\left(B_{n1}(n+2)(n+1)r^n+\frac{B_{n2}(n-2)(n-1)}{r^n}+B_{n3}n(n-1)r^{n-2}+\frac{B_{n4}n(n+1)}{r^{n+2}}\right)sin(n\theta)$$

$$\sigma_{r\theta}=\frac{A_{04}}{r^2}$$
$$+\left(2A_{11}r+\frac{A_{12}}{r}-\frac{2A_{14}}{r^3}\right)sin(\theta)$$
$$-\left(2B_{11}r+\frac{B_{12}}{r}-\frac{2B_{14}}{r^3}\right)cos(\theta)$$
$$+\sum_{n=2}^{+\infty}\left(A_{n1}n(n+1)r^n-\frac{A_{n2}n(n-1)}{r^n}+A_{n3}n(n-1)r^{n-2}-\frac{A_{n4}n(n+1)}{r^{n+2}}\right)sin(n\theta)$$
$$-\sum_{n=2}^{+\infty}\left(B_{n1}n(n+1)r^n-\frac{B_{n2}n(n-1)}{r^n}+B_{n3}n(n-1)r^{n-2}-\frac{B_{n4}n(n+1)}{r^{n+2}}\right)cos(n\theta)$$

Along with this strain field i.e. $\epsilon_{rr}, \epsilon_{\theta\theta}, \epsilon_{r\theta}\ and\ \epsilon_{zz}$ can be obtained using the generalised Hooke's Law. To obtain the displacement field i.e. $u_r\ and\ u_\theta$ we will use the differential relations between the strain and the displacement.

$$\epsilon_{rr}=\frac{\partial u_r}{\partial r}$$
$$\epsilon_{\theta\theta}=\frac{1}{r}\left(u_r+\frac{\partial u_\theta}{\partial\theta}\right)$$
$$\epsilon_{r\theta}=\frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta}+\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r}\right)$$

But on integration we will get arbitrary function of θ in u_r and similarly functions of r and functions of θ in u_θ.

2.4 Fourier Series

We are dealing with the problem in cylindrical coordinates and our boundary is circular. That means our boundary condition follow the following condition ⇒

$$P(\theta)=P(\theta+2\pi)\mid S(\theta)=S(\theta+2\pi)$$

This means that our boundary condition functions are 2π periodic. Apart from this our boundary condition functions are finite in the interval $[0,\ 2\pi]$, that means they are integrable in the interval $[0,\ 2\pi]$. With the above conditions satisfied we can obtain Fourier coefficients for our boundary condition functions using the following integral expressions ⇒

$$a_n=\frac{1}{\pi}\int_0^{2\pi}f(\theta)cos(n\theta)d\theta$$
$$b_n=\frac{1}{\pi}\int_0^{2\pi}f(\theta)sin(n\theta)d\theta$$

Using the above integrals and keeping in mind our general solution we obtain the following linear equations for our solution coefficients. For $n \neq 0, 1$ equation (1) and (3) on solving give A_{n2} and A_{n4} and similarly (2) and (4) give B_{n2} and B_{n4} .

$$A_{03} = -\frac{R^2}{2\pi}\int_0^{2\pi} P(\theta)d\theta$$

$$A_{14} =$$

$$B_{14} =$$

$$\frac{A_{n2}(n+2)(n-1)}{R^n} + \frac{A_{n4}n(n+1)}{R^{n+2}} = \frac{1}{\pi}\int_0^{2\pi} P(\theta)cos(n\theta)d\theta \; .....(1)$$

$$\frac{B_{n2}(n+2)(n-1)}{R^n} + \frac{B_{n4}n(n+1)}{R^{n+2}} = \frac{1}{\pi}\int_0^{2\pi} P(\theta)sin(n\theta)d\theta \; .....(2)$$

$$\frac{A_{n2}n(n-1)}{R^n} + \frac{A_{n4}n(n+1)}{R^{n+2}} = \frac{1}{\pi}\int_0^{2\pi} S(\theta)sin(n\theta)d\theta \; .....(3)$$

$$\frac{B_{n2}n(n-1)}{R^n} + \frac{B_{n4}n(n+1)}{R^{n+2}} = -\frac{1}{\pi}\int_0^{2\pi} S(\theta)cos(n\theta)d\theta \; .....(4)$$

For $n \neq 0, 1$ equation (1) and (3) on solving give A_{n2} and A_{n4} and similarly (2) and (4) give B_{n2} and B_{n4} .

3 Solving the problem using Maple programming

In order to solve the problem, a procedure can be written using Maple programming. The format of the procedure used here is as follows ⇒

```
SOLVE:= proc(E, υ, R, F, G, N, R_plot, θ_plot, K)
local .....;
      BODY OF THE CODE
      LOOPS
      EXTRACTING DATA
      PLOTS
end proc;
```

Here, E → Young's Modulus
υ → Poisson's ratio
R → Inner radius of the domain
F → Radial stress distribution
G → Shear stress distribution
N → To what coefficient we want to compute
(Since we are using Fourier series to solve our problem, the accuracy of our results depend on the number on coefficients we compute. This is what the value of N decides)
R_{plot} → Radius at which we want the variation of solution fields with respect to θ
θ_{plot} → θ at which we want the variation of solution fields with respect to r
K → This variable decides the y-range for the plots

After the computation is over, all the components of the solution field i.e. Radial stress, Hoop stress, Shear stress, Radial strain, Hoop strain, z-strain, Radial displacement, Hoop displacement and z-displacement are displayed as a function of r, θ. Along with the functions, plots showing the variations of solution fields with respect to θ at r = R_{plot} and with respect to r at $\theta = \theta_{plot}$.

Example

SOLVE(1, 0.3, 10, $\sin(4\theta)$, $\cos(4\theta)$, 4, 20, $\frac{\pi}{16}$, 2)

Results

Statics satisfied

$$S_{rr} = -\left(\frac{60000}{r^4} - \frac{5000000}{r^6}\right)\sin(4\,\theta)$$

$$S_{\theta^2} = \left(\frac{20000}{r^4} - \frac{5000000}{r^6}\right)\sin(4\,\theta)$$

$$S_{r\theta} = \left(\frac{40000}{r^4} - \frac{5000000}{r^6}\right)\cos(4\,\theta)$$

$$e_{rr} = -\left(\frac{60000}{r^4} - \frac{5000000}{r^6}\right)\sin(4\,\theta) - 0.3\left(\frac{20000}{r^4} - \frac{5000000}{r^6}\right)\sin(4\,\theta)$$

$$e_{\theta^2} = \left(\frac{20000}{r^4} - \frac{5000000}{r^6}\right)\sin(4\,\theta) + 0.3\left(\frac{60000}{r^4} - \frac{5000000}{r^6}\right)\sin(4\,\theta)$$

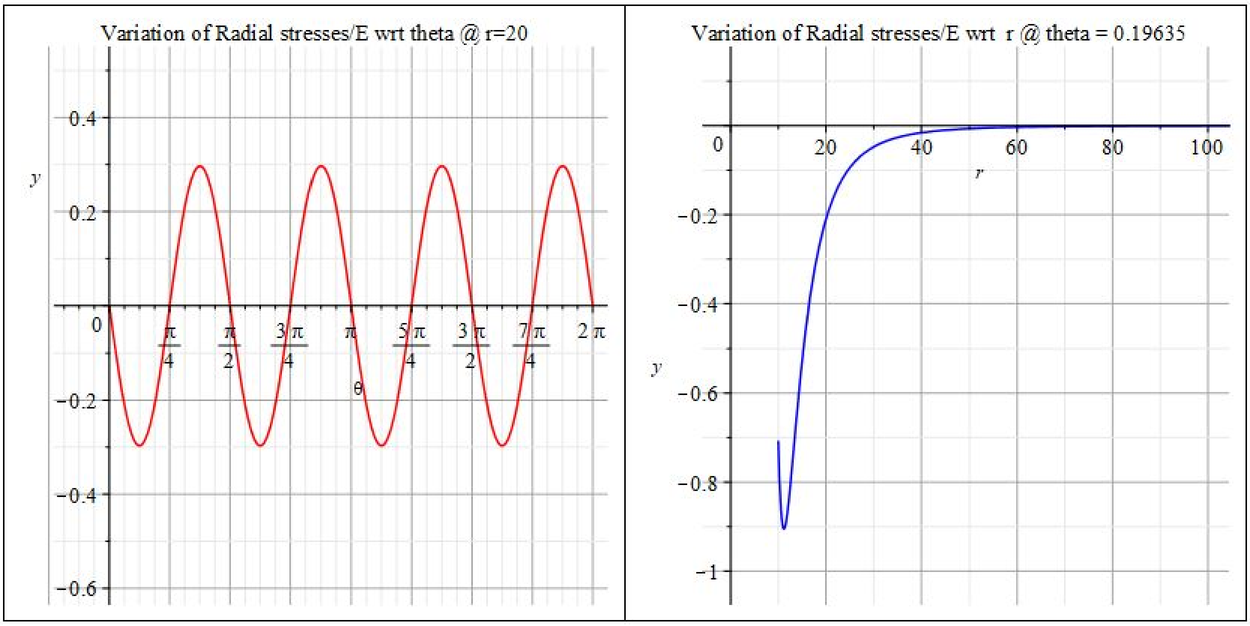
$$e_{r\theta} = 1.3\left(\frac{40000}{r^4} - \frac{5000000}{r^6}\right)\cos(4\,\theta)$$

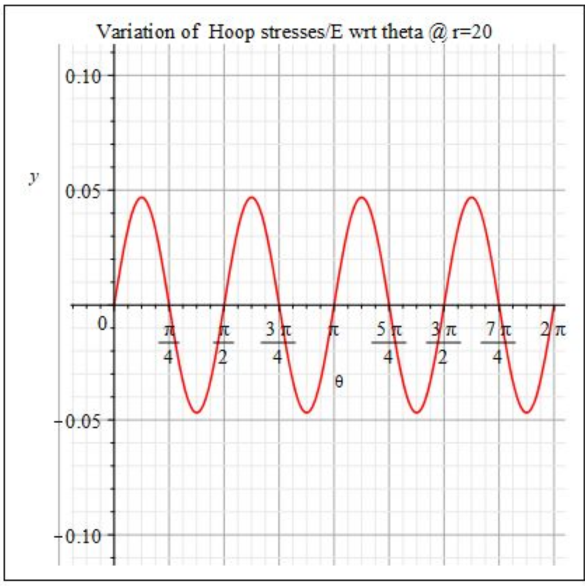
$$e_{zz} = 0.3\left(\frac{60000}{r^4} - \frac{5000000}{r^6}\right)\sin(4\,\theta) - 0.3\left(\frac{20000}{r^4} - \frac{5000000}{r^6}\right)\sin(4\,\theta)$$

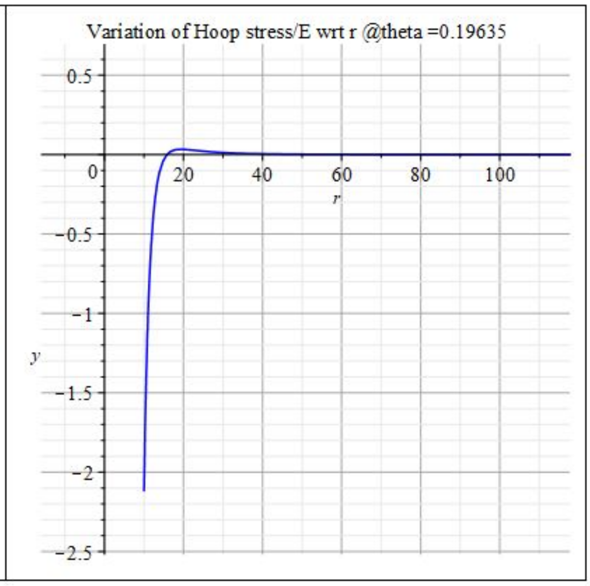
$$u_r = 1.3\left(\frac{16923.07692}{r^3} + \frac{13333.33333}{r^5}\right)\sin(4.\,\theta)$$

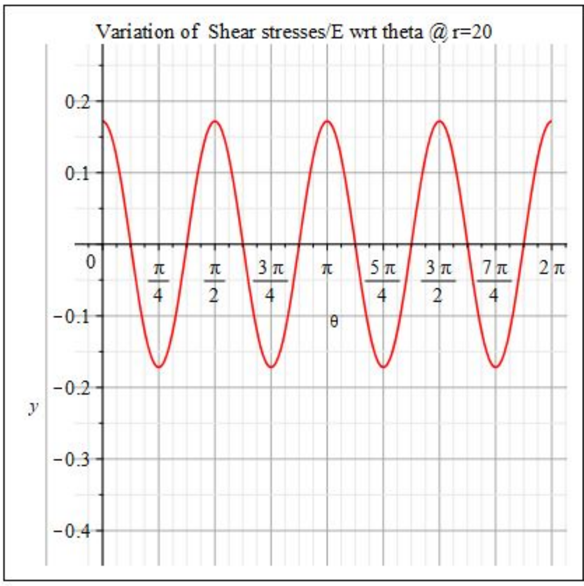
$$u_\theta = 1.3\left(-\frac{3076.923077}{r^3} + \frac{1.000000\;10^6}{r^5}\right)\cos(4.\,\theta)$$

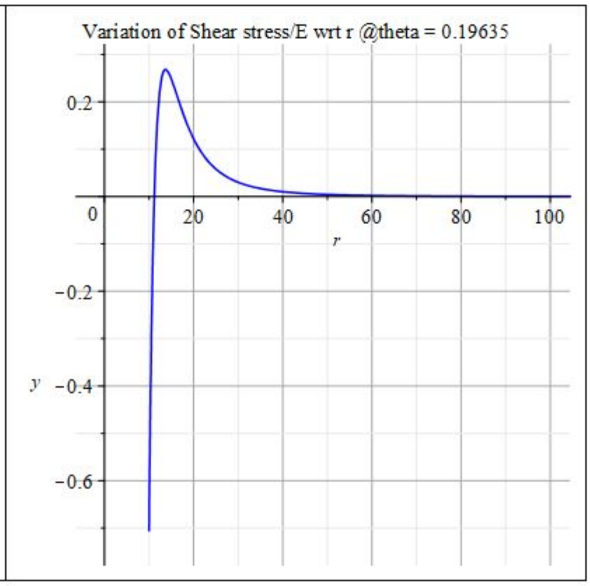
$$u_{zz} = \left(0.3\left(\frac{60000}{r^4} - \frac{5000000}{r^6}\right)\sin(4\,\theta) - 0.3\left(\frac{20000}{r^4} - \frac{5000000}{r^6}\right)\sin(4\,\theta)\right)z$$

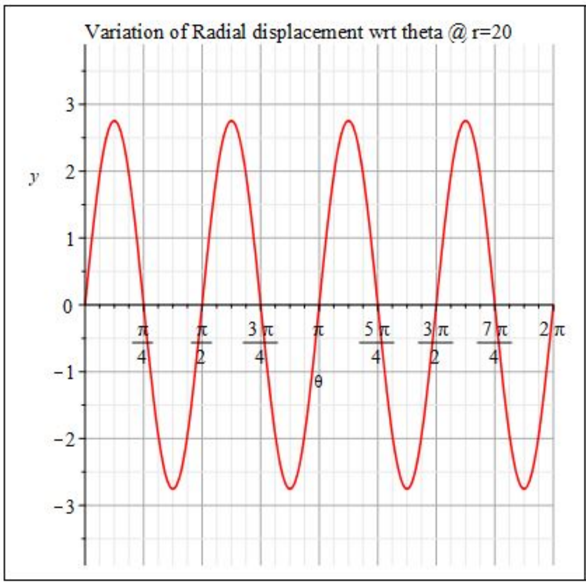


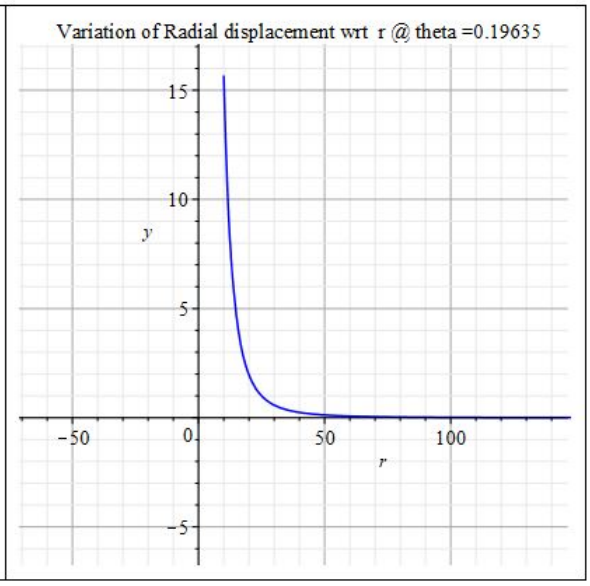

Variation of Hoop stresses/E wrt theta @ r=20
y
θ

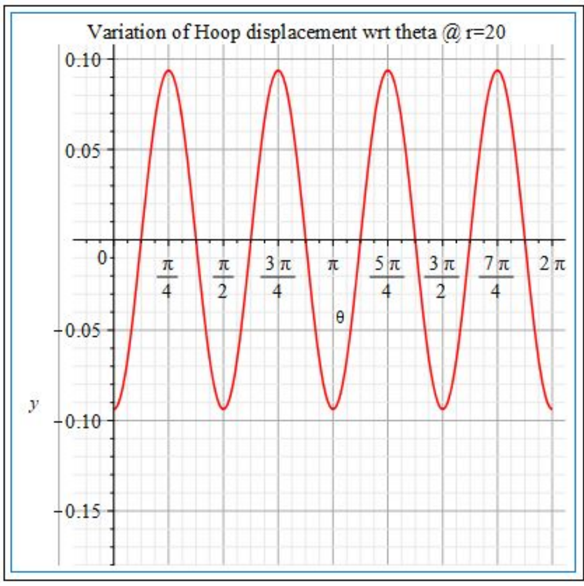

Variation of Hoop stress/E wrt r @theta =0.19635
y
r

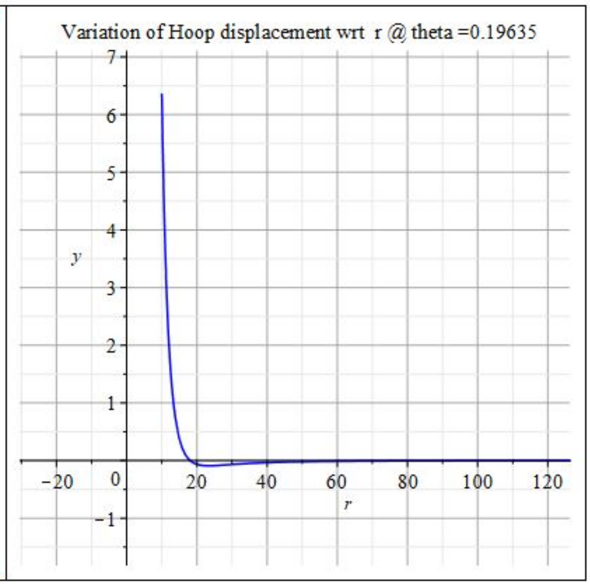

Variation of Shear stresses/E wrt theta @ r=20
y
θ

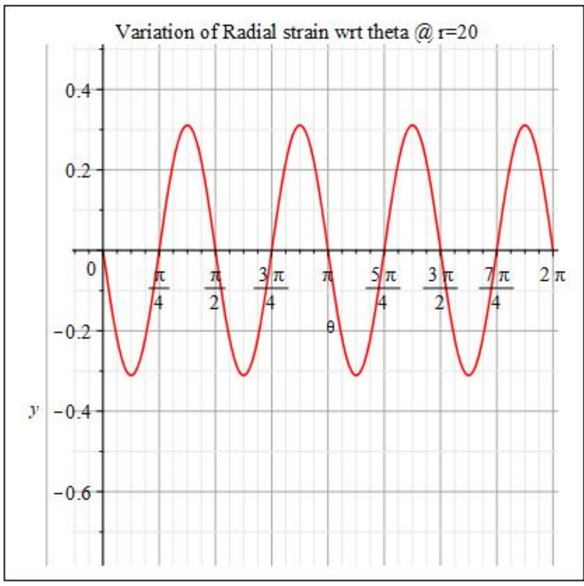

Variation of Shear stress/E wrt r @theta = 0.19635
y
r

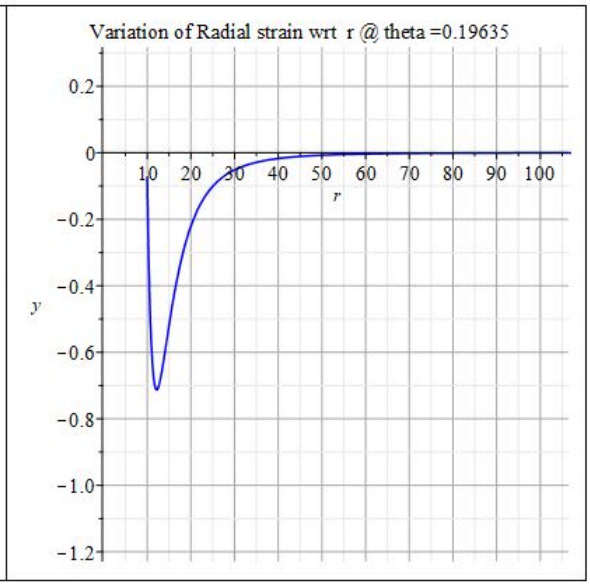

Variation of Radial displacement wrt theta @ r=20
y
θ

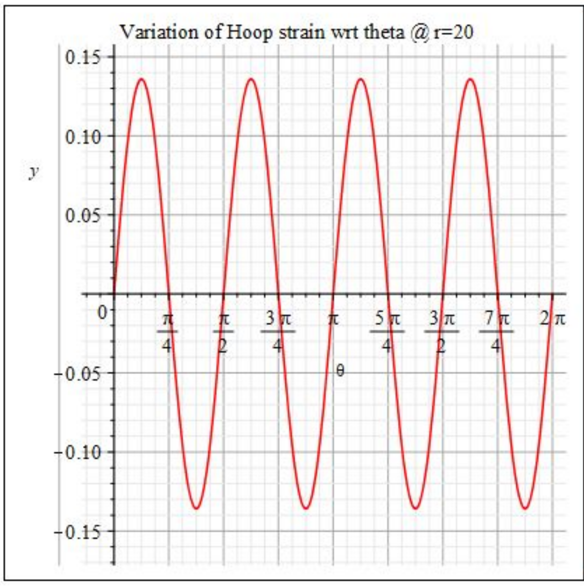

Variation of Radial displacement wrt r @ theta =0.19635
y
r


Variation of Hoop displacement wrt theta @ r=20
0.10
0.05
0
−0.05
−0.10
−0.15
y
π/4 π/2 3π/4 π 5π/4 3π/2 7π/4 2π
θ


Variation of Hoop displacement wrt r @ theta =0.19635
7
6
5
4
3
2
1
0
−1
y
−20 20 40 60 80 100 120
r


Variation of Radial strain wrt theta @ r=20
0.4
0.2
0
−0.2
−0.4
−0.6
y
π/4 π/2 3π/4 π 5π/4 3π/2 7π/4 2π
θ


Variation of Radial strain wrt r @ theta =0.19635
0.2
0
−0.2
−0.4
−0.6
−0.8
−1.0
−1.2
y
10 20 30 40 50 60 70 80 90 100
r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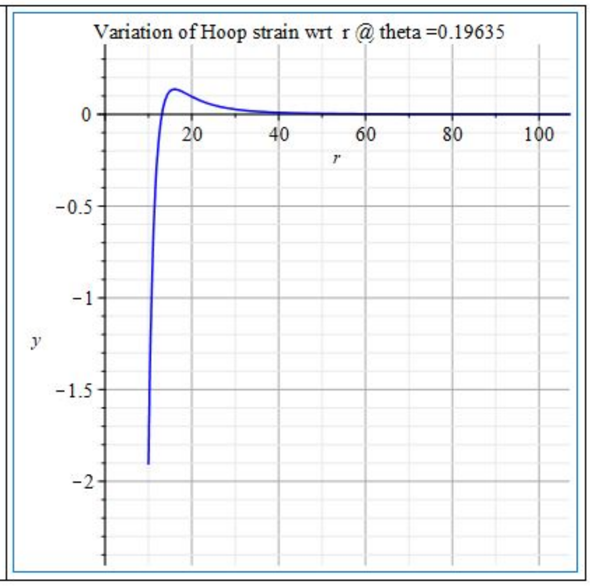
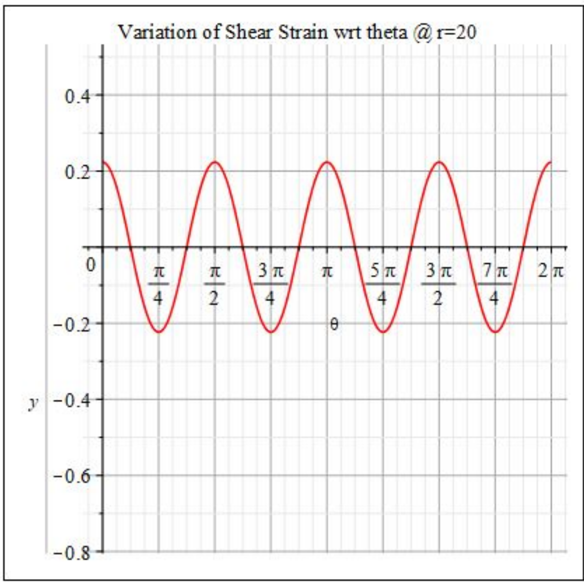
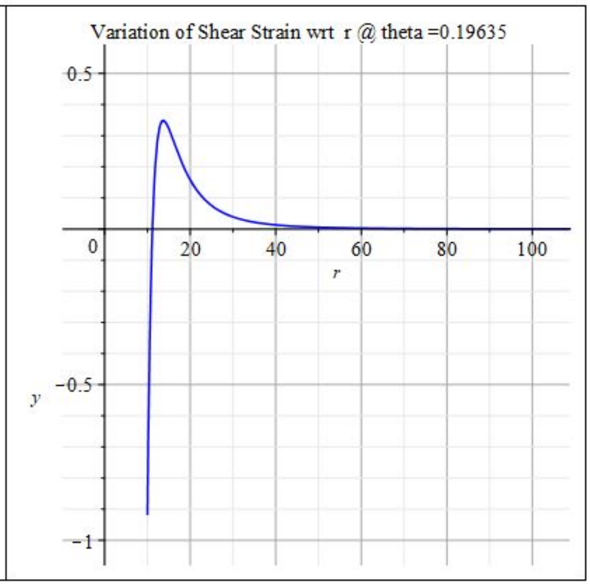
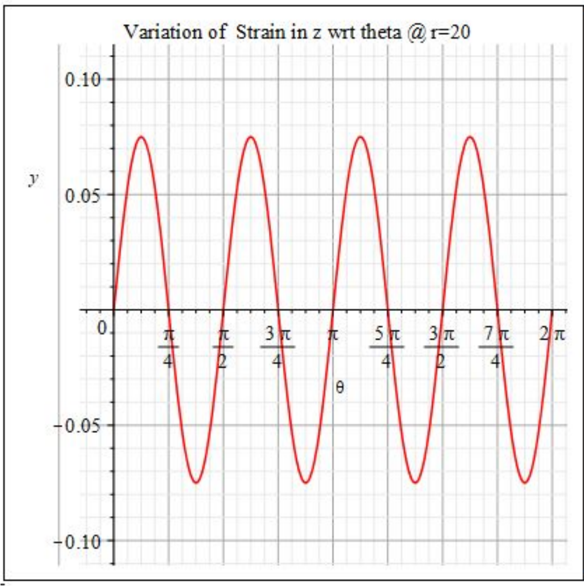
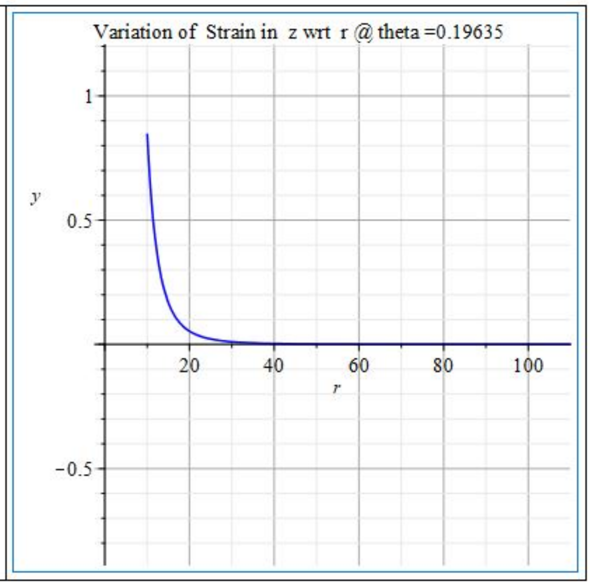
0.15
0.10
0.05
0
−0.05
−0.10
−0.15
y
π/4 π/2 3π/4 π 5π/4 3π/2 7π/4 2π
θ


Variation of Hoop strain wrt r @ theta =0.19635
0
−0.5
−1
−1.5
−2
y
20 40 60 80 100
r


Variation of Shear Strain wrt theta @ r=20
y
θ

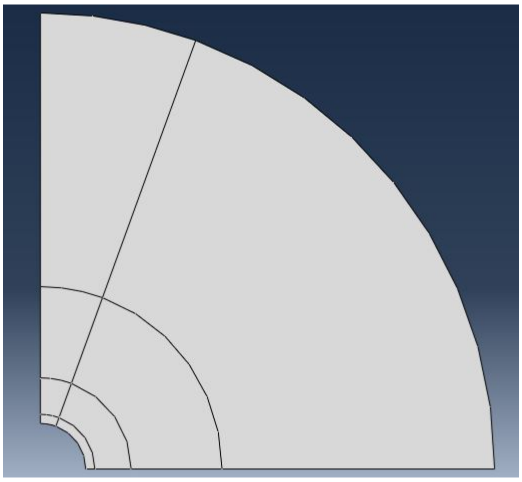

Variation of Shear Strain wrt r @ theta =0.19635
y
r


Variation of Strain in z wrt theta @ r=20
y
θ

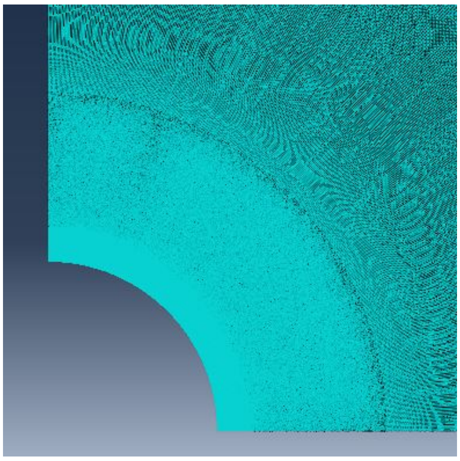

Variation of Strain in z wrt r @ theta =0.19635
y
r

4 Verification of Maple results using Finite Element Analysis

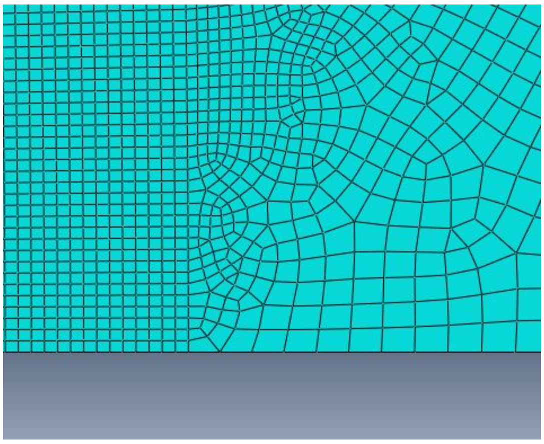
In order to verify the results obtained from Maple procedure, we will model our problem in **Abaqus** with proper loading and boundary conditions. It should be noted that our problem is symmetric in terms of loading, hence we can use quarter circle. In original problem we only had traction boundary conditions but since we are changing our domain we have to put relevant displacement boundary conditions to obtain same results.



To get appropriate results, one has to take great care of the mesh while modelling the problem. The meshing should be absolutely fine and **structured** around the region of loading. To do this, one can create partitions in the domain as shown and mesh them separately. For the rest of the region, we can gradually use **free** and coarser mesh as shown as we move outwards away the region of loading. This will give smooth and precise results near boundaries and region of applied load and will eliminate the possibilities of oscillations in the final result.



(a) Meshing of Domain

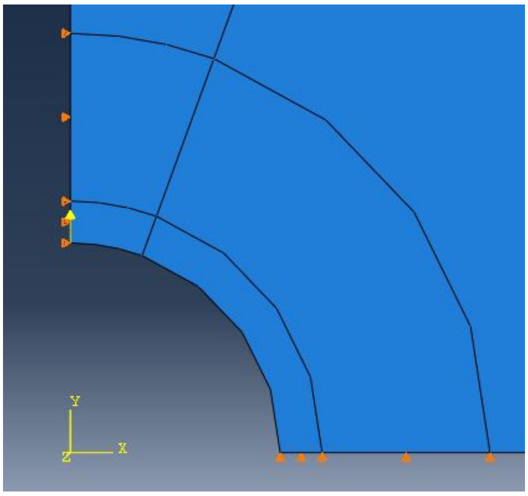


(b) Types of mesh - Free and Structured

Since we have radial and shear traction as our boundary condition, they will have a high chance of matching the FEM data. So, for complete verification we need to make sure that Maple data for $\sigma_{\theta\theta},\ u_r\ and\ u_\theta$ matches with the FEM data. Hence if the data for these three variables match then we can say that our Maple procedure works appropriately.
For verification purpose we will take E=1 and $\upsilon = 0.3$

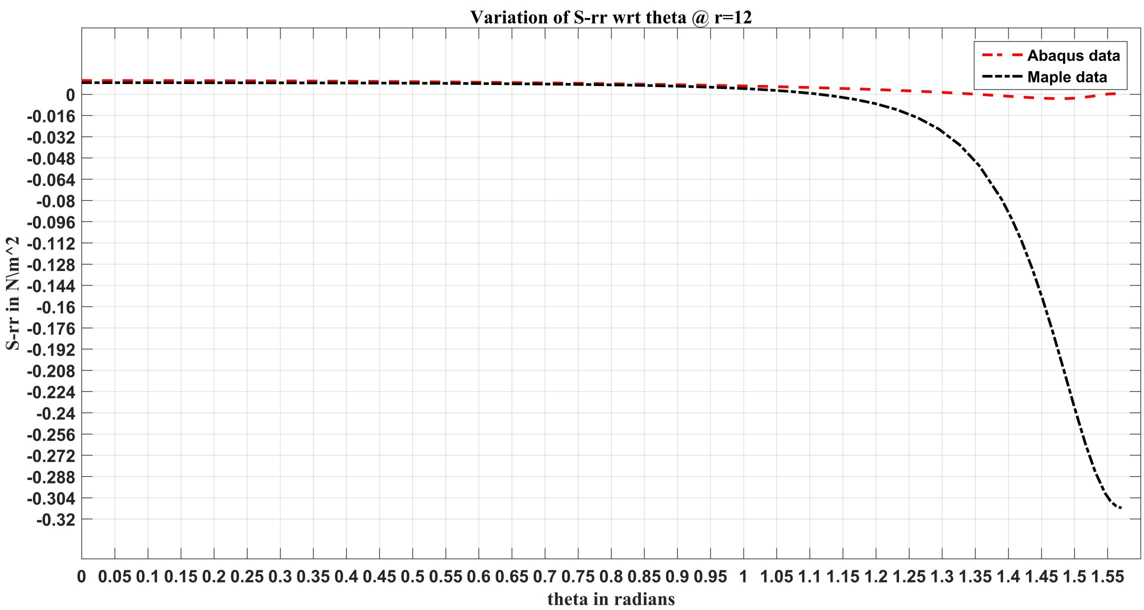
Example 1
In Maple let's take $P(\theta) = 0.1\big(Dirac(\theta - \frac{\pi}{2}) + Dirac(\theta - \frac{3\pi}{2})\big) \mid S(\theta) = 0$ for $\theta \in [0, 2\pi]$
N= 1000, $R_{plot} = 12$

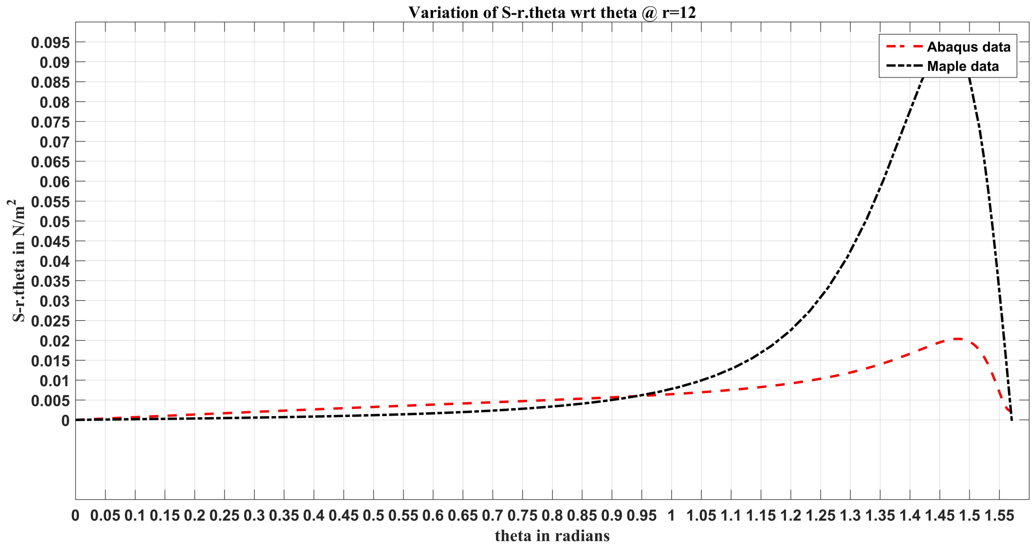
In Abaqus we will model our problem and set the traction and displacement boundary condition for this problem as shown $\Rightarrow$

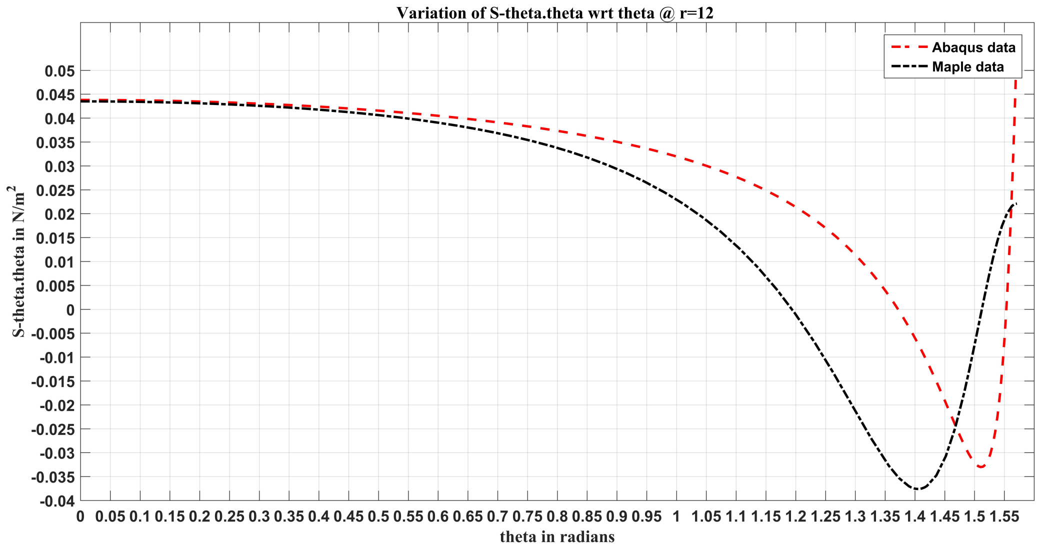


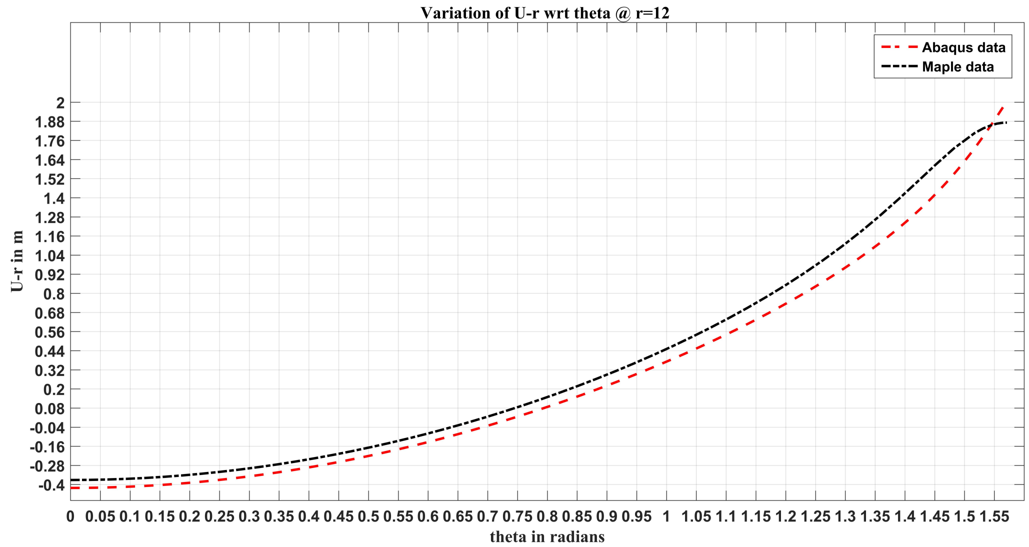
NOTE: Here we have multiplied the radial force distribution with 0.1 because the resulting stresses will be of the order of $\frac{F}{R}$ where F is the concentrated force whose magnitude here is taken to be 1 and R = 10. In Abaqus, we don't need to bother about this but since we are using the symmetry of the problem we have to give F=0.5 while simulating.

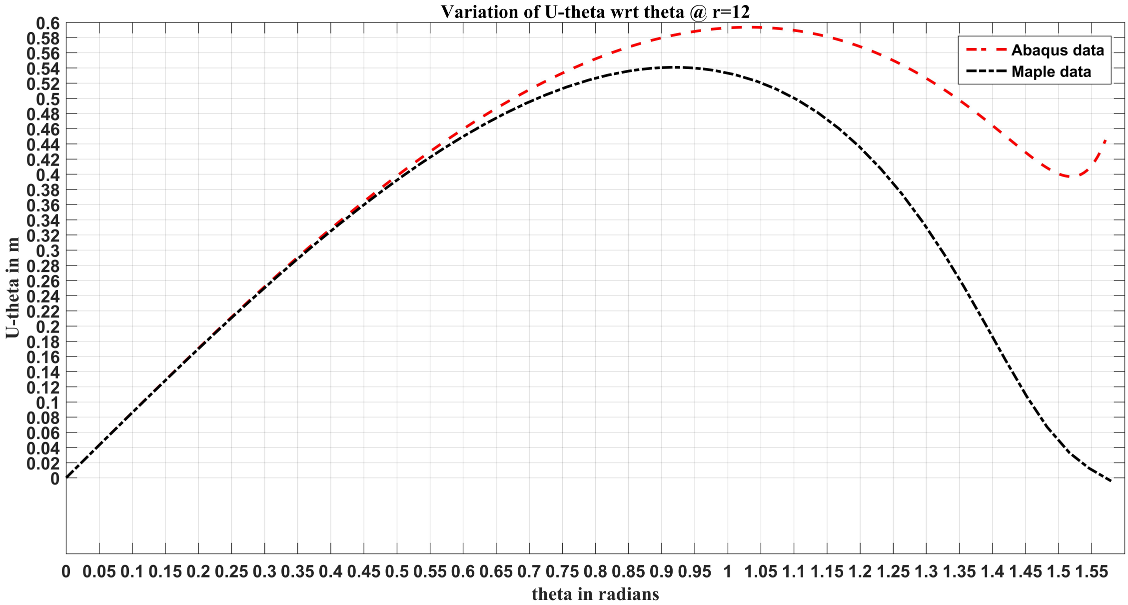
Results
The following plots show the comparison between the results obtained from Maple and Finite Element Analysis in Abaqus.




Variation of S-r.theta wrt theta @ r=12
Abaqus data
Maple data
S-r.theta in N/m^2
theta in radians


Variation of S-theta.theta wrt theta @ r=12
Abaqus data
Maple data
S-theta.theta in N/m^2
theta in radians


Variation of U-r wrt theta @ r=12
Abaqus data
Maple data
U-r in m
theta in radians



Inference from the plots

We can observe that upto 0.95 radians,solution data obtained from Maple matches almost exactly with the FEM data obtained from Abaqus simulation.As we know, we are applying a concentrated load at $r = 10 \ and \ \theta = \frac{\pi}{2}$ i.e. at this point we will have $\sigma_{rr} \to \infty$. But we can't get a infinite value from the software and hence our two data do not match as we approach this region of singularity.

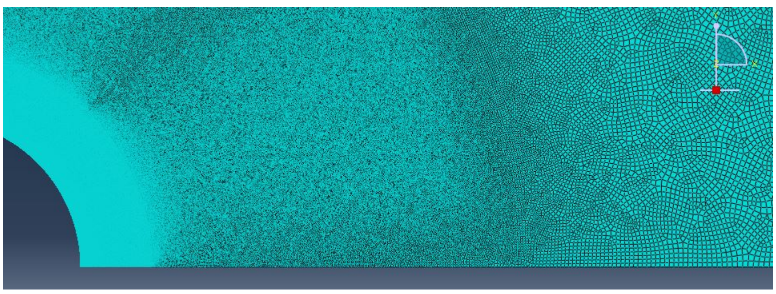
Example 2

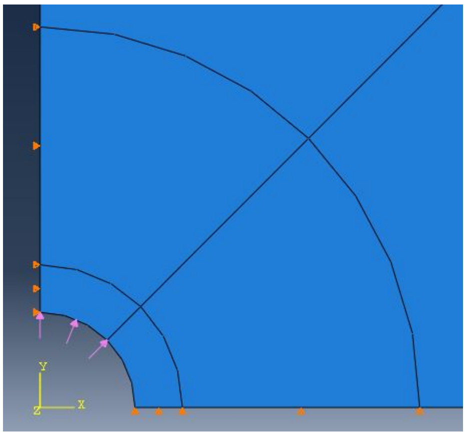
In Maple let's take

$$P(\theta) = \begin{cases} 0 & 0 < \theta < \frac{\pi}{4} \\ 1 & \frac{\pi}{4} \leq \theta \leq \frac{3\pi}{4} \\ 0 & \frac{3\pi}{4} < \theta < \frac{5\pi}{4} \\ 1 & \frac{5\pi}{4} \leq \theta \leq \frac{7\pi}{4} \\ 0 & \frac{7\pi}{4} < \theta < 2\pi \end{cases} \quad | \ S(\theta) = 0 for \ \theta \ \in [0, 2\pi]$$

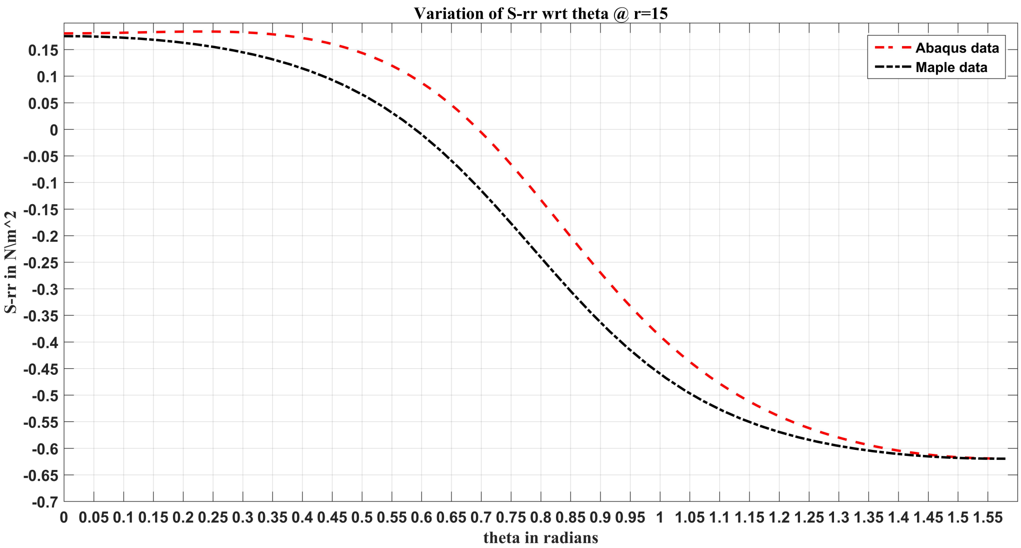
N= 1000, $R_{plot} = 12$

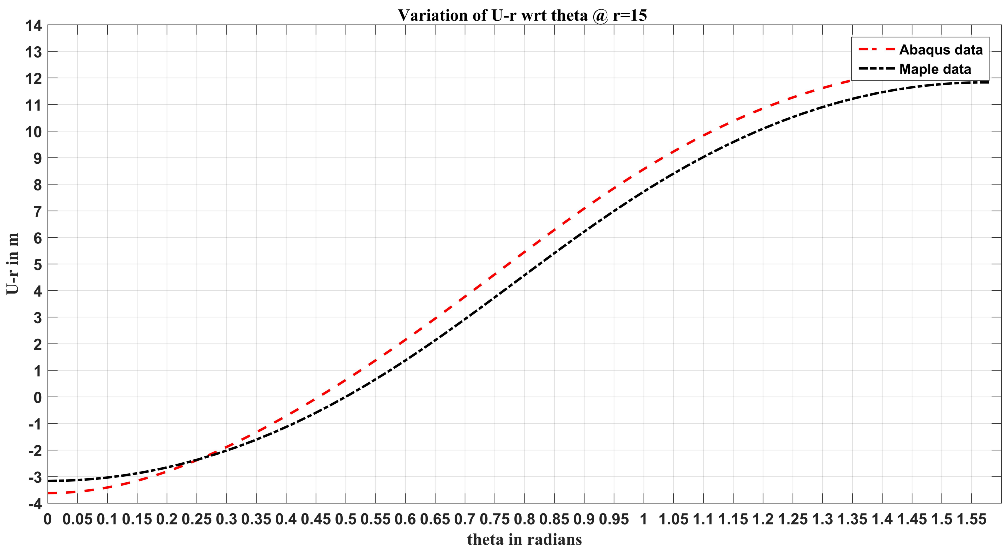
In Abaqus we will model our problem and set the traction and displacement boundary condition for this problem as shown. Here in this simulation we are using a quarter circle model because we know that the boundaries at $\theta = 0, 90$ have $u_\theta = 0$. Apart from this, similar partitions and meshing will be used for this simulation. Since we are applying a piecewise continuous load, we have to make partitions in the domain of the problem and apply the pressure accordingly. This is shown below



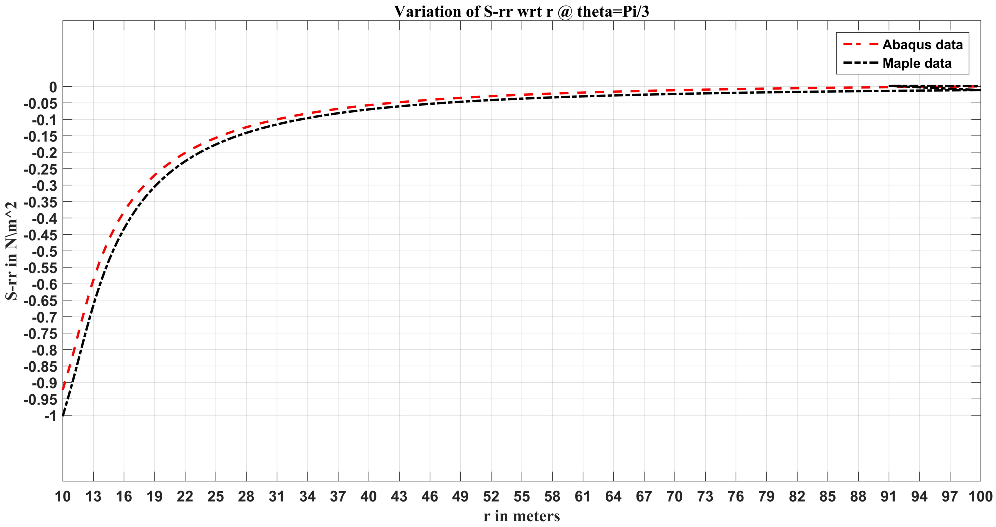


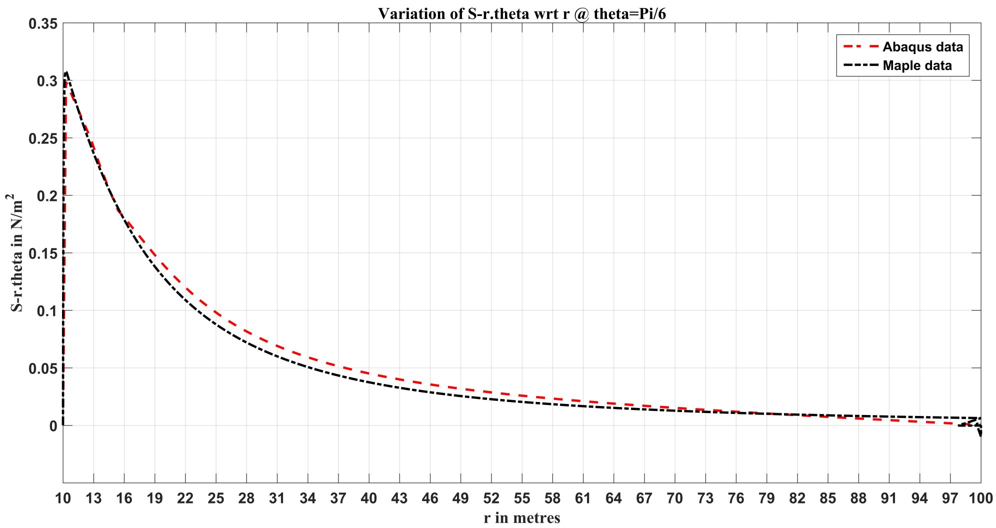
Results
The following plots show the comparison between the data obtained from Maple and Finite Element Analysis in Abaqus.

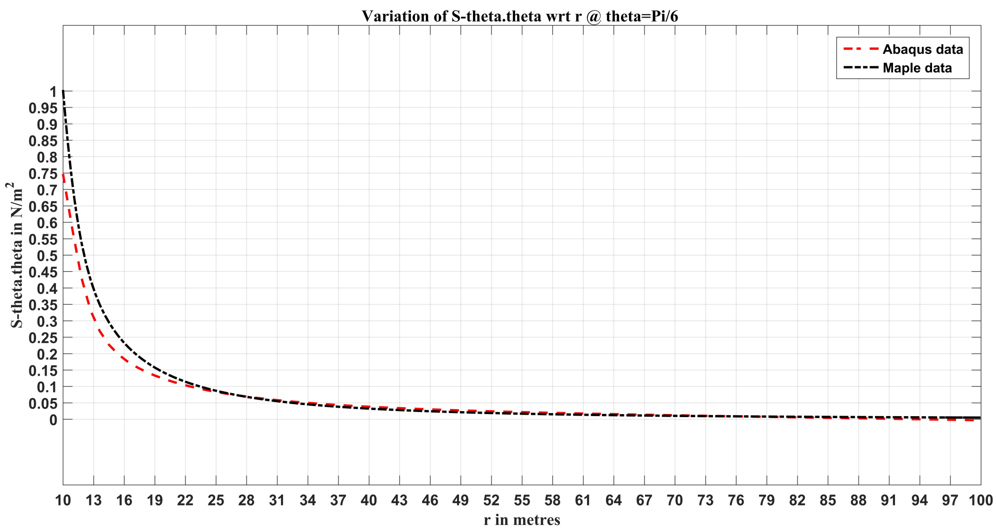


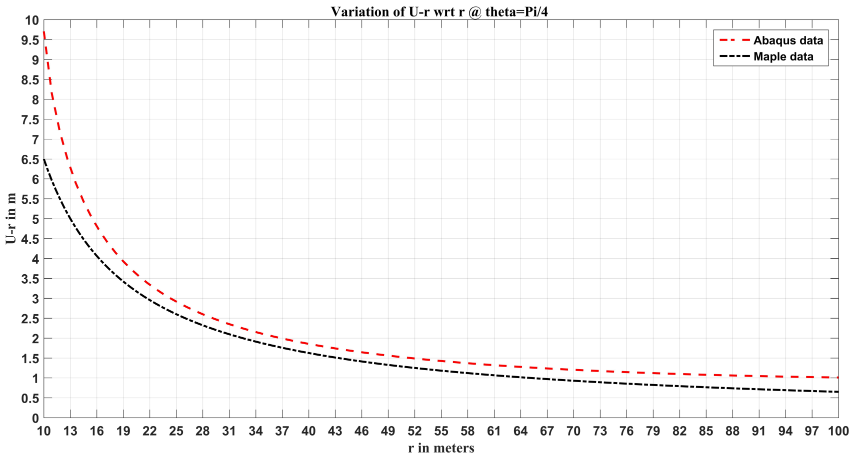


The above plots showed the variation of solution fields along a circular path i.e. wrt θ. Let's verify the variation of solution fields for the two data along the radial path i.e. wrt r.





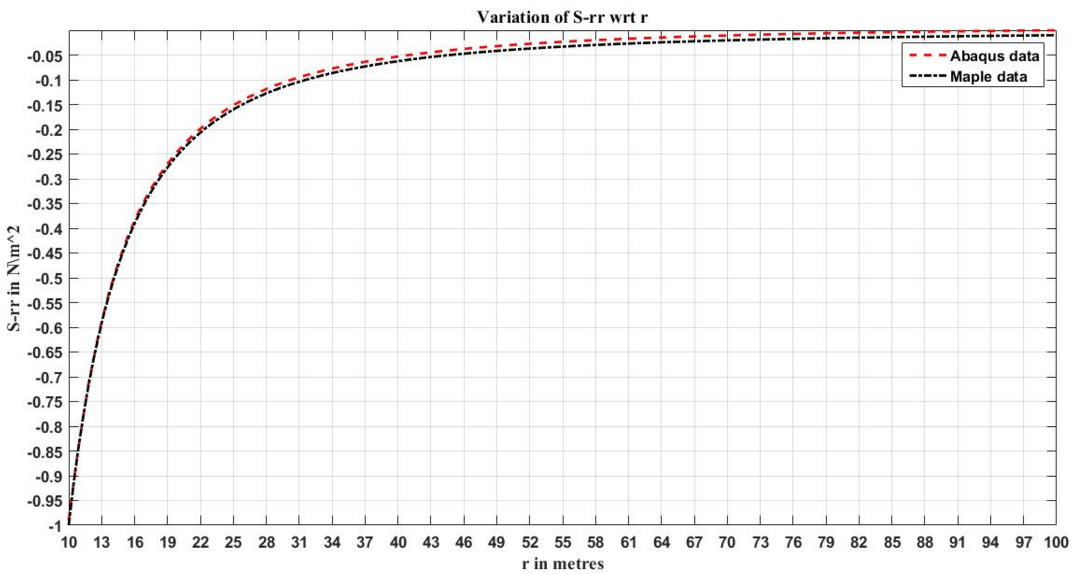


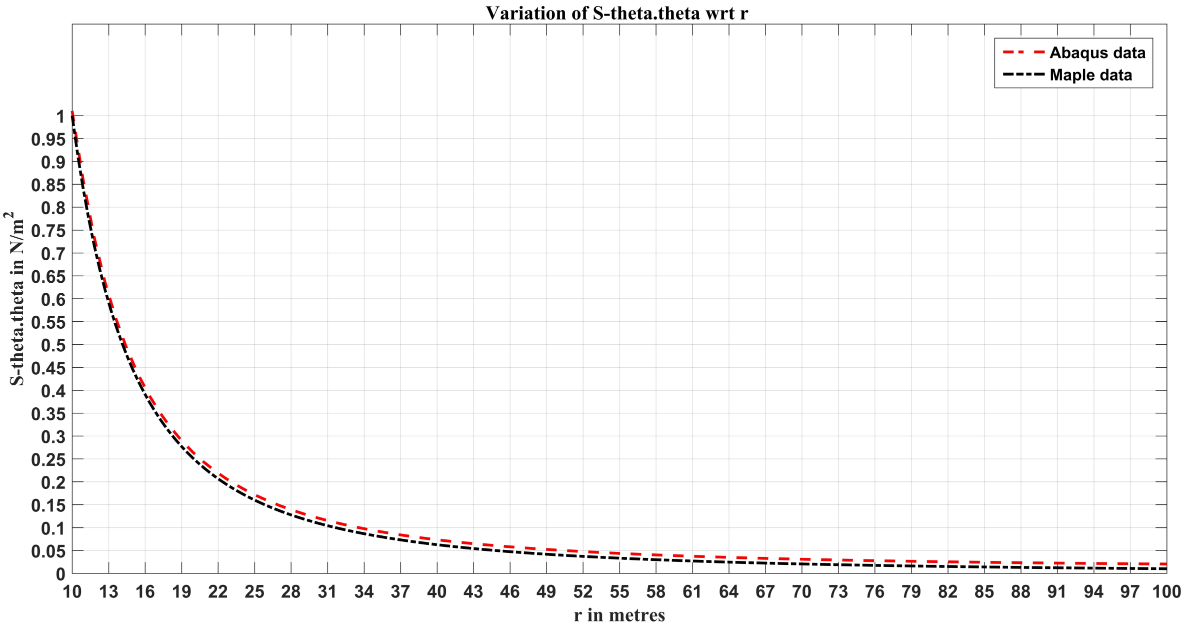


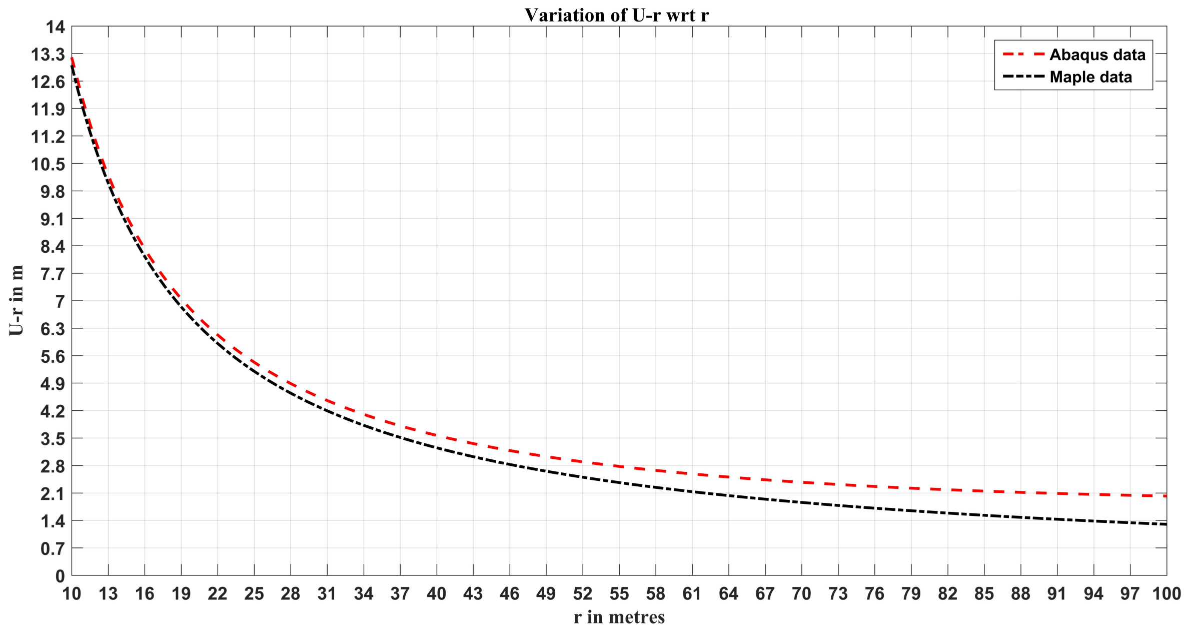
Inference from the plots
As it can be observed that the variation of solution field variables with respect to θ and r for the two data matches appropriately. But it can also be seen that the values from the two data show significant difference in value near the inner boundary at r = 10. Even if we are using reasonably fine mesh, the discontinuity in P(θ) at $\theta = \frac{\pi}{4}$ needs to be dealt with more care and hence a finer mesh is required. Hence a more finer mesh is required in that region. This effect can be seen clearly by comparing the radial plots for stresses plotted at $\theta \neq \frac{\pi}{4}$ and that for the displacement plotted at $\theta = \frac{\pi}{4}$

Example 3 - Lame's Problem
In Maple let's take $P(\theta) = 1 \mid S(\theta) = 0$ for $\theta \in [0, 2\pi]$
N= 1, $\theta_{plot} = \frac{\pi}{4}$ (For this problem, the value of θ for radial path will not matter since for Lame's problem, solution field variables are strictly a function of r)

In Abaqus we will again use the quarter circle domain and this time since we are having a very simple loading, a course mesh throughout the domain can be used. Let's verify the variation of solution fields for the two data along the radial path i.e. wrt r. Since we are applying uniform pressure inside the hole (only continuous radial traction boundary condition) , we have $u_\theta = 0 \; and \; \sigma_{r\theta} = 0$. Following plots show the difference between the two data.







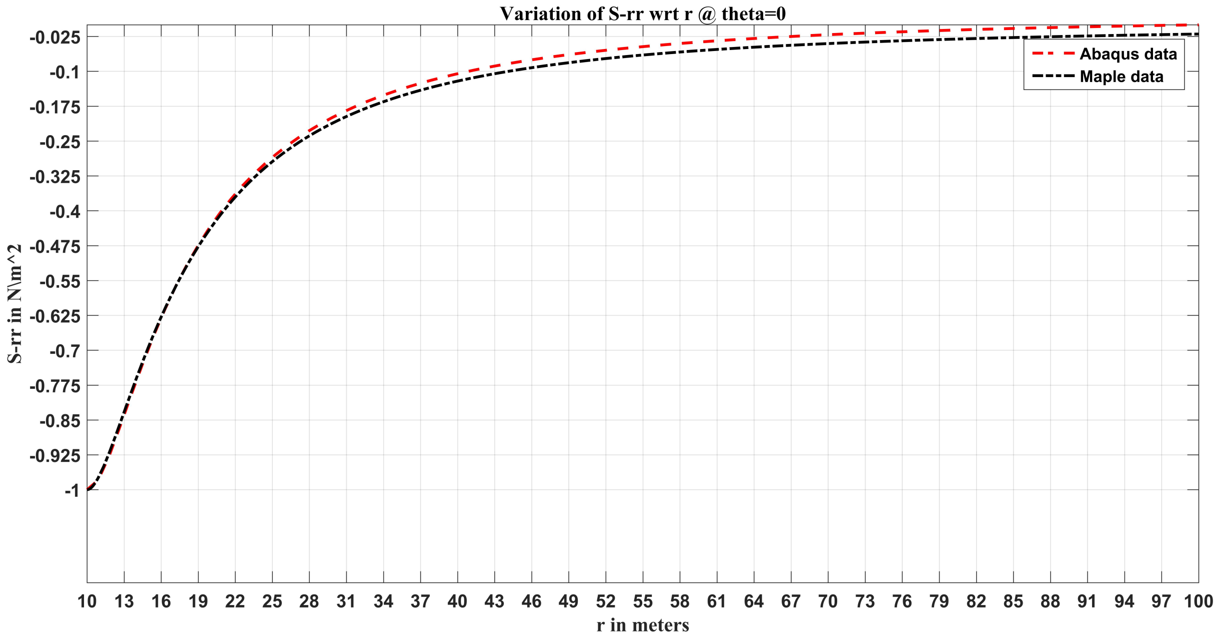
Inference from the plots

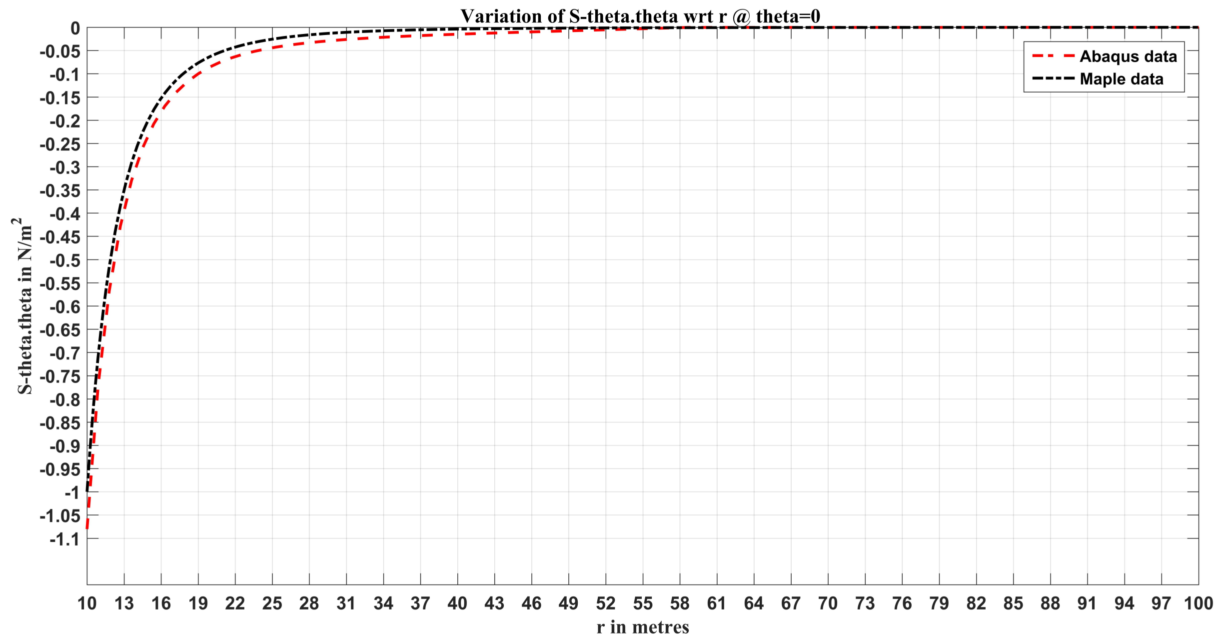
It can be observed that data for σ_{rr} *and* $\sigma_{\theta\theta}$ matches perfectly. For u_r the data matches near the inner boundary but the difference between the data goes on increasing as we proceed towards the outer boundary. This is because in **Maple** we are solving problem in which the outer boundary is at $r \to \infty$ but in **ABAQUS** we model our problem with outer boundary as r = 100. Hence, in **Maple** u_r has the tendency to go on decreasing and converge to zero at $r \to \infty$, but in case of **ABAQUS** since our boundary is finite, u_r has the tendency to converge to a finite value at r=100.

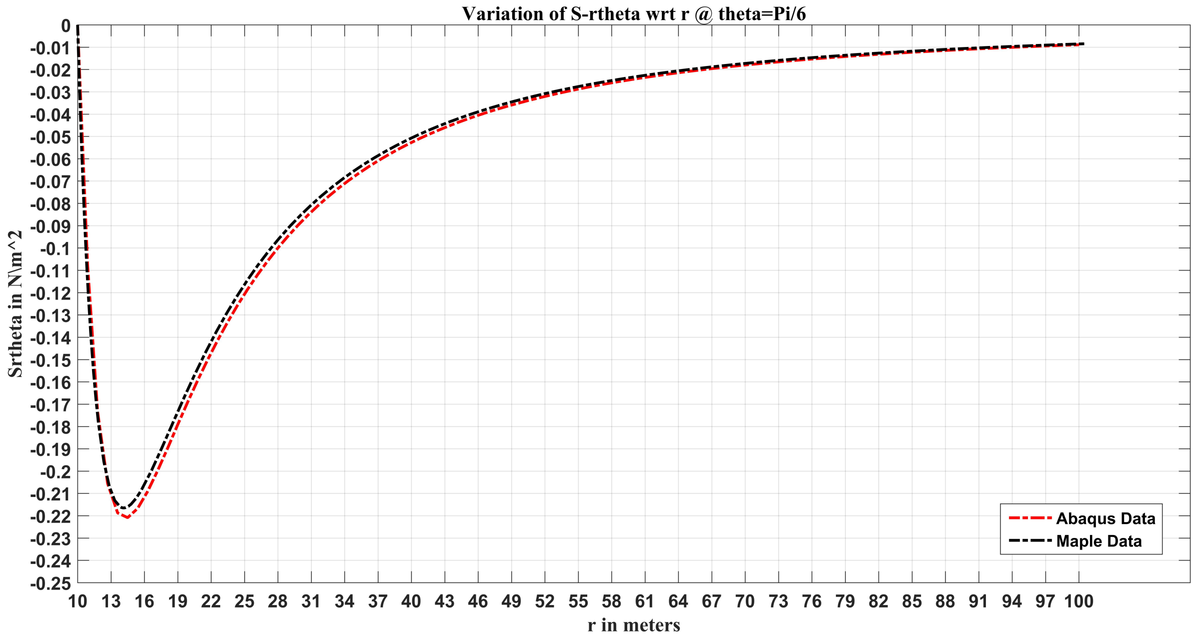
Example 4
In Maple, let's take $P(\theta) = cos(2\theta) \mid S(\theta) = 0$ for $\theta \in [0, 2\pi]$
N= 2

Results

Following plots show the comparison between the stress fields i.e. σ_{rr} *and* $\sigma_{\theta\theta}$ *and* $\sigma_{r\theta}$ for the FEM data and Maple data.







Inference from the plots
From the above results and those obtained from Lame's problem, it can be concluded that for continuous load distribution, the values of solution field variables obtained from FEM analysis in ABAQUS matches almost perfectly with the data obtained from the procedure written in Maple. Even at the region of application of load, the values are matching with almost zero error.
For continuous load distribution,

5 Verification of Saint Venant's Principle

The Saint Venant's principle states that the effect of loading with the same magnitude but different distribution dissipate quickly as distance increases. In other words as the distance from the loading becomes greater, the local effects are reduced such that they can be considered not to be present.
As an example let's consider the following two distribution of load:

$$P_1(\theta) = \begin{cases} 0 & 0 < \theta < \frac{\pi}{4} \\ 1 & \frac{\pi}{4} \leq \theta \leq \frac{3\pi}{4} \\ 0 & \frac{3\pi}{4} < \theta < \frac{5\pi}{4} \\ 1 & \frac{5\pi}{4} \leq \theta \leq \frac{7\pi}{4} \\ 0 & \frac{7\pi}{4} < \theta < 2\pi \end{cases} \quad \Bigg| \quad P_2(\theta) = \begin{cases} 0 & 0 < \theta < \frac{\pi}{3} \\ a_1 + a_2(\frac{\pi}{2} - \theta) & \frac{\pi}{3} \leq \theta \leq \frac{\pi}{2} \\ a_1 + a_2(\theta - \frac{\pi}{2}) & \frac{3\pi}{4} \leq \theta \leq \frac{5\pi}{4} \\ 0 & \frac{2\pi}{3} < \theta < \frac{4\pi}{3} \\ a_1 + a_2(\frac{3\pi}{2} - \theta) & \frac{4\pi}{3} \leq \theta \leq \frac{3\pi}{2} \\ a_1 + a_2(\theta - \frac{3\pi}{2}) & \frac{3\pi}{2} \leq \theta \leq \frac{5\pi}{3} \\ 0 & \frac{5\pi}{3} < \theta < 2\pi \end{cases}$$

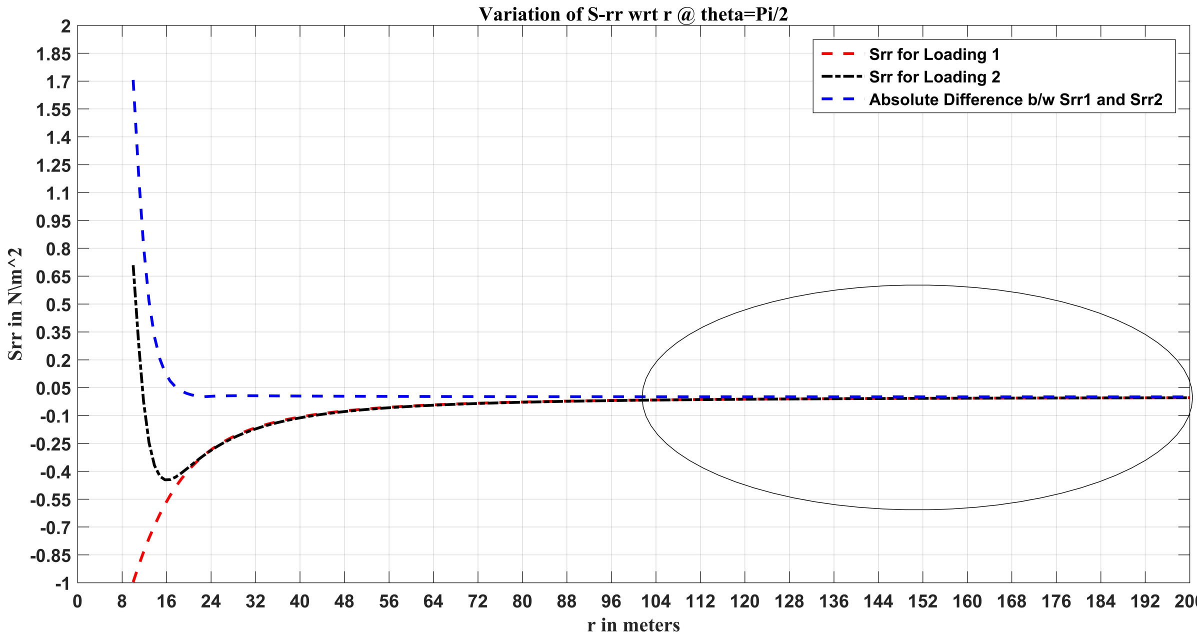
$$a_1 = -0.7354837867 \; and \; a_2 = 8.40876532.$$

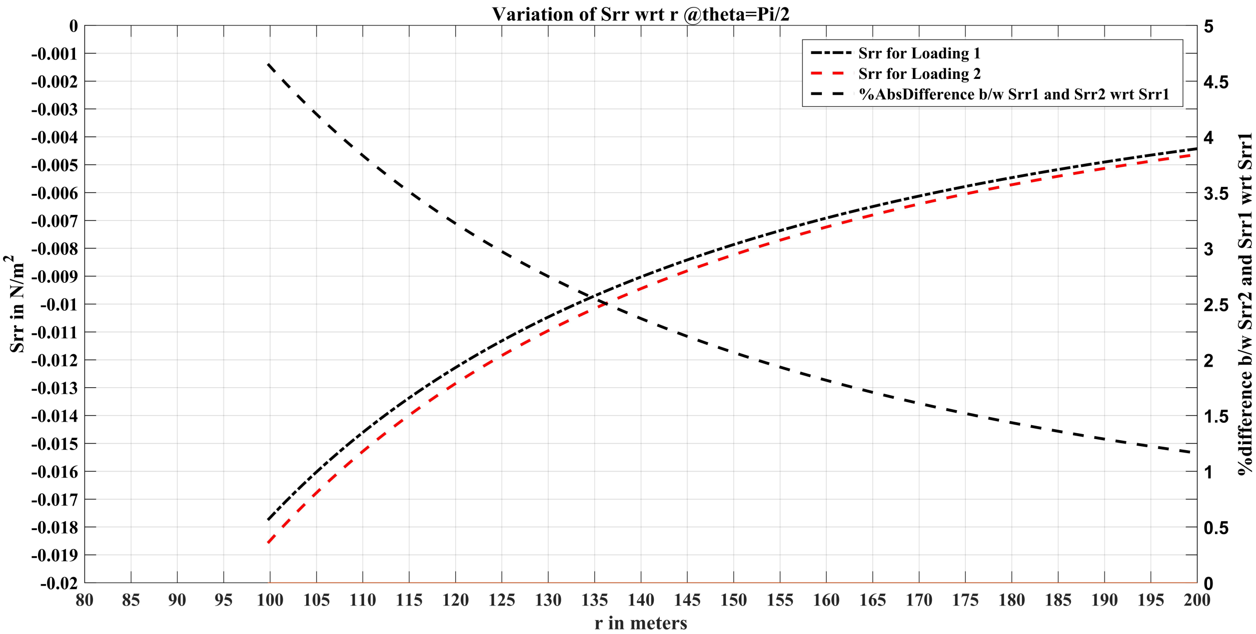
The above two loading conditions are statically equivalent i.e. net force generated in x and y directions by the field on a quarter part of the domain is equal for both the cases.
In fact, distribution for $P_2(\theta)$ has been obtained using the above condition which generates the following equations $\Rightarrow$

$$x \; : \; \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} P_1(\theta)cos(\theta)d\theta = \frac{1}{\sqrt{2}} = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} P_2(\theta)cos(\theta)d\theta$$

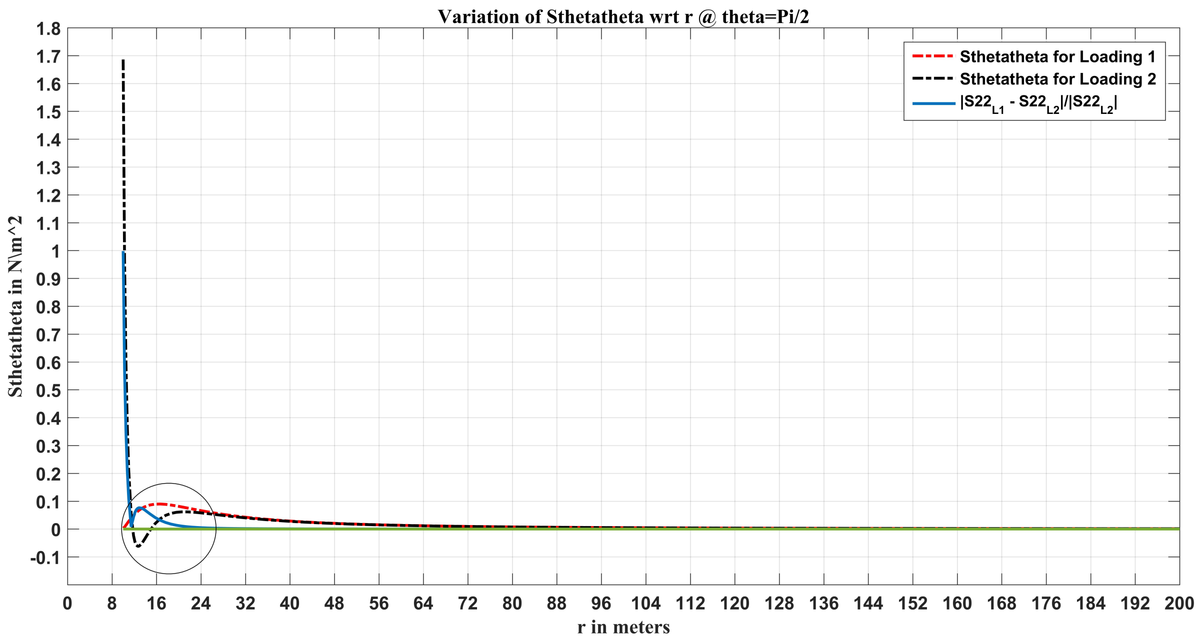
$$y \; : \; \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} P_1(\theta)sin(\theta)d\theta = 1 - \frac{1}{\sqrt{2}} = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} P_2(\theta)sin(\theta)d\theta$$

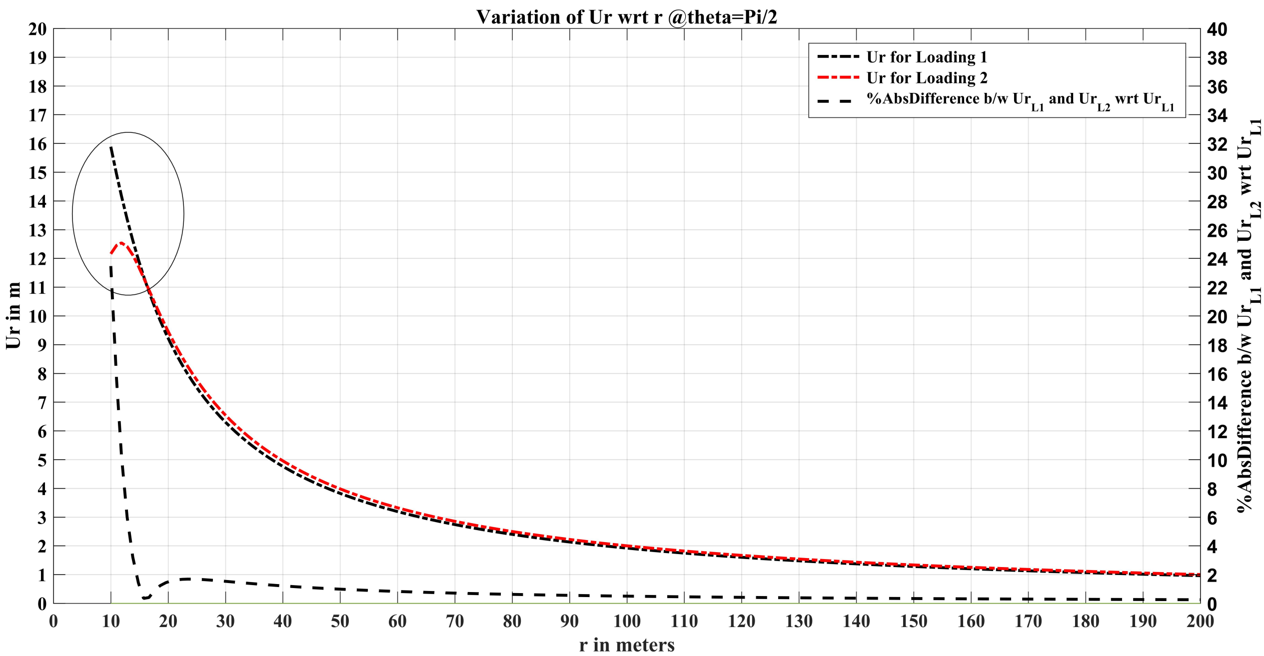
To obtain the distribution for loading 2, one can take $P_2(\theta) = a_1 + a_2\theta$ and obtain the values for the coefficients using above two equations.
Let's analyse simultaneous the solution field for the two loadings using the following plots:





From the first plot it can be observed that around the region of application of load, there is a large difference between the values of σ_{rr}. Even the direction of σ_{rr} is opposite for the two cases. But as we move away from the inner boundary, the values for both the cases start approaching zero and show high resemblance in the values. Taking a close look at the encircled region from the second plot, one can see that the values are now of the order of 10^{-3} and the two curves are still approaching each other. It can be seen that the percentage absolute difference between the two radial stresses with respect to σ_{rr} of loading 1, decreases as we move far away from the region of loading. All this follows in accordance with **Saint Venant's Principle** according to which two statically equivalent produce local effects only. These effect dissipate and the values start to resemble each other at a distance far from the region of loading. Following plots show similar trends for $\sigma_{\theta\theta}$ and u_r.





6 Conclusion from above analysis

As it was observed that the Maple data matches well with the ABAQUS data but still in some regions of the domain there were significant differences. Those differences may arise because of the following reasons $\Rightarrow$

1. Differences in the model considered while solving the problem using the two solvers. Here the difference, as discussed, was the position of the outer boundary of the domain.

2. Slight differences will always occur since we are using a exact appraoch in our code in Maple but on the other hand ABAQUS uses approximate numerical approach by calculating the values at the nodes. Because of this reason, there is a dependence of the results on the the meshing of the domain. Mathematically the errors due to the effects of meshing can be formulated using the following equation

$$A(h) = A(0) + ph^k$$

Here,
A(h) is the solution as a function of mesh size.
A(0) is the exact numerical solution
h is the mesh size
p and k are the hyperparameters for the equations

3. One can experience significantly high errors while dealing close to the regions having singularities, as discussed in Example 1. But as it was observed, the two data show significant resemblance as one moves away from the region of singularities.

4. Similar to the case of singularities, if one has discontinuity in the boundary condition i.e. the functions at the boundary are piecewise continuous then one needs to use very fine mesh around the region of discontinuity. Otherwise, the results will show greater difference around this region as compared to regions far away.

5. Unlike from the above cases, if we have a continuous distribution of load for $\theta \in [0, 2\pi]$ then the values of the solution field show almost perfect resemblance.

7 Applications of above analysis

The problem of pressurised hole is studied in various fields of engineering including mechanical, civil, marine and geothermal. The procedure written in Maple can be used in designing machines used for the purpose of **Grouting** and **Drilling of ground** in geothermal fields.
In grouting, apart from the normal load applied by the machine on the ground one can use the above Maple tool to judge the stresses developed using the displacement of the soil in the process. This can help in designing the material for drill bit for making the hole.
In geothermal engineering, large machine with wide drill heads are used to drill holes upto 4-5 km down the earth in order to exploit the geothermal fields for energy purposes. In this extensive drilling purpose, it equally important to erode the earthly surface from the from the lateral face of the hole as it to drive the drill bit into the ground. Eroding involves applying a specific shear and normal stress over the surface. For this purpose we can use the above results to find those optimal loading conditions keeping in mind the safety norms.
The above analysis can be extended by dealing with similar situation but involving multiple materials. This can be applied to determine the optimal flow pressure inside a thick metal or concrete pipe buried down inside the earth.

References

[1] Elasticity: J.R. Barber, Second Edition, Kluwer Academic Publishers

[2] Fourier Analysis and its Applications: Gerald B. Folland, Brooks/Cole Publishing Company