Eigen Decomposition & PCA

An introduction to Linear Algebra (Part 2)

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Special Kinds of Matrices and Vectors

Diagonal matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Symmetric matrix

$$\begin{bmatrix} 1 & 2 & 6 \\ 2 & 8 & 4 \\ 6 & 4 & 5 \end{bmatrix}$$

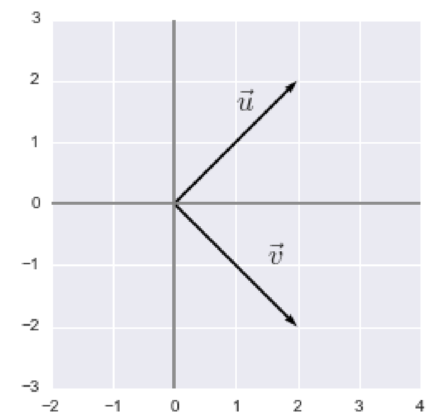
- A matrix $A_{i,j}$ is diagonal if its entries are all zeros except on the diagonal (when i=j)
- The diagonal matrix can be denoted *diag(**v**)* where **v** is the vector containing the diagonal values.
- The matrix **A** is symmetric if it is equal to its transpose. This concerns only square matrices

$$\mathbf{A}=\mathbf{A}^T$$

Special Kinds of Matrices and Vectors

- Unit vectors
  - A unit vector is a vector of length equal to 1. It can be denoted by a letter with a hat: $\hat{\boldsymbol{u}}$
- Orthogonal vectors
  - Two orthogonal vectors are separated by a 90° angle. The dot product of two orthogonal vectors gives 0.
  - Example: $\boldsymbol{x} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ $\boldsymbol{y} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$

$$\boldsymbol{x}^{\mathrm{T}}\boldsymbol{y} = \begin{bmatrix} 2 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \times 2 + 2 \times -2 \end{bmatrix} = 0$$

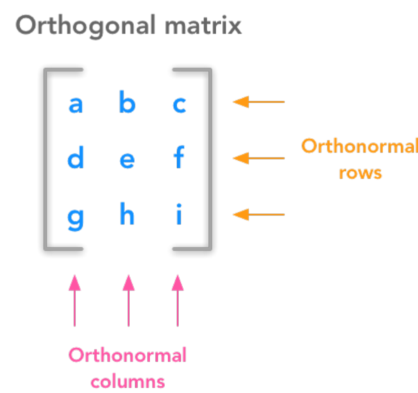


Special Kinds of Matrices and Vectors: Orthogonal matrices

- A matrix is orthogonal if columns are mutually orthogonal and have a unit norm (norm means length or magnitude of the vector) and rows are mutually orthonormal and have unit norm.

Property 1: $A^{\mathrm{T}} A = I$

Property 2: $A^{\mathrm{T}} = A^{-1}$



Special Kinds of Matrices and Vectors: Proof

$$\boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \boldsymbol{A}^{\mathrm{T}} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

$$\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} aa+cc & ab+cd \\ ab+cd & bb+dd \end{bmatrix}$$

$$= \begin{bmatrix} a^2+c^2 & ab+cd \\ ab+cd & b^2+d^2 \end{bmatrix}$$

$a^2 + c^2 = 1$ and $b^2 + d^2 = 1$. So we now have:

$$\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A} = \begin{bmatrix} 1 & ab+cd \\ ab+cd & 1 \end{bmatrix}$$

And we know that the columns are orthogonal which means that:

$$\begin{bmatrix} a & c \end{bmatrix} \begin{bmatrix} b \\ d \end{bmatrix} = 0$$

$$\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Matrices as linear transformations

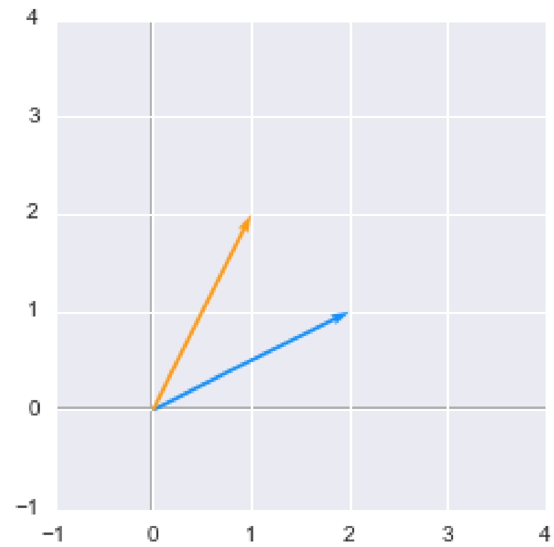
- Some matrices will rotate your space, others will rescale it.
- When we apply a matrix to a vector, we end up with a transformed version of the vector.
- When we say that we *apply* the matrix to the vector, it means that we calculate the dot product of the matrix with the vector

Example: Transformation using matrix

- Let **A**= $\begin{pmatrix} -1 & 3 \\ 2 & -2 \end{pmatrix}$

and **v**= $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

So applying **A** to **v** gives us $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$



In figure the blue line shows the older vector and orange shows the new vector

- Hence, matrix **A** transformed the vector with an angle.

Using matrices to describe the system

- If the transformation of the initial vector gives us a new vector that has the exact same direction and the scale is different , this special vector is called an *eigenvector* of the matrix.
- Mathematically, we have the following equation:

$$\boldsymbol{Av} = \lambda \boldsymbol{v}$$

- v- is the eigenvector and
- λ - is the eigenvalue

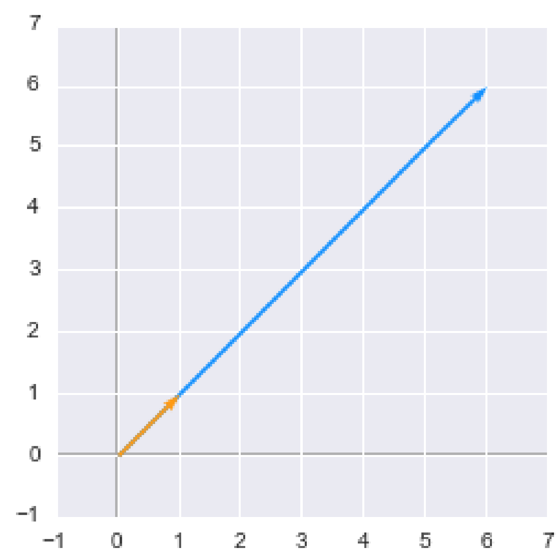
Example: Eigenvector and Eigenvalues

$$A = \begin{bmatrix} 5 & 1 \\ 3 & 3 \end{bmatrix} \qquad v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- Applying **A** to **v** gives us

$$\begin{bmatrix} 5 & 1 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$$

- Hence λ is 6



Find eigenvalues and eigenvectors in Python

- Numpy provides a function (*np.linalg.eig( )*) returning eigenvectors and eigenvalues.
- Two arrays re returned as a result from the function such that:
  - the first array corresponds to the eigenvalues and
  - the second to the eigenvectors concatenated in columns

Eigendecomposition

- Let **V** be a matrix such that all eigenvectors of a matrix **A** can be concatenated in a matrix with each column corresponding to each eigenvector.
- The vector **λ** can be created from all eigenvalues.
- Then the eigen decomposition is given by:

$$\boldsymbol{A} = \boldsymbol{V} \cdot diag(\boldsymbol{\lambda}) \cdot \boldsymbol{V}^{-1}$$

- For a symmetric matrix, the eigenvectors are orthogonal.