The Black-Scholes Equation

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Abstract

Partial differential equations (PDEs) of diffusion type are important tools for studying diffusion processes. Conversely, diffusion processes give insight into solutions of diffusion type partial differential equations. In this short article, we explain the physical phenomenon of Brownian Motion and use it to establish certain tools of Stochastic Calculus. We also introduce common terms in Finance and finally derive the Black-Scholes Equation. We also see how the Black-Scholes Equation is essentially a diffusion type PDE in a particular representation of variables and then solve it to obtain the famous Black-Scholes formula for Option Pricing. The Appendix contains some sample graphs that have been plotted in Jupyter (Python) assuming suitable values for the variables. The **Github** link for the codes has also been provided.

1. Introduction

The Heat Equation is a parabolic partial differential equation and is important in many fields of Science and Mathematics. It is connected with the study of Random Walks, Brownian Motion, the Black-Scholes Equation of Financial Mathematics and the Schrödinger Equation (Heat Equation in imaginary time). In this article, we introduce the Brownian Motion using the Heat Equation.

In the last 100 years, Brownian motion has not only revolutionized our fundamental understanding of the nature of thermal fluctuations in physical systems, but it has also explained many counterintuitive phenomena in earth and environmental sciences as well as in life sciences. In fact, the study of this phenomenon was one of the foremost applications of Stochastic Processes. Louis Bachalier was the first person to introduce the mathematical model of Brownian Motion and use it to value Stock Options. His was the first paper to use advanced mathematics in the study of Finance.

A Stochastic Process is a family of random variables. It is by nature a continuous process though it can also be expressed in discrete time using a time-series representation. The term random function is often used to describe a stochastic process because it can simply be interpreted as a random element of a function space. Stochastic processes have applications in a wide range of disciplines. Here we establish certain concepts of Stochastic Calculus and try to derive the Black-Scholes Formula using our knowledge of PDEs.

2. The Heat Equation

The heat equation is used to model things other than the flow of heat in a metal rod. It can also be used to model probability. For example, consider the flow of heat in a metal rod. Here we use $\rho(x,t)$ that gives the temperature at position x at time t. The temperature is modelled by

$$\frac{\partial \rho}{\partial t} = D\frac{\partial^2 \rho}{\partial x^2},$$

where the diffusion coefficient D depends on the material.

Now consider the initial condition $\rho(x,0) = \delta(x)$. We know that $\varphi(x,t) = \frac{1}{\sqrt{4\pi Dt}} \exp\left(-\frac{x^2}{4Dt}\right)$ satisfies the Heat Equation with the given condition where φ here is basically the Green's Function.

For a general solution, with the initial condition $\rho(x,0) = g(x)$ for $0 < t < \infty$, we can apply a convolution-

$$\rho(x,t) = \int g(y)\varphi(x-y,t)dy$$

$$\boxed{\rho(x,t) = \int_{-\infty}^{\infty} \frac{\rho(y,0)}{\sqrt{4\pi Dt}} \exp\left(-\frac{(x-y)^2}{4Dt}\right) dy} \qquad (1)$$

3. Brownian Motion

There exist collisions between the molecules of the fluid and the particles which are in suspension on it. Due to the fact that some molecules of the fluid have velocities larger than its mean thermal velocity, there is a net impulse acting on the particles in suspension. Hence, the particles in suspension move across the fluid and are braked by the viscous forces. This pattern of motion typically consists of random fluctuations in a particle's position inside a fluid sub-domain, followed by a relocation to another sub-domain. Each relocation is followed by more fluctuations within the new closed volume.

The many-body interactions that yield the Brownian pattern cannot be solved by a model accounting for every involved molecule. In consequence, we can employ only probabilistic models applied to molecular populations to describe it. Hence, we can say that Brownian Motion is a Stochastic Process.

3.1 Einstein's Theory

Let's assume that N particles start from position $x = 0$ at $t = 0$. The particles move in one-dimension. Let's consider the increment in particle positions in a time period τ to be a random variable θ with some probability density function $\varphi(\theta)$; $\varphi(\theta)$ is the probability for a jump of magnitude θ from x to $x + \theta$. Density i.e., number of particles per unit volume is denoted by $\rho(x,t)$. We also make the assumption that the **particle number is conserved**. Therefore, on Taylor expanding ρ-

$$\rho(x,t) + \tau\frac{\partial\rho(x)}{\partial t} + \cdots = \rho(x,t+\tau) = \int_{-\infty}^{\infty} \rho(x+\theta,t)\,\varphi(\theta)d\theta = \boldsymbol{E}_\theta[\rho(x+\theta,t)]$$

where $\boldsymbol{E}$ denotes the Expectation Value. Further expanding the integral term-

$$\rho(x,t) + \tau\frac{\partial\rho(x)}{\partial t} + \cdots$$
$$= \rho(x,t)\int_{-\infty}^{\infty}\varphi(\theta)d\theta + \frac{\partial\rho}{\partial x}\int_{-\infty}^{\infty}\theta\varphi(\theta)d\theta + \frac{\partial^2\rho}{\partial x^2}\int_{-\infty}^{\infty}\frac{\theta^2}{2}\varphi(\theta)d\theta + \cdots$$

The first integral on the RHS is by definition equal to 1. All the odd moments of θ vanish due to space symmetry (the integrand becomes an odd function for odd moments). Higher orders of τ are neglected since the time period is very small. Therefore, we have-

$$\frac{\partial \rho}{\partial t} = \frac{\partial^2 \rho}{\partial x^2} \int_{-\infty}^{\infty} \frac{\theta^2}{2\tau} \varphi(\theta) d\theta + (higher\ order\ moments)$$

We define $D = \int_{-\infty}^{\infty} \frac{\theta^2}{2\tau} \varphi(\theta) d\theta$ as the mass diffusivity. It is a constant. The density of Brownian particles at point x at time t satisfies the Heat Equation-

$$\frac{\partial \rho}{\partial t} = D \frac{\partial^2 \rho}{\partial x^2}$$

The second part of Einstein's theory deals with relating the constant D to physically measurable quantities but this is not relevant to our discussion. We have the initial condition-

$$\rho(x, 0) = \delta(x)$$

Therefore, the Heat Equation has the solution-

$$\rho(x, t) = \frac{N}{\sqrt{4\pi D t}} e^{-\frac{x^2}{4Dt}}$$

This is the Normal Distribution with mean $\mu = 0$ and variance $\sigma^2 = 2Dt$. From this we can directly calculate the position moments-

$\int x\rho(x, t) dx = 0$ – The first moment $\langle x \rangle$. This indicates that the Brownian particle is equally likely to move left or right.

$\int x^2 \rho(x, t) dx = 2Dt$ – The Second Moment $\langle x^2 \rangle$. From this expression Einstein argued that the displacement of a Brownian particle is not proportional to the elapsed time, but rather to its square root.

We also know from our discussion in the previous section the general solution of the Heat Equation is-

$$\rho(x, t) = \int_{-\infty}^{\infty} \frac{\rho(y, 0)}{\sqrt{4\pi D t}} e^{-\frac{(x-y)^2}{4Dt}} dy$$

By proper scaling of units, we can put $D = \frac{1}{2}$.

We have shown that the Probability Density of the Brownian Motion satisfies the Heat Equation and the joint density of x_0 and x_t (i.e., the particles start from x_0) is

$$\boxed{\rho(x_0, x_t, t) = \frac{\rho(x_0, 0)}{\sqrt{2\pi t}} e^{-\frac{(x-x_0)^2}{2t}}}$$

Suppose X_t is our Brownian Process. We have the following observations-

- $X_0 = 0$ since the particles start at the origin at $t = 0$
- Increment $\theta = X_{t+\tau} - X_t$ is a random variable: θ is independent of the past values of X_s with $s \leq t$.
- $X_{t+\tau} - X_t \sim \sqrt{2D}\mathcal{N}(0, \tau)$: X has Gaussian Increments; Normally Distributed with mean $\mu = 0$ and variance $\sigma^2 = \tau$.
- X has continuous paths i.e., X_t is continuous in t.

We see that Brownian Motion is in fact a **Wiener Process**. In mathematics, the Wiener process is a real valued continuous time stochastic process named in honour of American mathematician Norbert Wiener. It occurs frequently in pure and applied mathematics, Economics, Quantitative Finance, Evolutionary Biology, and Physics. A Wiener Process is characterized by the exact same properties listed above.

We can write for a Wiener Process W_t,

$$X_t = \sqrt{2D}W_t$$

Therefore, Wiener process is a Brownian Motion with Diffusion Constant $D = \frac{1}{2}$. The Wiener Process is continuous everywhere but nowhere differentiable. This is due to the nature of Brownian Motion. In a gas, there will be more than 10^{16} collisions per second and about 10^{20} collisions per second in a liquid. So essentially our function is not 'smooth' anywhere. It is very random. Even if we magnify the graph to look at a few milliseconds of data, we will obtain the same random nature.

$$\dot{W}(t) \sim \frac{\Delta W}{\Delta t} \sim \frac{\sqrt{\Delta t}}{\Delta t} \sim \frac{1}{\sqrt{\Delta t}},\text{ which is infinite as } \Delta t \text{ is very small.}$$

4. Certain Concepts of Stochastic Calculus

We can infer from the above discussion that for a Weiner Process W_t,

$$dW_t = \lim_{\tau \to 0}(W_{t+\tau} - W_t) \sim \sqrt{dt}$$

Hence, we have for the Quadratic Variation-

$$\boxed{\begin{gathered}[W,W](t) = dW_t\, dW_t = dt \\ dtdt \approx 0 \\ dW_t dt \approx 0\end{gathered}} \qquad (2)$$

4.1 Stochastic Differential Equation

A general Stochastic Differential Equation is written as follows,

$$dX_t = \beta(t, X_t)dt + \gamma(t, X_t)dW_t,$$

where X_t is a stochastic process. Here β and γ are given functions that represent the *drift* and *diffusion* terms respectively. A heuristic (but very helpful) interpretation of the stochastic differential equation is that in a small-time interval δ, the stochastic process changes its value by an amount that is normally distributed with expectation $\beta(t, X_t)\delta$ and variance $\gamma^2(t, X_t)\delta$ and is independent of the past behaviour of the process. This is because the increments of a Weiner Process are independent and normally distributed. X_t is called a Diffusion Process.

4.2 Itô's Lemma

Consider the diffusion process-

$$dX_t = \beta_t dt + \gamma_t dW_t$$

Let's take a twice-differentiable function $f \to f(t, x)$. Expanding in Taylor series,

$$df(t,x) = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dx + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}dx^2 + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}dt^2 + \cdots$$

Now substituting X_t for x, using equation 1 we get,

$$\boxed{df(t,X_t) = \left[\frac{\partial f}{\partial t} + \frac{\partial f}{\partial X_t}\beta_t + \frac{1}{2}\frac{\partial^2 f}{\partial {X_t}^2}{\gamma_t}^2\right]dt + \gamma_t\frac{\partial f}{\partial X_t}dW_t}$$

This is known as Itô's Lemma. The only difference from a normal derivative is the addition of the Quadratic variation term ${dW_t}^2 = dt$.

4.3 Geometric Brownian Motion

Consider a Diffusion Process-

$dX(t) = \beta X(t)dt + \gamma X(t)dW_t$, where β and γ are constants.

Now consider a function $f \to \ln(X_t)$. Applying Itô's Lemma with $\beta_t \to \beta X(t)$ and $\gamma_t \to \gamma X(t)$,

$$\frac{\partial f}{\partial t} = 0$$

$$df = \left(\beta - \frac{1}{2}\gamma^2\right)dt + \gamma dW_t$$

$$\int df = \int \left(\beta - \frac{1}{2}\gamma^2\right)dt + \int \gamma dW_t$$

The second integral on the RHS is actually an *Itô integral* which is different from our regular integral. But since γ is not a Stochastic process, the Itô Integral reduces to our regular integral. We have not defined what an Itô Integral is because its definition is much more technical and we will not encounter it ahead in our discussion.

We therefore have for the limit 0 to t,

$$\boxed{\begin{gathered}\ln\big(X(t)\big) = \ln(X_0) + \left(\beta - \frac{1}{2}\gamma^2\right)t + \gamma W_t \\ X(t) = X_0 \exp\left[\gamma W_t + \left(\beta - \frac{1}{2}\gamma^2\right)t\right]\end{gathered}}$$

This represents the Geometric Brownian Motion in which the logarithm of the randomly varying quantity $X(t)$ follows a Brownian Motion with a drift. It is an important example of stochastic processes satisfying an SDE; in particular, it is used in mathematical finance to model stock prices in the Black-Scholes Model. $X(t)$ is a log-normally distributed random variable and it always assumes positive values.

5. Common Finance Know-hows

Now that we have established the required mathematical concepts, let's look at some basic financial terms.

Stock: An asset that an individual or a corporation owns.

Shares: A stock is partitioned into shares. Shares represent a fraction of ownership in a business.

Interest: Suppose an amount P_0 is borrowed at time $t = 0$ at a fixed rate r. Then the Simple Interest is given by $I = P_0 rt$. The amount that has to be repaid at time t is given by $P(t) = (P_0 + I)$.

Compound Interest: Interest on Interest- Interest of the next period is earned on the principal sum plus the previously accumulated interest. Let's discretize the time period t into n equal time intervals. The interest rate over the whole period is a constant r. The return at time t is given by $P(t) = P_0 \left[1 + \frac{rt}{n}\right]^n, n \rightarrow number\ of\ periods$

Now for a continuous time interval,

$$P(t) = P_0 \lim_{n \to \infty} \left[1 + \frac{rt}{n}\right]^n = P_0 e^{rt}$$

Discounted Price at Time t: The value invested at time $t = 0$ so as to get a value $P(t)$ at some time t. This is given by $P_0 = P(t) \left[1 + \frac{rt}{n}\right]^{-n}$. For continuous time,

$P_0 = P(t) e^{-rt}$; P_0 is the **Discounted Price** at time t.

Market: A market is a place where buyers and sellers can meet to facilitate the exchange or transaction of goods and services.

Portfolio: A portfolio is any combination of financial assets like stocks, bonds, etc.

Risk: The potential for a financial loss. Any type of investment carries a risk. Risk is usually measure by variance.

Hedging: It is a method of reducing risk where an investment position is taken to offset and balance against the risk adopted by a companion investment.

Arbitrage: Arbitrage is the simultaneous purchase and sale of the same asset in different markets in order to profit from tiny differences in the asset's listed price.

Derivative: It is a secondary financial instrument whose price depends on the primary instrument (e.g., stocks). The price of the derivatives is derived from the price of the primary instruments as we shall see in the next section.

Option: Option is a financial derivative. It is a contract which gives the holder, the right, but not the obligation, to buy (**call option**) or sell (**put option**) an underlying asset or instrument at a specified strike price prior to or on a specified date, depending on the form of the option.

Long & Short Positions: A long position refers to the purchase of an asset with the expectation that it will increase in value. A long position in options contract indicates to the holder owning the underlying asset. A short position is the opposite; The investor sells the asset, expecting its value to fall, with a plan to buy it later

6. The Black-Scholes Equation

Consider two individuals A and B. Suppose A has a stock and its price value is $\{S(t)\}$. B intends to buy the stock from A. Both of them get into a contract- A sells a **Call Option** to B. So, B has the right but not the obligation to buy the stock from A within the expiration time of the contract at a price K. This is called the Strike Price which is fixed for the contract.

$$T \to Expiration\ Time; K \to Strike\ Price$$

Suppose anytime during the contract the market price of the stock is lesser than the strike price K, B will simply buy the stock from the market thus incurring a loss on A. Suppose the market price of the stock is greater than K, B will simply call the option and buy the stock from A at K again incurring a loss on A since A cannot sell the stock to anybody else for a greater price. So how do we ensure that A does not incur a loss?

For this, B has to pay a premium at time $t = 0$ when buying the contract. This premium is called the option price. So how should we value options?

Let's consider a simplistic market with one stock plus a bank (money market) with the following characteristics-

- Market is complete- Allows for perfect hedging- complete risk elimination.
- Market is free of arbitrage- Price discrepancies are removed. We do not consider alternate pricing of the same stock. Hence, simultaneous buying and selling without price discrepancies.
- Risk-neutral pricing- Does not matter in what security you invest (stock or bond), both have the same growth rate r. Hence you are neutral to that investment. In a real market stocks usually grow much faster than bonds (e.g., fixed deposit bank account).

Let's denote the option price by $C\big(t, S(t)\big)$. We therefore have,

$$C\big(T, S(T)\big) = \max\{S(T) - K, 0\} \text{ since price cannot be negative.}$$

This can also be read as B's gain. Since we have assumed a complete market, then for A to have no loss, the payoff to the call option at maturity must equal the worth of A's portfolio.

$$X(T) = C(T, S(T)), \text{ where } X(t) \text{ denotes the value of A's portfolio.}$$

Suppose A buys a stock S. Geometric Brownian Motion is used to model Stock Prices. This is more or less consistent with what is expected in reality. Hence, we have,

$$dS(t) = \beta S(t)dt + \gamma S(t)dW(t) \qquad \boxed{3}$$

$S_0 = S(t)e^{-rt}$; S_0 is the **Discounted Price** at time t. This is by consequence of risk-neutral pricing; Both stocks and interest received from bank grow at the same rate r. A uses the premium received to hedge against the risk adopted when buying the stock S; γ^2gives a measure of the risk.

Let's denote the following,

$$\Delta(t) \to No.\,of\ shares\ of\ stock\ S\ that\ A\ has\ bought\ at\ time\ t;$$

$$S(t) \to Cost\ of\ 1\ share\ of\ the\ stock\ S$$

Hence total worth of A's stock = $\Delta(t)S(t)$. Therefore, the amount of money left with A = $X(t) - \Delta(t)S(t)$. This money goes to the bank on which A earns interest.

At time t, $Interest = rt[X(t) - \Delta(t)S(t)]$.

Interest earned in dt time interval = $r[X(t) - \Delta(t)S(t)]dt$

Under ideal conditions, we assume that buying and selling happens instantaneously. Therefore,

$$X(t) = \Delta(t)S(t) + rt[X(t) - \Delta(t)S(t)]$$
$$\Rightarrow (worth\ of\ the\ stock\ at\ time\ t) + (interest\ earned\ at\ time\ t)$$

$$dX(t) = \Delta(t+dt)S(t+dt) - \Delta(t)S(t) + r[X(t) - \Delta(t)S(t)]dt$$

$$\Delta(t+dt) \approx \Delta(t)$$

Hence, $dX(t) = \Delta(t)dS(t) + r[X(t) - \Delta(t)S(t)]dt$

Therefore, from equation 3 we get,

$$dX(t) = [(\beta - r)\Delta(t)S(t) + rX(t)]dt + \Delta(t)\gamma S(t)dW(t)$$

Then, using Ito's Lemma the Discounted Portfolio value of A becomes,

$$d\left(e^{-rt}X(t)\right) = df\left(t, X(t)\right) = \left[\frac{\partial f}{\partial t} + \frac{\partial f}{\partial X_t}\beta_t + \frac{1}{2}\frac{\partial^2 f}{\partial {X_t}^2}{\gamma_t}^2\right]dt + \gamma_t \frac{\partial f}{\partial X_t}dW_t$$

$$\beta_t \to [(\beta - r)\Delta(t)S(t) + rX(t)]\ and\ \gamma_t \to \Delta(t)\gamma S(t)$$

$$d\left(e^{-rt}X(t)\right) = e^{-rt}[(\beta - r)\Delta(t)S(t)]dt + e^{-rt}\Delta(t)\gamma S(t)dW_t$$

Now, $d\left(e^{-rt}S(t)\right) = (\beta - r)S(t)e^{-rt}dt + e^{-rt}\gamma S(t)dW_t$ again, from Ito's Lemma

Therefore, $$\boxed{d\left(e^{-rt}X(t)\right) = \Delta(t)[e^{-rt}[(\beta - r)S(t)]dt + e^{-rt}\gamma S(t)dW_t] = \Delta(t)d\left(e^{-rt}S(t)\right)} \quad \boxed{4}$$

Now, let's consider the evolution of the Option value with time. Using Taylor expansion,

$$dC\left(t, S(t)\right) = \frac{\partial C\left(t, S(t)\right)}{\partial t} + \frac{\partial C\left(t, S(t)\right)}{\partial S_t}dS(t) + \frac{1}{2}\frac{\partial^2 C\left(t, S(t)\right)}{\partial {S_t}^2}dS(t)dS(t)$$

We know that the Quadratic Variation, $dS(t)dS(t) = \gamma^2\left(S(t)\right)^2 dt$. Therefore,

$$dC\left(t, S(t)\right) = \left[\frac{\partial C\left(t, S(t)\right)}{\partial t} + \beta\frac{\partial C\left(t, S(t)\right)}{\partial S_t}S(t) + \frac{1}{2}\frac{\partial^2 C\left(t, S(t)\right)}{\partial {S_t}^2}\gamma^2\left(S(t)\right)^2\right]dt + \gamma S(t)\frac{\partial C\left(t, S(t)\right)}{\partial S_t}dW(t)$$

Option Price is dependent on the stock price. Hence, option price will also evolve at the same rate r. Now using Ito's formula,

$$\boxed{\begin{aligned} &d\left(e^{-rt}C\big(t,S(t)\big)\right) \\ &\qquad = e^{-rt}\left[-rC\big(t,S(t)\big) + \frac{\partial C\big(t,S(t)\big)}{\partial t} + \beta\frac{\partial C\big(t,S(t)\big)}{\partial S_t}S(t) \right.\\ &\qquad \left. + \frac{1}{2}\frac{\partial^2 C\big(t,S(t)\big)}{\partial {S_t}^2}\gamma^2\big(S(t)\big)^2\right]dt + e^{-rt}\gamma S(t)\frac{\partial C\big(t,S(t)\big)}{\partial S_t}dW(t) \end{aligned}} \quad \boxed{5}$$

We want the option value and the portfolio value to evolve in the same way only then we can have perfect hedging.

$$X(T) = C\big(T,S(T)\big) \text{ and } X(0) = C\big(0,S(0)\big)$$

Therefore, equating equation 4 and equation 5,

$$d\big(e^{-rt}X(t)\big) = d\left(e^{-rt}C\big(t,S(t)\big)\right)$$

Comparing coefficients of on both sides of the equation,

$$\boxed{Coefficients\ of\ dW(t):\quad \Delta(t) = \frac{\partial C\big(t,S(t)\big)}{\partial S_t}, \forall\ t \in [0,T]} \quad \boxed{6}$$

$$\begin{aligned} &Coefficients\ of\ dt:\quad (\beta - r)S(t)\Delta(t) \\ &\qquad = \left[-rC\big(t,S(t)\big) + \frac{\partial C\big(t,S(t)\big)}{\partial t} + \beta\frac{\partial C\big(t,S(t)\big)}{\partial S_t}S(t) \right.\\ &\qquad \left. + \frac{1}{2}\frac{\partial^2 C\big(t,S(t)\big)}{\partial {S_t}^2}\gamma^2\big(S(t)\big)^2\right] \end{aligned}$$

$rC\big(t,S(t)\big) = \left[\frac{\partial C(t,S(t))}{\partial t} + rS(t)\frac{\partial C(t,S(t))}{\partial S_t} + \frac{1}{2}\frac{\partial^2 C(t,S(t))}{\partial {S_t}^2}\gamma^2\big(S(t)\big)^2\right]$, from equation 6.

We have the terminal condition $C\big(T,S(T)\big) = \max\{S(T) - K, 0\}$.

Let $S(t) = x$, ($x \geq 0$ since price cannot be negative). Therefore, we have,

$$\boxed{\frac{\partial C(t,x)}{\partial t} + rx\frac{\partial C(t,x)}{\partial x} + \frac{1}{2}\frac{\partial^2 C(t,x)}{\partial x^2}\gamma^2 x^2 - rC(t,x) = 0,\ \ \forall t \in [0,T]\ \&\ x \geq 0,}$$

This is known as the **Black-Scholes Partial Differential Equation**. The Terminal and Boundary Conditions for a Call Option are-

$$\boxed{\begin{gathered} C(T, x|_T) = \max\{x|_T - K, 0\};\ x|_T = S(T) \\ C(t,0) = 0, \forall t \in [0,T] \\ C(t,x) \to x\ as\ x \to \infty \end{gathered}}$$

7. Solving the Black-Scholes PDE

Notice that our PDE is a "backward" equation because we have considered the discounted value everywhere; We have a terminal condition instead of an initial condition. Let's convert this into a forward equation with the following variable transformation- $\tau = T - t$

$$\frac{\partial C}{\partial t} = \frac{\partial C}{\partial t}\frac{\partial \tau}{\partial \tau} = -\frac{\partial C}{\partial \tau}, \quad C \to C\big(t, S(t)\big)$$

Making further variable transformations,

$y = \ln(S) + \left(r - \frac{1}{2}\gamma^2\right)\tau$ ——— similar to the logarithm of the solution of GBM without the variance term.

and $u = Ce^{rt}$ ——— counteracting the e^{-rt} factor of the discounted value.

Representing our new initial condition (originally terminal condition) in terms of new variables- $u(0, y) = (e^y - K)^+$, The plus sign indicates that we only consider positive values.

Now, $C\big(t, S(t)\big) \stackrel{\text{def}}{=} c(\tau, y)$

$$\frac{\partial C}{\partial S} = \frac{\partial c}{\partial S}\frac{\partial y}{\partial y} = \frac{\partial c}{\partial y}\left[\frac{1}{S}\right]$$

$$\frac{\partial^2 C}{\partial S^2} = \frac{\partial}{\partial S}\frac{\partial c}{\partial S} = \frac{\partial}{\partial S}\left[\frac{\partial c}{\partial y}\left(\frac{1}{S}\right)\right] = -\frac{\partial c}{\partial y}\left(\frac{1}{S^2}\right) + \left(\frac{1}{S^2}\right)\frac{\partial^2 c}{\partial S^2}$$

$$\frac{\partial C}{\partial \tau} = \frac{\partial c}{\partial \tau} + \frac{\partial c}{\partial y}\frac{\partial y}{\partial \tau} = \frac{\partial c}{\partial \tau} + \frac{\partial c}{\partial y}\left(r - \frac{1}{2}\gamma^2\right)$$

Therefore, substituting all these equations in our PDE and simplifying, we get,

$$-\frac{\partial c}{\partial \tau} + \frac{1}{2}\gamma^2\frac{\partial^2 c}{\partial y^2} - rc = 0,$$

Now, $c = ue^{-r\tau}$. The above equation becomes,

$$\frac{\partial u}{\partial \tau} = \frac{1}{2}\gamma^2\frac{\partial^2 u}{\partial y^2}$$

Hence, we have converted the Black-Scholes PDE into the Heat Equation whose solution we already know. Therefore, the solution of the above heat equation is given by,

$$u(\tau, y) = \int_{-\infty}^{\infty} u(0, z)\frac{1}{\sqrt{2\pi\gamma^2\tau}}\exp\left(-\frac{(z-y)^2}{2\gamma^2\tau}\right)dz$$

Since we can only have positive values of $u(0, y)$, the lower limit of the integration changes to $(\ln K)$. Hence, we have,

$$u(\tau, y) = \int_{\ln(K)}^{\infty} (e^z - K)\frac{1}{\sqrt{2\pi\gamma^2\tau}}\exp\left(-\frac{(z-y)^2}{2\gamma^2\tau}\right)dz$$

$$u(\tau, y) = \int_{\ln(K)}^{\infty} e^z\frac{1}{\sqrt{2\pi\gamma^2\tau}}\exp\left(-\frac{(z-y)^2}{2\gamma^2\tau}\right)dz - \int_{\ln(K)}^{\infty} K\frac{1}{\sqrt{2\pi\gamma^2\tau}}\exp\left(-\frac{(z-y)^2}{2\gamma^2\tau}\right)dz$$

Consider the second integral first; Changing variables- $z = y - \gamma\sqrt{\tau}Z$. The limits change accordingly. We have,

$$\int_{\ln(K)}^{\infty} K\frac{1}{\sqrt{2\pi\gamma^2\tau}}\exp\left(-\frac{(z-y)^2}{2\gamma^2\tau}\right)dz = K\int_{-\infty}^{\frac{-\ln(K)+y}{\gamma\sqrt{\tau}}}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{Z^2}{2}\right)dZ$$

The integrand is analogous to the **Standard Normal Cumulative Distribution Function**. Let's represent this distribution by φ. Then we have,

$$I_2 = \int_{\ln(K)}^{\infty} K\frac{1}{\sqrt{2\pi\gamma^2\tau}}\exp\left(-\frac{(z-y)^2}{2\gamma^2\tau}\right)dz = K\varphi\left(\frac{-\ln(K)+y}{\gamma\sqrt{\tau}}\right)$$

Now let's come to the first integral.

$$\int_{\ln(K)}^{\infty} e^z\frac{1}{\sqrt{2\pi\gamma^2\tau}}\exp\left(-\frac{(z-y)^2}{2\gamma^2\tau}\right)dz$$

$$= \int_{\ln(K)}^{\infty}\frac{\exp\left[-\frac{(y^2-(y+\gamma^2\tau)^2)}{2\gamma^2\tau}\right]}{\sqrt{2\pi\gamma^2\tau}}\exp\left(-\frac{\left(z-(y+\gamma^2\tau)\right)^2}{2\gamma^2\tau}\right)dz$$

$$= e^{\left[y+\frac{\gamma^2\tau}{2}\right]}\int_{\ln(K)}^{\infty}\frac{1}{\sqrt{2\pi\gamma^2\tau}}\exp\left(-\frac{\left(z-(y+\gamma^2\tau)\right)^2}{2\gamma^2\tau}\right)dz$$

Exactly as done for the second integral, we have $y+\gamma^2\tau$ instead of just y. Hence, we get,

$$I_1 = \int_{\ln(K)}^{\infty} e^z\frac{1}{\sqrt{2\pi\gamma^2\tau}}\exp\left(-\frac{(z-y)^2}{2\gamma^2\tau}\right)dz = e^{\left[y+\frac{\gamma^2\tau}{2}\right]}\varphi\left(\frac{-\ln(K)+y+\gamma^2\tau}{\gamma\sqrt{\tau}}\right)$$

Therefore, our solution is $I_1 - I_2$,

$$u(\tau,y) = e^{\left[y+\frac{\gamma^2\tau}{2}\right]}\varphi\left(\frac{-\ln(K)+y+\gamma^2\tau}{\gamma\sqrt{\tau}}\right) - K\varphi\left(\frac{-\ln(K)+y}{\gamma\sqrt{\tau}}\right) \quad \boxed{7}$$

Now $c(\tau,y) = u(\tau,y)e^{-r\tau}$ and $y = \ln(S) + \left(r-\frac{1}{2}\gamma^2\right)\tau$. Substituting these in equation 7, we get-

$$c(\tau,S) = S(\tau)\varphi\left[\frac{\left(-\ln(K)+\ln\left(S(\tau)\right)+\left(r+\frac{1}{2}\gamma^2\right)\tau\right)}{\gamma\sqrt{\tau}}\right] - Ke^{-r\tau}\varphi\left[\frac{\left(-\ln(K)+\ln\left(S(\tau)\right)+\left(r-\frac{1}{2}\gamma^2\right)\tau\right)}{\gamma\sqrt{\tau}}\right]$$

$$C(t,S) = S(t)\varphi\left[\frac{\left(\ln\left(\frac{S(t)}{K}\right)+\left(r+\frac{1}{2}\gamma^2\right)(T-t)\right)}{\gamma\sqrt{(T-t)}}\right] - Ke^{-r(T-t)}\varphi\left[\frac{\left(\ln\left(\frac{S(t)}{K}\right)+\left(r-\frac{1}{2}\gamma^2\right)(T-t)\right)}{\gamma\sqrt{(T-t)}}\right]$$

$$\boxed{\begin{aligned} C(t,S) &= S(t)\varphi(d_1) - Ke^{-r(T-t)}\varphi(d_2) \\ \text{with } d_1 &= \frac{\left(\ln\left(\frac{S(t)}{K}\right) + \left(r + \frac{1}{2}\gamma^2\right)(T-t)\right)}{\gamma\sqrt{(T-t)}} \\ \text{and } d_2 &= \frac{\left(\ln\left(\frac{S(t)}{K}\right) + \left(r - \frac{1}{2}\gamma^2\right)(T-t)\right)}{\gamma\sqrt{(T-t)}} \end{aligned}}$$

This is the famous **Black-Scholes Formula**.

8. Concluding Remarks

The Black-Scholes Formula can also be derived using the Feynman-Kac formula. Feynman-Kac formula is used for backward parabolic PDEs similar to the form of the Black-Scholes equation with an extra added known function $f(t,x)$ which is 0 in our case. We have not used the formula here because its proof is long and difficult.

Backward Parabolic PDEs usually do not have well-posed boundary conditions and their solutions often grow unbounded in finite time or even fail to exist. But they arise repeatedly in the pricing of financial instruments.

For any general second-order parabolic PDE, the second order term describes diffusion, the first order term describes transport and the zeroth order term describes creation or depletion. In our case, we can rearrange the Black-Scholes PDE as-

$$rC - rS\frac{\partial C}{\partial S} = \frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\gamma^2 S^2$$

We make the following interpretation from the perspective of individual B- The time derivative term on the RHS indicates the time decay of the option value and the second ‘spatial’ derivative term represents the convexity of the option value with respect to the underlying stock S. On the LHS, the zeroth order term is the riskless (since we have assumed perfect hedging) return due to a long position (creation term) and the first order ‘spatial’ term represents a short position (depletion term); Equation 6 shows that $\frac{\partial C}{\partial S}$ is actually the number of shares of stock S.

References

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3. Shreve. E. Steven. Stochastic Calculus for Finance II Continuous-Time Models (Springer Finance) (v. 2).
4. Goodman. Jonathan. MATH-GA 2902.001. Courant Institute for Mathematical Sciences, New York University. Stochastic Calculus. Fall, 2012.

Appendix

Github Link- [Github Link]

Figure 1:

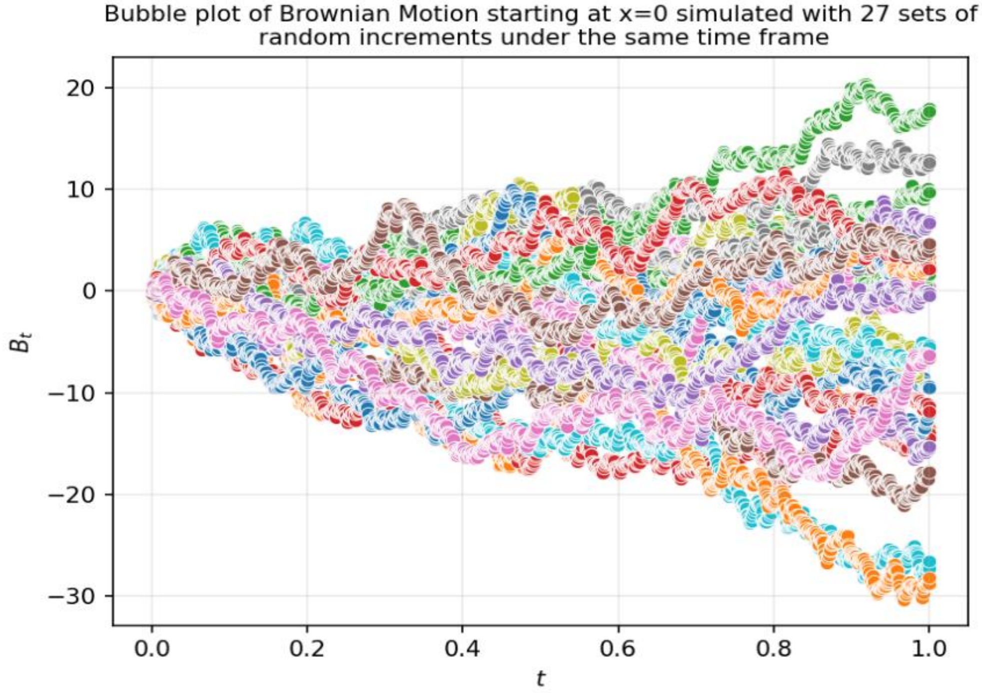
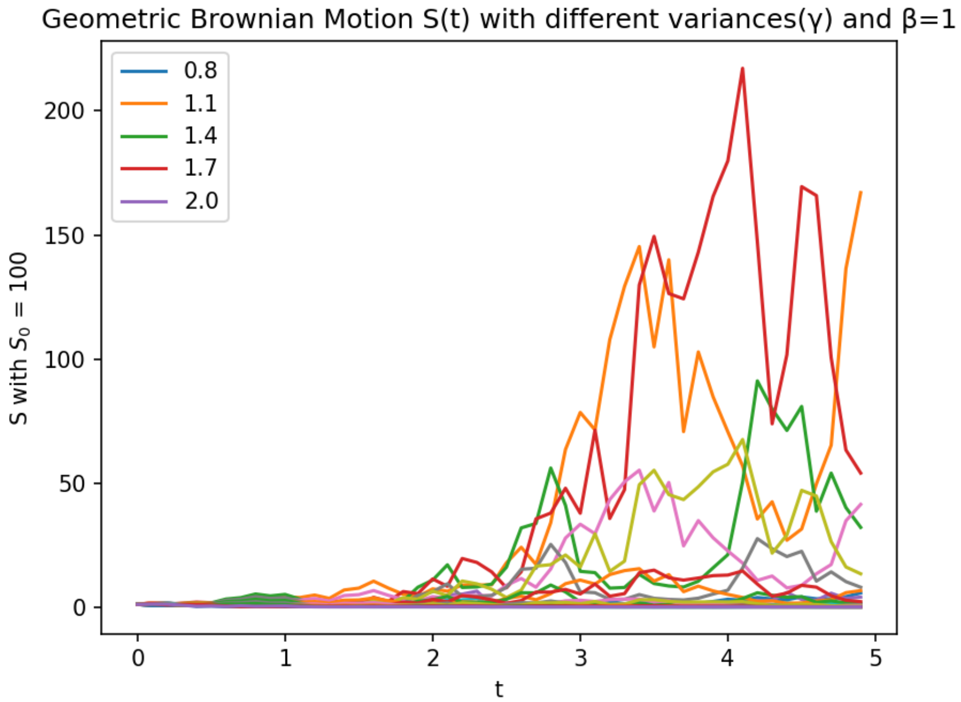


Figure 2:



<u>Figure 3:</u> For a Call Option-