B.Tech Project Report

Reduced-order controller for leader-follower consensus of Multi-agent systems

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1 Abstract

This technical note deals with output feedback based leader-follower state consensus of linear multi-agent systems (MAS). This report provides a comprehensive explanation of how the controller developed in [1] can be extended to solve the leader-follower state consensus problem of linear homogeneous MAS. In [1], the developed controller guaranteed convergence of agents to a common trajectory that depends on every agent's initial condition in the group, but in this note, all the agents (followers) will converge on the trajectory of the leader only. Unlike most papers on multi-agent consensus, this note utilizes the technique of frequency-domain based controllers. These controllers are capable of performing equally well in the presence of uniform and arbitrarily large communication and input delays in the network.

2 Introduction

Consider a multi-agent network with $N+1$ agents: 1 leader and N followers. The dynamics of each agent is governed by an $n^{th}-order$ single-input single-output (SISO) linear system whose transfer function $\mathbf{P}$ is marginally unstable, i.e. all its poles lie in the closed left-half of the complex plane. Each agent can access the relative output of its neighboring agents and only a subset of the follower agents has access to the output variable of the leader agent. The laplacian matrix of the communication graph describing this information exchange has N eigenvalues and all of them have a positive real part. The consensus problem considered in this work is to design an identical controller for each agent which, using the relative output data accessible to that agent, ensures that the states of all the followers converge to the leader's trajectory.

Motivation

The leader-follower consensus problem has been solved for homogeneous multi-agent network using state-space techniques. This problem has been solved in [7] using state feedback controllers, output feedback controllers, and observers, that were developed using an optimal design approach. In [6], a distributed observer type consensus protocol based on relative output measurements was proposed. Reduced-order observer-type output-feedback protocols has been developed in [8] to address the leader-follower consensus problem. It should be noted that all the aforementioned results are based on observer type protocols or full-order state/output feedback controllers. Well it was found that, in [1], low gain reduced-order frequency domain controller has been developed to solve the output feedback state consensus problem in a leaderless homogeneous multi-agent network.

Inspired by the above discussions, this paper is devoted to analysis and synthesis of leader-follower state consensus of linear multiagent systems based on output-feedback reduced-order frequency domain controller. It was found in [1] that if one can relate the leader-follower consensus problem to a simultaneous stabilization problem, then it is possible to construct a controller of order n that will solve the leader-follower consensus problem. In Theorem 3.3 of this note, the technique of simultaneous stabilization is used to arrive at a condition under which a stable distributed controller will solve the problem. Later, in this work results from [1](Theorem 4.1) will be used to develop a low-gain reduced-order controller to solve the leader-follower consensus problem.

Notation and Definition:

For two sets U and V, U/V is the set of all elements in U not in V. The sets of real numbers and complex numbers are denoted by $\mathbb{R}$ and $\mathbb{C}$. We let $\mathbb{R}^+ = \{a \in \mathbb{R} \mid a > 0\}$, $j\mathbb{R} = \{ja \in \mathbb{C} \mid a \in \mathbb{R}\}$, where $j = \sqrt{-1}$, $\mathbb{C}^- = \{z \in \mathbb{C} \mid Re(z) < 0\}$, $\overline{\mathbb{C}^-} = \{z \in \mathbb{C} \mid Re(z) \leq 0\}$,$\mathbb{C}^+ = \mathbb{C}\backslash\overline{\mathbb{C}^-}$ and $\overline{\mathbb{C}^+} = \mathbb{C}\backslash\mathbb{C}^-$. For $\alpha \in \mathbb{C}$, we denote its magnitude by $|\alpha|$ and principle phase angle by $\angle\alpha$ $(so\ \angle\alpha \in (-\pi, \pi])$. We denote the set of eigenvalues of a matrix $A \in \mathbb{C}^{n\times n}$ by $\sigma(A)$. For a polynomial $r(s) = \sum_{l=0}^{n} a_l s^l$, its degree is $\deg(r) = n$ and r is said to be Hurwitz if all its zeros lie in $\mathbb{C}^-$. Let $L^2([a,b))$ be the set of real-valued square integerable functions on the interval $[a,b)$.

Graph Theory:

A directed graph is denoted by $\mathcal{G} = (\mathcal{N}, \mathcal{E})$, where $\mathcal{N} = 1, 2, ..., N$ is a finite non-empty set of nodes and $\mathcal{E} \subset \mathcal{N} \times \mathcal{N}$ is an edge set of ordered pairs of nodes. A subgraph $(\mathcal{N}_1, \mathcal{E}_1)$ of $\mathcal{G}$ is a graph such that $\mathcal{N}_1 \subset \mathcal{N}$ and $\mathcal{E}_1 \subset \mathcal{E} \cap (\mathcal{N}_1 \times \mathcal{N}_1)$. An edge (i,j) in a directed graph denotes that the agent j obtains information from the agent i. For an edge (i,j) in a directed graph, i is the parent node and j is the child node. A directed path of length l is a sequence of edges in a directed graph of the form $((i_1, i_2), (i_2, i_3), ..., (i_l, i_{l+1}))$ where $(i_j, i_{j+1}) \in \mathcal{E}$ for $j = 1, 2, ..., l$ and $i_j \neq i_k$ for $j, k = 1, 2, ..., l$ and $j \neq k$. A directed tree is a directed graph where every node has exactly one parent except for one node called the root which has no parent and there exists a directed path from the root to every other node of $\mathcal{G}$. A directed spanning tree of $\mathcal{G}$ is a directed tree that contains all the nodes of $\mathcal{G}$. A directed graph contains a directed spanning tree if there exists a directed spanning tree as a subgraph of the directed graph.

The adjacency matrix $\mathcal{A} = [\alpha_{ij}] \in \mathbb{R}^{N\times N}$ of a directed graph with a node set $\mathcal{N} = 1, 2, ..., N$ is defined such that α_{ij} is a positive weight if $(j, i) \in \mathcal{E}$, while $\alpha_{ij} = 0$ if $(j, i) \notin \mathcal{E}$. The laplacian matrix $\mathcal{L} = [\beta_{ij}] \in \mathbb{R}^{N\times N}$ of a directed graph is defined as $\beta_{ii} = \sum_{j\neq i} \alpha_{ij}$ and $\beta_{ij} = -\alpha_{ij}$ for all $i \neq j$. By its construction, $\sum_{j=1}^{N} \beta_{ij} = 0$ for $i = 1, 2, ..., N$ and hence the Laplacian L has an eigenvalue 0 with associated eigenvector $\mathbf{1}_N$ where $\mathbf{1}_N$ is an $N \times 1$ column vector whose elements are all one.

3 Problem Statement

Consider a multi-agent system with $N + 1$ agents. Out of these $N + 1$ agents, there are N followers and one of them is the leader. The dynamics of each follower is governed by an n^{th}-order SISO linear system, i.e. for each $i \in \{1, 2, ...N\}$

$$\dot{x^i} = Ax^i + Bu^i; \quad y^i = Cx^i \ \forall \ t \geq 0$$

where $x_i = [x_{i,1}, ..., x_{i,n}]^T \in \mathbb{R}^n$ is the state, $u_i \in \mathbb{R}$ is the control input, and $y_i \in \mathbb{R}$ is the measured output, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$ and $C \in \mathbb{R}^{1\times n}$.

Assumption 1: (C, A, B) is stabilisable and detectable and all the eigenvalues of A lie in the closed left-half complex plane i.e. $\sigma(A) \subset \overline{\mathbb{C}^-}$.

Here, $n = n_s + n_u$, where n_s is the number of eigenvalues of A in $\mathbb{C}^-$ and n_u is the number of eigenvalues of A lying on the imaginary axis. Let the set of eigenvalues of A on the imaginary axis, without counting multiplicities, be $\sigma(A) = \{0, \pm j\omega_1, \pm j\omega_2, ... \pm j\omega_m\}$ with $0 < \omega_1 < \omega_2 < \ < \omega_m$. Then we have $n_u = \Sigma_{k=-m}^{m} m_k$, where $m_k = m_{-k}$ is the algebraic multiplicity of $j\omega_k \in \sigma(A)$. Since (C, A) is detectable and $C \in \mathbb{R}^{1\times n}$, the geometric multiplicity of each $j\omega_k \in \sigma(A)$ must be 1.

Dynamics of followers:

$$\dot{x}^i = Ax^i + Bu^i, \quad y^i = Cx^i \tag{1}$$

Here, i is the index of the follower and $i \in \{1, 2, ..., N\}$

Dynamics of the leader:

$$\dot{x}^0 = Ax^0, \quad y^0 = Cx^0 \tag{2}$$

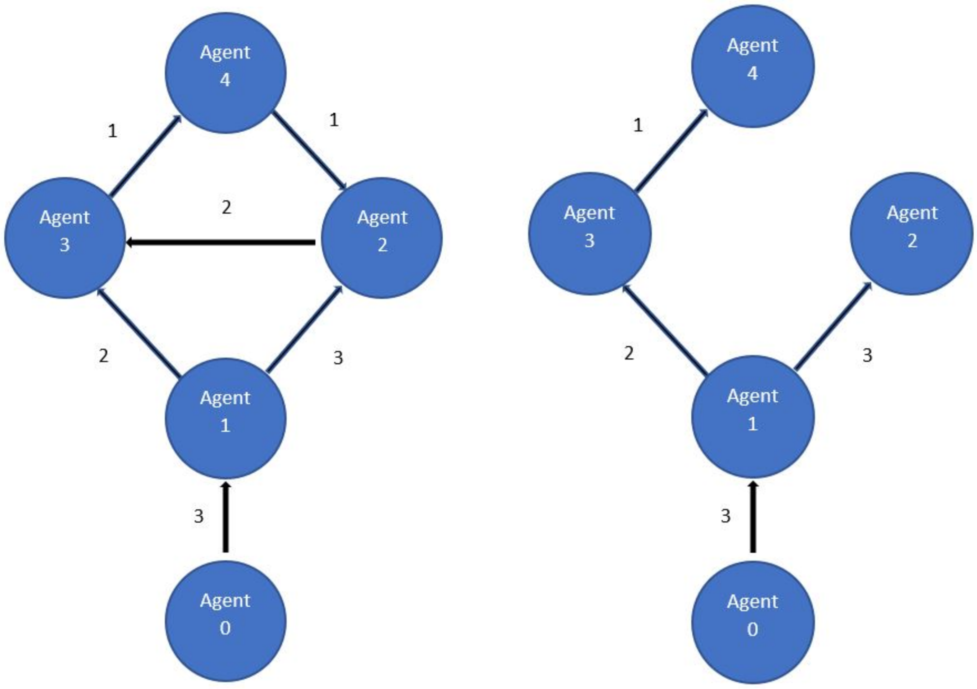
We have chosen '0' as the index of the leader.

Each agent can access the relative output of its neighboring agents and only a subset of the follower agents has access to the output variable of the leader agent, and hence i^{th} agent can compute the weighted sum:

$$\zeta_i = \sum_{l=1}^{N} \alpha_{il}(y^l - y^i) + g_{ii}(y^0 - y^i) \tag{3}$$

Here, α_{ij} is the weight of the edge connecting agent j and agent i. This has been discussed in detail in the topic of graph theory under the introductory section. We will focus more on the information exchange by the leader which is incorporated in the topology using the matrix G. G is a diagonal matrix whose elements are non-negative, i.e., $g_{ii} \geq 0 \ \forall \ i \in \{1, 2, ..., N\}$. It has been mentioned in the introduction that only a subset of the followers are able to access the output information of the leader. This means that $g_{ii} \neq 0$ if there exist an edge between the leader and the i^{th} agent, and $g_{ii} = 0$ otherwise. For example in Figure 1, only Agent 1 can access the output information of Agent 0 i.e. the leader. This implies that $g_{11} = 1$ and $g_{ii} = 0 \forall\ i \in \{2, 3, 4\}$ and the G matrix looks like

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$



(a) An example of weighted communication topology

(b) Directed spanning tree in weighted digraph

Figure 1: Leader-follower communication topology

Without loss of generality, it has been mentioned earlier that the agents indexed by $1, ..., N$, are the N followers and the agent indexed by 0 is the leader. Moreover, we assume that the communication graph $\mathcal{G}$ among the $N + 1$ agents satisfies the following assumption.

Assumption 2: The graph $\mathcal{G}$ contains a directed spanning tree with the leader as the root node. ([3])

After looking at the equations governing the dynamics of the agents, we now proceed to find the condition under which consensus problem will be solved. We try to do so by using the technique of simultaneous stabilization ([2]).

Theorem 3.1: Suppose x^i denotes the state of a follower with i^{th} index and x^0 is the state of the leader. The output feedback leader-follower consensus problem will be solved if $\forall i \in \{1, 2, ..., N\}$ we have

$$\lim_{t \to \infty} e^i = 0 \tag{4}$$

where $e^i = x^i(t) - x^0(t)$.

Problem 3.2: Find A_c, B_c, C_c and D_c, with $\sigma(A_c) \subset C^-$ such that the controller in (8) solves the output feedback leader-follower consensus problem for the multi-agent system described above in (1)-(3)

We now state the result that relates the consensus problem to the simultaneous stabilization problem. A similar result for leaderless consensus problem has been stated in [1]

Theorem 3.3: If a matrix quadruple $(A_c,\ B_c,\ C_c,\ D_c)$ is such that $\sigma(A_c) \subset C^-$ and $\sigma(\mathcal{A}_i) \subset C^-$ for all $i \in \{1, 2, ..., N\}$, where

$$\mathcal{A}_i = \begin{bmatrix} A - \hat{\lambda}^i_{\hat{L}} BD_cC & BC_c \\ -\hat{\lambda}^i_{\hat{L}} B_cC & A_c \end{bmatrix}, \tag{5}$$

and $\hat{\lambda}^i(\hat{L})\, \forall\, i \in \{1.2, ..., N\}$ is the eigenvalue of laplacian matrix $\hat{L} = L + G$, then $(A_c,\ B_c,\ C_c,\ D_c)$ solve Problem 3.2.

Proof: Consider the error variable introduced in Theorem 3.1,

$$e^i = x^i(t) - x^0(t)$$

The dynamics of this error term is as follows:

$$\dot{e}^i = \dot{x}^i - \dot{x}^0$$

$$\dot{e}^i = (Ax^i + Bu^i) - (Ax^0)$$
$$\dot{e}^i = A(x^i - x^0) + Bu^i$$
$$\dot{e}^i = Ae^i + Bu^i \tag{6}$$

From now onwards in this proof, we will consider a group of N agents whose state dynamics is governed by equation (5). The output equation for these agents is given by:

$$f_i = y^i - y^0$$

$$f^i = Cx^i - Cx^0$$
$$f^i = Ce^i \tag{7}$$

Note: The above variables are introduced solely for the purpose of simplifying the mathematical analysis. These variables will not be a part of the designed controller.

Consider the following controller which has stable dynamics i.e. $\sigma(A_c) \subset C^-$

$$\dot{x_c}^i = A_c {x_c}^i + B_c {u_c}^i \tag{8}$$

$${y_c}^i = C_c {x_c}^i + D_c {u_c}^i$$

Here, ${y_c}^i$ is the output of the controller and ${u_c}^i$ is the relative information of its neighboring agents accessed by the i^{th} agent. The output of the controller is essentially the control input u^i to the follower's dynamics i.e. $u^i = {y_c}^i$. By definition, one can note that ${u_c}^i$ is equal to information exchange signal defined in equation (3).

$${u_c}^i = \zeta^i$$

$${u_c}^i = \sum_{l=1}^{N} \alpha_{il}(y^l - y^i) + g_{ii}(y^0 - y^i)$$

$$u_c{}^i = -\sum_{l=1}^{N} \beta_{il} y^l - g_{ii} f^i \tag{9}$$

To proceed with the proof, we now utilize one property of Laplacian matrix. Sum of elements of a row in Laplacian matrix is zero, $\sum_{l=1}^{N} \beta_{il} = 0$. From this we can write $\sum_{l=1}^{N} \beta_{il} y^0 = 0$. Add this expression to Equation (8).

$$u_c{}^i = -\sum_{l=1}^{N} \beta_{il} y^l + \sum_{l=1}^{N} \beta_{il} y^0 - g_{ii} f^i$$

$$u_c{}^i = -\sum_{l=1}^{N} \beta_{il} (y^l - y^0) - g_{ii} f^i$$

$$u_c{}^i = -\sum_{l=1}^{N} \beta_{il} (f^l) - g_{ii} f^i \tag{10}$$

Now, we look at the closed loop dynamics of each agent in the multi-agent systems consisting of agents with states e^i.

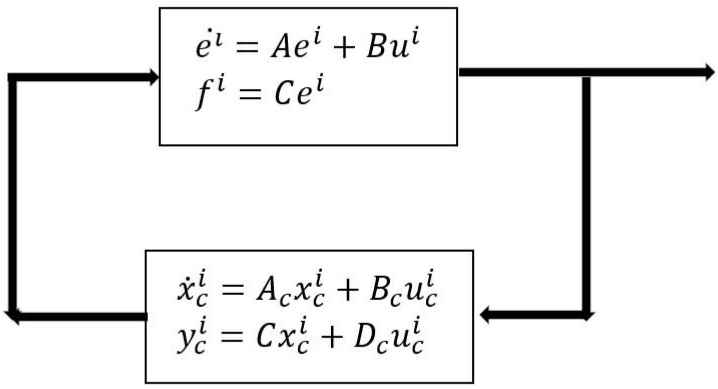


Figure 2: Closed loop error dynamics

Let $\bar{e}^i$ denote the closed loop state for the follower. With $\bar{e}^i = [e^i, x_c{}^i]^T$, the system dynamics of each individual agent can be written as

$$\dot{\bar{e}}^i = \begin{bmatrix} A & BC_c \\ 0 & A_c \end{bmatrix} \begin{bmatrix} e^i \\ x_c{}^i \end{bmatrix} + \begin{bmatrix} BD_c \\ B_c \end{bmatrix} u_c{}^i = \bar{A}\bar{e}^i + \bar{B} u_c{}^i \tag{11}$$

The output variable for the closed loop system will remain the same.

$$f^i = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} e^i \\ x_c{}^i \end{bmatrix} = \bar{C}\bar{e}^i \tag{12}$$

With $\tilde{e} = [\bar{e}^1, \bar{e}^2, ..., \bar{e}^N]^T$ and $\tilde{f} = [f^1, f^2, ..., f^N]^T$, the overall dynamics is written in the form:

$$\dot{\tilde{e}}(t) = (I_N \otimes \bar{A})\tilde{e}(t) + (I_N \otimes \bar{B})\tilde{u_c} \tag{13}$$

Here, $\tilde{u_c} = [u_c{}^1, u_c{}^2, ..., u_c{}^N]^T$.

$$\tilde{u_c} = \begin{bmatrix} -\sum_{l=1}^{N} \beta_{1l}(f^l) - g_{11} f^1 \\ -\sum_{l=1}^{N} \beta_{2l}(f^l) - g_{22} f^2 \\ . \\ . \\ . \\ -\sum_{l=1}^{N} \beta_{Nl}(f^l) - g_{NN} f^N \end{bmatrix}$$

$$\tilde{u_c} = -(L \otimes I_P)\tilde{f} - (G \otimes I_P)\tilde{f}$$
$$\tilde{u_c} = -[(L + G) \otimes I_P]\tilde{f} \tag{14}$$

In Equation (14) P = 1 because we are dealing with SISO systems. Substitute (14) in (13), we get

$$\dot{\tilde{e}}(t) = (I_N \otimes \bar{A})\tilde{e}(t) - (I_N \otimes \bar{B})[(L + G) \otimes I_1]\tilde{f} \tag{15}$$

Substituting (12) in (15),

$$\dot{\tilde{e}}(t) = (I_N \otimes \bar{A})\tilde{e}(t) - (I_N \otimes \bar{B})[(L + G) \otimes I_1](I_N \otimes \bar{C})\tilde{e}(t) \tag{16}$$

Simplifying it using the identity: $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$, we get

$$\dot{\tilde{e}}(t) = (I_N \otimes \bar{A} - (L + G) \otimes \bar{B}\bar{C})\tilde{e}(t) \tag{17}$$

Now, consider a linear transformation T such that $T\hat{L}T^{-1} = \hat{J}_L$, where $\hat{L} = L + G$ and $\hat{J}_L$ is the Jordan form of $\hat{L}$. With the transformed state, $\tilde{\xi} = (T \otimes I_{n+n_c})\tilde{e}$, we get the following overall dynamics (n_c is the order of the controller in (8)),

$$\dot{\tilde{\xi}} = (T \otimes I_{n+n_c})(I_N \otimes \bar{A} - (L + G) \otimes \bar{B}\bar{C})(T^{-1} \otimes I_{n+n_c})\tilde{\xi} \tag{18}$$

Using the above Kronecker product identity, we get

$$\dot{\tilde{\xi}} = (I_N \otimes \bar{A} - \hat{J}_{\hat{L}} \otimes \bar{B}\bar{C})\tilde{\xi} = \mathcal{A}\tilde{\xi} \tag{19}$$

with the initial condition $\tilde{\xi}(0) = (T \otimes I_{n+n_c})\tilde{e}(0)$. The equation (18) is nothing but collection of

$$\dot{\bar{\xi}^i} = (\bar{A} - \hat{\lambda}^i_{\hat{L}}\bar{B}\bar{C})\bar{\xi}^i = \mathcal{A}_i\bar{\xi}^i \tag{20}$$

$$\dot{\bar{\xi}^i} = (\bar{A} - \hat{\lambda}^i_{\hat{L}}\bar{B}\bar{C})\bar{\xi}^i - \bar{B}\bar{C}\bar{\xi}^{i+1} = \mathcal{A}_i\bar{\xi}^i - \bar{B}\bar{C}\bar{\xi}^{i+1}$$

where

$$\mathcal{A}_i = \begin{bmatrix} A - \hat{\lambda}^i_{\hat{L}}BD_cC & BC_c \\ -\hat{\lambda}^i_{\hat{L}}B_cC & A_c \end{bmatrix} \tag{21}$$

It should be noted that $\tilde{\xi} = [\bar{\xi}^1, \bar{\xi}^2, ..., \bar{\xi}^N]^T$ and $\hat{\lambda}^i(\hat{L})$ for $i \in \{1, 2, ..., N\}$ are the eigenvalues of $\hat{L}$. Also, $Re\{\hat{\lambda}^i(\hat{L})\} > 0 \; \forall \; i \in \{1, 2, ..., N\}$.

If we find matrix quadruple (A_c, B_c, C_c, D_c) such that A_c is hurwitz and $\mathcal{A}_i \; \forall \; i \in \{1, 2, ..., N\}$ is hurwitz, then Problem 3.2 is solved, i.e.

$$\lim_{t \to \infty} e^i = 0 \; \forall \; i \; \in \{1, 2, ..., N\}$$

$$\lim_{t \to \infty} x^i = x^0 \; \forall \; i \; \in \{1, 2, ..., N\}$$

Similar results can be found in Theorem 2.2 of [1], but in their case the above theorem was stated for $i \in \{2, 3, ..., N\}$ but in the above proof we have made that statement $\forall \; i \in \{1, 2, ..., N\}$. This is because in leaderless case the laplacian matrix L has zero eigenvalue (with algebraic multiplicity of 1 because it has directed spanning tree) and so it can be observed that $\sigma(\mathcal{A}_i) \in \overline{\mathbb{C}^-}$ for $\lambda = 0$ i.e. the matrix is stable for zero eigenvalue.

But for the leader-follower case, the communication topology is of the form $\hat{L} = L + G$. Here, L depicts the communication topology among followers and L has an eigenvalue $\lambda = 0$. Now, it has been mentioned before that on addition of the leader to the group of followers, there will be some changes that will occur in the communication topology of the leader-follower group. To incorporate those changes we add a matrix G which is a diagonal matrix which contains the edges' weight of information exchange between leader and a subset of followers. All elements of G are either positive or equal to zero and hence on adding G to L i.e. for $\hat{L} = L + G$ we observe that the real part of all eigenvalues of $\hat{L}$ are positive, $Re(\hat{\lambda}^i(\hat{L})) > 0 \; \forall \; i \in \{1, 2, ..., N\}$. This can be easily proved using Greshgorin Disk Theorem. This statement is supported by the fact the $\hat{L}$ is a non singular M-matrix ([4]).

Although there are some differences between the two cases, the frequency domain controller used for leaderless consensus of multi-agent system, can be applied to the leader-follower consensus problem with some changes in the procedure for controller construction. In the next section we will summarize some theorems from [1] discuss about these changes.

4 Low Gain Controller Design

Recall that $\hat{L} = L + G$ is the laplacian matrix of the graph associated with the leader-follower multi-agent system. Also, recall $\sigma(\hat{L}) = \{\hat{\lambda}^1, \hat{\lambda}^2, ..., \hat{\lambda}^N\}$.
Define $\tau > 0$ as follows:

$$\tau = \min_{\hat{\lambda}^i \in \sigma(\hat{L})} Re(\hat{\lambda}^i) \tag{22}$$

In Theorem 4.2 of this section, using ideas from the low-gain approach to output regulation in [5], we construct a stable low-gain controller of the form (8) which ensures that $\sigma(\mathcal{A}_i) \subset \mathbb{C}^-$ ($\mathcal{A}_i$ is defined in (5 & 21)) for all $i \in \{1, 2, ..., N\}$. It then follows from Theorem 3.3 that this controller solves Problem 3.2. While [5] suggest an approach for designing a controller such that $\sigma(\mathcal{A}_i) \subset \mathbb{C}^-$ for any fixed i, we must combine it with a robustness result for designing a controller such that $\sigma(\mathcal{A}_i) \subset \mathbb{C}^-$ for all $i \in \{1, 2, ..., N\}$. The small-gain theorem based lemma below presents such a robustness result which is used to prove Theorem 4.2.

Let $q \geq 2$be an integer. Let $\alpha, \beta \in \mathbb{R}^+$ be such that $2\alpha > \tau^{-1}$ and $\beta^2 \geq 2\alpha$. Choose $a_0, a_1, a_2, ..., a_{q-1} \in \mathbb{R}^+$ so that

$$\sum_{l=0}^{q-1} a_l s^{q-l} + \alpha = (s^2 + \beta s + \alpha)(s + 1)^{q-2} \tag{23}$$

Then for each non-zero $\hat{\lambda}^i \in \sigma(\hat{L})$ and any $\gamma \geq 1$, the transfer function $Q(s) = \frac{1}{(s^q + \hat{\lambda}^i \gamma \sum_{q-1}^{l=0} a_l s^l)}$ is stable, i.e. all the poles of Q are in $\mathbb{C}^-$.

Consider the transfer functions $P(s) = C(sI - A)^{-1}B$ and $C(s) = C_c(sI - A_c)^{-1}B_c + D_c$ of the agent dynamics (1 & 2) and controller (8), respectively. Then $\sigma(\mathcal{A})_i \subset \mathbb{C}^-$ for each $\hat{\lambda}^i \in \sigma(\hat{L})$ if and only if the negative feedback interconnection of $\hat{\lambda}^i P$ and C is internally stable for each $\hat{\lambda}^i \in \sigma(\hat{L})$. The next theorem is the main result of this section and it provides an algorithm for constructing a stable low-gain controller for which the internal stability conditions hold, and hence this controller solves Problem 3.2. Recall that $\sigma(A) \subset \overline{C^-}$ and $n = n_s + n_u$. The set of distinct eigenvalues of A on the imaginary axis is $\sigma_u(A) = \{\omega_0, \pm j\omega_1, \pm j\omega_2, ..., \pm j\omega_m\}$ with $0 = \omega_0 < \omega_1 < \omega_2 < ... < \omega_m$ and $n_u = \sum_{k=-m}^{m} m_k$, where $m_{-k} = m_k$ is the algebraic multiplicity of $j\omega_k$. Hence

$$P(s) = \frac{N(s)}{D(s)} \prod_{k=-m}^{m} \frac{1}{(s - j\omega_k)^{m_k}} \tag{24}$$

where $\omega_{-k} = -\omega_k$, N is a polynomial, D is a Hurwitz polynomial of degree $n - n_u$.

Consider the transfer function

$$C(s) = \epsilon \frac{G(s)}{H(s)} \prod_{k=0}^{m} G_k^{\epsilon}(s) \tag{25}$$

where $\epsilon > 0$ is a tuning parameter. For each $k \in \{0, 1, 2, ..., m\}$ and $s \in \mathbb{C}$, let

$$G_k^{\epsilon}(s) = \epsilon^{m_k - 1} j^{d_k} \sum_{i \in \{-k, k\}} (s + j\omega_i)^{m_k d_k} K_i\left(\frac{s - j\omega_i}{\epsilon}\right) \tag{26}$$

where $d_k = 0$ if $k = 0$ and $d_k = 1$ otherwise, $K_k(s) = \sum_{i=0}^{m_k-1} a_{k,i} s^i$ and $K_{-k}(s) = -K_k(s)$, with coefficients $a_{k,i} \in \mathbb{R}^+$ chosen as follows:
if $m_k = 1$, then $a_{k,0} = 1$ and if $m_k > 1$, then $a_{k,i}$ satisfy

$$\sum_{i=0}^{m_k-1} a_{k,i} s^{m_k - i} + \alpha = (s^2 + \beta s + \alpha)(s+1)^{m_k - 2} \tag{27}$$

Here $\alpha > 0.5\tau^{-1}$ and $\beta^2 \geq 2\alpha$ are constants. The polynomials G and H have real coefficients and are chosen such that $C(s)$ is proper, stable and for every $k \in \{0, 1, 2, ..., m\}$.

$$\angle\left(\frac{G(j\omega_k)}{H(j\omega_k)}\right) = -\angle\left(\frac{N(j\omega_k)}{D(j\omega_k)}\right) - \angle\left(\prod_{i=1, i\neq k}^{m} \frac{G_i^0(j\omega_k)}{(\omega_i^2 - \omega_k^2)^{m_i}}\right) \tag{28}$$

and for every k with $m_k > 1$,

$$\left|\frac{G(j\omega_k)}{H(j\omega_k)}\right| \geq \left|\frac{D(j\omega_k)}{N(j\omega_k)} \prod_{i=1, i\neq k}^{m} \frac{G_i^0(j\omega_k)}{(\omega_i^2 - \omega_k^2)^{m_i}}\right| \left|\frac{\omega_k^{m_0}}{G_0^0(j\omega_k)}\right|^{d_k} \tag{29}$$

Here, $G_k^0(s) = \lim_{\epsilon \to 0} G_k^{\epsilon}(s)\ \forall s \in \mathbb{C}$. Then for all $\epsilon > 0$ sufficiently small, every minimal realization (A_c, B_c, C_c, D_c) of the transfer function $C(s)$ solves the consensus problem.

Note that in (26), $i \in \{-k, k\}$ should be understood as $i \in \{0\}$ when $k = 0$. In the controller proposed above, apart from the tuning parameter ϵ, the only unknowns are the polynomials G and H which must be chosen to satisfy condition (28) and (29). In the next theorem, we present a method for choosing G and H such that they satisfy conditions (28) and (29).

There exists a proper and stable controller $C(s)$, where the polynomials G, H and $G_0^{\epsilon}, G_1^{\epsilon}, ..., G_m^{\epsilon}$ have real coefficients, G_k^{ϵ} is chosen according to Theorem 2 for each $k \in \{0, 1, ..., m\}$ and G and H satisfy magnitude and phase matching condition, such that the order of $C(s)$ is $n_u - 1$. For all $\epsilon > 0$ sufficiently small, every minimal realization (A_c, B_c, C_c, D_c) of this C solves the output feedback leader-follower consensus problem.

Proof: Choose G_k^{ϵ} for each $k \in \{0, 1, 2, ..., m\}$ according to Theorem 4.2. Note that $deg(G_k^{\epsilon} = (d_k + 1)(m_k - 1)$ and hence $\sum_{k=0}^{m} deg(G_k^{\epsilon}) = n_u - 2m - 1$. Fix a real Hurwitz polynomial H such that deg(H) $= n_u - 1$. For every $k \in \{0, 1, 2, ..., m\}$, define

$$\theta_k = -\angle\left(\frac{N(j\omega_k)}{D(j\omega_k)}\right) - \angle\left(\prod_{i=1, i\neq k}^{m} \frac{G_i^0(j\omega_k)}{(\omega_i^2 - \omega_k^2)^{m_i}}\right) \tag{30}$$

$$\Gamma_k = \left|\frac{D(j\omega_k)}{N(j\omega_k)} \prod_{i=1, i\neq k}^{m} \frac{G_i^0(j\omega_k)}{(\omega_i^2 - \omega_k^2)^{m_i}}\right| \left|\frac{\omega_k^{m_0}}{G_0^0(j\omega_k)}\right|^{d_k} \tag{31}$$

Note that the right-hand sides of (30) and (31) are well-defined, and $\theta_k \in (-\pi, \pi]$ and $0 < \Gamma_k < \infty$. Define $\phi_k = \angle H(j\omega_k) + \theta_k$ and $L_k = \Gamma_k |H(j\omega_k)|$. Let $z_k = L_k e^{j\phi_k}$ for each $k \in \{0, 1, 2, ..., m\}$. Clearly $z_0 \in \mathbb{R}$. Using Lagrange interpolation, define

$$G(s) = \sum_{i=-m}^{m} z_i \prod_{l=-m,\ l \neq i}^{m} \frac{(s - j\omega_l)}{(j\omega_i - j\omega_l)} \tag{32}$$

where z_{-k} is complex-conjugate of z_k, and $\omega_{-k} = -\omega_k$. Then G is a real polynomial with $deg(G) = 2m$ and $G(j\omega_k) = z_k$ for all $k \in \{0, 1, 2, ...m\}$. Note that $deg(G) + \sum_{k=0}^{m} deg(G_k^\epsilon) = n_u - 1$. It is easy to verify that G/H satisfies (28) & (29) and the C constructed using it is stable and proper. Therefore, from Theorem 4.2 we get that for each $\epsilon > 0$ sufficiently small, every minimal realization (A_c, B_c, C_c, D_c) of this C solves Problem 3.2 and, since the order of C is $deg(H) = n_u - 1 < n_u$, (A_c, B_c, C_c, D_c) also solves the leader-follower consensus problem.

Theorem 4.2 and 4.3 provide an algorithm for finding reduced-order controller to solve the leader-follower consensus problem in homogeneous multi-agent network.

Note: Proofs for above theorems are not discussed in this note because these are similar to Theorem 3.2 and 4.1 of [1]. For proofs, refer to the theorems of the journal cited here.

5 Numerical Example

Consider a multi-agent network without delays consisting of four identical agents. The agent dynamics (1) is a minimal realization of the transfer function

$$P(s) = \frac{s^3 + 2s^2 - 1}{(s^4 + s^2)(s + 3)^2} \tag{33}$$

So $n = 6$, $n_u = 4$, $m = 1$, $\omega_0 = 0$, $m_0 = 2$, $\omega_1 = 1$, and $m_1 = 1$. The communication graph of this network is

$$\hat{L} = L + G = \begin{bmatrix} 6 & -1 & -2 & -3 \\ -3 & 4 & -1 & 0 \\ 0 & 0 & 2 & -2 \\ -1 & -2 & -3 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 7 & -1 & -2 & -3 \\ -3 & 4 & -1 & 0 \\ 0 & 0 & 2 & -2 \\ -1 & -2 & -3 & 6 \end{bmatrix}$$

Clearly, $\sigma(\hat{L}) = \{0.0947, 3.5089, 7.6982 \pm 0.9607j\}$ and $\tau^{-1} = 10.56$. We fix $\alpha = 6$, $\beta = 3.4642$, and using Theorem 4.2 we find $K_0(s) = 3.4642s + 1$, $K_1(s) = 1$, $G_0^\epsilon = 3.4642s + \epsilon$ and $G_1^\epsilon = -2$. Next we design the transfer function $\frac{G(s)}{H(s)}$ satisfying (28) for $k \in \{0, 1\}$ and (29) for $k = 0$. We use Theorem 4.3 to find the phase and magnitude of $\frac{G(s)}{H(s)}$. We fix $H(s) = (s + 1)(s + 2)(s + 3)$ It was found that $\theta_0 = 0°$, $\theta_1 = -161.565°$, $\Gamma_0 = 4.5$ and $\Gamma_1 = 0.913$. Using θ and Γ, we can determine $\phi_0 = 0°$, $\phi_1 = -71.565°$, $L_0 = 27$, $L_1 = 9.13$. From here we determine $z_0 = 27$, $z_1 = 9.13e^{j\phi_1}$. Next we find G(s) using (32)

$$G(s) = 25.556s^2 - 4.33s + 27$$

From (25) we get

$$C(s) = \frac{-2\epsilon(\beta s + \epsilon)(25.556s^2 - 4.33s + 27)}{(s + 1)(s + 2)(s + 3)},\ \epsilon = 0.006 \tag{34}$$

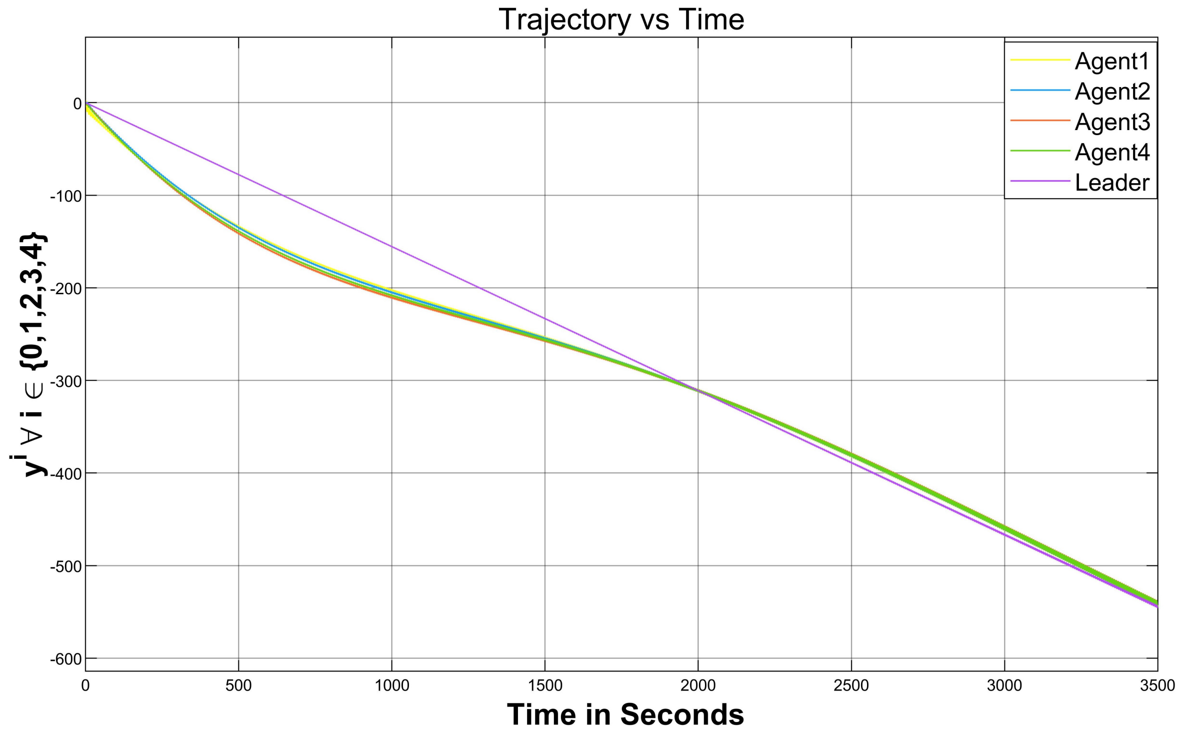


Figure 3: The output of the four agents in the multi-agent network converge to a common trajectory using the controller in (34) with $\epsilon = 0.006$.

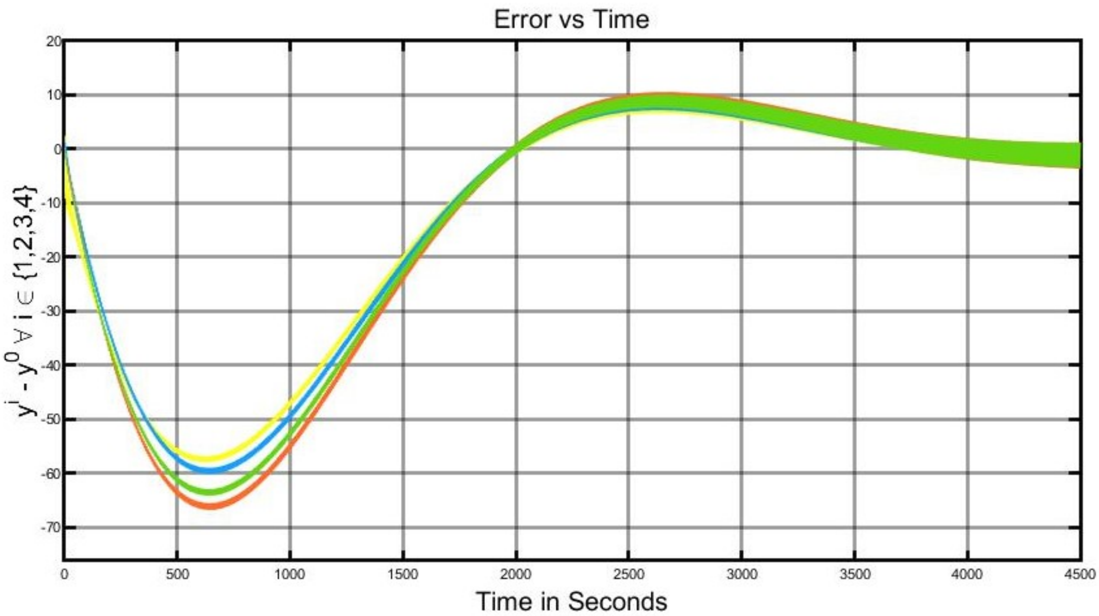


Figure 4: The output consensus error of the four agents w.r.t. to the leader

6 Conclusions

In this note, reduced-order controller developed in [1] was extended to solve the leader-follower consensus problem in a homogeneous multi-agent network with SISO agents. Simultaneous stabilisation was used to arrive at conditions under which consensus can be reached by developing a stable controller (A_c, B_c, C_c, D_c). The order of the controller (n_c) was significantly lower that the order of dynamics (n) of the agents. In particular, the order of controller was used was independent of the number of stable poles in the agent dynamics P. Numerical simulations showed that low-gain reduced-order controllers can solve the leader-follower consensus problem in a homoenous multi-agent network. We will use a similar approace to address this problem, i.e., first we sill use simulatneous stabiliaation to arrive at a condition that ensures consensus.

In future, we will try to address the problem involving a leader with different state matrix from the followers and see if it is solvable through frequency domain controller developed in [1].

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